POLYNOMIALLY CONVEX SETS WHOSE UNION HAS NONTRIVIAL HULL

ALEXANDER J. IZZO

Dedicated to the memory of Peter Duren

Abstract. Several results concerning pairs of polynomially convex sets whose union is not even rationally convex are given. It is shown that there is no restriction on how two spaces can be embedded in some $\mathbb{C}^N$ so as to be polynomially convex but have nonrationally convex union. It is shown that there exist two disjoint polynomially convex Cantor sets in $\mathbb{C}^3$ whose union is not rationally convex. The analogous assertion for arcs is also established. As an application, it is shown that every simple closed curve in $\mathbb{C}^N$, $N \geq 3$, can be approximated uniformly by locally polynomially convex simple closed curves that are not rationally convex.

1. Introduction

The motivation for the work presented here is a recent result of the author and Lee Stout [10, Theorem 3.1]: Given an arbitrary polynomially convex simple closed curve γ, every rectifiable simple closed curve σ sufficiently close to γ in the uniform metric is also polynomially convex. Note that although there is no regularity hypothesis on the curve γ, the curve σ is required to be rectifiable. In fact, it follows from the existence of nonpolynomially convex arcs that the statement becomes false without the rectifiability hypothesis; one can obtain a nonpolynomially convex simple closed curve arbitrarily close to γ by choosing a nonpolynomially convex arc contained in a small neighborhood of some point of γ and modifying γ so as to contain that nonpolynomially convex arc. (The existence of a nonpolynomially convex arc was

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first proved by John Wermer [18] and can be found in standard texts, e.g., [1, pp. 222-223], [3, pp. 29 and 70], and [17, pp. 53–54].) However, every *rectifiable* arc is polynomially convex, and hence every *rectifiable* simple closed curve is *locally* polynomially convex. One might, therefore, conjecture that the above result of the author and Stout could be generalized to assert that every locally polynomially convex simple closed curve sufficiently near a polynomially convex simple closed curve is itself polynomially convex. One of the main goals of the present paper is to show that, on the contrary, every simple closed curve γ in $\mathbb{C}^N$, $N \geq 3$, can be approximated uniformly by a locally polynomially convex, nonpolynomially convex simple closed curve. In fact, the approximating simple closed curve σ can be taken to be the union of *two* polynomially convex arcs whose interiors cover σ (and such that, in fact, σ is not even rationally convex) (Theorem 4.1). Whether the same holds in $\mathbb{C}^2$ remains open.

One possible approach to trying to prove the assertion just made regarding curves in $\mathbb{C}^N$, $N \geq 3$, is to attempt to construct two disjoint arcs near γ each of which is polynomially convex but whose union is not polynomially (or rationally) convex, and then to attach to these two disjoint arcs additional arcs to form a simple closed curve. We are thus led to the question of whether there exist two disjoint polynomially convex arcs whose union is not polynomially convex. More generally, the question arises as to what disjoint polynomially convex sets can have nonpolynomially convex union. A fundamental question here is whether there exist two disjoint polynomially convex Cantor sets whose union is not polynomially convex.

We will establish quite general results regarding the issue of pairs of polynomially convex sets whose union is not polynomially convex. Roughly, we will show that there is *no* restriction on how two spaces can be embedded in some $\mathbb{C}^N$ so as to be polynomially convex but have nonpolynomially convex union, and even nonrationally convex union (Theorem 3.13 and Corollary 3.14) . We will also obtain several results about embedding particular spaces in $\mathbb{C}^N$ for particular values of N. In particular, we will show that there exist two disjoint polynomially convex Cantor sets in $\mathbb{C}^3$ whose union is not rationally convex (Corollary 3.5). The same holds for arcs in place of Cantor sets (Corollary 3.6). We will also obtain two polynomially convex Cantor sets in $\mathbb{C}^2$ whose union is not rationally convex (Theorem 3.11); however, in that setting we do not know whether the polynomially convex Cantor sets can be taken to be disjoint.

An interesting related question is whether there exists a minimal nonpolynomially convex Cantor set in some $\mathbb{C}^N$, that is, whether there

is a nonpolynomially convex (or nonrationally convex) Cantor set every closed subset of which is polynomially convex. This question remains open for sets in a finite-dimensional $\mathbb{C}^N$. However, a theorem of Kenneth Hoffman and Isadore Singer [4, pp. 222–223], later generalized by I. Netuka, A. G. O'Farrell, and M. A. Sanabria-García [13, Theorem 4.1], shows that the question has an affirmative answer in a more general abstract uniform algebra setting. We will give a proof, different from those of Hoffman and Singer and of Netuka, O'Farrell, and Sanabria-García, of a special case of the result of Netuka, O'Farrell, and Sanabria-García that contains the result of Hoffman and Singer. This proof motivates the proof of Theorem 3.3.

Stout raised the question of whether the union of the closed unit ball in $\mathbb{C}^N$ and a polynomially convex arc that meets the ball precisely in a single end point must be polynomially convex. When the arc is rectifiable, polynomial convexity of the union is a consequence of a standard result [17, Theorem 3.1.1]. As noted by Stout, applying [11, Theorem 1.6] shows that polynomial convexity of the union holds more generally whenever the interior of the arc is locally rectifiable (Theorem 5.1). We will show that nevertheless, in general, the union of the closed unit ball and a polynomially convex arc that meets the ball in precisely a single end point need not be even rationally convex (Corollary 5.4). We will also give an example of a polynomially convex arc disjoint from the closed unit ball such that the union is not rationally convex.

It was once conjectured that whenever the polynomial hull $\widehat{X}$ of a compact set X in $\mathbb{C}^N$ is strictly larger than X, the complementary set $\widehat{X} \setminus X$ must contain an analytic disc. This conjecture was disproved by Gabriel Stolzenberg [16]. There are now many known counterexamples due to several mathematicians. By combining our results on pairs of polynomially convex sets whose union has nontrivial hull with a construction of nontrivial hulls without analytic discs in the paper of the author and Norman Levenberg [9], we will establish results on pairs of polynomially convex sets whose union has nontrivial hull without analytic discs (Theorems 6.1 and 6.2 and Corollary 6.3).

All of the examples we will produce rely on the same fundamental idea originally introduced by Wermer [18] to construct an arc with nontrivial hull in $\mathbb{C}^3$ and developed further by Walter Rudin [15] to construct in $\mathbb{C}^2$ both a Cantor set with nontrivial hull and an arc with nontrivial hull. In the next section, in addition to defining terminology and notation (some of which has already been used above) and presenting some preliminary results, we will introduce a certain algebra

A_J on which these constructions depend and summarize the facts we will need about A_J.

Our main results concerning pairs of polynomially convex sets whose union is not rationally convex are treated in Section 3. The approximation of simple closed curves by locally polynomially convex, nonrationally convex simple closed curves is treated in Section 4. The results concerning the union of the closed unit ball and a polynomially convex arc are in Section 5. The results on hulls without analytic discs are in the concluding Section 6.

It is with a mixture of joy and sorrow that I dedicate this paper to the memory of Peter Duren. Sorrow, of course, that he is no longer with us; joy that I had the privilege of knowing him.

2. Preliminaries

By definition an *arc* is a space homeomorphic to the closed unit interval, and a *simple closed curve* is a space homeomorphic to the unit circle. An *open arc* is a space homeomorphic to the open unit interval, and a *half-open arc* is a space homeomorphic to a half-open unit interval. For convenience we will also use the term *arc* (or *simple closed curve*) to refer to a topological embedding whose domain is an interval (or a circle). Frequently we will use the common abuse of notation and conflate such an embedding with its image. By a *Cantor set* we mean any space that is homeomorphic to the usual middle-thirds Cantor set. We will make repeated use of the well-known characterization of Cantor sets as the compact, totally disconnected, metrizable spaces without isolated points.

We call a space *simply coconnected* if its first Čech cohomology group with integer coefficients is zero. It is well known that a compact set in the plane is simply coconnected if and only if its complement is connected.

By the topological dimension of a space we mean the usual Lebesgue covering dimension.

