

Existence of Rotating Stars with Variable Entropy

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Abstract

We model a rotating star as a compressible fluid subject to gravitational forces. In almost all the mathematical literature the entropy is considered to be constant. Here we allow it to be variable. We consider a star that steadily rotates differentially around a fixed axis, say the z -axis. We prove the existence of a family of such stars with small angular velocity ω and small entropy variation s and with an equation of state $p = Ke^s \rho^\gamma$. Our analysis reduces to a hyperbolic equation for the modified entropy coupled to an elliptic equation for the modified density, together with a mass constraint. Due to the variable entropy and the consequent loss of both regularity and variational structure, all the methods in the previous literature fail. We develop a new ad hoc perturbative strategy that allows us to construct rotating stars that bifurcate from the non-rotating ones.

1 Introduction

1.1 Rotating Stars

The study of rotating stars is a classical topic in astrophysics and mathematics, and there has been a great deal of interest and activity for centuries. Early studies can be tracked back to Newton, Maclaurin, Jacobi, Poincaré, Liapunov *et al* who studied incompressible stars, while compressible stars began to be treated later by Lichtenstein [25] and Chandrasekhar [6]. We refer to [8, 22] for historical accounts on the topic. Other excellent general references are [33, 16]. The existence of rotating stars with a given angular velocity distribution is not a trivial task. In particular the support of a star is not known *a priori* and it is one of the unknowns.

In the search for rotating star solutions there are two modern approaches. One is based on variational methods where the rotating solutions are obtained

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via a constrained minimization problem [3, 4, 13, 14, 24, 27, 28, 35], and the other is based on perturbative methods where the solutions are obtained by an implicit function theorem or contraction mapping principle around non-rotating Lane-Emden stars [17, 19, 20, 21, 29, 30]. The second approach has been extended to construct a global set of rotating stars [31]. Both approaches have been particularly successful to prove the existence of rotating star solutions with constant entropy, in which case an integral reformulation can be effectively used.

When it comes to rotating star solutions with variable entropy, however, very few results are available, despite its physical importance [18, 33]. The goal of this paper is to establish the existence of rotating star solutions for self-gravitating perfect fluids with variable entropy. To the best of our knowledge, ours is the first mathematical result on rotating star solutions that allows variable entropy. Furthermore, ours is the first that allows the prescribed angular velocity ω to depend on both of the cylindrical coordinates r and z , while all prior works treated merely r -dependent angular velocity profiles. As we will see from (11), any z -dependence of the angular velocity implies that entropy is not constant. A system with such a nontrivial z -dependence is called a *barocline*. Actual stellar rotations typically exhibit such dependence. An example is our own sun [33].

Our problem does not have a variational structure. Although the implicit function theorem would seem like a natural technique, it does not work because of the loss of regularity. Instead we use an explicit iteration and we perform estimates in two different normed spaces in Sections 4 and 5. In order to make optimal use of the available regularity, most of our analysis is carried out in a bounded domain.

1.2 Euler-Poisson Model

Now we describe the main contents of the current work. We regard the star as a compressible inviscid fluid. The basic equations are those of Euler-Poisson which describe the conservation of mass, momentum and energy subject to self-gravity and the transport of entropy by the fluid. The equations are:

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = \rho \nabla U, \quad (2)$$

$$s_t + \mathbf{v} \cdot \nabla s = 0, \quad (3)$$

$$U(x, t) = G \int_{\mathbb{R}^3} \frac{\rho(x', t)}{|x - x'|} dx', \quad (4)$$

where ρ is the density, $\mathbf{v}$ the velocity, p the pressure, s the entropy, and G Newton's universal constant of gravitation. The first three equations are valid in the region $\{\rho > 0\}$ occupied by the star. We assume the pressure p is a function of ρ and s , specifically that the star is a simple ideal gas:

$$p = K e^s \rho^\gamma, \quad (5)$$

as in [9] (p.6-7), [5] (Section 3.4) and [12](p. 61-62), where $K > 0$ and $\gamma > 1$ are constants. Note that the entropy transport equation (3) is a consequence of

the other conservation laws together with the energy conservation law:

$$\left(\rho \left(\frac{1}{2} |\mathbf{v}|^2 + \varepsilon \right) \right)_t + \nabla \cdot \left(\rho \left(\frac{1}{2} |\mathbf{v}|^2 + \varepsilon \right) \mathbf{v} \right) + \nabla \cdot (p \mathbf{v}) = \rho \nabla U \cdot \mathbf{v}, \quad (6)$$

where

$$\varepsilon = \frac{K}{\gamma - 1} e^s \rho^{\gamma-1}$$

is the specific internal energy. For notational convenience we choose $K = \frac{\gamma - 1}{\gamma}$ and $G = 1$ in the following, but the same results hold for any positive K and G .

Our main result states that there exists a family of rotating solutions which are axisymmetric and even in z and which bifurcate from the non-rotating (Lane-Emden) radial solution ρ^0 that has a radius R^0 and zero entropy. (See Lemma 2.1 for details of such non-rotating star solutions.) We denote by B_{R^0} the ball with radius R^0 centered at the origin on which ρ^0 is supported. In the following theorem, let $\beta \in \left(0, \min \left(\frac{2-\gamma}{\gamma-1}, 1 \right) \right)$ be given. The parameter κ measures the intensity of rotation, and μ measures the size of the floor entropy. Here floor entropy refers to the entropy on the equatorial plane $z = 0$.

Theorem 1. *Assume $\frac{6}{5} < \gamma < 2$, and $\gamma \neq \frac{4}{3}$. Let B be an arbitrary open ball strictly containing B_{R^0} . Let $s_0 \in C^{1,\beta}(\overline{B} \cap \{z = 0\})$ be a given axisymmetric floor entropy profile on the equatorial plane $\{z = 0\}$, and $\omega \in C^{2,\beta}(\overline{B})$ be a given axisymmetric angular velocity profile that is even in z . Then there exists a family of solutions $(\rho_{\kappa,\mu}, s_{\kappa,\mu}) \in [C^{1,\beta}(\overline{B})]^2$ for $\kappa, \mu > 0$ sufficiently small, which depend continuously on (κ, μ) and converge to $(\rho^0, 0)$ as $(\kappa, \mu) \rightarrow 0$ such that each solution satisfies the following properties.*

- $(\rho_{\kappa,\mu}, s_{\kappa,\mu})$ is axisymmetric and even in z .
- $(\rho_{\kappa,\mu}, s_{\kappa,\mu})$ is a time independent solution of (1)-(5), where the velocity $\mathbf{v}$ is an axisymmetric rotation with angular velocity $\sqrt{\kappa} \omega(r, z)$.
- $s_{\kappa,\mu}$ satisfies the floor entropy condition

$$s_{\kappa,\mu}|_{z=0} = \mu s_0.$$

- $\rho_{\kappa,\mu}$ is nonnegative, compactly supported in B and has the same total mass

$$\int_B \rho_{\kappa,\mu} \, dx = \int_{B_{R^0}} \rho^0 \, dx$$

as the Lane-Emden solution.

The exceptional index $\gamma = 4/3$ is excluded from the theorem. It is the mass critical index for which all the radial Lane-Emden solutions have the identical total mass. In this exceptional case there is a trivial curve of non-rotating solutions which are related to each other by scaling. Moreover, in case $\gamma = 4/3$ with constant entropy and angular velocity, it is shown in [32] that there does not exist any slowly rotating solution close to a given Lane-Emden solution with the same total mass. So in that case the local solution space is exhausted by the trivial curve of non-rotating stars.

1.3 Related Literature

So far as we know, the only works in the literature that are related to the Euler-Poisson equations with variable entropy are [10, 11, 26, 34, 37]. *Our result differs from them in several important ways.* First of all, most of the earlier works that considered variable entropy were essentially restricted to solving only the divergence of the momentum equation (2) on a given domain with prescribed entropy s , while the curl part was largely ignored. Therefore such solutions do not necessarily satisfy the Euler-Poisson system. In fact, even if the curl part holds, as for instance in the case of radial symmetry, in order to recover the momentum equation from the divergence and the curl components, one would need an extra boundary condition (different from the Dirichlet condition of ρ) on the surface of the star. Without such a boundary condition, the two sides of the momentum equation would differ by the gradient of a harmonic function. See Section 2.1 for a more detailed discussion. *This key step of going from the divergence and the curl back to the original system seems to be missing in all the prior works, with the exception of [37].*

In [37], the author uses a variational method on an integral formulation of the momentum equation, in a spirit analogous to the prior variational results [3] with constant entropy. However, in order to fit variable entropy into this framework, the author has to work with functions that are constants on a given nested family of ellipsoids. As all the test functions for the Euler-Lagrange variation then have fixed ellipsoidal symmetry, the solutions obtained in [37] merely solve ellipsoidal averages of the momentum equation and do not solve the original system.

In summary, to the best of our knowledge, Theorem 1 is the first existence result of variable entropy rotating star solutions to the full Euler-Poisson system. In order to obtain our result, we predetermine neither the entropy s nor the fluid domain of the star. Both of them appear as unknowns in our formulation.

1.4 Significance of the dependence on z

Moreover, all of the above mentioned works require an angular velocity profile ω that depends only on r . *In contrast, our result allows an arbitrary axisymmetric rotation speed ω that is even in z .* In our opinion, when allowing variable entropy, an ω with nontrivial z -dependence presents the more interesting case, and in a certain sense the only interesting case. In fact, as we shall see in Section

2.1, the curl of the momentum equation implies that ω^2 depends only on r if and only if ∇s and $\nabla \rho$ are parallel vectors everywhere. In other words, if ω^2 depends only on r , then s must be constant on the level surfaces of ρ . This fact is known as the Poincaré-Wavre theorem in the physics literature ([33]). If the solution is close to a nonrotating Lane-Emden star, all level surfaces of ρ have different values. In this case there exists a one-variable function f such that $s = f(\rho)$ on the fluid domain. Such a solution is called a *barotrope*. Therefore the pressure p also depends only on ρ . Finally, we point out that the rotation speed of our own sun depends on both r and z ([33]), so that our result provides the first mathematical description in the Euler-Poisson model of our own star.

Before beginning to discuss our method, we set the convention here that all functions dealt with in this paper are axisymmetric and are even in the z variable. Therefore, every function space we will use is assumed to have the above mentioned symmetry without further declaration. In particular, a C^1 function V with such symmetry will have the properties that $\nabla V(0, 0, 0) = 0$, $\partial_{x_1} V(0, 0, z) = 0$, $\partial_{x_2} V(0, 0, z) = 0$, $\partial_z V(x_1, x_2, 0) = 0$. For the same reason, we identify a function defined for $r > 0, z > 0$ with a function defined on $\mathbb{R}^3$ with the above mentioned symmetry without further explanation.

2 Reformulation

The main difficulties come from the presence of the nontrivial entropy. First of all, the entropy cannot be prescribed arbitrarily but must be solved together with the other unknowns. In fact, by taking the curl of the momentum equation, one finds that the entropy satisfies a first order transport equation whose coefficients are given by the first derivatives of the density (or enthalpy). The z derivative of the angular velocity comes into the transport equation as a source term. This new transport structure highlights fundamental differences between constant and variable entropy. For variable entropy, we are led to studying the ensuing system of equations which is of mixed elliptic-hyperbolic type, while for constant entropy we would be led to a purely elliptic problem. The nontrivial coupling between the entropy and the enthalpy precludes an integral reformulation.

2.1 Reduction to the Div-Curl System

As a first step to solving this problem, we must choose certain good variables that yield a system with better structure. This is a crucial step in our analysis. We use cylindrical coordinates (r, z) where all the functions are independent of the cylindrical angle θ . Steady flow means that all the time derivatives in (1)-(3) vanish. Rotation with angular velocity ω means that the velocity is $\mathbf{v} = \sqrt{\kappa}\omega(r, z)r\mathbf{e}_\theta$, where $r\mathbf{e}_\theta = (-x_2, x_1, 0)$. As a result, the mass conservation equation (1) and the entropy transport equation (3) are automatically satisfied.

The momentum equation (2) reduces to

$$\frac{\nabla p}{\rho} = \nabla \left(\frac{1}{|\cdot|} * \rho \right) + \kappa \omega^2 r \mathbf{e}_r, \quad (7)$$

where $r \mathbf{e}_r = (x_1, x_2, 0)$. We introduce the new variables and constant

$$S = e^{s/\gamma}, \quad V = \rho^{\gamma-1} e^{(\gamma-1)s/\gamma}, \quad q = \frac{1}{\gamma-1}. \quad (8)$$

Then (7) takes the form

$$S \nabla V = \nabla \left(\frac{1}{|\cdot|} * \frac{V^q}{S} \right) + \kappa \omega^2 r \mathbf{e}_r. \quad (9)$$

This equation is originally only required to hold on the fluid domain, namely where $\rho > 0$.

However, we will look for a solution to the slightly modified equation

$$S \nabla V = \nabla \left(\frac{1}{|\cdot|} * \frac{V_+^q}{S} \right) + \kappa \omega^2 r \mathbf{e}_r, \quad (10)$$

where $V_+ = \max(V, 0)$, in a large ball B_R containing the fluid domain, where R will be chosen later. The density ρ will then be recovered as $V_+^q S^{-1}$. *In this way, the fluid domain is not prescribed in advance, but is obtained as the positivity set of the solution V .* Taking the curl and the divergence of (10), we obtain the two scalar equations

$$S_r V_z - S_z V_r = -\kappa (\omega^2 r)_z, \quad (11)$$

$$\nabla \cdot (S \nabla V) = -4\pi \frac{V_+^q}{S} + \kappa \nabla \cdot (\omega^2 r \mathbf{e}_r). \quad (12)$$

We will regard (11) as a transport equation for S with coefficients given by components of ∇V and source term given by its right hand side. Although we will treat (11) structurally as a transport equation, it has nothing directly to do with the actual entropy transport by fluid velocity given in (3). Equation (11) illustrates the effect of the dependence on z of the angular velocity. In fact, it is immediately clear that any z -dependence of ω implies that the entropy is not constant.

Equation (12), on the other hand, will be treated basically as a semilinear elliptic equation for V with S appearing as a coefficient. However, as we will see more clearly later, the coupling of S with V causes the whole problem to have a quasilinear flavor.

We have to supplement (11) and (12) with suitable boundary conditions. The first of these is the *floor entropy condition* (on the equatorial plane)

$$S(r, 0) = \exp \left(\frac{\mu s_0(r)}{\gamma} \right), \quad (13)$$

where $s_0(r)$ is the floor entropy profile which we treat as essentially arbitrary, and μ is a small parameter. We will use this condition to uniquely solve the transport equation (11). Secondly, the bounded domain B_R is employed in order to make maximal use of the elliptic regularity estimates. Consequently, we will need a key boundary condition of V on ∂B_R that will help us recover (10) from its curl and divergence. Such a boundary condition can be much simplified if we cut off ω^2 and s_0 near ∂B_R . Such a cut-off will have no effect to our final solution.

2.2 Non-rotating Solution

For clarification, let us be a bit more specific about the non-rotating Lane-Emden solution around which we perturb.

Lemma 2.1. *Given $\frac{6}{5} < \gamma < 2$ and $R^0 > 0$, there exists a unique function V^0 which is continuous on $\mathbb{R}^3$, positive in the ball B_{R^0} and negative outside B_{R^0} , as well as a unique constant α^0 such that*

$$V^0 = \frac{1}{|\cdot|} * (V^0)_+^q + \alpha^0 \quad (14)$$

in all of $\mathbb{R}^3$. Furthermore (V^0, α^0) has the following properties: $\alpha^0 < 0$, $V^0 \in C^{3, \beta_0}(\mathbb{R}^3)$, where $\beta_0 = \min(q-1, 1)$, and $V^0|_{B_{R^0}} \in C^\infty(B_{R^0})$. Furthermore, V^0 is radially symmetric and strictly decreasing, and

$$\lim_{\tau \rightarrow 0^+} \frac{(V^0)'(\tau)}{\tau} < 0. \quad (15)$$

Here $(V^0)'$ means the radial derivative of V^0 .

Proof. The classical reference is Chapter IV of [7]. See also Lemmas 3.2 and 3.3 of [30]. $\square$

Obviously $V = V^0$ is a solution to (10) with $S \equiv 1$ and $\kappa = 0$. Fixing a radius $R > R^0$, we will perturb V^0 to a solution V corresponding to a rotating star. Since V^0 is strictly negative near ∂B_R , the support of V_+ will stay away from ∂B_R if the perturbation is sufficiently small. The following useful observation is obvious.

