



Equilibria in the Large-Scale Competition for Market Share in a Commodity with Resource-Buying

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Abstract

We study a mean field game model of Cournot/Bertrand competition between firms. Chan and Sircar introduced such a mean field model of competition in natural resource extraction. In their model, each firm has a finite reserve of a commodity and may choose to extract a positive quantity per unit time. We instead treat the situation in which firms compete to purchase raw materials, rather than produce the raw material. With this change, we arrive at the same nonlinear system of partial differential equations, but what corresponds to the positive rate of resource extraction in the Chan–Sircar model is instead negative in our setting. We prove existence of stationary solutions, using a Lyapunov–Schmidt decomposition and multiple applications of the implicit function theorem.

Keywords Partial differential equations · Stationary solutions · Cournot competition · Bertrand competition

1 Introduction

In [5], Chan and Sircar derive and examine a mean field representation of the Bertrand and Cournot oligopoly models, of the type introduced by Guéant et al. [24] representing the large-scale competition between firms for market share in an exhaustible resource through a coupled pair of partial differential equations. This model and its applications was then subsequently generalized somewhat in by Chan and Sircar in [6]. We propose an alternate game, based on competition to acquire raw materials rather than to produce raw materials; however, we demonstrate that our game has the same representation as a mean field game. The difference between the two games is related to the requirement in [5] that the production quantity q be nonnegative for each player; our model instead features nonpositive q . We demonstrate how this is meaningful in the context of the classical Cournot game and in the N -player game of Chan and Sircar, before studying the associated mean field games PDE system.

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The Bertrand competition model, as used in [5], has N firms competing for market share in the production of interchangeable (although not indistinguishable) goods by simultaneously setting the price p per unit of their good. Of course, similarly, Cournot competition instead has the firms competing by setting their production quantity, q . In either case, a linear demand law determines the relationship between q and p . The profit yielded by the firm that sets price p is then $\Pi = (p - s)q$ where s is the constant cost of production per unit. The competitors are assumed to be rational in the sense that their logic is interchangeable; they simultaneously seek to maximize profit, and the system exhibits a Nash equilibrium [5].

Chan and Sircar show that Bertrand and Cournot competition are equivalent in the continuum case $N \rightarrow \infty$ and proceed to derive the following mean field game system of the competition, which we shall call the Chan–Sircar system.

$$\begin{cases} u_t + \frac{\sigma^2}{2} u_{xx} - ru + \mathcal{H}(t, x, u_x, m) = 0 \\ m_t - \frac{\partial}{\partial x} [mG(t, x, u_x, m)] - \frac{\sigma^2}{2} m_{xx} = 0 \\ u(t, 0) = m(t, 0) = 0 \\ t \in [0, \infty), x \in [0, L] \end{cases} \quad (1)$$

where

$$\begin{aligned} \mathcal{H}(t, x, u_x, m) &= \frac{1}{4} \left(\frac{2}{2 + \epsilon \eta(t)} + \frac{\epsilon}{2 + \epsilon \eta(t)} \int_0^L u_x(t, s) m(t, s) \, ds - u_x(t, x) \right)^2, \\ G(t, x, u_x, m) &= \frac{1}{2} \left(\frac{2}{2 + \epsilon \eta(t)} + \frac{\epsilon}{2 + \epsilon \eta(t)} \int_0^L u_x(t, s) m(t, s) \, ds - u_x(t, x) \right), \end{aligned}$$

and

$$\eta(t) = \int_0^L m(t, s) \, ds.$$

In this model $\epsilon \in [0, \infty)$ is an interchangeability parameter, $r > 0$ is an economic drag coefficient, and σ is market volatility. We shall consider $\epsilon = 0$ to be the monopoly condition, in which the goods of distinct firms are non-interchangeable, and we shall consider $\epsilon = \infty$ to represent the case in which the goods are indistinguishable. The firms are indexed by the amount of remaining resource $x \in [0, L]$. The firm with x units of resource remaining at time t has utility $u(t, x)$, and $m(t, x)$ is the evolving density of firms. The trajectory of each firm is described by the stochastic differential equation

$$dX_t = -q(t, X_t) \, dt + \sigma \, dW_t \quad (2)$$

where W_t is an adapted Brownian motion, so production quantity q is the control used to guide the resource level X_t . From the derivation of the Fokker–Planck equation we find that $q(t, x) = G(t, x, u_x, m)$ for any firm in state x at time t where (u, m) is a solution of the system. Note that at present an incomplete set of boundary conditions has been given for the system (1); the solutions we will find will therefore form a two-parameter family of solutions. We will use one of these degrees of freedom to insist that m be a probability measure, meaning that $\int_0^L m(t, x) \, dx = 1$, for all t , and the other degree of freedom will allow a boundary condition to be set at $x = L$.

We will, in Sect. 2.2 below, give a new interpretation of the Chan–Sircar N -player game, in which the roles of costs and revenues are reversed. The firms in this case compete to buy quantities of a raw material which is needed for production; as a firm or its competitors produce more goods for sale, there is necessarily higher demand for the raw material, leading

to an increased cost for the raw material. Then the finished goods are sold at a given price. We then consider the mean field limit of this N -player game, arriving at the same mean field game as in [5]. In this new interpretation, the domain size, L , represents a maximum amount of resource able to be consumed. This may arise, for instance, from a regulatory quota system (such as fishing quotas or air pollution limitations).

It is important to note that while we employ a new interpretation of the Chan–Sircar mean field game model, the equations determining the value function and the distribution of agents remain the same. We explore decidedly different solutions, however, since the quantity, q , in the solutions studied at present is negative, while the Chan–Sircar interpretation requires non-negative q . We are unaware of a reason that solutions to the Chan–Sircar system in the original interpretation would keep q non-negative a priori. This suggests that to use the Chan–Sircar system for its original purpose, the model should be modified, introducing a positive part operation, to only allow values of q which are non-negative. For the present interpretation, the time-dependent model would similarly require the introduction of a negative part operation. We do not bother to introduce this in the present work, as we consider here only stationary solutions for which we can verify that the quantity is in fact negative. For consideration of the time-dependent problem, however, this change should be made.

Since their development by Lasry and Lions in [29] and [30], and by Caines, Huang, and Malhalmé in [26] and [25], mean field games have become valuable tools for efficiently modeling the large-scale interaction of many agents. To represent the competition described above between N players traditionally, a coupled system of N Hamilton–Jacobi equations would be required to obtain the utility function $u_i(t)$ of each firm i , but this approach becomes burdensome as N becomes large. Drawing on the techniques of statistical mechanics, a mean field game (MFG) model reduces the continuum case $N \rightarrow \infty$ to a system of two PDE’s, coupled through a nonlinear term (the Hamiltonian $\mathcal{H} = \mathcal{H}(t, x, p, m)$) that governs the interaction of agents. The first equation is a Hamilton–Jacobi–Bellman (HJB) equation, solved backwards in time, that governs the utility function $u(t, x)$ common to all agents, and the second equation is a Fokker–Planck (or Forwards Kolmogorov) equation that governs the evolution forwards in time of the density $m(t, x)$. For a general exposition of mean field games and their applications, we direct the reader to [31].

Many existence results for mean field games rely on separable structure of the Hamiltonian, in which the Hamiltonian may be decomposed as $\mathcal{H}(t, x, p, m) = F(t, x, p) + V(t, x, m)$. In such systems, V is referred to as the coupling, which may depend on m locally or nonlocally (in the nonlocal case, V may depend only on integrals of m). When the Hamiltonian is separable, assumptions such as convexity or monotonicity of F or V are typically made; then mathematical tools related to these structures may be used. For the stationary mean field games problem, there are many prior works which rely on separable structure [3, 7, 8, 12, 14, 15, 32]. For stationary solutions of nonseparable mean field games, prior works tend to focus on nonseparable Hamiltonians which arise in specific applications; the greatest number of these are related to models with congestion effects [9–11, 13]. Another existence theorem for stationary solutions of nonseparable mean field games is for a model of household wealth [1]. (We note that this is by no means an exhaustive list of prior results on stationary solutions for mean field games in either the separable or nonseparable cases.)

Prior work on the Chan–Sircar system includes existence of solutions for the time-dependent problem. In [18], Bensoussan and Graber establish the existence and uniqueness of classical solutions to the time-dependent problem, assuming that the terminal data u^T and the initial data m^0 are sufficiently smooth. These regularity requirements were lowered by Graber and Mouzouni in [21]. Graber and Sircar have also studied solutions of the related master equation [23]. Graber and Laurel studied the dependence of solutions on the inter-

changability parameter ϵ in [20]. We also refer the reader also to [19] where Graber, Ignazio, and Neufeld examine the mean field system arising from a generalized form of the game studied in [5]. The authors use a generalized Hamiltonian that allows their results to be generalized to cover a range of examples arising in the literature, they generalize the demand schedule, and jumps are allowed in the trajectory X_t through the addition of a Lévy process to (2). The model (1) is of a kind known as a mean field game of controls, or an extended mean field game, because the distribution is on the players strategies rather than simply their states [4, 16]. Other than the previously mentioned prior works on the Chan–Sircar system, other analysis for mean field games of controls includes [17, 22, 28].