For X a compact Hausdorff space, we denote by $C(X)$ the algebra of all continuous complex-valued functions on X with the supremum norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$. A *uniform algebra* on X is a closed subalgebra of $C(X)$ that contains the constant functions and separates the points of X. The maximal ideal space of a uniform algebra A will be denoted by $\mathfrak{M}_A$.

For a compact set X in $\mathbb{C}^N$, the *polynomial hull* $\widehat{X}$ of X is defined by

$$\widehat{X} = \{z \in \mathbb{C}^N : |p(z)| \leq \max_{x \in X} |p(x)| \text{ for all polynomials } p\},$$

and the *rational hull* $h_r(X)$ of X is defined by

$$h_r(X) = \{z \in \mathbb{C}^N : p(z) \in p(X) \text{ for all polynomials } p\}.$$

An equivalent formulation of the definition of $h_r(X)$ is that $h_r(X)$ consists precisely of those points $z \in \mathbb{C}^N$ such that every polynomial that vanishes at z also has a zero on X. The set X is said to be *polynomially convex* if $\widehat{X} = X$ and *rationally convex* if $h_r(X) = X$. We say that a polynomial hull $\widehat{X}$ (or rational hull $h_r(X)$) is *nontrivial* if $\widehat{X} \setminus X$ (or $h_r(X) \setminus X$) is nonempty.

We denote by $P(X)$ the uniform closure on $X \subset \mathbb{C}^N$ of the polynomials in the complex coordinate functions $z_1, \ldots, z_N$, and we denote by $R(X)$ the uniform closure of the rational functions holomorphic on (a neighborhood of) X. Both $P(X)$ and $R(X)$ are uniform algebras, and it is well known that the maximal ideal space of $P(X)$ can be naturally identified with $\widehat{X}$, and the maximal ideal space of $R(X)$ can be naturally identified with $h_r(X)$.

As usual, we denote the Gelfand transform of a uniform algebra element f by $\hat{f}$. Given a uniform algebra A and a compact subset E of the maximal ideal space $\mathfrak{M}_A$ of A, the *A-convex hull* of E is the set

$$\widehat{E} = \{\phi \in \mathfrak{M}_A : |\hat{f}(\phi)| \leq \|\hat{f}\|_E \text{ for all } f \in A\}.$$

The set E is said to be A-convex if $\widehat{E} = E$. The A-convex hull $\widehat{E}$ of E can be naturally identified with the maximal ideal space of the uniform algebra $\overline{A|E}$ obtained by restricting the functions in A to the set E and taking uniform closure. The A-convex hull of a compact set can be thought of as an abstract polynomial hull. Indeed, when $E \subset X$ are compact sets in $\mathbb{C}^N$ and A is taken to be $P(X)$, the A-convex hull of E coincides with the polynomial hull of E.

There is also an analogous abstract rational hull defined as follows.

Definition 2.1. Given a uniform algebra A and a compact subset E of the maximal ideal space $\mathfrak{M}_A$ of A, we define the *A-rational hull* $h_r(E)$ *of* E to be the set

$$h_r(E) = \{\phi \in \mathfrak{M}_A : \hat{f}(\phi) \in \hat{f}(E) \text{ for all } f \in A\}.$$

We say that E is *A-rationally convex* if $h_r(E) = E$.

An equivalent formulation of the definition of the A-rational hull $h_r(E)$ is that $h_r(E)$ consists precisely of those points $\phi \in \mathfrak{M}_A$ such that if $f \in A$ satisfies $\hat{f}(\phi) = 0$, then $\hat{f}$ has a zero on E. When $E \subset X$ are compact sets in $\mathbb{C}^N$, and A is taken to be $P(X)$, the A-rational hull of E coincides with the usual rational hull of E.

For A a uniform algebra on a compact space X, the *Shilov boundary* for A is the smallest closed subset Γ of X such that $\|f\|_\Gamma = \|f\|_X$ for every function f in A. Thus, the Shilov boundary for A is the smallest closed subset whose A-convex hull is $\mathfrak{M}_A$. (For the existence of the Shilov boundary see [2, pp. 94–95], [3, pp. 9–10], or [17, pp. 15–16].)

By an *analytic disc* in $\mathbb{C}^N$, we mean an injective holomorphic map from an open disc in the complex plane into $\mathbb{C}^N$. By the statement that a subset S of $\mathbb{C}^N$ contains no analytic discs, we mean that there is no analytic disc in $\mathbb{C}^N$ whose image is contained in S.

The following theorem and its corollaries are useful for proving the polynomial or rational convexity of certain sets.

Theorem 2.2. *Let A be a uniform algebra on a compact space X with maximal ideal space $\mathfrak{M}_A$. Suppose $\mathfrak{M}_A \setminus X$ is homeomorphic to a subspace of a space that is a finite union of closed subspaces each of which is a one-dimensional manifold. Then $\mathfrak{M}_A = X$.*

Proof. Assume to get a contradiction that $\mathfrak{M}_A \setminus X$ is nonempty. By hypothesis, there exists a space L that contains $\mathfrak{M}_A \setminus X$ and is of the form $L = L_1 \cup \cdots \cup L_k$ where each L_j is closed in L and is a one dimensional manifold. We may assume that $\mathfrak{M}_A \setminus X$ is contained in the union of no proper subcollection of $\{L_1, \ldots, L_k\}$. Fix a point x of $\mathfrak{M}_A \setminus X$ that is in L_1 but is not in $L_2 \cup \cdots \cup L_k$. Let U be a neighborhood of x in $\mathfrak{M}_A$ whose closure is contained in $\mathfrak{M}_A \setminus X$. By Rossi's local maximum modulus principle [3, Theorem III.8.2], if f is a function in A whose Gelfand transform $\hat{f}$ vanishes identically on ∂U, then $\hat{f}$ must vanish also at x. Since U can be chosen so that its boundary consists of at most two points, this is a contradiction. □

Corollary 2.3. *If the polynomial hull of a compact set $E \subset \mathbb{C}^N$ is contained in an arc or a simple closed curve or a finite union of such sets, then E is polynomially convex. The same statement holds with polynomial convexity replaced by rational convexity.*

Proof. Apply the theorem to the uniform algebra $P(E)$ or $R(E)$. □

As an immediate consequence we obtain the following.

Corollary 2.4. *Suppose $\Sigma \subset \mathbb{C}^N$ is an arc or a simple closed curve or a finite union of such sets. If Σ is polynomially convex, then so is*

every closed subset of Σ. *The same statement holds with polynomial convexity replaced by rational convexity.*

When Σ is an arc or a simple closed curve, this corollary is the finitely generated case of a result of Wilken [20, Lemma 3.1].

Given a compact planar set J, we denote by A_J the algebra of continuous complex-valued functions on the Riemann sphere S^2 that are holomorphic on $S^2 \setminus J$. We summarize here the facts we need about A_J. More detail can be found in many texts, for instance [3, pp. 28–31] and [17, pp. 53–54]. In general, A_J need not contain nonconstant functions. However, whenever A_J does contain a nonconstant function, then there are three functions in A_J that separate points on S^2, and consequently, A_J is a uniform algebra. Indeed, if g is a nonconstant function in A_J and a_1 and a_2 are points in $\mathbb{C} \setminus J$ such that $g(a_1) \neq g(a_2)$, then, regarding $\big((g - g(a_j)\big)/(z - a_j)$, $j = 1, 2$, as functions in A_J, it is well known (and easily verified) that the three functions g, $\big((g - g(a_1)\big)/(z - a_1)$, and $\big((g - g(a_2)\big)/(z - a_2)$ separate points on S^2. The algebra A_J contains a nonconstant function whenever J has positive planar measure, for then the function g defined by

$$g(\zeta) = \iint_J \frac{dx\, dy}{z - \zeta}$$

is such a function. By a theorem of Richard Arens, whenever A_J contains a nonconstant function, the maximal ideal space of A_J is S^2. See [3, Theorem II.1.9] for a proof. It is immediate from the maximum principle that every function in A_J takes its maximum modulus on J, so the A_J-convex hull of J is S^2. In fact, for every function $f \in A_J$, the inclusion $f(S^2) \subset f(J)$ holds, so the A_J-rational hull of J is S^2. For a proof, see [17, p. 54]. It follows immediately that given $f_1, \ldots, f_N \in A_J$ and letting $\pi : S^2 \to \mathbb{C}^N$ be defined by $\pi(z) = \big(f_1(z), \ldots, f_N(z)\big)$, the inclusion $\pi(S^2) \subset h_r(\pi(J))$ holds, and hence $\pi(J)$ fails to be rationally convex whenever $\pi(S^2) \not\subset \pi(J)$.