Remark 2.1. *There exists a $\delta > 0$ such that if $\|V - V^0\|_{C^0(\overline{B_R})} < \delta$, then the support of V_+ is contained in $B_{\frac{R^0+R}{2}}$.*

In our perturbative method, V will always stay close to V^0 on B_R . Therefore, without loss of generality, we can first multiply ω^2 and s_0 by a smooth cutoff function χ such that $\chi \equiv 1$ on $B_{\frac{R^0+R}{2}}$ and $\chi \equiv 0$ outside $B_{\frac{R^0+2R}{3}}$, and then solve the whole problem. Since the fluid domain will always be contained in $B_{\frac{R^0+R}{2}}$, the cutoff will be invisible to the solution. In the following, we will

always assume that ω^2 and s_0 are prepared in this way, and when appropriate, we will even extend these functions outside B_R by zero. Such an extension is obviously smooth as well.

2.3 Boundary Condition and Main Theorem

As we shall see in Section 3, the solution S to the transport equation (11) will be identically equal to 1 near ∂B_R , as a consequence of the cutoff we performed on ω^2 and s_0 . The same cutoff and (10) imply that

$$\nabla V = \nabla \left(\frac{1}{|\cdot|} * \frac{V_+^q}{S} \right) \quad (16)$$

near ∂B_R . We thus obtain the key *boundary condition* on V :

$$V = \frac{1}{|\cdot|} * \frac{V_+^q}{S} + \alpha \quad \text{on } \partial B_R \quad (17)$$

for some constant α . Note that V_+ will be supported on $B_{\frac{R^0+R}{2}}$ by Remark 2.1. Thus the function $V_+^q S^{-1}$ that appears under the convolution in (17) can be regarded as the zero extension from its values on B_R to $\mathbb{R}^3$.

Finally, we prescribe the *mass constraint*

$$\int_{B_R} \frac{V_+^q}{S} dx = M = \int_{B_R} (V^0)_+^q dx, \quad (18)$$

so that our rotating star will have the same mass M as the unperturbed Lane-Emden star.

Our method to work with the div-curl system is summarized in the following lemma.

Lemma 2.2. *Let $\omega^2 \in C^{2,\beta}(\overline{B_R})$ and suppose that $V \in C^{2,\beta}(\overline{B_R})$, $0 < S \in C^{1,\beta}(\overline{B_R})$ solve (11), (12), (17). Also suppose that $S = 1$, $\omega = 0$ in a neighborhood of ∂B_R . Then $s = \gamma \log S \in C^{1,\beta}(\overline{B_R})$, $\rho = V_+^q S^{-1} \in C^{1,\beta}(\overline{B_R})$ solve (7) and (5) in the set $\{\rho > 0\}$.*

Proof. Let

$$W = S \nabla V - \nabla \left(\frac{1}{|\cdot|} * \frac{V_+^q}{S} \right) - \kappa \omega^2 r \mathbf{e}_r$$

on $\overline{B_R}$. We have

$$\nabla \times W = [S_r V_z - S_z V_r + \kappa(\omega^2 r)_z] \mathbf{e}_\theta. \quad (19)$$

$$\nabla \cdot W = \nabla \cdot (S \nabla V) + 4\pi V_+^q S^{-1} - \kappa \nabla \cdot (\omega^2 r \mathbf{e}_r). \quad (20)$$

By (19) and (11), W is a $C^{1,\beta}$ conservative vector field on $\overline{B_R}$. Thus there exists $\psi \in C^{2,\beta}(\overline{B_R})$ such that $\nabla \psi = W$. By (20) and (12), $\Delta \psi = \nabla \cdot W = 0$. So ψ is harmonic on B_R . Since $S = 1$ and $\omega^2 = 0$ in a neighborhood of ∂B_R ,

$$\nabla \psi = W = \nabla \left(V - \frac{1}{|\cdot|} * \frac{V_+^q}{S} \right) \quad (21)$$

in the same neighborhood. By (21) and (17), the tangential derivative of ψ on ∂B_R is zero. Thus ψ is a constant on ∂B_R . It follows that ψ is identically equal to a constant on $\overline{B_R}$. Thus $W = 0$ on B_R . In other words, (10) holds on B_R . It is now straightforward to see that (7) and (5) are equivalent to (10) under the given change of variables where $\rho > 0$. $\square$

Our problem is thus reduced to finding solutions of (11) and (12) subject to the boundary data on the floor (13), (17) and the mass constraint (18) on the large ball B_R . The existence result for our reformulated div-curl system is stated as follows.

Theorem 2. *Let $s_0 \in C^{1,\beta}(\overline{B_R} \cap \{z = 0\})$ and $\omega^2 \in C^{2,\beta}(\overline{B_R})$ be given. There exist $\epsilon > 0$, $\epsilon_1 > 0$ such that if $|\kappa| + |\mu| < \epsilon_1$, then there exists a unique solution $(V^*, \alpha^*) \in C^{2,\beta}(\overline{B_R}) \times \mathbb{R}$, and $S^* \in C^{1,\beta}(\overline{B_R})$ to (11), (12), (13), (17), (18), with $\|(V^*, \alpha^*) - (V^0, \alpha^0)\|_{C^{2,\beta}(\overline{B_R}) \times \mathbb{R}} \leq \epsilon$. Furthermore, the solution has the properties that V_+^* is supported on $B_{\frac{R_0+R}{2}}$, and $S^* = 1$ outside B_{R_1} for some fixed $R_1 \in (R_0, R)$.*

Theorem 1 is a direct consequence of Lemma 2.2 and Theorem 2, as we now show.

Proof of Theorem 1: We may assume $B = B_R$ where $R > R_0$. By Theorem 2, we have a family of solutions $V_{\kappa,\mu} \in C^{2,\beta}(\overline{B_R})$ and $S_{\kappa,\mu} \in C^{1,\beta}(\overline{B_R})$ which satisfy all the assumptions in Lemma 2.2. Therefore $s_{\kappa,\mu} = \gamma \log S_{\kappa,\mu} \in C^{1,\beta}(\overline{B_R})$ and $\rho_{\kappa,\mu} = V_{\kappa,\mu}^q S_{\kappa,\mu}^{-1} \in C^{1,\beta}(\overline{B_R})$ solve (7) and (5) on $\{\rho > 0\}$. Together with the velocity profile $\mathbf{v} = \sqrt{\kappa} \omega(r, z)(-x_2, x_1, 0)$, it is then easy to see that $(\rho_{\kappa,\mu}, s_{\kappa,\mu})$ is a time independent solution of (1)-(5) and that $(\rho_{\kappa,\mu}, s_{\kappa,\mu})$ satisfies all the properties in Theorem 1. $\square$

The rest of the paper is devoted to the proof of Theorem 2. Before going any further, we outline our strategy as well as the difficulties in the proof of Theorem 2.

2.4 Iteration Scheme and Regularity Difficulties

Our solution to the div-curl system will be constructed as a fixed point of the mapping $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ defined as follows. We write

$$\mathcal{F}_1(V, \alpha, \kappa, \mu) = V^\# \quad \text{and} \quad \mathcal{F}_2(V, \alpha, \kappa, \mu) = \alpha^\#,$$

where $V^\#$ is the solution to the linear elliptic equation

$$\nabla \cdot (S \nabla V^\#) = -4\pi V_+^q S^{-1} + \kappa \nabla \cdot (\omega^2 r \mathbf{e}_r) \quad (22)$$

on B_R subject to the Dirichlet boundary condition

$$V^\# = \frac{1}{|\cdot|} * \frac{V_+^q}{S} + \alpha \quad (23)$$

on ∂B_R . Here $S = \mathcal{S}(V, \kappa, \mu)$ is the solution to the transport equation

$$S_r V_z - S_z V_r = -\kappa(\omega^2 r)_z \quad (24)$$

with the equatorial boundary condition

$$S(r, 0) = e^{\frac{\mu s_0(r)}{\gamma}}. \quad (25)$$

Finally, we define

$$\alpha^\# = \alpha + \int V_+^q S^{-1} - M. \quad (26)$$

The Lane-Emden solution given in Lemma 2.1 can now be formulated as

$$\mathcal{F}(V^0, \alpha^0, 0, 0) = (V^0, \alpha^0).$$

Our goal is to look for (V, α) close to (V^0, α^0) solving $\mathcal{F}(V, \alpha, \kappa, \mu) = (V, \alpha)$, for given small parameters κ, μ . We will work in the space $(V, \alpha) \in C^{2,\beta}(\overline{B_R}) \times \mathbb{R}$ where $0 < \beta < \min(q-1, 1)$.

Most of the difficulty of the problem lies in the subtle loss of regularity of the V -component of the mapping $\mathcal{F}$, which is a consequence of the loss of regularity in the S transport equation (24). As we will see in Section 3, since the coefficients of the S transport terms in (24) involve first derivatives of V , S will lose a derivative and belong only to $C^{1,\beta}(\overline{B_R})$ in general. Since at most one derivative of S appears in (22), we will be able to recover $V^\#$ in exactly the same space $C^{2,\beta}(\overline{B_R})$ using Schauder theory. In fact, this is one of the main reasons we chose S and V to be our good variables. By comparison, for instance, if we were to use s and ρ as variables, the corresponding elliptic equation would involve second derivatives of s and could not be used to recover a solution in the same regularity space.

However, even with the choice of our good variables, $\mathcal{F}$ still suffers a loss of regularity at the level of its Fréchet derivative. Of course, the natural method to solve the perturbed fixed point problem is to apply the implicit function theorem. Such an approach requires the existence of the Fréchet differential in the same regularity space. If one then tries to vary V by a variation δV , the corresponding variation δS satisfies

$$(\delta S)_r V_z - (\delta S)_z V_r = -(S + \delta S)_r (\delta V)_z + (S + \delta S)_z (\delta V)_r,$$

which linearizes to

$$(\delta S)_r V_z - (\delta S)_z V_r = -S_r (\delta V)_z + S_z (\delta V)_r. \quad (27)$$

The right hand side of (27) involves first derivatives of S , which in general belong only to $C^{0,\beta}(\overline{B_R})$. As a consequence, one can only obtain $C^{0,\beta}(\overline{B_R})$ estimates on δS . δS will be smoother along the characteristics, but not so along the transverse direction. This is not enough to recover the Fréchet differential of $\mathcal{F}$ in $C^{2,\beta}(\overline{B_R})$. This loss of derivative is an essential feature of the entropy

transport equation and cannot be remedied by discovering any hidden structure in the problem. In fact, $\mathcal{F}$ does not even depend continuously on V in the space $C^{2,\beta}(\overline{B_R})$ as soon as one shifts away from the special Lane-Emden solution V^0 . The essential reason is that the C^β Hölder norm is discontinuous with respect to smooth inner variations of a C^β function, such as a horizontal shift: $f(x) \mapsto f(x+h)$. As a result of (24), the characteristics along which S is transported are level curves of V . Thus a smooth variation of V can cause these characteristics to be shifted horizontally, resulting in a discontinuous variation of S in $C^{1,\beta}(\overline{B_R})$. Since the first derivative of S appears in (22), the discontinuity would be passed on to $V^\#$.

We overcome this difficulty by working directly with the standard Newton iteration scheme (as in the proof of the implicit function theorem). For simplicity, we denote by $D\mathcal{F}^0$ the Fréchet derivative of $\mathcal{F}$ with respect to (V, α) at $(V^0, \alpha^0, 0, 0)$. Recall that the entropy is constant ($S \equiv 1$) for the Lane-Emden solution. Thus the above mentioned loss of regularity on the Fréchet differential does not happen at this particular point in function space. We write the equation $(V, \alpha) = \mathcal{F}(V, \alpha, \kappa, \mu)$ as

$$(V, \alpha) - (V^0, \alpha^0) = \mathcal{F}(V, \alpha, \kappa, \mu) - \mathcal{F}(V^0, \alpha^0, 0, 0),$$

and further as

$$\begin{aligned} & (V - V^0, \alpha - \alpha^0) - D\mathcal{F}^0(V - V^0, \alpha - \alpha^0) \\ &= \mathcal{F}(V, \alpha, \kappa, \mu) - \mathcal{F}(V^0, \alpha^0, 0, 0) - D\mathcal{F}^0(V - V^0, \alpha - \alpha^0). \end{aligned}$$

At this point, we will show invertibility of $I - D\mathcal{F}^0$. This step is fairly non-trivial, as it amounts to working with an elliptic operator with an unfavorable sign for the maximum principle. Inverting $I - D\mathcal{F}^0$, we obtain

$$\begin{aligned} & (V - V^0, \alpha - \alpha^0) \\ &= (I - D\mathcal{F}^0)^{-1}[\mathcal{F}(V, \alpha, \kappa, \mu) - \mathcal{F}(V^0, \alpha^0, 0, 0) - D\mathcal{F}^0(V - V^0, \alpha - \alpha^0)]. \end{aligned} \quad (28)$$

As is explained above, we will not attempt to show that the right hand side of (28) is a contraction in $C^{2,\beta}(\overline{B_R}) \times \mathbb{R}$ (as is commonly done in the proof of the implicit function theorem). Instead, we produce a sequence (V_n, α_n) of approximate solutions by iterating the right hand side of (28), and show that:

- (i) The sequence is uniformly bounded and remains near (V^0, α^0) in the space $C^{2,\beta}(\overline{B_R}) \times \mathbb{R}$.
- (ii) $(V_{n+1} - V_n, \alpha_{n+1} - \alpha_n)$ contracts in the weaker space $C^{1,\beta}(\overline{B_R}) \times \mathbb{R}$ as n grows.

Note that the loss of derivative of δS in (27) described above can now be tolerated if the goal is to only get the weaker $C^{1,\beta}(\overline{B_R})$ estimates on $V_{n+1} - V_n$. We will then combine the smooth norm estimates with the rough ones using interpolation of Hölder norms, and conclude that the iteration sequence converges

in $C^{2,\beta'}(\overline{B_R}) \times \mathbb{R}$ for some $0 < \beta' < \beta$, which will be enough to get a solution to our problem.

The whole problem is of quasilinear type, because S has the same regularity as ∇V , which then appears as the coefficient of the second order elliptic term in (22). Of course, first derivatives of S also appear in (22), which seem like fully nonlinear combinations of second derivatives of V . However, when the solutions are close to Lane-Emden, the first derivatives of S are small in size, and can be absorbed by the main elliptic term. The method of combining estimates of smooth norms with rough norms may also prove useful in treating other quasilinear hyperbolic problems.

2.5 Outline of Paper

The rest of the paper will be organized as follows. In Section 3, we obtain the $C^{1,\beta}(\overline{B_R})$ estimates on S by solving (24) using the method of characteristics. Although the solution is easy to write down formally, one cannot automatically get the required estimates, since the characteristic vector field is degenerate near the origin. In order to properly estimate S near the origin, we must prove certain weighted Hölder estimates on the characteristic coordinates. This makes Section 3 rather technical. In Section 4, we prove the invertibility of the linearized operator at the Lane-Emden solution and then construct the iteration map using this inverse. In Section 5, we prove convergence of the iteration scheme in $C^{2,\beta'}(\overline{B_R}) \times \mathbb{R}$ for any $0 < \beta' < \beta$, by first showing uniform boundedness of the smooth norms, and then showing contraction of the rough norms of the iteration sequence. We also show eventually that the solution thus obtained actually lies in the smoother space $C^{2,\beta}(\overline{B_R})$, and is unique.

3 Entropy Estimates

In this section, we derive the $C^{1,\beta}(\overline{B_R})$ estimates for S via the method of characteristics. It is immediate from (24) that the characteristics are the level curves of V . A minor problem arises when one tries to solve (24) with boundary condition (25). When V is perturbed away from V^0 , some of the characteristics near ∂B_R may end on ∂B_R and not on $\{z = 0\}$. In this case, the boundary data (25) on the equatorial plane is not enough to uniquely determine S . To avoid such ambiguity in the definition of S , we solve (24) in a slightly larger ball.