The plan of the paper is as follows. In Sect. 2 below we develop our proposed game; we begin with revisiting very classical Cournot competition in Sect. 2.1, and the Chan–Sircar N -player game in Sect. 2.2. We then progress to the mean field game in the rest of Sect. 2. We give the stationary system and develop some function spaces and mapping properties in Sect. 3. We prove existence of stationary solutions in our first main theorem, Theorem 4.11, in Sect. 4 by making a Lyapunov–Schmidt decomposition, and making two applications of the implicit function theorem. Many works on existence of stationary solutions for mean field games are adapted to a moving reference frame; the solutions of Sect. 4, however, are genuinely stationary. We can also treat the moving frame case, though, and prove existence of stationary solutions in this case in our second main theorem, Theorem 5.8, in Sect. 5. Finally, we close with some discussion in Sect. 6.

2 Proposed Game

We now define the game that we examine in this paper, and show that it yields the same mean field representation as the Bertrand–Cournot competition examined in [5].

2.1 The Classical Cournot Model with 2 Firms

To begin, we review the classical Cournot model with 2 firms, and we offer a reinterpretation. We follow the textbook [34] for the classical model.

Each of two firms seeks to maximize their profit, $\Pi_i(q_1, q_2) = q_i P(q_1 + q_2) - cq_i$, where $P(q_1 + q_2)$ is the market price the firms receive from selling their produced quantity q_i of goods, and c represents a fixed cost per unit produced. The price function follows a linear model, $P(Q) = \alpha - Q$, for some $\alpha > 0$ and for $Q \leq \alpha$. The unique Nash equilibrium can then be found, which is $q_1 = q_2 = \frac{\alpha - c}{3}$, as long as $\alpha > c$.

In this classical model, firms choose the quantity they produce, knowing this affects the price they receive for the goods being sold. In this way it appears to make sense only for q_i to be non-negative. However, we may treat exactly the same model by taking q_i to be non-positive, if we reinterpret what had been costs as revenue and what had been revenue as costs. With q_i non-positive, the firms still each seek to maximize $\Pi_i(q_1, q_2) = q_i P(q_1 + q_2) - cq_i$. Now $-cq_i \geq 0$ is the revenue, with c being the fixed price received for each unit sold. Likewise, $q_i P(q_1 + q_2) \leq 0$ represents each firm's costs, which depends upon the quantity that all firms choose to produce (we again take $P(Q) = \alpha - Q > 0$ for some $\alpha > 0$). The unique Nash equilibrium is again $q_1 = q_2 = \frac{\alpha - c}{3}$, but now with the assumption that $c > \alpha$.

In general terms, the idea of a fixed price with variable costs appears relevant when the price for the final good is regulated or subject to contractual obligations over some time period, while raw materials may need to be purchased on the open market with prices

subject to demand over shorter time scales. We view electricity generation as an example of a situation in which these circumstances may generally arise. That is, the owner of an electricity generation plant may need to buy fossil fuels on the open market, and the price for these may depend on the overall demand. Meanwhile the price received per unit of electricity generated may be fixed (at least in the medium term) as it is subject to regulation. Production of electronics hardware may similarly be subject to such phenomena, as the costs for raw materials (such as rare earth elements) may fluctuate based on demand on a faster timescale than contracted prices for the finished product.

2.2 N -Player Game

We now review and reinterpret the N -player game of [5]. Suppose that N players participate in a 1-period game based on Cournot competition. Player i purchases raw material from Supplier i and converts it into a commodity to be sold on a common market. The quantity x of raw material purchased by a firm is allowed to range from 0 to an upper limit L , and the strategy possessed by Firm i is the rate $q_i \in \mathbb{R}$ at which raw material is acquired. While $q = q^+ \geq 0$ always in the game presented by Chan and Sircar, in our game we expect $q_i \leq 0$, which means that Player i purchases raw material from Supplier i at a rate of $|q_i|$.

Assumption 2.1 We assume that the agents are rational in the sense that their logic is interchangeable. In particular, assume that there exists a continuously differentiable utility function $u(x)$ where $u(x)$ reflects the expected profit Π yielded by a firm with x units of raw material in reserve, assuming that the firm chooses an optimal strategy. In this case, the marginal utility $u'(x)$ represents the expected increase in profit if a firm's supply is increased from x to $x + 1$.

For each i let $q_{\sim i}$ denote the vector of production quantities set by all players other than Player i , and let $\epsilon \in [0, \infty)$ be the same interchangeability parameter employed in [5]. If the firms choose strategies $(q_1, \dots, q_N)$ then the price of Commodity i on the common market is given by

$$p_i = p(q_i, q_{\sim i}) = 1 - \left(q_i + \frac{\epsilon}{N-1} \sum_{j \neq i} q_j \right);$$

in our interpretation, this price is a cost to Player i . Then the profit achieved by Player i is given by

$$\Pi_i = \Pi_i(q_i, q_{\sim i}; x) = (p_i - u'(x))q_i.$$

Note that in the original case with $q_i > 0$, Player i depletes their material, yielding a revenue of $p_i q_i$ at a "depletion cost" (that is, the cost given by the quantity of resource depleted times the marginal utility of that resource) of $u'(x)q_i$. In our case instead we have $q_i < 0$, and Player i purchases material at a cost of $p_i |q_i|$, yielding a revenue of $u'(x)|q_i|$.

2.3 Nash Equilibrium

The following employs the same method used in [5] to obtain the Chan–Sircar system. We fix $q_{\sim i}$ and optimize Π_i using q_i . Now

$$\Pi_i = q_i - q_i^2 - \frac{\epsilon q_i}{N-1} \sum_{j \neq i} q_j - u'(x)q_i$$

so

$$\partial_{q_i} \Pi_i = 1 - 2q_i - \frac{\epsilon}{N-1} \sum_{j \neq i} q_j - u'(x)$$

and therefore $\partial_{q_i} \Pi_i(q_i^*, q_{\sim i}; x_i) = 0$ implies

$$2q_i^* = 1 - \frac{\epsilon}{N-1} \sum_{j \neq i} q_j - u'(x),$$

which is to say

$$q_i^* = \frac{1}{2} \left(1 - \frac{\epsilon}{N-1} \sum_{j \neq i} q_j - u'(x_i) \right).$$

Let $J = (1, \dots, 1) \in \mathbb{R}^N$ be the all-ones vector. We then obtain the matrix equation

$$2 \begin{pmatrix} q_1^* \\ \vdots \\ q_N^* \end{pmatrix} = J - \frac{\epsilon}{N-1} Z \begin{pmatrix} q_1^* \\ \vdots \\ q_N^* \end{pmatrix} - \begin{pmatrix} u'(x_1) \\ \vdots \\ u'(x_N) \end{pmatrix}$$

where

$$[Z]_{ij} = \begin{cases} 1, & i \neq j \\ 0, & i = j \end{cases}.$$

Define

$$A = A_{\epsilon, N} = 2I_N + \frac{\epsilon}{N-1} Z$$

so that we may obtain the linear equilibrium equation

$$A \begin{pmatrix} q_1^* \\ \vdots \\ q_N^* \end{pmatrix} = J - \begin{pmatrix} u'(x_1) \\ \vdots \\ u'(x_N) \end{pmatrix}.$$

Observe that $A_{\epsilon, N}$ is invertible if and only if $\epsilon \neq 2N - 2$. We will disregard this possibility as it becomes irrelevant when we take $N \rightarrow \infty$. We may rewrite

$$A = \left(2 - \frac{\epsilon}{N-1} \right) I_N + \frac{\epsilon}{N-1} J J^T$$

and apply the Sherman-Morrison formula [35] to obtain

$$A^{-1} = \frac{N-1}{2N-2-\epsilon} I_N - \frac{\epsilon}{(2N-2-\epsilon)(2+\epsilon)} J J^T.$$

For players with resource levels $(x_1, \dots, x_N)$ in a 1-period game, we obtain

$$\begin{aligned} \begin{pmatrix} q_1^* \\ \vdots \\ q_N^* \end{pmatrix} &= \left(\frac{N-1}{2N-2-\epsilon} - \frac{\epsilon N}{(2N-2-\epsilon)(2+\epsilon)} \right) J + \frac{\epsilon}{(2N-2-\epsilon)(2+\epsilon)} \sum_{j=1}^N u'(x_j) J \\ &\quad - \frac{N-1}{2N-2-\epsilon} \begin{pmatrix} u'(x_1) \\ \vdots \\ u'(x_N) \end{pmatrix}. \end{aligned}$$

The equilibrium strategy for Player i is then

$$q_i^* = \frac{1}{2 + \epsilon} + \frac{\epsilon N}{(2N - 2 - \epsilon)(2 + \epsilon)} \bar{c} - \frac{N - 1}{2N - 2 - \epsilon} u'(x_i), \quad (3)$$

where, in keeping with the notation of Chan and Sircar, the average $\bar{c}$ is calculated as the average value of $u'(x)$ over the distribution of N players. In this case,

$$\bar{c} = \frac{1}{N} \sum_{j=1}^N u'(x_j).$$

It is worth noting that the optimal strategy given above requires that each firm has global knowledge of the distribution $(x_1, \dots, x_N)$ of the other firms, which is also required by the game analyzed in [5].

