Trace and extension theorems for homogeneous Sobolev and Besov spaces for unbounded uniform domains in metric measure spaces

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Dedicated to Professor O. V. Besov on the occasion of his 90th birthday.

Abstract

In this paper we fix $1 \leq p < \infty$ and consider (Ω, d, μ) be an unbounded, locally compact, non-complete metric measure space equipped with a doubling measure μ supporting a p-Poincaré inequality such that Ω is a uniform domain in its completion $\overline{\Omega}$. We realize the trace of functions in the Dirichlet-Sobolev space $D^{1,p}(\Omega)$ on the boundary $\partial\Omega$ as functions in the homogeneous Besov space $HB^{\alpha}_{p,p}(\partial\Omega)$ for suitable α; here, $\partial\Omega$ is equipped with a non-atomic Borel regular measure ν. We show that if ν satisfies a θ-codimensional condition with respect to μ for some $0 < \theta < p$, then there is a bounded linear trace operator $T : D^{1,p}(\Omega) \to HB^{1-\theta/p}(\partial\Omega)$ and a bounded linear extension operator $E : HB^{1-\theta/p}(\partial\Omega) \to D^{1,p}(\Omega)$ that is a right-inverse of T.

Key words and phrases: Besov spaces, traces, Newton-Sobolev spaces, unbounded uniform domain, doubling measure, Poincaré inequality

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1 Introduction

In investigating the extension to which Dirichlet problems on a Euclidean domain can be posed in the study of partial differential equations, O. V. Besov [3, 4] formulated the notion of Besov spaces, thus extending the work of Nikolskiĭ [34]. It was seen in [3, 4, 8, 9] and the series of papers [5, 6, 7] that for certain bounded Euclidean Lipschitz domains $\Omega \subset \mathbb{R}^n$, traces of Sobolev functions $W^{1,p}(\Omega)$ belong to the Besov space $B^{1-1/p}_{p,p}(\partial\Omega)$, where Ω denotes the boundary of Ω, see also [17]. In his papers, Besov refers to the Besov spaces as *Lipschitz-type spaces.* The subsequent work of Jonsson and Wallin [23, 24] extended this identification of certain Besov spaces as trace class of Sobolev spaces for more irregular Euclidean domains, namely uniform domains whose boundaries are Ahlfors regular sets. Strictly speaking, [23, 24] only considered traces of functions in $W^{1,p}(\mathbb{R}^n)$ to Ahlfors regular compact subsets of $\mathbb{R}^n$; however, when Ω is a uniform domain and hence is a Sobolev extension domain, we can view the results of [23, 24] in the sense described above, with the

Ahlfors regular set being the boundary of Ω. We point out here a related work of Vodopyanov and Tyulenev [38] that considers traces of $W^{1,p}$-functions on $\mathbb{R}^n$ to d-thick subsets of $\mathbb{R}^n$ when $d < n$. In [33, Chapter 10], Maz'ya gives a detailed account of Besov capacities of Euclidean sets, using both the homogeneous and inhomogeneous versions of Besov spaces. Thus, every Besov function on the boundary of such a domain is permissible as a boundary condition in the study of elliptic Dirichlet problems.

The exploration of traces of Sobolev function has been extended to the setting of metric measure spaces where the measure is doubling and supports a suitable Poincaré inequality. There, uniform domains Ω whose boundary $\partial\Omega$ satisfies a natural θ-codimensional Hausdorff measure condition for $0 < \theta < p$ were shown in [30] to satisfy the condition that the traces of functions in the Newton-Sobolev space $N^{1,p}(\Omega)$ are in the Besov class $B^{1-\theta/p}_{p,p}(\partial\Omega)$. The preprint [30], however, required the domain to be bounded. Subsequently, this result was extended for unbounded uniform domains with *bounded* boundaries in [18].

In the event that the boundary of the uniform domain is unbounded, it is natural to ask what trace theorems hold true when $N^{1,p}(\Omega)$ is replaced by its homogeneous analogue, the Dirichlet-Sobolev space $D^{1,p}(\Omega)$, and $B^{1-\theta/p}_{p,p}(\partial\Omega)$ is replaced by the homogeneous Besov space, $HB^{1-\theta/p}_{p,p}(\partial\Omega)$. The primary goal of the present paper is to address this question, the answer to which is summarized in the following, the main theorem of the present note.

Theorem 1.1. *Let (Ω, d, μ) be an unbounded, locally compact, non-complete doubling metric measure space that supports a p-Poincaré inequality for some $1 \le p < \infty$, and in addition Ω be a uniform domain in its completion $\overline{\Omega}$. Suppose also that $\partial\Omega := \overline{\Omega} \setminus \Omega$, the boundary of Ω, is unbounded and supports a non-atomic Borel regular measure ν satisfying the following θ-codimensional condition for some $0 < \theta < p$: there exists a constant $C \ge 1$ such that for each $\zeta \in \partial\Omega$ and $r > 0$, we have*

$$\frac{1}{C}\,\nu(B(\zeta, r) \cap \partial\Omega) \le \frac{\mu(B(\zeta, r) \cap \Omega)}{r^\theta} \le C\,\nu(B(\zeta, r) \cap \partial\Omega).$$

Then there is a bounded linear trace operator $T : D^{1,p}(\Omega) \to HB^{1-\theta/p}_{p,p}(\partial\Omega)$ such that we have

$$\lim_{r\to 0^+} \fint_{B(\zeta,r)\cap\Omega} |u - Tu(\zeta)|\, d\mu = 0$$

for ν-a.e. $\zeta \in \partial\Omega$ whenever $u \in D^{1,p}(\Omega)$. Moreover, there is a bounded linear extension operator $E : HB^{1-\theta/p}_{p,p}(\partial\Omega) \to D^{1,p}(\Omega)$ such that $T \circ E$ is the canonical identity operator on $HB^{1-\theta/p}_{p,p}(\partial\Omega)$.

The proof of the above theorem, adapting the technique of [30], will be in two parts; the trace part is proved in Theorem 3.8, and the extension part is proved in Theorem 4.14. The reader might also be interested in [20] for a discussion of trace classes of Hajłasz-Sobolev functions on Euclidean domains satisfying a John-type condition.

We do not know whether choice of homogeneous versions of Besov and Sobolev spaces in the above theorem can be replaced with their inhomogeneous counterparts. In [30], where Ω is bounded, the homogeneous spaces in the above theorem coincide with the corresponding inhomogeneous spaces; in this case, there is even control of the L^p-norms of the respective functions. When Ω is an unbounded uniform domain but with $\partial\Omega$ bounded, then the identity of certain *inhomogeneous* Besov classes of functions on $\partial\Omega$ as the trace of Dirichlet-Sobolev classes of functions on Ω follows from [18]. Thus, the novelty in the present work is the ability to handle the possibility that $\partial\Omega$ is unbounded.

Note that when $p = 1$, the theorem forces $0 < \theta < 1$. This is necessary as, when $\theta = 1$, the trace class of $N^{1,1}(\Omega)$ is known to be $L^1(\partial\Omega)$. Indeed, in the case that $\theta = 1$, there is a *linear* extension operator from $B^0_{1,1}(\partial\Omega)$ to $N^{1,1}(\Omega)$, but the trace operator from $N^{1,1}(\Omega)$ is onto $L^1(\partial\Omega)$, with the extension from $L^1(\partial\Omega)$ being necessarily non-linear. See [17] for the existence of such non-linear extension operator in the Euclidean setting and [35] for the proof that no bounded linear extension operator can exist in the Euclidean setting; we refer the reader to [31] for the setting of metric measure spaces. In the case that $0 < \theta < 1$, the extension operator we obtain *is* bounded and linear. For the case of Euclidean domains, there is a nice discussion of alternate definitions of trace given in [14, 15], and an accessible version of this can also be found in [33, Chapter 9.5].

Slight modifications throughout the paper show that the theorem still holds if we regard Ω as a domain living inside a larger complete metric measure space X (as opposed to $\overline{\Omega}$). Since the problem of traces in that setting can be reduced to the case that the ambient space is merely $\overline{\Omega}$, we leave this detail to the interested reader.

The link between Newton-Sobolev or Dirichlet-Sobolev spaces and the homogeneous or inhomogeneous Besov spaces give us a handy way of analyzing the behavior of potentials related to Besov energy, a non-local energy, by utilizing the now well-known behavior of potentials related to Dirichlet-Sobolev energy, see for example [16, 29]. Conversely, the identification of Besov spaces as traces of Sobolev-type spaces also gives us the limit on the type of boundary data that give rise to finite-energy solutions, on the domain, of certain Dirichlet boundary value problems, see for instance [11]. We hope the results given in this paper help further this endeavor of connecting non-local energies to local energies.

We conclude this introductory section with a brief discussion on a paper on traces of Sobolev functions in the Euclidean setting of $\mathbb{R}^n$, where more minimal assumptions are made on the Euclidean domain. In [36], Shvartsman considers Euclidean domains Ω that satisfy a so-called (A_α)–condition, and gives a characterization of traces of continuous functions in the inhomogeneous Sobolev spaces $W^{1,p}(\Omega)$ and the homogeneous Sobolev spaces, denoted in [36] as $L^1_p(\Omega)$, when p is larger than the ambient dimension, that is, $p > n$. Taking advantage of the Hölder continuity of representative functions from $W^{1,p}(\Omega)$ at sub-Whitney scales when $p > n$, a characterization of functions on $\partial\Omega$ as traces of functions in $W^{1,p}(\Omega)$ is given in [36, Theorem 1.8–1.12]. These results are different from the ones considered in the present paper as the metric on $\partial\Omega$ considered in [36] is *not* the restriction of the Euclidean metric to $\partial\Omega$ but an extension of an α-subquasihyperbolic metric on Ω to $\partial\Omega$, with such an extension well-defined because Ω satisfies the (A_α)-condition. Moreover, the norm on the trace-class considered in [36] does not yield Besov classes of functions in $\partial\Omega$, and only the range of $p > n$ is considered there. However, certain non-uniform domains such as a planar domain with an external cusp can be handled using the intriguing results of [36]. Since our goal is to preserve the metric on the boundary of the uniform domain in metric setting, with the long-term focus on studying Besov-type spaces in complete doubling metric measure spaces, we do not pursue the line of investigation indicated in [36] into the non-Euclidean realm.