First of all, as in Lemma 2.1, we regard V^0 as defined not only on B_R , but on the entire space $\mathbb{R}^3$. A function $V \in C^{2,\beta}(\overline{B_R})$ defined on B_R can be extended to B_{R+1} in the following way. Let

$$E : C^{2,\beta}(\overline{B_R}) \rightarrow C^{2,\beta}(\overline{B_{R+1}})$$

be a bounded linear extension operator such that the support of Eu is contained in $B_{R+\frac{1}{2}}$ for all $u \in C^{2,\beta}(\overline{B_R})$. We extend V by

$$V_{\text{ext}} = V^0 + E(V - V^0).$$

Such an extended V has the property that if $\|V - V^0\|_{C^{2,\beta}(\overline{B_R})} < \delta$, then

$$\|V_{\text{ext}} - V^0\|_{C^{2,\beta}(\overline{B_{R+1}})} < 2\delta \quad \text{and} \quad V_{\text{ext}} \equiv V^0 \quad \text{on} \quad \overline{B_{R+1}} \setminus B_{R+\frac{1}{2}}.$$

Slightly abusing the notation, we still denote V_{ext} by V . As is explained in the discussion following Lemma 2.1, s_0 and ω^2 can be regarded as defined on B_{R+1} . Then we define S to be the solution to (24) and (25) on B_{R+1} . Since $V = V^0$ outside $B_{R+\frac{1}{2}}$, none of the characteristics will penetrate ∂B_{R+1} , and S is uniquely defined.

We are now ready to state the main result of this section.

Theorem 3. *Fix $\beta \in (0, 1)$. Let ω^2 and s_0 be prepared as above. Assume $s_0 \in C^{1,\beta}(\overline{B_{R+1}} \cap \{z = 0\})$ and $\omega^2 \in C^{2,\beta}(\overline{B_{R+1}})$.*

(i) Then there exists $\delta > 0$ such that for every V with $\|V - V^0\|_{C^{2,\beta}(\overline{B_R})} < \delta$, and extended to B_{R+1} as explained above, the equations (24), (25) admit a unique solution $S \in C^{1,\beta}(\overline{B_{R+1}})$. Moreover, there is a radius $R_1 \in (R^0, R)$ such that $S(r, z) \equiv 1$ for $|(r, z)| > R_1$.

(ii) If $|\kappa| + |\mu| < 1$, then there exists a constant $C > 0$ such that the function S satisfies

$$\|S - 1\|_{C^{1,\beta}(\overline{B_{R+1}})} < C(|\kappa| + |\mu|). \quad (29)$$

In order to prove Theorem 3, we will of course construct the solution S by the method of characteristics. The characteristic ODEs associated with the transport equation (24) are given by

$$\begin{cases} \frac{\partial r}{\partial t}(t, \tau) = V_z(r(t, \tau), z(t, \tau)), \\ \frac{\partial z}{\partial t}(t, \tau) = -V_r(r(t, \tau), z(t, \tau)), \\ r(0, \tau) = \tau, \quad z(0, \tau) = 0. \end{cases} \quad (30)$$

We are using t as a parameter along the characteristic curves and τ as a label of the individual curves. (t has nothing to do with time.) These characteristic equations define a mapping $\phi : (t, \tau) \rightarrow (r, z)$.

The remainder of Section 3 constitutes the proof of Theorem 3. It involves technical estimates of S by using the characteristic coordinates (t, τ) . They may be skipped on first reading if the reader is only interested in the main existence proof.

3.1 Characteristic Coordinates

We now construct the characteristic coordinates on

$$\Omega = B_{R+1} \cap \{r > 0, z > 0\}, \quad (31)$$

which amounts to the construction of the diffeomorphism which maps a proper open set in (t, τ) coordinates onto Ω . We will consistently use $\|\cdot\|_k$ and $\|\cdot\|_{k,\beta}$ to denote the $C^k(\overline{B_R})$ and $C^{k,\beta}(\overline{B_R})$ norms. We begin by constructing the mapping ϕ on a precise set.

Proposition 3.1. *There exists $\delta > 0$ sufficiently small such that, for any function V for which $\|V - V^0\|_2 < \delta$, there exists an open subset $O \subset \{(t, \tau) \mid t > 0, 0 < \tau < R+1\}$ and a C^1 diffeomorphism $\tilde{\phi} : O \rightarrow \Omega$ such that $(r(t, \tau), z(t, \tau)) = \tilde{\phi}(t, \tau)$ extends to a C^1 map on $\bar{O} \setminus \{\tau = 0\}$, and solves (30).*

We will prove this proposition by means of a few lemmas. We first write down the trivial coordinates (r^0, z^0) induced by the nonrotating V^0 .

Lemma 3.1. *Let $(r^0(t, \tau), z^0(t, \tau)) = \phi^0(t, \tau)$ be the solution to (30) for the function $V = V^0(\sqrt{r^2 + z^2})$. Then*

$$(r^0(t, \tau), z^0(t, \tau)) = \tau \left(\cos \left(-\frac{(V^0)'(\tau)t}{\tau} \right), \sin \left(-\frac{(V^0)'(\tau)t}{\tau} \right) \right) \quad (32)$$

with its corresponding domain

$$O^0 = \left\{ (t, \tau), 0 < \tau < R+1, 0 < t < \frac{\pi\tau}{-2(V^0)'(\tau)} \right\}.$$

In particular, $|\phi^0(t, \tau)| = \tau$.

Proof. Since $V^0(r, z) = V^0(\sqrt{r^2 + z^2})$, we can easily check that (30) leads to $\frac{\partial}{\partial t} [(r^0)^2 + (z^0)^2] = 0$, which in turn implies $(r^0)^2 + (z^0)^2 = \tau^2$. Hence $r^0(t, \tau) = \tau \cos \theta(t)$ and $z^0(t, \tau) = \tau \sin \theta(t)$ for some $\theta(t)$. The first equation of (30) gives $\frac{d\theta}{dt} = -\frac{(V^0)'(\tau)}{\tau}$. Since $r^0(0, \tau) = \tau$, $\theta(0) = 0$. Thus $\theta(t) = -\frac{(V^0)'(\tau)}{\tau}t$. In order to cover the first quadrant in r^0, z^0 coordinates, we want $0 < \theta(t) < \frac{\pi}{2}$. This verifies the domain O^0 specified above. The last assertion trivially follows from $(r^0)^2 + (z^0)^2 = \tau^2$. $\square$

For general V , the domain will be distorted from O^0 . For convenience, we will actually construct the map ϕ to a slightly extended domain O^a such that $\phi(t, \tau) = (r(t, \tau), z(t, \tau))$ solves (30). To this end, we define

$$O^a = \left\{ (t, \tau), 0 < \tau < R+1, -\frac{\pi}{4a} < t < \frac{\pi\tau}{-2(V^0)'(\tau)} + \frac{\pi}{4a} \right\}$$

where

$$a = \sup_{0 < \tau < R+1} \frac{-(V^0)'(\tau)}{\tau} > 0$$

by Lemma 2.1.

Lemma 3.2. *Define ϕ to be the solution to (30). Then ϕ is a C^1 map on O^a , with a range avoiding the third quadrant, and there exists a constant $C > 0$ such that for $(t, \tau), (t_1, \tau), (t_2, \tau), (t, \tau_1), (t, \tau_2) \in O^a$,*

$$|\phi(t, \tau) - \phi^0(t, \tau)| \leq C\|V - V^0\|_2 t \tau, \quad (33)$$

$$\begin{aligned}
& |\phi(t_1, \tau) - \phi^0(t_1, \tau) - \phi(t_2, \tau) + \phi^0(t_2, \tau)| \\
& \leq C \|V - V^0\|_2 |\phi^0(t_1, \tau) - \phi^0(t_2, \tau)|, \quad (34)
\end{aligned}$$

$$\begin{aligned}
& |\phi(t, \tau_1) - \phi^0(t, \tau_1) - \phi(t, \tau_2) + \phi^0(t, \tau_2)| \\
& \leq C \|V - V^0\|_2 |\phi^0(t, \tau_1) - \phi^0(t, \tau_2)|. \quad (35)
\end{aligned}$$

Remark 3.1. Lemma 3.2 concerns solutions to (30) starting at $(\tau, 0)$, but similar estimates hold for solutions starting at a general point (r, z) in O^a .

Remark 3.2. An obvious consequence of (33) is that

$$\frac{1}{1 + C\delta} \tau \leq |\phi(t, \tau)| \leq (1 + C\delta) \tau \quad (36)$$

when $\|V - V^0\|_2 < \delta$.

Proof. We begin by noting that the solution to (30) is defined for all t if $0 < \tau < R + 1$, because the characteristics are confined to a compact region $\overline{B_{R+1}}$. ϕ is a C^1 map as $\nabla V \in C^1$. We denote $\phi^0 = (r^0, z^0)$.

Proof of (33). Writing

$$\begin{aligned}
\frac{\partial r}{\partial t}(t, \tau) - \frac{\partial r^0}{\partial t}(t, \tau) &= V_z(r, z) - (V^0)_z(r^0, z^0) \\
&= V_z(r, z) - V_z(r^0, z^0) + (V - V^0)_z(r^0, z^0),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial t}(t, \tau) - \frac{\partial z^0}{\partial t}(t, \tau) &= V_r(r, z) - (V^0)_r(r^0, z^0) \\
&= V_r(r, z) - V_r(r^0, z^0) + (V - V^0)_r(r^0, z^0),
\end{aligned}$$

and $\nabla(V - V^0)(r^0, z^0) = \nabla(V - V^0)(r^0, z^0) - \nabla(V - V^0)(0, 0)$, we can integrate and estimate

$$\begin{aligned}
|\phi(t, \tau) - \phi^0(t, \tau)| &\leq 2\|V\|_2 \int_0^t |\phi(s, \tau) - \phi^0(s, \tau)| \, dt \\
&\quad + 2\|V - V^0\|_2 \int_0^t |\phi^0(s, \tau)| \, ds. \quad (37)
\end{aligned}$$

By Gronwall's inequality, this implies

$$|\phi(t, \tau) - \phi^0(t, \tau)| \leq 2e^{2\|V\|_2 t} \|V - V^0\|_2 \tau t \leq C \|V - V^0\|_2 \tau t. \quad (38)$$

Here we have used $|\phi^0(s, \tau)| = \tau$ as shown in Lemma 3.1.

Proof of (34). We write

$$\begin{aligned}
& \frac{\partial}{\partial t} (r(t, \tau) - r^0(t, \tau) - r(t_1, \tau) + r^0(t_1, \tau)) \\
&= V_z(r, z) - (V^0)_z(r^0, z^0) \\
&= V_z(r, z) - V_z(r^0, z^0) + (V - V^0)_z(r^0, z^0),
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial t} (z(t, \tau) - z^0(t, \tau) - z(t_1, \tau) + z^0(t_1, \tau)) \\
&= V_r(r, z) - (V^0)_r(r^0, z^0) \\
&= V_r(r, z) - V_r(r^0, z^0) + (V - V^0)_r(r^0, z^0),
\end{aligned}$$

and use

$$\begin{aligned}
|\nabla V(r, z) - \nabla V(r^0, z^0)| &\leq \|V\|_2 |\phi(t, \tau) - \phi^0(t, \tau)| \\
&\leq \|V\|_2 |\phi(t, \tau) - \phi^0(t, \tau) - \phi(t_1, \tau) + \phi^0(t_1, \tau)| \\
&\quad + \|V\|_2 |\phi(t_1, \tau) - \phi^0(t_1, \tau)| \\
&\leq \|V\|_2 |\phi(t, \tau) - \phi^0(t, \tau) - \phi(t_1, \tau) + \phi^0(t_1, \tau)| \\
&\quad + C\|V - V^0\|_2 \tau.
\end{aligned}$$

Here we have used (33) on the last term. As a consequence,

$$\begin{aligned}
& |\phi(t, \tau) - \phi^0(t, \tau) - \phi(t_1, \tau) + \phi^0(t_1, \tau)| \\
&\leq C\|V\|_2 \int_{t_1}^t |\phi(s, \tau) - \phi^0(s, \tau) - \phi(t_1, \tau) + \phi^0(t_1, \tau)| ds \\
&\quad + C\|V - V^0\|_2 \tau(t - t_1).
\end{aligned}$$

Now we use Gronwall on $[t_1, t_2]$ to get

$$\begin{aligned}
& |\phi(t_1, \tau) - \phi^0(t_1, \tau) - \phi(t_2, \tau) + \phi^0(t_2, \tau)| \\
&\leq C\|V - V^0\|_2 \tau(t_2 - t_1).
\end{aligned} \tag{39}$$

Finally, observe that the range of t_1, t_2 and (32) imply

$$\tau(t_2 - t_1) \leq C|\phi^0(t_1, \tau) - \phi^0(t_2, \tau)|, \tag{40}$$

and the proof of (34) is complete.

Proof of (35). To prove (35), we consider the equations satisfied by ϕ_τ :

$$\begin{cases} \frac{\partial r_\tau}{\partial t} = V_{zr}(r, z)r_\tau + V_{zz}(r, z)z_\tau, \\ \frac{\partial z_\tau}{\partial t} = -V_{rr}(r, z)r_\tau - V_{rz}(r, z)z_\tau, \\ r_\tau(0, \tau) = 1, \quad z_\tau(0, \tau) = 0, \end{cases} \tag{41}$$

and similar equations satisfied by ϕ_τ^0 with V^0 replacing V . Taking differences of the ϕ_τ equations and ϕ_τ^0 equations as before, we get

$$\begin{aligned}
\left| \frac{\partial}{\partial t} (\phi_\tau(t, \tau) - \phi_\tau^0(t, \tau)) \right| &\leq C\|V\|_2 |\phi_\tau - \phi_\tau^0| + C\|V - V^0\|_2 \\
&\quad + C\|V^0\|_3 |\phi - \phi^0| \\
&\leq C\|V\|_2 |\phi_\tau - \phi_\tau^0| + C\|V - V^0\|_2.
\end{aligned} \tag{42}$$

Here we have used (33) to absorb the last term by the second term. We integrate and use Gronwall as before to conclude

$$|\phi_\tau(t, \tau) - \phi_\tau^0(t, \tau)| \leq C \|V - V^0\|_2. \quad (43)$$

We integrate in τ on $[\tau_1, \tau_2]$ to get

$$|\phi(t, \tau_1) - \phi^0(t, \tau_1) - \phi(t, \tau_2) + \phi^0(t, \tau_2)| \leq C \|V - V^0\|_2 |\tau_1 - \tau_2|.$$

Finally observing

$$|\tau_1 - \tau_2| \leq |\phi^0(t, \tau_1) - \phi^0(t, \tau_2)|$$

completes the proof. $\square$

Proof of Proposition 3.1. First we will show that ϕ is one-to-one on O^a ; secondly we will show that $\phi(O^a)$ covers Ω ; and finally we will show that ϕ has non-zero Jacobian on O^a . Then we will just define $\tilde{\phi}$ to be the restriction of ϕ to $\phi^{-1}(\Omega)$.

Step 1: ϕ is 1-1 on O^a . We begin by observing that $\phi(t_1, \tau_1) \neq \phi(t_2, \tau_2)$ whenever $\tau_1 \neq \tau_2$. Suppose this were false and $t_1 \geq t_2$. Then $\phi(t_1, \tau_1) = \phi(t_2, \tau_2)$, $\tau_1 \neq \tau_2$ and $t_1 \geq t_2$. Tracing the solution backwards in t , we would have $\phi(t_1 - t_2, \tau_1) = \phi(0, \tau_2) = (\tau_2, 0)$ for $t_1 - t_2 \geq 0$. This is not possible, because the solution never enters the third quadrant of the (r, z) plane and can only go from the fourth quadrant to the first quadrant by the positivity of $-V_r$ on the positive r -axis. To see the latter fact, write $V_r = (V^0)_r + (V - V^0)_r$ and observe that $\inf_{0 < r < R+1} \frac{-(V^0)'(r)}{r} > 0$ and

$$|(V - V^0)_r(r, 0)| = |(V - V^0)_r(r, 0) - (V - V^0)_r(0, 0)| \leq \|V - V^0\|_2 \cdot r. \quad (44)$$

Thus

$$-V_r(r, 0) > 0 \quad \text{for } 0 < r < R + 1 \quad (45)$$

if $\|V - V^0\|_2$ is sufficiently small.

Now we consider $\phi(t_1, \tau)$ and $\phi(t_2, \tau)$ with $t_1 \neq t_2$. We obviously have $\phi^0(t_1, \tau) \neq \phi^0(t_2, \tau)$. Inequality (34) now implies $\phi(t_1, \tau) \neq \phi(t_2, \tau)$.