2.4 Continuum Game

It is now a simple matter to see that the continuous, time-bound form of the N -player game given above has the same Nash equilibrium as the game analyzed by Chan and Sircar, and that therefore both games admit the same mean field representation.

Lemma 2.2 *Suppose that a continuum of firms are distributed in resource level according to a continuous probability density $m(x)$ on $[0, L]$, and that for each N an ordered N -player sampling $x_1 < \dots < x_N$ is produced, and that the sampling is rich enough that*

$$\bigcup_{N=1}^{\infty} \{x_1, \dots, x_N\}$$

is dense in $[0, L]$. Note that this is equivalent to choosing a sequence of partitions of $[0, L]$ that we may use to construct a Riemann sum. For each N , set $x_{N+1} = L$ and define

$$\xi_N(x) = \sum_{j=1}^N \frac{1}{N(x_{j+1} - x_j)} I_{[x_j, x_{j+1})}(x)$$

where $I_{[x_j, x_{j+1})}$ is the characteristic function on $[x_j, x_{j+1})$. Then, ξ_N is the probability density of the N -player sampling, and its distributional limit as $N \rightarrow \infty$ is $m(x)$.

Proof Because we have sampled firms that do not repeat resource levels within the continuous distribution, we see that $\xi_N(x)$ is defined for all $x \in [0, L]$, and clearly $\xi_N \geq 0$. Let μ denote Lebesgue measure and let P denote probability. Out of the N -player sampling, suppose we choose a firm at random with resource level x (i.e., $x = x_i$ for some $i = 1, \dots, N$), and let $0 \leq a < b \leq L$.

$$\begin{aligned} \int_a^b \xi_N(x) \, dx &= \sum_{j=1}^N \frac{\mu([x_j, x_{j+1}] \cap [a, b])}{N(x_{j+1} - x_j)} = \sum_{j=1}^N \frac{1}{N} P\left(x \in [a, b] \mid x \in [x_j, x_{j+1})\right) \\ &= \sum_{j=1}^N P\left(x \in [x_j, x_{j+1})\right) P\left(x \in [a, b] \mid x \in [x_j, x_{j+1})\right) = P\left(x \in [a, b]\right). \end{aligned}$$

In particular

$$\int_0^L \xi_N(x) dx = \sum_{j=1}^N \frac{1}{N} = 1.$$

Observe that for any integrable function f on $[0, L]$,

$$\lim_{N \rightarrow \infty} \int_0^L f(s) \xi_N(s) ds = \int_0^L f(s) m(s) ds$$

by the Bounded Convergence Theorem. $\square$

Suppose a continuum of firms are sampled in the manner described in the Lemma above, and let $\bar{c}_N$ denote the average value of $u'(x)$ over the N -player sampling, as given in Eq. 3. Now

$$\bar{c}_N = \sum_{j=1}^N \frac{u'(x_j)}{N} = \sum_{j=1}^N \frac{u'(x_j)}{N(x_{j+1} - x_j)} (x_{j+1} - x_j) = \sum_{j=1}^N u'(x_j) \xi_N(x_j) (x_{j+1} - x_j)$$

is a Riemann sum approximating

$$\int_0^L u'(s) m(s) ds$$

as $N \rightarrow \infty$. Taking the limit of (3) as $N \rightarrow \infty$ we obtain the continuous equilibrium

$$q^*(x) = \frac{1}{2} \left(\frac{2}{2 + \epsilon} + \frac{\epsilon}{2 + \epsilon} \int_0^L u'(s) m(s) ds - u'(x) \right) \quad (4)$$

which matches exactly the 1-period continuous equilibrium obtained in [5, Section 2.2].

Suppose we construct a sequence of 1-period games indexed by time $t \in [0, T]$ so that we remove any firms that have exhausted their quota of resources from the competition, and let $\eta(t) = \int_0^L m(t, x) dx$ denote the proportion of firms remaining at time t . If we repeat the same 1-period calculation performed above at time t , over a proportion $\eta(t)$ of the original firms, then we obtain (4) with the substitutions $\epsilon \mapsto \epsilon \eta$ and

$$\bar{p} \mapsto \frac{1}{\eta} \int_0^L p(t, x) m(t, x) dx$$

yielding

$$q^*(t, x) = \frac{1}{2} \left(\frac{2}{2 + \epsilon \eta(t)} + \frac{\epsilon}{2 + \epsilon \eta(t)} \int_0^L u_x(t, s) m(t, s) ds - u_x(t, x) \right).$$

As in [5], the optimal profit is given by $\Pi^* = q^{*2}$ which can be found by examination of the linear demand law relating p and q . The projected future revenue for an agent with resource level x at time t , discounted according to the interest rate $r > 0$, assuming that they choose an optimal strategy is given by

$$u(t, x) = \sup_p \left\{ E \left[\int_t^\infty e^{-r(s-t)} p(s, t) q[p](s, t) I_{X_s \geq 0} ds \mid X_t = x \right] \right\}.$$

Because (p^*, q^*) in the time-bound mean field case of our game is identical to the equilibrium obtained by Chan and Sircar in [5], $u(t, x)$ satisfies the same HJB equation, namely,

$$u_t + \frac{\sigma^2}{2} u_{xx} - ru + \Pi^* = 0.$$

The Fokker–Planck equation for the distribution of agents is merely the one-dimensional Fokker–Planck equation with drift term $q^*(t, x)$. The system (1) is then yielded in the mean field limit of our time-bound game. For more information on the derivation of mean field games, we direct the reader to [31].

3 The Stationary System, Function Spaces, and Mapping Properties

For convenience, let $\mathcal{H} = \mathcal{H}_\epsilon$ and $G = G_\epsilon$ represent the Chan–Sircar Hamiltonian and $-\mathcal{H}_p$ obtained for a chosen value of ϵ . We first seek solutions to the Chan–Sircar system of the form $u(t, x) = u(x)$ and $m(t, x) = m(x)$ yielding the following stationary MFG system:

$$\begin{cases} \frac{\sigma^2}{2}u'' - ru + \mathcal{H}_\epsilon(t, x, u', m) = 0 \\ \frac{\sigma^2}{2}m'' + \left(mG_\epsilon(t, x, u', m)\right)' = 0 \\ x \in [0, L] \end{cases} \quad (5)$$

Let

$$M[\epsilon, r, u, m] = \begin{pmatrix} \frac{\sigma^2}{2}u'' - ru + \mathcal{H}_\epsilon(t, x, u', m) \\ \frac{\sigma^2}{2}m'' + \left(mG_\epsilon(t, x, u', m)\right)' \end{pmatrix}.$$

In what follows, we will introduce the function spaces and provide a definition of a solution.

Definition 3.1 Let

$$R_0 = \left\{ v \in H^2([0, L]) : v(0) = 0 \right\}$$

be H^2 with an imposed Dirichlet condition, endowed with the typical inner product

$$\langle v, w \rangle_{R_0} = \langle v, w \rangle_{H^2([0, L])} = \sum_{k=0}^2 \int_0^L v^{(k)} w^{(k)} \, ds$$

for $i = 1, 2$. Let $R = R_0 \oplus R_0$ be endowed with the natural inner product,

$$\langle (u, m), (v, n) \rangle_R = \langle u, v \rangle_{R_0} + \langle m, n \rangle_{R_0}.$$

Definition 3.2 A solution to system (5) over R is an ordered pair $(u, m) \in R$ such that for some $r \geq 0$ and $\epsilon \geq 0$, $M[\epsilon, r, u, m] = 0$, (u, m) satisfies $u', m \geq 0$, $u'' < 0$, and

$$\eta = \int_0^L m \, ds = 1.$$

We regard m as a stationary density of the Chan–Sircar system. Let $\Omega \subset R$ denote the set of all such solutions. We have set $\eta = 1$, although in what follows, we will see that we could choose a different value for $\eta \in [0, 1]$ to obtain results that are not fundamentally different. The stipulation that u' be a decreasing function reflects our desire that the stationary solutions satisfy the law of diminishing marginal utility.

Definition 3.3 We define the target space. Let $S = L^2([0, L]) \times L^2([0, L])$ be equipped with the inner product

$$\langle (u, m), (v, n) \rangle_S = \langle u, v \rangle_{L^2} + \langle m, n \rangle_{L^2}.$$

We will use the following standard result on Sobolev spaces, which follows directly from Sobolev embedding [2].

Lemma 3.4 *If $\{u_n\}_{n \in \mathbb{Z}^+}$ converges in $H^s([0, L])$ for any $s < \frac{1}{2}$ then it converges uniformly on $[0, L]$.*

Finally, we have our mapping property.

