Short Proof of the Asymptotic Confirmation of the Faudree-Lehel Conjecture

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Abstract

Given a simple graph G, the *irregularity strength* of G, denoted $s(G)$, is the least positive integer k such that there is a weight assignment on edges $f : E(G) \to \{1, 2, \ldots, k\}$ for which each vertex weight $f^V(v) := \sum_{u:\{u,v\}\in E(G)} f(\{u, v\})$ is unique amongst all $v \in V(G)$. In 1987, Faudree and Lehel conjectured that there is a constant c such that $s(G) \leqslant n/d + c$ for all d-regular graphs G on n vertices with $d > 1$, whereas it is trivial that $s(G) \geqslant n/d$. In this short note we prove that the Faudree-Lehel Conjecture holds when $d \geqslant n^{0.8+\epsilon}$ for any fixed $\epsilon > 0$, with a small additive constant $c = 28$ for n large enough. Furthermore, we confirm the conjecture asymptotically by proving that for any fixed $\beta \in (0, 1/4)$ there is a constant C such that for all d-regular graphs G, $s(G) \leqslant \frac{n}{d}(1 + \frac{C}{d^\beta}) + 28$, extending and improving a recent result of Przybyło that $s(G) \leqslant \frac{n}{d}(1 + \frac{1}{\ln^{\epsilon/19} n})$ whenever $d \in [\ln^{1+\epsilon} n, n/\ln^{\epsilon} n]$ and n is large enough.

Mathematics Subject Classifications: 05C15, 05C78

1 Introduction

Let G be a simple graph with n vertices. For a positive integer k, an edge-weighting function $f : E(G) \to \{1, 2, \ldots, k\}$ is called *k-irregular* if the weighted degrees, denoted by $f^V(v) = \sum_{u\in N(v)} f(\{v, u\})$ are distinct for $v \in V(G)$; we will call $f(\{u, v\})$ and $f^V(v)$ simply the *weights* of $\{u, v\}$ and v. The *irregularity strength* of G, denoted $s(G)$, is the least k, if exists, for which there is such a k-irregular edge-weighting function f; we set $s(G) = \infty$ otherwise. It is easy to see that $s(G) < \infty$ if and only if G has no isolated edges and at most one isolated vertex [7].

The irregularity strength was first introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz, and Saba [3]. Later an optimal general bound $s(G) \leqslant n - 1$ was proved in [1, 12] for all graphs with finite irregularity strength except for K_3. It turned out

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that this bound was far from optimum for graphs with larger minimum degree. Special concern was in this context devoted to d-regular graphs. In [6] Faudree and Lehel showed $s(G) \leqslant \lceil n/2 \rceil + 9$ for these. By a simple counting argument, it is easy to see that on the other hand,

$$s(G) \geqslant \lceil (n + d + 1)/d \rceil.$$

This lower bound motivated Faudree and Lehel to conjecture that n/d is close to optimal, as proposed in [6] in 1987. In fact this conjecture was first posed by Jacobson, as mentioned in [10].

Conjecture 1 ([6])**.** There is a constant $C > 0$ such that for all d-regular graphs G on n vertices and with $d > 1$, $s(G) \leqslant \frac{n}{d} + C$.

It is this conjecture that "energized the study of the irregularity strength", as stated in [4], and many related subjects throughout the following decades. It remains open after more than thirty years since its formulation. A significant step forward towards solving it was achieved in 2002 by Frieze, Gould, Karoński, and Pfender, who used the probabilistic method to prove the first linear bound $s(G) \leqslant 48(n/d) + 1$ for $d \leqslant \sqrt{n}$, and a super-linear one $s(G) \leqslant 240(\log n)(n/d) + 1$ in the remaining cases. The linear bound in n/d was further extended to the case when $d \geqslant 10^{4/3} n^{2/3} \log^{1/3} n$ by Cuckler and Lazebnik [4]. The first general and unified linear bound in n/d for the full spectrum of (n, d) was delivered by Przybyło [13, 14], who used a constructive rather than random approach to prove the bound $s(G) \leqslant 16(n/d) + 6$. Since then several works based on inventive new algorithms have been conducted to improve the multiplicative constant in front of n/d, see e.g. [8, 9, 11]. The best result among these for any value of d is due to Kalkowski, Karoński, and Pfender [9], who showed that in general $s(G) \leqslant 6\lceil n/\delta \rceil$ for graphs with minimum degree $\delta \geqslant 1$ and without isolated edges. Only just recently it was proved by Przybyło [15] that the Faudree-Lehel Conjecture holds asymptotically almost surely for random graphs $G(n, p)$ (which are typically "close to" regular graphs), for any constant p, and holds asymptotically (in terms of d and n) for d not in extreme values.

Theorem 2 (Przybyło [15])**.** *Given any $\epsilon > 0$, for every d-regular graph G with n vertices and $d \in [\ln^{1+\epsilon} n, n/\ln^{\epsilon} n]$, if n is sufficiently large,*

$$s(G) \leqslant \frac{n}{d}\left(1 + \frac{1}{\ln^{\epsilon/19} n}\right).$$

In [15], Przybyło moreover mentioned that "a poly-logarithmic in n lower bound on d is unfortunately unavoidable" within his approach. In this paper we present an argument which is firstly quite short, secondly bypasses the mentioned poly-logarithmic in n lower bound and extends the asymptotic bound to all possible cases $1 \leqslant d \leqslant n - 1$ and thirdly, the upper bound we present is stronger than the one in Theorem 2 (where in particular $\ln^{\epsilon/19} n \ll \ln^{(1+\epsilon)/19} n \leqslant d^{1/19}$).

