BV functions and fractional Laplacians on Dirichlet spaces

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December 28, 2022

The authors dedicate this paper to the memory of Ka-Sing Lau, whose important contributions to the general theory of Markov processes and Dirichlet forms on fractals and metric measure spaces, especially in the papers [43, 44, 48, 49], influenced the development of this work.

Abstract

We study bounded variation (BV) and fractional Sobolev functional spaces, L^p Besov critical exponents and isoperimetric and Sobolev inequalities associated with fractional Laplacians on metric measure spaces. The main tool is the theory of heat semigroup based Besov classes in Dirichlet Metric Measure Spaces that was introduced by the authors in previous works.

Contents

[*] P.A.R. was partly supported by the NSF grants DMS 1855349, 1951577, 2140664.
[†] F.B. was partly supported by the NSF grants DMS 660031, 901315 and the Simons Foundation.
[‡] L.C. was partly supported by Simons collaboration grant #853249.
[§] L.R. was partly supported by the NSF grants DMS 1659643, 1950543.
[¶] N.S. was partly supported by the NSF grant DMS 2054960.
[‖] A.T. was partly supported by the NSF grant DMS 1613025 and the Simons Foundation.

1 Introduction

Two classical characterizations of the space of bounded variation functions (BV) have been used to define BV on more general metric spaces: Caccioppoli's characterization via relaxed convergence of Sobolev functions was used by Miranda [57] in the setting of metric measure spaces from the point of view of energy given by the local Lipschitz "constant" functions related to Lipschitz continuous functions, while a version of the characterization via heat semigroups due to De Giorgi [33] (see also Ledoux [55]) was used by the present authors in the setting of metric measure spaces admitting a Dirichlet form with suitable estimates [5]. In the latter case, one of the properties assumed for the Dirichlet form was locality.

The first purpose of the present work is to show that a BV space can also be characterized using a non-local form, namely that obtained by subordination of a local form like that studied in [5], for a range of values of the subordination parameter δ. The second is to investigate the properties of the fractional Sobolev space which arises when the subordination parameter δ is outside that range. This approach is complementary to existing work on fractional Sobolev spaces and BV functions in various contexts, including manifolds and Carnot groups, domains with fractal boundary as in [52, 56, 65], and the distributional approach in [31]. We hope that the approach may steer further connections to research in PDE, for example as in [34].

Our setting is a locally compact metric space (X, d) with a Radon measure μ and a strongly local regular Dirichlet form $\mathcal{E}$ corresponding to a Laplacian L for which the associated heat flow $P_t = e^{tL}$ admits a kernel $p_t(x, y)$ with sub-Gaussian estimates having parameters d_H (Hausdorff dimension) and d_W (walk dimension). Definitions and further details are provided in Section 2. The study of function spaces in this context is an active area of research, with substantial published literature, see for instance [2, 9–11, 13–19, 36, 41, 42, 45, 46, 51, 53, 54, 58, 59, 63].

In what follows, we consider the non-local fractional Laplacian $(-L)^\delta$ for some $0 < \delta < 1$ and the corresponding subordinated semigroup $P_t^{(\delta)}$ with generator $p_t^{(\delta)}$. The heat semigroup Besov spaces $\mathbf{B}_\delta^{p,\alpha}(X)$ are defined via finiteness of the seminorm

$$\|f\|_{\delta,p,\alpha} := \sup_{t>0} t^{-\alpha} \left(\int_X \int_X |f(x) - f(y)|^p p_t^{(\delta)}(x, y) d\mu(y) d\mu(x) \right)^{1/p}. \tag{1}$$

If in addition P_t satisfies the weak Bakry-Émery estimate (6), then also the subordinated semigroup does. Namely, there is some $\kappa > 0$ such that for all $x, y \in X$ and $f \in L^\infty(X, \mu)$

$$|P_t^{(\delta)} f(x) - P_t^{(\delta)} f(y)| \leq C \frac{d(x,y)^\kappa}{t^{\frac{\kappa}{\delta d_W}}} \|f\|_{L^\infty(X,\mu)},$$

c.f. Lemma 2.8. When $\lambda_1^\# = d_W - \kappa$, then it is natural to define BV as the Korevaar-Schoen space $\mathrm{KS}^{\lambda_1^\#,1}(X)$, where, following [50], $\mathrm{KS}^{\lambda,p}$ is the subspace of L^p determined by finiteness of the seminorm

$$\limsup_{r \to 0^+} \left(\int_X \int_{B(x,r)} \frac{|f(x) - f(y)|^p}{r^{\lambda p} \mu(B(x,r))} d\mu(y) d\mu(x) \right)^{1/p}, \tag{2}$$

and $\lambda_p^\#$ is the critical exponent defined as

$$\lambda_p^\# := \sup\{\lambda : \mathrm{KS}^{\lambda,p}(X) \text{ contains non-constant functions}\}. \tag{3}$$

As we will see in Section 5, the space $\mathrm{KS}^{\lambda_1^\#,1}$ enjoys many classical BV properties, including Sobolev embedding, isoperimetric inequalities, and a co-area formula, as well as an interpretation of the variation as a "BV measure". These properties will be established by comparing the seminorm (2) with the heat Besov seminorm (1) for $\alpha = \lambda/(\delta d_W)$ and fixed $0 < \delta < 1$, and the corresponding critical exponents $\lambda_p^\#$ of the Korevaar-Schoen spaces and $\alpha_{\delta,p}^\#$ of the heat Besov spaces. Similar associations were studied in [3] for the non-fractional version of the heat Besov spaces, where $p_t^{(\delta)}$ was replaced by the standard heat kernel p_t associated with $\mathcal{E}$.

In Section 3 we classify the spaces $\mathbf{B}_\delta^{p,\alpha}(X)$ in terms of the parameter α: when $\alpha < \frac{1}{p}$ they agree with Korevaar-Schoen spaces, when $\alpha = \frac{1}{p}$ they are fractional Sobolev spaces, and when $\alpha > \frac{1}{p}$ they are trivial. We also compare them to metric Besov spaces studied by other authors.

Typically the smaller the Besov spaces, the more useful they are. The smallest in the present context are those that contain the smoothest functions subject to a non-triviality condition leading to the study of the spaces $\mathbf{B}_\delta^{p,\alpha_{p,\delta}^\#}(X)$ with the critical exponent

$$\alpha_{p,\delta}^\# = \sup\{\alpha : \mathbf{B}_\delta^{p,\alpha}(X) \text{ contains non-constant functions}\}.$$

The results of Section 3 ensure $\alpha_{p,\delta}^\# \leq \frac{1}{p}$ so that the critical heat Besov space is either a Korevaar-Schoen space or a fractional Sobolev space. In general it may be difficult to determine the critical exponent $\alpha_{p,\delta}^\#$, but we show in Section 4 that the a-priori assumption of a weak Bakry-Émery inequality for the original (non-subordinated) semigroup provides estimates for this exponent, and determines it completely in the case $p = 1$.

The preceding results provide circumstances under which we can identify the heat Besov space $\mathbf{B}_\delta^{1,\alpha_{1,\delta}^\#}(X)$ for the subordinated semigroup with the critical Korevaar Schoen space $\mathrm{KS}^{\lambda_1^\#,1}(X)$ which we consider to be the BV function space. Specifically, we need $\lambda_1^\# = d_W - \kappa$ so that $\mathrm{KS}^{\lambda_1^\#,1}(X)$ is BV by the results of [5], and $\delta > 1 - \frac{\kappa}{d_W}$ to ensure this latter coincides with the critical heat Besov space, see Theorem 5.3. A point of particular interest is that this characterization is independent of the subordination parameter δ in the interval

$(\lambda_1^{\#}/d_W, 1) = (1-\kappa/d_W, 1)$. Intuitively, this result says that although being in BV appears to be a local property and was initially defined in terms of a local Dirichlet form and semigroup, we can recover it from the non-local fractional Laplacian operator and semigroup, as long as the latter it is not "too non-local" (meaning $\delta > \lambda_1^{\#}/d_W$). This property of stability in the subordination parameter is consistent with the possibility of a non-trivial connection between the condition $\lambda_1^{\#}/d_W = 1-\kappa/d_W$ and the topological and metric structure of the space. In Conjecture 5.4 of [5] the authors suggest that in some examples this quantity may depend not only on the Hausdorff and walk dimensions but also on the topological Hausdorff dimension defined in [8].

The characterization of BV by the critical heat Besov space $\mathbf{B}_\delta^{1,\alpha_{1,\delta}^{\#}}(X)$ when $\delta > 1-\frac{\kappa}{d_W}$ is presented in Section 5.1, where we use results of [5] to express properties of BV in terms of $\mathbf{B}_\delta^{1,\alpha_{1,\delta}^{\#}}$ and the subordinated heat Besov seminorm, including the notions of variation, co-area formula, Sobolev and isoperimetric inequalities and existence of BV measures. Section 5.2 contains the corresponding results for the fractional Sobolev space that arises when the subordination parameter instead satisfies $\delta \le 1-\frac{\kappa}{d_W}$. In this case, one has $\alpha_{\delta,1}^{\#} = 1$ and $\mathbf{B}_\delta^{1,1}(X)$ is an instance of the classical fractional Sobolev space $W^{\lambda,p}(X)$ defined by Gagliardo in [39] for $X = \mathbb{R}^n$, whose seminorm is

$$\|f\|_{W^{\lambda,p}} := \left(\int_X \int_X \frac{|f(x)-f(y)|^p}{d(x,y)^{d_H+\lambda p}} d\mu(x)d\mu(y)\right)^{1/p}.$$

An interesting observation is that the dominant role of the non-locality in the case $\delta \le 1-\frac{\kappa}{d_W}$ means that the proofs all require only elementary estimates, and the weak Bakry-Émery condition plays no role beyond determining the critical value $1-\frac{\kappa}{d_W}$. This contrasts strongly with the central role of the weak Bakry-Émery inequality in Section 5.1, where the local behavior of the semigroup is dominant.

We close the paper by determining, in Section 6, some properties of the fractional Sobolev spaces in our setting as applications of this correspondence, including density of $W^{\delta d_W/p,p}(X)$ in $L^p(X,\mu)$ and when $p \ge 2$, a Sobolev embedding theorem.

Acknowledgments

The authors thank Martin Barlow, Alexander Grigory'an, and Takashi Kumagai for questions and suggestions leading to improvements in the authors' work.

2 Preliminaries: Fractional Laplacians on metric measure spaces

In general, for expressions a and b, we write $a \simeq b$ if there exist constants $c_1, c_2 > 0$ such that $c_1 a \le b \le c_2 a$.