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2 Preliminaries

In this section, we develop the background material needed for the remainder of the paper. In what follows, (Z, d_Z, μ_Z) is an arbitrary metric measure space unless otherwise stated.

2.1 Sobolev spaces

In a metric measure space with no *a priori* smooth structure, let alone linear structure, there is no one natural candidate for the notion of derivative. One possibility, which generalizes the fundamental theorem of calculus and exploits the geometry of curves in a metric measure space, is the notion of upper gradients, first proposed by Heinonen and Koskela [21].

We say that a non-negative Borel function g on Z is an *upper gradient* of a measurable function u on Z if

$$|u(y) - u(x)| \leq \int_\gamma g\, ds \tag{2.1}$$

holds for all rectifiable cures in Z joining x to y. The right-hand side is meant to be interpreted as infinity if at least one of $u(x)$ or $u(y)$ is infinite. Every function trivially has $g \equiv \infty$ as an upper gradient, and for each upper gradient g the function $g + \tilde{g}$ is also an upper gradient for every non-negative Borel function $\tilde{g}$. For $1 \leq p < \infty$, we say that g is a *p-weak upper gradient* of u if the collection Γ of rectifiable curves for which inequality (2.1) fails has p-modulus zero. Here, by a family Γ of curves having p-modulus zero we mean that there is a non-negative Borel function $\rho \in L^p(Z)$ such that $\int_\gamma \rho\, ds = \infty$ for each $\gamma \in \Gamma$.

Of special importance in the context of metric measure spaces are the Lipschitz functions, which, in some sense, play a similar role to that of the smooth functions in real analysis. If u is L–Lipschitz, then it is immediate that $g \equiv L$ is an upper gradient for u. For a merely locally Lipschitz function u, then its *local Lipschitz constant function*

$$\operatorname{Lip} u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_Z(y, x)}$$

is an upper gradient for u, see for example [22, Lemma 6.2.6].

For a fixed measurable function u on Z, consider the collection $D_p(u)$ of all p-weak upper gradients of u. The set $D_p(u) \cap L^p(Z)$ is a closed convex subset of $L^p(Z)$ and so, if it is non-empty, has a unique element of smallest L^p-norm. We denote this element by g_u and call it the *minimal p-weak upper gradient* of u.

We say that a measurable function u on Z is in the *Dirichlet-Sobolev space* $D^{1,p}(Z)$ for $1 \leq p < \infty$ if the following semi-norm is finite: $\|u\|_{D^{1,p}} := \|g_u\|_{L^p}$. If, in addition, u satisfies $\int_Z |u|^p\, d\mu_Z < \infty$, then u is said to be in the *Newton-Sobolev space* $N^{1,p}(Z)$ with semi-norm $\|u\|_{N^{1,p}} := \|u\|_{L^p} + \|u\|_{D^{1,p}}$.

In the context of Euclidean domains, $N^{1,p}(Z)$ corresponds to the classical Sobolev spaces $W^{1,p}(Z)$. We invite the interested reader to consult [22] for details and proofs regarding the statements made in this subsection.

2.2 Sobolev p-capacities

Let $1 \leq p < \infty$. Given a set $E \subset Z$, we set its *Sobolev p-capacity* to be the number

$$\operatorname{Cap}_p^Z(E) := \inf_{u \in \mathcal{A}(E)} \|u\|_{N^{1,p}}$$

where $\mathcal{A}(E)$ is the collection of all functions $u \in N^{1,p}(Z)$ such that $u \geq 1$ on E.

A function in $L^p(Z)$ is well-defined only up to sets of μ_Z-measure zero. Newton-Sobolev functions are more constrained, for they are well-defined up to sets of Sobolev p-capacity zero in the sense that if $u \in N^{1,p}(Z)$, then $\|u\|_{N^{1,p}} = 0$ if and only if the p-capacity of the set $\{z \in Z : u(z) \neq 0\}$ is zero, see for instance [22, Corollary 7.2.10].

2.3 Doubling property of measure

The metric measure space (Z, d_Z, μ_Z) is said to be *doubling* if there is a constant $C \geq 1$ such that for all $z \in Z$ and $r > 0$ we have

$$\mu_Z(B(z, 2r)) \leq C\, \mu_Z(B(z, r))\,.$$

Given a ball $B \subset Z$, there may be more than one choice of center z and radius r; we will assume that a generic ball B is identified together with its center and radius. For a ball $B = B(z, r)$, we will denote by τB the ball $B(z, \tau r)$ when $\tau > 0$.

The doubling property of μ_Z implies that (Z, d_Z) is a *doubling metric space*. That is, there exists a positive integer N, depending only on the doubling constant of μ_Z, such that for each $r > 0$ and $x \in Z$, every $r/2$–separated set $A \subset B(x, r)$ has at most N elements. A set being $r/2$–separated means that for each $y, z \in A$ with $y \neq z$, we have $d_Z(y, z) \geq r/2$.

2.4 Poincaré inequalities

Let $1 \leq p < \infty$. The metric measure space (Z, d_Z, μ_Z) is said to support a *p-Poincaré inequality* if there are constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset Z$ and upper gradients g of function $u \in D^{1,p}(Z)$,

$$⨍_B |u - u_B|\, d\mu_Z \leq C\, \mathrm{rad}(B) \left(⨍_{\lambda B} g^p\, d\mu_Z \right)^{1/p}.$$

Support of a Poincaré inequality implies some strong geometric connectivity properties of Z; see [22] and the references therein for more on this topic. The validity of p-Poincaré inequality automatically implies that functions in $D^{1,p}(Z)$ are necessarily in $L^1_{loc}(Z)$.

When (Z, d_Z, μ_Z) is doubling and supports a p-Poincaré inequality, a stronger version of the Lebesgue differentiation theorem is known for Newton-Sobolev functions. Recall that if μ_Z is doubling, then μ_Z-almost every point in Z is a Lebesgue point of a function $u \in L^p(Z)$. From [22, Theorem 9.2.8] for the case $p > 1$ and from [26] for the case $p = 1$ (see also [27] for a related Sobolev function-space called the Hajłasz-Sobolev space), we have the following result. Note that when $u \in D^{1,p}(Z)$, for each ball $B \subset Z$ we have that $u\,\eta_B$ is in the Newton-Sobolev class $N^{1,p}(Z)$ where η_B is a Lipschitz function on Z with support in $2B$ such that $\eta_B = 1$ on B and $0 \leq \eta_B \leq 1$.

Proposition 2.2. *If (Z, d_Z, μ_Z) is complete, doubling, and supports a p-Poincaré inequality, then for each $u \in D^{1,p}(Z)$ the set of non-Lebesgue points of u is of Sobolev p-capacity zero.*

The homogeneous space $D^{1,p}(Z)$ is, in some instances, different from $N^{1,p}(Z)$. Note that $N^{1,p}(Z) + \mathbb{R} \subset D^{1,p}(Z)$ in the sense that adding constants to functions in $N^{1,p}(Z)$ gives a function that is in $D^{1,p}(Z)$. However, $D^{1,p}(Z)$ could be larger than $N^{1,p}(Z) + \mathbb{R}$, see for example [12, Theorem 1.4, Proposition 7.3, Example 7.1]. Currently, to the best our knowledge, no potential non-trivial criteria are known that characterize when $D^{1,p}(Z) = N^{1,p}(Z) + \mathbb{R}$. A similar question for the homogeneous and inhomogeneous Besov spaces $HB^{\alpha}_{p,p}(Z)$, $B^{\alpha}_{p,p}(Z)$ can be posed; these spaces are defined in the next subsection below. The above-mentioned relationships between the homogeneous and inhomogeneous spaces have implications to existence problems related to global energy minimizers and potential theory.

2.5 Besov spaces

The study of a specific sub-class of Besov spaces was first initiated, in the context of Z being a smooth Euclidean space, by Nikolskiĭ [34] in relation to "fractional derivatives" of functions in a generalized Zygmund class. These were then extended to more general Besov classes $B^{\alpha}_{p,q}(Z)$ by O. V. Besov [3, 4]. Motivated by Dirichlet problems for Lipschitz domains in Euclidean spaces, the papers [4, 6] developed the theory of Besov spaces as traces, to the boundary of the domain, of certain Sobolev function classes on the domain. In [4], one can also find the identification of Besov spaces as interpolation spaces, interpolated between L^p and Sobolev spaces, in the context of Euclidean spaces, see also [25, 37, 39] for some discussion on this aspect of the theory. From the point of view of interpolation in the context of metric measure spaces, Besov spaces were first studied in [19]. The context of traces in the metric setting, under various limitations on the shape of the domain in the metric space, can be found in [18, 30, 31] for instance. The aim of the present note is to extend this aspect of traces to the case where both the domain and its boundary are unbounded.