Step 2: $\Omega \subset \phi(O^a)$. We pick $(r_0, z_0) \in \Omega$. Let $\tau_0 = \sqrt{r_0^2 + z_0^2}$. Denote the polar coordinates of (r_0, z_0) by (τ_0, θ_0) . We solve the ODE in (30) backwards in t for an interval of length $t_0 = \frac{\theta\tau_0}{-(V^0)'(\tau_0)} + \frac{\pi}{8a}$, with initial value at (τ_0, θ_0) . If $V = V^0$, the solution reaches the final point with polar coordinates $(\tau_0, \theta_f) = (\tau_0, -\frac{-(V^0)'(\tau_0)}{\tau_0} \frac{\pi}{8a})$. Note that since $\inf_{0 < \tau < R+1} \frac{-(V^0)'(\tau)}{\tau} = b > 0$, we have $|\theta_f| > \frac{\pi b}{8a}$. We now use Remark 3.1 and use a modified version of (33) with starting point at (τ_0, θ_0) to conclude that for a general V , we have

$$|(r, z)(-t_0) - \tau_0(\cos \theta_f, \sin \theta_f)| \leq C \|V - V^0\|_2 \cdot \tau_0. \quad (46)$$

Together with the lower bound on $|\theta_t|$, this implies that $(r, z)(-t_0)$ is in the fourth quadrant, as long as $\|V - V^0\|_2$ is sufficiently small. Since the only way to enter the fourth quadrant is to pass the positive r -axis, we conclude that the solution $(r, z)(t)$ must intersect the segment $0 < r < R + 1$ at some value $r = \tau_1$ for some $t = -t_1 \in (-t_0, 0)$. By uniqueness, $\phi(t_1, \tau_1) = (r_0, z_0)$. It remains to show that $(t_1, \tau_1) \in O^a$. In fact, we use Remark 3.1 again to get

$$|(\tau_1, 0) - \tau_0(\cos \theta_1, \sin \theta_1)| \leq C\|V - V^0\|_2 \cdot \tau_0, \quad (47)$$

where θ_1 is the polar angle of $(r, z)(-t_1)$ when $V = V^0$. This implies

$$|\tau_0 - \tau_1| \leq C\|V - V^0\|_2. \quad (48)$$

By the smoothness of V^0 , we have

$$\left| \frac{\tau_0}{(V^0)'(\tau_0)} - \frac{\tau_1}{(V^0)'(\tau_1)} \right| \leq C|\tau_0 - \tau_1| \leq C\|V - V^0\|_2. \quad (49)$$

We now note that

$$\begin{aligned} 0 < t_1 < t_0 < \frac{\pi\tau}{-2(V^0)'(\tau)} + \frac{\pi}{8a} &\leq C\|V - V^0\|_2 + \frac{\pi\tau_1}{-2(V^0)'(\tau_1)} + \frac{\pi}{8a} \\ &\leq \frac{\pi\tau_1}{-2(V^0)'(\tau_1)} + \frac{\pi}{4a}, \end{aligned} \quad (50)$$

provided $\|V - V^0\|_2$ is sufficiently small. Thus $(t_1, \tau_1) \in O^a$.

Step 3: ϕ has non-zero Jacobian on O^a . We now compute the Jacobian of $\phi(t, \tau) = (r(t, \tau), z(t, \tau))$.

$$\begin{aligned} r_t z_\tau - r_\tau z_t &= V_z(r(t, \tau), z(t, \tau))z_\tau + V_r(r(t, \tau), z(t, \tau))r_\tau \\ &= \frac{d}{d\tau} V(r(t, \tau), z(t, \tau)). \end{aligned} \quad (51)$$

On the other hand,

$$\frac{d}{dt} V(r(t, \tau), z(t, \tau)) = V_r r_t + V_z z_t = V_r V_z - V_z V_r = 0, \quad (52)$$

so that

$$V(r(t, \tau), z(t, \tau)) = V(r(0, \tau), z(0, \tau)) = V(\tau, 0). \quad (53)$$

Thus

$$r_t z_\tau - r_\tau z_t = \frac{d}{d\tau} V(r(t, \tau), z(t, \tau)) = V_r(\tau, 0) < 0 \quad (54)$$

by (45). We finish the proof by defining $O = \phi^{-1}(\Omega)$, and $\tilde{\phi} = \phi|_O$. $\square$

Our next task is to derive some Hölder type estimates of the first derivatives of the characteristic coordinates.

3.2 Regularity of the Characteristics

Lemma 3.3. Fix $\beta \in (0, 1)$. Let $\|V - V^0\|_{2,\beta}$ be sufficiently small and $\phi : O \rightarrow \Omega$ be the diffeomorphism given in Proposition 3.1 and let $(t, \tau), (t_1, \tau_1), (t_2, \tau_2) \in O$, $(r, z), (r_1, z_1), (r_2, z_2) \in \Omega$. Denote $\phi^{-1}(r, z) = (t(r, z), \tau(r, z))$. There exists a constant $C > 0$ such that

$$|\phi_t(t, \tau)| \leq C\tau, \quad |\phi_\tau(t, \tau)| \leq C, \quad (55)$$

$$|\nabla t(r, z)| \leq \frac{C}{\sqrt{r^2 + z^2}}, \quad |\nabla \tau(r, z)| \leq C. \quad (56)$$

$$|\phi_\tau(t_1, \tau_1) - \phi_\tau(t_2, \tau_2)| \leq C(|t_1 - t_2| + |\tau_1 - \tau_2|^\beta), \quad (57)$$

$$\left| \frac{1}{\tau_1} \phi_t(t_1, \tau_1) - \frac{1}{\tau_2} \phi_t(t_2, \tau_2) \right| \leq C(|t_1 - t_2|^\beta + |\tau_1 - \tau_2|^\beta), \quad (58)$$

$$|(r_1^2 + z_1^2)^{\frac{1+\beta}{2}} \nabla t(r_1, z_1) - (r_2^2 + z_2^2)^{\frac{1+\beta}{2}} \nabla t(r_2, z_2)| \leq C|(r_1, z_1) - (r_2, z_2)|^\beta, \quad (59)$$

$$\left| (r_1^2 + z_1^2)^{\frac{\beta}{2}} \nabla \tau(r_1, z_1) - (r_2^2 + z_2^2)^{\frac{\beta}{2}} \nabla \tau(r_2, z_2) \right| \leq C|(r_1, z_1) - (r_2, z_2)|^\beta \quad (60)$$

Proof. **Proof of (55).** By (30) and (36),

$$|\phi_t(t, \tau)| \leq \|V\|_2 |\phi(t, \tau)| \leq C\|V\|_2 \tau.$$

The bound on ϕ_τ is easily obtained from (41). In fact, (43) gives such a bound.

Proof of (56). To estimate $\nabla \phi^{-1}$, we note that

$$t_r = \frac{z_\tau}{r_t z_\tau - r_\tau z_t}, \quad t_z = \frac{-r_\tau}{r_t z_\tau - r_\tau z_t}, \quad (61)$$

$$\tau_r = \frac{-z_t}{r_t z_\tau - r_\tau z_t}, \quad \tau_z = \frac{r_t}{r_t z_\tau - r_\tau z_t}. \quad (62)$$

Let us estimate t_r . First recall (54) and write

$$t_r = \frac{z_\tau}{\frac{1}{\tau} V_r(\tau, 0)} \frac{1}{\tau}. \quad (63)$$

Now $|z_\tau| \leq C$ by (55), and since $(V_r - V_r^0)(0, 0) = 0$,

$$\frac{1}{\tau} V_r(\tau, 0) = \frac{(V^0)_r(\tau, 0)}{\tau} + \int_0^1 (V - V^0)_{rr}(s\tau, 0) ds.$$

Thus

$$\left| \frac{V_r(\tau, 0)}{\tau} - \frac{(V^0)_r(\tau, 0)}{\tau} \right| \leq \|V - V^0\|_2 \quad (64)$$

Since $\inf_{0 < \tau < R+1} \frac{-(V^0)_r(\tau, 0)}{\tau} = b > 0$, we have $\left| \frac{V_r(\tau, 0)}{\tau} \right| > \frac{b}{2} > 0$ when $\|V - V^0\|_2$ is sufficiently small. We deduce from (63), (55) and (36) that

$$|t_r| \leq \frac{C}{\tau} \leq \frac{C}{|\phi(t, \tau)|} = \frac{C}{\sqrt{r^2 + z^2}}. \quad (65)$$

The expressions t_z, τ_r, τ_z can be estimated similarly. The only difference for τ_r, τ_z is that the numerators in (62) involve ϕ_t , which can be estimated with an extra factor of τ , as is seen in (55).

Proof of (57). We next obtain Hölder estimates of the first derivatives of ϕ and ϕ^{-1} . It is easy to see that one just needs to prove it first for the case $\tau_1 = \tau_2$, then for the case $t_1 = t_2$. Assuming for the moment that $\tau_1 = \tau_2 = \tau$, we note that $\phi_\tau(t_2, \tau) - \phi_\tau(t_1, \tau) = \int_{t_1}^{t_2} \phi_{t\tau}(s, \tau) ds$ and use (41) for $\phi_{t\tau}$ to get

$$|\phi_\tau(t_2, \tau) - \phi_\tau(t_1, \tau)| \leq \|V\|_2 \|\phi_\tau\|_0 |t_2 - t_1| \leq C |t_2 - t_1|.$$

We assume $t_1 = t_2 = t$ and use (41) again to get

$$\begin{aligned} \left| \frac{\partial}{\partial t} (\phi_\tau(t, \tau_1) - \phi_\tau(t, \tau_2)) \right| &\leq \|V\|_2 |\phi_\tau(t, \tau_1) - \phi_\tau(t, \tau_2)| \\ &\quad + \|V\|_{2,\beta} |\phi(t, \tau_1) - \phi(t, \tau_2)|^\beta \\ &\leq \|V\|_2 |\phi_\tau(t, \tau_1) - \phi_\tau(t, \tau_2)| \\ &\quad + C \|V\|_{2,\beta} |\tau_1 - \tau_2|^\beta. \end{aligned}$$

Here the bound on $\phi(t, \tau_1) - \phi(t, \tau_2)$ can be obtained by using (55) as

$$|\phi(t, \tau_1) - \phi(t, \tau_2)| = \left| \int_{\tau_2}^{\tau_1} \phi_\tau(t, s) ds \right| \leq C |\tau_1 - \tau_2|.$$

Now we integrate and use Gronwall to conclude

$$|\phi_\tau(t, \tau_1) - \phi_\tau(t, \tau_2)| \leq C |\tau_1 - \tau_2|^\beta.$$

This proves (57).

Proof of (58). Let us now compute

$$\begin{aligned} \frac{1}{\tau} r_t(t, \tau) &= \frac{1}{\tau} V_z(\phi(t, \tau)) \\ &= \int_0^1 [V_{zr}(\phi(t, s\tau)) r_\tau(t, s\tau) + V_{zz}(\phi(t, s\tau)) z_\tau(t, s\tau)] ds, \end{aligned} \quad (66)$$

for which we used $V_z(\phi(t, 0)) = 0$. It follows that

$$\begin{aligned} &\frac{1}{\tau_1} r_t(t_1, \tau_1) - \frac{1}{\tau_2} r_t(t_2, \tau_2) \\ &= \int_0^1 [V_{zr}(\phi(t_1, s\tau_1)) r_\tau(t_1, s\tau_1) - V_{zr}(\phi(t_2, s\tau_2)) r_\tau(t_2, s\tau_2)] ds \\ &\quad + \int_0^1 [V_{zz}(\phi(t_1, s\tau_1)) z_\tau(t_1, s\tau_1) - V_{zz}(\phi(t_2, s\tau_2)) z_\tau(t_2, s\tau_2)] ds, \end{aligned}$$

We can write similar expressions for $\frac{1}{\tau}z_t$. Inequality (58) now follows from the following estimates:

$$\begin{aligned}
|\nabla^2 V(\phi(t_1, s\tau_1)) - \nabla^2 V(\phi(t_2, s\tau_2))| &\leq \|V\|_{2,\beta} |\phi(t_1, s\tau_1) - \phi(t_2, s\tau_2)|^\beta \\
&\leq C \|\nabla \phi\|_0^\beta (|t_1 - t_2| + |\tau_1 - \tau_2|)^\beta \\
&\leq C (|t_1 - t_2| + |\tau_1 - \tau_2|)^\beta, \\
|\phi_\tau(t, s\tau)| &\leq C, \\
|\phi_\tau(t_1, s\tau_1) - \phi_\tau(t_2, s\tau_2)| &\leq C (|t_1 - t_2| + |\tau_1 - \tau_2|)^\beta.
\end{aligned}$$

Note that the last estimate above is the previously proven (57).

Proof of (59). We now recall (63) to get

$$(r^2 + z^2)^{\frac{1+\beta}{2}} t_r(r, z) = \frac{(r^2 + z^2)^{\frac{1+\beta}{2}}}{\tau} \frac{z_\tau}{\int_0^1 V_{rr}(s\tau, 0) ds}. \quad (67)$$

Denoting by $(t_i, \tau_i) = \phi^{-1}(r_i, z_i)$, $i = 1, 2$, we have

$$\begin{aligned}
&(r_1^2 + z_1^2)^{\frac{1+\beta}{2}} t_r(r_1, z_1) - (r_2^2 + z_2^2)^{\frac{1+\beta}{2}} t_r(r_2, z_2) \\
&= \frac{|\phi(t_1, \tau_1)|^{1+\beta}}{\tau_1} \frac{z_\tau(t_1, \tau_1)}{\int_0^1 V_{rr}(s\tau_1, 0) ds} - \frac{|\phi(t_2, \tau_2)|^{1+\beta}}{\tau_2} \frac{z_\tau(t_2, \tau_2)}{\int_0^1 V_{rr}(s\tau_2, 0) ds} \\
&= \frac{|\phi(t_1, \tau_1)|^{1+\beta}}{\tau_1} \left(\frac{z_\tau(t_1, \tau_1)}{\int_0^1 V_{rr}(s\tau_1, 0) ds} - \frac{z_\tau(t_2, \tau_2)}{\int_0^1 V_{rr}(s\tau_2, 0) ds} \right) \quad (68)
\end{aligned}$$

$$+ \frac{z_\tau(t_2, \tau_2)}{\int_0^1 V_{rr}(s\tau_2, 0) ds} \left(\frac{|\phi(t_1, \tau_1)|^{1+\beta}}{\tau_1} - \frac{|\phi(t_2, \tau_2)|^{1+\beta}}{\tau_2} \right). \quad (69)$$

To estimate (68), we use (36) for the size of $\phi(t_1, \tau_1)$, (55), (57) for $z_\tau(t_1, \tau_1) - z_\tau(t_2, \tau_2)$, and Hölder estimates on V_{rr} for the integrand to infer that the size of (68) is bounded by

$$C\tau_1^\beta (|t_1 - t_2| + |\tau_1 - \tau_2|)^\beta. \quad (70)$$

Without loss of generality, assume $0 < |(r_1, z_1)| \leq |(r_2, z_2)|$. If $\frac{1}{2}|(r_2, z_2)| \leq |(r_1, z_1)| \leq |(r_2, z_2)|$, use (36), (56) to deduce that (70) is bounded by

$$\begin{aligned}
&C(r_1^2 + z_1^2)^{\frac{\beta}{2}} \left(\frac{1}{\sqrt{r_1^2 + z_1^2}} |(r_1, z_1) - (r_2, z_2)| + |(r_1, z_1) - (r_2, z_2)|^\beta \right) \\
&\leq C |(r_1, z_1) - (r_2, z_2)|^\beta \left(\frac{|(r_1, z_1) - (r_2, z_2)|^{1-\beta}}{|(r_1, z_1)|^{1-\beta}} + |(r_1, z_1)|^\beta \right) \\
&\leq C |(r_1, z_1) - (r_2, z_2)|^\beta. \quad (71)
\end{aligned}$$

If $|r_1| \leq \frac{1}{2}|r_2|$, then $|r_1| \leq |r_1 - r_2|$. In this case, (70) is bounded by

$$C|(r_1, z_1)|^\beta (1 + |(r_1, z_1) - (r_2, z_2)|^\beta) \leq C|(r_1, z_1) - (r_2, z_2)|^\beta.$$

Next we estimate the terms in the parentheses of (69). If $\frac{1}{2}|(r_2, z_2)| \leq |(r_1, z_1)| \leq |(r_2, z_2)|$, we write them as

$$\frac{|(r_1, z_1)|^{1+\beta} - |(r_2, z_2)|^{1+\beta}}{\tau_1} + |(r_2, z_2)|^{1+\beta} \frac{\tau_2 - \tau_1}{\tau_1 \tau_2}, \quad (72)$$

which can be bounded by

$$\begin{aligned} & C \left(\frac{|(r_1, z_1)|^\beta |(r_1, z_1) - (r_2, z_2)|}{|(r_1, z_1)|} + \frac{|(r_1, z_1) - (r_2, z_2)|}{|(r_2, z_2)|^{1-\beta}} \right) \\ & \leq C |(r_1, z_1) - (r_2, z_2)|^\beta \frac{|(r_1, z_1) - (r_2, z_2)|^{1-\beta}}{|(r_1, z_1)|^{1-\beta}} \\ & \leq C |(r_1, z_1) - (r_2, z_2)|^\beta. \end{aligned} \quad (73)$$

If $|r_1, z_1| \leq \frac{1}{2}|r_2, z_2|$, the terms in the parentheses of (69) is bounded by

$$C (|(r_1, z_1)|^\beta + |(r_2, z_2)|^\beta) \leq C |(r_2, z_2)|^\beta \leq C |(r_1, z_1) - (r_2, z_2)|^\beta.$$

The proof of (59) for t_r is now complete. The estimates on t_z are completely analogous.