Lemma 3.5 *The mapping M takes the Hilbert space R to the Hilbert space S , i.e. $M[\epsilon, r, \cdot] : R \rightarrow S$ for any $(\epsilon, r) \in \mathbb{R}^2$.*

Proof It is easy to see that by Sobolev embedding, $u' \in C([0, L])$ for any $u \in R_0$ and $m \in C^1([0, L])$ for any $m \in R_0$, and $H^2([0, L])$ is closed under scalar multiplication. Thus

$$\mathcal{H}_\epsilon(t, x, u', m) \in L^2([0, L])$$

for any $(u, m) \in R$. Furthermore m and u' both belong to $H^1([0, L])$ so m and $G_\epsilon(t, x, u', m)$ both belong to $H^1([0, L])$, which is closed under pointwise multiplication [2]. Thus

$$mG_\epsilon(t, x, u', m) \in H^1([0, L])$$

and therefore

$$\left(mG_\epsilon(t, x, u', m)\right)' \in L^2([0, L]).$$

Thus for any $(\epsilon, r) \in \mathbb{R}^2$ and $(u, m) \in R$, $M[\epsilon, r, u, m] \in S$. □

Proposition 3.6 (A Proposition on Regularity) *Let (u, m) belong to the solution set Ω . Then*

$$u, m \in C^\infty([0, L]).$$

Proof By Sobolev embedding

$$u \in C^1([0, L])$$

and because $M[\epsilon, r, u, m] = 0$,

$$\frac{\sigma^2}{2}u'' = ru - \mathcal{H}(t, x, u', m) \in C([0, L])$$

so $u \in C^2([0, L])$. We repeat this reasoning iteratively to obtain $u \in C^\infty([0, L])$. Similarly Sobolev embedding yields

$$m \in C^1([0, L]).$$

We find

$$\frac{\sigma^2}{2}m'' = -\left(mG(t, x, u', m)\right)' = m'G(t, x, u', m) + mG(t, \cdot, u', m)' \in C([0, L])$$

so $m \in C^2([0, L])$. As before, we repeat this reasoning iteratively to obtain that $m \in C^\infty([0, L])$. □

4 Existence of Stationary Solutions

In this section, we will prove the existence of stationary solutions, culminating in Theorem 4.11 below. Many works on mean field games demonstrate the existence of solutions which are stationary in a moving reference frame; in the present section, we instead develop solutions which are genuinely stationary. However, below in Sect. 5, we will also develop solutions which are stationary in a moving frame.

4.1 Base-Case Solution

We now seek an ordered pair $(\phi, \psi) \in R$ (corresponding to $u(x, t) = \phi(x)$ and the stationary density $m(x, t) = \psi(x)$) such that $M[0, 0, \phi, \psi] = 0$. The HJB equation becomes

$$\frac{\sigma^2}{2}\phi'' + \frac{1}{4}(1 - \phi')^2 = 0.$$

Let $\theta = 1 - \phi'$. Now

$$2\sigma^2\theta' = \theta^2$$

and by integrating we obtain

$$\frac{-2\sigma^2}{\theta} = x + c$$

for some $c \in \mathbb{R}$. Thus

$$\phi' = 1 + \frac{2\sigma^2}{x + c}$$

and therefore $\phi(0) = 0$ yields

$$\phi = x + 2\sigma^2 \ln\left(\frac{x + c}{c}\right).$$

Equation 2 in (5) yields

$$\frac{\sigma^2}{2}\psi'' + \left(\frac{\psi}{2}(1 - \phi')\right)' = 0$$

and therefore, by integrating,

$$\frac{\sigma^2}{2}\psi' + \frac{1}{2}(1 - \phi')\psi = \frac{\sigma^2}{2}\psi'(0).$$

Thus

$$\left(\psi e^{(x-\phi)/\sigma^2}\right)' = \psi'(0)e^{(x-\phi)/\sigma^2}$$

so

$$\psi = \psi'(0)e^{(\phi-x)/\sigma^2} \int_0^x e^{(s-\phi)/\sigma^2} ds.$$

Observe that

$$\frac{x - \phi}{\sigma^2} = \ln \left[\left(\frac{x + c}{c} \right)^{-2} \right]$$

so if $\psi(0) = 0$, then

$$\psi = \psi'(0) \left(\frac{x+c}{c} \right)^2 \int_0^x \left(\frac{s+c}{c} \right)^{-2} ds = \psi'(0)(x+c)^2 \left[\frac{x}{(x+c)c} \right] = \frac{\psi'(0)}{c} x(x+c).$$

Note that as long as $\psi'(0) \geq 0$ and $c \geq 0$ we also have that $\psi \geq 0$. In order to satisfy the requirement that $\eta = 1$, we simply choose

$$\psi'(0) = \frac{6c}{L^2(2L+3c)}$$

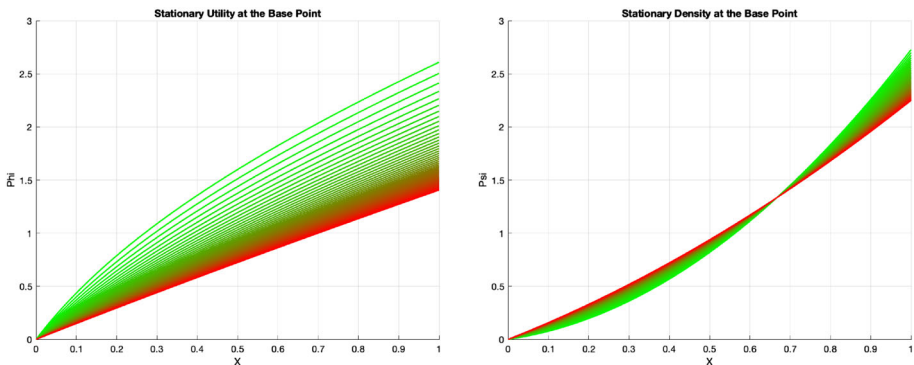
yielding the base-case solutions

$$\phi = x + 2\sigma^2 \ln \left(\frac{x+c}{c} \right)$$

and

$$\psi = \frac{6x(x+c)}{L^2(2L+3c)}$$

for $c > 0$. It is clear that ϕ' is a decreasing function, satisfying the law of diminishing marginal utility. In the following figure, we see samples of the base-case solutions obtained above, in this case corresponding to $L = 1$, $\sigma = 1/\sqrt{2}$, and c ranging from $1/4$ (in green) to 2 (in red).



Remark 4.1 Observe that the base-case utility ϕ satisfies the law of diminishing marginal utility with ϕ' approaching 1 util per unit of resource as $x \rightarrow \infty$.

Remark 4.2 In the original Chan–Sircar model, a firm with x units of an exhaustible resource remaining at time t extracts the resource at a rate of $q(t, x)$, meeting market demand [5]. By contrast, in our model, in the case $\epsilon = r = 0$, we observe that $q < 0$ for all x and t , meaning that firms instead acquire the relevant resource, and the relationship $dX_t = -q(t, X_t)dt + \sigma dW_t$ is driving the firms to the right along the x -axis, forming the distribution seen above. To see this we calculate

$$q(t, x; \epsilon = r = 0) = G_{\epsilon=0} \left(t, x, \phi', \psi \right) = \frac{1}{2} \left(1 - \phi' \right) < \frac{-\sigma^2}{L+c} < 0.$$

As we have remarked in the introduction, for time-dependent problems, it seems problematic that the Chan–Sircar system allows solutions with negative q and with positive q . This suggests that the model should be modified, with the original application requiring the taking of

a positive part to disallow solutions with negative q . The corresponding operation, taking the negative part, is unnecessary at present only because we are considering stationary solutions for which we can verify $q < 0$ as above.

Let $\Theta = (u, m)$ and $V = (v, w)$ denote generic points in R . For our implicit function theorem argument, we will need to work with the linearization $D_{\Theta}M$ about the point $(0, 0, \phi, \psi)$. A straightforward calculation yields the formula which we give in the next definition.

Definition 4.3 We will use the following notation to denote the linearization of the system about $(0, 0, \phi, \psi)$, in the monopoly condition with a zero interest rate:

$$B(V) = D_{\Theta}M[0, 0, \phi, \psi](V) = \begin{pmatrix} \frac{\sigma^2}{2}v'' - \frac{1}{2}(1 - \phi')v' \\ \frac{\sigma^2}{2}w'' + \frac{1}{2}\left((1 - \phi')w - \psi v'\right)' \end{pmatrix}.$$

4.2 Lyapunov–Schmidt Decomposition

We will demonstrate that the linearization is not injective at the base-case. We will then apply Lyapunov–Schmidt decomposition to yield a local system of solutions about the base-case $(0, 0, \phi, \psi)$.