From now on we only consider measures μ_Z that satisfy $0 < \mu_Z(B(z,r)) < \infty$ for each $z \in Z$ and $r > 0$. Here, and in the rest of the paper, $B(z,r) = \{w \in Z : d_Z(w,z) < r\}$. We say that a function $f \in L^p_{\mathrm{loc}}(Z)$ is in the *homogeneous Besov space* $HB^{\alpha}_{p,q}(Z)$ for $0 \le \alpha < \infty$, $1 \le p < \infty$, and $1 \le q \le \infty$ if the following semi-norm is finite:

$$\|f\|_{HB^{\alpha}_{p,q}} := \begin{cases} \left(\displaystyle\int_0^\infty \left(\int_Z \fint_{B(y,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p}} \, d\mu_Z(x) \, d\mu_Z(y) \right)^{\frac{q}{p}} \frac{dr}{r} \right)^{\frac{1}{q}}, & q < \infty \\ \displaystyle\sup_{r>0} \left(\int_Z \fint_{B(y,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p}} \, d\mu_Z(x) \, d\mu_Z(y) \right)^{\frac{1}{p}}, & q = \infty \end{cases}.$$

If, in addition, $f \in L^p(Z)$, then f is said to be in the *inhomogeneous Besov space* $B^{\alpha}_{p,q}(Z)$ with semi-norm $\|f\|_{B^{\alpha}_{p,q}} = \|f\|_{L^p} + \|f\|_{HB^{\alpha}_{p,q}}$. Note that the case $q = \infty$ is related to the so-called Korevaar-Schoen spaces, see for instance [28] or [2, Section 4]. In the present paper, we focus on the classes $B^{\alpha}_{p,p}(Z)$ for suitable choice of α, as these spaces arise in the theory of traces of Sobolev functions on Z. Such Besov spaces enjoy the following characterization, the proof of which is included for the benefit of the interested reader, see also [13, 19].

Lemma 2.3. *Assume that μ_Z is doubling and has no atoms. For $\alpha > 0$, $1 \le p < \infty$, and $f \in L^p_{loc}(Z)$,*

$$\|f\|^p_{HB^{\alpha}_{p,p}} \approx \int_Z \int_Z \frac{|f(y)-f(x)|^p}{d_Z(y,x)^{\alpha p} \mu_Z(B(y, d_Z(y,x)))} \, d\mu_Z(x) \, d\mu_Z(y) \,, \tag{2.4}$$

and, for each $C > 0$, we have

$$\|f\|^p_{HB^{\alpha}_{p,p}} \approx \sum_{l \in \mathbb{Z}} \frac{1}{2^{l\alpha p}} \int_Z \fint_{B(y, C2^l)} |f(y)-f(x)|^p \, d\mu_Z(x) \, d\mu_Z(y) \,. \tag{2.5}$$

Proof. Let $f \in L^p_{\mathrm{loc}}(Z)$. Fix $y \in Z$ and partition $(0,\infty)$ into intervals of the form $(C2^{l-1}, C2^l)$ for some $C > 0$ and $l \in \mathbb{Z}$. For $C2^{l-1} < r < C2^l$, we have, using the doubling property of μ_Z, that

$$\begin{aligned} &\frac{1}{2^{l(\alpha p+1)}} \fint_{B(y,C2^{l-1})} |f(y)-f(x)|^p \, d\mu_Z(x) \\ &\qquad \lesssim \fint_{B(y,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p+1}} \, d\mu_Z(x) \lesssim \frac{1}{2^{(l-1)(\alpha p+1)}} \fint_{B(y,C2^l)} |f(y)-f(x)|^p \, d\mu_Z(x)\,. \end{aligned} \tag{2.6}$$

Integrating the above over the interval $[C2^{l-1}, C2^l)$ with respect to dr, summing up over $l \in \mathbb{Z}$, and then applying Fubini's theorem gives (2.5).

Setting $A_i = B(y, C2^i) \setminus B(y, C2^{i-1})$, since μ_Z is non-atomic, we have that $\mu_Z(Z \setminus \bigcup_{i\in\mathbb{Z}} A_i) = 0$. Therefore, (2.6) also tells us, using the doubling property of μ_Z, that

$$\fint_{B(y,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p+1}} \, d\mu_Z(x) \approx \frac{1}{2^{l(\alpha p+1)} \mu_Z(B(y,2^l))} \sum_{i=-\infty}^{l} \int_{A_i} |f(y)-f(x)|^p \, d\mu_Z(x)\,.$$

Hence,

$$\begin{aligned} \int_0^\infty \fint_{B(y,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p}} \, d\mu_Z(x) \, \frac{dr}{r} &\approx \sum_{l\in\mathbb{Z}} \frac{1}{2^{l\alpha p} \mu_Z(B(y,2^l))} \sum_{i=-\infty}^{l} \int_{A_i} |f(y)-f(x)|^p \, d\mu_Z(x) \\ &= \sum_{i\in\mathbb{Z}} \left(\sum_{l=i}^{\infty} \frac{1}{2^{l\alpha p} \mu_Z(B(y,2^l))} \right) \int_{A_i} |f(y)-f(x)|^p \, d\mu_Z(x). \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{2^{i\,\alpha p} \mu_Z(B(y,2^i))} \le \sum_{l=i}^{\infty} \frac{1}{2^{l\alpha p} \mu_Z(B(y,2^l))} &\le \frac{1}{2^{i\alpha p} \mu_Z(B(y,2^i))} \sum_{l=i}^{\infty} \frac{1}{2^{(l-i)\alpha p}} \\ &\lesssim \frac{1}{2^{i\alpha p} \mu_Z(B(y,2^i))}\,, \end{aligned}$$

we have that

$$\begin{aligned} \int_0^\infty \fint_{B(y,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p}} \, d\mu_Z(x) \, \frac{dr}{r} &\approx \sum_{i\in\mathbb{Z}} \frac{1}{2^{i\alpha p} \mu_Z(B(y,2^i))} \int_{A_i} |f(y)-f(x)|^p \, d\mu_Z(x) \\ &\approx \sum_{i\in\mathbb{Z}} \frac{1}{2^{i\alpha p} \mu_Z(B(y,2^i))} \int_{A_i} |f(y)-f(x)|^p \, d\mu_Z(x). \end{aligned}$$

For $x \in A_i$, we have that $d_Z(y,x) \approx 2^i$ and so $\mu_Z(B(y, d_Z(y,x))) \approx \mu_Z(B(y,2^i))$ by doubling and monotonicity of measure; therefore,

$$\begin{aligned} \int_0^\infty \fint_{B(y,r)} \frac{|f(y)-f(x)|^p}{r^{\alpha p}} \, d\mu_Z(x) \, \frac{dr}{r} &\approx \sum_{i\in\mathbb{Z}} \int_{A_i} \frac{|f(y)-f(x)|^p}{d_Z(y,x)^{\alpha p} \mu_Z(B(y, d_Z(y,x)))} \, d\mu_Z(x) \\ &= \int_Z \frac{|f(y)-f(x)|^p}{d_Z(y,x)^{\alpha p} \mu_Z(B(y, d_Z(y,x)))} \, d\mu_Z(x). \end{aligned}$$

An application of Fubini's theorem then yields (2.4). $\square$

2.6 Uniform domains

Uniform domains were first introduced by Martio and Sarvas [32] in the context of quasiconformal mappings between Euclidean domains, and since then, they have been used extensively in different contexts, including quaisconformal mapping theory, potential theory, and PDEs.

If (Z, d_Z) is a complete metric space and Ω is a locally compact, non-complete domain in Z, its boundary is the set $\partial\Omega := \overline{\Omega} \setminus \Omega$, where $\overline{\Omega}$ is the metric completion of the non-complete space Ω with respect to the metric d_Z. For $z \in Z$, we set

$$d_\Omega(z) := \operatorname{dist}(z, \partial\Omega) := \min\{d_Z(z, x) \,:\, x \in \partial\Omega\}.$$

Since Ω is locally compact, it follows that Ω is open in $\overline{\Omega}$ and so $d_\Omega(z) > 0$ when $z \in \Omega$.

The domain Ω is said to be a *uniform domain* if there is a constant $A \geq 1$ such that whenever $x, y \in \Omega$, we can find a curve γ in Ω with end points x, y such that

(i) the length $\ell(\gamma) \leq A\, d_Z(x, y)$,

(ii) for each point z in the trajectory of γ we have

$$\min\{\ell(\gamma_{x,z}), \ell(\gamma_{z,y})\} \leq A\, d_\Omega(z),$$

where $\gamma_{x,z}$ denotes any segment of γ with end points x, z, and a similar interpretation for $\gamma_{z,y}$ holds.

A curve γ that satisfies the above two conditions is said to be an A-uniform curve. We also say that Ω is an A*-uniform domain* if each pair of points $x, y \in \Omega$ have an A-uniform curve in Ω connecting them as described above.

From [10] we know that if Ω is a uniform domain in a metric measure space (Z, d_Z, μ_Z) such that the metric measure space is doubling and supports a p-Poincaré inequality, then the restriction of the measure μ_Z and the metric d_Z to Ω also yields a metric measure space that is doubling and supports a p-Poincaré inequality.

2.7 Standing assumptions

Let (Ω, d, μ) be an unbounded, locally compact metric measure space such that Ω is uniform in its completion $\overline{\Omega}$. We assume (Ω, d, μ) is doubling and satisfies a p-Poincaré inequality, $1 \leq p < \infty$, and that there exists a non-atomic Borel regular measure ν on $\partial\Omega$ that is θ–codimensional to μ, $0 < \theta < p$, in the sense that there exists $C \geq 1$ for which

$$C^{-1}\frac{\mu(B(\zeta, r) \cap \Omega)}{r^\theta} \leq \nu(B(\zeta, r) \cap \partial\Omega) \leq C\frac{\mu(B(\zeta, r) \cap \Omega)}{r^\theta} \tag{2.7}$$

holds for all $\zeta \in \partial\Omega$ and $r > 0$. Note that μ being doubling on Ω implies that ν is doubling on $\partial\Omega$.