Proof of (60). We write

$$(r^2 + z^2)^{\frac{\beta}{2}} \tau_r(r, z) = (r^2 + z^2)^{\frac{\beta}{2}} \frac{-\frac{z_t}{\tau}}{\int_0^1 V_{rr}(s\tau, 0) ds} \quad (74)$$

and estimate as before. We only have to use (58) to replace (57) in the argument. $\square$

3.3 Weighted Hölder spaces and S estimates

To facilitate the proof of Theorem 3, we introduce the following weighted Hölder spaces. For simplicity we denote (r, z) by x . Given $\beta \in (0, 1)$, $k \in \mathbb{R}$, for a continuous function f on B_R , we define the norms

$$\|f\|_{C_{(k)}^{0,\beta}} = \sup_{x \in B_R} |x|^k |f(x)| + \sup_{x, y \in B_R, x \neq y} \frac{||x|^{k+\beta} f(x) - |y|^{k+\beta} f(y)|}{|x - y|^\beta}. \quad (75)$$

$$\|f\|_{C_{[k]}^{0,\beta}} = \sup_{x \in B_R} |x|^k |f(x)| + \sup_{x, y \in B_R, x \neq y} \min(|x|^{k+\beta}, |y|^{k+\beta}) \frac{|f(x) - f(y)|}{|x - y|^\beta}. \quad (76)$$

The spaces $C_{(k)}^{0,\beta}$ and $C_{[k]}^{0,\beta}$ are defined accordingly as sets of continuous functions with finite norms. We may now reformulate (56), (59), (60) as

$$\nabla t \in C_{(1)}^{0,\beta}, \quad \nabla \tau \in C_{(0)}^{0,\beta}. \quad (77)$$

The spaces $C_{(k)}^{0,\beta}$ and $C_{[k]}^{0,\beta}$ actually coincide, as we show in the next lemma.

Lemma 3.4. *For each k , there exists a constant $C > 0$ such that*

$$\frac{1}{C} \|f\|_{C_{(k)}^{0,\beta}} \leq \|f\|_{C_{[k]}^{0,\beta}} \leq C \|f\|_{C_{(k)}^{0,\beta}}. \quad (78)$$

Proof. First consider the case that $k + \beta \geq 0$. Assume x, y are in the first octant and $|x| \leq |y|$ without loss of generality, so that $\min(|x|^{k+\beta}, |y|^{k+\beta}) = |x|^{k+\beta}$. We estimate

$$\begin{aligned} & \frac{|x|^{k+\beta} f(x) - |y|^{k+\beta} f(y)}{|x - y|^\beta} \\ &= \frac{|x|^{k+\beta} (f(x) - f(y))}{|x - y|^\beta} + \frac{f(y) (|x|^{k+\beta} - |y|^{k+\beta})}{|x - y|^\beta} \\ &= \text{(I)} + \text{(II)} \end{aligned}$$

We have

$$|\text{I}| \leq \|f\|_{C_{[k]}^{0,\beta}}.$$

If $\frac{|y|}{2} \leq |x| \leq |y|$,

$$|\text{II}| \leq C \frac{|f(y)| |y|^{k+\beta-1} |x - y|}{|x - y|^\beta} \leq C \sup_{y \in B_R} |y|^k |f(y)|.$$

If $|x| \leq \frac{|y|}{2}$,

$$|\text{II}| \leq C |f(y)| \frac{|y|^{k+\beta}}{|y|^\beta} \leq C \sup_{y \in B_R} |y|^k |f(y)|.$$

These estimates imply $\|f\|_{C_{(k)}^{0,\beta}} \leq C \|f\|_{C_{[k]}^{0,\beta}}$. The case when $k + \beta < 0$ can be estimated in a similar way by just assuming $|x| \geq |y|$ in the above estimates instead.

Let us now focus on the reverse direction and assume $k + \beta \geq 0$ again. Assuming $|x| \leq |y|$ without loss of generality, we write

$$\begin{aligned} & |x|^{k+\beta} \frac{f(x) - f(y)}{|x - y|^\beta} \\ &= \frac{|x|^{k+\beta} f(x) - |y|^{k+\beta} f(y)}{|x - y|^\beta} + \frac{f(y) (|y|^{k+\beta} - |x|^{k+\beta})}{|x - y|^\beta} \\ &= \text{(I)} + \text{(II)}. \end{aligned}$$

Obviously

$$|\text{I}| \leq \|f\|_{C_{(k)}^{0,\beta}},$$

and (II) can be estimated as above. The case $k + \beta < 0$ is analogous. $\square$

The weighted Hölder spaces $C_{(k)}^{0,\beta}$ enjoy the following algebraic property, which will be useful for the Hölder estimates of the entropy.

Lemma 3.5. For given $\beta \in (0, 1)$, $k, l \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$\|fg\|_{C_{(k+l)}^{0,\beta}} \leq C \|f\|_{C_{(k)}^{0,\beta}} \|g\|_{C_{(l)}^{0,\beta}}. \quad (79)$$

Proof. One obviously has

$$|x|^{k+l} f(x)g(x) \leq \sup_{x \in B_R} |x|^k |f(x)| \cdot \sup_{x \in B_R} |x|^l |g(x)|.$$

On the other hand, for $x, y \in B_R$, $x \neq y$,

$$\begin{aligned} & \frac{|x|^{k+l+\beta} f(x)g(x) - |y|^{k+l+\beta} f(y)g(y)}{|x-y|^\beta} \\ &= \frac{|x|^k f(x) (|x|^{l+\beta} g(x) - |y|^{l+\beta} g(y))}{|x-y|^\beta} + \frac{|y|^l g(y) (|x|^{k+\beta} f(x) - |y|^{k+\beta} f(y))}{|x-y|^\beta} \\ & \quad + |x|^k f(x) |y|^l g(y) \frac{|y|^\beta - |x|^\beta}{|x-y|^\beta}. \end{aligned}$$

It is obvious that every term in the above sum is bounded by $C \|f\|_{C_{(k)}^{0,\beta}} \|g\|_{C_{(l)}^{0,\beta}}$. $\square$

We are now prepared to prove Theorem 3.

Proof of Theorem 3. Note the obvious bound $\|\nabla S\|_{0,\beta} \leq \|\nabla S\|_{C_{(-\beta)}^{0,\beta}}$. Thus the key in the following proof is to obtain $C_{(-\beta)}^{0,\beta}$ estimates on ∇S . Indeed, it is easy to see that the unique solution S to (24), (25) is given by

$$S(\phi(t, \tau)) = e^{\frac{\mu s_0(\tau)}{\gamma}} - \kappa \int_0^t (\omega^2 r)_z(\phi(t', \tau)) dt'. \quad (80)$$

Since κ appears as a linear coefficient in (80) and the same is true for μ in the estimate $|e^{\frac{\mu s_0(\tau)}{\gamma}} - 1| \leq C |\mu| |s_0(\tau)| / \gamma$ for bounded s_0 , it will only be necessary to provide uniform estimates on s_0 and on the integral in (80). Thus (29) will be a consequence of such estimates.

More specifically, (80) defines S in the first quadrant, while the values of S in the other quadrants are determined by symmetry. We observe that $S(r, z) \equiv 1$ near the boundary of B_R . In fact, by the set up of Theorem 3 we know that $s_0(\tau) \equiv 0$ for $\tau > \frac{2R^0 + 4R}{6}$, $\omega^2(r, z) \equiv 0$ for $\sqrt{r^2 + z^2} > \frac{2R^0 + 4R}{6}$, while (36) implies that

$$\sqrt{\frac{2R^0 + 4R}{R^0 + 5R}} \tau < |\phi(t, \tau)| < \sqrt{\frac{R^0 + 5R}{2R^0 + 4R}} \tau \quad (81)$$

if $\|V - V^0\|_{C^2} < \delta$ is sufficiently small. Now if $\sqrt{r^2 + z^2} = |\phi(t, \tau)| > R_1 = \frac{R^0 + 5R}{6}$, then $\tau > \sqrt{\frac{2R^0 + 4R}{R^0 + 5R}} R_1 > \frac{2R^0 + 4R}{6}$, and $|\phi(t', \tau)| > \frac{2R^0 + 4R}{R^0 + 5R} R_1 = \frac{2R^0 + 4R}{6}$. As a consequence, we see that $S(r, z) = 1$ for $\sqrt{r^2 + z^2} > R_1$.

By Proposition 3.1, ϕ is a C^1 diffeomorphism. Thus $S(r, z)$ is obviously C^1 on Ω . We will now obtain Hölder estimates for the first derivatives. We only show details for the r -derivative because the z -derivative is analogous.

$$S_r(r, z) = \mu e^{\frac{\mu s_0(\tau)}{\gamma}} s'_0(\tau) \tau_r \quad (82)$$

$$- \kappa(\omega^2 r)_z(r, z) t_r \quad (83)$$

$$- \kappa \tau_r \int_0^t [(\omega^2 r)_{zr}(\phi(t', \tau)) r_\tau(t', \tau) + (\omega^2 r)_{zz}(\phi(t', \tau)) z_\tau(t', \tau)] dt'. \quad (84)$$

We estimate each term one by one. First of all,

$$\left| e^{\frac{\mu s_0(\tau(r_1, z_1))}{\gamma}} - e^{\frac{\mu s_0(\tau(r_2, z_2))}{\gamma}} \right| \leq C \|s'_0\|_0 \|\nabla \tau\|_0 |(r_1, z_1) - (r_2, z_2)| \\ \leq C |(r_1, z_1) - (r_2, z_2)|$$

by (56). Thus $e^{\frac{\mu s_0(\tau)}{\gamma}} \in C^{0,1}(\overline{B_R})$. By (77) and Lemma 3.5, we only need to bound $s'_0(\tau(r, z))$ in $C^{0,\beta}_{(-\beta)}$ in order to estimate (82). Indeed, since $s'_0(0) = 0$,

$$(r^2 + z^2)^{-\frac{\beta}{2}} |s'_0(\tau(r, z))| \leq \|s_0\|_{1,\beta} \frac{|\tau(r, z)|^\beta}{(r^2 + z^2)^{\frac{\beta}{2}}} \leq C$$

by (81), and

$$|s'_0(\tau(r_1, z_1)) - s'_0(\tau(r_2, z_2))| \leq C \|s_0\|_{1,\beta} \|\nabla \tau\|_0^\beta |(r_1, z_1) - (r_2, z_2)|^\beta \\ \leq C |(r_1, z_1) - (r_2, z_2)|^\beta.$$

In order to estimate (83), we must bound $(\omega^2 r)_z$ in $C^{0,\beta}_{(-1-\beta)}$, as can be seen from (77) and Lemma 3.5. By Lemma 3.4, we may compute the $C^{0,\beta}_{[-1-\beta]}$ norm of $(\omega^2 r)_z$ instead. In fact, we show that it belongs to $C^{0,1}_{[-2]} \subset C^{0,\beta}_{[-1-\beta]}$, as follows.

$$|(r, z)|^{-2} |(\omega^2)_z(r, z) r| \leq \|\omega^2\|_2 \frac{r \sqrt{r^2 + z^2}}{r^2 + z^2} \leq C.$$

Let $r_1^2 + z_1^2 \geq r_2^2 + z_2^2$.

$$|(r_1, z_1)|^{-1} \frac{|(\omega^2)_z(r_1, z_1) r_1 - (\omega^2)_z(r_2, z_2) r_2|}{|(r_1, z_1) - (r_2, z_2)|} \\ \leq (r_1^2 + z_1^2)^{-1/2} \|\omega^2\|_2 \sqrt{r_1^2 + z_1^2} \\ \leq C.$$

Now to estimate (84), we must bound the integral term in $C^{0,\beta}_{(-\beta)}$. Indeed,

$$|(\omega^2 r)_{zr}(\phi(t', \tau))| \leq \|\omega^2\|_2 |\phi(t', \tau(r, z))| \leq C \tau(r, z) \leq C |(r, z)| \quad (85)$$

by (81). The same estimate holds for $(\omega^2 r)_{zz}$. Together with (55), this implies that the integral in (84) is bounded by $C|(r, z)| \leq C|(r, z)|^\beta$.

We now turn to the Hölder estimate on the integral. Denote $(t_i, \tau_i) = \phi^{-1}(r_i, z_i)$, $i = 1, 2$. We assume $|(r_1, z_1)| \leq |(r_2, z_2)|$ and write

$$\begin{aligned} & \int_0^{t_1} [(\omega^2 r)_{zr}(\phi(t', \tau_1))r_\tau(t', \tau_1) + (\omega^2 r)_{zz}(\phi(t', \tau_1))z_\tau(t', \tau_1)] dt' \\ & - \int_0^{t_2} [(\omega^2 r)_{zr}(\phi(t', \tau_2))r_\tau(t', \tau_2) + (\omega^2 r)_{zz}(\phi(t', \tau_2))z_\tau(t', \tau_2)] dt' \\ & = \int_{t_2}^{t_1} [(\omega^2 r)_{zr}(\phi(t', \tau_1))r_\tau(t', \tau_1) + (\omega^2 r)_{zz}(\phi(t', \tau_1))z_\tau(t', \tau_1)] dt' \end{aligned} \quad (86)$$

$$+ \int_0^{t_2} [(\omega^2 r)_{zr}(\phi(t', \tau_1))r_\tau(t', \tau_1) - (\omega^2 r)_{zr}(\phi(t', \tau_2))r_\tau(t', \tau_2) \quad (87)$$

$$+ (\omega^2 r)_{zz}(\phi(t', \tau_1))z_\tau(t', \tau_1) - (\omega^2 r)_{zz}(\phi(t', \tau_2))z_\tau(t', \tau_2)] dt'. \quad (88)$$

By (85) and a similar estimate on $(\omega^2 r)_{zz}$, (86) is bounded by

$$C|(r_1, z_1)||t_1 - t_2| \leq C|(r_1, z_1)| \frac{|(r_1, z_1) - (r_2, z_2)|}{|(r_1, z_1)|} \leq C|(r_1, z_1) - (r_2, z_2)|,$$

where we have used (56) to estimate ∇t . Now we write the integrand of (87) as

$$[(\omega^2 r)_{zr}(\phi(t', \tau_1)) - (\omega^2 r)_{zr}(\phi(t', \tau_2))]r_\tau(t', \tau_2) \quad (89)$$

$$+ (\omega^2 r)_{zr}(\phi(t', \tau_1))[r_\tau(t', \tau_1) - r_\tau(t', \tau_2)]. \quad (90)$$

By (55) and (56), we deduce that (89) is bounded by

$$\|\omega^2\|_{2,\beta} \|\phi_\tau\|_0^{1+\beta} \|\nabla \tau\|_0^\beta |(r_1, z_1) - (r_2, z_2)|^\beta \leq C|(r_1, z_1) - (r_2, z_2)|^\beta.$$

Also, by (85), (57), (56), we obtain that (90) is bounded by

$$C|(r_1, z_1)||\tau_1 - \tau_2|^\beta \leq C\|\nabla \tau\|_0^\beta |(r_1, z_1) - (r_2, z_2)|^\beta \leq C|(r_1, z_1) - (r_2, z_2)|^\beta.$$

Thus we conclude that (87) is bounded by $C|(r_1, z_1) - (r_2, z_2)|^\beta$. Similar estimates can be obtained for (88). We have shown that the integral in (84) is in $C_{(-\beta)}^{0,\beta}$, and we have completed the $C^{1,\beta}$ estimates of S on $B_R \cap \{r > 0, z > 0\}$.