Lemma 4.4 *The kernel $\text{Ker } B$ is 2-dimensional over $\mathbb{R}$ given any $c > 0$.*

Proof If $B[v, w] = 0$ then

$$v = v'(0) \int_0^x e^{(s-\phi)/\sigma^2} ds$$

and

$$\frac{\sigma^2}{2}w'' + \frac{1}{2}\left(w(1 - \phi') - \psi v'\right)' = 0$$

so

$$w' + \frac{1}{\sigma^2}(1 - \phi')w = w'(0) + \frac{1}{\sigma^2}v'(0)e^{(x-\phi)/\sigma^2}\psi.$$

Applying an integrating factor

$$\left(we^{(x-\phi)/\sigma^2}\right)' = w'(0)e^{(x-\phi)/\sigma^2} + \frac{1}{\sigma^2}v'(0)e^{2(x-\phi)/\sigma^2}\psi$$

and therefore

$$w = \frac{v'(0)}{\sigma^2}e^{(\phi-x)/\sigma^2} \int_0^x e^{2(s-\phi)/\sigma^2}\psi ds + w'(0)e^{(\phi-x)/\sigma^2} \int_0^x e^{(s-\phi)/\sigma^2} ds.$$

Observe then that $\text{Ker } B$ has basis $\{\tilde{V}, \tilde{W}\}$ where

$$\tilde{V} = \left(\int_0^x e^{(s-\phi)/\sigma^2} ds, \frac{1}{\sigma^2}e^{(\phi-x)/\sigma^2} \int_0^x e^{2(s-\phi)/\sigma^2}\phi ds \right)$$

and

$$\tilde{W} = \left(0, e^{(\phi-x)/\sigma^2} \int_0^x e^{(s-\phi)/\sigma^2} ds \right) = \left(0, \frac{\psi}{\psi'(0)} \right).$$

Each component in the two vectors above are C^∞ on $[0, L]$ and therefore belong to $H^2([0, L])$. We see that $\tilde{V}, \tilde{W} \in R$. $\square$

Lemma 4.5 *The linearization $B : R \rightarrow S$ is bounded and surjective.*

Proof We have shown that B has a finite-dimensional kernel; we now show that B is bounded and surjective. Let $V = (v, w) \in R$ and observe that

$$\left\| \frac{\sigma^2}{2} v'' + \frac{1}{2} (\phi' - 1) v' \right\|_{L^2} \leq \frac{\sigma^2}{2} \|v''\|_{L^2} + \frac{1}{2} \|\phi' - 1\|_\infty \|v'\|_{L^2}$$

and

$$\begin{aligned} & \left\| \frac{\sigma^2}{2} w'' + \frac{1}{2} \left((1 - \phi') w - \psi v' \right)' \right\|_{L^2} \\ & \leq \frac{\sigma^2}{2} \|w''\|_{L^2} + \frac{1}{2} \|1 - \phi'\|_\infty \|w'\|_{L^2} + \frac{1}{2} \|\phi''\|_\infty \|w\|_{L^2} + \frac{1}{2} \|\psi'\|_\infty \|v'\|_{L^2} \\ & \quad + \frac{1}{2} \|\psi\|_\infty \|v''\|_{L^2} \end{aligned}$$

so there exists $b = b(\sigma^2, c, L) > 0$ such that

$$\|B(V)\|_S \leq b \|V\|_R.$$

The linearization is bounded; we will show that it is surjective. Suppose that $B(V) = (f, g)$ for some $V = (v, w) \in R$ and $(f, g) \in S$. Then

$$\frac{\sigma^2}{2} v'' - \frac{1}{2} (1 - \phi') v' = f$$

and therefore

$$\left(e^{(\phi-x)/\sigma^2} v' \right)' = \frac{2}{\sigma^2} e^{(\phi-x)/\sigma^2} f.$$

Integrating we obtain

$$e^{(\phi-x)/\sigma^2} v' = v'(0) + \frac{2}{\sigma^2} \int_0^x e^{(\phi-s)/\sigma^2} f \, ds$$

so

$$v = \int_0^x \left(\frac{c}{s+c} \right)^2 \left\{ v'(0) + \frac{2}{\sigma^2} \int_0^s \left(\frac{z+c}{c} \right)^2 f \, dz \right\} ds \in H^2([0, L])$$

with $v(0) = 0$ and therefore $v \in R_0$. Similarly $B(V)_2 = g$ yields

$$\frac{\sigma^2}{2} w'' + \frac{1}{2} \left((1 - \phi') w - \psi v' \right)' = g.$$

Now

$$\frac{\sigma^2}{2} w' + \frac{1}{2} (1 - \phi') w - \frac{1}{2} \psi v' = \frac{\sigma^2}{2} w'(0) + \int_0^x g(s) \, ds$$

so by applying an integrating factor,

$$\left(e^{(x-\phi)/\sigma^2} w' \right)' = \left\{ w'(0) + \frac{2}{\sigma^2} \int_0^x g(s) \, ds + \frac{1}{\sigma^2} \psi v' \right\} e^{(x-\phi)/\sigma^2},$$

$$w = e^{(\phi-x)/\sigma^2} \int_0^x \left\{ w'(0) + \frac{2}{\sigma^2} \int_0^s g(z) dz + \frac{1}{\sigma^2} \psi v' \right\} e^{(s-\phi)/\sigma^2} ds \in H^2([0, L])$$

with $w(0) = 0$ implying R_0 . Thus for every $(f, g) \in S$ there exists $V = (v, w) \in R$ such that $B(V) = (f, g)$. B is a surjective bounded linear mapping between Hilbert spaces with a finite-dimensional kernel, and trivially it has a finite-dimensional cokernel. It is then an index-2 Fredholm operator. $\square$

Definition 4.6 Let

$$P : R \rightarrow \text{Ker } B$$

be a projection and let $Q = I - P$ so that for any $\Theta \in R$, we may decompose Θ as

$$\Theta = P(\Theta) + Q(\Theta) = \beta_1 \tilde{V} + \beta_2 \tilde{W} + Q(\Theta).$$

Define

$$K[\epsilon, r, P(\Theta), Q(\Theta)] = M[\epsilon, r, P(\Theta) + Q(\Theta)] = M[\lambda, r, \Theta].$$

Now let

$$\tilde{B} = B|_{(\text{Ker } B)^\perp}$$

so $\tilde{B}$ is surjective from $Q(R)$ to S and may be calculated as the linearization of K in $Q(\Theta)$ at the base point $(0, 0, P(\Theta_0), Q(\Theta_0))$, where $\Theta_0 = (\phi, \psi)$.

In what follows we use the following form of the Implicit Function Theorem [27, Theorem 13.22].

Theorem 4.7 (Implicit Function Theorem) Suppose that X, Y , and Z are Banach spaces, and that $F : U \subset X \times Y \rightarrow Z$ is a continuously differentiable function defined on an open subset U of $X \times Y$. If $(x_0, y_0) \in U$ is a point such that $F(x_0, y_0) = 0$, and $D_y F(x_0, y_0) : Y \rightarrow Z$ is a one-to-one, onto, bounded linear map, then there is an open neighborhood $V \subset X$ of x_0 , an open neighborhood $W \subset Y$ of y_0 , and a unique function $f : V \rightarrow W$ such that

$$F(x, f(x)) = 0$$

for all $x \in V$. The function f is continuously differentiable.

Lemma 4.8 There exists a function

$$\Xi \in C^1([0, \infty) \times (0, \infty) \times \mathbb{R}^2; Q(R))$$

such that for any $\epsilon \geq 0$ and $r > 0$ sufficiently small, and for any $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$\|\beta_1 \tilde{V} + \beta_2 \tilde{W} - P(\Theta_0)\|_R$$

is sufficiently small,

$$M[\epsilon, r, \beta_1 \tilde{V} + \beta_2 \tilde{W} + \Xi(\epsilon, r, \beta_1, \beta_2)] = 0.$$

Proof Clearly B is surjective from $Q(R)$ onto S , so $\tilde{B}$ is surjective. B has a trivial kernel when restricted to $Q(R)$, so by construction $\tilde{B}$ is injective. We apply the Implicit Function Theorem in the sense above taking $X = \mathbb{R}^2 \times \text{Ker } B$ and $Y = (\text{Ker } B)^\perp$ to obtain the C^1 function Ξ described. $\square$

4.3 Properties of the Solutions

We have established a C^1 family of functions $(u, m) \in R$, indexed by $(\epsilon, r, \beta_1, \beta_2)$, that contains $\Theta_0 = (\phi, \psi)$. We now demonstrate that a C^1 branch of these solutions satisfy the conditions we require of solutions to system (5), namely, that $u', m \geq 0$, $u'' < 0$, and $\eta = 1$.