We will often consider Ω as a domain living within the metric measure space $(\overline{\Omega}, d, \mu)$, where μ is extended by zero to $\partial\Omega$. Since μ is doubling on Ω, its extension is doubling on $\overline{\Omega}$. Moreover, since Ω supports a p-Poincaré inequality, then so does $\overline{\Omega}$, see [1, Proposition 7.1]. Hence we have that $D^{1,p}(\Omega) = D^{1,p}(\overline{\Omega})$ and $N^{1,p}(\Omega) = N^{1,p}(\overline{\Omega})$.

As discussed in Subsection 2.2, under the above standing assumptions, it follows from Proposition 2.2 that for $u \in D^{1,p}(\overline{\Omega})$ the complement of the Lebesgue points of u has Sobolev p-capacity zero. Hence, by [18, Proposition 3.11, Lemma 8.1] we know that ν-almost every point in $\partial\Omega$ is a Lebesgue point of u. For greater details on the above, we refer the reader to Subsection 3.2 below.

3 On traces

In this section, we consider the trace of Dirichlet-Sobolev functions under the aforementioned standing assumptions. We can assume also without loss of generality that Ω is A-uniform domain with $A \geq 2$; see Subsection 2.6.

3.1 Constructing cones

Here we fix a parameter $\tau \geq 1$ (in our application, we will choose $\tau = \lambda$ where λ is the scaling factor on the right-hand side of the Poincaré inequality). The following construction is based on [10, Lemma 4.3]. Let $\xi, \zeta \in \partial\Omega$, and let γ be a uniform curve in Ω with end points ζ, ξ, that is, $\gamma : [0, \ell(\gamma)] \to \overline{\Omega}$ such that $\gamma(0) = \xi$, $\gamma(\ell(\gamma)) = \zeta$, and $\gamma((0, \ell(\gamma))) \subset \Omega$ is a uniform curve. Let $x_0 = \gamma(\ell(\gamma)/2)$, the mid-point of the curve γ. We focus on the subcurve γ_{ξ,x_0} of γ to construct the balls B_k for $k \geq 0$, with the similar construction for $\gamma_{x_0,\zeta}$ giving balls B_k for $k < 0$.

Let $r_0 = \frac{d_\Omega(x_0)}{16\tau}$, and set $B_0 = B(x_0, r_0)$. Next, let x_1 be the point in the trajectory of γ_{ξ,x_0} to be the last point at which γ_{ξ,x_0} leaves the ball B_0 so that γ_{ξ,x_1} does not intersect B_0. If $d_\Omega(x_1) \geq \frac{d_\Omega(x_0)}{2}$, then we choose $r_1 = r_0$; if $d_\Omega(x_1) < \frac{d_\Omega(x_0)}{2}$, then we set $r_1 = \frac{d_\Omega(x_1)}{16\tau}$. We then set $B_1 = B(x_1, r_1)$. Continuing inductively, once x_k and r_k have been selected for some positive integer k, let x_{k+1} be the last point in γ_{ξ,x_0} at which γ_{ξ,x_0} leaves $\bigcup_{j=0}^k B_k$, namely, $\gamma_{\xi,x_{k+1}}$ does not intersect $\bigcup_{j=0}^k B_k$, but $\gamma_{x_{k+1},x_k} \setminus \{x_{k+1}\} \subset \bigcup_{j=0}^k B_k$. We then set $r_{k+1} = r_k$ if $d_\Omega(x_{k+1}) \geq 8\tau r_k$, and $r_{k+1} = \frac{d_\Omega(x_{k+1})}{16\tau}$ otherwise. Note that in either case,

$$d_\Omega(x_{k+1}) \geq 8\tau r_{k+1}. \tag{3.1}$$

Now we consider the properties of the chain of balls B_k, $k = 0, 1, \ldots$. Clearly, $B_k \cap B_{k+1}$ is not empty. We fix a non-negative integer j such that $r_j = \frac{d_\Omega(x_j)}{16\tau}$, and let $k > j$ be such that $r_k = r_j$. Let $L = \ell(\gamma_{\xi,x_j})$, and $l = \ell(\gamma_{x_k,x_j})$. As $r_k = r_j$, we have that $d_\Omega(x_k) \geq 8\tau r_j = d_\Omega(x_j)/2$. It then follows from the A-uniformity of γ that

$$\frac{d_\Omega(x_j)}{2} \leq d_\Omega(x_k) \leq d(x_k, \xi) \leq L - l \leq A d_\Omega(x_j) - l.$$

Thus, $l \leq (A - \frac{1}{2}) d_\Omega(x_j)$, and so

$$k - j \leq \frac{l}{r_j} \leq (A - \tfrac{1}{2}) \frac{d_\Omega(x_j)}{d_\Omega(x_j)/(16\tau)} \leq 16\tau A. \tag{3.2}$$

If $k = j + 1$ such that $r_k \neq r_j$, then $r_k < r_j/2$, and so we have that $\lim_k r_k = 0$. Indeed, as γ_{ξ,x_0} is of finite length and terminates at the point $\xi \in \partial\Omega$, it follows from the construction of the balls that $d(x_k, \xi) \to 0$ as $k \to \infty$. Thus, for each $j \in \mathbb{N}$ there is some positive integer $k > j$ such that $d_\Omega(x_k) \leq d(x_k, \xi) < r_j$; for such k we necessarily have that $r_k < r_j/2$ and, in addition, we have that the sequence $\{r_k\}_k$ is monotone decreasing. It follows that for each positive integer n there is some positive integer $k > j$ such that $r_k \leq 2^{-n} r_j$. As each x_k lies on the curve γ_{ξ,x_0}, it follows that $\lim_k x_k = \xi$.

Next, fix a non-negative integer j for which $r_j = \frac{d_\Omega(x_j)}{16\tau}$, and let $k > j$ be the smallest integer for which $r_k \neq r_j$. Then we know that $r_k = \frac{d_\Omega(x_k)}{8\tau}$ and $d_\Omega(x_k) < \frac{d_\Omega(x_j)}{2}$, with $d_\Omega(x_{k-1}) \geq \frac{d_\Omega(x_j)}{2}$.

It follows from triangle inequality that

$$d_\Omega(x_k) \geq d_\Omega(x_{k-1}) - d(x_k, x_{k-1}) \geq \frac{d_\Omega(x_j)}{2} - r_j = \left[1 - \frac{1}{8\tau}\right] \frac{d_\Omega(x_j)}{2},$$

and so

$$\left[1 - \frac{1}{8\tau}\right] \frac{d_\Omega(x_j)}{2} \leq d_\Omega(x_k) < \frac{d_\Omega(x_j)}{2}. \tag{3.3}$$

From (3.2) and (3.3) we see that there is some constant $K > 1$ (which depends only on A and τ) such that for each $k \geq 0$ we have

$$8\tau r_k \leq d_\Omega(x_k) \leq K\, r_k. \tag{3.4}$$

By the A-uniformity of γ we also have that

$$d_\Omega(x_k) \leq d(\xi, x_k) \leq A\, d_\Omega(x_k).$$

It follows that for $x \in 4\tau B_k$, we have that

$$d_\Omega(x) \approx d_\Omega(x_k) \approx d(x, \xi) \approx d(x_k, \xi).$$

Now suppose that k and j are non-negative integers with $k > j$ such that $4\tau B_k \cap 4\tau B_j$ is non-empty and that $r_j \neq r_k$. Then by (3.1), we have

$$d_\Omega(x_j) - d_\Omega(x_k) \leq d(x_j, x_k) \leq 4\tau(r_j + r_k) \leq \frac{d_\Omega(x_j) + d_\Omega(x_k)}{2}, \tag{3.5}$$

from which we obtain $d_\Omega(x_j) \leq 3\, d_\Omega(x_k)$. Note that for positive integers m, n with $m \neq n$, it is possible to have $r_n = r_m$. As pointed out in the discussion above, if $r_m \neq r_{m-1} = r_n = \frac{d_\Omega(x_n)}{8\tau}$, then $d_\Omega(x_m) < \frac{d_\Omega(x_n)}{2}$. If in the string of positive integers between j and k we had N distinct values for r_m, $j \leq m \leq k$, then by (3.3), necessarily we have $d_\Omega(x_k) < \frac{d_\Omega(x_j)}{2^{N-1}} \leq \frac{3\, d_\Omega(x_k)}{2^{N-1}}$, and so we must have $2^{N-1} < 3$, that is, $N \leq 2$. It follows now from (3.2) that $2^{-j} \leq 2^{-k+N_0}$ for some positive integer N_0 that depends solely on A and τ, that is, $k \leq j + N_0$. Thus we have shown that if k, j are non-negative integers so that $4\tau B_j \cap 4\tau B_k$ is non-empty, then $|k - j| \leq N_0$. Combining this with (3.2) we see that there is a constant $C \geq 1$ such that

$$\sum_{k=0}^{\infty} \chi_{4\tau B_k} \leq C, \tag{3.6}$$

that is, we have a bounded overlap of the enlarged balls $4\tau B_k$.

The cones $C[\xi, \zeta]$ and $C[\zeta, \xi]$ are the sets

$$C[\xi, \zeta] := \bigcup_{k=0}^{\infty} 4\tau B_k, \qquad C[\zeta, \xi] := \bigcup_{k=0}^{\infty} 4\tau B_{-k}. \tag{3.7}$$

Note also that $d_\Omega(x_0) \approx d(\zeta, \xi)$ with the comparison constant depending solely on A.

3.2 The co-dimensional measure on $\partial\Omega$ and the existence of traces

Recall that we assume Ω to support a p-Poincaré inequality. It follows that given a function $u \in D^{1,p}(\Omega)$, and a compactly supported Lipschitz function η on Ω, the function ηu is in the inhomogeneous Sobolev class $N^{1,p}(\Omega)$. It follows from [1, Proposition 7.1] that ηu has an extension to $\partial\Omega$ such that the extended function lies in $N^{1,p}(\overline{\Omega})$. As the minimal p-weak upper gradient of a function is determined by the local behavior of the function, it follows that u itself has an extension to $\partial\Omega$ such that the extended function lies in $D^{1,p}(\overline{\Omega})$; that is, $D^{1,p}(\Omega) = D^{1,p}(\overline{\Omega})$.