Theorem 3. *Given any $0 < \beta < 1/4$, for every d-regular graph G with n vertices, if d is sufficiently large in terms of β,*

$$s(G) < \frac{n}{d}\left(1 + \frac{14}{d^{\beta}}\right) + 28.$$

Corollary 4. *Given any $0 < \beta < 1/4$, there is a constant C such that for every d-regular graph G with n vertices, $s(G) < \frac{n}{d}\left(1 + \frac{C}{d^\beta}\right) + 28$.*

The second contribution of this paper is a confirmation that the Faudree-Lehel Conjecture, i.e. Conjecture 1, holds literally (not only asymptotically) for "dense" graphs, i.e., whenever $d \geqslant n^{0.8+\epsilon}$ for any fixed $\epsilon > 0$.

Theorem 5. *Given any $0 < \beta < 1/4$, for every d-regular graph G on n vertices with $d^{1+\beta} \geqslant n$, if d is sufficiently large in terms of β, then*

$$s(G) < n/d + 28.$$

Corollary 6. *Given any $0 < \beta < 1/4$, there is a constant C such that for every d-regular graph G on n vertices with $d^{1+\beta} \geqslant n$, $s(G) < n/d + C$.*

We remark that similar conclusions as the ones above can also be derived from [16], which describes in more than 30 pages a very long, multistage and technically complex random construction yielding general results for all graphs (not only regular graphs). Taking into account that Conjecture 1 remains a central open question of the related field, cf. [16] for more comprehensive exposition of the history and relevance of this problem, we decided to present separately this very concise argument concerning the conjecture itself, which is also dramatically easier to follow. Moreover, the present proof is a local lemma based argument, and thus is different from the one in [16], which might also be beneficial for further research. Lastly, unlike in [16], we also provide a specific additive constant in the obtained bounds for regular graphs, in particular in Theorem 5, which is relatively small.

2 Proof of main results

2.1 Preliminaries

For a set $U \subset V(G)$ and a vertex $v \in V(G)$, we use $\deg_U(v)$ to denote the number of neighbors of v in U. For a positive constant x, let $\{x\}$ stand for $x - \lfloor x \rfloor$. We will use the following tools.

Lemma 7 (Chernoff Bound)**.** *Let $X_1, \ldots, X_n$ be i.i.d. random variables such that $\Pr(X_i = 1) = p$ and $\Pr(X_i = 0) = 1 - p$ for each i. Then for any $t \geqslant 0$,*

$$\Pr\left(\left|\sum_{i=1}^{n} X_i - np\right| > t\right) \leqslant 2e^{-t^2/(3np)}, \quad \text{for } 0 \leqslant t \leqslant np,$$
$$\Pr\left(\left|\sum_{i=1}^{n} X_i - np\right| > t\right) \leqslant 2e^{-t/3}, \quad \text{for } t > np.$$

Lemma 8 (Lovász Local Lemma)**.** *[5, 2] Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be n events in any given probability space. Let H be a simple graph with vertex set $[n]$ such that for each $i \in [n]$, the event $\mathcal{E}_i$*

is mutually independent from the remaining events corresponding to non-neighbors of the vertex i, i.e., $\{\mathcal{E}_j : j \neq i, \{i,j\} \notin E(H)\}$. Suppose there exist values $x_1, \ldots, x_n \in (0,1)$ such that for each $i \in [n]$,

$$\Pr(\mathcal{E}_i) \leqslant x_i \prod_{\{i,j\} \in E(H)} (1 - x_j).$$

Then the probability that none of the events $\mathcal{E}_i$ happens is positive, i.e., $\Pr(\bigcap_{i=1}^n \bar{\mathcal{E}}_i) > 0$.

2.2 Random vertex partition through local lemma

Some part of our construction builds on ideas from [15]. In order to bypass the $\log n$ barrier for d and be able to analyze the algorithm for all $1 \leqslant d \leqslant n$, we however need to phrase our construction differently, using quantization and the Lovász Local Lemma (Lemma 8).

The idea is to partition $V(G)$ into a big set $B = \{v_1, \ldots, v_{|B|}\}$ and a small set S, where $|S| = (n/d) \cdot o(d)$. At the end, we will assure that $f^V(v_{i+1}) = f^V(v_i) + 1$ in B, and that vertices in S have larger weights than those in B. Our argument divides into three steps. Step 1 includes a random construction positioning weights in B close to expected values, which are relatively sparsely distributed. In Step 2 we modify the weights of edges between B and S to make vertices in B have the desired weights. This is also the main purpose of singling out the set S. One benefit of S being small compared to B is that if we assign heavy weights between S and B, then weights of vertices in S are expected to increase more significantly than those in B. Step 3 is to modify weights in S in order to make them all pairwise distinct.

Fix parameters ϵ, γ such that $\epsilon \in (0, 1/4)$ and $0 < 2\gamma < \epsilon$. Let G be an n-vertex d-regular graph. Set

$$s^* = 13\lceil d^{1/2+\epsilon}/13 \rceil,$$

note that $s^* \in [d^{1/2+\epsilon}, d^{1/2+\epsilon} + 13)$ and $13 | s^*$. Unless specified, we always assume d is sufficiently large in terms of γ.