Proposition 2.5. *Let* $J \subset \mathbb{C}$ *be a compact set such that* A_J *contains a nonconstant function, and let* Γ *denote the Shilov boundary for* A_J. *Then* $A_\Gamma = A_J$.

Proof. This is [3, exercise II.1(a)]. □

Proposition 2.6. *Let* $J \subset \mathbb{C}$ *be a compact set such that* A_J *contains a nonconstant function, and let* Γ *denote the Shilov boundary for* A_J. *Then* $J \setminus \Gamma$ *has planar measure zero.*

Proof. Let $x_0 \in J$ be arbitrary, and suppose that the intersection of each disc centered at x_0 with J has positive measure. We will show that then x_0 is in the Shilov boundary for A_J. The lemma follows.

Let $r > 0$ be arbitrary, and denote the disc of radius r centered at x_0 by $D(x_0, r)$. Choose a compact subset E of $J \cap D(x_0, r)$ of positive measure. Define a function f on S^2 by

$$f(\zeta) = \iint_E \frac{dx\, dy}{z - \zeta}.$$

It is well known that f is in $A_E \subset A_J$ and f is nonconstant in the unbounded component of the complement of E. Thus $|f(\zeta)| < \|f\|_{S^2}$ for every point $\zeta \in S^2 \setminus D(x_0, r)$. Consequently, the Shilov boundary for A_J must intersect $D(x_0, r)$. Since $r > 0$ was arbitrary, this shows that x_0 is in the Shilov boundary for A_J. □

Corollary 2.7. *Let $J \subset \mathbb{C}$ be a compact set that is locally of positive planar measure. Then J is the Shilov boundary for A_J.*

3. Nonrationally convex unions

In this section we establish several results regarding pairs of polynomially convex sets whose union is not even rationally convex. We will begin with a previously known result giving, in the abstract uniform algebra setting, minimal sets with nontrivial hull. Note that the condition that a nonrationally convex set Γ is a minimal nonpolynomially convex set is stronger than the condition that Γ is the union of two polynomially convex sets. We will then give a general result about pairs of polynomially convex sets in $\mathbb{C}^3$ whose union is not rationally convex from which we will obtain several concrete examples. After that we will give some examples in $\mathbb{C}^2$. We will conclude with a general result about embedding a pair of spaces in some $\mathbb{C}^N$ so as to be polynomially convex but have nonrationally convex union.

The main content of the following result is contained in [13, Theorem 4.1]. We will give a new proof of the result as this proof motivates the proof of Theorem 3.3. By the complement and interior of a planar set, we mean its complement and interior relative to the Riemann sphere.

Theorem 3.1. *Let J be a compact planar set such that A_J contains a nonconstant function and such that J is the boundary of each component of the complement of J. Let Γ denote the Shilov boundary for A_J. Then the A_Γ-rational hull of Γ is S^2, but every proper closed subset K of Γ is A_Γ-convex and satisfies $\overline{A_\Gamma|K} = C(K)$.*

Note the analogy with the disc algebra: the polynomial hull of the unit circle in the plane is the closed unit disc while every proper closed subset of the circle is polynomially convex and the polynomials are dense in the continuous functions there. A similar analogy holds with the big disc algebra. However, in contrast to the situation with the disc algebra and big disc algebra, Γ has A_Γ-*rational* hull, not just A_Γ-convex hull.

There are examples of the situation in Theorem 3.1 with Γ a Cantor set or an arc. To see this, simply take J to be a Cantor set or arc that is locally of positive planar measure. Then $\Gamma = J$ by Corollary 2.7.

Our proof of Theorem 3.1 will make use of the the following topological lemma.

Lemma 3.2. *Let J be a compact planar set such that J is the boundary of each component of the complement of J. Let K be a proper closed subset of J. Then K has empty interior and connected complement.*

Proof. Obviously J has empty interior, and hence the same is true of K. Since J has empty interior, each component of the complement of K intersects the complement of J and hence contains a component of the complement of J. Let x be a point of $J \setminus K$. Then some disc Δ centered at x is contained in $\mathbb{C} \setminus K$. This disc intersects every component of the complement of J and hence intersects every component of the complement of K. Consequently, the disc Δ is contained in every component of the complement of K. Thus the complement of K consists of a single component. □

Proof of Theorem 3.1. By Proposition 2.5, $A_\Gamma = A_J$, and as mentioned just before the proposition, for any compact planar set E for which A_E contains a nonconstant function, the A_E-rational hull of E is S^2.

Now let K be a proper closed subset of Γ. Since Γ is the Shilov boundary for A_Γ, there is a function $f \in A_\Gamma$ such that $\|f\|_K < \|f\|_{S^2} = \|f\|_\Gamma$. Choose a point $z_0 \in \Gamma \setminus K$ such that $|f(z_0)| = \|f\|_{S^2}$. Set $\varepsilon = |f(z_0)| - \|f\|_K > 0$. By [3, Theorem II.1.8], there is a function $g \in A_\Gamma$ such that g is holomorphic in a neighborhood of z_0 and $\|f-g\|_{S^2} < \varepsilon/2$. Then $|g(z_0)| > \|g\|_K$. In particular, g never takes the value $g(z_0)$ on the A_Γ-convex hull $\widehat{K}$ of K. Therefore, the function $\big(g - g(z_0)\big)|K$ is invertible as an element of the uniform algebra $\overline{A_\Gamma|K}$.

Since g is holomorphic in a neighborhood of z_0, we can regard $\big(g - g(z_0)\big)/(z - z_0)$ as a continuous function on S^2 that is holomorphic on $S^2 \setminus \Gamma$ and thus belongs to A_Γ. On K we have

$$\left(\frac{g - g(z_0)}{z - z_0}\right)\big(g - g(z_0)\big)^{-1} = \frac{1}{z - z_0}$$

so $(z - z_0)^{-1}$ is in $\overline{A_\Gamma|K}$. Since by Lemma 3.2, K has empty interior and connected complement, Lavrentiev's theorem [3, Theorem II.8.7] gives that every continuous function on K can be approximated uniformly by polynomials. By Runge's theorem, every polynomial can be approximated uniformly on K by rational functions whose only pole is at z_0. Thus $\overline{A_\Gamma|K} = C(K)$ and, in particular, K is A_Γ-convex. □

We turn now to hulls in $\mathbb{C}^3$.

Theorem 3.3. *Let J be a compact planar set with empty interior such that A_J contains a nonconstant function. Let K_1 and K_2 be compact sets whose union is J neither of which contains the Shilov boundary for A_J and each of which has connected complement. Then there is an embedding π of J into $\mathbb{C}^3$ such that $\pi(J)$ has nontrivial rational hull, but $\pi(K_1)$ and $\pi(K_2)$ are each polynomially convex and satisfy $P(\pi(K_j)) = C(\pi(K_j))$, $j = 1, 2$.*

This theorem can be reformulated in the following more intrinsic manner.

Corollary 3.4. *Let (J, K_1, K_2) be a triple with J an uncountable, compact space of topological dimension at most 1 that embeds in the plane and K_1 and K_2 closed subspaces of J whose union is J and such that each of K_1 and K_2 is simply coconnected and each of $J \setminus K_1$ and $J \setminus K_2$ is uncountable. Then there is an embedding π of J into $\mathbb{C}^3$ such that $\pi(J)$ has nontrivial rational hull, but $\pi(K_1)$ and $\pi(K_2)$ are each polynomially convex and satisfy $P(\pi(K_j)) = C(\pi(K_j))$, $j = 1, 2$.*

Before proving Theorem 3.3 and Corollary 3.4, we present several other corollaries. In all these corollaries the polynomially convex sets can be chosen to be sets on which the polynomials are dense in the continuous functions. This has been omitted from the statements to avoid excessive repetitiveness.