From [18, Proposition 3.11] (with $U = \overline{\Omega}$ and μ_Z the measure μ, and with $A = E \subset \partial\Omega \subset U$) we know that whenever $E \subset \partial\Omega$ is a set such that the Sobolev capacity $\mathrm{Cap}_p^{\overline{\Omega}}(E) = 0$, then necessarily the codimensional Hausdorff measure of E, $\mathcal{H}^{-t}(E)$, is zero for $0 < t < p$. From [18, Lemma 8.1] we know that when $0 < \theta < p$, necessarily $\mathcal{H}^{-\theta}|_{\partial\Omega} \approx \nu$, and so it follows that $\nu(E) = 0$.

Note that we assume the codimensionality parameter θ in (2.7) to be positive. This is forced upon us by the facts that Ω is a uniform domain and μ is doubling on $\overline{\Omega}$. Indeed, for each $z \in \partial\Omega$ and $0 < r < 1$ we know that there is an A-uniform curve from any point $x \in \Omega$ that terminates at z, and so there is a point $w \in B(z, r/2) \cap \Omega$ such that $d_\Omega(w) \geq r/4$, and so $B(w, r/4) \subset B(z, r/2) \cap \Omega$, that is, $\partial\Omega$ is porous in $\overline{\Omega}$. As μ is doubling, we must have that μ-a.e. point in $\overline{\Omega}$ is a Lebesgue point of the function χ_Ω; thus, we must have $\mu(B(z, r/2) \cap \partial\Omega) = 0$. This is not possible if $\theta = 0$ in (2.7), as then we must have $\nu(\partial\Omega) = 0$. Therefore, we do not consider the case $\theta = 0 = t$ here.

Having established that $u \in D^{1,p}(\Omega)$ has an extension to a function in $D^{1,p}(\overline{\Omega})$ and that (by the Poincaré inequality on $\overline{\Omega}$) p-capacity almost every point in $\partial\Omega$ is a Lebesgue point of u, and that p-capacity zero subsets of $\partial\Omega$ are ν-null, see [22, Theorem 9.2.8], we have that ν-a.e. point in $\partial\Omega$ is a Lebesgue point of $u \in D^{1,p}(\overline{\Omega})$. Thus for each $u \in D^{1,p}(\Omega)$ there is a set $E_u \subset \partial\Omega$ with $\nu(E_u) = 0$ such that whenever $\zeta \in \partial\Omega \setminus E_u$, there is a real number, denoted $Tu(\zeta)$, such that

$$\lim_{r\to 0^+} \fint_{B(\zeta,r)\cap\Omega} |u - Tu(\zeta)|\, d\mu = 0.$$

3.3 The trace theorem

In this subsection we finally identify the trace relationship, the first part of Theorem 1.1.

Theorem 3.8. *Let $1 \leq p < \infty$ and $0 < \theta < p$. Then there is a bounded linear trace operator $T : D^{1,p}(\Omega) \to HB^{1-\theta/p}_{p,p}(\partial\Omega)$ such that when $u \in D^{1,p}(\Omega)$, we have*

$$Tu(\zeta) = \lim_{r\to 0^+} \fint_{B(\zeta,r)} u\, d\mu$$

for ν-almost every $\zeta \in \partial\Omega$.

In the above setting, we can also consider ν to be a measure on $\overline{\Omega}$ by extending ν by zero to Ω; a similar null-extension of μ to $\partial\Omega$ would allow us to simplify notation (by not needing to use $B(\zeta, r) \cap \partial\Omega$ but merely using $B(\zeta, r)$ for instance in talking about the measure ν of the balls).

Proof of Theorem 3.8. For $\zeta, \xi \in \partial\Omega$ that are μ-Lebesgue points of u, we use the chain of balls B_k, $k \in \mathbb{Z}$ from Subsection 3.1 above, with the choice of $\tau = \lambda$, where λ is the scaling constant associated with the Poincaré inequality.

Let $u \in D^{1,p}(\Omega) = D^{1,p}(\overline{\Omega})$; the discussion of Subsection 3.2 tells us that ν-almost every point $\zeta \in \partial\Omega$ is a μ-Lebesgue point of such u, and hence $Tu(\zeta)$ is well-defined. For the rest of the proof, we will continue to denote $Tu(\zeta)$ by $u(\zeta)$ as this does not give rise to conflict of notation. Then, fixing $\varepsilon > 0$ such that $\theta + \varepsilon < p$,

$$\begin{aligned}
|u(\zeta) - u(\xi)| \le \sum_{k\in\mathbb{Z}} |u_{B_k} - u_{B_{k+1}}| &\lesssim \sum_{k\in\mathbb{Z}} r_k \left(\fint_{4\lambda B_k} g_u^p \, d\mu \right)^{1/p} \\
&= \sum_{k\in\mathbb{Z}} r_k^{1-(\theta+\varepsilon)/p} \, r^{(\theta+\varepsilon)/p} \left(\fint_{4\lambda B_k} g_u^p \, d\mu \right)^{1/p} \\
&\le \left(\sum_{k\in\mathbb{Z}} r_k^{\theta+\varepsilon} \fint_{4\lambda B_k} g_u^p \, d\mu \right)^{1/p} \left(\sum_{k\in\mathbb{Z}} r_k^{\frac{p-\theta-\varepsilon}{p-1}} \right)^{1-1/p}.
\end{aligned}$$

Note that

$$\sum_{k\in\mathbb{Z}} r_k^{\frac{p-\theta-\varepsilon}{p-1}} \approx d(\xi,\zeta)^{\frac{p-\theta-\varepsilon}{p-1}} \sum_{k\in\mathbb{Z}} 2^{-|k| \, \frac{p-\theta-\varepsilon}{p-1}}.$$

Since the sum on the right-hand side of the above expression is finite (and independent of ξ, ζ), it follows that

$$\left(\sum_{k\in\mathbb{Z}} r_k^{\frac{p-\theta-\varepsilon}{p-1}} \right)^{1-1/p} \approx d(\zeta,\xi)^{1-\frac{\theta+\varepsilon}{p}}.$$

Hence,

$$|u(\zeta) - u(\xi)|^p \lesssim d(\xi,\zeta)^{p-\theta-\varepsilon} \sum_{k\in\mathbb{Z}} r_k^{\theta+\varepsilon} \fint_{4\lambda B_k} g_u^p \, d\mu \approx d(\xi,\zeta)^{p-\theta-\varepsilon} \sum_{k\in\mathbb{Z}} \frac{r_k^{\theta+\varepsilon}}{\mu(B_k)} \int_{4\lambda B_k} g_u^p \, d\mu.$$

By the codimensionality condition on $\partial\Omega$, and by the doubling property of μ, we have

$$\mu(B_k) \approx \mu(4AB_k) \approx \mu(B(\omega_k, r_k) \cap \Omega) \approx r_k^\theta \, \nu(B(\omega_k, r_k) \cap \partial\Omega),$$

where $\omega_k = \zeta$ for $k > 0$ and $\omega_k = \xi$ for $k \le 0$. It follows that

$$\frac{|u(\zeta) - u(\xi)|^p}{d(\xi,\zeta)^{p-\theta}} \lesssim d(\xi,\zeta)^{-\varepsilon} \sum_{k\in\mathbb{Z}} \frac{r_k^\varepsilon}{\nu(B(\omega_k, r_k) \cap \partial\Omega)} \int_{4\lambda B_k} g_u^p \, d\mu.$$

Setting

$$I := \sum_{k=0}^{\infty} \frac{r_k^\varepsilon}{\nu(B(\zeta, r_k) \cap \partial\Omega)} \int_{4\lambda B_k} g_u^p \, d\mu$$

and

$$II := \sum_{k=1}^{-\infty} \frac{r_k^\varepsilon}{\nu(B(\xi, r_k) \cap \partial\Omega)} \int_{4\lambda B_k} g_u^p \, d\mu,$$

from the doubling property of ν and the bounded overlap of the balls B_k, $k \in \mathbb{N}$ as observed in (3.6), we have the following:

$$I \approx \sum_{k=1}^{\infty} \int_{4\lambda B_k} \frac{d(x,\zeta)^{\varepsilon} g_u(x)^p}{\nu(B(\zeta, d(\zeta,x)) \cap \partial\Omega)} \, d\mu(x) \lesssim \int_{C[\zeta,\xi]} \frac{d(x,\zeta)^{\varepsilon} g_u(x)^p}{\nu(B(\zeta, d(\zeta,x)) \cap \partial\Omega)} \, d\mu(x),$$
$$II \approx \sum_{k=0}^{-\infty} \int_{4\lambda B_k} \frac{d(x,\xi)^{\varepsilon} g_u(x)^p}{\nu(B(\xi, d(\xi,x)) \cap \partial\Omega)} \, d\mu(x) \lesssim \int_{C[\xi,\zeta]} \frac{d(x,\xi)^{\varepsilon} g_u(x)^p}{\nu(B(\xi, d(\xi,x)) \cap \partial\Omega)} \, d\mu(x),$$

where we have used the fact that $r_k \approx d(x,\omega_k) \approx d_\Omega(x_k)$ for each $x \in 4\lambda B_k$, see (3.5) together with (3.4). We have denoted in the above $C[\zeta,\xi] = \bigcup_{k=0}^{\infty} 4\lambda B_k$ and $C[\xi,\zeta] = \bigcup_{k=1}^{-\infty} 4\lambda B_k$, as in Subsection 3.1, and used the fact that the balls $4\lambda B_k$ are of bounded overlap, see (3.6).