2.1 Standing assumptions

Throughout the paper, (X, d, μ) will denote a proper metric measure space (i.e. where closed and bounded subsets are compact) whose measure μ is Radon and supported on X. The space X is equipped with a Dirichlet form $(\mathcal{E}, \mathcal{F} = \mathrm{dom}(\mathcal{E}))$, that is a closed, symmetric and Markovian bilinear form on $L^2(X, \mu)$ whose domain $\mathcal{F} = \{u \in L^2(X) : \mathcal{E}(u, u) < \infty\}$ is dense in $L^2(X)$, see [29,38]. The vector space of continuous functions with compact support in X is denoted by $C_c(X)$, and $C_0(X)$ is its closure with respect to the supremum norm. Recall, see for example [38, p.6], that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *regular* if it admits a core, which is a subset of $C_c(X) \cap \mathcal{F}$ that is dense in $C_c(X)$ in the supremum norm and dense in $\mathcal{F}$ in the norm

$$\|f\|_{\mathcal{E}_1} := \Big(\|f\|^2_{L^2(X,\mu)} + \mathcal{E}(f, f)\Big)^{1/2}.$$

Also, $(\mathcal{E}, \mathcal{F})$ is said to be *local* if $\mathcal{E}(f, g) = 0$ for any two functions $f, g \in \mathcal{F}$ that have compact support and f vanishes on the support of g. The Dirichlet form $\mathcal{E}$ is said to be *strongly local* if $\mathcal{E}(f, g) = 0$ for any two functions $f, g \in \mathcal{F}$ that have compact support and such that f is constant in a neighborhood of the support of g. The heat semigroup associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is denoted by $\{P_t\}_{t>0}$, and its associated infinitesimal generator is denoted by L.
The following assumptions on the Dirichlet space $(X, \mathcal{E}, \mathcal{F}, \mu)$ will be required throughout the paper.

Assumption 2.1 (Regularity)**.**

(A1) *For any $x \in X$ and $r > 0$, $\overline{B}(x, r) := \{y \in X \mid d(x, y) \le r\}$ is compact;*

(A2) *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular and strongly local.*

Assumption 2.2 (Sub-Gaussian Heat Kernel Estimates)**.** *The semigroup $\{P_t\}_{t>0}$ satisfies $P_t\mathbf{1} = 1$ and there are constants $c_3, c_4, c_5, c_6 \in (0, \infty)$ and $2 \le d_W < \infty$ such that the semigroup has a continuous heat kernel $p_t(x, y)$ satisfying*

$$c_5 t^{-d_H/d_W} \exp\left(-c_6 \Big(\frac{d(x, y)^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\right) \le p_t(x, y) \le c_3 t^{-d_H/d_W} \exp\left(-c_4 \Big(\frac{d(x, y)^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\right)$$

for $\mu\times\mu$-a.e. $(x, y) \in X \times X$ and each $t \in \big(0, +\infty\big)$.

The parameter d_H is the Hausdorff dimension of X, and the parameter d_W is usually called the *walk dimension* of X associated with the Dirichlet form $\mathcal{E}$. Note that for the standard Dirichlet form associated with the Euclidean spaces, the walk dimension $d_W = 2$ while the Hausdorff dimension d_H is the dimension of the Euclidean space. Under these assumptions, the Dirichlet form satisfies

$$\mathcal{E}(f, f) \simeq \limsup_{r\to 0^+} \int_X \int_{B(x,r)} \frac{|f(y) - f(x)|^2}{r^{d_W+d_H}}\, d\mu(y)\, d\mu(x)$$

for any $f \in \mathcal{F}$, c.f. [41, Section 5.3] and [21, Theorem 3.3.1].

Remark 2.3. *Assumption 2.1 together with Assumption 2.2 imply that $\mu(X) = +\infty$. They also imply that (X, d, μ) is Ahlfors d_H-regular, that is, $\mu\big(B(x,r)\big) \simeq r^{d_H}$, and that $(\mathcal{E}, \mathcal{F})$ is strongly local, see [42, Theorem 4.1].*

Remark 2.4. *While $d_W \geq 2$ always, it is possible that $d_H > d_W$: if (X, d, μ) has Hausdorff dimension $d'_H \leq d_W$, then for sufficiently large positive integer n the Cartesian product X^n will have the same walk dimension d_W but Hausdorff dimension $d_H = nd'_H > d_W$.*

2.2 Korevaar-Schoen spaces and weak Bakry-Émery inequality

The following class of Korevaar-Schoen spaces was introduced in [50], see also [51].

Definition 2.5. *For $\lambda > 0$ and $1 \leq p < \infty$, set*

$$KS^{\lambda,p}(X) := \{f \in L^p(X, \mu) : \|f\|_{KS^{\lambda,p}(X)} < \infty\}, \tag{4}$$

where

$$\|f\|^p_{KS^{\lambda,p}(X)} := \limsup_{r \to 0^+} \int_X \int_{B(x,r)} \frac{|f(x) - f(y)|^p}{r^{\lambda p} \mu(B(x,r))} d\mu(y)\, d\mu(x).$$

The space $\mathcal{KS}^{\lambda,p}(X)$ is defined analogously to (4) but with the semi-norm

$$\|f\|^p_{\mathcal{KS}^{\lambda,p}(X)} := \sup_{r>0} \int_X \int_{B(x,r)} \frac{|f(x) - f(y)|^p}{r^{\lambda p} \mu(B(x,r))} d\mu(y)\, d\mu(x). \tag{5}$$

A priori we have that $\mathcal{KS}^{\lambda,p}(X) \subset KS^{\lambda,p}(X)$ and $KS^{\lambda_1,p}(X) \subset KS^{\lambda_2,p}(X)$ when $\lambda_1 > \lambda_2$, so that the following critical value of λ is well-defined.

Definition 2.6. *The critical exponent for the spaces $KS^{\lambda,p}(X)$ is*

$$\lambda_p^{\#} = \sup\{\lambda > 0 : KS^{\lambda,p}(X) \text{ contains non-constant functions}\}.$$

Our discussion of BV functions relies heavily on the heat flow satisfying a Hölder continuity condition that we called a weak Bakry-Émery estimate, see [5, Definition 3.1].

Definition 2.7. *The heat flow P_t satisfies weak Bakry-Émery estimate with exponent $\kappa \in (0, d_W)$ if there is $C_\kappa > 0$ such that for every $t > 0$, $g \in L^\infty(X, \mu)$ and $x, y \in X$,*

$$|P_t g(x) - P_t g(y)| \leq C_\kappa \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|g\|_{L^\infty(X,\mu)}. \tag{6}$$

Note that if this estimate holds for $\kappa < d_W$ then it is also valid for $\kappa' \in (0, \kappa]$ with constant $\max(2, C_\kappa)$ by [5, Lemma 3.3]. In the context of our article, from [5, Remark 3.12] one has

$$0 < \kappa \leqslant d_W/2.$$

We also note that if (X, d) satisfies a chain condition, then $\kappa \leq 1$, see [5, Remark 3.2] and [47, page 81].

2.3 Subordination and fractional Laplacians

Let $0 < \delta < 1$ be a parameter that is fixed throughout the paper. The fractional power $(-L)^\delta$, of the infinitessimal generator $-L$ of the Dirichlet form $\mathcal{E}$, can be defined via the following formula, see (5) in [64, p. 260],

$$(-L)^\delta f = -\frac{\delta}{\Gamma(1-\delta)} \int_0^\infty t^{-\delta-1}(P_t f - f)\, dt.$$

From Proposition 1 in [64, p.260] it is known that $(-L)^\delta$ is the generator of a Markovian semigroup $\{P_t^{(\delta)}\}_{t>0}$ is related to $\{P_t\}_{t>0}$ by the *subordination formula*

$$P_t^{(\delta)} f(x) := \int_0^\infty \eta_t^{(\delta)}(s)\, P_s f(x)\, ds, \tag{7}$$

where $\eta_t^{(\delta)}(s)$ is the non-negative continuous function (the subordinator) such that

$$\int_0^\infty \eta_t^{(\delta)}(s) e^{-s\lambda} ds = e^{-t\lambda^\delta} \tag{8}$$

for every $\lambda > 0$. The heat kernel $p_t^{(\delta)}$ for $(-L)^\delta$ is related to the heat kernel p_t for $-L$ by the formula

$$p_t^{(\delta)}(x,y) = \int_0^\infty \eta_t^{(\delta)}(s)\, p_s(x,y)\, ds.$$

In addition, the subordinator $\eta_t^{(\delta)}(s)$ satisfies the following upper bound

$$\eta_t^{(\delta)}(s) \le C \left(\frac{1}{t^{1/\delta}} \wedge \frac{t}{s^{1+\delta}} \right). \tag{9}$$

For $-\infty < \alpha < \delta$ we have

$$\int_0^{+\infty} \eta_t^{(\delta)}(s) s^\alpha ds = \frac{\Gamma(1-\alpha/\delta)}{\Gamma(1-\alpha)} t^{\alpha/\delta}, \tag{10}$$

and for $\alpha \ge \delta$ we have

$$\int_0^{+\infty} \eta_t^{(\delta)}(s) s^\alpha ds = +\infty.$$

The Markov semigroup $P_t^{(\delta)}$ is associated with the Dirichlet form

$$\mathcal{E}^{(\delta)}(f,f) := \lim_{t\to 0^+} \frac{1}{t} \int_X f(f - P_t^{(\delta)} f) d\mu. \tag{11}$$

The domain $\mathcal{F}^{(\delta)}$ of $\mathcal{E}^{(\delta)}$ is the set of all functions $f \in L^2(X,\mu)$ for which the limit (11) is finite. For details and proofs regarding the above statements, we refer the reader to [64, p.260]. From [41, Lemma 5.4], under Assumptions 2.1 and 2.2, the Dirichlet form $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$ associated with $P_t^{(\delta)}$ satisfies

$$\mathcal{E}^{(\delta)}(f,f) \simeq \int_X \int_X \frac{|f(x)-f(y)|^2}{d(x,y)^{d_H+\delta d_W}}\, d\mu(y) d\mu(x), \tag{12}$$

and that for the corresponding heat kernel $p_t^{(\delta)}(x,y)$ admits the estimates

$$c_5 t^{-\frac{d_H}{\delta d_W}}\left(1+c_6\frac{d(x,y)}{t^{\frac{1}{\delta d_W}}}\right)^{-d_H-\delta d_W} \le p_t^{(\delta)}(x,y) \le c_3 t^{-\frac{d_H}{\delta d_W}}\left(1+c_4\frac{d(x,y)}{t^{\frac{1}{\delta d_W}}}\right)^{-d_H-\delta d_W}. \tag{13}$$

From (7) and (10) one readily obtains the following estimate.

Lemma 2.8. *If P_t satisfies a weak Bakry-Émery condition with exponent κ as in (6), then there exists a constant $C>0$ such that for every $t>0$, $g\in L^\infty(X,\mu)$ and $x,y\in X$,*

$$|P_t^{(\delta)}g(x)-P_t^{(\delta)}g(y)| \le C\frac{d(x,y)^\kappa}{t^{\frac{\kappa}{\delta d_W}}}\|g\|_{L^\infty(X,\mu)}.$$

That is, $P_t^{(\delta)}$ satisfies a weak Bakry-Émery condition with d_W replaced by δd_W.