We first describe the main random ingredient of the construction. Let X_v for $v \in V(G)$ be i.i.d. uniform random variables, $X_v \sim U[0,1]$. We use the values of X_v's to separate the vertices into d bins B_i where

$$B_i = \{v \in V(G) : (i-1)/d \leqslant X_v < i/d\}$$

for $1 \leqslant i < d$ and $B_d = \{v \in V(G) : 1 - 1/d \leqslant X_v \leqslant 1\}$; note that in expectation, each B_i includes n/d vertices. Let the big set, consisting of most of the bins be defined as $B = \bigcup_{1 \leqslant i \leqslant d - s^*} B_i$. The remaining bins form a small set S, which we partition into 13 subsets $S_i = \{\bigcup B_j : d - (14-i)s^*/13 < j \leqslant d - (13-i)s^*/13\}$ for $1 \leqslant i \leqslant 13$, hence $S = V(G) \setminus B = \bigcup_{1 \leqslant i \leqslant 13} S_i$.

Finally, we label some edges as "corrected" to satisfy a subtle technical issue (and guarantee later that the average weight of edges weighted $\lfloor n/d \rfloor + 1$ and $\lfloor n/d \rfloor + 2$ is exactly $(n/d)+1$). More precisely, we randomly label an edge with both end vertices in B *"corrected"* independently with probability $\max(\{n/d\}, 1 - \{n/d\})$ (where $\{n/d\}$ denotes the fractional part of n/d), which is at least $1/2$.

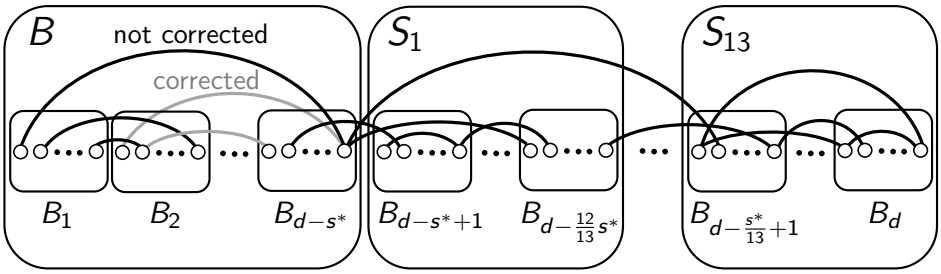

Figure 1: Key random vertex partition. Grey edges in B are corrected.

Lemma 9. *With positive probability, the following statements hold simultaneously if d is large enough.*

(I) *(C_{vS_i}) For each $v \in V(G)$ and $1 \leqslant i \leqslant 13$, $\deg_{S_i}(v) \in [s^*/13 - d^{1/2+\gamma}, s^*/13 + d^{1/2+\gamma}]$.*

(II) *(C_{vS}) For each $v \in V(G)$, $\deg_S(v) \in [s^* - 13d^{1/2+\gamma}, s^* + 13d^{1/2+\gamma}]$, or equivalently, $\deg_B(v) \in [d - s^* - 13d^{1/2+\gamma}, d - s^* + 13d^{1/2+\gamma}]$.*

(III) *(C_{vB}) For each $v \in V(G)$, if $v \in B_i \cap B$ for some i, then the number of edges between v and $\{\bigcup B_j, d - s^* - i + 1 < j \leqslant d - s^*\}$ is in the interval $[(i-1) - d^{1/2+\gamma}, (i-1) + d^{1/2+\gamma}]$.*

(IV) *(C'_{vB}) For each $v \in V(G)$, if $v \in B_i \cap B$ for some i, then the number of edges between v and $\{\bigcup B_j, d - s^* - i + 1 < j \leqslant d - s^*\}$ that are labeled "corrected" is in the interval $[(i-1)\alpha - \alpha d^{1/2+\gamma}, (i-1)\alpha + \alpha d^{1/2+\gamma}]$ where $\alpha = \max(\{n/d\}, 1 - \{n/d\})$.*

(V) *(C_i) For each $1 \leqslant i \leqslant d$, $|\bigcup_{j \leqslant i} B_j| \in [in/d - nd^\gamma/\sqrt{d}, in/d + nd^\gamma/\sqrt{d}]$.*

(VI) *(C_{S_i}) For each $1 \leqslant i \leqslant 13$, $|S_i| \in [s^*n/(13d) - 2nd^\gamma/\sqrt{d}, s^*n/(13d) + 2nd^\gamma/\sqrt{d}]$.*

Proof. Let $\mathcal{E}_{vS_i}$ be the bad event that C_{vS_i} does not hold for given $v \in V(G)$, $1 \leqslant i \leqslant 13$. We analogously denote by $\mathcal{E}_{vS}, \mathcal{E}_{vB}, \mathcal{E}'_{vB}, \mathcal{E}_i, \mathcal{E}_{S_i}$ the remaining bad events. We first bound the probability of each of these, and then use Lovász Local Lemma to show that with positive probability none of these bad events happen.

Fix $v \in V(G)$ and let us consider $\mathcal{E}_{vS_i}$ for any given $1 \leqslant i \leqslant 13$. Since each of d neighbors of v is independently included in S_i with probability exactly $s^*/(13d)$, where $d^{1/2+\gamma} < s^*/13 < d$ for d large enough, by the Chernoff Bound,

$$\Pr(\mathcal{E}_{vS_i}) < 2e^{-d^{1+2\gamma}/(3d)} < 2e^{-d^{2\gamma}/6}.$$

As the events C_{vS_i} imply C_{vS}, we proceed to compute the conditional probabilities $\Pr(\mathcal{E}_{vB} | v \in B_i)$ and $\Pr(\mathcal{E}'_{vB} | v \in B_i)$. These are trivially 0 for $i = 1$. Thus we next assume $2 \leqslant i \leqslant d - s^*$. As each of d neighbors u of v has probability $(i-1)/d$ to

be in $\bigcup_{d-s^*-i+1<j\leqslant d-s^*} B_j$ and probability $\alpha(i-1)/d$ to be in $\bigcup_{d-s^*-i+1<j\leqslant d-s^*} B_j$ and simultaneously form a corrected edge uv, by the two Chernoff Bounds, since $\alpha \geqslant 1/2$,

$$\Pr(\mathcal{E}_{vB}|v \in B_i) \leqslant 2\exp\left(-\frac{d^{1+2\gamma}}{3\max(i-1, d^{1/2+\gamma})}\right) < 2e^{-d^{1+2\gamma}/(3d)} < 2e^{-d^{2\gamma}/6},$$
$$\Pr(\mathcal{E}'_{vB}|v \in B_i) \leqslant 2\exp\left(-\frac{\alpha^2 d^{1+2\gamma}}{3\max(\alpha(i-1), \alpha d^{1/2+\gamma})}\right) < 2e^{-\alpha d^{1+2\gamma}/(3d)} \leqslant 2e^{-d^{2\gamma}/6}.$$