Now, we write

$$\frac{|u(\zeta)-u(\xi)|^p}{d(\xi,\zeta)^{p-\theta}} \lesssim \int_{C[\zeta,\xi]} \frac{d(\xi,\zeta)^{-\varepsilon} d(x,\zeta)^{\varepsilon} g_u(x)^p}{\nu(B(\zeta, d(\zeta,x)) \cap \partial\Omega)} \, d\mu(x) + \int_{C[\xi,\zeta]} \frac{d(\xi,\zeta)^{-\varepsilon} d(x,\xi)^{\varepsilon} g_u(x)^p}{\nu(B(\xi, d(\xi,x)) \cap \partial\Omega)} \, d\mu(x).$$

Hence,

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(\zeta)-u(\xi)|^p}{d(\xi,\zeta)^{p-\theta}\nu(B(\zeta,d(\zeta,\xi))\cap\partial\Omega)} \, d\nu(\zeta)\, d\nu(\xi) \lesssim E + F,$$

where

$$E := \int_{\partial\Omega}\int_{\partial\Omega}\int_{C[\zeta,\xi]} \frac{d(\xi,\zeta)^{-\varepsilon} d(x,\zeta)^{\varepsilon} g_u(x)^p}{\nu(B(\zeta,d(\zeta,x))\cap\partial\Omega)\,\nu(B(\zeta,d(\zeta,\xi))\cap\partial\Omega)} \, d\mu(x)\, d\nu(\zeta)\, d\nu(\xi)$$

and

$$F := \int_{\partial\Omega}\int_{\partial\Omega}\int_{C[\xi,\zeta]} \frac{d(\xi,\zeta)^{-\varepsilon} d(x,\xi)^{\varepsilon} g_u(x)^p}{\nu(B(\xi,d(\xi,x))\cap\partial\Omega)\,\nu(B(\zeta,d(\zeta,\xi))\cap\partial\Omega)} \, d\mu(x)\, d\nu(\zeta)\, d\nu(\xi).$$

We estimate E as follows. We first note that for $x \in \Omega$, if $x \in C[\zeta,\xi]$ then necessarily $d(\xi,\zeta) \geq d_\Omega(x)/A$ by the uniformity of the curve that was used to generate $C[\zeta,\xi]$, and, moreover, $A d_\Omega(x) \geq d(\zeta,x)$. Therefore,

$$\begin{aligned}
E &= \int_{\partial\Omega}\int_{\partial\Omega}\int_{\Omega} \frac{d(\xi,\zeta)^{-\varepsilon} d(x,\zeta)^{\varepsilon} g_u(x)^p \,\chi_{C[\zeta,\xi]}(x)}{\nu(B(\zeta,d(\zeta,x))\cap\partial\Omega)\,\nu(B(\zeta,d(\zeta,\xi))\cap\partial\Omega)} \, d\mu(x)\, d\nu(\zeta)\, d\nu(\xi) \\
&\leq \int_{\partial\Omega}\int_{\partial\Omega}\int_{\Omega} \frac{d(\xi,\zeta)^{-\varepsilon} d(x,\zeta)^{\varepsilon} g_u(x)^p \,\chi_{B(x,Ad_\Omega(x))}(\zeta)\,\chi_{\partial\Omega\setminus B(\zeta,d_\Omega(x)/A)}(\xi)}{\nu(B(\zeta,d(\zeta,x))\cap\partial\Omega)\,\nu(B(\zeta,d(\zeta,\xi))\cap\partial\Omega)} \, d\mu(x)\, d\nu(\zeta)\, d\nu(\xi) \\
&= \int_{\Omega} g_u(x)^p \int_{\partial\Omega}\int_{\partial\Omega\setminus B(\zeta,d_\Omega(x)/A)} \frac{d(\xi,\zeta)^{-\varepsilon} d(x,\zeta)^{\varepsilon}\,\chi_{B(x,Ad_\Omega(x))}(\zeta)}{\nu(B(\zeta,d(\zeta,x))\cap\partial\Omega)\,\nu(B(\zeta,d(\zeta,\xi))\cap\partial\Omega)} \, d\nu(\xi)\, d\nu(\zeta)\, d\mu(x).
\end{aligned}$$

In the last line above we used Tonelli's theorem. To estimate the inner-most integral, for each positive integer j we set

$$A_j = \big(B(\zeta, 2^j d_\Omega(x)/A) \setminus B(\zeta, 2^{j-1} d_\Omega(x)/A)\big) \cap \partial\Omega,$$

and see from the doubling property of ν that

$$\int_{\partial\Omega\setminus B(\zeta,d_\Omega(x)/A)} \frac{d(\xi,\zeta)^{-\varepsilon}}{\nu(B(\zeta,d(\zeta,\xi))\cap\partial\Omega)}\,d\nu(\xi) = \sum_{j=1}^{\infty}\int_{A_j}\frac{d(\xi,\zeta)^{-\varepsilon}}{\nu(B(\zeta,d(\zeta,\xi))\cap\partial\Omega)}\,d\nu(\xi)$$
$$\approx \sum_{j=1}^{\infty}(2^j d_\Omega(x)/A)^{-\varepsilon}$$
$$\approx d_\Omega(x)^{-\varepsilon}.$$

Hence, as $d(x,\zeta)\approx d_\Omega(x)$ when $\zeta\in B(x,Ad_\Omega(x))\cap\partial\Omega\subset(B(x,Ad_\Omega(x))\setminus B(x,d_\Omega(x)))\cap\partial\Omega$, it follows that

$$\begin{aligned} E &\lesssim \int_\Omega g_u(x)^p \int_{B(x,Ad_\Omega(x))\cap\partial\Omega}\frac{d(x,\zeta)^{\varepsilon}\,d_\Omega(x)^{-\varepsilon}}{\nu(B(\zeta,d(\zeta,x))\cap\partial\Omega)}\,d\nu(\zeta)\,d\mu(x)\\ &\approx \int_\Omega g_u(x)^p \int_{B(x,Ad_\Omega(x))\cap\partial\Omega}\frac{1}{\nu(B(\zeta,d(\zeta,x))\cap\partial\Omega)}\,d\nu(\zeta)\,d\mu(x). \end{aligned}$$

Again, by the above observation about $d(\zeta,x)$ for $\zeta\in B(x,Ad_\Omega(x))\cap\partial\Omega$, it follows that for each such ζ we have $\nu(B(\zeta,d(\zeta,x))\cap\partial\Omega)\approx\nu(B(x,2Ad_\Omega(x))\cap\partial\Omega)$ via the doubling property of ν and μ. Indeed, as $d_\Omega(x)\le d(x,\zeta)<Ad_\Omega(x)$, it follows that $B(\zeta,d(\zeta,x))\subset B(x,2Ad_\Omega(x))$ and $B(x,2Ad_\Omega(x))\subset B(\zeta,3Ad(\zeta,x))$, and so

$$\nu(B(x,2Ad_\Omega(x)))\le\nu(B(\zeta,3Ad(\zeta,x)))\approx\nu(B(\zeta,d(\zeta,x))\cap\partial\Omega)\le\nu(B(x,2Ad_\Omega(x))).$$

Hence,

$$E\lesssim\int_\Omega g_u(x)^p\,d\mu(x).$$

A similar treatment of the term F yields

$$F\lesssim\int_\Omega g_u(x)^p\,d\mu(x).$$

In conclusion, we obtain the desired estimate

$$\int_{\partial\Omega}\int_{\partial\Omega}\frac{|u(\zeta)-u(\xi)|^p}{d(\xi,\zeta)^{p-\theta}\nu(B(\zeta,d(\zeta,\xi))\cap\partial\Omega)}\,d\nu(\zeta)\,d\nu(\xi)\lesssim\int_\Omega g_u(x)^p\,d\mu(x).$$

□

4 On extensions

In this section, we consider the extension of Besov functions from the boundary to the entire domain. To construct the extension operator, we consider a partition of unity subordinate to a Whitney cover.

4.1 Whitney coverings and constructing the extension operator

Recall that we assume that $\partial\Omega$ is unbounded. It follows that for each integer n, the set $\Omega_n := \{z \in \Omega : 2^n \le d_\Omega(z) < 2^{n+1}\}$ is a non-empty set. This is because we can find pairs of points $\zeta_n, \xi_n \in \partial\Omega$ with $d(\zeta_n, \xi_n) \ge A\,2^{n+2}$, and the A-uniform curve in Ω with endpoints at ζ_n, ξ_n has to intersect Ω_n. Moreover, for each integer n and positive integer m, we can choose ζ_n, ξ_n such that $d(\zeta_n, \xi_n) \ge 3A\,2^{|n|+m}$, and it follows that there are points in Ω_n that are a distance at least 2^m from each other, that is, $\mathrm{diam}(\Omega_m) = \infty$.

Since $\partial\Omega$ is non-empty and (Ω, d) is a doubling metric space, we are able to construct a Whitney decomposition of Ω; that is, a countable collection W_Ω of balls $B_{i,j} = B(x_{i,j}, r_{i,j})$, $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, in Ω satisfying the following:

(i) $\Omega = \bigcup_{i,j} B_{i,j}$;

(ii) there exists a constant $C > 0$ such that $\sum_{i,j} \chi_{2B_{i,j}} \le C$;

(iii) for each $i \in \mathbb{Z}$, $2^{i-1} < r_{i,j} \le 2^i$ for all $j \in \mathbb{N}$;

(iv) and, for each $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $r_{i,j} = \frac{1}{8}\, d_\Omega(x_{i,j})$.

The elements of W_Ω are called Whitney balls. See for example [22, Proposition 4.1.15]; a simple modification of the proof found there yields our desired Whitney decomposition.