Proof. By virtue of (6),

$$\begin{aligned}|P_t^{(\delta)}g(x)-P_t^{(\delta)}g(y)| &\le \int_0^\infty \eta_t^{(\delta)}(s)|P_sg(x)-P_sg(y)|\,ds \le C_\kappa\int_0^\infty \eta_t^{(\delta)}(s)d(x,y)^\kappa s^{-\kappa/d_W}\|g\|_\infty ds\\ &\le C_\kappa d(x,y)^\kappa\|g\|_\infty\int_0^\infty s^{-\kappa/d_W}\eta_t^{(\delta)}(s)\,ds = C_\kappa\frac{\Gamma\left(1+\frac{\kappa}{\delta d_W}\right)}{\Gamma\left(1+\frac{\kappa}{d_W}\right)}\frac{d(x,y)^\kappa}{t^{\frac{\kappa}{\delta d_W}}}\|g\|_\infty,\end{aligned}$$

where the last equality follows from (10) since $-\kappa/d_W<\delta$. □

2.4 Examples

In this section we give two examples of Dirichlet forms and metric measure spaces that satisfy our standing assumptions.

Poisson kernel in $\mathbb{R}^d$

The first classical example of non-local Dirichlet form is that associated with the fractional Laplacian $(-\Delta)^{1/2}$. The corresponding Poisson kernel $q_t^{(\frac{1}{2})}:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ is given for any $t>0$ by

$$q_t^{(\frac{1}{2})}(x,y)=\frac{\Gamma\left(\frac{d+1}{2}\right)\pi^{-\frac{d+1}{2}}}{t^d}\left(1+\frac{|x-y|^2}{t^2}\right)^{-\frac{d+1}{2}}$$

and provides the fundamental solution to the *Poisson equation* in $\mathbb{R}^d$, $\partial_t^2 f+\Delta f=0$.

Non-local forms on nested fractals

The concept of fractal metric measure space with fractional diffusion was introduced by Barlow in [9], to which we refer the reader for a precise definition. Both Assumption 2.1 and Assumption 2.2 are valid on such spaces. Nested fractals like the Vicsek set and the Sierpinski gasket are examples that fall into this class of spaces and also satisfy the weak

Bakry-Émery estimate with $\kappa = d_W - \lambda_1^{\#}$, which is the largest possible exponent for a weak Bakry-Émery condition can be valid, see [5, Theorem 5.1].
The non-local Dirichlet form obtained through subordination from such a fractional diffusion has been studied in the literature by many authors, see e.g. [24, 25, 30, 48, 53, 54, 62]. In particular, it was proved in [53, 62] that for some values of δ, the associated stable-like process whose corresponding Dirichlet form satisfies (12) can also be obtained as the trace of a d-dimensional Brownian motion on X, assuming that $X \subset \mathbb{R}^d$.
The Sierpinski carpet is a fractal metric measure space with a unique natural fractional diffusion that satisfies a weak Bakry-Émery estimate [12,14]. The best exponent for the weak Bakry-Émery estimate is not known, see [5, Conjecture 5.4] and the discussion preceding Assumption 4.2 below.

3 Heat Besov classes for the fractional Laplacian

Recall that we have fixed a choice of $\delta \in (0, 1)$. As in [3–5], for any $p \geq 1$ and $\alpha \geq 0$, we consider the Besov seminorm

$$\|f\|_{\delta,p,\alpha} := \sup_{t>0} t^{-\alpha} \left(\int_X \int_X |f(x) - f(y)|^p p_t^{(\delta)}(x, y) d\mu(x) d\mu(y) \right)^{1/p}$$

and the associated heat semigroup-based Besov class

$$\mathbf{B}_\delta^{p,\alpha}(X) = \{f \in L^p(X, \mu) \,:\, \|f\|_{\delta,p,\alpha} < \infty\}.$$

We refer to [3] for some of the basic properties of these spaces. The remainder of this section is devoted to identifying and classifying them depending on the relation between the parameters p and α.

3.1 The case $\alpha < 1/p$

We start by comparing the space $\mathbf{B}_\delta^{p,\alpha}(X)$ to the Korevaar-Schoen classes defined in (4) and (5).

Theorem 3.1. *Let $1 \leq p < \infty$ and $0 < \alpha < \frac{1}{p}$. Then,*

$$\mathbf{B}_\delta^{p,\alpha}(X) = \mathcal{KS}^{\lambda,p}(X) = KS^{\lambda,p}(X),$$

where

$$\lambda = \alpha \, \delta \, d_W.$$

Moreover, $\mathbf{B}_\delta^{p,\alpha}(X)$ and $\mathcal{KS}^{\lambda,p}(X)$ have equivalent seminorms for this choice of λ.

Proof. The inclusion $\mathcal{KS}^{\alpha\delta d_W,p}(X) \subset KS^{\alpha\delta d_W,p}(X)$ follows directly from the definition, while $KS^{\alpha\delta d_W,p}(X) \subset \mathcal{KS}^{\alpha\delta d_W,p}(X)$ is obtained in exactly the same way as in [5, Proposition 4.1]. Thus it suffices now to prove that $\mathbf{B}_\delta^{p,\alpha}(X) = \mathcal{KS}^{\alpha\delta d_W,p}(X)$.
Let

$$\Phi(s) = (1 + c_6 s)^{-d_H - \delta d_W},$$

so that the heat kernel estimate (13) implies $p_t^{(\delta)}(x,y) \geq c\,\Phi(1) t^{\frac{-d_H}{\delta d_W}}$ for $x \in B(y, t^{\frac{1}{\delta d_W}})$. Then Besov seminorm, $\|f\|_{p,\alpha}$ is given by

$$\|f\|_{\delta,p,\alpha}^p = \sup_{t>0} \frac{1}{t^{\alpha p}} \int_X \int_X |f(x)-f(y)|^p p_t^{(\delta)}(x,y) d\mu(x) d\mu(y).$$

Note that

$$\begin{aligned}
&\frac{1}{t^{\alpha p}} \int_X \int_X |f(x)-f(y)|^p p_t^{(\delta)}(x,y) d\mu(x) d\mu(y) \\
&\qquad \geq \frac{1}{t^{\alpha p}} \int_X \int_{B(y,t^{\frac{1}{\delta d_W}})} |f(x)-f(y)|^p p_t^{(\delta)}(x,y) d\mu(x) d\mu(y) \\
&\qquad \geq c\,\Phi(1) \int_X \int_{B(y,t^{\frac{1}{\delta d_W}})} \frac{|f(x)-f(y)|^p}{t^{\alpha p + \frac{d_H}{\delta d_W}}} d\mu(x) d\mu(y).
\end{aligned}$$

Taking the supremum over $t>0$ yields $\mathbf{B}_\delta^{p,\alpha}(X) \subset \mathcal{KS}^{\alpha\delta d_W, p}(X)$. For the reverse inclusion, fix $r>0$ and set

$$\begin{aligned}
A(t,r) &:= \int_X \int_{X \setminus B(y,r)} |f(x)-f(y)|^p p_t^{(\delta)}(x,y)\, d\mu(x)\, d\mu(y), \\
B(t,r) &:= \int_X \int_{B(y,r)} |f(x)-f(y)|^p p_t^{(\delta)}(x,y)\, d\mu(x)\, d\mu(y).
\end{aligned}$$

For each positive integer i we set $A_i(y) := B(y, 2^i r) \setminus B(y, 2^{i-1} r)$. By the Ahlfors d_H-regularity of μ (see Remark 2.3), the form of the function Φ, and (13),

$$\begin{aligned}
\int_{X \setminus B(y,r)} p_t^{(\delta)}(x,y)\, d\mu(x) &\approx \sum_{i\in\mathbb{N}} \int_{A_i(y)} t^{-d_H/\delta d_W} \Phi(2^i r / t^{1/\delta d_W})\, d\mu \\
&\approx \sum_{i\in\mathbb{N}} t^{-d_H/\delta d_W} (2^i r)^{d_H}\, \Phi(2^i r/t^{1/\delta d_W}) \;\approx\; \sum_{i\in\mathbb{N}} s_i^{d_H} \Phi(s_i),
\end{aligned}$$

where $s_i = 2^i r / t^{1/\delta d_W}$. Note that we have used $\mu(A_i(y) \leq C(2^i r)^{d_H}$ here. It follows that

$$\int_{X \setminus B(y,r)} p_t^{(\delta)}(x,y)\, d\mu(x) \leq C \int_{\frac{1}{2} r t^{-\frac{1}{\delta d_W}}}^{\infty} s^{d_H} \Phi(s) \frac{ds}{s}.$$

Applying the inequality $|f(x)-f(y)|^p \leq 2^{p-1}(|f(x)|^p + |f(y)|^p)$, the symmetry of $p^{(\delta)}(x,y)$, the Fubini theorem, and the preceding inequality, we obtain

$$\begin{aligned}
A(t,r) &\leq 2^p \int_X \int_{X \setminus B(y,r)} |f(y)|^p p_t^{(\delta)}(x,y)\, d\mu(x)\, d\mu(y) \\
&\leq 2^p C \|f\|_{L^p(X,\mu)}^p \int_{\frac{1}{2} r t^{-\frac{1}{\delta d_W}}}^{\infty} s^{d_H} \Phi(s) \frac{ds}{s} \\
&\leq C t^{p\alpha} r^{-\alpha\delta d_W p} \|f\|_{L^p(X,\mu)}^p \int_{\frac{1}{2} r t^{-\frac{1}{\delta d_W}}}^{\infty} s^{d_H + \alpha\delta d_W p} \Phi(s) \frac{ds}{s}. \qquad (14)
\end{aligned}$$

On the other hand, for $B(t,r)$, writing $r_k = 2^{-k}r$ and $A_k(y) = B(y, r_k) \setminus B(y, r_{k+1})$, we have by virtue of (13) that

$$\begin{aligned}
B(t,r) &\le Ct^{-\frac{d_H}{\delta d_W}} \sum_{k=0}^{\infty} \Phi\Big(r_{k+1} t^{-\frac{1}{\delta d_W}}\Big) r_k^{d_H+\alpha\delta d_W p} \int_X \int_{A_k(y)} \frac{|f(x)-f(y)|^p}{r_k^{\alpha\delta d_W p + d_H}}\, d\mu(x)\, d\mu(y) \\
&\le Ct^{-\frac{d_H}{\delta d_W}} \sum_{k=0}^{\infty} \Phi\Big(r_{k+1} t^{-\frac{1}{\delta d_W}}\Big) r_k^{d_H+\alpha\delta d_W p} \int_X \int_{B(y,r_k)} \frac{|f(x)-f(y)|^p}{r_k^{\alpha\delta d_W p + d_H}}\, d\mu(x)\, d\mu(y) \\
&\le Ct^{\alpha p} \|f\|^p_{\mathcal{KS}^{\alpha\delta d_W, p}(X)} \int_0^{\infty} s^{d_H+\alpha\delta d_W p} \Phi(s) \frac{ds}{s}. \qquad (15)
\end{aligned}$$