Corollary 3.5. *There exists a Cantor set in $\mathbb{C}^3$ that has nontrivial rational hull and is the union of two disjoint polynomially convex Cantor sets.*

Proof. Take J to be a Cantor set in the plane that is locally of positive planar measure, take K_1 and K_2 to be a separation of J, and apply Theorem 3.3. □

Taking J in Theorem 3.3 to be an arc that is locally of positive planar measure we obtain:

Corollary 3.6. *There exists an arc in $\mathbb{C}^3$ that has nontrivial rational hull and is the union of two polynomially convex arcs.*

Note that the subarcs can be chosen to have only a point in common, or alternatively, they can be chosen so as to overlap so that the nonrationally convex arc is locally polynomially convex.

Corollary 3.7. *There exist two disjoint polynomially convex arcs in* $\mathbb{C}^3$ *whose union is not rationally convex.*

Corollary 3.8. *There exists a simple closed curve in* $\mathbb{C}^3$ *that has nontrivial rational hull and is the union of two polynomially convex arcs.*

Of course the existence of two polynomially convex arcs whose union is a nonpolynomially convex simple closed curve is trivial; just take two arcs in the plane whose union is the unit circle. However, that the union can fail to be rationally convex as in the above corollary does not seem to be obvious. Note that the arcs in the corollary can be chosen so that their interiors cover the nonrationally convex simple closed curve and hence the nonrationally convex simple closed curve is locally polynomially convex.

In preparation for the proof of Theorem 3.3 we establish a lemma. For A a uniform algebra on a compact space X, a point x_0 of X is said to be a *peak point* for A if there is a function f in A such that $f(x_0) = 1$ and $|f(x)| < 1$ for every $x \in X \setminus \{x_0\}$. Such a function is said to *peak* at x_0. We denote the real part of a complex number or function z by $\Re z$.

Lemma 3.9. *Let A be a uniform algebra on a compact space X. Let $x_0 \in X$ be a peak point for A, and let K be a compact subset of X that does not contain x_0. Then there is a function f in A such that*

(i) $f(x_0) = 1$
(ii) $\Re f \geq 0$ *everywhere on* X
(iii) $\Re f \leq 1/4$ *on* K.

Proof. Choose a function $h \in A$ that peaks at x_0. Let $r = \|h\|_K < 1$, and let α be the real number such that the conformal automorphism of the disc defined by $\varphi_\alpha(z) = (z - \alpha)/(1 - \alpha z)$ takes r to $-1/2$. Then the function $f = \big(1 + (\varphi_\alpha \circ h)\big)/2$ has the desired properties. □

Proof of Theorem 3.3. Since neither of K_1 and K_2 contains the Shilov boundary for A_J, there exist peak points a_1 and a_2 for A_J with $a_1 \notin K_1$ and $a_2 \notin K_2$. By Lemma 3.9 there are functions f_1 and f_2 in A_J such that for each $j = 1, 2$, we have

(i) $f_j(a_j) = 1$
(ii) $\Re f_j \geq 0$ everywhere on J
(iii) $\Re f_j \leq 1/4$ on K_j.

Set $f = f_1 - f_2$. Then

(i) $\Re f(a_1) \geq 3/4$
(ii) $\Re f \leq 1/4$ on K_1
(iii) $\Re f(a_2) \leq -3/4$
(iv) $\Re f \geq -1/4$ on K_2.

Repeated application of [3, Theorem II.1.8] yields a function $g \in A_J$ such that g is holomorphic in neighborhoods of each of a_1 and a_2 and satisfies $\|f - g\|_{S^2} < 1/4$. Then

(i) $\Re g(a_1) > 1/2$
(ii) $\Re g < 1/2$ on K_1
(iii) $\Re g(a_2) < -1/2$
(iv) $\Re g > -1/2$ on K_2.

Note that $\big(g - g(a_1)\big)/(z - a_1)$ and $\big(g - g(a_2)\big)/(z - a_2)$ can be regarded as functions in A_J. Let $\pi : S^2 \to \mathbb{C}^3$ be defined by $\pi(z) = \big(g(z), (g(z) - g(a_1))/(z - a_1), (g(z) - g(a_2))/(z - a_2)\big)$. Then (see the paragraph preceding Proposition 2.5) π is injective and hence is an embedding, and $\pi(S^2)$ is contained in the rational hull of $\pi(J)$, so the rational hull of $\pi(J)$ is nontrivial.

Let A denote the uniform algebra on S^2 generated by the three functions g, $\big(g - g(a_1)\big)/(z - a_1)$, and $\big(g - g(a_2)\big)/(z - a_2)$. Because on K_1 we have $\Re g < 1/2 < \Re g(a_1)$, the function $1/\big(z - g(a_1)\big)$ can be approximated uniformly on $g(K_1)$ by polynomials. Consequently, the function $\big(g - g(a_1)\big)|K_1$ is invertible as an element of the uniform algebra $\overline{A|K_1}$. (Alternatively, one can obtain this by noting that because on K_1 we have $\Re g < 1/2 < \Re g(a_1)$, the function g never takes the value $g(a_1)$ on the A-convex hull of K_1.) On K_1 we have

$$\left(\frac{g - g(a_1)}{z - a_1}\right)\big(g - g(a_1)\big)^{-1} = \frac{1}{z - a_1}$$

so $(z - a_1)^{-1}$ is in $\overline{A|K_1}$. Now exactly as in the proof of Theorem 3.1, we get that $\overline{A|K_1} = C(K_1)$. It follows that $P(\pi(K_1)) = C(\pi(K_1))$ and hence $\pi(K_1)$ is polynomially convex. The proof that the same holds for K_2 is similar. □

The proof of Corollary 3.4 uses the following standard result which can be found in [12, Chapter 13].

Theorem 3.10. *Given Cantor sets E and E^* in $\mathbb{C}$, there exists a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h(E) = E^*$.*

Proof of Corollary 3.4. By hypothesis each of $J \setminus K_1$ and $J \setminus K_2$ is an uncountable open set of J. Every open set in a metrizable space is a

countable union of closed subsets of the space. Consequently, each of $J \setminus K_1$ and $J \setminus K_2$ contains an uncountable closed, and hence compact, subset of J. Every uncountable compact metrizable space contains a Cantor set. Thus we can choose Cantor sets G_1 and G_2 in $J \setminus K_1$ and $J \setminus K_2$, respectively. Note that then $G_1 \cup G_2$ is also a Cantor set.

Choose an embedding $g : J \to \mathbb{C}$. Choose a Cantor set L in $\mathbb{C}$ that is locally of positive planar measure. By Theorem 3.10, there is a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $h\big(g(G_1 \cup G_2)\big) = L$. Let $f : J \to \mathbb{C}$ be the embedding given by $f = h \circ g$. Then each of $f(G_1)$ and $f(G_2)$ is a Cantor set of positive planar measure. Consequently, each of $f(J) \setminus f(K_1)$ and $f(J) \setminus f(K_2)$ has positive planar measure and hence intersects the Shilov boundary for $A_{f(J)}$, by Theorem 2.6.

Since J has topological dimension at most 1, the set $f(J)$ has empty interior in $\mathbb{C}$. The simple coconnectivity of K_1 and K_2 implies that each of $f(K_1)$ and $f(K_2)$ has connected complement in $\mathbb{C}$.

Now Theorem 3.3 applies to yield the desired conclusion. □

We turn next to examples in $\mathbb{C}^2$.

Theorem 3.11. *There exists a Cantor set in $\mathbb{C}^2$ that has nontrivial rational hull and is the union of two polynomially convex Cantor sets.*

Whether the two polynomially convex Cantor sets in this theorem can be taken to be disjoint, as was shown in the case of sets in $\mathbb{C}^3$, remains open. Before proving the theorem we present a corollary.

