



A Positive Fraction Erdős–Szekeres Theorem and Its Applications

Andrew Suk¹ · Ji Zeng¹

Received: 17 August 2022 / Revised: 26 March 2023 / Accepted: 1 April 2023 /

Published online: 7 July 2023

© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract

A famous theorem of Erdős and Szekeres states that any sequence of n distinct real numbers contains a monotone subsequence of length at least \sqrt{n} . Here, we prove a positive fraction version of this theorem. For $n > (k - 1)^2$, any sequence A of n distinct real numbers contains a collection of subsets $A_1, \dots, A_k \subset A$, appearing sequentially, all of size $s = \Omega(n/k^2)$, such that every subsequence (a_1, \dots, a_k) , with $a_i \in A_i$, is increasing, or every such subsequence is decreasing. The subsequence $S = (A_1, \dots, A_k)$ described above is called *block-monotone of depth k and block-size s* . Our theorem is asymptotically best possible and follows from a more general Ramsey-type result for monotone paths, which we find of independent interest. We also show that for any positive integer k , any finite sequence of distinct real numbers can be partitioned into $O(k^2 \log k)$ block-monotone subsequences of depth at least k , upon deleting at most $(k - 1)^2$ entries. We apply our results to mutually avoiding planar point sets and biarc diagrams in graph drawing.

Keywords Erdős–Szekeres theorem · Block-monotone sequences · Mutually avoiding sets · Biarc diagrams

Mathematics Subject Classification 05B25 · 05D99

Editor in Charge: Kenneth Clarkson

A.S.: supported by NSF CAREER award DMS-1800746, NSF award DMS-1952786, and an Alfred Sloan Fellowship. J.Z.: supported by NSF grant DMS-1800746.

Andrew Suk
asuk@ucsd.edu

Ji Zeng
jzeng@ucsd.edu

¹ Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA

1 Introduction

In 1935, Erdős and Szekeres [7] proved that any sequence of n distinct real numbers contains a monotone subsequence of length at least \sqrt{n} . This is a classical result in combinatorics and its generalizations and extensions have many important consequences in geometry, probability, and computer science. See Steele [15] for seven different proofs along with several applications.

In this paper, we prove a positive fraction version of the Erdős–Szekeres theorem. We state this theorem using the following notion: A sequence $(a_1, a_2, \dots, a_{ks})$ of ks distinct real numbers is said to be *block-increasing* (resp. *block-decreasing*) with *depth* k and *block-size* s if every subsequence $(a_{i_1}, a_{i_2}, \dots, a_{i_k})$, for $(j-1)s < i_j \leq js$, is increasing (resp. decreasing). We call a sequence *block-monotone* if it's either block-increasing or block-decreasing. In other words, such a sequence consists of k parts, each of size s and appearing sequentially, such that the parts (i.e., blocks) are in a monotone position.

Theorem 1.1 *Let k and $n > (k-1)^2$ be positive integers. Then every sequence of n distinct real numbers contains a block-monotone subsequence of depth k and block-size $s = \Omega(n/k^2)$. Furthermore, such a subsequence can be computed within $O(n^2 \log n)$ time.*

We prove Theorem 1.1 by establishing a more general Ramsey-type result for monotone paths, which we describe in detail in the next section. The theorem is also asymptotically best possible, see Remark 2.5.

By a repeated application of Theorem 1.1, we can decompose any sequence of n distinct real numbers into $O(k \log n)$ block-monotone subsequences of depth k upon deleting at most $(k-1)^2$ entries. Our next result shows that we can obtain such a partition, where the number of parts doesn't depend on n .

Theorem 1.2 *For any positive integer k , every finite sequence of distinct real numbers can be partitioned into at most $O(k^2 \log k)$ block-monotone subsequences of depth at least k upon deleting at most $(k-1)^2$ entries.*

Our proof of Theorem 1.2 is constructive and implies an algorithm for the claimed partition whose time complexity is polynomial in k and n , where n is the length of the given sequence. Theorem 1.2 is inspired by a similar problem of partitioning planar point sets into convex-positioned clusters studied by Pór and Valtr [14]. A positive fraction Erdős–Szekeres-type result for convex polygons is given previously by Bárány and Valtr [3].

We give two applications of Theorems 1.1 and 1.2.

Mutually avoiding sets. Let A and B be finite point sets of \mathbb{R}^2 in *general position*, that is, no three points are collinear. We say that A and B are *mutually avoiding* if no line generated by a pair of points in A intersects the convex hull of B , and vice versa. Aronov et al. [1] used the Erdős–Szekeres Theorem to show that every n -element planar point set P in general position contains subsets $A, B \subset P$, each of size $\Omega(\sqrt{n})$, s.t. A and B are mutually avoiding. Valtr [16] showed that this bound is asymptotically best possible by slightly perturbing the points in an $\sqrt{n} \times \sqrt{n}$ grid. Following the same ideas of Aronov et al., we can use Theorem 1.1 to obtain the following.

Theorem 1.3 *For every positive integer k there is a constant $\epsilon_k = \Omega(1/k^2)$ s.t. every sufficiently large point set P in the plane in general position contains $2k$ disjoint subsets $A_1, \dots, A_k, B_1, \dots, B_k$, each of size at least $\epsilon_k |P|$, s.t. every pair of sets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$, with $a_i \in A_i$ and $b_i \in B_i$, are mutually avoiding.*

This improves an earlier result of Mirzaei and the first author [11], who proved the theorem above with $\epsilon_k = \Omega(1/k^4)$. The result above is asymptotically best possible for both k and $|P|$: Consider a $k \times k$ grid G and replace each point with a cluster of $|P|/k^2$ points placed very close to each other so that the resulting point set P is in general position. If we can find subsets A_i 's and B_i 's as in Theorem 1.3, but each of size $\epsilon'_k |P|$ with $\epsilon'_k = \omega(1/k^2)$, then we can find mutually avoiding subsets in G of size $\omega(k)$, contradicting Valtr's result in [16].

Finally, let us remark that a recent result due to Pach et al. [13] shows that every n -element planar point set in general position determines at least $n/e^{O(\sqrt{\log n})}$ pairwise crossing segments. By using Theorem 1.3 instead of Lemma 3.3 from their paper, one can improve the constant hidden in the O -notation.

Monotone biarc diagrams. A *proper arc diagram* is a drawing of a graph in the plane, whose vertices are points placed on the x -axis, called the *spine*, and each edge is drawn as a half-circle. A classic result of Bernhard and Kainen [5] shows that a planar graph admits a *planar proper arc diagram* if and only if it's a subgraph of a planar Hamiltonian graph. A *monotone biarc diagram* is a drawing of a graph in the plane, whose vertices are placed on a spine, and each edge is drawn either as a half-circle or two half-circles centered on the spine, forming a continuous x -monotone biarc. See Fig. 6 for an illustration. In [6], Di Giacomo et al. showed that every planar graph can be drawn as a *planar monotone biarc diagram*.

Using the Erdős–Szekeres Theorem, Bar-Yehuda and Fogel [2] showed that every graph $G = (V, E)$, with a given order on V , has a *double-paged book embedding* with at most $O(\sqrt{E})$ pages. That is, E can be partitioned into $O(\sqrt{|E|})$ parts, s.t. for each part E_i , (V, E_i) can be drawn as a planar monotone biarc diagram, and V appears on the spine with the given order. Our next result shows that we can significantly reduce the number of pages (parts), if we allow a small fraction of the pairs of edges to cross on each page.

Theorem 1.4 *For any $\epsilon > 0$ and a graph $G = (V, E)$, where V is an ordered set, E can be partitioned into $O(\epsilon^{-2} \log \epsilon^{-1} \log |E|)$ subsets E_i s.t. each (V, E_i) can be drawn as a monotone biarc diagram having no more than $\epsilon |E_i|^2$ crossing edge-pairs, and V appears on the spine with the given order.*

This paper is organized as follows: In Sect. 2, we prove Theorem 1.1 in the setting of monotone paths in multicolored ordered graphs. Section 3 is devoted to the proof of Theorem 1.2. In Sect. 4, we present proofs for the applications claimed above. Section 5 lists some remarks.