The integrals in both (14) and (15) are bounded because $\alpha < 1/p$ by assumption (note that when $\alpha < 1/p$ we have $\int_1^{\infty} s^{d_H+\alpha\delta d_W p}\Phi(s)\frac{ds}{s} \approx \int_1^{\infty} s^{\delta d_W(\alpha p - 1)}\frac{ds}{s} < \infty$), and the bound on $A(t,r) + B(t,r)$ yields

$$\frac{1}{t^{\alpha p}} \int_X \int_X |f(x)-f(y)|^p p_t^{(\delta)}(x,y) d\mu(x)\, d\mu(y) \le C_{p,\alpha} \Big(\frac{1}{r^{d_W \alpha\delta p}} \|f\|^p_{L^p(X,\mu)} + \|f\|^p_{\mathcal{KS}^{\alpha\delta d_W, p}(X)} \Big)$$

for some $C_{p,\alpha} > 0$ and any $t > 0$. Taking the supremum over $t > 0$, we obtain the inclusion $\mathcal{KS}^{\alpha\delta d_W, p}(X) \subset \mathbf{B}^{p,\alpha}_{\delta}(X)$ and letting $r \to \infty$ gives the equivalence of the seminorms. □

3.2 The case $\alpha = 1/p$

When $\alpha = 1/p$, the space $\mathbf{B}^{p,\alpha}_{\delta}(X)$ does not compare to a Korevaar-Schoen class. We now show that instead, it coincides with a suitable *fractional Sobolev space*, defined in general as

$$W^{\lambda,p}(X) := \left\{ f \in L^p(X,\mu) : \|f\|_{W^{\lambda,p}(X)} < +\infty \right\},$$

where the seminorm is

$$\|f\|_{W^{\lambda,p}(X)} := \Big(\int_X \int_X \frac{|f(x)-f(y)|^p}{d(x,y)^{d_H+\lambda p}}\, d\mu(y) d\mu(x) \Big)^{1/p}.$$

Remark 3.2. *As pointed out in Section 1, in the case $X = \mathbb{R}^d$, we have $d_W = 2$ and the fractional Sobolev space $\mathbf{B}^{p,1/p}_{\delta}(\mathbb{R}^d)$ coincides with Gagliardo's Euclidean fractional Sobolev space [39]*

$$W^{2\delta/p,p}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d, dx) : \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{d+2\delta}} dy\, dx \right)^{1/p} < \infty \right\}.$$

This motivates our next proposition.

Proposition 3.3. *Let $1 \le p < \infty$. Then $\mathbf{B}^{p,1/p}_{\delta}(X) = W^{\delta d_W/p,p}(X)$ with equivalent seminorms.*

Proof. Observe from (13) that

$$c_5 \big(t^{\frac{1}{\delta d_W}} + c_6 d(x,y) \big)^{-d_H-\delta d_W} \le \frac{1}{t} p_t^{(\delta)}(x,y) \le c_3 \big(t^{\frac{1}{\delta d_W}} + c_4 d(x,y) \big)^{-d_H-\delta d_W}.$$

The upper bound gives

$$\begin{aligned}\frac{1}{t}\int_X\int_X |f(x)-f(y)|^p p_t^{(\delta)}(x,y)\,d\mu(x)\,d\mu(y) &\le C\int_X\int_X \frac{|f(x)-f(y)|^p}{d(x,y)^{d_H+\delta d_W}}d\mu(y)\,d\mu(x)\\ &= C\|f\|^p_{W^{\delta d_W/p,p}(X)},\end{aligned}\tag{16}$$

and by taking the supremum over $t>0$ we obtain $\|f\|_{p,1/p}\le C\|f\|^p_{W^{\delta d_W/p,p}(X)}$. To obtain the lower bound, we argue as follows. For each $t>0$,

$$\begin{aligned}\int_X\int_X \frac{|f(x)-f(y)|^p}{(t^{\frac{1}{\delta d_W}}+d(x,y))^{d_H+\delta d_W}}d\mu(y)\,d\mu(x) &\le \frac{C}{t}\int_X\int_X |f(x)-f(y)|^p p_t^{(\delta)}(x,y)\,d\mu(x)\,d\mu(y)\\ &\le C\|f\|^p_{\delta,p,1/p}.\end{aligned}\tag{17}$$

Taking $\liminf_{t\to 0^+}$ and applying the Fatou lemma we obtain $\|f\|^p_{W^{\delta d_W/p,p}(X)}\le C\|f\|_{\delta,p,1/p}$ as desired. □

In the course of the above proof we established the following locality-in-time estimate for the Besov norm which will be useful later.

Corollary 3.4. *Let $1\le p<\infty$. There exists $C>0$ such that for any $f\in \mathbf{B}^{p,1/p}_\delta(X)$,*

$$\|f\|_{\delta,p,1/p}\le C\liminf_{t\to 0^+}\frac{1}{t^{1/p}}\left(\int_X\int_X |f(x)-f(y)|^p p_t^{(\delta)}(x,y)\,d\mu(x)\,d\mu(y)\right)^{1/p}.$$

This condition was previously considered in [3, Definition 6.7], where it was called property $(P_{p,1/p})$.

3.3 The case $\alpha>1/p$

In this subsection we will show that the spaces $\mathbf{B}^{p,\alpha}_\delta(X)$ for $\alpha>1/p$ are trivial and thus not interesting for further analysis. This completes the classification of these spaces.

Proposition 3.5. *Let $1\le p<\infty$ and $\alpha>1/p$. Then, $\mathbf{B}^{p,\alpha}_\delta(X)$ only contains the zero function.*

Proof. The estimate (17) gives

$$\int_X\int_X \frac{|f(x)-f(y)|^p}{(t^{\frac{1}{\delta d_W}}+d(x,y))^{d_H+\delta d_W}}d\mu(y)\,d\mu(x)\le Ct^{\alpha p-1}\|f\|^p_{\delta,p,\alpha}.$$

Applying Fatou's lemma, we conclude from $\alpha p>1$ and $\|f\|_{\delta,p,\alpha}<\infty$ that

$$\int_X\int_X \frac{|f(x)-f(y)|^p}{d(x,y)^{d_H+\delta d_W}}d\mu(y)\,d\mu(x)\le \liminf_{t\to 0^+}\int_X\int_X \frac{|f(x)-f(y)|^p}{(t^{\frac{1}{\delta d_W}}+d(x,y))^{d_H+\delta d_W}}d\mu(y)\,d\mu(x)=0,$$

which implies that f is constant. Since $\mu(X)=\infty$ (see Remark 2.3) and $f\in L^p(X)$ we conclude that $f\equiv 0$. □

3.4 Critical exponent for the heat Besov spaces

Since the Besov spaces are nested and decreasing in the parameter α, it will be useful to focus our attention on the Besov space with the largest exponent subject the condition that $\mathbf{B}^{p,\alpha}_\delta(X)$ contains functions other than constants. The corresponding critical exponent is thus defined as

$$\alpha^\#_{\delta,p} := \sup\{\alpha > 0 :\ \mathbf{B}^{p,\alpha}_\delta(X) \text{ contains non-constant functions}\}.$$

Corollary 3.6. *For any $1 \le p < \infty$,*

(i) $\alpha^\#_{\delta,p} \le \frac{1}{p}$*;*

(ii) $\alpha^\#_{\delta,p} = \frac{\lambda^\#_p}{\delta d_W}$ *if* $\alpha^\#_{\delta,p} < \frac{1}{p}$.

Proof. The first statement is an immediate consequence of Proposition 3.5 and the second follows from the correspondence in Theorem 3.1. □

3.5 Comparison to metric Besov spaces previously considered in the literature

To compare the spaces $\mathbf{B}^{p,\alpha}_\delta(X)$ to the metrically-defined Besov type spaces previously considered by Grigor'yan in [41], let us define, for any $f \in L^p(X,\mu)$ and $r > 0$,

$$N^\alpha_p(f,r) := \frac{1}{r^{\alpha + d_H/p}} \left(\iint_{\{d(x,y)<r\}} |f(x) - f(y)|^p \, d\mu(x)\, d\mu(y) \right)^{1/p}.$$

For any real number q with $\max\{1,p\} \le q < \infty$ we now set

$$N^\alpha_{p,q}(f) := \left(\int_0^\infty \left(N^\alpha_p(f,r) \right)^q \frac{dr}{r} \right)^{1/q},$$

and for $q = \infty$

$$N^\alpha_{p,\infty}(f) := \sup_{r>0} N^\alpha_p(f,r).$$

In [41], a version of the Besov space $\mathfrak{B}^\alpha_{p,q}(X)$ is defined as

$$\mathfrak{B}^\alpha_{p,q}(X) := \left\{ f \in L^p(X,\mu) :\ N^\alpha_{p,q}(f) < \infty \right\}.$$

With these notations, we see from the definitions that $\mathfrak{B}^\alpha_{p,\infty}(X) = \mathcal{KS}^{\alpha,p}(X)$ for all $\alpha > 0$. On the other hand, from [40, Theorem 5.2] and the fact that μ is Ahlfors d_H-regular, we also know that $\mathfrak{B}^\alpha_{p,p}(X) = W^{\alpha,p}(X)$ for all $\alpha > 0$. Thus we can rewrite Theorem 3.1 and Proposition 3.3 as follows:

Proposition 3.7. *Let $p \ge 1$.*

(i) *If $0 \le \alpha < 1/p$ we have $\mathbf{B}^{p,\alpha}_\delta(X) = \mathfrak{B}^{\alpha\delta d_W}_{p,\infty}(X)$ and $\|f\|_{\delta,p,\alpha} \simeq N^{\alpha\delta d_W}_{p,\infty}(f)$.*

(ii) $\mathbf{B}^{p,1/p}_\delta(X) = \mathfrak{B}^{\delta d_W/p}_{p,p}(X)$ *and* $\|f\|_{p,1/p} \simeq N^{\delta d_W/p}_{p,p}(f)$.

