Phenomenology of the Prethermal Many-Body Localized Regime

David M. Long,[1,*] Philip J. D. Crowley,[2] Vedika Khemani,[3] and Anushya Chandran[1]
[1]Department of Physics, Boston University, Boston, Massachusetts 02215, USA
[2]Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
[3]Department of Physics, Stanford University, Stanford, California 94305, USA

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The dynamical phase diagram of interacting disordered systems has seen substantial revision over the past few years. Theory must now account for a large *prethermal many-body localized* regime in which thermalization is extremely slow, but not completely arrested. We derive a quantitative description of these dynamics in short-ranged one-dimensional systems using a model of successive many-body resonances. The model explains the decay timescale of mean autocorrelators, the functional form of the decay—a stretched exponential—and relates the value of the stretch exponent to the broad distribution of resonance timescales. The Jacobi method of matrix diagonalization provides numerical access to this distribution, as well as a conceptual framework for our analysis. The resonance model correctly predicts the stretch exponents for several models in the literature. Successive resonances may also underlie slow thermalization in strongly disordered systems in higher dimensions, or with long-range interactions.

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Localized systems fail to thermalize under their own dynamics due to strong spatial inhomogeneities [1]. Without interactions, a stable fully localized phase exists in any dimension [2–4]. Including interactions, the existence of *many-body localization* (MBL) becomes difficult to confirm. The consensus from the last decade and a half [5–11] is that sufficiently strong random disorder produces stable MBL in short-ranged one dimensional systems only. Specifically, the best-studied model—the Heisenberg chain with random fields [8]—was believed to have a direct transition from a thermalizing phase to a fully MBL phase at a critical disorder strength $W = W_c$ between 3 and 6 (in units of the Heisenberg coupling) [8,12].

However, numerical evidence has been accumulating that this understanding is wrong—there is no transition near $W = 3$ [13–21]. In fact, recent studies suggest $W_c \gtrsim 20$ [19,20], which is larger than numerically or experimentally observable [19,22]. The phase diagram must thus be modified to contain a large *prethermal MBL* regime, in which the system appears to be localized for a long time [Fig. 1(a)].

Prethermal MBL phenomenology has been studied in short chains using exact diagonalization techniques [14,17,18,41,42]. Three key features have emerged: exponential growth of the thermalization time τ with disorder, approximately logarithmic decay of autocorrelators up to a time $O(\tau)$, and apparent localization when τ exceeds the Heisenberg time $\tau_{\text{Heis}} = O(2^L)$ in finite chains. Rare regions of anomalously high disorder can neither explain the slow decay, nor are they empirically observed in this parameter regime [15,16,26]. Rather, the observed decay [43–49] and apparent localization can be partially explained [22,50] through *resonances* between many-body states [22,43,50–52].

At short times, a product state may *resonate* with another state with a locally different magnetization pattern.

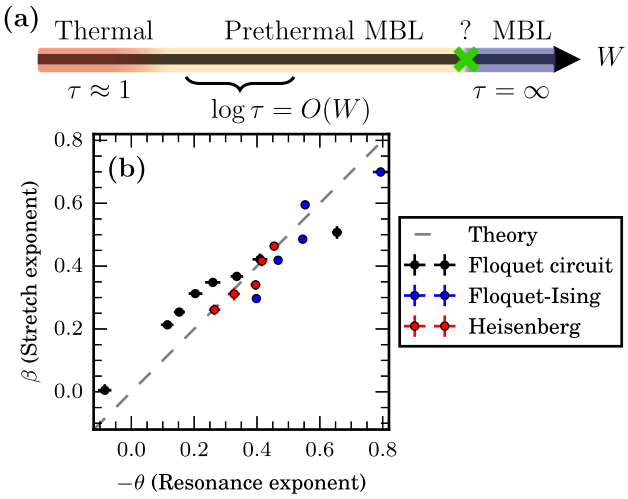

FIG. 1. (a) Disordered many-body systems cross over from a well-thermalizing regime into a prethermal many-body localized regime, where the local equilibration time τ grows exponentially with the disorder strength W. Any transition to an MBL phase must occur at much larger disorder strength. (b) The successive resonance model predicts that the stretch exponent (β) appearing in stretched exponential decay of autocorrelators equals another exponent ($-\theta$) which describes the broad distribution of resonance timescales. Data from a one-dimensional Floquet circuit model of MBL [19], a Floquet-Ising model [23], and the usual disordered Heisenberg model [24–26] are broadly consistent the prediction $\beta = -\theta$ in the prethermal MBL regime.