Since by definition $\Pr(\mathcal{E}_{vB}|v \in S) = 0$ and $\Pr(\mathcal{E}'_{vB}|v \in S) = 0$, thus by the law of total probability, $\Pr(\mathcal{E}_{vB}) < 2e^{-d^{2\gamma}/6}$ and $\Pr(\mathcal{E}'_{vB}) < 2e^{-d^{2\gamma}/6}$.

To finally estimate $\Pr(\mathcal{E}_i)$, we note that for any i each of n vertices is independently included in $\bigcup_{j\leqslant i} B_i$ with probability i/d. Thus by the Chernoff Bound,

$$\Pr(\mathcal{E}_i) \leqslant 2\exp\left(-\frac{n^2 d^{2\gamma}/d}{3\max\left(in/d, nd^{\gamma}/\sqrt{d}\right)}\right) \leqslant 2e^{-\frac{n^2 d^{2\gamma-1}}{3n}} = 2e^{-n/(3d^{1-2\gamma})}.$$

Since conditions (I) and (V) of the lemma imply conditions (II) and (VI), respectively, we just need to show that with positive probability none of $\mathcal{E}_{vS_i}, \mathcal{E}_{vB}, \mathcal{E}'_{vB}, \mathcal{E}_i$ holds. We will apply the Lovász Local Lemma (Lemma 8). There are $13n$ events of type $\mathcal{E}_{vS_i}$ (for each $v \in V(G)$ and $1 \leqslant i \leqslant 13$), n events of type $\mathcal{E}_{vB}$, n events of type $\mathcal{E}'_{vB}$ and d events of type $\mathcal{E}_i$. Note that for any given v and i, each of the events $\mathcal{E}_{vS_i}, \mathcal{E}_{vB}, \mathcal{E}'_{vB}$ is mutually independent of all other events $\mathcal{E}_{uS_j}, \mathcal{E}_{uB}, \mathcal{E}'_{uB}$ with u at distance at least 3 from v in G, i.e. all but most $13(d^2+1)+(d^2+1)+(d^2+1) < 16d^2$ such events (while each $\mathcal{E}_i$ may depend on any other event). We assign value $x = d^{-2}/1600$ to each $\mathcal{E}_{vS_i}, \mathcal{E}_{vB}, \mathcal{E}'_{vB}$, and assign value $y = d^{-1}/100$ to all $\mathcal{E}_i$. Therefore, in order to apply Lemma 8 we just need to check that

$$\begin{cases} 2e^{-d^{2\gamma}/6} \leqslant x(1-x)^{16d^2}(1-y)^d \\ 2e^{-n/(3d^{1-2\gamma})} \leqslant y(1-y)^d(1-x)^{15n} \end{cases}.$$

Note that $1 - a \geqslant e^{-10a}$ for $0 \leqslant a \leqslant 0.5$. Thus it is sufficient to show that:

$$\begin{cases} 2e^{-d^{2\gamma}/6} \leqslant e^{\ln(d^{-2}/1600)} e^{-160d^2\cdot(d^{-2}/1600)} e^{-10d\cdot d^{-1}/100} \\ 2e^{-n/(3d^{1-2\gamma})} \leqslant e^{\ln(d^{-1}/100)} e^{-10\cdot d\cdot d^{-1}/100} e^{-150n\cdot d^{-2}/1600} \end{cases},$$

which is equivalent to:

$$\begin{cases} d^{2\gamma}/6 \geqslant \ln 2 + \ln(1600d^2) + 1/10 + 1/10 \\ n/(3d^{1-2\gamma}) \geqslant \ln 2 + \ln(100d) + 1/10 + 15n/(160d^2) \end{cases}.$$

As $n \geqslant d+1$, these two inequalities above hold when d is sufficiently large in terms of γ. The conclusion thus follows by Lemma 8. □

2.3 Assigning weights

Suppose all statements in Lemma 9 hold. We will assign and modify edge weights in G in three steps. Whenever needed we assume d is large enough in terms of γ.

Step 1. The purpose of this step is to construct an initial weighting function $f_1 : E(G) \to \mathbb{N}$ so that all $v \in B_i$ have weights very close to $(n/d)i$ for each i. Let us set

$$\omega := \max(\lceil n/d^{1+\epsilon-2\gamma} \rceil, 2).$$

We define $f_1(\{u,v\})$ for all $\{u,v\} \in E(G)$ as follows:

$$f_1(\{u,v\}) := \begin{cases} \lfloor n/d \rfloor + 1, & \text{if } v \in B_i \cap B,\ u \in B_j \cap B,\ d - s^* - i + 1 < j \leqslant d - s^* \\ & \quad \text{and } (\{n/d\} \geqslant 1/2 \text{ and } \{u,v\} \text{ is not a corrected edge}) \\ & \quad\quad \text{or } (\{n/d\} < 1/2 \text{ and } \{u,v\} \text{ is a corrected edge})), \\ \lfloor n/d \rfloor + 2, & \text{if } v \in B_i \cap B,\ u \in B_j \cap B,\ d - s^* - i + 1 < j \leqslant d - s^* \\ & \quad \text{and } (\{n/d\} < 1/2 \text{ and } \{u,v\} \text{ is not a corrected edge}) \\ & \quad\quad \text{or } (\{n/d\} \geqslant 1/2 \text{ and } \{u,v\} \text{ is a corrected edge})), \\ i\omega + \lceil n/d \rceil, & \text{if } v \in B,\ u \in S_i, \text{ for } 1 \leqslant i \leqslant 13, \\ 1, & \text{otherwise.} \end{cases}$$