Finally, in order to extend the $C^{1,\beta}$ estimates of S to B_R , we must show that $S_r(0, z) = 0$, $S_z(r, 0) = 0$. In fact, by (30), (54), (62),

$$\tau_r(r, z) = \frac{V_r(r, z)}{V_r(\tau(r, z), 0)}.$$

So for $\tau > 0$

$$\tau_r(0, z) = \frac{V_r(0, z)}{V_r(\tau(0, z), 0)} = 0$$

by the axisymmetry of V . On the other hand, we obviously have $(\omega^2 r)_z(0, z) = 0$. The formula of S_r given by (82), (83), (84) now shows that $S_r(0, z) = 0$. The proof that $S_z(r, 0) = 0$ is similar. $\square$

4 Construction of the Iteration Map

Recall the definition of the mapping $\mathcal{F}$ in Section 2.4. The next lemma establishes the basic properties of $\mathcal{F}$.

Lemma 4.1. *Let $\beta \in (0, \min(q, 1))$, $s_0 \in C^{1,\beta}(\overline{B_R} \cap \{z = 0\})$, and $\omega^2 \in C^{2,\beta}(\overline{B_R})$. Then there exist $\kappa_0 > 0$, $\mu_0 > 0$ and a neighborhood $B_\delta(V^0)$ of V^0 in $C^{2,\beta}(\overline{B_R})$ such that $\mathcal{F}$ maps $B_\delta(V^0) \times \mathbb{R} \times (-\kappa_0, \kappa_0) \times (-\mu_0, \mu_0)$ into a bounded subset of $C^{2,\beta}(\overline{B_R}) \times \mathbb{R}$.*

Proof. We let $B_\delta(V^0)$ be the neighborhood of V^0 in $C^{2,\beta}(\overline{B_R})$ specified in Theorem 3. By that theorem, for each $V \in B_\delta(V^0)$ there exists a unique solution $S \in C^{1,\beta}(\overline{B_R})$ to (24) and (25). We fix such a pair V and S .

We have to show that (i) the solution $V^\#$ to (22), (23) exists, is unique and $V^\# \in C^{2,\beta}(\overline{B_R})$ and (ii) $\alpha^\#$ given in (26) is a well-defined finite number. The latter claim (ii) is clear since $V_+^q S^{-1}$ is integrable. To verify (i), we invoke the Schauder theory for second-order linear elliptic equations. By (29), if we limit κ_0 and μ_0 suitably, then $S \in C^{1,\beta}(\overline{B_R})$ is bounded away from 0, the function $-4\pi V_+^q S^{-1} + \kappa \nabla \cdot (\omega^2 \text{re}_r)$ belongs to $C^{0,\beta}(\overline{B_R})$, and the function $\frac{1}{|\cdot|} * (V_+^q S^{-1}) + \alpha$ belongs to $C^{2,\beta}(\overline{B_R})$. By Corollary 6.9 of [15], the Dirichlet problem (22), (23) has a unique solution $V^\# \in C^{2,\beta}(\overline{B_R})$ whose $C^{2,\beta}$ norm depends only on $\|S\|_{1,\beta}$, $\|V\|_{2,\beta}$ and $\|\omega^2\|_{2,\beta}$. $\square$

In the next lemma we estimate a typical semilinear term in a Hölder norm.

Lemma 4.2. *Let $U, V \in C^1(\overline{B_R})$ and $\|V\|_1 < 1$. Let $g \in C^{1,\beta_1}([0, \|U\|_0 + 1])$ for some $0 < \beta < \beta_1 < 1$. Then there is a constant $C > 0$ such that*

$$\|g(U + V) - g(U) - g'(U)V\|_0 \leq \|g\|_{1,\beta_1} \|V\|_0^{1+\beta_1}, \quad (91)$$

$$\|g(U + V) - g(U) - g'(U)V\|_{0,\beta} \leq C \|g\|_{1,\beta_1} (\|U\|_1 + 1)^\beta \|V\|_{0,\beta}^{1+\beta_1-\beta}. \quad (92)$$

Proof. Since $g \in C^1$, we can write

$$\begin{aligned} [g(U + V) - g(U) - g'(U)V](x) &= \int_0^1 [g'(U + tV) - g'(U)](x) dt \cdot V(x) \\ &:= I(x) \cdot V(x). \end{aligned} \quad (93)$$

Since $\|U + tV\|_0 \leq \|U\|_0 + 1$ for $\|V\|_1 < 1$, we have

$$\|I\|_0 \leq \|g\|_{1,\beta_1} \|V\|_0^{\beta_1}, \quad (94)$$

so that (91) follows. Also

$$\begin{aligned} [g(U + V) - g(U) - g'(U)V]_{0,\beta} &\leq [I]_{0,\beta} \|V\|_0 + \|I\|_0 [V]_{0,\beta} \\ &\leq [I]_{0,\beta} \|V\|_{0,\beta} + \|g\|_{1,\beta_1} \|V\|_0^{\beta_1} [V]_{0,\beta}. \end{aligned} \quad (95)$$

To estimate $[I]_{0,\beta}$, we compute

$$\begin{aligned}
|I(x) - I(y)| &= \left| \int_0^1 [g'(U(x) + tV(x)) - g'(U(y) + tV(y))] dt \right. \\
&\quad \left. - [g'(U(x)) - g'(U(y))] \right| \\
&\leq \|g\|_{1,\beta_1} (|U(x) + tV(x) - U(y) - tV(y)|^{\beta_1} + |U(x) - U(y)|^{\beta_1}) \\
&\leq C \|g\|_{1,\beta_1} (\|U\|_1 + \|V\|_1)^{\beta_1} |x - y|^{\beta_1}.
\end{aligned} \tag{96}$$

On the other hand,

$$|I(x) - I(y)| \leq 2\|I\|_0 \leq 2\|g\|_{1,\beta_1} \|V\|_0^{\beta_1}. \tag{97}$$

Since $\beta < \beta_1$, we can interpolate between (96) and (97) to get

$$|I(x) - I(y)| \leq C \|g\|_{1,\beta_1} (\|U\|_1 + \|V\|_1)^\beta |x - y|^\beta \|V\|_0^{\beta_1 - \beta}. \tag{98}$$

Thus

$$[I]_{0,\beta} \leq C \|g\|_{1,\beta_1} (\|U\|_1 + 1)^\beta \|V\|_0^{\beta_1 - \beta}, \tag{99}$$

and (92) follows from (95). $\square$

Next we show that the mapping $\mathcal{F}$ is Fréchet differentiable and also that $I - D\mathcal{F}$ is invertible in case $\kappa = \mu = 0$.

Lemma 4.3. *Let $\frac{6}{5} < \gamma < 2$, $\gamma \neq \frac{4}{3}$. Fix $\beta \in (0, \min(q-1, 1))$. Then $\mathcal{F}(V, \alpha, 0, 0)$ is Fréchet differentiable with respect to (V, α) on $C^{2,\beta}(\overline{B_R}) \times \mathbb{R}$, and*

$$\Lambda = I - D_{(V,\alpha)} \mathcal{F}(V^0, \alpha^0, 0, 0) \tag{100}$$

is invertible with a bounded inverse. Furthermore, Λ^{-1} is also bounded in the $C^{1,\beta}(\overline{B_R}) \times \mathbb{R}$ norm, that is,

$$\|\Lambda^{-1}(\delta V, \delta \alpha)\|_{C^{1,\beta}(\overline{B_R}) \times \mathbb{R}} \leq \|(\delta V, \delta \alpha)\|_{C^{1,\beta}(\overline{B_R}) \times \mathbb{R}} \tag{101}$$

for all $(\delta V, \delta \alpha) \in C^{2,\beta}(\overline{B_R}) \times \mathbb{R}$.

Proof. It is easy to see from (24) and (25) that $S \equiv 1$ in case $\kappa = \mu = 0$. Thus by (22), (23), (26), $(V^\#, \alpha^\#) = \mathcal{F}(V, \alpha, 0, 0)$ solves

$$\Delta V^\# = -4\pi V_+^q \quad \text{on } B_R, \tag{102}$$

$$V^\# = \frac{1}{|\cdot|} * V_+^q + \alpha \quad \text{on } \partial B_R, \tag{103}$$

$$\alpha^\# = \alpha + \int_{B_R} V_+^q dx - M. \tag{104}$$

From (102), (103), it follows that $V^\# - \frac{1}{|\cdot|} * V_+^q - \alpha$ is harmonic on B_R with zero boundary values. Thus

$$V^\# = \frac{1}{|\cdot|} * V_+^q + \alpha \quad (105)$$

actually holds in the whole ball B_R . It is then not hard to show using Lemma 4.2 that $(V, \alpha) \mapsto (V^\#, \alpha^\#)$ is Fréchet differentiable with

$$D_{(V, \alpha)} V^\#(\delta V, \delta \alpha) = \frac{q}{|\cdot|} * \left(V_+^{q-1} \delta V \right) + \delta \alpha, \quad (106)$$

$$D_{(V, \alpha)} \alpha^\#(\delta V, \delta \alpha) = \delta \alpha + q \int_{B_R} V_+^{q-1} \delta V \, dx. \quad (107)$$

It is also easy to see that the Fréchet derivative of $(V, \alpha) \mapsto (V^\#, \alpha^\#)$ is compact on $C^{2, \beta}(\overline{B_R}) \times \mathbb{R}$, so that Λ defined by (100) is Fredholm of index zero. As for invertibility, we only need to prove its injectivity. So assume that $(\delta V, \delta \alpha)$ satisfies $\Lambda(\delta V, \delta \alpha) = 0$. That is,

$$\delta V = \frac{q}{|\cdot|} * \left((V^0)_+^{q-1} \delta V \right) + \delta \alpha, \quad (108)$$

$$\int_{B_R} (V^0)_+^{q-1} \delta V \, dx = 0. \quad (109)$$

We must prove that these equations admit only the zero solution. This will follow in a way similar to Lemma 4.3 of [31]; the argument is also related to section 4 of [30].

In fact, we extend δV to the entire $\mathbb{R}^3$ by (108). It follows that

$$\Delta(\delta V) = -4\pi q (V^0)_+^{q-1} \delta V. \quad (110)$$

Let $Y_{lm}(\theta)$ ($l = 0, 1, 2, \dots$, $m = -l, \dots, l$, $\theta \in \mathbb{S}^2$) be the standard spherical harmonics. For any function $f(x)$ with $x \in \mathbb{R}^3$, denote its (l, m) component by

$$f_{lm}(|x|) = \langle f, Y_{lm} \rangle(|x|) = \int_{\mathbb{S}^2} f(|x|\theta) Y_{lm}(\theta) \, d_{\mathbb{S}^2} \theta.$$

We will first show $(\delta V)_{lm} = 0$ if $l \geq 1$, so that only a radial component remains. Indeed, taking the (l, m) component of (110), and using

$$\begin{aligned} \Delta &= \partial_{|x|}^2 + \frac{2}{|x|} \partial_{|x|} + \frac{1}{|x|^2} \Delta_{\mathbb{S}^2}, \\ \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} f) g \, d_{\mathbb{S}^2} \theta &= \int_{\mathbb{S}^2} f (\Delta_{\mathbb{S}^2} g) \, d_{\mathbb{S}^2} \theta, \\ \Delta_{\mathbb{S}^2} Y_{lm} &= -l(l+1) Y_{lm}, \end{aligned}$$

we obtain

$$\Delta[(\delta V)_{lm}] - \frac{l(l+1)}{|x|^2}(\delta V)_{lm} = -4\pi q(V^0)_+^{q-1}(\delta V)_{lm}. \quad (111)$$

Since $(V^0)_+ = 0$ for $|x| > R^0$, we can solve (111) explicitly there and obtain

$$(\delta V)_{lm}(x) = C|x|^{-(l+1)} + D|x|^l. \quad (112)$$

Since δV is bounded near infinity, we obtain

$$(\delta V)_{lm}(x) = C|x|^{-(l+1)} \quad \text{if } |x| \geq R^0. \quad (113)$$

By Lemma 2.1, $(V^0)' = \partial_{|x|}V^0 < 0$ for $|x| > 0$. We define for $|x| > 0$

$$\psi_{lm} = \frac{(\delta V)_{lm}}{(V^0)'}. \quad (114)$$

Note that for $l \geq 1$,

$$\begin{aligned} |\psi_{lm}(x)| &:= \left| \frac{1}{(V^0)'(x)} \int_{\mathbb{S}^2} [\delta V(|x|\theta) - \delta V(0)] Y_{lm}(\theta) d_{\mathbb{S}^2}\theta \right| \\ &\leq \frac{C|x|}{|(V^0)'(x)|} \sup_{|y| \leq |x|} |\nabla(\delta V)(y)|. \end{aligned}$$

Inequality (15) and $\nabla(\delta V)(0) = 0$ imply that

$$\lim_{|x| \rightarrow 0^+} \psi_{lm}(x) = 0. \quad (115)$$

By (14),

$$\Delta V^0 = -4\pi(V^0)_+^q, \quad (116)$$

from which it follows that

$$\Delta[(V^0)'] - \frac{2}{|x|^2}(V^0)' = -4\pi q(V^0)^{q-1}(V^0)'. \quad (117)$$

From (111), (114), (117) we get

$$\Delta\psi_{lm} + \frac{2\nabla(V^0)' \cdot \nabla\psi_{lm}}{(V^0)'} + \frac{2-l(l+1)}{|x|^2}(V^0)'\psi_{lm} = 0. \quad (118)$$

If $l \geq 1$, then $2-l(l+1) \leq 0$. Letting $\Psi = \sup_{0 < |x| < R^0} \psi_{lm}(x) > 0$ and using the strong maximum principle on (118), we know that Ψ cannot be attained at any interior point on the punctured ball $0 < |x| < R^0$. By the condition (115) at the origin, Ψ can only be attained for $|x| = R^0$, so that $\psi_{lm}(R^0) = \Psi$. By the Hopf maximum principle, $\partial_{|x|}\psi_{lm}(R^0) > 0$. Now the C^1 continuity of ψ_{lm} across the surface at $|x| = R^0$ and (113), (114), (116) implies

$$0 < \Psi = \psi_{lm}(R^0) = \frac{C}{(R^0)^{l+1}(V^0)'(R^0)}, \quad (119)$$

$$0 < \partial_{|x|} \psi_{lm}(R^0) = \frac{C(1-l)}{(R^0)^{l+2} (V^0)'(R^0)}. \quad (120)$$

These equations force $1-l > 0$, which contradicts the assumption that $l \geq 1$. We therefore conclude that $\Psi = \sup_{0 < |x| < R^0} \psi_{lm}(x) \leq 0$. By similar reasoning, $\inf_{0 < |x| < R^0} \psi_{lm}(x) \geq 0$. It follows that ψ_{lm} , as well as $(\delta V)_{lm}$, vanish on B_{R^0} . (113) now implies $\delta V_{lm} = 0$ everywhere.

Since δV is now restricted to be a radial function, (110) can be regarded as the following ODE (using $'$ to denote $\partial_{|x|}$):

$$(\delta V)'' + \frac{2}{|x|} (\delta V)' + 4\pi q (V^0)_+^{q-1} \delta V = 0. \quad (121)$$

We also have the obvious condition

$$(\delta V)'(0) = 0 \quad (122)$$

due to symmetry. On the other hand, we note that by scaling symmetry of (116), $V(x; a) = a^{\frac{2}{q-1}} V^0(ax)$ solves

$$\Delta V + 4\pi V_+^q = 0. \quad (123)$$

It follows that $U = \partial_a V(x; 1)$ solves

$$\Delta U + 4\pi q (V^0)_+^{q-1} U = 0,$$

or

$$U'' + \frac{2}{|x|} U' + 4\pi q (V^0)_+^{q-1} U = 0, \quad (124)$$

with

$$U'(0) = 0. \quad (125)$$

So δV and U satisfy the same ODE with vanishing derivative at zero. By uniqueness, it follows that $\delta V = CU$ for some constant multiple C . Also, $V(x; a)$ satisfies

$$\int_{\mathbb{R}^3} V_+^q dx = a^{\frac{3-q}{q-1}} \int_{\mathbb{R}^3} (V^0)_+^q dx. \quad (126)$$

Taking the derivative with respect to a and setting $a = 1$, we get

$$\int_{\mathbb{R}^3} q (V^0)_+^{q-1} U dx = \frac{3-q}{q-1} \int_{\mathbb{R}^3} (V^0)_+^q dx \neq 0 \quad (127)$$

if $q \neq 3$. In this case, (109) and the condition $\delta V = CU$ obtained above imply that $C = 0$. Hence δV and $\delta \alpha$ are zero. The proof is complete as we observe that $q = 3$ if and only if $\gamma = \frac{4}{3}$.