4 Critical exponents and the weak Bakry-Émery inequality

Recall that the weak Bakry-Émery inequality (6) is the Hölder regularity estimate

$$|P_t g(x) - P_t g(y)| \le C_\kappa \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|g\|_{L^\infty(X,\mu)}.$$

It was established in [5] that a weak Bakry-Émery inequality for the semigroup constrains the possible values of the critical exponent introduced in Section 3.4. It is convenient to define a parameter

$$\beta_p := \Big(1 - \frac{2}{p}\Big)\frac{\kappa}{d_W} + \frac{1}{p}.$$

Theorem 4.1. *The following hold:*

1. *If $\frac{1}{2} \le \delta < 1$, then*
   (i) *for $1 \le p < 2\delta$ we have $\frac{1}{2\delta} \le \alpha^{\#}_{\delta,p} \le \min\{\frac{\beta_p}{\delta}, \frac{1}{p}\}$, and*
   (ii) *for $p \ge 2\delta$ we have $\alpha^{\#}_{\delta,p} = \frac{1}{p}$.*
2. *If $0 < \delta < \frac{1}{2}$, then for every $p \ge 1$, $\alpha^{\#}_{\delta,p} = \frac{1}{p}$.*

In particular, for every $\delta \in (0,1)$ and $p \ge 2\delta$, $\alpha^{\#}_{\delta,p} = \frac{1}{p}$ and $\mathbf{B}^{p,\alpha^{\#}_{\delta,p}}_{\delta}(X) = W^{\frac{\delta d_W}{p},p}(X)$.

Proof. Corollary 3.6 says that $\alpha^{\#}_{\delta,p} \le \frac{1}{p}$. If $\alpha^{\#}_{\delta,p} < \frac{1}{p}$, then by Theorem 3.1 we have that $\mathbf{B}^{p,\alpha^{\#}_{\delta,p}}_{\delta}(X) = \mathrm{KS}^{\delta d_W \alpha^{\#}_{\delta,p},p}(X)$. As increasing the value of α slightly beyond $\alpha^{\#}_{\delta,p}$ leads to $\mathbf{B}^{p,\alpha}_{\delta}(X)$ being trivial, it follows also that increasing λ slightly beyond $\delta d_W \alpha^{\#}_{\delta,p}$ only results in $\mathrm{KS}^{\lambda,p}(X)$ being trivial. It follows that $\delta d_W \alpha^{\#}_{\delta,p} = \lambda^{\#}_p$, where $\lambda^{\#}_p$ is the critical exponent for the Korevaar-Schoen space, see (3). Combining this with the estimates for $\lambda^{\#}_p$ obtained in [5, Theorem 3.11] yields the claim. (Note that in the latter we have $\lambda^{\#}_p = d_W \alpha^{\#}_p$, where $\alpha^{\#}_p$ is the critical exponent for the heat Besov space of the original, local, heat flow.) $\square$

4.1 Critical weak Bakry-Émery inequality

From [5, Remark 3.12] it follows that the maximal possible value of κ in a weak Bakry-Émery inequality (6) is $d_W - \lambda^{\#}_1$, where $\lambda^{\#}_1$ is the critical exponent for non-triviality of the L^1 Korevaar-Schoen space, see Definition 2.6. In the remainder of this section and all of the next section we assume that the weak Bakry-Émery condition is valid at this critical value.

Assumption 4.2 (Critical Weak Bakry-Émery estimate)**.** *The heat flow P_t satisfies the weak Bakry-Émery estimate (6) with exponent $\kappa = d_W - \lambda^{\#}_1$.*

Under this assumption we obtain the precise value of the critical exponent when $p = 1$.

Theorem 4.3. *Suppose that Assumption 4.2 holds. Then*

$$\alpha^{\#}_{\delta,1} = \min\Big\{1, \frac{1}{\delta}\big(1 - \frac{\kappa}{d_W}\big)\Big\}.$$

Proof. From Theorem 4.1 with $p = 1$, we see that if $\frac{1}{2} < \delta < 1$, then (noting that $\beta_1 = 1 - \frac{\kappa}{d_W}$), we have $\frac{1}{2\delta} \le \alpha^{\#}_{\delta,1} \le \min\{1, \left(1 - \frac{\kappa}{d_W}\right)/\delta\}$. If $0 < \delta \le \frac{1}{2}$, then we have $\alpha^{\#}_{\delta,1} = 1$.
If $\alpha^{\#}_{\delta,1} < 1$, (and recalling that we now have $p = 1$), then by Theorem 3.1 we have that $\mathbf{B}^{1,\alpha}_{\delta}(X) = \mathrm{KS}^{\delta d_W \alpha,1}(X)$ for each $0 < \alpha < \alpha^{\#}_{\delta,1}$, and just as in the proof of Theorem 3.1 we see that $\delta d_W \alpha^{\#}_{\delta,1} = \lambda^{\#}_1 = d_W - \kappa$. Thus when $\frac{1}{2} < \delta < 1$, we have the desired identity.
Now we consider the case that $0 < \delta \le \frac{1}{2}$. In this case, as observed above, $\alpha^{\#}_{\delta,1} = 1$, and so by Theorem 3.1 we have that for each $0 < \alpha < 1$ the space $\mathbf{B}^{p,\alpha}_{\delta}(X) = KS^{\lambda,1}(X)$ is non-trivial. It follows that $\lambda^{\#}_1 \ge \alpha\delta d_W$ for each $\alpha < 1$, and so $\lambda^{\#}_1 \ge \delta d_W$. Hence, as $\kappa = d_W - \lambda^{\#}_1$, we have that $\frac{1}{\delta}\left(1 - \frac{\kappa}{d_W}\right) \ge 1$, and so the desired inequality holds in this case as well. □

Example 4.4. *Let $\delta \in (0, 1)$ and consider the Dirichlet form $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$ on $\mathbb{R}^d$ associated with the fractional Laplacian $(-\Delta)^{\delta}$, namely*

$$\mathcal{E}^{(\delta)}(f, f) \simeq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2\delta}}\, dy\, dx.$$

Then, $\kappa = 1$ and $d_W = 2$, so that $\beta_p = \frac{1}{2}$, $\alpha^{\#}_p = \min\left\{\frac{1}{p}, \frac{1}{2\delta}\right\}$ and

$$\mathbf{B}^{p,\alpha^{\#}_{\delta,p}}_{\delta}(X) = \begin{cases} \mathcal{KS}^{1,p}(\mathbb{R}^d) & \text{if } 1 \le p < 2\delta, \\ W^{2\delta/p,p}(\mathbb{R}^d) & \text{if } p \ge \max\{1, 2\delta\}. \end{cases}$$

Note that the spaces $\mathcal{KS}^{1,p}(\mathbb{R}^d)$ coincide with the usual Sobolev spaces $W^{1,p}(\mathbb{R}^d)$ when $p > 1$, while for $p = 1$ we have

$$\mathbf{B}^{1,\alpha^{\#}_{\delta,1}}_{\delta}(\mathbb{R}^d) = \begin{cases} \mathcal{KS}^{1,1}(\mathbb{R}^d) = BV(\mathbb{R}^d) & \text{if } \delta > 1/2, \\ W^{2\delta,1}(\mathbb{R}^d) & \text{if } \delta \le 1/2. \end{cases}$$

Interestingly, for $\delta = \frac{1}{2}$, one has

$$\mathbf{B}^{1,\alpha^{\#}_{\delta,1}}_{\delta}(\mathbb{R}^d) = W^{2\delta,1}(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d, dx) : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|^{d+1}} dy\, dx < +\infty \right\}$$

which, by [26, Proposition 1], is a trivial space containing only the zero function.

Example 4.5. *Let $\delta \in (0, 1)$ and consider the Dirichlet form $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$ on an unbounded nested fractal X associated with the fractional Laplacian $(-\Delta)^{\delta}$, namely*

$$\mathcal{E}^{(\delta)}(f, f) \simeq \int_X\int_X \frac{|f(x) - f(y)|^2}{d(x, y)^{d_H + \delta d_W}}\, d\mu(y) d\mu(x).$$

Then, $\kappa = d_W - d_H$ by [5, Theorem 3.7] and $\lambda_1^\# = d_H$ by [5, Theorem 5.1]. In particular, also Assumption 4.2 is valid. The bounds in Theorem 4.1 do not determine $\alpha_{\delta,p}^\#$ for $1 < p < 2$, but for $p = 1$ we obtain from Theorem 4.3 that $\alpha_{\delta,1}^\# = \min\left\{1, \frac{d_H}{\delta d_W}\right\}$ and

$$\mathbf{B}_\delta^{1,\alpha_{\delta,1}^\#}(X) = \begin{cases} \mathcal{KS}^{d_H,1}(X) & \text{if } \delta > d_H/d_W, \\ W^{\delta d_W,1}(X) & \text{if } \delta \le d_H/d_W. \end{cases}$$

Example 4.6. *Let X denote the unbounded Vicsek fractal. There is a Dirichlet form on X which satisfies Assumptions 2.1, 2.2, 4.2 with $d_H = \frac{\ln 5}{\ln 3}$ and $d_W = \frac{\ln 15}{\ln 3} = d_H + 1$. For $\delta \in (0,1)$ let $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$ be the form of the subordinated semigroup, so that*

$$\mathcal{E}^{(\delta)}(f,f) \simeq \int_X \int_X \frac{|f(x) - f(y)|^2}{d(x,y)^{d_H + \delta d_W}}\, d\mu(y) d\mu(x).$$

The critical exponents of the Korevaar-Schoen spaces for this fractal have been computed in [23] and we have

$$\mathbf{B}_\delta^{p,\alpha_{\delta,p}^\#}(X) = \begin{cases} \mathcal{KS}^{\beta_p d_W,p}(X) & \text{if } 1 \le p < 2 - (1-\delta) d_W, \\ W^{d_W \delta/p,p}(X) & \text{if } p \ge \max\{1, 2 - (1-\delta) d_W\}. \end{cases}$$

For the Vicsek fractal, the spaces $\mathcal{KS}^{\beta_p d_W,p}(X)$ are a natural class of Sobolev spaces and were studied in detail in [22, 23].

5 Properties of BV and $W^{\delta d_W,1}$ from heat Besov spaces

Under Assumption 4.2, it was argued in [5] that one should consider $\mathrm{KS}^{\lambda_1^\#,1}$ to be the space BV of functions of bounded variation. The justification was that there is a natural notion of variation which admits a co-area formula, and measures that play the role of the classical total variation of a gradient, and that functions in the space satisfy Sobolev and isoperimetric inequalities. The proofs formed the bulk of the paper [5] and relied heavily on Assumption 4.2. The variation is defined as follows.