2 A Positive Fraction Result for Monotone Paths

Several authors [8, 10, 12] observed that the Erdős–Szekeres theorem generalizes to the following graph-theoretic setting. Let G be a graph with vertex set $[n] = \{1, \dots, n\}$. A *monotone path of length k* in G is a k -tuple (v_1, \dots, v_k) of vertices s.t. $v_i < v_j$ for all $i < j$ and all edges $v_i v_{i+1}$, for $i \in [k-1]$, are in G .

Theorem 2.1 *Let χ be a q -coloring of the pairs of $[n]$. Then there must be a monochromatic monotone path of length at least $n^{1/q}$.*

Given subsets $A, B \subset [n]$, we write $A < B$ if every element in A is less than every element in B .

Definition 2.2 Let G be a graph with vertex set $[n]$ and let $V_1, \dots, V_k \subset [n]$ and $p_1, \dots, p_{k+1} \in [n]$. Then we say that $(p_1, V_1, p_2, V_2, p_3, \dots, p_k, V_k, p_{k+1})$ is a *block-monotone path of depth k and block-size s* if

- $|V_i| = s$ for all i ,
- we have $p_1 < V_1 < p_2 < V_2 < p_3 < \dots < p_k < V_k < p_{k+1}$,
- and every $(2k+1)$ -tuple of the form

$$(p_1, v_1, p_2, v_2, \dots, p_k, v_k, p_{k+1}),$$

where $v_i \in V_i$, is a monotone path in G .

Our main result in this section is the following Ramsey-type theorem.

Theorem 2.3 *There is an absolute constant $c > 0$ s.t. the following holds. Given integers $q \geq 2$, $k \geq 1$, and $n \geq (ck)^q$, let χ be a q -coloring of the pairs of $[n]$. Then χ produces a monochromatic block-monotone path of depth k and block-size $s \geq n/(ck)^q$.*

A careful calculation shows that we can take $c = 40$ in the theorem above. We will need the following lemma.

Lemma 2.4 *Let $q \geq 2$ and $N > 3^q$. Then for any q -coloring of the pairs of $[N]$, there is a monochromatic block-monotone path of depth 1 and block-size $s \geq N/(q3^{3q})$.*

Proof Let χ be a q -coloring of the pairs of $[N]$, and set $r = 3^q$. By Theorem 2.1, every subset of size r of $[N]$ gives rise to a monochromatic monotone path of length 3. Hence, χ produces at least

$$\frac{\binom{N}{r}}{\binom{N-3}{r-3}} \geq \frac{6}{r^3} \binom{N}{3}$$

monochromatic monotone paths of length 3 in $[N]$. Hence, there are at least $6\binom{N}{3}/(qr^3)$ monochromatic monotone paths of length 3, all of which have the same color. By

averaging, there are two vertices $p_1, p_2 \in [N]$, s.t. at least $N/(qr^3)$ of these monochromatic monotone paths of length 3 start at vertex p_1 and ends at vertex p_2 . By setting V_1 to be the “middle” vertices of these paths, (p_1, V_1, p_2) is a monochromatic block-monotone path of depth 1 and block-size $s \geq N/(qr^3) = N/(q3^{3q})$. \square

Proof of Theorem 2.3 Let χ be a q -coloring of the pairs of $[n]$ and let c be a sufficiently large constant that will be determined later. Set $s = \lceil n/(ck)^q \rceil$. For the sake of contradiction, suppose χ does not produce a monochromatic block-monotone path of depth k and block-size s . For each element $v \in [n]$, we label v with $f(v) = (b_1, \dots, b_q)$, where b_i denotes the depth of the longest block-monotone path with block-size s in color i , ending at v . By our assumption, we have $0 \leq b_i \leq k-1$, which implies that there are at most k^q distinct labels. By the pigeonhole principle, there is a subset $V \subset [n]$ of size at least n/k^q , s.t. the elements of V all have the same label.

By Lemma 2.4, there are vertices $p_1, p_2 \in V$, a subset $V' \subset V$, and a color α s.t. (p_1, V', p_2) is a monochromatic block-monotone path in color α , with block-size $t \geq |V|/(q3^{3q})$. By setting c to be sufficiently large, we have

$$t \geq \frac{|V|}{q3^{3q}} \geq \frac{n}{k^q q 3^{3q}} \geq \left\lceil \frac{n}{(ck)^q} \right\rceil = s.$$

However, this contradicts the fact that $f(p_1) = f(p_2)$, since the longest supported monotone path with block-size s in color α ending at vertex p_1 can be extended to a longer one ending at p_2 . This completes the proof. \square

Proof of Theorem 1.1 Let $A = (a_1, \dots, a_n)$ be a sequence of distinct real numbers. Let χ be a red/blue coloring of the pairs of A s.t. for $i < j$, we have $\chi(a_i, a_j) = \text{red}$ if $a_i < a_j$ and $\chi(a_i, a_j) = \text{blue}$ if $a_i > a_j$. In other words, we color the increasing pairs by red and the decreasing pairs by blue.

If $n < (ck)^2$, notice that $n/(ck)^2 < 1$. By our assumption $n > (k-1)^2$, the classical Erdős–Szekeres theorem gives us a monotone subsequence in A of length at least k , which can be regarded as a block-monotone subsequence of depth at least k and block-size $s = 1 > n/(ck)^2$.

If $n \geq (ck)^2$, by Theorem 2.3, there is a monochromatic block-monotone path of depth k and block-size $s \geq n/(ck)^2$ in the complete graph on A , which can be regarded as a block-monotone subsequence of A with the claimed depth and block-size.

Now we focus on computing such a block-monotone subsequence. If $n < (ck)^2$, it suffices to compute the longest monotone subsequence of A . It's well known that the longest increasing subsequence can be computed within $O(n \log n)$ time, see [9], so we are done with this case.

If $n \geq (ck)^2$, we set $s = \lceil n/(ck)^2 \rceil$. We call a pair (a_i, a_j) s -gapped if there exist s other entries a_x with $i < x < j$ satisfying $a_i < a_x < a_j$ or $a_i > a_x > a_j$. We describe an $O(n^2 \log n)$ -time algorithm that computes the longest increasing subsequence with consecutive entries s -gapped.

Firstly, we preprocess A into a data structure s.t. we can answer within $O(\log n)$ time whether any given pair (a_i, a_j) is s -gapped or not. The classical data structure

for 2-D orthogonal range counting works for our purpose and its preprocessing time is $O(n \log n)$, see [4, Exer. 5.10].

Next, let $l(i)$ be the length of the longest increasing subsequence of A with consecutive entries s -gapped ending at a_i . We compute each $l(i)$ as i proceeds from 1 to n as follows: After $l(1), \dots, l(i-1)$ are all determined, we have

$$l(i) = \max_{j < i} \{l(j) : (a_j, a_i) \text{ is } s\text{-gapped}\} + 1.$$

Here, we consider $\max(\emptyset) := 0$. Hence we can compute $l(i)$ by checking which pairs in $\{(a_j, a_i) : j < i\}$ are s -gapped using our preprocessed data structure. Clearly, this computation of all $l(i)$ takes $O(n^2 \log n)$ time.

While computing $l(i)$, let the algorithm record $p(i)$, which is the index $j < i$ with the largest $l(j)$ s.t. (a_j, a_i) is s -gapped. This recording process won't increase the magnitude of time complexity. After all $l(i)$ and $p(i)$ are determined, we find the index i_1 with the largest $l(i) =: L$, and inductively set $i_{j+1} = p(i_j)$ for $j \in [L-1]$. Then $a_{i_L}, a_{i_{L-1}}, \dots, a_{i_1}$ is the longest increasing subsequence of A with consecutive entries s -gapped.