It remains to reprove the lemma in the weaker space $C^{1,\beta}(\overline{B_R})$. Observe that the right hand sides of (106) and (107) are well-defined for $\delta V \in C^{1,\beta}(\overline{B_R})$, so

that we can extend the definition of Λ to $C^{1,\beta}(\overline{B_R}) \times \mathbb{R}$. Once again Λ is Fredholm with index zero on $C^{1,\beta}(\overline{B_R}) \times \mathbb{R}$. Thus we only need to show that (108), (109) has unique zero solution assuming merely that $\delta V \in C^{1,\beta}(\overline{B_R})$. But due to the gain of regularity of the right hand side of (108), we can recover $\delta V \in C^{2,\beta}(\overline{B_R})$. The result thus follows. $\square$

We define the sequence of approximate solutions (V_n, α_n) by

$$(V_0, \alpha_0) = (V^0, \alpha^0), \quad (128)$$

$$(V_{n+1}, \alpha_{n+1}) = (V_n, \alpha_n) - \Lambda^{-1}[(V_n, \alpha_n) - \mathcal{F}(V_n, \alpha_n, \kappa, \mu)]. \quad (129)$$

Here (128) simply means that we use the Lane-Emden solution as the zeroth step in the iteration. In order for such iterations to be defined for all n , we require V_n to remain close to V^0 in $C^{2,\beta}(\overline{B_R})$. This will be established in Lemma 5.1.

5 Convergence of Iterations and Uniqueness

In this section, we fix $\beta < \beta_1 := \min(q-1, 1)$ and will prove convergence of the iteration sequence in $C^{2,\beta'}(\overline{B_R}) \times \mathbb{R}$ for every $0 < \beta' < \beta$. The argument consists mainly of two steps. In the first step we show that all (V_n, α_n) remain in a small neighborhood of (V^0, α^0) in the $C^{2,\beta}(\overline{B_R}) \times \mathbb{R}$ norm, so that the next term in the sequence is always well-defined by applying Lemma 4.1. In the second step we show that the $C^{1,\beta}(\overline{B_R}) \times \mathbb{R}$ norm of $(V_{n+1} - V_n, \alpha_{n+1} - \alpha_n)$ contracts, so that the iterates form a Cauchy sequence in the low norm. We then use interpolation in the hierarchy of Hölder spaces to prove convergence.

To avoid clutter in the equations, we further simplify the notation as follows. Denote

$$g(V) = V_+^q$$

and

$$\mathcal{F}(V, \alpha, \kappa, \mu) \text{ by } \mathcal{F}(V, \alpha), \quad \mathcal{F}(V, \alpha, 0, 0) \text{ by } \mathcal{F}^0(V, \alpha).$$

Recall that the Lane-Emden solution (V^0, α^0) is a fixed point: $\mathcal{F}^0(V^0, \alpha^0) = (V^0, \alpha^0)$. As before, we also use the notation $(V^\#, \alpha^\#) = \mathcal{F}(V, \alpha)$. Moreover, we denote

$$D_{(V,\alpha)}\mathcal{F}(V^0, \alpha^0, 0, 0) \text{ by } D\mathcal{F}^0$$

and

$$(\delta V^\dagger, \delta \alpha^\dagger) = D\mathcal{F}^0(\delta V, \delta \alpha).$$

From Lemma 4.3 and (102), (103), (104) we may write

$$\Delta(\delta V^\dagger) = -4\pi g'(V^0)\delta V \quad \text{on } B_R, \quad (130)$$

$$\delta V^\dagger = \frac{1}{|\cdot|} * (g'(V^0)\delta V) + \delta \alpha \quad \text{on } \partial B_R, \quad (131)$$

$$\delta \alpha^\dagger = \delta \alpha + \int_{B_R} g'(V^0)\delta V \, dx. \quad (132)$$

Thus we use $\#$ to denote components of the nonlinear mapping $\mathcal{F}$, and $\dagger$ to denote components of the linearized operator $D\mathcal{F}^0$. We also denote the exponentiated entropy S constructed in Theorem 3 by $\mathcal{S}(V)$ to emphasize its dependence on V . Finally, we denote by X the space $C^{2,\beta}(\overline{B_R}) \times \mathbb{R}$.

5.1 Boundedness in the High Norm

We now prove the uniform boundedness of the X norm of the iterates.

Lemma 5.1. *For all $\epsilon > 0$, $\epsilon_1 > 0$ sufficiently small, if $\|(V_n, \alpha_n) - (V^0, \alpha^0)\|_X < \epsilon$ and $|\kappa| + |\mu| < \epsilon_1$, then $\|(V_{n+1}, \alpha_{n+1}) - (V^0, \alpha^0)\|_X < \epsilon$.*

Proof. We first take $\epsilon < \delta$ as given in Lemma 4.1, so that (V_{n+1}, α_{n+1}) is well-defined. From (129) and $\mathcal{F}^0(V^0, \alpha^0) = (V^0, \alpha^0)$ we get

$$\begin{aligned} & (V_{n+1} - V^0, \alpha_{n+1} - \alpha^0) \\ &= (V_n - V^0, \alpha_n - \alpha^0) - \Lambda^{-1}\{(V_n - V^0, \alpha_n - \alpha^0) - \mathcal{F}(V_n, \alpha_n) + \mathcal{F}^0(V^0, \alpha^0)\} \\ &= (V_n - V^0, \alpha_n - \alpha^0) - \Lambda^{-1}\{\Lambda(V_n - V^0, \alpha_n - \alpha^0) - \mathcal{R}_n\} \\ &= \Lambda^{-1}\mathcal{R}_n, \end{aligned} \tag{133}$$

where the remainder $\mathcal{R}_n := (V_n^{\mathcal{R}}, \alpha_n^{\mathcal{R}})$ is defined as

$$\begin{aligned} (V_n^{\mathcal{R}}, \alpha_n^{\mathcal{R}}) &= \mathcal{F}(V_n, \alpha_n) - \mathcal{F}^0(V^0, \alpha^0) - D\mathcal{F}^0(V_n - V^0, \alpha_n - \alpha^0) \\ &= (V_n^{\#} - V^0, \alpha_n^{\#} - \alpha^0) - ((V_n - V^0)^{\dagger}, (\alpha_n - \alpha^0)^{\dagger}). \end{aligned} \tag{134}$$

By the definitions of the respective terms, we have

$$\nabla \cdot (\mathcal{S}(V_n) \nabla V_n^{\#}) = -\frac{4\pi g(V_n)}{\mathcal{S}(V_n)} + \kappa \nabla \cdot (\omega^2 r e_r) \quad \text{on } B_R, \tag{135}$$

$$\begin{aligned} V_n^{\#} &= \frac{1}{|\cdot|} * \frac{g(V_n)}{\mathcal{S}(V_n)} + \alpha_n \quad \text{on } \partial B_R \\ \alpha_n^{\#} &= \alpha_n + \int_{B_R} \frac{g(V_n)}{\mathcal{S}(V_n)} dx - M. \end{aligned}$$

$$\begin{aligned} \nabla \cdot (\nabla V^0) &= -4\pi g(V^0) \quad \text{on } B_R, \\ V^0 &= \frac{1}{|\cdot|} * g(V^0) + \alpha^0 \quad \text{on } \partial B_R \\ \alpha^0 &= \alpha^0 + \int_{B_R} g(V^0) dx - M. \end{aligned}$$

$$\begin{aligned} \nabla \cdot (\nabla (V_n - V^0)^{\dagger}) &= -4\pi g'(V^0)(V_n - V^0) \quad \text{on } B_R, \\ (V_n - V^0)^{\dagger} &= \frac{1}{|\cdot|} * (g'(V^0)(V_n - V^0)) + \alpha_n - \alpha^0 \quad \text{on } \partial B_R \\ (\alpha_n - \alpha^0)^{\dagger} &= \alpha_n - \alpha^0 + \int_{B_R} g'(V^0)(V_n - V^0) dx. \end{aligned} \tag{136}$$

Combining the preceding equations, we obtain equations for $(V_n^{\mathcal{R}}, \alpha_n^{\mathcal{R}})$, namely,

$$\begin{aligned}\Delta V_n^{\mathcal{R}} &= \nabla \cdot (\{1 - \mathcal{S}(V_n)\} \nabla V_n^{\#}) - 4\pi g(V_n) \left\{ \frac{1}{\mathcal{S}(V_n)} - 1 \right\} \\ &\quad - 4\pi \{g(V_n) - g(V^0) - g'(V^0)(V_n - V^0)\} + \kappa \nabla \cdot (\omega^2 r e_r) \\ &:= I_1 + I_2 + I_3 + I_4 \quad \text{in } B_R,\end{aligned}\tag{137}$$

$$\begin{aligned}V_n^{\mathcal{R}} &= \frac{1}{|\cdot|} * \left(\left\{ \frac{1}{\mathcal{S}(V_n)} - 1 \right\} g(V_n) \right) + \frac{1}{|\cdot|} * \{g(V_n) - g(V^0) - g'(V^0)(V_n - V^0)\} \\ &:= I_5 + I_6 \quad \text{on } \partial B_R,\end{aligned}\tag{138}$$

$$\begin{aligned}\alpha_n^{\mathcal{R}} &= \int_{B_R} \left\{ \frac{1}{\mathcal{S}(V_n)} - 1 \right\} g(V_n) \, dx + \int_{B_R} \{g(V_n) - g(V^0) - g'(V^0)(V_n - V^0)\} \, dx \\ &:= I_7 + I_8.\end{aligned}\tag{139}$$

Now let $\|(V_n, \alpha_n) - (V^0, \alpha^0)\|_X < \epsilon$ and $|\kappa| + |\mu| < \epsilon_1$, where ϵ, ϵ_1 are to be determined. We first estimate the $C^{2,\beta}(\overline{B_R})$ norm of $V_n^{\mathcal{R}}$ by means of Schauder estimates. This amounts to estimating the $C^{0,\beta}(\overline{B_R})$ norm of (137) and the $C^{2,\beta}(\partial B_R)$ norm of (138). Indeed, we have

$$\|I_1\|_{0,\beta} \leq \|V_n^{\#}\|_{2,\beta} \|\mathcal{S}(V_n) - 1\|_{1,\beta} \leq C\epsilon_1 \tag{140}$$

by Lemma 4.1 and Theorem 3. Similarly,

$$\|I_2\|_{0,\beta} \leq C\|V_n\|_{0,\beta} \|\mathcal{S}(V_n) - 1\|_{0,\beta} \leq C\epsilon_1. \tag{141}$$

On the other hand, by Lemma 4.2, we have

$$\|I_3\|_{0,\beta} \leq C\|V_n - V^0\|_{0,\beta}^{1+\delta_1} = C\epsilon^{1+\delta_1}, \tag{142}$$

where $\delta_1 = \beta_1 - \beta$. It is obvious that $\|I_4\|_{0,\beta} \leq C\epsilon_1$. By standard potential estimates, the convolution with $\frac{1}{|x|}$ is bounded from $C^{0,\beta}(\overline{B_{R+1}})$ to $C^{2,\beta}(\overline{B_R})$. Because all the functions appearing under the convolution in (138) are supported in the interior of B_R , we only have to bound their $C^{0,\beta}(\overline{B_R})$ norms. It follows as before that $\|I_5\|_{2,\beta} \leq C\epsilon_1$, and $\|I_6\|_{2,\beta} \leq C\epsilon^{1+\delta_1}$. We now use the Schauder estimates to conclude that

$$\|V_n^{\mathcal{R}}\|_{2,\beta} \leq C\epsilon_1 + C\epsilon^{1+\delta_1}. \tag{143}$$

We also use the previous estimates directly to estimate I_7 and I_8 , thereby obtaining

$$|\alpha_n^{\mathcal{R}}| \leq C\epsilon_1 + C\epsilon^{1+\delta_1}. \tag{144}$$

Estimation of (133) yields

$$\begin{aligned}\|(V_{n+1}, \alpha_{n+1}) - (V^0, \alpha^0)\|_X &\leq \|\Lambda^{-1}\|_{X \rightarrow X} C(\epsilon_1 + \epsilon^{1+\delta_1}) \\ &\leq C(\epsilon_1 + \epsilon^{1+\delta_1}).\end{aligned}$$

We now choose ϵ so small that $C\epsilon^{1+\delta_1} < \frac{\epsilon}{2}$, and ϵ_1 so small that $C\epsilon_1 < \frac{\epsilon}{2}$. The proof is complete. $\square$

5.2 Contraction in the Low Norm

For notational convenience we denote $\mathcal{S}(V_n)$ by S_n . We denote by Y the space $C^{1,\beta}(\overline{B_R}) \times \mathbb{R}$. By Lemma 5.1, V_n and α_n are well-defined, and $\|V_n - V^0\|_{2,\beta} < \epsilon$ for all n if $|\kappa| + |\mu| < \epsilon_1$. We want to prove that the Y -norm of $(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})$ decays. To that end, we first estimate the difference $S_n - S_{n-1}$.

Lemma 5.2. *Let $S_n = \mathcal{S}(V_n)$ be given as above. Then*

$$\|S_n - S_{n-1}\|_0 \leq C \|\nabla S_{n-1}\|_0 \|\nabla(V_n - V_{n-1})\|_0 \quad (145)$$

$$\|S_n - S_{n-1}\|_{C_{[1-\beta]}^{0,\beta}} \leq C \|\nabla S_{n-1}\|_\beta \|\nabla(V_n - V_{n-1})\|_{0,\beta} \quad (146)$$

where $C_{[1-\beta]}^{0,\beta}$ denotes the weighted Hölder space defined in (76).

Proof. We note that the difference $S_n - S_{n-1}$ is the solution to the transport equation

$$(\partial_z V_n) \partial_r (S_n - S_{n-1}) - (\partial_r V_n) \partial_z (S_n - S_{n-1}) = h_n, \quad (147)$$

where

$$h_n = (\partial_z S_{n-1}) \partial_r (V_n - V_{n-1}) - (\partial_r S_{n-1}) \partial_z (V_n - V_{n-1}), \quad (148)$$

with the zero floor data $(S_n - S_{n-1})(r, 0) = 0$. Then $S_n - S_{n-1}$ satisfies

$$(S_n - S_{n-1})(r, z) = \int_0^t h_n(\phi(t', \tau)) dt', \quad (149)$$

where $(r, z) = \phi(t, \tau)$ is the C^1 diffeomorphism given by the characteristic coordinates associated with V_n , as in Section 3.1. The C^0 estimate (145) follows directly from (149) since t is bounded.