Consider any $v \in B_i \cap B$. We assume $\{n/d\} \geqslant 1/2$, as the analysis and result in the opposite case is essentially the same. By the definition of f_1 and Lemma 9, since $1 < n/d$ and $\omega \leqslant \lceil n/d \rceil < 2n/d$,

$$\begin{aligned} f_1^V(v) &= \sum_{u:\{u,v\}\in E(G), u\in S} f_1(\{u,v\}) + \sum_{u:\{u,v\}\in E(G), u\in B} f_1(\{u,v\}) \\ &\geqslant \left(\sum_{j=1}^{13} (j\omega + \lceil n/d \rceil)(s^*/13 - d^{1/2+\gamma}) \right) \\ &\quad + \left(\lfloor n/d \rfloor ((i-1) - d^{1/2+\gamma}) + \{n/d\}((i-1) - d^{1/2+\gamma}) + \left(d - s^* - 13d^{1/2+\gamma}\right) \right) \\ &> \left((7\omega + \lceil n/d \rceil)\, s^* - d^{1/2+\gamma}(91\omega + 13\lceil n/d \rceil) \right) \\ &\quad + \left((n/d)(i-1) + d - s^* - 16n/d^{1/2-\gamma} \right) \\ &\geqslant ((n/d)(i-1) + d + (7\omega + \lceil n/d \rceil - 1)\, s^*) - 224n/d^{1/2-\gamma}. \end{aligned}$$

By almost the same reasoning we may obtain an analogous upper bound for $f_1^V(v)$, implying that

$$\left| f_1^V(v) - ((n/d)(i-1) + d + (7\omega + \lceil n/d \rceil - 1)\, s^*) \right| \leqslant 224n/d^{1/2-\gamma}. \tag{1}$$

Moreover, the following claim holds.

Claim 10. *For any edge $e \in E(B)$, $f_1(e) \in [1, \lfloor n/d \rfloor + 2]$. For any edge e between B and S, $f_1(e) \in [\lceil n/d \rceil, \lceil n/d \rceil + 13\omega]$. For any edge $e \in E(S)$, $f_1(e) = 1$.*

Step 2. Consider a linear ordering $v_1, v_2, \ldots$ of the vertices in B such that $X_{v_j} \geqslant X_{v_i}$ if $j \geqslant i$ (where X_v's refer to values of the random variables used within the proof of Lemma 9 for which all conditions of the lemma hold; we may assume these are all distinct, as this is true with probability 1). To adjust edge and vertex weights we will provide $f_2 : E(G) \to \mathbb{N}$ supported on edges between B and S (i.e. equal to 0 for the remaining edges) so that as a result, for $f_{12} := f_1 + f_2$, the following conditions hold:

- each $v_k \in B$ has weight $k + d + (7\omega + \lceil n/d \rceil - 1)\, s^* + \lceil 250n/d^{1/2-\gamma} \rceil$;
- for any $u \in S, v \in B$, $f_{12}^V(u) - f_{12}^V(v) > 0$; and finally:
- for $u \in S_{i+1}, v \in S_i$ with $1 \leqslant i \leqslant 12$, $f_{12}^V(u) - f_{12}^V(v)$ is large enough to provide a buffer for weight adjustments in Step 3.

Suppose $v_k \in B_i \cap B$. Then

$$|\bigcup_{j \leqslant i-1} B_j| \leqslant k \leqslant |\bigcup_{j \leqslant i} B_j|,$$

and thus, by Lemma 9 (V), $(i-1)n/d - n/d^{1/2-\gamma} \leqslant k \leqslant in/d + n/d^{1/2-\gamma}$. Therefore,

$$|k - (i-1)n/d| \leqslant 2n/d^{1/2-\gamma}. \tag{2}$$

By (1), (2) and the triangle inequality,

$$|f_1^V(v_k) - (k + d + (7\omega + \lceil n/d \rceil - 1)\, s^*)\,| \leqslant 226n/d^{1/2-\gamma}. \tag{3}$$

Claim 11. *There exists $f_2 : E(G) \to \mathbb{N}$ supported on edges between B and S such that $\|f_2\|_\infty \leqslant \lceil 10^3 n/d^{1+\epsilon-\gamma} \rceil$ and for each $v_k \in B$, $f_{12}^V(v_k) = k + d + (7\omega + \lceil n/d \rceil - 1)\, s^* + \lceil 250n/d^{1/2-\gamma} \rceil$, provided d is sufficiently large in terms of ϵ, γ.*

Proof. Note that by (3), the weight of every $v_k \in B$ is smaller than the target value $k+d+(7\omega + \lceil n/d \rceil - 1)\, s^* + \lceil 250n/d^{1/2-\gamma} \rceil$, while we need to add no more than $500n/d^{1/2-\gamma}$ to achieve it. This discrepancy can be leveled up by adding appropriate quantities to weights of edges between v_k and S, thereby defining f_2. As by Lemma 9 (II), $d_S(v_k) \geqslant s^* - 13d^{1/2+\gamma} > s^*/2$, it is sufficient to add to every edge weight between v_k and S at most $\lceil (500n/d^{1/2-\gamma})/(s^*/2) \rceil \leqslant \lceil 10^3 n/d^{1+\epsilon-\gamma} \rceil$. □