Let's return to computing the block-monotone subsequence. By the previous argument on block-monotone paths, there exists a monotone subsequence $S \subset A$ with consecutive entries s -gapped whose length is at least $k+1$. We can use the algorithm above to compute S within $O(n^2 \log n)$ time. Clearly, the entries of A "gapped" by consecutive entries of S form a block-monotone subsequence as claimed, and they can be found within $O(n)$ time. Hence we conclude the theorem. \square

Remark 2.5 For each $k, q, s > 0$, the simple construction below shows Theorem 2.3 is tight up to the constant factor c^q . We first construct $K(k, q)$, for each k and q , a q -colored complete graph on $[k^q]$, whose longest monochromatic monotone path has length k : $K(k, 1)$ is just a monochromatic copy of the complete graph on $[k]$. To construct $K(k, q)$ from $K(k, q-1)$, take k copies of $K(k, q-1)$ with the same set of $q-1$ colors, place them in order and color the remaining edges by a new color. Now replace each point in $K(k, q)$ by a cluster of s points, where within each cluster one can arbitrarily color the edges. The resulting q -colored complete graph has no k subsets $V_1, V_2, \dots, V_k \subset [n]$ each of size $s+1$ and edges between them monochromatic, otherwise $K(k, q)$ would have a monochromatic monotone path with length larger than k . One example of the sharpness of the classical Erdős–Szekeres theorem is the sequence

$$S(k) = (k, k-1, \dots, 1, 2k, 2k-1, \dots, k+1, \dots, k^2, k^2-1, k(k-1)+1).$$

Notice that if we color the increasing pairs of $S(k)$ by red and the decreasing pairs of $S(k)$ by blue, we obtain the graph $K(k, 2)$. If we replace each entry $s_i \in S(k)$ by a cluster of s distinct real numbers very close to s_i , we obtain an example showing that Theorem 1.1 is asymptotically best possible.

3 Block-Monotone Sequence Partition

This section is devoted to the proof of Theorem 1.2. We shall consider this problem geometrically by identifying each entry a_i of a given sequence $A = (a_i)_{i=1}^n$ as a planar point $(i, a_i) \in \mathbb{R}^2$. As we consider sequences of distinct real numbers, throughout this section, we assume that all point sets have the property that no two members share the same x -coordinate or the same y -coordinate.

Thus, we analogously define block-monotone point sets as follows: A set of ks planar points is said to be *block-increasing* (resp. *block-decreasing*) with *depth* k and *block-size* s if it can be written as $\{(x_i, y_i)\}_{i=1}^{ks}$ s.t. $x_i < x_{i+1}$ for all i and every sequence $(y_{i_1}, y_{i_2}, \dots, y_{i_k})$, for $(j-1)s < i_j \leq js$, is increasing (resp. decreasing). We say that a point set is *block-monotone* if it's either block-increasing or block-decreasing. For each $j \in [k]$ we call the subset $\{(x_i, y_i)\}_{i=(j-1)s+1}^{js}$ the *j -th block* of this block-monotone point set. Hence, Theorem 1.2 immediately follows from the following.

Theorem 3.1 *For any positive integer k , every finite planar point set can be partitioned into at most $O(k^2 \log k)$ block-monotone point subsets of depth at least k and a remaining set of size at most $(k-1)^2$.*

Given a point set $P \subset \mathbb{R}^2$, let

$$U(P) := \{(x, y) \in \mathbb{R}^2 : y > y', \forall (x', y') \in P\}, \quad (\text{up})$$

$$D(P) := \{(x, y) \in \mathbb{R}^2 : y < y', \forall (x', y') \in P\}, \quad (\text{down})$$

$$L(P) := \{(x, y) \in \mathbb{R}^2 : x < x', \forall (x', y') \in P\}, \quad (\text{left})$$

$$R(P) := \{(x, y) \in \mathbb{R}^2 : x > x', \forall (x', y') \in P\}. \quad (\text{right})$$

Our proof of Theorem 3.1 relies on the following definitions. The constant c below (and throughout this section) is from Theorem 2.3. See Fig. 1 for an illustration.

Definition 3.2 A point set P is said to be a (k, t) -configuration if P can be written as a disjoint union of subsets $P = Y_1 \cup Y_2 \cup \dots \cup Y_{2t+1}$ s.t.

- for every $i \in [t]$, Y_{2i} is a block-monotone point set of depth k and block-size at least $|Y_{2j+1}|/(3ck)^2$ for all $j \in \{0\} \cup [t]$;
- either $\bigcup_{j=i+1}^{2t+1} Y_j$ is located entirely in $R(Y_i) \cap U(Y_i)$ for all $i \in [2t]$, or $\bigcup_{j=i+1}^{2t+1} Y_j$ is located entirely in $R(Y_i) \cap D(Y_i)$ for all $i \in [2t]$.

Definition 3.3 A point set P is said to be a (k, l, t) -pattern if P can be written as a disjoint union of subsets $P = S_1 \cup S_2 \cup \dots \cup S_l \cup Y$ s.t.

- Y is a (k, t) -configuration;
- for every $i \in [l]$, S_i is a block-monotone point set of depth k and block-size at least $|Y|/(3ck)^2$;
- for every $i \in [l]$, the set $Y \cup \bigcup_{j=i+1}^l S_j$ is located entirely in one of the following regions: $U(S_i) \cap L(S_i)$, $U(S_i) \cap R(S_i)$, $D(S_i) \cap L(S_i)$, and $D(S_i) \cap R(S_i)$.

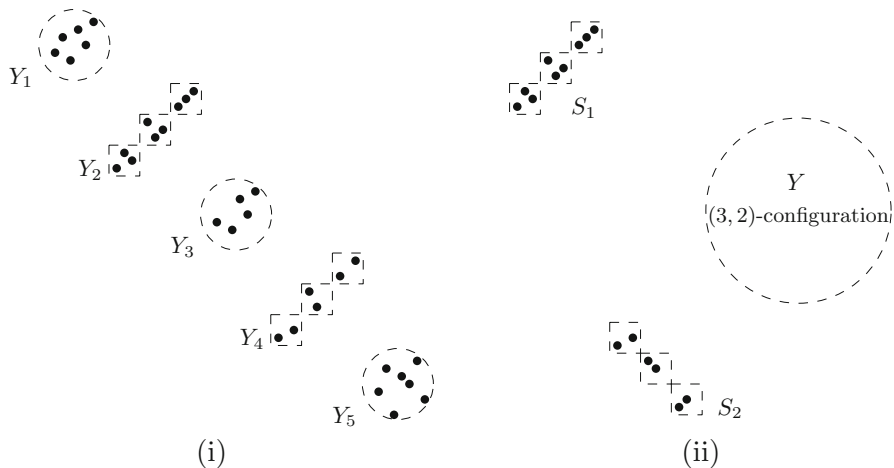


Fig. 1 (i) a $(3, 2)$ -configuration. (ii) a $(3, 2, 2)$ -pattern

If a planar point set P is a $(k, 4k, t)$ -pattern or a (k, l, k) -pattern, the next two lemmas state that we can efficiently partition P into few block-monotone point sets and a small remaining set.

Lemma 3.4 *If P is a $(k, 4k, t)$ -pattern, then P can be partitioned into $O(k \log k)$ block-monotone point sets of depth at least k and a remaining set of size $O(k^2)$.*

Lemma 3.5 *If P is a (k, l, k) -pattern, then P can be partitioned into $O(k^2 \log k + l)$ block-monotone point sets of depth at least k and a remaining set of size $O(k^3)$.*

Starting with an arbitrary point set P , which can be regarded as a $(k, 0, 0)$ -pattern, we will repeatedly apply the following lemma until P is partitioned into few block-monotone point sets, a set P' that is either a $(k, 4k, t)$ -pattern or a (k, l, k) -pattern, and a small remaining set.

Lemma 3.6 *For $l < 4k$ and $t < k$, a (k, l, t) -pattern P can be partitioned into $O(k \log k)$ block-monotone point sets of depth at least k , a point set P' , and a remaining set of size $O(k^2)$, s.t. either (i) $|P'| \leq k(3k - 1)^2$; or (ii) P' is a $(k, l, t + 1)$ -pattern; or (iii) P' is a $(k, l + t, 0)$ -pattern. Moreover, P' always satisfies either (i) or (ii) when $t = 0$.*

Before we prove the lemmas above, let us use them to prove Theorem 3.1.

Proof of Theorem 3.1 Let P be the given point set. For $i \geq 0$, we inductively construct a partition $\mathcal{F}_i \cup \{P_i, E_i\}$ of P s.t.