Definition 5.1. *Under assumption 4.2 (so $\lambda_1^\# = d_W - \kappa$) let $BV(X) := KS^{\lambda_1^\#,1}(X)$ and for $f \in BV(X)$ let*

$$\mathbf{Var}(f) := \liminf_{r \to 0^+} \int_X \int_{B(x,r)} \frac{|f(y) - f(x)|}{r^{\lambda_1^\#} \mu(B(x,r))}\, d\mu(y)\, d\mu(x).$$

Observe that the crucial difference between this and the $\mathrm{KS}^{d_W - \kappa,1}(X)$ seminorm is that $\mathbf{Var}(f)$ is a limit infimum rather than a limit supremum.
We saw in Theorem 4.3 that under Assumption 4.2, the critical parameter $\alpha_{\delta,1}^\#$ is determined, and hence so is the critical heat Besov space. From Theorem 3.1, if $\alpha_{\delta,1}^\# < 1$, or equivalently $1 - \frac{\kappa}{d_W} < \delta < 1$, then this space is $KS^{\lambda_1^\#,1} = KS^{d_W - \kappa,1}(X)$, which we consider to be the space $BV(X)$ in Definition 5.1. However, when $\alpha_{\delta,1}^\# = 1$, or equivalently $0 < \delta \le 1 - \frac{\kappa}{d_W}$,

we saw in Proposition 3.3 that the critical heat Besov space was the nonlocal Sobolev space $W^{\delta d_W,1}(X)$. Thus, in the case $\delta > 1-\frac{\kappa}{d_W}$ the local behavior of the semigroup predominates, while in the case $\delta \leq 1-\frac{\kappa}{d_W}$ it is the nonlocal structure that is determinative. An interesting dichotomy is that all arguments about the structure of functions in the critical space for the case $\delta > 1-\frac{\kappa}{d_W}$ depend heavily on the weak Bakry-Émery inequality, while the only use of the inequality in the case $\delta \leq 1-\frac{\kappa}{d_W}$ is to determine that then $\alpha^{\#}_{\delta,1} = 1$. Of course, the fact that $1-\frac{\kappa}{d_W}$ is the value of δ at which the behavior changes from local to non-local does depend on the weak Bakry-Émery condition.
In this section we discuss the properties of the spaces BV and $W^{\delta d_W,1}$ that can be derived from the characterization by heat Besov spaces.

5.1 The case $\delta > 1-\frac{\kappa}{d_W}$: Characterizing BV properties of heat Besov spaces

In this section we record how the main results of [5] regarding the space BV can be written in terms of the heat Besov spaces for the subordinated semigroup. Most proofs are short because the results are obtained directly from the seminorm correspondence in Theorem 3.1 and results in [5]. The validity of the weak Bakry-Émery inequality at the critical exponent $\kappa = d_W - \lambda_1^{\#}$ from Assumption 4.2 is used throughout this section.

Remark 5.2. *By Assumption 4.2, the condition $\delta > 1-\frac{\kappa}{d_W}$ is equivalent to $\alpha^{\#}_{\delta,1} < 1$, for in this case $\alpha^{\#}_{\delta,1} = \frac{1}{\delta}\left(1-\frac{\kappa}{d_W}\right)$.*

Theorem 5.3. *For $\delta > 1-\frac{\kappa}{d_W}$ we have $\mathbf{B}_\delta^{1,\alpha^{\#}_{\delta,1}}(X) = BV(X)$ and there exist constants $c, C > 0$ such that for every $f \in BV(X)$,*

$$\mathbf{Var}(f) \simeq \|f\|_{\delta,1,\alpha^{\#}_{\delta,1}}.$$

Proof. By Theorem 3.1 we know that when $\alpha^{\#}_{\delta,1} < 1$, we have $\mathbf{B}_\delta^{1,\alpha^{\#}_{\delta,1}}(X) = KS^{\lambda,1}(X) = \mathcal{KS}^{\lambda,1}(X)$ with $\lambda = \alpha^{\#}_{\delta,1}\delta d_W$. Note that $BV(X) = \mathbf{B}_\delta^{1,\alpha^{\#}_{\delta,1}}(X)$. By considering the non-triviality of the spaces $\mathbf{B}_\delta^{1,\alpha}(X)$ for $\alpha < \alpha^{\#}_{\delta,1}$, we see that $\lambda_1^{\#} = \alpha^{\#}_{\delta,1}\delta d_W$.
According to [5, Proposition 4.1, Theorem 4.9], the weak Bakry-Émery estimate implies that the Korevaar-Schoen norm $\|f\|_{KS^{d_W-\kappa,1}(X)}$ is bounded above and below by $\mathbf{Var}(f)$. Since $d_W - \kappa = \lambda_1^{\#}$ by Assumption 4.2 and $\alpha^{\#}_{\delta,1} < 1$ we obtain the result from Corollary 3.6. $\square$

It is important to note that generally $\mathbf{B}_\delta^{1,\alpha}(X)$ depends on both α and the subordination parameter δ, but from Theorem 3.1 and Theorem 4.3 we know that $\mathbf{B}_\delta^{1,\alpha^{\#}_{\delta,1}}(X) = \mathcal{KS}^{d_W-\kappa,1}(X)$ does not depend on δ when $\delta > 1-\frac{\kappa}{d_W}$, though the Besov norm could depend on δ. Accordingly, we can restate the characterization in Theorem 5.3 in a manner that emphasizes the distinction between this result and [5, Theorem 4.9].
A key feature of any function space that is considered for candidacy of BV class is that almost every super-level set of a BV function is in $BV(X)$ and the BV norm can be reconstructed

from that of the level sets (the co-area formula). This feature is verified for our candidate $BV(X)$ next. In addition, we have a pseudo-Poincaré inequality for $p=1$. Pseudo-Poincaré inequalities were introduced in [32, 60] and are a useful tool to prove Sobolev inequalities, see e.g. [61] and [1].

Theorem 5.4. *Let $\delta > 1 - \frac{\kappa}{d_W}$.*

(i) *For any non-negative $f \in L^1(X,\mu)$,*

$$\int_0^\infty \|\mathbf{1}_{E_t(f)}\|_{\delta,1,\alpha^{\#}_{\delta,1}}\,dt \simeq \int_0^\infty \mathbf{Var}(\mathbf{1}_{E_t(f)})\,dt \simeq \|f\|_{\delta,1,\alpha^{\#}_{\delta,1}}.$$

where $E_t(f) := \{x \in X : f(x) > t\}$ is a super-level set. In particular, $\mathbf{1}_{E_t(f)} \in \mathbf{B}^{1,\alpha^{\#}_{\delta,1}}_\delta(X)$ for any $f \in \mathbf{B}^{1,\alpha^{\#}_{\delta,1}}_\delta(X)$ and almost every $t>0$.

(ii) *For every $f \in \mathbf{B}^{1,\alpha^{\#}_{\delta,1}}_\delta(X)$, we have the following pseudo-Poincaré inequality*

$$\|P^{(\delta)}_t f - f\|_{L^1(X,\mu)} \le C t^{\alpha^{\#}_{\delta,1}} \mathbf{Var}(f).$$

Proof. By virtue of Theorem 5.3, the first two claims follow from [5, Theorem 4.15].
To prove the claim related to the pseudo-Poincaré inequalities, we argue as follows. By our assumptions, the subordinated heat semigroup $P^{(\delta)}_t$ is also conservative. Hence, applying (7), Fubini's theorem, and [5, Proposition 3.10] (where β_1 stands in for $1-\frac{\kappa}{d_W}$), we get

$$\begin{aligned}
\|P^{(\delta)}_t f - f\|_{L^1(X,\mu)} &= \int_X \Big| \int_0^\infty \eta^{(\delta)}_t(s) P_s(f - f(x))(x)\,ds \Big|\, d\mu(x) \\
&\le \int_0^\infty \eta^{(\delta)}_t(s) \|P_s f - f\|_{L^1(X,\mu)}\,ds \\
&\le C \int_0^\infty \eta^{(\delta)}_t(s)\, s^{1-\frac{\kappa}{d_W}}\,ds \left(\liminf_{\tau\to 0^+} \frac{1}{\tau^{1-\frac{\kappa}{d_W}}} \int_X P_\tau(|f-f(x)|)(x)\,d\mu(x) \right).
\end{aligned}$$

Since $\alpha^{\#}_{\delta,1} < 1$, we have that $\alpha^{\#}_{\delta,1} = \frac{1}{\delta}(1-\kappa/d_W)$. From [5, Lemma 4.13] we have

$$\mathbf{Var}(f) \approx \liminf_{t\to 0^+} \frac{1}{t^{1-\frac{\kappa}{d_W}}} \int_X \int_X p_t(x,y)|f(y)-f(x)|\,d\mu(y)\,d\mu(x).$$

Now, by using (10) and Theorem 5.3, we see that the expression on the right-hand side of the above displayed ineqality is bounded by $C\,\frac{\Gamma(1-\alpha^{\#}_{\delta,1})}{\Gamma(\kappa/d_W)} t^{\alpha^{\#}_{\delta,1}} \mathbf{Var}(f)$. □

We next prove the Sobolev embedding property and isoperimetric inequality under an additional assumption on the dimension d_H.

Theorem 5.5. *Assume that $d_W - \kappa < d_H$. Then $\mathbf{B}^{1,d_W-\kappa}_\delta(X) = BV(X) \subset L^{d_H/(d_H-d_W+\kappa)}(X)$ and there exists $C>0$ such that for $f \in BV(X)$,*

$$\|f\|_{L^{d_H/(d_H-d_W+\kappa)}(X)} \le C\|f\|_{\delta,1,d_W-\kappa}.$$

In particular, for any set E for which $\mathbf{1}_E \in BV$, we have

$$\mu(E)^{\frac{d_H-d_W+\kappa}{d_H}} \le C\|\mathbf{1}_E\|_{\delta,1,d_W-\kappa}.$$

Proof. This theorem follows from combining [5, Theorem 4.18] with Theorem 5.3. ☐

In classical analysis the gradient of a BV function is a vector-valued signed Radon measure. The natural candidate in our setting, when $\alpha^{\#}_{\delta,1} < 1$, was identified in [5] in the following manner. For $f \in L^1(X,\mu)$, let

$$\mathcal{M}_r f(x) := \frac{1}{r^{d_W-\kappa}\mu(B(x,r))}\int_{B(x,r)} |f(x)-f(y)|d\mu(y) \tag{18}$$

so that $\mathbf{Var}(f) = \liminf_{r\to 0^+}\int_X \mathcal{M}_r f(x)\,d\mu(x)$. A BV measure is then a weak*-limit of a subsequence of the family of Radon measures $\mathcal{M}_r f d\mu$, $r>0$; note that as $\mathbf{Var}(f)$ is finite, the total mass of each of these Radon measures is uniformly bounded, and so by the Riesz representation theorem such a subsequence and limit exists. This limit measure can be considered to be a BV measure of f.
Recall that we assume $\alpha^{\#}_{\delta,1} < 1$.

Definition 5.6. *Let $f \in \mathbf{B}^{1,\alpha^{\#}_{\delta,1}}_{\delta}(X) = BV(X)$. A BV measure γ_f is a Radon measure on X such that there exists a sequence $r_n \to 0^+$, such that for every $g \in C_0(X)$*

$$\lim_{n\to+\infty}\int_X g\mathcal{M}_{r_n} f d\mu = \int_X g d\gamma_f. \tag{19}$$

The existence of at least one BV measure associated with a given function implies that the function is in $BV(X)$.