Corollary 3.12. *There exist two polynomially convex arcs in $\mathbb{C}^2$ whose union is not rationally convex.*

Proof. By the preceding theorem there exist polynomially convex Cantor sets K_1 and K_2 in $\mathbb{C}^2$ whose union is not rationally convex. Every Cantor set in a Euclidean space is contained in an arc [19, Theorem p. 57]. Therefore, by [7, Corollary 1.2] there are polynomially convex arcs J_1 and J_2 containing K_1 and K_2, respectively. Since $K_1 \cup K_2$ is not rationally convex, Corollary 2.4 shows that $J_1 \cup J_2$ is not rationally convex either. □

Whether Corollary 3.12 can be strengthened to show that Corollaries 3.6–3.8 hold with $\mathbb{C}^2$ in place of $\mathbb{C}^3$ remains open.

Proof of Theorem 3.11. Let K_1 and K_2 be two Cantor sets in $\mathbb{C}$ that are locally of positive planar measure. Define functions f_1 and f_2 on S^2 by

$$f_1(\zeta) = \iint_{K_1} \frac{dx\,dy}{z-\zeta} \qquad \text{and} \qquad f_2(\zeta) = \iint_{K_2} \frac{dx\,dy}{z-\zeta}.$$

As noted in Section 2, each f_j is in A_{K_j}. Furthermore, $f_j'(\infty) = \lim_{\zeta\to\infty} \zeta f(\zeta) = -\iint_{K_j} dx\,dy \neq 0$. Thus by replacing K_2 by a translate of itself that is sufficiently far away from K_1, we can arrange to have f_1 be one-to-one on K_2 and f_2 be one-to-one on K_1. Let $J = K_1 \cup K_2$. Note that of course f_1 and f_2 belong to A_J.

Define $\pi : J \to \mathbb{C}^2$ by $\pi(z) = \big(f_1(z), f_2(z)\big)$. Note that π maps each K_j homeomorphically onto $\pi(K_j)$. Consequently, each of $\pi(K_1)$, $\pi(K_2)$, and $\pi(J)$ is a Cantor set. Furthermore, because the coordinate function z_2 is one-to-one on $\pi(K_1)$, Lavrentiev's theorem gives that $P(\pi(K_1)) = C(\pi(K_1))$. The same holds with the roles of 1 and 2 reversed. In particular, each of $\pi(K_1)$ and $\pi(K_2)$ is polynomially convex. As noted in Section 2, $\pi(S^2)$ is contained in the rational hull of $\pi(J)$, and hence the rational hull of $\pi(J)$ is nontrivial. □

We conclude this section by showing that it follows from the existence of disjoint polynomially convex Cantor sets whose union has nontrivial rational hull that any uncountable compact subspace of a Euclidean space can be embedded in some $\mathbb{C}^N$ as a nonrationally convex set that is the union of two polynomially convex sets. First we consider an abstract form of this result that applies to more general compact spaces.

Theorem 3.13. *Let (J, K_1, K_2) be a triple with J a compact Hausdorff space and K_1 and K_2 closed subspaces of J whose union is J and such that each of $J \setminus K_1$ and $J \setminus K_2$ contains a Cantor set. Then there exists a uniform algebra A on J such that J has nontrivial A-rational hull, but each of K_1 and K_2 is A-convex and $\overline{A|K_j} = C(K_j)$, $j = 1, 2$.*

Proof. Choose Cantor sets G_1 and G_2 in $J \setminus K_1$ and $J \setminus K_2$, respectively. Then Corollary 3.5 yields a uniform algebra B on $G_1 \cup G_2$ such that $G_1 \cup G_2$ has nontrivial B-rational hull, but G_1 and G_2 are each B-convex. Furthermore, B can be chosen so that $\overline{B|G_1} = C(G_1)$ and $\overline{B|G_2} = C(G_2)$. Let A be the uniform algebra on J defined by $A = \{f \in C(J) : f|(G_1 \cup G_2) \in B\}$.

Each of J and $\mathfrak{M}_B$ can be regarded as subsets of $\mathfrak{M}_A$ in standard ways. Then $\mathfrak{M}_B$ is the A-convex hull of $G_1 \cup G_2$ in $\mathfrak{M}_A$, and it follows that $J \cap \mathfrak{M}_B = G_1 \cup G_2$. Since the B-rational hull of $G_1 \cup G_2$ is easily seen to be contained in the A-rational hull of $G_1 \cup G_2$, we obtain that the A-rational hull of $G_1 \cup G_2$ is not contained in J. Consequently, J has nontrivial A-rational hull.

Since $\overline{B|G_2} = C(G_2)$, it follows from the Tietze extension theorem that the functions in $C(K_1)$ whose restrictions to G_2 lie in $B|G_2$ form a dense subset of $C(K_1)$. Thus to show that $\overline{A|K_1} = C(K_1)$, it suffices to show that each function $f \in C(K_1)$ that satisfies $f|G_2 \in B|G_2$ extends

to a member of A. To do so, choose $g \in B \subset C(G_1 \cup G_2)$ such that $g|G_2 = f|G_2$, note that the formula

$$h(x) = \begin{cases} f(x), & x \in K_1 \\ g(x), & x \in G_1 \end{cases}$$

yields a well-defined continuous function on $K_1 \cup G_1$, extend h to a continuous function $\tilde{h}$ on J, and note that then $\tilde{h}$ is in A (because $\tilde{h}|(G_1 \cup G_2) = g \in B$) and $\tilde{h}|K_1 = f$. Thus $\overline{A|K_1} = C(K_1)$. The verification that $\overline{A|K_2} = C(K_2)$ is similar. The equation $\overline{A|K_j} = C(K_j)$ implies A-convexity of K_j. □

Corollary 3.14. *Let (J, K_1, K_2) be a triple with J a compact subspace of $\mathbb{R}^N$ and K_1 and K_2 closed subspaces of J whose union is J and such that each of $J \setminus K_1$ and $J \setminus K_2$ is uncountable. Then there exists an embedding π of J into $\mathbb{C}^{N+4}$ such that $\pi(J)$ has nontrivial rational hull, but each of $\pi(K_1)$ and $\pi(K_2)$ is polynomially convex and $P(\pi(K_j)) = C(\pi(K_j))$, $j = 1, 2$.*

Since every compact metrizable space of topological dimension m embeds in $\mathbb{R}^{2m+1}$ [5, Theorem V.2], Corollary 3.14 shows, in particular, that there is a homeomorphic copy of every uncountable such space in $\mathbb{C}^{2m+5}$ that has nontrivial rational hull but is the union of two polynomially convex sets. As another immediate consequence of Corollary 3.14, if K_1 and K_2 are any two uncountable compact subsets of $\mathbb{R}^N$, then there are disjoint polynomially convex sets in $\mathbb{C}^{N+4}$ that are homeomorphic to K_1 and K_2 but whose union is not rationally convex.

Proof. As in the proof of Corollary 3.4, we can choose Cantor sets G_1 and G_2 in $J \setminus K_1$ and $J \setminus K_2$, respectively. As in the proof of Theorem 3.13, let B be a uniform algebra on $G_1 \cup G_2$ such that $G_1 \cup G_2$ has nontrivial B-rational hull, but G_1 and G_2 are each B-convex, and $\overline{B|G_j} = C(G_j)$, $j = 1, 2$. By Corollary 3.5, the uniform algebra B can be chosen so as to be generated by three functions f_1, f_2, f_3.

Set $A = \{f \in C(J) : f|(G_1 \cup G_2) \in B\}$. Extend each of f_1, f_2, f_3 to continuous complex-valued functions $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ on J. Let $x_1, \ldots, x_N$ denote the real coordinate functions of $\mathbb{R}^N$. Choose a continuous real-valued function ρ on J whose zero set is precisely $G_1 \cup G_2$. Then the $N + 4$ functions $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \rho, \rho x_1, \ldots, \rho x_N$ generate the uniform algebra A by [8, Lemma 3.8]. The proof of Theorem 3.13 shows that J has nontrivial A-rational hull and $\overline{A|K_j} = C(K_j)$, $j = 1, 2$. It follows that the map $\pi : J \to \mathbb{C}^{N+4}$ whose component functions are the functions $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \rho, \rho x_1, \ldots, \rho x_N$ has the properties asserted in the corollary. □

4. Approximation by pathological curves

In this section we prove that every simple closed curve in $\mathbb{C}^N$, $N \geq 3$, can be approximated uniformly by a nonrationally convex simple closed curve that is locally polynomially convex, and is, in fact, covered by the interiors of *two* polynomially convex arcs. We will use $\|\cdot\|_\infty$ to denote the supremum norm.