- P_i is a (k, l_i, t_i) -pattern or $|P_i| \leq k(3k - 1)^2$,
- $|E_i| = O(k^2)$,
- \mathcal{F}_i is a disjoint family of block-monotone point sets of depth at least k , and $|\mathcal{F}_i| = O(k \log k)$.

We start with $P_0 = P$, which is a $(k, 0, 0)$ -pattern, and $\mathcal{F}_0 = E_0 = \emptyset$. Suppose we have constructed the i -th partition $\mathcal{F}_i \cup \{P_i, E_i\}$ of P . If $|P_i| \leq k(3k-1)^2$, or $l_i \geq 4k$, or $t_i \geq k$, we end this inductive construction process, otherwise, we construct the next partition $\mathcal{F}_{i+1} \cup \{P_{i+1}, E_{i+1}\}$ as follows: According to Lemma 3.6, P_i can be partitioned into $r_i = O(k \log k)$ block-monotone point sets with depth at least k , denoted as $\{A_{i,1}, \dots, A_{i,r_i}\}$, a point set P'_i , and a remaining set E'_i of size $O(k^2)$. We define $\mathcal{F}_{i+1} := \mathcal{F}_i \cup \{A_{i,1}, \dots, A_{i,r_i}\}$, $P_{i+1} := P'_i$, and $E_{i+1} := E_i \cup E'_i$. Clearly, we have $|\mathcal{F}_{i+1}| = |\mathcal{F}_i| + r_i = O((i+1)k \log k)$ and $|E_{i+1}| = |E_i| + |E'_i| = O((i+1)k^2)$ as claimed. By Lemma 3.6, we have either (i) $|P_{i+1}| \leq k(3k-1)^2$, so the construction ends; or (ii) P_{i+1} is a $(k, l_i, t_i + 1)$ -pattern, so $l_{i+1} = l_i$ and $t_{i+1} = t_i + 1$; or (iii) P_{i+1} is a $(k, l_i + t_i, 0)$ -pattern, so $l_{i+1} = l_i + t_i$ and $t_{i+1} = 0$. Moreover, when $t_i = 0$, Lemma 3.6 guarantees that P_{i+1} always satisfies (i) or (ii).

Let $\mathcal{F}_w \cup \{P_w, E_w\}$ be the last partition of P constructed in this process. Here, P_w is a (k, l_w, t_w) -pattern. We must have either $|P_w| \leq k(3k-1)^2$, or $l_w \geq 4k$, or $t_w \geq k$. Since $t_{i+1} \leq t_i + 1$ and $l_{i+1} \leq l_i + t_i$ for all i , we have $t_w \leq k$ and $l_w \leq 5k$. Since we always construct the $(i+1)$ -th partition with P_{i+1} satisfying either (i) or (ii) when $t_i = 0$, the sum $l_i + t_i$ always increases by at least 1 after two inductive steps. So we have $w/2 \leq t_w + l_w \leq 6k$ and hence $w \leq 12k$.

Now we handle $\mathcal{F}_w \cup \{P_w, E_w\}$ based on how the construction process ends. If the construction process ended with $|P_w| \leq k(3k-1)^2$, we define $E_{w+1} = E_w \cup P_w$ and $\mathcal{F}_{w+1} = \mathcal{F}_w$. Since $w \leq 12k$, we have $|\mathcal{F}_{w+1}| = O(k^2 \log k)$ and $|E_{w+1}| = O(k^3)$.

If the construction process ended with $l_w \geq 4k$, by Definition 3.3, we can partition P_w into $l_w - 4k$ many block-monotone point sets of depth k , denoted as $\{S_1, \dots, S_{l_w-4k}\}$, and a $(k, 4k, t_w)$ -pattern P'_w . Then, by Lemma 3.4, P'_w can be partitioned into $r_w = O(k \log k)$ block-monotone point sets of depth at least k , denoted as $\{A_{w,1}, \dots, A_{w,r_w}\}$, and a remaining set E'_w of size $O(k^2)$. We define $E_{w+1} = E_w \cup E'_w$ and

$$\mathcal{F}_{w+1} = \mathcal{F}_w \cup \{S_1, \dots, S_{l_w-4k}, A_{w,1}, \dots, A_{w,r_w}\}.$$

Using $w \leq 12k$, $l_w \leq 5k$, and other bounds we mentioned above, we can check that $|\mathcal{F}_{w+1}| = O(k^2 \log k)$ and $|E_{w+1}| = O(k^3)$.

If the construction process ended with $t_w \geq k$, we actually have $t_w = k$ and $l_w < 4k$. By Lemma 3.5, we can partition P_w into $r_w = O(k^2 \log k + l_w)$ block-monotone point sets of depth at least k , denoted as $\{A_{w,1}, \dots, A_{w,r_w}\}$, and a remaining set E'_w of size $O(k^3)$. We define $E_{w+1} = E_w \cup E'_w$ and $\mathcal{F}_{w+1} = \mathcal{F}_w \cup \{A_{w,1}, \dots, A_{w,r_w}\}$. Again, we have $|\mathcal{F}_{w+1}| = O(k^2 \log k)$ and $|E_{w+1}| = O(k^3)$.

Overall, we can always obtain a partition $\mathcal{F}_{w+1} \cup \{E_{w+1}\}$ of P with $|\mathcal{F}_{w+1}| = O(k^2 \log k)$ and $|E_{w+1}| = O(k^3)$. Using the classical Erdős–Szekeres theorem, we can always find a monotone sequence of length at least k in E_{w+1} when $|E_{w+1}| > (k-1)^2$. By a repeated application of this fact, we can partition E_{w+1} into $O(k^2)$ block-monotone point sets of depth k and block-size 1, and a remaining set E of size at most $(k-1)^2$. We define \mathcal{F} to be the union of \mathcal{F}_{w+1} and these block-monotone sequences. The partition $\mathcal{F} \cup \{E\}$ of P has the desired properties, completing our proof. \square

We now give proofs for Lemmas 3.4, 3.5, and 3.6. We need the following two facts.

Fact 3.7 For any positive integer k , every point set P can be partitioned into $O(k \log k)$ block-monotone point sets of depth k and a remaining set P' with $|P'| \leq \max\{|P|/k, (k-1)^2\}$.

This fact can be established by repeatedly using Theorem 1.1 to pull out block-monotone point sets and applying the elementary inequality $(1 - x^{-1})^{x \log x} \leq x^{-1}$ for any $x > 1$.

Fact 3.8 For any positive integer k and m , every block-monotone point set P with depth k and $|P| \geq m$ can be partitioned into a block-monotone point set of depth k , a subset of size exactly m , and a remaining set of size less than k .

This fact can be established by taking out $\lceil m/k \rceil$ points from each block of P . Then we have taken out $k \cdot \lceil m/k \rceil = m + r$ points, where $0 \leq r < k$.

Proof of Lemma 3.4 Write the given $(k, 4k, t)$ -pattern $P = S_1 \cup \dots \cup S_{4k} \cup Y$ as in Definition 3.3. By definition, each block-monotone point set S_i is contained in one of the four regions: $U(Y) \cap L(Y)$, $U(Y) \cap R(Y)$, $D(Y) \cap L(Y)$, and $D(Y) \cap R(Y)$. By the pigeonhole principle, there are k point sets among S_1, \dots, S_{4k} all contained in one of the regions above. Without loss of generality, we assume there are k among them all contained in $U(Y) \cap L(Y)$. In fact, we can further assume that these point sets are S_1, \dots, S_k as the proof also works for other cases.

We have $S_{i'} \subset D(S_i) \cap R(S_i)$ for all $1 \leq i < i' \leq k$. Indeed, since $Y \subset D(S_i) \cap R(S_i)$, Definition 3.3 guarantees that $Y \cup \bigcup_{j=i+1}^k S_j$ is contained in $D(S_i) \cap R(S_i)$ and, in particular, $S_{i'}$ is contained in this region. See Fig. 2 for an illustration.