Proposition 5.7. *Let $f \in L^1(X,\mu)$ and assume that there is a monotone decreasing sequence of real numbers $r_n \to 0^+$ and a finite Radon measure γ_f on X such that (19) holds for all $g \in C_0(X)$. Then, $f \in BV(X)$ and $\mathbf{Var}(f) \le \gamma_f(X)$.*

Proof. The weak*-convergence to a finite Radon measure implies that the sequence of measures $\mathcal{M}_{r_n} f d\mu$ is tight. Thus $\liminf_{r\to 0^+}\int_X \mathcal{M}_r f d\mu \le \gamma_f(X) < \infty$ and the result follows by Theorem 4.9 in [5]. ☐

The existence and properties of BV measures when $\alpha^{\#}_{\delta,1} < 1$ can be proved by subordination of results for the original semigroup. The following result was proved as Theorem 4.24 of [5] for the situation where $\delta = 1$ and $p_t^{(\delta)}$ is replaced by p_t; what is added here are the estimates (20). Note that the functions $\mathcal{D} = \{P_\epsilon u : \epsilon > 0, u \in C_c(X)\}$ are supremum-norm dense in $C_0(X)$ because P_t is Feller.

Theorem 5.8. *Let $\delta > 1 - \frac{\kappa}{d_W}$. Then,*

(i) *there is at least one BV measure associated to $f \in \mathbf{B}^{1,\alpha^{\#}_{\delta,1}}_{\delta}(X)$;*

(ii) *the class of all such BV measures associated to $f \in \mathbf{B}^{1,\alpha^{\#}_{\delta,1}}_{\delta}(X) \cap L^\infty(X,\mu)$ are mutually absolutely continuous;*

(iii) *for any* $f \in \mathbf{B}_\delta^{1,\alpha_{\delta,1}^\#}(X) \cap L^\infty(X,\mu)$ *and any associated BV measure* γ_f*, if* $g \in \mathcal{D}$ *with* $g \geq 0$*, then*

$$\limsup_{t\to 0^+} \int_X g(y)\mathcal{Q}_t^{(\delta)} f(y) d\mu(y) \simeq \int_X g(y) d\gamma_f(y), \tag{20}$$

where

$$\mathcal{Q}_t^{(\delta)} f(y) = t^{-\frac{1}{\delta}\left(1-\frac{\kappa}{d_W}\right)} \int_X p_t^{(\delta)}(x,y)|f(x)-f(y)|\, d\mu(x).$$

Proof. Since $\mathbf{B}_\delta^{1,\alpha_{\delta,1}^\#}(X) = BV(X)$ the existence of a BV measure and the mutual absolute continuity of BV measures are established in Lemma 4.22 and Theorem 4.24 of [5]. Theorem 4.24 of [5] also provides the stated bounds for the case where $\delta = 1$, $p_t^{(\delta)}$ is replaced by p_t and the corresponding operator from (20) is denoted $\mathcal{Q}_t$. If we now expand the subordination (7) and use the Fubini theorem we have

$$\begin{aligned}
\int_X g(y)\mathcal{Q}_t^{(\delta)} f(y)\, d\mu(y) &= t^{-\frac{1}{\delta}\left(1-\frac{\kappa}{d_W}\right)} \int_X\int_X p_t^{(\delta)}(x,y)|f(x)-f(y)|g(y)\, d\mu(x)\, d\mu(y) \\
&= t^{-\frac{1}{\delta}\left(1-\frac{\kappa}{d_W}\right)} \int_0^\infty \eta_t^{(\delta)}(s) \int_X\int_X p_s(x,y)|f(x)-f(y)|g(y)\, d\mu(x)\, d\mu(y)\, ds \\
&= \int_0^\infty \eta_t^{(\delta)}(s) \left(\frac{s}{t^{1/\delta}}\right)^{1-\frac{\kappa}{d_W}} \int_X g(y)\mathcal{Q}_s f(y)\, d\mu(y)\, ds.
\end{aligned}$$

However, for some constant $C > 0$, $C\eta_t^{(\delta)}(s)\left(\frac{s}{t^{1/\delta}}\right)^{1-\frac{\kappa}{d_W}}$ is an approximate identity at 0 as $t \to 0$ by virtue of (8), (9) and (10); the latter is applicable because $\frac{1}{\delta}\left(1-\frac{\kappa}{d_W}\right) = \alpha_1^\# < 1$ by hypothesis. Moreover, $g \in \mathcal{D}$ and it is immediate from the definition of the BV norm that

$$\left|\int_X g(y)\mathcal{Q}_t f(y)\, d\mu(y)\right| \leq \|g\|_\infty \|f\|_{1,1-\kappa/d_W}.$$

so we conclude that

$$\begin{aligned}
\liminf_{t\to 0+} \int_X g(y)\mathcal{Q}_t^{(\delta)} f(y)\, d\mu(y) &= C \liminf_{t\to 0+} \int_X g(y)\mathcal{Q}_t f(y)\, d\mu(y) \\
\limsup_{t\to 0+} \int_X g(y)\mathcal{Q}_t^{(\delta)} f(y)\, d\mu(y) &= C \limsup_{t\to 0+} \int_X g(y)\mathcal{Q}_t f(y)\, d\mu(y)
\end{aligned}$$

and the result follows from the estimate in Theorem 4.24 of [5], in which the constants do not depend on δ. □

5.2 The case $\delta < \left(1 - \frac{\kappa}{d_W}\right)$: Characterizing $W^{\delta d_W,1}$ by heat Besov spaces.

In this section we assume Assumption 4.2, but use it *only* to determine (from Theorem 4.3) that when $\delta \leq \left(1 - \frac{\kappa}{d_W}\right)$ we have $\alpha_{\delta,1}^\# = 1$. Then from Proposition 3.3 we see that $\mathbf{B}_\delta^{1,1}(X) = W^{\delta d_W,1}(X)$. We will show in this section that the space $\mathbf{B}_\delta^{1,1}(X)$ has all of the properties established for the critical Besov space with $\alpha^{\#}_{\delta,1} < 1$, with appropriate modifications. However, rather than relying on the weak Bakry-Émery property via the results of [5], the proofs here are more elementary, and rest mostly on the proof of Proposition 3.3. We begin by considering the notion of variation and the co-area formula. In a Sobolev space one might expect that the variation is just the Sobolev norm. That this is the correct interpretation here may be seen by comparing Definition 5.1 for $\mathbf{Var}(f)$ to the left side of the following equation, the validity of which was established in the proof of Proposition 3.3.

$$\liminf_{t\to 0^+} \frac{1}{t}\int_X\int_X |f(x)-f(y)|p_t^{(\delta)}(x,y)d\mu(x)d\mu(y) \simeq \int_X\int_X \frac{|f(x)-f(y)|}{d(x,y)^{d_H+\delta d_W}}\,d\mu(y)d\mu(x) = \|f\|_{W^{\delta d_W,1}}.$$

When $X = \mathbb{R}^d$, one has $d_H = d$, $d_W = 2$, $\kappa = 1$ and the above notion of variation of a function related to the fractional Laplacian $(-\Delta)^\delta$, $\delta \le 1/2$, coincides with the notion of fractional variation and associated fractional perimeter which are extensively studied in relation to the theory of non-local minimal surfaces, see for instance [6,20,27,28,35,37] and the references therein.

Under this interpretation of the variation, the co-area formula and the pseudo-Poincaré inequality read as follows.

Theorem 5.9. *Let $\delta \le 1 - \frac{\kappa}{d_W}$. Then,*

(i) *for any non-negative $f \in L^1(X,\mu)$ and $t > 0$,*

$$\|f\|_{\delta,1,1} = \int_0^\infty \|\mathbf{1}_{E_t(f)}\|_{\delta,1,1}\,dt.$$

In particular, for each $f \in \mathbf{B}^{1,1}_\delta(X)$ we have that $\mathbf{1}_{E_t(f)} \in \mathbf{B}^{1,1}_\delta(X)$ almost every $t > 0$;

(ii) *if $\mathbf{1}_{E_t(f)} \in \mathbf{B}^{1,1}_\delta(X)$ for almost every $t > 0$ and $\int_0^\infty \|\mathbf{1}_{E_t(f)}\|_{\delta,1,1}\,dt < \infty$, then $f \in \mathbf{B}^{1,1}_\delta(X)$;*

(iii) *for any $f \in \mathbf{B}^{1,1}_\delta(X)$,*

$$\|P_t^{(\delta)} f - f\|_{L^1(X,\mu)} \le Ct\int_X\int_X \frac{|f(x)-f(y)|}{d(x,y)^{d_H+\delta d_W}}\,d\mu(y)d\mu(x).$$

Proof. Since $f \ge 0$, for μ-almost every $x, y \in X$ we can write

$$|f(y)-f(x)| = \int_0^{+\infty} |\mathbf{1}_{E_t(f)}(x) - \mathbf{1}_{E_t(f)}(y)|dt.$$

Therefore,

$$W_{\delta d_W,1}(f) = \int_X\int_X \frac{|f(x)-f(y)|}{d(x,y)^{d_H+\delta d_W}}\,d\mu(y)d\mu(x) = \int_X\int_X\int_0^{+\infty} \frac{|\mathbf{1}_{E_t(f)}(x) - \mathbf{1}_{E_t(f)}(y)|}{d(x,y)^{d_H+\delta d_W}}\,dt\,d\mu(y)d\mu(x),$$

and (i) follows from Fubini's theorem and Proposition 3.3 for non-negative functions f. To prove (iii), note that

$$\|P_t^{(\delta)} f - f\|_{L^1(X,\mu)} \le \int_X\int_X p_t^{(\delta)}(x,y)|f(x)-f(y)|\,d\mu(y)\,d\mu(x).$$

The assertion follows thus as in (16). ∎

To establish Sobolev and isoperimetric inequalities for $W^{\delta d_W,1}(X)$ we begin with the Sobolev embedding established for a heat Besov space in [3].

Theorem 5.10. *Suppose that $d_H > \delta d_W$. Then there exists a constant $C > 0$ such that for every $f \in \mathbf{B}^{1,1}_\delta(X)$,*

$$\|f\|_{L^q(X,\mu)} \le C \int_X \int_X \frac{|f(x)-f(y)|}{d(x,y)^{d_H+\delta d_W}}\, d\mu(y) d\mu(x) = C\|f\|_{W^{\delta d_W,1}},$$

where $q = \frac{d_H}{d_H - \delta d_W}$.