Theorem 4.1. *Let γ be a simple closed curve in $\mathbb{C}^N$, $N \geq 3$. Given $\varepsilon > 0$, there exists a simple closed curve σ satisfying $\|\gamma - \sigma\|_\infty < \varepsilon$ that is not rationally convex but is the union of two polynomially convex arcs σ_1 and σ_2 satisfying $P(\sigma_j) = C(\sigma_j)$, $j = 1, 2$, and whose interiors cover σ.*

Since every simple closed curve in Euclidean space can be approximated uniformly by a smooth, and hence rectifiable, simple closed curve, the above theorem is a consequence of the following sharper result for rectifiable simple closed curves.

Theorem 4.2. *Given a rectifiable simple closed curve γ in $\mathbb{C}^N$, $N \geq 3$, given $\varepsilon > 0$, and given an open ball B of $\mathbb{C}^N$ that intersects γ, there is a simple closed curve γ_a that satisfies $\|\gamma - \gamma_a\|_\infty < \varepsilon$ and $\gamma_a \setminus B = \gamma \setminus B$ such that γ_a is not rationally convex but is the union of two polynomially convex arcs γ_1 and γ_2 that satisfy $P(\gamma_j) = C(\gamma_j)$, $j = 1, 2$, and whose interiors cover γ_a.*

For the reader's convenience we quote here two results we will use in the proof.

Theorem 4.3. [11, Theorem 1.6] *Let Y be a compact polynomially convex subset of $\mathbb{C}^N$, and let Γ be a subset of $\mathbb{C}^N$ such that $Y \cup \Gamma$ is compact and such that for every neighborhood U of Y in $\mathbb{C}^N$, the set $\Gamma \setminus U$ is contained in a compact connected set of finite length. Suppose also that the map $\check{H}^1(Y \cup \Gamma; \mathbb{Z}) \to \check{H}^1(Y; \mathbb{Z})$ induced by the inclusion $Y \hookrightarrow Y \cup \Gamma$ is a monomorphism. Then $Y \cup \Gamma$ is polynomially convex, and $P(Y \cup \Gamma) = \{f \in C(Y \cup \Gamma) : f|Y \in P(Y)\}$.*

Lemma 4.4. [7, Lemma 2.1] *Let λ be a closed set in $\mathbb{R}^N$, $N \geq 3$, of topological dimension at most 1, let a and a' be two points in λ, and let Ω be a connected open set of $\mathbb{R}^N$ that contains a and a'. Then there is an arc J from a to a' contained in Ω that intersects λ only in the end points a and a' of J and is such that the open arc $J \setminus \{a, a'\}$ is C^∞-smooth.*

The following elementary lemma will also be needed.

Lemma 4.5. *Let $X \subset \mathbb{C}^N$ be a compact set such that $P(X) = C(X)$, and let $x_1, \ldots, x_k$ be points in $\mathbb{C}^N$. Then $P(X \cup \{x_1, \ldots, x_k\}) = C(X \cup \{x_1, \ldots, x_k\})$.*

Proof. By induction it suffices to consider the case $k = 1$. Assume that $x_1 \notin X$ since otherwise there is nothing to prove. Since $P(X) = C(X)$, the set X is polynomially convex, so there exists a polynomial p such that $p(x_1) = 1 > \|p\|_X$. The function f that is 1 at x_1 and identically 0 on X is the uniform limit of the sequence $(p^n)_{n=1}^{\infty}$ and hence is in $P(X \cup \{x_1\})$.

Given $g \in C(X \cup \{x_1\})$, choose a sequence of polynomials (g_n) such that $g_n \to g$ uniformly on X. Then $g_n(1 - f) + g(x_1)f \to g$ uniformly on $X \cup \{x_1\}$, so g is in $P(X \cup \{x_1\})$. □

Proof of Theorem 4.2. We may assume without loss of generality that the diameter of B is less than ε. The simple closed curve γ_a will be obtained from γ by removing an arc from γ contained in B and replacing it with a new arc contained in B having the same end points. It is then immediate that $\gamma_a \setminus B = \gamma \setminus B$ and that under a suitable parametrization $\|\gamma - \gamma_a\|_\infty < \varepsilon$. We must show that the removal and insertion can be done in such a way that the other conditions are satisfied.

Remove from γ an open arc lying in B to obtain a rectifiable arc κ whose end points we will denote by p and q. As noted immediately after Corollary 3.6, there exists an arc τ, which can be taken to lie in B and disjoint from κ, such that τ is not rationally convex and such that, denoting the end points of τ by a and d, there are distinct points b and c in the interior of τ such that traversing τ from a to d, one encounters b before c and the arcs ac and bd are polynomially convex and satisfy $P(ac) = C(ac)$ and $P(bd) = C(bd)$.

The set $B \setminus (\kappa \cup \tau)$ is connected (because a connected manifold of real dimension greater than or equal to three cannot be disconnected by a subspace of topological dimension one [5, Corollary 1, p. 48]). Therefore, by repeated application of Lemma 4.4, we can obtain arcs ℓ_1 and ℓ_2 in B such that the arc ℓ_1 has end points p and a, the arc ℓ_2 has end points q and d, the open arcs $\ell_1 \setminus \{p, a\}$ and $\ell_2 \setminus \{q, d\}$ are C^∞-smooth, and the set

$$\gamma_a = \kappa \cup \ell_1 \cup \tau \cup \ell_2$$

is a simple closed curve.

Since τ is not rationally convex, γ_a is not rationally convex by Corollary 2.4.

Let $\gamma_1 = \kappa \cup \ell_1 \cup \ell_2 \cup ac$ and $\gamma_2 = \kappa \cup \ell_1 \cup \ell_2 \cup bd$. Then the interiors of γ_1 and γ_2 cover γ_a. Set $Y = ac \cup \{d, p, q\}$ and $\Gamma = \kappa \cup \ell_1 \cup \ell_2$. Then Y is

polynomially convex and $P(Y) = C(Y)$ by Lemma 4.5, so Theorem 4.3 applies and yields that γ_1 is polynomially convex and satisfies $P(\gamma_1) = C(\gamma_1)$. Similarly, γ_2 is polynomially convex and satisfies $P(\gamma_2) = C(\gamma_2)$. □

Although the nonrationally convex simple closed curve γ_a constructed in the above proof is locally polynomially convex, it contains the nonrationally convex, and hence nonpolynomially convex, arc τ. In contrast, every arc contained in a *rectifiable* simple closed curve is polynomially convex. This observation, in conjunction with the theorem of the author and Stout mentioned at the beginning of the paper, suggests the following open question.

Question 4.6. *Given a polynomially convex simple closed curve γ, must there exist an $\varepsilon > 0$ such that if σ is a simple closed curve with the property that every arc contained in σ is polynomially convex and σ satisfies $\|\gamma - \sigma\|_\infty < \varepsilon$, then σ is polynomially convex?*

5. A nonrationally convex ball and chain

In this section we construct examples of polynomially convex arcs whose union with the closed unit ball $\overline{\mathbb{B}}$ is not rationally convex, and in particular, we answer the question of Stout mentioned in the introduction. First though we present Stout's observation regarding polynomial convexity under a local rectifiability hypothesis.

Theorem 5.1. *Let J be an arc in $\mathbb{C}^N$ that meets $\overline{\mathbb{B}}$ in a single end point, say a. If the half-open arc $J \setminus \{a\}$ is locally rectifiable, then $\overline{\mathbb{B}} \cup J$ is polynomially convex.*

Note that the hypotheses do not require that J itself be rectifiable.