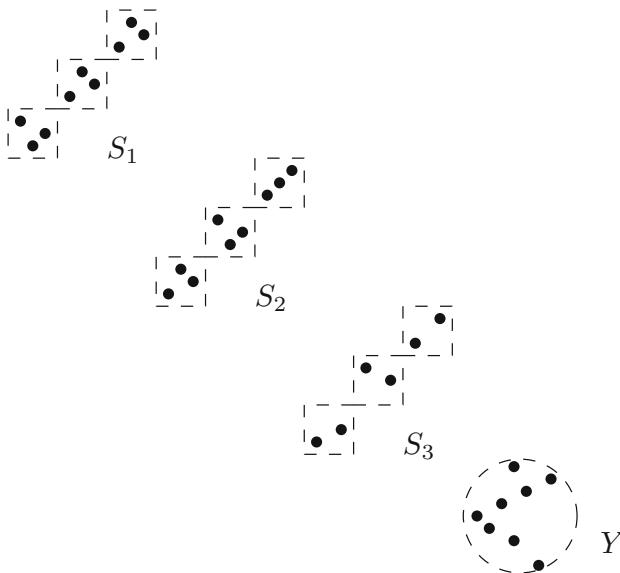


Fig. 2 In proof of Lemma 3.4, $S_{i'} \subset D(S_i) \cap R(S_i)$ for $i < i'$

Now by Fact 3.7, we can partition Y into $\{A_1, \dots, A_r, Y'\}$, where $r = O(k \log k)$, s.t. each A_i is block-monotone of depth $9c^2k$, and either $|Y'| \leq |Y|/(9c^2k)$ or $|Y'| \leq (9c^2k - 1)^2$. If $|Y'| \leq (9c^2k - 1)^2$, we have partitioned P into $O(k \log k)$ block-monotone point sets of depth at least k , which are $\{A_1, \dots, A_r, S_1, \dots, S_{4k}\}$, and a remaining set Y' of size $O(k^2)$, as wanted.

If $|Y'| \leq |Y|/(9c^2k)$, by Definition 3.3 we have $|Y'| \leq |S_i|$ for $i \in [k]$. We can apply Fact 3.8 with $m := |Y'|$ to obtain a partition $S_i = S'_i \cup B_i \cup E_i$ where S'_i is block-monotone of depth k , $|B_i| = |Y'|$, and $|E_i| \leq k$. Observe that $X := B_1 \cup B_2 \cup \dots \cup B_k \cup Y'$ is block-monotone of depth $k + 1$ by its construction. Then we have partitioned P into $O(k \log k)$ many block-monotone point sets, which are $\{A_1, \dots, A_r, S'_1, \dots, S'_k, S_{k+1}, \dots, S_{4k}, X\}$, and a remaining set $E := \bigcup_{i=1}^k E_i$ of size $O(k^2)$, as wanted. \square

Proof of Lemma 3.5 Write the given (k, l, k) -pattern $P = S_1 \cup \dots \cup S_l \cup Y$ as in Definition 3.3 and the (k, k) -configuration $Y = Y_1 \cup \dots \cup Y_{2k+1}$ as in Definition 3.2. Since each S_i is block-monotone of depth k , it suffices to partition Y into $O(k^2 \log k)$ many block-monotone point sets of depth at least k and a remaining set of size $O(k^3)$.

For each $j \in \{0\} \cup [k]$, we apply Fact 3.7 to obtain a partition of Y_{2j+1} into $O(k \log k)$ many block-monotone point sets of depth $9c^2k$ and a remaining set Y'_{2j+1} of size either at most $|Y_{2j+1}|/(9c^2k)$ or at most $(9c^2k - 1)^2$. We can apply Fact 3.7 again to partition Y'_{2j+1} into $O(k \log k)$ many block-monotone point sets of depth $k + 1$ and a remaining set Y''_{2j+1} with

$$|Y''_{2j+1}| \leq \max \left\{ \frac{|Y_{2j+1}|}{9c^2k(k+1)}, (9c^2k - 1)^2 \right\}. \quad (3.1)$$

Denote the block-monotone point sets produced in this process as $\{A_{j,1}, \dots, A_{j,r_j}\}$ where $r_j = O(k \log k)$.

Next we denote $J_1 := \{j \in \{0\} \cup [k] : |Y''_{2j+1}| > (9c^2k - 1)^2\}$ and $J_2 := (\{0\} \cup [k]) \setminus J_1$. For each $j \in J_1$ and $i \in [k]$, we must have

$$|Y''_{2j+1}| \leq \frac{|Y_{2j+1}|}{9c^2k(k+1)} \leq \frac{|Y_{2i}|}{k+1},$$

where the second inequality is by Definition 3.2. Hence $|Y_{2i}| \geq |\bigcup_{j \in J_1} Y''_{2j+1}|$. We can apply Fact 3.8 with $m := |\bigcup_{j \in J_1} Y''_{2j+1}|$ to obtain a partition $Y_{2i} = Y'_{2i} \cup B_i \cup E_i$ where Y'_{2i} is block-monotone of depth k , $|B_i| = m$, and $|E_i| \leq k$. Since $|B_i| = |\bigcup_{j \in J_1} Y''_{2j+1}|$, we can take a further partition $B_i = \bigcup_{j \in J_1} B_{j,i}$ with $|B_{j,i}| = |Y''_{2j+1}|$ for each $j \in J_1$. Then we observe that

$$X_j := B_{j,1} \cup \dots \cup B_{j,j} \cup Y''_{2j+1} \cup B_{j,j+1} \cup \dots \cup B_{j,k}$$

is block-monotone of depth $k + 1$ for each $j \in J_1$ by its construction.

Finally, let $E := \bigcup_{i=1}^k E_i \cup \bigcup_{j \in J_2} Y''_{2j+1}$, it easy to check that $|E| = O(k^3)$. So we have partitioned Y into $O(k^2 \log k)$ many block-monotone point sets, which

are $\bigcup_{j=0}^k \{A_{j,1}, \dots, A_{j,r_j}\} \cup \{X_j\}_{j \in J_1} \cup \{Y'_2, Y'_4, \dots, Y'_{2k}\}$, and a remaining set E of size $O(k^3)$, as wanted. \square

Proof of Lemma 3.6 Write the given (k, l, t) -pattern $P = S_1 \cup \dots \cup S_l \cup Y$ as in Definition 3.3 and its constituent (k, t) -configuration $Y = Y_1 \cup \dots \cup Y_{2t+1}$ as in Definition 3.2. Denote a set with largest size among $\{Y_{2j+1} : j \in \{0\} \cup [t]\}$ as Y_{i_0} . By Definition 3.2, we can assume without loss of generality that

$$\bigcup_{j=i+1}^{2t+1} Y_j \subset R(Y_i) \cap U(Y_i) \quad \text{for all } i \in [2t]. \quad (3.2)$$

If $|Y_{i_0}| \leq (3k-1)^2$, we can partition P into $l+t = O(k)$ many block-monotone point sets of depth k , which are $\{S_1, \dots, S_l, Y_2, Y_4, \dots, Y_{2t}\}$, and a set $P' := \bigcup_{j=0}^l Y_{2j+1}$ of size at most $k(3k-1)^2$ since $t < k$. So we conclude the lemma with P' as described in (i).

Now we assume $|Y_{i_0}| > (3k-1)^2$. Apply Theorem 1.1 to extract a block-monotone point set $X \subset Y_{i_0}$ of depth $3k$ and block-size at least $|Y_{i_0}|/(3ck)^2$ and name the i -th block of X as B_i for $i \in [3k]$. Notice that X splits into three parts

$$X_1 := B_1 \cup \dots \cup B_k, \quad X_2 := B_{k+1} \cup \dots \cup B_{2k}, \quad X_3 := B_{2k+1} \cup \dots \cup B_{3k}.$$

Our proof splits into two cases: X being block-increasing or X being block-decreasing.