Definition 4.9 Let $Y(\epsilon, r, \beta_1, \beta_2) = -1 + \eta(\epsilon, r, \beta_1, \beta_2)$ be difference between 1 and the total integral of m , that is

$$Y(\epsilon, r, \beta_1, \beta_2) = -1 + \int_0^L \{ \beta_1 \tilde{V}_2 + \beta_2 \tilde{W}_2 + \Xi(\epsilon, r, \beta_1, \beta_2)_2 \} dx.$$

Let $(\tilde{\beta}_1, \tilde{\beta}_2)$ denote the coefficients of the component of (ϕ, ψ) in the kernel of B so that

$$P(\Theta_0) = \tilde{\beta}_1 \tilde{V} + \tilde{\beta}_2 \tilde{W}.$$

Lemma 4.10 At $(\epsilon, r, \beta_1, \beta_2) = (0, 0, \tilde{\beta}_1, \tilde{\beta}_2)$,

$$\frac{\partial \Xi}{\partial \beta_1} = \frac{\partial \Xi}{\partial \beta_2} = 0.$$

Proof This can be seen by differentiating in the sense of Frechet. Observe

$$\begin{aligned} 0 &= D_{\beta_1} M[\epsilon, r, \beta_1 \tilde{V} + \beta_2 \tilde{W} + \Xi(\epsilon, r, \beta_1, \beta_2)](z) \\ &= D_{P(\Theta)} K[\epsilon, r, \beta_1 \tilde{V} + \beta_2 \tilde{W}, \Xi(\epsilon, r, \beta_1, \beta_2)](z \tilde{V}) \\ &\quad + D_{Q(\Theta)} K[\epsilon, r, \beta_1 \tilde{V} + \beta_2 \tilde{W}, \Xi(\epsilon, r, \beta_1, \beta_2)] \left(D_{\beta_1} \Xi(\epsilon, r, \beta_1, \beta_2)(z) \right) \end{aligned}$$

so at $(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)$,

$$B|_{P(R)} \left(z \tilde{V} \right) + \tilde{B} \left(D_{\beta_1} \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2) \right) = 0.$$

But $B|_{P(R)} = 0$ and $\tilde{B}$ is injective by construction so

$$D_{\beta_1} \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)(z) = 0$$

for all z . Similarly we see that

$$\begin{aligned} 0 &= D_{\beta_2} M[0, 0, \tilde{\beta}_1 \tilde{V} + \tilde{\beta}_2 \tilde{W} + \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)](z) \\ &= D_{P(\Theta)} K[0, 0, \tilde{\beta}_1 \tilde{V} + \tilde{\beta}_2 \tilde{W}, \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)](z \tilde{W}) \\ &\quad + D_{Q(\Theta)} K[0, 0, \tilde{\beta}_1 \tilde{V} + \tilde{\beta}_2 \tilde{W}, \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)] \left(D_{\beta_2} \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)(z) \right), \end{aligned}$$

that is,

$$B|_{P(R)}(z \tilde{W}) + \tilde{B} \left(D_{\beta_2} \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)(z) \right) = 0.$$

But again, $B|_{P(R)} = 0$ and $\tilde{B}$ is injective by construction, so

$$D_{\beta_2} \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)(z) = 0$$

for all $z \in \mathbb{R}$.

We have shown that

$$\frac{\partial \Xi}{\partial \beta_1}(0, 0, \tilde{\beta}_1, \tilde{\beta}_2) = D_{\beta_1} \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)(1) = 0$$

and

$$\frac{\partial \Xi}{\partial \beta_2}(0, 0, \tilde{\beta}_1, \tilde{\beta}_2) = D_{\beta_2} \Xi(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)(1) = 0.$$

□

We are now in the position to state our first main theorem.

Theorem 4.11 Choose any $1 < k < 1 + 2\sigma^2/L$. There exists a neighborhood U of $(0, 0)$ and

$$\Phi \in C^1(U; \mathbb{R})$$

such that for any $(\epsilon, r) \in U$, $M[\epsilon, r, \Phi(\epsilon, r)] = 0$ and $(u, m) = \Phi(\epsilon, r)$ satisfies $u' \geq 0$, $\eta = 1$, $u'(L) = k$, u' is decreasing, and

$$u, m \in C^\infty([0, L]).$$

Proof For any $1 < k < 1 + 2\sigma^2/L$ we may choose $c > 0$ such that $\phi'(L) = k$. Taking this value of c , let

$$T(\epsilon, r, \beta_1, \beta_2) = \left(\left[\begin{array}{c} Y(\epsilon, r, \beta_1, \beta_2) \\ \beta_1 \tilde{V}_1 + \beta_2 \tilde{W}_1 + \Xi(\epsilon, r, \beta_1, \beta_2) \end{array} \right]' (L) - k \right)$$

so $T(0, 0, \tilde{\beta}_1, \tilde{\beta}_2) = 0$. Now, Lemma 4.10 may be employed to find that

$$\begin{aligned} D_{(\beta_1, \beta_2)} T(0, 0, \tilde{\beta}_1, \tilde{\beta}_2) &= \left(\int_0^L \tilde{V}_2 dx \int_0^L \tilde{W}_2 dx \quad \tilde{V}'_1(L) \tilde{W}'_1(L) \right) \\ &= \left(\frac{\int_0^L \tilde{V}_2 dx}{c^2/(L+c)^2} \quad \frac{L^2(2L+3c)/6c}{0} \right). \end{aligned}$$

Clearly $\text{Rank } D_{(\beta_1, \beta_2)} T(0, 0, \tilde{\beta}_1, \tilde{\beta}_2) = 2$. Now we may apply the Implicit Function Theorem to T and obtain that in a neighborhood of $(0, 0, \tilde{\beta}_1, \tilde{\beta}_2)$, $(\beta_1, \beta_2) = \beta(\epsilon, r)$ may be chosen in a continuously differentiable manner such that $\eta = 1$ and $u'(L) = k$ where

$$(u, m) = \beta_1 \tilde{V} + \beta_2 \tilde{W} + \Xi(\epsilon, r, \beta_1, \beta_2).$$

Let

$$\Phi(\epsilon, r) = \beta_1(\epsilon, r) \tilde{V} + \beta_2(\epsilon, r) \tilde{W} + \Xi(\epsilon, r, \beta(\epsilon, r))$$

be our C^1 parametrization of solutions. We now argue that sufficiently close to the base case, these functions satisfy $u' \geq 0$. Because Φ is continuous at $(0, 0)$ we know

$$\Phi(\epsilon, r) \rightarrow \Phi(0, 0) = (\phi, \psi)$$

in $H^2 \times H^2$ and therefore $\Phi(\epsilon, r)_1'$ and $\Phi(\epsilon, r)_2$ converge to ϕ' and ψ , respectively, uniformly on $[0, L]$ by Lemma 3.4. Note that $\phi' \geq 1$ and by uniform convergence, there exists $b > 0$ such that for all (ϵ, r) satisfying $\sqrt{\epsilon^2 + r^2} < b$ we have

$$\|\Phi(\epsilon, r)_1' - \phi'\|_\infty < 1$$

which implies that for (ϵ, r) sufficiently small, $\Phi(\epsilon, r)_1' > 0$.

Similarly, we may apply uniform convergence to the stationary density. Observe that $\psi'(0) > 0$ and fix $\delta \in (0, \psi'(0))$. By uniform convergence and continuity due to Sobolev embedding [2], there exist $b_1, h > 0$ such that for any (ϵ, r) satisfying $\sqrt{\epsilon^2 + r^2} < b_1$, $\Phi(\epsilon, r)_2 > \delta$ on $[0, h]$. By continuity, then, $\Phi(\epsilon, r)_2(x) \geq 0$ in a neighborhood of $x = 0$ with $\Phi(\epsilon, r) = 0$ only at $x = 0$ given any (ϵ, r) sufficiently small. Choose $h > 0$ such that $\Phi(\epsilon, r)_2' > \delta$ on $[0, h]$ and let $z(\delta) = \psi(h) > 0$. Here we use that $\psi' > 0$ and therefore $\psi(h)$ is the minimum of ψ on $[h, L]$. By uniform convergence we may choose $b_2 \geq b_1$ such that $\sqrt{\epsilon^2 + r^2} < b_2$ implies

$$\|\Phi(\epsilon, r)_2 - \psi\|_{L^\infty([h, L])} < z(\delta).$$

Now for any (ϵ, r) smaller than b_2 , $\Phi(\epsilon, r)_2 \geq 0$ on $[0, h]$ because $\Phi(\epsilon, r)_2(0) = 0$ together with $\Phi(\epsilon, r)_2' > 0$ on $[0, h]$, and $\Phi(\epsilon, r)_2 > 0$ on $[h, L]$ due to uniform convergence. It follows that $m \geq 0$ and $\eta = 1$ for (ϵ, r) sufficiently small. (The choice of δ was not significant except that it determined where we broke the interval into $[0, h]$ and $[h, L]$.)

The regularity follows from Proposition 3.6.

Note that

$$\phi'' \leq \frac{-2\sigma^2 c}{(L+c)^2} < 0$$

and $\Phi(\epsilon, r)_1$, $\Phi(\epsilon, r)_1'$, and $\Phi(\epsilon, r)_2$ converge uniformly to ϕ , ϕ' , and ψ by Lemma 3.4. Every term on the right-hand-side, then, of the relationship

$$\frac{\sigma^2}{2} \Phi(\epsilon, r)_1'' = r \Phi(\epsilon, r) - \mathcal{H}(\cdot, \Phi(\epsilon, r))$$

converges uniformly as $(\epsilon, r) \rightarrow (0, 0)$ and therefore $\Phi(\epsilon, r)_1''$ converges uniformly to its L^2 -limit ϕ'' , which is negative and bounded away from zero, so $\Phi(\epsilon, r)_1' < 0$ for (ϵ, r) small enough. The lemma follows. $\square$

Remark 4.12 Note that our choice of $\eta = 1$ was arbitrary. We could have fixed any $\eta \in [0, 1]$ and found a corresponding system of stationary solutions with η representing a proportion of firms active in the market. For instance, we could define

$$\tilde{Y} = \frac{-1}{2} + \int_0^L \{\beta_1 \tilde{V}_2 + \beta_2 \tilde{W}_2 + \Xi(\epsilon, r, \beta_1, \beta_2)_2\} dx,$$

choose $\psi'(0)$ so that $\eta = 1/2$, redefine $\tilde{T}$ using $\tilde{Y}$, and find that

$$D_\beta \tilde{T}(0, 0, \tilde{\beta}) = D_\beta T(0, 0, \tilde{\beta})$$

because they only differ by a constant. The proof of the theorem above yields a system of solutions for which $\eta = 1/2$.