Remark 4.1. The constant C in (ii) above depends only on N from the definition of a doubling metric space, which in turn depends only on the doubling constant of μ. In fact, we can choose this cover in such a way that for each $\sigma \ge 1$, there is a constant N_σ which depends only on σ and the doubling constant of μ, such that for each $i \in \mathbb{Z}$ and $j \in \mathbb{N}$ there are at most N_σ many indices $k \in \mathbb{N}$ for which $\sigma B_{i,j} \cap \sigma B_{i,k}$ is non-empty.

Note also that by construction, for $x \in 2B_{i,j}$ the triangle inequality gives that

$$d_\Omega(x) \ge d_\Omega(x_{i,j}) - d(x, x_{i,j}) > 8r_{i,j} - 2r_{i,j} > 0,$$

and so $2B_{i,j} \subset \Omega$ for each $i \in \mathbb{Z}$ and $j \in \mathbb{N}$.

Lemma 4.2. *If $2B_{i,j} \cap B_{l,m} \ne \emptyset$, then $|i - l| \le 3$.*

Proof. We begin by assuming that $i > l$. Then, by the triangle inequality, and properties (iii) and (iv), we have that

$$d(x_{l,m}, x_{i,j}) \ge d_\Omega(x_{i,j}) - d_\Omega(x_{l,m}) = 8r_{i,j} - 8r_{l,m} > 2^{i+2} - 2^{l+3}.$$

This implies, still using property (iii), that

$$\mathrm{dist}(B_{l,m}, 2B_{i,j}) \ge d(x_{l,m}, x_{i,j}) - 2r_{i,j} - r_{l,m} > 2^{i+2} - 2^{l+3} - 2^{i+1} - 2^l > 2^{i+1} - 2^{l+4},$$

which is positive if $i - l > 3$.

On the other hand, when $i < l$, similar calculations yield

$$d(x_{i,j}, x_{l,m}) \ge d_\Omega(x_{l,m}) - d_\Omega(x_{i,j}) = 8r_{l,m} - 8r_{i,j} > 2^{l+2} - 2^{i+3}$$

and so

$$\operatorname{dist}(2B_{i,j}, B_{l,m}) \geq d(x_{i,j}, x_{l,m}) - 2r_{i,j} - r_{l,m} > 2^{l+2} - 2^{i+3} - 2^{i+1} - 2^{l} > 2^{l+1} - 2^{i+4},$$

which is positive if $l - i > 3$. $\square$

We now form a partition of unity subordinate to the Whitney decomposition W_Ω. Select functions $\varphi_{i,j}$ satisfying the following:

(i') $\chi_\Omega = \sum_{i,j} \varphi_{i,j}$;

(ii') for each $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $0 \leq \varphi_{i,j} \leq \chi_{2B_{i,j}}$;

(iii') and, there is a constant $C \geq 1$ such that for each $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, the function $\varphi_{i,j}$ is $C/r_{i,j}$–Lipschitz.

Given a center $x_{i,j} \in \Omega$ of the Whitney ball $B_{i,j}$, denote by $\hat{x}_{i,j}$ a closest point in $\partial\Omega$; there may be more than one such choice, but we fix one choice for each $B_{i,j}$. Then set $U_{i,j} := B(\hat{x}_{i,j}, r_{i,j}) \cap \partial\Omega$ and $U^*_{i,j} := B(\hat{x}_{i,j}, 2^8 r_{i,j})$. The number 2^8 in the construction of $U^*_{i,j}$ looks strange at this point of the discourse, but it is forced upon us in the proof in the next subsection, see for instance (4.7) and (4.8). But at this juncture, the reader will not go astray by replacing 2^8 with any large constant in visualizing $U^*_{i,j}$ and in the following lemma.

Lemma 4.3. *There is a positive integer N that depends only on the doubling constant of μ such that for each fixed $i \in \mathbb{Z}$ and for each $j \in \mathbb{N}$, there are at most N number of sets $U^*_{i,k}$, $k \in \mathbb{N}$, for which $U^*_{i,j} \cap U^*_{i,k}$ is non-empty.*

Proof. Indeed, if $U^*_{i,j}$ intersects $U^*_{i,k}$, then we have

$$d(x_{i,j}, x_{i,k}) \leq d_\Omega(x_{i,j}) + d_\Omega(x_{i,k}) + d(\hat{x}_{i,j}, \hat{x}_{i,k}) = 8(r_{i,j} + r_{i,k}) + 2^8(r_{i,j} + r_{i,k}) \leq 2^9(r_{i,j} + r_{i,k}).$$

As $r_{i,k} \leq 2^i \leq 2r_{i,j}$, it follows that $d(x_{i,j}, x_{i,k}) \leq 2^{10}\, r_{i,j}$, and hence $2^{11}B_{i,j} \cap 2^{11}B_{i,k}$ is non-empty. By the construction of the Whitney cover, it follows that there are at most $N = N_{2^{11}}$ number of such positive integers k – see Remark 4.1. $\square$

We are finally able to construct the extension operator. Beginning with a function $f \in L^1_{\mathrm{loc}}(\partial\Omega)$, we construct an extension F on Ω by writing

$$F(x) = \sum_{i,j} f_{U_{i,j}} \varphi_{i,j}(x), \tag{4.4}$$

where $U_{i,j} := B(\hat{x}_{i,j}, r_{i,j}) \cap \partial\Omega$ and $f_{U_{i,j}} := \fint_{U_{i,j}} f \, d\nu$.

4.2 The extension theorem

First, we show that the extension F of a function in $f \in HB^{1-\theta/p}_{p,p}(\partial\Omega)$ will be in $D^{1,p}(\Omega)$.

Proposition 4.5. *If $f \in HB^{1-\theta/p}_{p,p}(\partial\Omega)$, $1 \leq p < \infty$, and $0 < \theta < p$, then F is locally Lipschitz continuous in Ω, and $\|\operatorname{Lip} F\|_{L^p} \lesssim \|f\|_{HB^{1-\theta/p}_{p,p}(\partial\Omega)}$. In particular, $F \in D^{1,p}(\Omega)$ with*

$$\|F\|_{D^{1,p}} \lesssim \|f\|_{HB^{1-\theta/p}_{p,p}(\partial\Omega)}.$$

Proof. Fix a Whitney ball $B_{l,m} \in W_\Omega$. For any two points $x, y \in B_{l,m}$, we have from property (i') that

$$\sum_{i,j}(\varphi_{i,j}(y) - \varphi_{i,j}(x)) = 0,$$

and so

$$|F(y) - F(x)| = \left|\sum_{i,j} f_{U_{i,j}}(\varphi_{i,j}(y) - \varphi_{i,j}(x))\right| = \left|\sum_{i,j}(f_{U_{i,j}} - f_{U_{l,m}})(\varphi_{i,j}(y) - \varphi_{i,j}(x))\right|.$$

Denoting by $I(l,m)$ the collection of all (i,j) such that $2B_{i,j} \cap B_{l,m} \neq \emptyset$, we have from properties (ii) and (iii') that

$$|F(y) - F(x)| \le \sum_{I(l,m)} |f_{U_{i,j}} - f_{U_{l,m}}| \frac{d(y,x)}{r_{i,j}}. \tag{4.6}$$

From Lemma 4.2 we know that if $(i,j) \in I(l,m)$, then $|i - l| \le 3$. From this and (iii) it follows that

$$\min_{I(l,m)} r_{i,j} \approx r_{l,m} \approx 2^l. \tag{4.7}$$

Lemma 4.2 also implies that for $(i,j) \in I(l,m)$, $U_{i,j} \subset B(\hat{x}_{l,m}, 2^8 r_{l,m}) \cap \partial\Omega =: U^*_{l,m}$ and, from the doubling property of ν we then also have that $\nu(U^*_{l,m}) \approx \nu(U_{l,m}) \approx \nu(U_{i,j})$. Thus,

$$|f_{U_{i,j}} - f_{U_{l,m}}| \le \fint_{U_{i,j}} \fint_{U_{l,m}} |f(\zeta) - f(\xi)|\, d\nu(\xi) d\nu(\zeta) \lesssim \fint_{U^*_{l,m}} \fint_{U^*_{l,m}} |f(\zeta) - f(\xi)|\, d\nu(\xi) d\nu(\zeta). \tag{4.8}$$

Hence, for $x \in B_{l,m}$, property (ii) along with equations (4.6), (4.7), and (4.8) imply that

$$\operatorname{Lip} F(x) \lesssim \frac{1}{2^l} \fint_{U^*_{l,m}} \fint_{U^*_{l,m}} |f(\zeta) - f(\xi)|\, d\nu(\xi) d\nu(\zeta).$$

Applying the doubling property of μ along with the codimensionality condition on ν, we have that $\mu(B_{l,m}) \approx \mu(U_{l,m}) \lesssim r^\theta_{l,m} \nu(U_{l,m}) \approx 2^{l\theta} \nu(U^*_{l,m})$, and so

$$\begin{aligned}
\int_\Omega (\operatorname{Lip} F)^p\, d\mu \le \sum_{l,m} \int_{B_{l,m}} (\operatorname{Lip} F)^p\, d\mu &\lesssim \sum_{l,m} \frac{\mu(B_{l,m})}{(2^l)^p} \left(\fint_{U^*_{l,m}} \fint_{U^*_{l,m}} |f(\zeta) - f(\xi)|\, d\nu(\xi) d\nu(\zeta) \right)^p \\
&\lesssim \sum_{l,m} \frac{\nu(U^*_{l,m})}{(2^l)^{p-\theta}} \fint_{U^*_{l,m}} \fint_{U^*_{l,m}} |f(\zeta) - f(\xi)|^p\, d\nu(\xi) d\nu(\zeta) \\
&= \sum_{l,m} \frac{1}{(2^l)^{p-\theta}} \int_{U^*_{l,m}} \fint_{U^*_{l,m}} |f(\zeta) - f(\xi)|^p\, d\nu(\xi) d\nu(\zeta).
\end{aligned}$$