Case 1. Suppose X is block-increasing, we define

$$P' = (S_1 \cup \dots \cup S_l) \cup (Y_2 \cup Y_4 \cup \dots \cup Y_{i_0-1}) \cup (Y_{2t} \cup Y_{2t-2} \cup \dots \cup Y_{i_0+1}) \\ \cup (Y_{i_0} \setminus X).$$

By Definition 3.3, we can check that P' is a $(k, k+l, 0)$ -pattern with $Y_{i_0} \setminus X$ being its constituent $(k, 0)$ -configuration. Let $Z_1 := Y_1 \cup Y_3 \cup \dots \cup Y_{i_0-2}$ and $Z_3 := Y_{i_0+2} \cup Y_{i_0+4} \cup \dots \cup Y_{2t+1}$. We claim that $X_i \cup Z_i$ can be partitioned into $O(k \log k)$ block-monotone point sets of depth at least k and a remaining set of size $O(k^2)$ for $i = 1, 3$. Given this claim and the fact that $P = P' \cup X \cup Z_1 \cup Z_3$, we conclude the lemma with P' as described in (iii).

Now we justify this claim for $X_1 \cup Z_1$ and the justification for $X_3 \cup Z_3$ is similar. By an argument similar to (3.1), we can apply Fact 3.7 three times to partition Z_1 into $\{A_1, \dots, A_r, Z'_1\}$, where $r = O(k \log k)$, s.t. each A_i is block-monotone of depth at least k and $|Z'_1| \leq \max\{|Z_1|/(9c^2k^3), (9c^2k-1)^2\}$. If $|Z'_1| \leq (9c^2k-1)^2$, we have partitioned $X_1 \cup Z_1$ into $O(k \log k)$ block-monotone point sets of depth at least k , which are $\{A_1, \dots, A_r, X_1\}$, and a remaining set Z'_1 of size $O(k^2)$ as claimed.

If $|Z'_1| \leq |Z_1|/(9c^2k^3)$, noticing that $|Z_1| \leq t|Y_{i_0}| \leq k|Y_{i_0}|$, we have $|Z'_1| \leq |Y_{i_0}|/(3ck)^2 \leq |B_i|$ for all $i \in [k]$. We can take a partition $B_i = B'_i \cup B''_i$ with $|B'_i| = |Z'_1|$. We observe that $X'_1 := Z'_1 \cup B'_1 \cup \dots \cup B'_k$ is block-increasing of depth $k+1$ and $X''_1 := B''_1 \cup \dots \cup B''_k$ is block-increasing of depth k by their constructions.

So we have partitioned $X_1 \cup Z_1$ into $O(k \log k)$ block-monotone point sets of depth at least k , which are $\{A_1, \dots, A_r, X'_1, X''_1\}$, as claimed.

Case 2. Suppose X is block-decreasing, we choose two points in the following regions:

$$\begin{aligned}(x_1, y_1) &\in R(B_k) \cap D(B_k) \cap L(B_{k+1}) \cap U(B_{k+1}), \\(x_2, y_2) &\in R(B_{2k}) \cap D(B_{2k}) \cap L(B_{2k+1}) \cap U(B_{2k+1}).\end{aligned}$$

Also we require x_1 or x_2 isn't the x -coordinate of any element in P , and y_1 or y_2 isn't the y -coordinate of any element in P . We use the lines $x = x_i$ and $y = y_i$ for $i = 1, 2$ to divide the plane into a 3×3 grid and label the regions $R_i, i = 1, \dots, 9$, as in Fig. 3.

We define

$$Y' := Y_1 \cup Y_2 \cup \dots \cup Y_{i_0-1} \cup (R_7 \cap Y_{i_0}) \cup X_2 \cup (R_3 \cap Y_{i_0}) \cup Y_{i_0+1} \cup \dots \cup Y_{2t+1}.$$

Using condition (3.2) and Definition 3.2, we can check that Y' is a $(k, t+1)$ -configuration. And $P' := S_1 \cup \dots \cup S_l \cup Y'$ is a $(k, l, t+1)$ -pattern according to Definition 3.3. Let $Z_1 := (Y_{i_0} \setminus X) \cap (R_5 \cup R_6 \cup R_8 \cup R_9)$, and $Z_2 := (Y_{i_0} \setminus X) \cap (R_1 \cup R_2 \cup R_4)$. We claim that $X_i \cup Z_i$ can be partitioned into $O(k \log k)$ block-monotone point sets of depth at least k and a remaining set of size $O(k^2)$ for $i = 1, 3$. Given this claim and the fact that $P = P' \cup X_1 \cup X_3 \cup Z_1 \cup Z_3$, we conclude the lemma with P' as described in (ii).

Now we justify this claim for $X_1 \cup Z_1$ and the justification for $X_3 \cup Z_3$ is similar. By an argument similar to (3.1), we can apply Fact 3.7 twice to partition Z_1 into $\{A_1, \dots, A_r, Z'_1\}$, where $r = O(k \log k)$, s.t. each A_i is block-monotone of depth at least k and $|Z'_1| \leq \max\{|Z_1|/(3ck)^2, (9c^2k-1)^2\}$. If $|Z'_1| \leq (9c^2k-1)^2$, we have

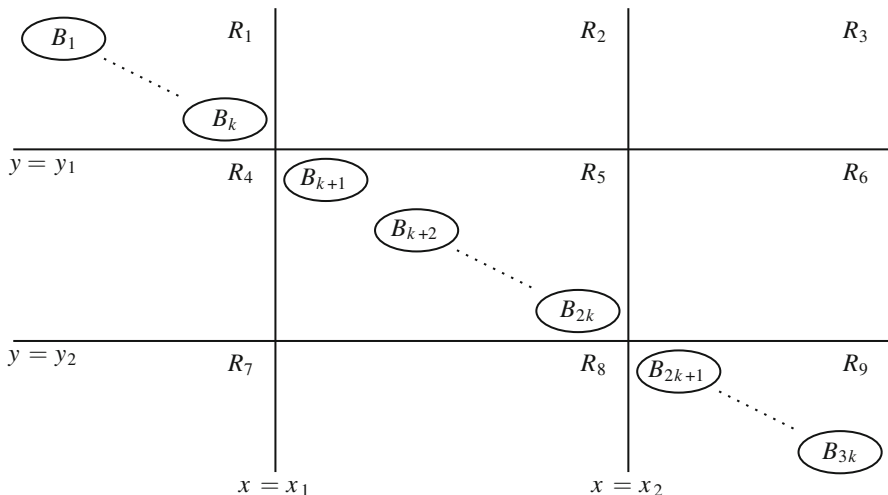


Fig. 3 Division of the plane into nine regions according to $(x_i, y_i), i = 1, 2$. Each ellipse represents a cluster of points as defined in the proof

partitioned $X_1 \cup Z_1$ into $O(k \log k)$ block-monotone point sets of depth at least k , which are $\{A_1, \dots, A_r, X_1\}$, and a remaining set Z'_1 of size $O(k^2)$ as claimed.

If $|Z'_1| \leq |Z_1|/(3ck)^2$, since $|Z_1| \leq |Y_{i_0}|$, we have $|Z'_1| \leq |Y_{i_0}|/(3ck)^2 \leq |B_i|$ for all $i \in [k]$. We can take a partition $B_i = B'_i \cup B''_i$ with $|B'_i| = |Z'_1|$. We observe that $X'_1 := B'_1 \cup \dots \cup B'_k \cup Z'_1$ is block-decreasing of depth $k+1$ and $X''_1 := B''_1 \cup \dots \cup B''_k$ is block-decreasing of depth k by their constructions. So we have partitioned $X_1 \cup Z_1$ into $O(k \log k)$ block-monotone point sets of depth at least k , which are $\{A_1, \dots, A_r, X'_1, X''_1\}$, as claimed.

Finally, when $t = 0$ is given in the hypothesis, the condition (3.2) and its opposite, i.e., $\bigcup_{j=i+1}^{2t+1} Y_j \subset R(Y_i) \cap D(Y_i)$ for all $i \in [2t]$, are both trivially true. Hence, when $|Y_{i_0}| > (3k-1)^2$, no matter whether X is block-increasing or block-decreasing, we can always use the arguments in Case 2 to conclude the lemma as described in (ii). \square

4 Applications

4.1 Mutually Avoiding Sets

We devote this subsection to the proof of Theorem 1.3. The proof is essentially the same as in [1], but we include it here for completeness. Given a non-vertical line L in the plane, we denote L^+ to be the closed upper-half plane defined by L , and L^- to be the closed lower-half plane defined by L . We need the following result, which is [1, Lemma 1].