Claim 12. *For every $u \in S$ and $v \in B$, $f_{12}^V(u) > f_{12}^V(v)$. For each $2 \leqslant i \leqslant 13$ and every $u \in S_i$, $u' \in S_{i-1}$, we have $f_{12}^V(u) - f_{12}^V(u') \geqslant 0.4\omega d$.*

Proof. Consider $u \in S_i$ for any fixed $1 \leqslant i \leqslant 13$. By the definition of f_{12}, we have $f_{12}(e) \geqslant f_1(e)$ for every edge e. Hence, as due to Lemma 9 (II), $\deg_B(u) \geqslant d - s^* - 13d^{1/2+\gamma} \geqslant d - 2s^*$, by the definition of f_1 and the fact that $\omega \leqslant \lceil n/d \rceil$,

$$f_{12}^V(u) \geqslant (i\omega + \lceil n/d \rceil)\,(d - 2s^*) \geqslant i\omega d + \lceil n/d \rceil d - 2s^*(i+1)\lceil n/d \rceil. \tag{4}$$

By Claim 11, for every $v \in B$,

$$f_{12}^V(v) \leqslant f_{12}^V(v_{|B|}) = |B| + d + (7\omega + \lceil n/d \rceil - 1)\, s^* + \lceil 250n/d^{1/2-\gamma} \rceil < n + d + 9\lceil n/d \rceil s^*.$$

Thus for $u \in S_i$ and $v \in B$, as $1 \leqslant i \leqslant 13$ and $d \leqslant 0.5\omega d$, together with (4),

$$f_{12}^V(u) - f_{12}^V(v) \geqslant i\omega d - 2s^*(i+1)\lceil n/d \rceil - d - 9\lceil n/d \rceil s^* \geqslant 0.5\omega d - 37\lceil n/d \rceil s^* > 0,$$

where the last inequality holds, as $\omega d \gg (n/d)s^*$ when $d \to \infty$. The first claim is thus proved.

Consider now any $u' \in S_{i-1}$ for a given $2 \leqslant i \leqslant 13$. By Claim 11 and the definition of f_1, $f_{12}(e) \leqslant (i-1)\omega + \lceil n/d \rceil + 10^3 n/d^{1+\epsilon-\gamma} + 1$ for every edge e incident with u'. Thus

$$f_{12}^V(u') \leqslant \left((i-1)\omega + \lceil n/d \rceil + 10^3 n/d^{1+\epsilon-\gamma} + 1\right) d = i\omega d + \lceil n/d \rceil d + d - \omega d + 10^3 n/d^{\epsilon-\gamma}. \tag{5}$$

Thus combining (4) and (5), for $u' \in S_{i-1}$ and $u \in S_i$, as $\omega d \gg ns^*/d, \omega d \gg n/d^{\epsilon-\gamma}$ when $d \to \infty$,

$$\begin{aligned} f_{12}^V(u) - f_{12}^V(u') &\geqslant \omega d - d - 10^3 n/d^{\epsilon-\gamma} - 2s^*(i+1)\lceil n/d \rceil \\ &\geqslant 0.5\omega d - 10^3 n/d^{\epsilon-\gamma} - 28\lceil n/d \rceil s^* \geqslant 0.4\omega d. \end{aligned}$$

□

By Claims 10 and 11, as a summary, the following holds after Step 2.

Claim 13. *For any edge $e \in E(B)$, $f_{12}(e) \in [1, \lfloor n/d \rfloor + 2]$. For any edge e between B and S, $f_{12}(e) \in [\lceil n/d \rceil, \lceil n/d \rceil + 13\omega + \lceil 10^3 n/d^{1+\epsilon-\gamma} \rceil]$. For any edge $e \in E(S)$, $f_{12}(e) = 1$.*

Step 3. In this step, we introduce $f_3 : E(G) \to \mathbb{N}$ that is only supported on $E(S)$ such that all vertices in S have distinct weights with respect to $f = f_1 + f_2 + f_3$. We will moreover show that Claim 12 implies that f attributes distinct weights to all vertices in G. For this aim we will adapt the algorithm from [15], which was modeled on the idea of Kalkowski, Karoński, and Pfender [9].

Let AP be the family of sets of the following form $\mathcal{AP} = \{(2\lambda)\lceil n/(3d) \rceil + a, (2\lambda + 1)\lceil n/(3d) \rceil + a\}$ where λ, a are integers with $\lambda \geqslant 0$, $a \in [0, \lceil n/(3d) \rceil - 1]$. Note the sets in AP with all possible values of λ, a partition the non-negative integers, where different values of a correspond to $\lceil n/(3d) \rceil$ different congruence classes, denoted by $\mathcal{C}_a = \{a + k\lceil n/(3d) \rceil, k \in \mathbb{N}\}$. Our primary goal is to attribute the weight of every vertex $v \in S$ to appropriately chosen $\mathcal{AP}_v \in \mathsf{AP}$ so that for each $1 \leqslant \ell \leqslant 13$, vertices in S_ℓ have associated pairwise distinct $\mathcal{AP}_v$'s, which are thus disjoint.