For $\zeta \in U^*_{l,m}$, we have $U^*_{l,m} \subset B(\zeta, 2^9 r_{l,m}) \cap \partial\Omega \subset B(\zeta, C\, 2^l) \cap \partial\Omega$ with $\nu(U^*_{l,m}) \approx \nu(B(\zeta, C\, 2^l) \cap \partial\Omega)$ from the doubling property of ν. This implies that

$$\begin{aligned}
\int_\Omega (\operatorname{Lip} F)^p\, d\mu &\lesssim \sum_{l,m} \frac{1}{(2^l)^{p-\theta}} \int_{U^*_{l,m}} \fint_{B(\zeta, C\, 2^l) \cap \partial\Omega} |f(\zeta) - f(\xi)|^p\, d\nu(\xi) d\nu(\zeta) \\
&\lesssim \sum_{l} \frac{1}{(2^l)^{p-\theta}} \int_{\partial\Omega} \fint_{B(\zeta, C\, 2^l) \cap \partial\Omega} |f(\zeta) - f(\xi)|^p\, d\nu(\xi) d\nu(\zeta)
\end{aligned}$$

using the fact that $\{U^*_{l,m}\}_m$ has bounded overlap for each fixed l, see Lemma 4.3. Finally, by (2.5),

$$\int_\Omega (\operatorname{Lip} F)^p\, d\mu \lesssim \int_0^\infty \frac{1}{r^{p-\theta}} \int_{\partial\Omega} \fint_{B(\zeta,r)\cap\partial\Omega} |f(\zeta)-f(\xi)|^p\, d\nu(\xi) d\nu(\zeta) \frac{dr}{r} \approx \|f\|^p_{HB^{1-\theta/p}_{p,p}(\partial\Omega)}.$$

□

Next, we show that this extension is really an extension.

Proposition 4.9. *If $f \in L^p_{loc}(\partial\Omega)$, $1 \le p < \infty$, then*

$$\lim_{r\to 0^+} \fint_{B(\zeta,r)\cap\Omega} |F - f(\zeta)|^p\, d\mu = 0$$

for ν-a.e. $\zeta \in \partial\Omega$. That is, the trace of F exists and equals f ν-a.e..

We will prove the above proposition using Lebesgue's differentiation theorem for the function f with respect to ν, together with the following lemmas.

Lemma 4.10. *If $f \in L^p_{loc}(\partial\Omega)$, then*

$$\int_{B_{l,m}} |F|^p\, d\mu \lesssim 2^{l\theta} \int_{U^*_{l,m}} |f|^p\, d\nu. \tag{4.11}$$

Proof. Fix a Whitney ball $B_{l,m} \in W_\Omega$. Lemma 4.2 implies that $U_{i,j} \subset U^*_{l,m}$ for all $(i,j) \in I(l,m)$ and that $\nu(U^*_{l,m}) \approx \nu(U_{i,j})$. Thus, by property (i') and by Hölder's inequality,

$$\begin{aligned}\int_{B_{l,m}} |F|^p\, d\mu = \int_{B_{l,m}} \left| \sum_{I(l,m)} \left(\fint_{U_{i,j}} f\, d\nu \right) \varphi_{i,j} \right|^p d\mu &\le \int_{B_{l,m}} \left(\left(\fint_{U^*_{l,m}} |f|\, d\nu \right) \sum_{i,j} \varphi_{i,j} \right)^p d\mu \\ &\le \mu(B_{l,m}) \fint_{U^*_{l,m}} |f|^p\, d\nu.\end{aligned}$$

As $\mu(B_{l,m}) \lesssim 2^{l\theta}\nu(U^*_{l,m})$ (this follows from the doubling property of μ and the θ-codimensionality of ν), we have the desired inequality. □

Lemma 4.12. *If $f \in L^p_{loc}(\partial\Omega)$ and $r > 0$, then*

$$\int_{B(\zeta,r)\cap\Omega} |F|^p\, d\mu \lesssim r^\theta \int_{B(\zeta,2^8 r)\cap\partial\Omega} |f|^p\, d\nu.$$

Proof. Fix $\zeta \in \partial\Omega$ and $r > 0$. Suppose that some Whitney ball $B_{i,j}$ intersects $B(\zeta, r)$. Then by the construction of the Whitney cover, we have $8r_{i,j} = d_\Omega(x_{i,j}) \le d(\zeta, x_{i,j})$, and so it follows that $r_{i,j} \le r/7$. Let $I(r)$ be the collection of all integers i for which there is some positive integer j with $B_{i,j} \cap B(\zeta, r)$ non-empty. Note that $I(r) = \{i \in \mathbb{Z} : i \le i_0\}$ for some i_0 that depends on r. For each $i \in I(r)$, denote by $\mathcal{J}(i)$ the collection of $j \in \mathbb{N}$ such that $B_{i,j} \cap B(\zeta, r) \ne \emptyset$. Then (4.11) implies that

$$\int_{B(\zeta,r)\cap\Omega} |F|^p\, d\mu \le \sum_{i\in I(r)} \sum_{j\in\mathcal{J}(i)} \int_{B_{i,j}} |F|^p\, d\mu \lesssim \sum_{i\in I(r)} \sum_{j\in\mathcal{J}(i)} 2^{i\theta} \int_{U^*_{i,j}} |f|^p\, d\nu.$$

For $i \in I(r)$ and $j \in \mathcal{J}(i)$, we have that $B_{i,j} \cap B(\zeta, r) \neq \emptyset$ and so $d(\zeta, x_{i,j}) \leq r + r_{i,j}$. Recall that for each $i \in I(r)$ and $j \in \mathcal{J}(i)$ we have

$$2^{i-1} \leq r_{i,j} \leq r/7. \tag{4.13}$$

The triangle inequality then implies that $U^*_{i,j} \subset B(\zeta, r + 2^9 r_{i,j}) \subset B(\zeta, 2^8 r)$. By the bounded overlap property of $\{U^*_{i,j}\}_{j\in\mathcal{J}(i)}$ for each fixed $i \in I(r)$, see Lemma 4.3, we have that

$$\begin{aligned}
\sum_{i\in I(r)} 2^{i\theta} \sum_{j\in\mathcal{J}(i)} \int_{U^*_{i,j}} |f|^p \, d\nu &\lesssim \sum_{i\in I(r)} 2^{i\theta} \int_{B(\zeta,2^8 r)\cap\partial\Omega} |f|^p \, d\nu \\
&= 2^{i_0\theta} \left(\sum_{i=0}^{\infty} 2^{-i\theta} \right) \int_{B(\zeta,2^8 r)\cap\partial\Omega} |f|^p \, d\nu \\
&\approx 2^{i_0\theta} \int_{B(\zeta,2^8 r)\cap\partial\Omega} |f|^p \, d\nu,
\end{aligned}$$

where $i_0 = \max I(r)$. Since for $i \in I(r)$ we have $2^{i-1} \leq r/7$, see (4.13) above, it follows that $2^{i_0\theta} \lesssim r^\theta$; the claim of the lemma now follows. $\square$

Proof of Proposition 4.9. Fix $\zeta \in \partial\Omega$ and write $f_\zeta(\xi) = f(\xi) - f(\zeta)$ for $\xi \in \partial\Omega$. We have that $f_\zeta \in L^p_{\mathrm{loc}}(\partial\Omega)$ and its extension F_ζ satisfies $F_\zeta(x) = F(x) - f(\zeta)$ for every $x \in \Omega$. An application of Lemma 4.12 to f_ζ yields

$$\begin{aligned}
\int_{B(\zeta,r)\cap\Omega} |F(x) - f(\zeta)|^p \, d\mu(x) &= \int_{B(\zeta,r)\cap\Omega} |F_\zeta(x)|^p \, d\mu(x) \\
&\lesssim r^\theta \int_{B(\zeta,2^8 r)\cap\partial\Omega} |f_\zeta(\xi)|^p \, d\nu(\xi) = r^\theta \int_{B(\zeta,2^8 r)\cap\partial\Omega} |f(\xi) - f(\zeta)|^p \, d\nu(\xi).
\end{aligned}$$

From the doubling and codimensionality of ν it follows that

$$\fint_{B(\zeta,r)\cap\Omega} |F(x) - f(\zeta)|^p \, d\mu(x) \lesssim \fint_{B(\zeta,r)\cap\partial\Omega} |f(\xi) - f(\zeta)|^p \, d\nu(\xi).$$

By the local p-integrability of f, ν-almost every $\zeta \in \partial\Omega$ is a Lebesgue point of f, and so the right-hand side of the above inequality tends to 0 as $r \to 0^+$ for ν-almost every $\zeta \in \partial\Omega$. $\square$

Now we are ready to prove the second part of Theorem 1.1.

Theorem 4.14. *Let $1 \leq p < \infty$ and $0 < \theta < p$. There is a bounded linear extension operator $E : HB^{1-\theta/p}_{p,p}(\partial\Omega) \to D^{1,p}(\Omega)$ such that $T \circ E$ is the identity map on $HB^{1-\theta/p}_{p,p}(\partial\Omega)$, where T is the trace operator constructed in the proof of Theorem 3.8.*

Proof. For $f \in HB^{1-\theta/p}_{p,p}(\partial\Omega)$, take $Ef = F$, where F is as in (4.4). Then E is linear by construction and is bounded from $HB^{1-\theta/p}_{p,p}(\partial\Omega)$ to $D^{1,p}(\Omega)$ by Proposition 4.5. Consider the trace operator T from Theorem 3.8. Then $T \circ Ef = TF = f$ ν-almost everywhere by Proposition 4.9. $\square$

This completes the proof of the main theorem of this note, Theorem 1.1.

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