Lemma 4.1 *Let $P, Q \subset \mathbb{R}^2$ be two n -element point sets with P and Q separated by a non-vertical line L and $P \cup Q$ in general position. Then for any positive integer $m \leq n$, there is another non-vertical line H s.t. $|H^+ \cap P| = |H^+ \cap Q| = m$ or $|H^- \cap P| = |H^- \cap Q| = m$.*

Proof of Theorem 1.3 Let k be as given and $n > 24k^2$. Let P be an n -element point set in the plane in general position. We start by taking a non-vertical line L to partition the plane s.t. each half-plane contains $\lfloor n/2 \rfloor$ points from P . Then by Lemma 4.1, we obtain a non-vertical line H with, say, $H^+ \cap (L^+ \cap P) = H^+ \cap (L^- \cap P) = \lfloor n/6 \rfloor$. Next, we find a third line N , by first setting $N = H$, and then sweeping N towards the direction of H^- , keeping it parallel with H , until $H^- \cap N^+ \cap L^+$ or $H^- \cap N^+ \cap L^-$ contains $\lfloor n/6 \rfloor$ points from P . Without loss of generality, let us assume $Q := P \cap (H^- \cap N^+ \cap L^+)$ first reaches $\lfloor n/6 \rfloor$ points, and the region $H^- \cap N^+ \cap L^-$ has less than $\lfloor n/6 \rfloor$ points from P . Hence, both $Q_l := P \cap (H^+ \cap L^-)$ and $Q_r := P \cap (N^- \cap L^-)$ have at least $\lfloor n/6 \rfloor$ points. See Fig. 4 for an illustration. \square

We can apply an affine transformation so that L and H are perpendicular, and N is on the right side of H . Think of L as the x -axis, H as the y -axis, and N as a vertical line with a positive x -coordinate. After ordering the elements in Q according to their x -coordinates, we apply Theorem 1.1 to Q to obtain disjoint subsets $Q_1, \dots, Q_{2k+1} \subset Q$ s.t. (Q_1, \dots, Q_{2k+1}) is block-monotone of depth $2k+1$ and block-size $\Omega(n/k^2)$, where each entry represents its y -coordinate. Without loss of generality, we can assume

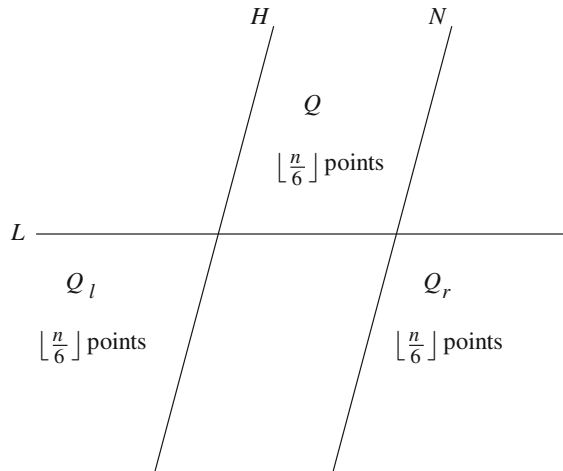


Fig. 4 The division of plane into regions according to L, H, N

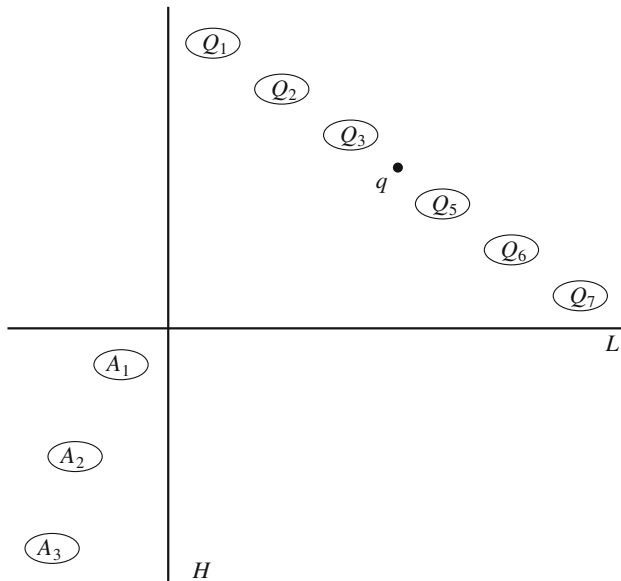


Fig. 5 An example when A_i 's are increasing. Each ellipse represents a cluster of points as defined in the proof

it is block-decreasing, otherwise we can work with Q_r rather than Q_l in the following arguments.

Now fix a point $q \in Q_{k+1}$. We express the points in Q_l in polar coordinates (ρ, θ) with q being the origin. We can assume no two points in Q_l are at the same distance to q , otherwise a slight perturbation may be applied. By ordering the points in Q_l with respect to θ , in counter-clockwise order, we apply Theorem 1.1 to Q_l to obtain disjoint subsets $A_1, \dots, A_k \subset Q_l$ s.t. (A_1, \dots, A_k) is block-monotone of depth k and block-size $\Omega(n/k^2)$, where each entry represents its distance to q . If it's block-decreasing, take $B_i = Q_i$ for $i \in [k]$, and if it's block-increasing, take $B_i = Q_{k+1+i}$. It is easy to

check that the sets $\{A_1, \dots, A_k\}$ and $\{B_1, \dots, B_k\}$ have the claimed properties. See Fig. 5 for an illustration. \square

4.2 Monotone Biarc Diagrams

We devote this subsection to the proof of Theorem 1.4. Our proof is constructive, hence implying a recursive algorithm for the claimed outcome. We start by making the simple observation that our main results hold for sequences of (not necessarily distinct) real numbers, if the term “*block-monotone*” now refers to being block-nondecreasing or block-nonincreasing. More precisely, a sequence $(a_1, a_2, \dots, a_{ks})$ of real numbers is said to be *block-nondecreasing* (resp. *block-nonincreasing*) with *depth* k and *block-size* s if every subsequence $(a_{i_1}, a_{i_2}, \dots, a_{i_k})$, for $(j-1)s < i_j \leq js$, is nondecreasing (resp. nonincreasing).

Theorem 4.2 *For any positive integer k , every finite sequence of (not necessarily distinct) real numbers can be partitioned into at most $O(k^2 \log k)$ block-monotone subsequences of depth at least k upon deleting at most $(k-1)^2$ entries.*

To see our main results imply the above variation, it suffices to slightly perturb the possibly equal entries of a given sequence until all entries are distinct. Algorithms for our main results can also be applied after such a perturbation.

We need the following lemma from [2] for Theorem 1.4.

Lemma 4.3 *For any graph $G = (V, E)$ with $V = [n]$, there exists $b \in [n]$ s.t. both the induced subgraphs of G on $\{1, 2, \dots, b\}$ and $\{b+1, b+2, \dots, n\}$ have no more than $|E|/2$ edges.*

Proof For $U \subset [n]$, let G_U denote the induced subgraph of G on U . Let b be the largest among $[n]$ s.t. $E(G_{[b]}) \leq |E|/2$, so $E(G_{[b+1]}) > |E|/2$. Notice that $E(G_{[b+1]})$ and $E(G_{[n] \setminus [b]})$ are two disjoint subsets of E , so $E(G_{[n] \setminus [b]}) \leq |E| - E(G_{[b+1]}) < |E|/2$, as wanted. \square

Proof of Theorem 1.4 We prove by induction on $|E|$. The base case when $|E| = 1$ is trivial. For the inductive step, by the given order on V , we can identify V with $[n]$. We find such a b according to Lemma 4.3. Consider the set E' of edges between $[b]$ and $[n] \setminus [b]$. By writing each edge $e \in E'$ as (x, y) , where $x \in [b]$ and $y \in [n] \setminus [b]$, we order the elements in E' lexicographically: for $(x, y), (x', y') \in E$, we have $(x, y) < (x', y')$ when $x < x'$ or when $x = x'$ and $y < y'$.