We initialize f_3 by setting $f_3(e) = \lceil n/(3d) \rceil$ for all $e \in E(S)$. Given an ordering $v_1, \ldots, v_{|S|}$ of vertices in S (specified later), each edge $\{v_i, v_j\}$ with $i < j$ is called a *forward edge* of v_i and a *backward edge* of v_j. The algorithm will sequentially process v_i's modifying f_3 on edges in S incident to currently analyzed v_i. For v_1 no modifications are needed – we simply let $\mathcal{AP}_{v_1}$ be the set in AP that contains the current value of $f^V(v_1)$ and move on (to v_2). Then for every consecutive $i \geqslant 2$, we will choose a special set $\mathcal{AP}_{v_i} \in \mathsf{AP}$ and guarantee that $f^V(v_i)$ belongs in $\mathcal{AP}_{v_i}$ from the end of step i until the end of the algorithm. We admit two options to modify f_3 on backward edges of the given v_i: either by adding 0 or one of the values in $\{\pm\lceil n/(3d) \rceil\}$. Specifically, say $\{v_i, u\}$ is a backward edge of v_i. If the current value of $f^V(u)$ is the smaller value in $\mathcal{AP}_u$, we admit adding 0 or $\lceil n/(3d) \rceil$ to $f_3(\{v_i, u\})$; if $f^V(u)$ is the larger value in $\mathcal{AP}_u$, we in turn admit subtracting 0 or $\lceil n/(3d) \rceil$ from $f_3(\{v_i, u\})$. Thereby the updated $f^V(u)$ will always remain in $\mathcal{AP}_u$, as desired. We finally admit adding any value in $\{0, 1, \ldots, \lceil n/(3d) \rceil\}$ to

the weights $f_3(e)$ of all forward edges of v_i, which will in particular allow us to determine the congruence class $f^V(v_i)$ will eventually land in.

We now specify the ordering $v_1, v_2, \dots$ of the vertices in S. At the beginning we arrange the vertices in $\bigcup_{1\leqslant i\leqslant 12} S_i$ according to the values of X_{v_i}, from the smallest to the largest, and thus consistently with the order of $S_1, \dots, S_{12}$. The last in the ordering are vertices from S_{13}, which are ordered differently due to some technical subtlety concerning vertices without forward edges. Suppose $C_1, \dots, C_K$ are the connected components in S_{13}, ordered arbitrarily. Each component has at least two vertices by Lemma 9(I). For each C_i, we use reversed BFS to order its vertices and denote r_i, t_i the last two vertices in C_i. (Thus t_i is the root of the tree in BFS; $\{r_i, t_i\} \in E(S)$.) Let $R = \{r_1, \dots, r_K\}$ and $T = \{t_1, \dots, t_K\}$. We finally define the ordering in S_{13} by concatenating the orderings of $C_1, \dots, C_K$. Note that by Lemma 9(I), the set of *terminal vertices*, i.e., vertices with no forward edges in the obtained ordering in S, is T.

We now show specific procedures which will allows us to achieve the desired goal. Suppose we are in step i, i.e. we are analyzing $v_i \in S_t$, where $1 \leqslant t \leqslant 13$, and that $v_i \notin R \cup T$, hence v_i has at least one forward edge, say e_i. The existing sets $\mathcal{AP}_u$ for u prior to v_i in S_t correspond to at most $|S_t|$ congruence classes with possible duplicates. Therefore, there must be a congruence class $\mathcal{C}_a$ that includes at most $|S_t|/\lceil n/(3d)\rceil$ prior sets $\mathcal{AP}_u$ with $u \in S_t$. Thus we may include the weight of $f^V(v_i)$ in $\mathcal{C}_a$ by adding one of admissible values in $\{0, 1, \dots, \lceil n/(3d)\rceil\}$ to the weight $f_3(e_i)$. We then modify the rest of the forward edges of v_i by adding 0 or $\lceil n/(3d)\rceil$ and change the weights of some backward edges of v_i by $\lceil n/(3d)\rceil$ according to the specified rules, if necessary. Note that this way we may obtain $\deg_S(v_i)$ consecutive terms in $\mathcal{C}_a$ as potential weights of v_i (in entire G). Since each prior set $\mathcal{AP}_u$ blocks at most two consecutive terms in $\mathcal{C}_a$, we can find this way an attainable $f^V(v_i) \in \mathcal{C}_a$ which is not blocked if $\deg_S(v_i) > 2|S_t|/\lceil n/(3d)\rceil$. This is however implied by an even stronger inequality, which holds by Lemma 9 (VI)(II):

$$\begin{aligned} 4|S_t|/\lceil n/(3d)\rceil + 2 \leqslant & 4(s^* n/(13d) + 2n/d^{1/2-\gamma})/(n/(3d)) + 2 = 12s^*/13 + 24d^{1/2+\gamma} + 2 \\ < & s^* - 13d^{1/2+\gamma} \leqslant \deg_S(v_i). \end{aligned} \tag{6}$$

We finally set $\mathcal{AP}_{v_i}$ as the only set in AP containing the attained weight of v_i.

We are left to show how to handle $r_j, t_j \in R \cup T$, where $\{r_j, t_j\}$ is the only forward edge of r_j. We analyze both vertices simultaneously in a similar manner as above. Recall $r_j, t_j \in S_{13}$. First, by an averaging argument, we can choose an admissible addition from $\{0, 1, \dots, \lceil n/(3d)\rceil\}$ to $f_3(\{r_j, t_j\})$ such that the two new congruence classes of $f^V(r_j), f^V(t_j)$ each includes at most $2|S_{13}|/\lceil n/(3d)\rceil$ prior sets $\mathcal{AP}_u$ with $u \in S_{13}$, disregarding temporarily $\mathcal{AP}_{r_j}$ from the point of view of t_j. Next, analogously as above, by (6), we can change the weights of backward edges of r_j by $\pm\lceil n/(3d)\rceil$ so that the resulting $f^V(r_j)$ belongs to $\mathcal{AP}_{r_j} \in \mathsf{AP}$ disjoint from those of the prior vertices in S_{13}. Finally, we analogously adjust the weights of all backward edges of t_j except $\{r_j, t_j\}$ so that the resulting $f^V(t_j)$ belongs to $\mathcal{AP}_{t_j} \in \mathsf{AP}$ disjoint from those of the prior vertices in S_{13} including $\mathcal{AP}_{r_j}$, which is again feasible by (6) (where "+2" was incorporated in this inequality to facilitate distinguishing $\mathcal{AP}_{t_j}$ from $\mathcal{AP}_{r_j}$).