Proposition 4.13 *We shall consider the demand function $q(t, x)$.*

- (1) *For (ϵ, r) sufficiently small, $q < 0$ along the solution $\Phi(\epsilon, r)$.*
- (2) *For any $(\epsilon, r) \in U$, q is increasing in x .*

Proof (1) If $(u, m) = \Phi(\epsilon, r)$ then $u' > 1$ due to Theorem 4.11 so if $\epsilon = 0$ then

$$q(t, x) = \frac{1}{2} \left(1 - u' \right) < 0$$

is bounded away from zero. In fact, by Remark 4.2, q is bounded away from zero at the base case $\epsilon = r = 0$. The Implicit Function Theorem yields that Φ is continuous with respect to the $H^2 \times H^2$ -norm, so by Sobolev embedding [2], u' and m vary continuously with respect to the uniform norm for $(\epsilon, r) \in U$. The mapping

$$(\epsilon, r, u, m) \in \Sigma \mapsto q_{\epsilon, r, u, m}(x) = \frac{1}{2} \left(\frac{2}{2 + \epsilon} + \frac{\epsilon}{2 + \epsilon} \int_0^L u'(s) m(s) \, ds - u'(x) \right)$$

is continuous, where Σ is the set of all tuples (ϵ, r, u, m) satisfying $M[\epsilon, r, u, m] = 0$, $u', m \geq 0$, $\eta = 1$, and $u'' < 0$.

□

5 Stationary Within a Moving Reference Frame

We now examine a family of solutions of the time-dependent Chan–Sircar system of the form $u(t, x) = v(x) - \alpha t$ and $m(t, x) = \mu(x)$ where $\alpha > 0$. This yields the system

$$\begin{cases} \frac{\sigma^2}{2} v'' - rv + \mathcal{H}_\epsilon(t, x, v', \mu) = \alpha(1 - rt) \\ \frac{\sigma^2}{2} \mu'' + \left(\mu G_\epsilon(t, x, v', \mu) \right)' = 0 \\ v(0) = \mu(0) = 0. \end{cases} \quad (6)$$

These solutions have stationary densities, but the utility functions are only stationary with respect to a reference frame moving with constant velocity. The differential of the utility function, however, is truly stationary. We prove that such solutions exist in our second main theorem, Theorem 5.8.

5.1 Zero Interest Rate

In what follows, we consider the zero-interest-rate case. This enables system (6) to support solutions of the form described, in which v and μ are independent of t . Taking $r = 0$ yields

$$\begin{cases} \frac{\sigma^2}{2} v'' + \mathcal{H}_\epsilon(t, x, v', \mu) = \alpha \\ \frac{\sigma^2}{2} \mu'' + \left(\mu G_\epsilon(t, x, v', \mu) \right)' = 0 \end{cases}.$$

As before, we seek base-case solutions (v_0, μ_0) to (6) corresponding to $\epsilon = r = 0$. In this case the HJB equation becomes

$$\frac{\sigma^2}{2} v_0'' + \frac{1}{4} (1 - v_0')^2 = \alpha.$$

Let $\theta = 1 - v'_0$. Now

$$\theta' = \frac{1}{2\sigma^2}(\theta^2 - 4\alpha)$$

and by separation of variables and integration we obtain

$$c - \frac{\sqrt{\alpha}}{\sigma^2}x = \tanh^{-1}\left(\frac{\theta}{2\sqrt{\alpha}}\right)$$

for some $c \in \mathbb{R}$. Here we make use of the fact that

$$\int \frac{d\theta}{\theta^2 - 4\alpha} = \frac{-1}{4\alpha} \int \frac{d\theta}{1 - \left(\frac{\theta}{2\sqrt{\alpha}}\right)^2}$$

and apply

$$\left(\tanh^{-1} z\right)' = \frac{1}{1 - z^2}$$

to obtain the conclusion above. Now

$$\frac{\theta}{2\sqrt{\alpha}} = \tanh\left(c - \frac{\sqrt{\alpha}}{\sigma^2}x\right)$$

and therefore

$$v'_0 = 1 + 2\sqrt{\alpha} \tanh\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right).$$

Integrating and applying $v_0(0) = 0$ we obtain

$$v_0 = x + 2\sigma^2 \ln \cosh\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right) - 2\sigma^2 \ln \cosh(c).$$

The second equation in (6) yields

$$\frac{\sigma^2}{2} \mu''_0 + \left(\frac{\mu_0}{2}(1 - v'_0)\right)' = 0$$

and therefore, by integrating,

$$\mu'_0 + \frac{1}{\sigma^2}(1 - v'_0)\mu_0 = \mu'_0(0).$$

Thus

$$\left(e^{(x-v_0)/\sigma^2} \mu_0\right)' = \mu'_0(0)e^{(x-v_0)/\sigma^2}$$

so

$$\mu_0 = \mu'_0(0)e^{(v_0-x)/\sigma^2} \int_0^x e^{(s-v_0)/\sigma^2} ds.$$

Observe that

$$x - v_0 = \sigma^2 \ln \cosh^2(c) - \sigma^2 \ln \operatorname{sech}^2\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right)$$

so

$$\begin{aligned}\mu_0 &= \mu'_0(0) \cosh^2\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right) \int_0^x \operatorname{sech}^2\left(\frac{\sqrt{\alpha}}{\sigma^2}s - c\right) ds \\ &= \frac{\sigma^2}{\sqrt{\alpha}} \mu'_0(0) \left\{ \sinh\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right) \cosh\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right) + \tanh(c) \cosh^2\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right) \right\} \\ &= \frac{\sigma^2}{2\sqrt{\alpha}} \mu'_0(0) \left\{ \sinh\left(\frac{2\sqrt{\alpha}}{\sigma^2}x - 2c\right) + 2 \tanh(c) \cosh^2\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right) \right\}.\end{aligned}$$

Note that no matter our choice of $c \in \mathbb{R}$,

$$\mu_0 = \mu'_0(0) \cosh^2\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right) \int_0^x \operatorname{sech}^2\left(\frac{\sqrt{\alpha}}{\sigma^2}s - c\right) ds \geq 0$$

as long as $\mu'_0(0) \geq 0$, and that for $\mu'_0(0) > 0$ we only have $\mu_0(x) = 0$ at $x = 0$. In order to satisfy the requirement that $\eta = \int_0^L \mu_0 ds = 1$, choose

$$\mu'_0(0) = \left(\int_0^L \mu_0 dx \right)^{-1}$$

yielding the following base-case solutions.

Definition 5.1 For $\alpha, \sigma > 0$ and $c \in \mathbb{R}$ let

$$v_0 = x + 2\sigma^2 \ln \cosh\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right) - 2\sigma^2 \ln \cosh(c)$$

and

$$\mu_0 = \frac{\sigma^2}{2\sqrt{\alpha}} \mu'_0(0) \left\{ \sinh\left(\frac{2\sqrt{\alpha}}{\sigma^2}x - 2c\right) + 2 \tanh(c) \cosh^2\left(\frac{\sqrt{\alpha}}{\sigma^2}x - c\right) \right\}.$$

We now define the mapping and solution set corresponding to (6) in the zero-interest-rate case.

Definition 5.2 Let $F : \mathbb{R} \times R \rightarrow S$ be given by

$$F[\epsilon, \Theta] = \left(\begin{array}{l} \frac{\sigma^2}{2} v'' + \mathcal{H}_\epsilon(t, x, v', \mu) - \alpha \\ \frac{\sigma^2}{2} \mu'' + \left(\mu G_\epsilon(t, x, v', \mu) \right)' \end{array} \right)'$$

where $\Theta = (v, \mu)$. We define a zero-interest-rate solution of (6) to be an ordered pair $(u, m) \in R$ such that $F[\epsilon, u, m] = 0$ for $r = 0$ and for some $\epsilon \geq 0$, $u', m \geq 0$ and $\eta = \int_0^L m ds = 1$. (In this case, we remove the expectation that u satisfy the law of diminishing marginal utility.) For any $k \in (1 - 2\sqrt{\alpha}, 1 + 2\sqrt{\alpha})$ such that $k \geq 0$, let $\Omega(k)$ denote the set of all such solutions satisfying $u'(L) = k$.