Given the order on E' described above, consider the sequence of right-endpoints in E' . We apply Theorem 4.2 with parameter $k = \lceil \epsilon^{-1} \rceil$ to this sequence, to decompose it into $C_k = O(k^2 \log k)$ many block-monotone sequences of depth k , upon deleting at most $(k-1)^2$ entries. For each block-monotone subsequence of depth k , we draw the corresponding edges on a single page as follows. If the subsequence is block-nonincreasing of depth k and block-size s , we draw the corresponding edges as semicircles above the spine. Then, two edges cross only if they come from the same block. Since there are a total of $\binom{ks}{2}$ pairs of edges, and only $k \binom{s}{2}$ such pairs from the same block, the fraction of pairs of edges that cross in such a drawing is at

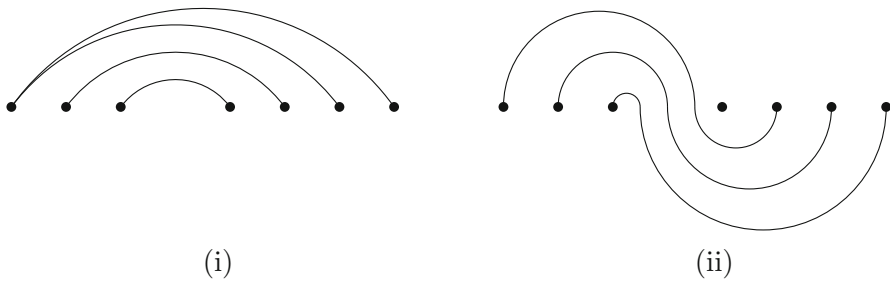


Fig. 6 (i) a proper arc diagram. (ii) a monotone biarc diagram

most $1/k$. See Fig. 6(i). Similarly, if the subsequence is block-nonincreasing of depth k and block-size s , we draw the corresponding edges as monotone biarcs, consisting of two semicircles with the first (left) one above the spine, and the second (right) one below the spine. Furthermore, we draw the monotone biarc s.t. it crosses the spine at $b + 1 - \ell/n - r/(2n^2)$, where ℓ and r are the left and right endpoints of the edge respectively. See Fig. 6(ii). By the same argument above, the fraction of pairs of edges that cross in such a drawing is at most $1/k$.

Hence, E' can be decomposed into $C_k + (k - 1)^2$ many monotone biarc diagrams, s.t. each monotone biarc diagram has at most $1/k$ -fraction of pairs of edges that are crossing. \square

For edges within $[b]$, Lemma 4.3 and the inductive hypothesis tell us that they can be decomposed into $(C_k + (k - 1)^2)(\log |E| - 1)$ monotone biarc diagrams, s.t. the fraction of pairs of edges that are crossing in each diagram is at most $1/k$. The same argument applies to the edges within $[n] \setminus [b]$. However, notice that two such monotone biarc diagrams, one in $[b]$ and another in $[n] \setminus [b]$, can be drawn on the same page without introducing more crossings. Hence, we can decompose $E \setminus E'$ into at most $(C_k + (k - 1)^2)(\log |E| - 1)$ such monotone biarc diagrams, giving us a total of $(C_k + (k - 1)^2) \log |E|$ monotone biarc diagrams.

5 Final Remarks

1. We call a sequence (a_1, a_2, \dots, a_n) of n distinct real numbers ϵ -increasing (resp. ϵ -decreasing) if the number of decreasing (resp. increasing) pairs (a_i, a_j) , where $i < j$, is less than ϵn^2 . And we call a sequence ϵ -monotone if it's either ϵ -increasing or ϵ -decreasing. Clearly, a block-monotone sequence of depth k is an ϵ -monotone sequence with $\epsilon = k^{-1}$. Hence, Theorem 1.1 implies the following.

Corollary 5.1 *For all $n > 0$ and $\epsilon > 0$, every sequence of n distinct real numbers contains an ϵ -monotone subsequence of length at least $\Omega(\epsilon n)$.*

This corollary is also asymptotically best possible. To see this, for $n > (k - 1)^2$ and a sequence $A = (a_i)_{i=1}^n$ of distinct real numbers, we can apply Corollary 5.1 with $\epsilon = (64k)^{-1}$ to A and obtain an ϵ -monotone subsequence $S \subset A$ and then apply [13, Lemma 2.1] to S to obtain a block-monotone subsequence of depth k and block-size $\Omega(n/k^2)$. So Corollary 5.1 implies Theorem 1.1.

2. Let $f(k)$ be the smallest number N s.t. every finite sequence of distinct real numbers can be partitioned into at most N block-monotone subsequences of depth at least k upon deleting $(k-1)^2$ entries. Our Theorem 1.2 is equivalent to saying $f(k) = O(k^2 \log k)$. The $K(k, 2)$ -type construction in Remark 2.5 implies $f(k) \geq k$. What is the asymptotic order of $f(k)$?

3. We suspect our algorithm presented in Theorem 1.1 can be improved. How fast can we compute a block-monotone subsequence as large as asserted in Theorem 1.1? Can we do it within time almost linear in n for all k ?

Acknowledgements We wish to thank the anonymous SoCG referees for their valuable suggestions.

References

1. Aronov, B., Erdős, P., Goddard, W., Kleitman, D.J., Klugerman, M., Pach, J., Schulman, L.J.: Crossing families. In: 7th Annual Symposium on Computational Geometry (North Conway 1991), pp. 351–356. ACM, New York (1991)
2. Bar Yehuda, R., Fogel, S.: Partitioning a sequence into few monotone subsequences. *Acta Inform.* **35**(5), 421–440 (1998)
3. Bárány, I., Valtr, P.: A positive fraction Erdős–Szekeres theorem. *Discrete Comput. Geom.* **19**(3), 335–342 (1998)
4. de Berg, M., Cheong, O., van Kreveld, M., Overmars, M.: *Computational Geometry: Algorithms and Applications*. Springer, Berlin (2008)
5. Bernhart, F., Kainen, P.C.: The book thickness of a graph. *J. Combin. Theory Ser. B* **27**(3), 320–331 (1979)
6. Di Giacomo, E., Didimo, W., Liotta, G., Wismath, S.K.: Curve-constrained drawings of planar graphs. *Comput. Geom.* **30**(1), 1–23 (2005)
7. Erdős, P., Szekeres, G.: A combinatorial problem in geometry. *Compos. Math.* **2**, 463–470 (1935)
8. Fox, J., Pach, J., Sudakov, B., Suk, A.: Erdős–Szekeres-type theorems for monotone paths and convex bodies. *Proc. Lond. Math. Soc.* **105**(5), 953–982 (2012)
9. Fredman, M.L.: On computing the length of longest increasing subsequences. *Discrete Math.* **11**(1), 29–35 (1975)
10. Milans, K.G., Stolee, D., West, D.B.: Ordered Ramsey theory and track representations of graphs. *J. Comb.* **6**(4), 445–456 (2015)
11. Mirzaei, M., Suk, A.: A positive fraction mutually avoiding sets theorem. *Discrete Math.* **343**(3), # 111730 (2020)
12. Moshkovitz, G., Shapira, A.: Ramsey theory, integer partitions and a new proof of the Erdős–Szekeres theorem. *Adv. Math.* **262**, 1107–1129 (2014)
13. Pach, J., Rubin, N., Tardos, G.: Planar point sets determine many pairwise crossing segments. *Adv. Math.* **386**, # 107779 (2021)
14. Pór, A., Valtr, P.: The partitioned version of the Erdős–Szekeres theorem. *Discrete Comput. Geom.* **28**(4), 625–637 (2002)
15. Steele, J.M.: Variations on the monotone subsequence theme of Erdős and Szekeres. In: *Discrete Probability and Algorithms* (Minneapolis 1993). IMA Vol. Math. Appl., vol. 72, pp. 111–131. Springer, New York (1995)
16. Valtr, P.: On mutually avoiding sets. *The Mathematics of Paul Erdős*, vol. 2. *Algorithms Combin.*, vol. 14, pp. 324–332. Springer, Berlin (1997)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.