To show (146), it suffices to estimate the weighted Hölder semi-norm. To this end, let $(r_1, z_1), (r_2, z_2)$ be given. We may assume $0 < |(r_1, z_1)| \leq |(r_2, z_2)|$ without loss of generality. By letting $(t_i, \tau_i) = \phi^{-1}(r_i, z_i)$, $i = 1, 2$, we see that

$$\begin{aligned} & (S_n - S_{n-1})(r_1, z_1) - (S_n - S_{n-1})(r_2, z_2) \\ &= \int_{t_2}^{t_1} h_n(\phi(t', \tau_1)) dt' + \int_0^{t_2} h_n(\phi(t', \tau_1)) - h_n(\phi(t', \tau_2)) dt' \\ &:= I_1 + I_2 \end{aligned} \quad (150)$$

For the first integral I_1 we have by (56)

$$|I_1| \leq |t_1 - t_2| \|h_n\|_0 \leq C \frac{|(r_1, z_1) - (r_2, z_2)|}{|(r_1, z_1)|} \|h_n\|_0 \quad (151)$$

For I_2 , using the Hölder regularity of h_n and (55), (56), we obtain

$$\begin{aligned} |I_2| &\leq C \|\phi_\tau\|_0^\beta |\tau_1 - \tau_2|^\beta \|h_n\|_{0,\beta} \\ &\leq C \|\nabla \tau\|_0^\beta |(r_1, z_1) - (r_2, z_2)|^\beta \|h_n\|_{0,\beta} \\ &\leq C |(r_1, z_1) - (r_2, z_2)|^\beta \|h_n\|_{0,\beta}. \end{aligned} \quad (152)$$

Combining the above estimates, we deduce that

$$\begin{aligned} & |(r_1, z_1)| \cdot |(S_n - S_{n-1})(r_1, z_1) - (S_n - S_{n-1})(r_2, z_2)| \\ & \leq C |(r_1, z_1) - (r_2, z_2)|^\beta \|h_n\|_{0,\beta}. \end{aligned} \quad (153)$$

By the definition (76) of the $C_{[1-\beta]}^{0,\beta}$ norm, we have proven (146). $\square$

We now write the recursive equation for the difference $(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})$. From (133), we have

$$(V_{n+1} - V_n, \alpha_{n+1} - \alpha_n) = \Lambda^{-1}(\mathcal{R}_n - \mathcal{R}_{n-1}), \quad (154)$$

where

$$\begin{aligned} \mathcal{R}_n - \mathcal{R}_{n-1} &= \mathcal{F}(V_n, \alpha_n) - \mathcal{F}(V_{n-1}, \alpha_{n-1}) - D\mathcal{F}^0(V_n - V_{n-1}, \alpha_n - \alpha_{n-1}) \\ &= (V_n^\# - V_{n-1}^\# - (V_n - V_{n-1})^\dagger, \alpha_n^\# - \alpha_{n-1}^\# - (\alpha_n - \alpha_{n-1})^\dagger) \end{aligned} \quad (155)$$

$$:= (\mathcal{W}_n, \alpha_n^\mathcal{R} - \alpha_{n-1}^\mathcal{R}). \quad (156)$$

Here we denoted the first component of $\mathcal{R}_n - \mathcal{R}_{n-1}$ by $\mathcal{W}_n$. We obtain the equation for $\mathcal{W}_n$ from (135), (136):

$$\nabla \cdot (S_n \nabla \mathcal{W}_n) = -\nabla \cdot H_n - 4\pi G_n \quad \text{on } B_R, \quad (157)$$

where

$$H_n = \{S_n - 1\} \nabla(V_n - V_{n-1})^\dagger + [S_n - S_{n-1}] \nabla V_{n-1}^\# \quad (158)$$

and

$$G_n = \frac{g(V_n)}{S_n} - \frac{g(V_{n-1})}{S_{n-1}} - g'(V^0)(V_n - V_{n-1}), \quad (159)$$

and on the surface ∂B_R we have

$$\mathcal{W}_n = \frac{1}{|\cdot|} * G_n. \quad (160)$$

Similarly, we have

$$\alpha_n^\mathcal{R} - \alpha_{n-1}^\mathcal{R} = \int_{B_R} G_n \, dx. \quad (161)$$

Lemma 5.3. *There exist $\epsilon > 0, \epsilon_1 > 0$ such that if $\sup_{n \geq 1} \|(V_n, \alpha_n) - (V^0, \alpha^0)\|_X < \epsilon$ for $|\kappa| + |\mu| < \epsilon_1$, then*

$$\|(V_{n+1} - V_n, \alpha_{n+1} - \alpha_n)\|_Y \leq \frac{1}{2} \|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y \quad (162)$$

for all n .

Proof. We first claim the estimates

$$\|H_n\|_{0,\beta} \leq C\epsilon_1 \|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y, \quad (163)$$

and

$$\|G_n\|_0 \leq C(\epsilon_1 + \epsilon^{\beta_1}) \|V_n - V_{n-1}\|_1 + C \|V_n - V_{n-1}\|_0^{1+\beta_1}. \quad (164)$$

Indeed, by (130), (131) with $\delta V = V_n - V_{n-1}$, $\delta\alpha = \alpha_n - \alpha_{n-1}$ and standard potential theoretic estimates, we have

$$\begin{aligned} \|(V_n - V_{n-1})^\dagger\|_{1,\beta} &\leq C(\|V_n - V_{n-1}\|_0 + |\alpha_n - \alpha_{n-1}|) \\ &= C\|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y. \end{aligned} \quad (165)$$

We denote the two terms in (158) by H_{n1} and H_{n2} respectively. We estimate

$$\begin{aligned} \|H_{n1}\|_{0,\beta} &\leq \|\{S_n - 1\} \nabla(V_n - V_{n-1})^\dagger\|_{0,\beta} \\ &\leq C\|S_n - 1\|_{0,\beta} \|(V_n - V_{n-1})^\dagger\|_{1,\beta} \\ &\leq C\epsilon_1 \|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y, \end{aligned} \quad (166)$$

where we have used (29). On the other hand,

$$\begin{aligned} \|H_{n2}\|_{0,\beta} &\leq \|[S_n - S_{n-1}] \nabla V_{n-1}^\#\|_{0,\beta} \\ &\leq \|[S_n - S_{n-1}] \nabla V_{n-1}^\#\|_{C_{[-\beta]}^{0,\beta}} \end{aligned} \quad (167)$$

$$\leq C\|S_n - S_{n-1}\|_{C_{[1-\beta]}^{0,\beta}} \|\nabla V_{n-1}^\#\|_{C_{[-1]}^{0,\beta}} \quad (168)$$

$$\leq C\|\nabla(S_{n-1} - 1)\|_{0,\beta} \|V_n - V_{n-1}\|_{1,\beta} \quad (169)$$

$$\leq C\epsilon_1 \|V_n - V_{n-1}\|_{1,\beta}. \quad (170)$$

Here we used the definition of the weighted Hölder spaces and Lemmas 3.4 and 3.5 regarding their properties to get (167) and (168). We used Lemma 5.2 and the estimate $\|\nabla V_{n-1}^\#\|_{C_{[-1]}^{0,\beta}} \leq \|\nabla V_{n-1}^\#\|_1 \leq C$ to get (169). That $V_{n-1}^\#$ is bounded in C^2 follows from the last assertion of Lemma 4.1. Combining the above estimates, we infer that

$$\|H_n\|_{0,\beta} \leq C\epsilon_1 \|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y. \quad (171)$$

We may rewrite G_n as

$$\begin{aligned} G_n &= [g(V_n) - g(V_{n-1}) - g'(V_{n-1})(V_n - V_{n-1})] + \{g'(V_{n-1}) - g'(V^0)\} (V_n - V_{n-1}) \\ &\quad + \left\{ \frac{1}{S_n} - 1 \right\} [g(V_n) - g(V_{n-1})] + \left[\frac{1}{S_n} - \frac{1}{S_{n-1}} \right] g(V_{n-1}) \\ &:= G_{n1} + G_{n2} + G_{n3} + G_{n4}. \end{aligned}$$

By (91),

$$\|G_{n1}\|_0 \leq C \|V_n - V_{n-1}\|_0^{1+\beta_1}. \quad (172)$$

We also have

$$\|G_{n2}\|_0 \leq C\|V_{n-1} - V^0\|_0^{\beta_1} \|V_n - V_{n-1}\|_0 \leq C\epsilon^{\beta_1} \|V_n - V_{n-1}\|_0. \quad (173)$$

From (29),

$$\|G_{n3}\|_0 \leq C\|S_n - 1\|_0 \|V_n - V_{n-1}\|_0 \leq C\epsilon_1 \|V_n - V_{n-1}\|_0. \quad (174)$$

From Lemma 5.2,

$$\begin{aligned} \|G_{n4}\|_0 &\leq C\|S_n - S_{n-1}\|_0 \leq C\|S_{n-1} - 1\|_1 \|V_n - V_{n-1}\|_1 \\ &\leq C\epsilon_1 \|V_n - V_{n-1}\|_1. \end{aligned} \quad (175)$$

(164) follows from combining the above estimates.

By standard potential estimates, we have

$$\left\| \frac{1}{|\cdot|} * G_n \right\|_{1,\beta} \leq C\|G_n\|_0. \quad (176)$$

We now use the $C^{1,\beta}$ Schauder estimates given by Theorem 8.33 of [15] on the equations (157), (160) to get

$$\begin{aligned} &\|\mathcal{W}_n\|_{1,\beta} \\ &\leq C(\|H_n\|_{0,\beta} + \|G_n\|_0) \\ &\leq C(\epsilon_1 + \epsilon^{\beta_1})\|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y + C\|V_n - V_{n-1}\|_0^{1+\beta_1} \\ &\leq C(\epsilon_1 + \epsilon^{\beta_1})\|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y \end{aligned} \quad (177)$$

where $\|V_n - V_{n-1}\|_0 \leq 2\epsilon$, which follows from the assumption that $\|(V_n, \alpha_n) - (V^0, \alpha^0)\|_X < \epsilon$ for all $n \geq 1$. We also have from (161) that

$$|\alpha_n^{\mathcal{R}} - \alpha_{n-1}^{\mathcal{R}}| \leq C(\epsilon_1 + \epsilon^{\beta_1})\|V_n - V_{n-1}\|_1 \quad (178)$$

From (154), (156) and by Lemma 4.3 we get

$$\begin{aligned} &\|(V_{n+1} - V_n, \alpha_{n+1} - \alpha_n)\|_Y \\ &\leq C\|\Lambda^{-1}\|_{Y \rightarrow Y}(\epsilon_1 + \epsilon^{\beta_1})\|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y \\ &\leq C(\epsilon_1 + \epsilon^{\beta_1})\|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y. \end{aligned} \quad (179)$$

We now choose ϵ_1 so small that $C\epsilon_1 < \frac{1}{4}$, and ϵ so small that $C\epsilon^{\beta_1} < \frac{1}{4}$, which completes the proof. $\square$

5.3 Existence and Uniqueness

We need the following interpolation lemma on Hölder spaces.

Lemma 5.4. *Let $0 < \beta' < \beta$. There exists a $C > 0$ such that*

$$\|u\|_{2,\beta'} \leq C \|u\|_{1,\beta}^{\lambda_1} \|u\|_{2,\beta}^{1-\lambda_1}, \quad (180)$$

$$\|v\|_{1,\beta'} \leq C \|v\|_0^{\lambda_2} \|v\|_{1,\beta}^{1-\lambda_2}, \quad (181)$$

for all $u \in C^{2,\beta}(\overline{B_R})$ and $v \in C^{1,\beta}(\overline{B_R})$. Here

$$\begin{aligned} 2 + \beta' &= \lambda_1(1 + \beta) + (1 - \lambda_1)(2 + \beta), \\ 1 + \beta' &= (1 - \lambda_2)(1 + \beta). \end{aligned}$$

Proof. See [2] or Ex. 3.2.6 in [23]. $\square$

We are now ready to prove the main existence theorem, which we repeat for the reader's convenience.

Theorem 2. *Let $s_0 \in C^{1,\beta}(\overline{B_R} \cap \{z = 0\})$ and $\omega^2 \in C^{2,\beta}(\overline{B_R})$ be given. There exist $\epsilon > 0$, $\epsilon_1 > 0$ such that if $|\kappa| + |\mu| < \epsilon_1$, then there exists a unique solution $(V^*, \alpha^*) \in X$, and $S^* \in C^{1,\beta}(\overline{B_R})$ to (11), (12), (13), (17), (18), with $\|(V^*, \alpha^*) - (V^0, \alpha^0)\|_X \leq \epsilon$. Furthermore, the solution satisfies that V_+^* is supported on $B_{\frac{R^0+R}{2}}$, and $S^* = 1$ outside B_{R_1} for some fixed $R_1 \in (R^0, R)$.*

Proof. We take ϵ and ϵ_1 small enough that Lemmas 5.1 and 5.3 hold. It follows by iteration that

$$\|(V_n - V_{n-1}, \alpha_n - \alpha_{n-1})\|_Y \leq \frac{C}{2^n}.$$

Lemma 5.2 now gives

$$\|S_n - S_{n-1}\|_0 \leq \frac{C}{2^n}.$$

Pick a $\beta' \in (0, \beta)$. Lemmas 5.1, 5.4 together with the preceding estimates imply that

$$\|V_n - V_{n-1}\|_{2,\beta'} \leq \frac{C}{2^{\lambda_1 n}}, \quad (182)$$

$$\|S_n - S_{n-1}\|_{1,\beta'} \leq \frac{C}{2^{\lambda_2 n}}. \quad (183)$$

It follows that $\{V_n\}$ is a Cauchy sequence in $C^{2,\beta'}(\overline{B_R})$, $\{S_n\}$ is a Cauchy sequence in $C^{1,\beta'}(\overline{B_R})$, and $\{\alpha_n\}$ is a Cauchy sequence in $\mathbb{R}$. Denote their limits by V^* , S^* and α^* . By (129), we have

$$(V_n, \alpha_n) - \mathcal{F}(V_n, \alpha_n, \kappa, \mu) = \Lambda(V_{n+1} - V_n, \alpha_{n+1} - \alpha_n). \quad (184)$$

Since Λ is bounded on $C^{2,\beta'}(\overline{B_R}) \times \mathbb{R}$ by Lemma 4.3, $(V_n, \alpha_n) - \mathcal{F}(V_n, \alpha_n, \kappa, \mu)$ converges to zero, and $(V_n^\#, \alpha_n^\#) = \mathcal{F}(V_n, \alpha_n, \kappa, \mu)$ converges to (V^*, α^*) in

$C^{2,\beta'}(\overline{B_R}) \times \mathbb{R}$. We know that $V_n^\#, \alpha_n^\#$ and S_n satisfy the equations

$$\begin{aligned} (S_n)_r(V_n)_z - (S_n)_z(V_n)_r &= -\kappa(\omega^2 r)_z \quad \text{on } B_R, \\ S_n(r, 0) &= e^{\frac{\mu s_0}{\gamma}}, \\ \nabla \cdot (S_n \nabla V_n^\#) &= -\frac{4\pi g(V_n)}{S_n} + \kappa \nabla \cdot (\omega^2 r e_r) \quad \text{on } B_R, \\ V_n^\# &= \frac{1}{|\cdot|} * \frac{g(V_n)}{S_n} + \alpha_n \quad \text{on } \partial B_R \\ \alpha_n^\# &= \alpha_n + \int_{B_R} \frac{g(V_n)}{S_n} dx - M. \end{aligned}$$

Taking the limit as n tends to infinity, we see that (11), (12), (13), (17), (18) are satisfied by V^*, α^*, S^* .

To see that $V^* \in C^{2,\beta}(\overline{B_R})$ and $\|(V^*, \alpha^*) - (V^0, \alpha^0)\|_X \leq \epsilon$, we recall from Lemma 5.1 that $\|(V_n, \alpha_n) - (V^0, \alpha_0)\|_X < \epsilon$. So

$$\|V_n - V^0\|_2 + \sum_{i,j} \frac{|\partial_i \partial_j [V_n - V^0](x) - \partial_i \partial_j [V_n - V^0](y)|}{|x - y|^\beta} + |\alpha_n - \alpha^0| < \epsilon.$$

Taking the limit as n tends to infinity, we get $V^* \in C^{2,\beta}(\overline{B_R})$ and $\|(V^*, \alpha^*) - (V^0, \alpha^0)\|_X \leq \epsilon$. Choosing $\epsilon < \sup_{|x| > (R^0 + R)/2} V^0(x)$, and recalling that V^0 is positive precisely on B_{R^0} , it follows that V_+^* is supported on $B_{(R^0 + R)/2}$ (see Remark 2.1). That $S^* = 1$ outside B_{R_1} was already proven in Theorem 3.

Finally in order to prove uniqueness, suppose there are two such fixed points $(V_1^*, \alpha_1^*, S_1^*)$ and $(V_2^*, \alpha_2^*, S_2^*)$. We can write $S_i^* = \mathcal{S}(V_i^*)$, and

$$(V_i^*, \alpha_i^*) = (V_i^*, \alpha_i^*) - \Lambda^{-1}[(V_i^*, \alpha_i^*) - \mathcal{F}(V_i^*, \alpha_i^*)] \quad (185)$$

for $i = 1, 2$. We can now simply repeat on (185) the low-norm estimates on (129) obtained in Lemma 5.3 to get

$$\|(V_1^* - V_2^*, \alpha_1^* - \alpha_2^*)\|_Y \leq \frac{1}{2} \|(V_1^* - V_2^*, \alpha_1^* - \alpha_2^*)\|_Y. \quad (186)$$

Thus $\|(V_1^* - V_2^*, \alpha_1^* - \alpha_2^*)\|_Y = 0$ and the two solutions coincide. $\square$

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