Proof. Since Corollary 3.4 establishes the property $(P_{1,1})$ defined in [3, Definition 6.7], the assertion follows from [3, Theorem 6.9] and Proposition 3.3. $\square$

In the above theorem, we could have merely required that $f \in L^1_{loc}(X)$, for if $f \notin \mathbf{B}^{1,1}_\delta(X)$, then the right-hand side of the claimed inequality there would be infinite and so the inequality would trivially hold. Therefore, as an immediate corollary, we have the following fractional isoperimetric inequality, which is the global analogue in our setting of the known fractional relative isoperimetric inequalities noted in the Euclidean setting in [37], see also [35].

Corollary 5.11. *Assume that $d_H > \delta d_W$. There is $C > 0$ such that for every measurable set $E \subset X$,*

$$\mu(E)^{\frac{d_H - \delta d_W}{d_H}} \le C \int_E \int_{X \setminus E} \frac{1}{d(x,y)^{d_H+\delta d_W}}\, d\mu(y) d\mu(x).$$

The endpoint case $d_H = \delta d_W$ is that at which we get an embedding into L^∞. In the following theorem, note that since we always assume $\delta < 1$, the condition $d_H = \delta d_W$ can only hold when $d_H < d_W$. Moreover, let us stress that we actually do not know examples of X for which $\mathbf{B}^{1,1}_\delta(X)$ is non-trivial when $d_H = \delta d_W$ (see the remark at the end of Example 4.4).

Proposition 5.12. *Assume $d_H = \delta d_W$. Then, $\mathbf{B}^{1,1}_\delta(X) \subset L^\infty(X,\mu)$ and there exists a constant $C > 0$ such that for every $f \in \mathbf{B}^{1,1}_\delta(X)$*

$$\|f\|_{L^\infty(X,\mu)} \le C \int_X \int_X \frac{|f(x)-f(y)|}{d(x,y)^{d_H+\delta d_W}}\, d\mu(y) d\mu(x).$$

Proof. Let $f \in \mathbf{B}^{1,1}_\delta(X)$. Without loss of generality, we assume $f \ge 0$ almost everywhere. For $t \ge 0$ we define the set $E_t(f) = \{x \in X : f(x) > t\}$. Since $d_H = \delta d_W$, according to [3, Corollary 6.6], there is $c > 0$ such that for every set E of positive measure satisfying $\|\mathbf{1}_E\|_{1,1} < +\infty$, one has $\|\mathbf{1}_E\|_{1,1} \ge c$. However, from Theorem 5.9,

$$\int_0^\infty \|\mathbf{1}_{E_t(f)}\|_{1,1}\, dt = \|f\|_{1,1} < +\infty.$$

Therefore, the set $\Sigma(f) := \{t > 0 :\ \mu(E_t(f)) > 0\}$ has finite Lebesgue measure. Indeed, we have

$$c\,\mathcal{H}^1(\Sigma(f)) \le \|f\|_{1,1}.$$

Note that if $0 < t_1 < t_2$, then $E_{t_2}(f) \subset E_{t_1}(f)$; and so $\Sigma(f)$ is an interval of the form $[0, \|f\|_{L^\infty(X,\mu)})$; and so we obtain the truth of the claim in the proposition from the above inequality. $\square$

From Theorem 5.10, Corollary 5.11 and Proposition 5.12 we deduce the following result for the Sobolev space $W^{\delta d_W,1}$. Note that the fact we identify the critical range for δ distinguishes our result from the corresponding one that appears, with a different proof, in [7, Theorem 9.1].

Corollary 5.13. *Suppose $\delta \le 1 - \frac{\kappa}{d_W}$. If $\delta < \frac{d_H}{d_W}$ then there is $C > 0$ so that any $f \in W^{\delta d_W,1}(X)$ satisfies $\|f\|_{L^q} \le C\|f\|_{W^{\delta d_W,1}}$ and any measureable set satisfies*

$$\mu(E)^{\frac{1}{q}} \le C \int_E \int_{X\setminus E} \frac{1}{d(x,y)^{d_H+\delta d_W}}\, d\mu(y) d\mu(x),$$

where $q = \frac{d_H}{d_H - \delta d_W}$. Moreover, $W^{d_H,1}(X) \subset L^\infty(X)$ with $\|f\|_{L^\infty} \le \|f\|_{W^{d_H,1}}$.

Finally, we observe that the notion of a BV measure from Definition 5.6 was intended to be an analogue of the absolute value measure associated with the gradient. In an L^1 Sobolev space we would expect this to be an L^1 function multiplied by the measure $d\mu$. Again, this expectation is fulfilled. If we consider a quantity analogous to $\mathcal{M}_r f(y)$ from (18) it should be

$$\tilde{\mathcal{M}}_r f(x) = \frac{1}{r^{\delta d_W} \mu(B(x,r))} \int_{B(x,r)} |f(x) - f(y)|\, d\mu(y),$$

although the corresponding non-local variation is not a limit but the integral

$$\int_0^\infty \int_X \tilde{\mathcal{M}}_r f(x)\, d\mu(x) \frac{dr}{r};$$

this is readily verified by comparing it to the $W^{\delta d_W,1}$ norm of f. The natural definition of a corresponding BV measure γ_f would then be absolutely continuous with respect to μ, with the Radon-Nikodym derivative

$$\frac{d\gamma_f}{d\mu}(x) = \int_0^\infty \tilde{\mathcal{M}}_r f(x) \frac{dr}{r} = \int_X \left(\int_{d(x,y)}^\infty \frac{1}{r^{\delta d_W} \mu(B(x,r))} \frac{dr}{r} \right) |f(x) - f(y)|\, d\mu(y) \tag{21}$$

so that by Ahlfors regularity $\frac{d\gamma_f}{d\mu}(x) \simeq \int_X |f(x) - f(y)| d(x,y)^{-(\delta d_W + d_H)}\, d\mu(y)$. Since $f \in W^{\delta d_W,1}(X)$ we see $\frac{d\gamma_f}{d\mu} \in L^1(\mu)$. The preceding justifies defining a BV measure in the situation where the non-locality dominates as follows.

Definition 5.14. *If $\alpha^\#_{\delta,1} = 1$ and $f \in \mathbf{B}^{1,1}_\delta(X) = W^{\delta d_W,1}(X)$, the unique BV measure γ_f corresponding to f is the measure γ_f with density as in (21).*

6 Further properties of $W^{\delta d_W/p,p}$

We saw in Theorem 4.1 that whenever $p \ge 2\delta$ the critical exponent $\alpha^\#_{\delta,p} = \frac{1}{p}$. According to Proposition 3.3 the corresponding space $\mathbf{B}^{p,1/p}_\delta(X)$ is then the fractional Sobolev space $W^{\delta d_W/p,p}(X)$, and the seminorms are equivalent. The purpose of this section is to record some properties of these fractional Sobolev spaces that can be established using their heat semigroup characterization.

6.1 $W^{\delta d_W/p,p}(X)$ is dense in $L^p(X,\mu)$ for $p \geq 2$.

Proposition 6.1. *Let $p \geq 2$. There exists $C > 0$ such that for any $t > 0$ and $f \in L^p(X,\mu)$*

$$\|P_t^{(\delta)} f\|_{\delta,p,1/p} \leq \frac{C}{t^{1/p}} \|f\|_{L^p(X,\mu)}.$$

Proof. When $f \in L^\infty(X,\mu)$, we have for a.e. $x, y \in X$,

$$|P_t f(x) - P_t f(y)| \leq 2\|f\|_{L^\infty(X,\mu)}.$$

It follows that, for any $s > 0$, the operator $\mathcal{P}_t^{(\delta)} : L^\infty(X,\mu) \to L^\infty(X \times X, p_s^{(\delta)} \mu \otimes \mu)$ defined by $\mathcal{P}_t^{(\delta)} f(x,y) = P_t^{(\delta)} f(x) - P_t^{(\delta)} f(y)$ has its operator norm bounded by 2. On the other hand, we know from [3, Theorem 5.1] that

$$\|P_t^{(\delta)} f\|_{\delta,2,1/2} \leq \frac{C}{t^{1/2}} \|f\|_{L^2(X,\mu)},$$

i.e. $\mathcal{P}_t^{(\delta)} : L^2(X,\mu) \to L^2(X \times X, p_s^{(\delta)} \mu \otimes \mu)$ is bounded by $C(s/t)^{1/2}$. The Riesz-Thorin interpolation theorem now yields that $\mathcal{P}_t^{(\delta)} : L^p(X,\mu) \to L^p(X \times X, p_s \mu \otimes \mu)$ is bounded by $C(s/t)^{1/p}$, hence

$$\frac{1}{s^{1/p}} \left(\int_X \int_X p_t^{(\delta)}(x,y) |P_t^{(\delta)} f(x) - P_t^{(\delta)} f(y)|^p d\mu(y)\, d\mu(x) \right)^{1/p} \leq \frac{C}{t^{1/p}} \|f\|_{L^p(X,\mu)}.$$

Taking the supremum over $s > 0$ on the left hand side yields the result. □

A consequence of the previous proposition is that the Sobolev space $W^{\delta d_W/p,p}(X)$ is large.

Corollary 6.2. *For $p \geq 2$, $\mathbf{B}_\delta^{p,1/p}(X) = W^{\delta d_W/p,p}(X)$ is dense in $L^p(X,\mu)$.*

Proof. Let $f \in L^p(X,\mu)$. Proposition 6.1 provides that for $t > 0$, $P_t f \in W^{\delta d_W/p,p}(X)$, but by the L^p-strong continuity of the heat semigroup we have $\|P_t f - f\|_{L^p(X,\mu)} \to 0$ when $t \to 0^+$. □

6.2 Sobolev and isoperimetric inequalities

The following result generalizes Theorem 5.10, and has the same proof.

Theorem 6.3. *Let $p \geq 1$. Assume $d_H > \delta d_W$. There is $C > 0$ such that for every $f \in \mathbf{B}_\delta^{p,1/p}(X)$,*

$$\|f\|_{L^q(X,\mu)} \leq C \left(\int_X \int_X \frac{|f(x) - f(y)|^p}{d(x,y)^{d_H + \delta d_W}} d\mu(y) d\mu(x) \right)^{1/p},$$

where $q = \frac{p d_H}{d_H - \delta d_W}$.

Proof. In Corollary 3.4 we saw that the condition $(P_{p,1/p})$ of [3, Definition 6.7] holds. By assumption $\frac{d_H}{\delta d_W} > 1 \geq \frac{1}{p}$, hence [3, Theorem 6.9] yields the desired inequality. □

This provides a corresponding result for the Sobolev space.

Corollary 6.4. *Let $p \geq 1$, $\delta \leq 1 - \frac{\kappa}{d_W}$ and $\delta < \frac{d_H}{d_W}$. Every $f \in W^{\delta d_W/p,p}(X)$ satisfies*

$$\|f\|_{L^q(X,\mu)} \leq C \|f\|_{W^{\delta d_W/p,p}}$$

for $q = \frac{p d_H}{d_H - \delta d_W}$.

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