Remark 5.3 Observe that while $\mu_0 \geq 0$ always holds, all values of (α, c) do not allow $v'_0 \geq 0$. In particular,

$$\min \{v'_0(x) : x \in [0, L]\} = v'_0(0) = 1 - 2\sqrt{\alpha} \tanh(c).$$

If $v'_0 \geq 0$, then, we must have $2\sqrt{\alpha} \tanh(c) \leq 1$, so either $\alpha \leq 1/4$ and any value of c will do or $\alpha > 1/4$ and

$$c \leq \tanh^{-1}\left(\frac{1}{2\sqrt{\alpha}}\right).$$

In what follows, we will choose to take $0 < \alpha \leq 1/4$ so that our choice of c is unrestricted, and therefore we may choose to fix any $u'(L)$ in $(1 - 2\sqrt{\alpha}, 1 + 2\sqrt{\alpha})$. That is to say, $u_x(t, L) = v'(L) > 0$ will hold in all of the solutions considered in Theorem 5.8.

Observe that F differs from $M[\epsilon, 0, u, m]$ only in the addition of a constant, and therefore when we calculate the linearization of F at $(0, v_0, \mu_0)$, we obtain the same linearization that M yielded at the base point, with the substitution of (v_0, μ_0) for (ϕ, ψ) .

Definition 5.4 We define the following linearization of F at the base point:

$$A(V) = D_{\Theta} F[0, v_0, \mu_0](V) = \begin{pmatrix} \frac{\sigma^2}{2} v'' - \frac{1}{2}(1 - v'_0)v' \\ \frac{\sigma^2}{2} \mu'' + \frac{1}{2} \left((1 - v'_0)\mu - \mu_0 v' \right)' \end{pmatrix}$$

where $V = (v, \mu) \in R$.

Lemma 5.5 *The linearization A is bounded and surjective with a 2-dimensional kernel. It is then an index-2 Fredholm operator; that is, substituting (v_0, μ_0) for (ϕ, ψ) has not fundamentally changed the linearization.*

Proof We repeat the calculation from Lemma 4.4 to obtain that $\text{Ker}(A)$ has a basis consisting of the vectors

$$\bar{V} = \left(\int_0^x e^{(s-v_0)/\sigma^2} ds, \frac{2}{\sigma^2} e^{(v_0-x)/\sigma^2} \int_0^x e^{2(s-v_0)/\sigma^2} v_0 ds \right)$$

and

$$\bar{W} = (0, \mu_0).$$

Now

$$\begin{aligned} \|A(V)\|_S &\leq \frac{1}{2} \left(\|\mu'_0\|_{\infty} + \|v'_0 - 1\|_{\infty} \right) \|v'\|_{L^2} + \frac{1}{2} \left(\sigma^2 + \frac{1}{2} \|\mu_0\|_{\infty} \right) \|v''\|_{L^2} \\ &\quad + \frac{1}{2} \|v''_0\|_{\infty} \|\mu\|_{L^2} + \frac{1}{2} \|v_0 - 1\|_{\infty} \|\mu'\|_{L^2} + \frac{\sigma^2}{2} \|\mu''\|_{L^2} \end{aligned}$$

so A has an upper bound. We may also set $A(v, \mu) = (f, g)$ for any $(f, g) \in L^2 \times L^2$ and perform the same calculation employed in the proof of 4.4 to obtain

$$v = \int_0^x \left\{ v'(0) e^{(s-v_0)/\sigma^2} + \frac{2}{\sigma^2} e^{(s-v_0)/\sigma^2} \int_0^s e^{(v_0-z)/\sigma^2} f dz \right\} ds$$

and

$$\mu = e^{(x-v_0)/\sigma^2} \int_0^x \left\{ \mu'(0) + \frac{2}{\sigma^2} \int_0^s g dz + \frac{1}{\sigma^2} \mu_0 v' \right\} e^{(s-v_0)/\sigma^2} ds.$$

Observe that $(v, \mu) \in H^2([0, L])$ and $v(0) = \mu(0) = 0$, demonstrating that A is surjective onto S . $\square$

Definition 5.6 Let P_{α} and Q_{α} be projections from R onto $\text{Ker}(A)$ and $\text{Ker}(A)^{\perp}$, respectively.

We repeat the Lyapunov–Schmidt decomposition performed in the $\alpha = 0$ case to obtain the following.

Lemma 5.7 *There exists a function*

$$\tilde{\Xi} \in C^1\left([0, \infty) \times \mathbb{R}^2; Q_\alpha(R)\right)$$

such that for any $\epsilon \geq 0$ sufficiently small and any $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$\left\| \beta_1 \bar{V} + \beta_2 \bar{W} - P_\alpha\left((v_0, \mu_0)\right) \right\|_R$$

is sufficiently small,

$$F\left[\epsilon, \beta_1 \bar{V} + \beta_2 \bar{W} + \tilde{\Xi}\left(\epsilon, \beta_1, \beta_2\right)\right] = 0.$$

We employ the same construction, and therefore we may repeat the calculation performed previously to obtain that

$$\nabla_\beta \tilde{\Xi}(0, \bar{\beta}) = (0, 0)$$

where $P((v_0, \mu_0)) = \bar{\beta}_1 \bar{V} + \bar{\beta}_2 \bar{W}$.

We now are able to state our second main theorem.

Theorem 5.8 *Choose any $k \in (1 - 2\sqrt{\alpha}, 1 + 2\sqrt{\alpha})$. There exists $\epsilon' > 0$ and*

$$\Phi \in C^1\left([0, \epsilon'); R\right)$$

such that for any $\epsilon \in [0, \epsilon']$, $F[\epsilon, \Phi(\epsilon)] = 0$ and $(u, m) = \Phi(\epsilon)$ satisfies $u', m \geq 0$, $\eta = 1$, $u'(L) = k$, and

$$u, m \in C^\infty\left([0, L]\right).$$

Proof Allowing the repeated notation, let

$$Y(\epsilon, \beta_1, \beta_2) = -1 + \int_0^L \left\{ \beta_1 \bar{V}_2 + \beta_2 \bar{W}_2 + \tilde{\Xi}\left(\epsilon, \beta\right)_2 \right\} dx$$

and

$$T(\epsilon, \beta_1, \beta_2) = \left(\begin{array}{c} Y(\epsilon, \beta) \\ \left[\beta_1 \bar{V}_1 + \beta_2 \bar{W}_1 + \tilde{\Xi}\left(\epsilon, \beta\right) \right]'(L) \end{array} \right).$$

Now $T(0, \bar{\beta}) = 0$ and

$$D_\beta T(0, \bar{\beta}) = \left(\begin{array}{cc} \int_0^L \bar{V}_2 dx & 1 \\ \bar{V}_1'(L) & 0 \end{array} \right) = \left(\begin{array}{cc} \int_0^L \bar{V}_2 dx & 1 \\ e^{(x-v_0(x))/\sigma^2} & 0 \end{array} \right)$$

so $\text{Rank } D_\beta T(0, \bar{\beta}) = 2$. We apply the Implicit Function Theorem to obtain that for $\epsilon \geq 0$ sufficiently close to zero, $(u, m) \in R$ may be chosen in a continuously differentiable manner so that $\eta = 1$ and $u'(L) = k$. Regularity follows from Proposition 3.6 Our proof that $u', m \geq 0$ in a neighborhood of the base case may be repeated exactly from our proof of Theorem 4.11, with uniform convergence on $[0, L]$ yielded by H^2 -convergence on a compact interval. $\square$

6 Discussion

In addition to time-dependent and stationary solutions of mean field games, quasistationary solutions have also been studied [33]. The quasistationary analogue of (5) is

$$\begin{cases} \frac{\sigma^2}{2} u_{xx} - ru + \mathcal{H}(t, x, u_x, m) = 0 \\ m_t = \frac{\sigma^2}{2} m_{xx} + \left(m G(t, x, u_x, m) \right)_x \end{cases} \quad (7)$$

where $u, m : [0, T] \times [0, L] \rightarrow \mathbb{R}$ with $m(0, x) = m^0(x)$. Note that the system is now a time-dependent approximation of (1). In particular, it is a generalization of the stationary system in that while (7) supports non-stationary solutions, any stationary solution of the Chan–Sircar model is a solution of (7). In [33], Mouzouni examines an N -player competition with separable Hamiltonian and nonlocal coupling and shows that, under certain assumptions, the competition has a quasistationary mean field limit as $N \rightarrow \infty$ and that furthermore, given a quadratic Hamiltonian and a coupling that satisfies a monotonicity property, solutions of the quasistationary system converge exponentially fast to stationary solutions of the competition. We propose a similar study of (1) allowing $q < 0$, focusing on the convergence of solutions of (7) to solutions of (5).

The nonlocal dependence of $\mathcal{H}$ on m in the Chan–Sircar system invites the use of distribution-valued initial data $m(0, \cdot)$. In [21], Graber and Mouzouni establish existence of solutions to the Chan–Sircar model with an initial density in the tempered distributions. We propose a more general examination of Chan–Sircar-like mean field games, that is, of mean field games with nonseparable Hamiltonians depending nonlocally on m , that can be applied to other forms of competition arising in the literature. The well-posedness of time-dependent Chan–Sircar-like mean field games with density-valued initial data is highly relevant to developing rigorous asymptotics for N -player games as N goes to infinity.

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