Proof. The cohomology group $\check{H}^1(\overline{\mathbb{B}} \cup J; \mathbb{Z})$ is the zero group, so the natural map $\check{H}^1(\overline{\mathbb{B}} \cup J; \mathbb{Z}) \to \check{H}^1(\overline{\mathbb{B}}; \mathbb{Z})$ is a monomorphism, whence application of Theorem 4.3 yields the result. □

Variations on Theorem 5.1 can be proved similarly. For instance, it is enough to require only that the interior of J be locally rectifiable. Also the arc J could be taken to be disjoint from $\overline{\mathbb{B}}$ or could intersect $\overline{\mathbb{B}}$ in an interior point of J with the two half-open arcs making up $J \setminus \overline{\mathbb{B}}$ each required to be locally rectifiable.

We turn now to the nonrationally convex examples.

Theorem 5.2. *There exists in $\mathbb{C}^3$ a polynomially convex arc J that is disjoint from the closed unit ball $\overline{\mathbb{B}}$ such that $\overline{\mathbb{B}} \cup J$ is not rationally convex.*

Since there exists an arc in the plane having positive planar measure (a theorem of Osgood [14]), the above result is a special case of the following one.

Theorem 5.3. *Let Γ be a compact subset of $\mathbb{C}$ of positive planar measure with empty interior and connected complement. There exists in $\mathbb{C}^3$ a polynomially convex set E homeomorphic to Γ and disjoint from the closed unit ball such that $\overline{\mathbb{B}} \cup E$ is not rationally convex.*

Proof. Set

$$g(\zeta) = \iint_\Gamma \frac{dx\,dy}{z-\zeta}.$$

Then g is a nonconstant, continuous function on the Riemann sphere S^2 that is holomorphic off Γ and vanishes at ∞. (See Section 2.) Choose a point $z_0 \in \mathbb{C} \setminus \Gamma$ such that $g(z_0) \neq 0$. Set

$$h(z) = \frac{g(z) - g(z_0)}{z - z_0}.$$

Then h is also a nonconstant, continuous function on the Riemann sphere S^2 that is holomorphic off Γ and vanishes at ∞. Note that ∞ is the only common zero of g and h. In particular, g and h have no common zero on Γ, so there is a positive constant k such that $|(kg, kh)| > 1$ everywhere on Γ, where (kg, kh) denotes the function $z \mapsto \big(kg(z), kh(z)\big)$.

Since $(kg, kh)(\infty) = 0$, we have for sufficiently large $R > 0$ that $|(kg, kh)| < 1/2$ everywhere on the circle $\{|z| = R\}$. Fix $R > 0$ large enough that, in addition, the circle $\{|z| = R\}$ encloses the set Γ.

Define $F : \mathbb{C} \to \mathbb{C}^3$ by

$$F(z) = \big(z/2R, kg(z), kh(z)\big),$$

and set $E = F(\Gamma)$. Then F maps Γ homeomorphically onto E. Furthermore, E is polynomially convex because polynomials in z_1 are dense in $C(E)$, since polynomials in z are dense in $C(\Gamma)$ by Lavrentiev's theorem [3, Theorem II.8.7]. In addition, because $|(kg, kh)| > 1$ everywhere on Γ, the set E is disjoint from $\overline{\mathbb{B}}$.

Note that $F(\{|z| = R\})$ is contained in $\overline{\mathbb{B}}$. Thus the connected set $F(\{|z| \leq R\})$ intersects each of the two disjoint closed sets $\overline{\mathbb{B}}$ and E, and hence $F(\{|z| \leq R\})$ cannot be contained in the union of $\overline{\mathbb{B}}$ and E. Thus to show that $\overline{\mathbb{B}} \cup E$ is not rationally convex, it suffices to show that $F(\{|z| \leq R\})$ lies in the rational hull of $\overline{\mathbb{B}} \cup E$.

Suppose p is a polynomial on $\mathbb{C}^3$ having no zeros on $\overline{\mathbb{B}} \cup E$. Since each of $\overline{\mathbb{B}}$ and E is simply coconnected, p has a continuous logarithm in a neighborhood of $\overline{\mathbb{B}} \cup E$. Consequently, $p \circ F$ has a continuous

logarithm on a neighborhood of $\{|z| = R\} \cup \Gamma$ in $\mathbb{C}$. By the argument principle it follows that $p \circ F$ has no zeros on $\{|z| \leq R\}$. Thus p has no zeros on $F(\{|z| \leq R\})$. Consequently, $F(\{|z| \leq R\})$ is contained in the rational hull of $\overline{\mathbb{B}} \cup E$. □

Corollary 5.4. *There exists a polynomially convex arc J in $\mathbb{C}^3$ that meets the closed unit ball $\overline{\mathbb{B}}$ in a single end point such that $\overline{\mathbb{B}} \cup J$ is not rationally convex.*

Proof. By Theorem 5.2 there is a polynomially convex arc E in $\mathbb{C}^3$ disjoint from $\overline{\mathbb{B}}$ such that $\overline{\mathbb{B}} \cup E$ is not rationally convex. One can construct an arc σ from a point of ∂B to an end point of E such that the interior of σ is disjoint from $\overline{\mathbb{B}} \cup E$ and such that σ is smooth except possibly at the end point where it meets E. The details of the construction are similar to the proof of [11, Theorem 1.2] so we omit them; the basic idea is to choose a sequence of points in the complement of $\overline{\mathbb{B}} \cup E$ converging to an end point of E, and then choose smooth arcs connecting successive points of the sequence and fitting together so as to form a smooth arc.

Set $J = E \cup \sigma$. Then J is an arc and is polynomially convex by [11, Theorem 1.7] (a corollary of Theorem 4.3). Of course $h_r(\overline{\mathbb{B}} \cup J) \supset h_r(\overline{\mathbb{B}} \cup E)$, and applying Theorem 2.2 to the uniform algebra $R(\overline{\mathbb{B}} \cup E)$ shows that $h_r(\overline{\mathbb{B}} \cup E)$ cannot be contained in $(\overline{\mathbb{B}} \cup E) \cup \sigma = \overline{\mathbb{B}} \cup J$. Thus $h_r(\overline{\mathbb{B}} \cup J) \setminus (\overline{\mathbb{B}} \cup J)$ is nonempty. □

6. Unions with hull without analytic discs

In this section we establish the existence of various pairs of polynomially convex sets whose union has nontrivial polynomial hull without analytic discs.

Theorem 6.1. *There exist two polynomially convex Cantor sets K_1 and K_2 in $\mathbb{C}^3$ such that the polynomial hull of $K_1 \cup K_2$ is nontrivial but contains no analytic discs.*

In $\mathbb{C}^4$ the two polynomially convex Cantor sets can be taken to be disjoint.

Theorem 6.2. *There exist two disjoint polynomially convex Cantor sets K_1 and K_2 in $\mathbb{C}^4$ such that the polynomial hull of $K_1 \cup K_2$ is nontrivial but contains no analytic discs.*

In connection with these results, note that the union of two Cantor sets is itself a Cantor set.

As a corollary of Theorem 6.2 we will prove a general result about unions with polynomial hull without analytic discs.

Corollary 6.3. *Let (J, K_1, K_2) be a triple with J a compact subspace of $\mathbb{R}^N$ and K_1 and K_2 closed subspaces of J whose union is J and such that each of $J \setminus K_1$ and $J \setminus K_2$ is uncountable. Then there exists an embedding π of J into $\mathbb{C}^{N+5}$ such that the polynomial hull of $\pi(J)$ is not trivial but contains no analytic discs while each of $\pi(K_1)$ and $\pi(K_2)$ is polynomially convex and $P(\pi(K_j)) = C(\pi(K_j))$, $j = 1, 2$.*

The proofs of these results are based on a result from the paper of the author and Norman Levenberg [9] which associates to each compact set with nontrivial polynomial hull in $\mathbb{C}^N$ a compact set in $\mathbb{C}^{N+1}$ with a nontrivial polynomial hull that contains no analytic discs. We quote the needed result here for the reader's convenience.