Claim 14. *For every edge e of G, $1 \leqslant f(e) \leqslant \lceil n/d \rceil + 13\omega + \lceil 10^3 n/d^{1+\epsilon-\gamma} \rceil$.*

Proof. By Claim 13 all edge weights were in the interval $[1, \lceil n/d \rceil + 13\omega + \lceil 10^3 n/d^{1+\epsilon-\gamma} \rceil]$ prior to Step 3, while edges in $E(S)$ were assigned 1. Within Step 3 we first added $\lceil n/(3d) \rceil$ to the weights of the edges in $E(S)$ and only these edges could have been further modified, each at most twice (once as a forward edge, when its weight could be increased by a non-negative integer not exceeding $\lceil n/(3d) \rceil$, and once as a backward edge, whose weight could be modified by at most $\lceil n/(3d) \rceil$). Thus $f_3(e) \in [0, 3\lceil n/(3d) \rceil]$, and hence the result follows. □

2.4 Proof of Theorems 3 and 5

Note that by the algorithm applied above, $f^V(v)$'s are pairwise distinct for vertices in the same S_i for $1 \leqslant i \leqslant 13$. We first show that if $u \in S_i$ and $u' \in S_{i-1}$, where $2 \leqslant i \leqslant 13$, then $f^V(u) > f^V(u')$. By Claim 12, $f^V_{12}(u) - f^V_{12}(u') \geqslant 0.4\omega d$. Moreover, by the algorithm in Step 3, $f^V_{12}(u'') \leqslant f^V(u'') \leqslant f^V_{12}(u'') + 3\lceil n/(3d) \rceil \deg_S(u'')$ for every $u'' \in S$, as $0 \leqslant f_3(e) \leqslant 3\lceil n/(3d) \rceil$ for each $e \in E(S)$. Hence, by Lemma 9 (II), since $\omega d \gg ns^*/d$ as $d \to \infty$,

$$f^V(u) - f^V(u') \geqslant 0.4\omega d - 3\lceil n/(3d) \rceil \deg_S(u') \geqslant 0.4\omega d - 4(n/d)2s^* > 0.$$

Thus all vertices in S have pairwise distinct weights. For any vertices $u \in S$ and $v \in B$, since $f^V_3(v) = 0$ and $f^V_3(u) \geqslant 0$, by Claim 12, $f^V(u) - f^V(v) > 0$. Finally, as by Claim 11, the weights of the vertices in B formed a $|B|$-element segment of integers after Step 2 and have not changed ever since, all vertices in G have distinct weights.

Suppose $d^{1+\epsilon-2\gamma} \geqslant n$. Then $\omega = 2$ and $\lceil 10^3 n/d^{1+\epsilon-\gamma} \rceil = 1$ for d large enough. Thus by Claim 14, there is d_0 such that $\|f\|_\infty \leqslant \lceil n/d \rceil + 27 < n/d + 28$ for $d \geqslant d_0$. As there are only finitely many graphs with $d < d_0$ and $d^{1+\epsilon-2\gamma} \geqslant n$ (while e.g. by [9], $s(G) \leqslant 6\lceil n/d \rceil$), Theorem 5 and Corollary 6 follow due to taking $\beta = \epsilon - 2\gamma$, as $\epsilon \in (0, 1/4)$ while γ can be chosen arbitrarily small.

On the other hand, by Claim 14, regardless of the proportion of n to d, f is upper bounded by

$$\begin{aligned} \lceil n/d \rceil + 13\omega + \lceil 10^3 n/d^{1+\epsilon-\gamma} \rceil &< (n/d+1) + 13(n/d^{1+\epsilon-2\gamma} + 2) + (n/d^{1+\epsilon-2\gamma} + 1) \\ &= n/d(1 + 14/d^{\epsilon-2\gamma}) + 28 \end{aligned} \tag{7}$$

when d is sufficiently large, say $d \geqslant d_0$ (where d_0 is a constant dependent on ϵ, γ). Hence Theorem 3 follows by taking $\beta = \epsilon - 2\gamma$ analogously as above. For $d < d_0$, we may finally again use the result in [9] implying that $s(G) < (n/d)(1+5) + 6$, where $5 \leqslant C/d^\beta$ for large enough (in terms of d_0) constant C. Thus Corollary 4 is proved as well due to (7).

3 Conclusion and remarks

In this note, we proved a uniform upper bound $s(G) \leqslant \frac{n}{d}(1 + C/d^\beta) + 28$, which confirms the Faudree-Lehel Conjecture for $d \geqslant n^\alpha$ for any fixed $\alpha > 0.8$. Our primary goal was to present a relatively short proof, hence we did not strive to optimize all constants and

auxiliary functions within our argument. In particular, using a slightly more detailed analysis concerning r_i, t_i in the algorithm in Step 3 (applied already e.g. in [9, 11]) and a few other minor alterations, one may easily reduce the constant 28 to 16 in all our main results (and the constant 14 to 8).

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