Theorem 6.4 ([9], Theorem 1.1)**.** *Let $X \subset \mathbb{C}^N$ be a compact set whose polynomial hull is nontrivial. Then there exists a compact set $Y \subset \mathbb{C}^{N+1}$ such that, letting π denote the restriction to $\widehat{Y}$ of the projection $\mathbb{C}^{N+1} \to \mathbb{C}^N$ onto the first N coordinates, the following conditions hold:*
(i) $\pi(Y) = X$
(ii) $\pi(\widehat{Y} \setminus Y) = \widehat{X} \setminus X$
(iii) *$\widehat{Y}$ contains no analytic discs*
(iv) *each fiber $\pi^{-1}(z)$ for $z \in \widehat{X}$ is totally disconnected.*

We will also need two lemmas concerning perfect subsets. Recall that a subset of a space is said to be *perfect* if it is closed and has no isolated points. Every space contains a unique largest perfect subset (which can be empty), namely the closure of the union of all perfect subsets of the space.

Lemma 6.5. *Let Y be a polynomially convex set in $\mathbb{C}^N$, and let K be the largest perfect subset of Y. Then K is polynomially convex.*

Proof. Note that $\widehat{K} \subset \widehat{Y} = Y$. Thus since K is the *largest* perfect subset of Y, if $\widehat{K}$ were strictly larger than K, then $\widehat{K}$ would have an isolated point, and the isolated point would necessarily be a point of $\widehat{K} \setminus K$. But this is impossible, for it follows from the Oka-Weil theorem that every component of $\widehat{K}$ must intersect K. □

Lemma 6.6. *Let Y be a space, and let Y_1 and Y_2 be closed subspaces of Y whose union is Y. Let K, K_1, and K_2 be the largest perfect subsets of Y, Y_1, and Y_2, respectively. Then $K_1 \cup K_2 = K$.*

In case Y_1 and Y_2 are disjoint, this lemma is essentially obvious. Only that case is needed for the proof of Theorem 6.2, but the general case is needed for the proof of Theorem 6.1.

Proof. The proof is an application of Zorn's lemma. Let $\mathscr{P}$ be the collection of ordered pairs (Σ_1, Σ_2) of closed subsets of Y such that $K_j \subset \Sigma_j \subset Y_j$ for $j = 1, 2$ and $\Sigma_1 \cup \Sigma_2 = K$. Note that $\mathscr{P}$ is nonempty since $(Y_1 \cap K, Y_2 \cap K)$ is in $\mathscr{P}$. Partially order $\mathscr{P}$ by declaring $(\Sigma_1, \Sigma_2) \leq (\Sigma_1', \Sigma_2')$ if $\Sigma_1 \subset \Sigma_1'$ and $\Sigma_2 \subset \Sigma_2'$. Given a totally ordered subcollection $\mathscr{Q}$ of $\mathscr{P}$, set

$$Q_1 = \bigcap\nolimits_{(\Gamma_1, \Gamma_2) \in \mathscr{Q}} \Gamma_1 \quad \text{and} \quad Q_2 = \bigcap\nolimits_{(\Gamma_1, \Gamma_2) \in \mathscr{Q}} \Gamma_2.$$

It is easily verified that (Q_1, Q_2) is a lower bound for $\mathscr{Q}$ in $\mathscr{P}$. Thus by Zorn's lemma, $\mathscr{P}$ has a minimal element (P_1, P_2). To conclude the proof, it suffices to show that $(P_1, P_2) = (K_1, K_2)$ since then $K_1 \cup K_2 = P_1 \cup P_2 = K$.

Assume to get a contradiction that $(P_1, P_2) \neq (K_1, K_2)$. Then without loss of generality, P_1 properly contains K_1. Then P_1 is not perfect and hence has an isolated point x. Since K is perfect, and $P_1 \cup P_2 = K$, it must be that x is a limit point of P_2, and hence x is in P_2. Consequently, the closed sets $P_1 \setminus \{x\}$ and P_2 have union K. Also since x is isolated in P_1, the point x can not be in K_1. Thus $K_1 \subset P_1 \setminus \{x\}$. We conclude that $(P_1 \setminus \{x\}, P_2)$ is a member of $\mathscr{P}$ that is strictly smaller than (P_1, P_2), contrary to the minimality of (P_1, P_2). □

Proof of Theorems 6.1 and 6.2. Set $N = 2$ for the proof of Theorem 6.1, or $N = 3$ for the proof of Theorem 6.2. By Corollary 3.5 and Theorem 3.11 there exist polynomially convex Cantor sets X_1 and X_2 in $\mathbb{C}^N$ such that their union, which we will denote by X, has nontrivial polynomial hull. Furthermore, when $N = 3$, we can choose X_1 and X_2 to be disjoint. Let Y be the set in $\mathbb{C}^{N+1}$ obtained from X by applying Theorem 6.4. As in Theorem 6.4, let $\pi : \widehat{Y} \to \mathbb{C}^N$ be the restriction to $\widehat{Y}$ of the projection $\mathbb{C}^{N+1} \to \mathbb{C}^N$ onto the first N coordinates. Set $Y_1 = \pi^{-1}(X_1)$ and $Y_2 = \pi^{-1}(X_2)$. Let K, K_1, and K_2 be the largest perfect subsets of Y, Y_1, and Y_2, respectively. Then $K = K_1 \cup K_2$ by Lemma 6.6. Also $\widehat{K} \setminus K \supset \widehat{Y} \setminus Y$ by [6, Lemma 4.2], so condition (ii) of Theorem 6.4 gives that $\widehat{K}$ is nontrivial. Since $\widehat{K} \subset \widehat{Y}$, condition (iii) gives that $\widehat{K}$ contains no analytic discs. It is easily verified that the polynomial convexity of X_1 and X_2 implies that Y_1 and Y_2 are polynomially convex. Consequently, K_1 and K_2 are polynomially convex by Lemma 6.5. Finally, conditions (i) and (iv) of Theorem 6.4 imply that Y is totally disconnected, and hence the same is true of the subsets K_1 and K_2. Consequently, K_1 and K_2 are Cantor sets by the usual characterization. Note that K_1 and K_2 are disjoint when $N = 3$. □

Proof of Corollary 6.3. As in the proof of Corollary 3.4, we can choose Cantor sets G_1 and G_2 in $J \setminus K_1$ and $J \setminus K_2$, respectively. Let $f_1, \ldots, f_4$ be the components of a homeomorphism of $G_1 \cup G_2$ onto the nonpolynomially convex set in Theorem 6.2 that maps G_1 onto the first Cantor set in that theorem and G_2 onto the second Cantor set there.

Extend $f_1, \ldots, f_4$ to continuous complex-valued functions $\tilde{f}_1, \ldots, \tilde{f}_4$ on J. Let $x_1, \ldots, x_N$ denote the real coordinate functions of $\mathbb{R}^N$. Choose a continuous real-valued function ρ on J whose zero set is precisely $G_1 \cup G_2$. Let $\pi : J \to \mathbb{C}^{N+5}$ be the mapping whose components are the functions $\tilde{f}_1, \ldots, \tilde{f}_4, \rho, \rho x_1, \ldots, \rho x_N$.

Because $\pi(J)$ is contained in $\mathbb{C}^4 \times \mathbb{R}^{N+1}$, the polynomial hull $\widehat{\pi(J)}$ is the union of the polynomial hulls of the slices $\pi(J) \cap (\mathbb{C}^4 \times \{r\})$ for $r \in \mathbb{R}^{N+1}$ (by [8, Proposition 3.1] for instance). Each nonempty such slice is a single point with the exception of the slice given by $r = 0$. That slice is the set $\pi(G_1 \cup G_2)$, which is exactly the image, under the canonical embedding of $\mathbb{C}^4$ into $\mathbb{C}^{N+5}$, of the set in Theorem 6.2 with nontrivial hull containing no analytic discs. Consequently, $\pi(J)$ has nontrivial hull containing no analytic discs.

Application of the Bishop antisymmetric decomposition ([2, Theorem 2.7.5] or [3, Theorem II.13.1]) shows that $P(\pi(K_j)) = C(\pi(K_j))$, $j = 1, 2$, and hence each of $\pi(K_1)$ and $\pi(K_2)$ is polynomially convex. □

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Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403
Email address: aizzo@bgsu.edu