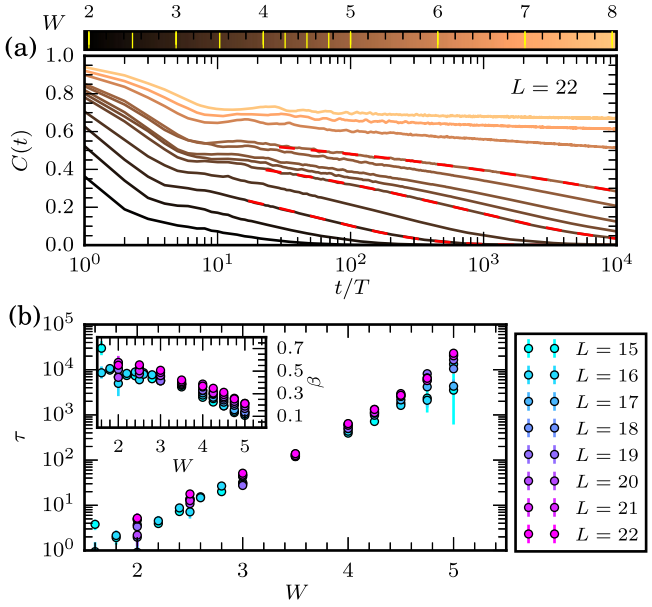

FIG. 2. (a) Within a broad regime of disorder strengths (specific values marked yellow in the color bar), local autocorrelation functions (2) for the Floquet circuit model decay very slowly. Fits to a stretched exponential (red, dashed for $W \in \{3, 4, 5\}$) show excellent agreement with the numerical data. (b) The decay times τ extracted from stretched exponential fits (3) grow exponentially with disorder W. (Fits with $\tau \gtrsim 10^4$ exceed our maximum simulation time, and are unreliable.) (Inset) The stretch exponent β decreases with disorder, and increases weakly with system size [26].

Physically, this manifests as large oscillations in local autocorrelators. In this Letter we propose that, in the prethermal regime, states become involved in still more resonances at longer times. Thus, a hierarchy emerges of *successive resonances* forming at progressively longer timescales. Our statistical treatment of this resonance formation predicts exponentially long thermalization times, and stretched exponential decay of disorder averaged autocorrelators. Numerically, we find that autocorrelators indeed decay as a stretched exponential [23] (Fig. 2).

The Jacobi algorithm for iterative matrix diagonalization [53,54] is the basis of our analytical framework, and allows us to extract the distribution of resonance frequencies. The distribution is described by a power law with exponent $-1 + \theta$ [22]. The successive resonance model predicts that the stretch exponent β for autocorrelator decay is linearly related to θ:

$$\beta = -\theta. \tag{1}$$

Both our own numerics and previously published data show good agreement with this prediction [Fig. 1(b)].

Dynamics of autocorrelators.—Infinite-temperature autocorrelation functions are a measurable probe of thermalization, and their slow decay is a notable characteristic of the prethermal regime [23,25,55,56]. In a disordered model, Fig. 2 shows that autocorrelators decay as a stretched exponential (3) with a decay constant that is exponential in the disorder strength (4).

We consider autocorrelators of operators which are diagonal in the disorder basis (the z basis). The numerics presented in the main text use

$$C(t) = \frac{1}{2^L}[\mathrm{Tr}(\sigma_0^z(t)\sigma_0^z(0))] \tag{2}$$

in the Floquet circuit model of Ref. [19] with periodic boundary conditions, described in detail in the Supplemental Material [26]. Here, σ_0^z is the z spin operator on an arbitrary site (labeled 0), L is the number of qubit degrees of freedom, $\sigma_0^z(t) = U(t)^\dagger \sigma_0^z U(t)$, $U(t)$ is the unitary evolution operator, and square brackets denote a disorder average. In this model, every σ^z operator is conserved in the $W \to \infty$ limit [26].

The Floquet circuit model has a well-thermalizing regime for $W \ll 1$, where $C(t)$ rapidly decays to zero. In any MBL phase, $C(t)$ acquires a nonzero late time value. In the intermediate regime of prethermal MBL, $1 \lesssim W \lesssim 25$, $C(t)$ decays slowly to zero in the $L \to \infty$ limit [19].

In more detail, in the intermediate regime, $C(t)$ first drops to some $O(1)$ value within a few tens of periods, and then decays very slowly. The functional form of this decay appears logarithmic at small system sizes or short times [17], but a better fit for larger system sizes is to a stretched exponential (Fig. 2) [26],

$$C(t) \sim Ae^{-(t/\tau)^\beta}. \tag{3}$$

(It is notoriously difficult to distinguish stretched exponential relaxation from a logarithm at intermediate times [57].)

The timescale for decay (τ) is extracted from a fit of this functional form to the late-time data for $C(t)$. Consistent with other recent observations [14,17,41], τ increases exponentially in the disorder strength.

$$\log \tau = O(W). \tag{4}$$

In a Hamiltonian system, local equilibration on the timescale τ would be followed by slow hydrodynamic decay.

The observations (3) and (4) are the primary features that the model of successive resonances explains.

Jacobi algorithm.—In the prethermal MBL regime, eigenstates of large systems should be expected to obey the eigenstate thermalization hypothesis (ETH) [58–60]. This makes them a poor basis for predicting finite time dynamics of local correlators. It is more revealing to use a short time expansion in a dressed basis.

The Jacobi algorithm for matrix diagonalization [53,54] provides a convenient numerical tool for constructing such

a dressed basis. It also provides a more concrete framework within which to understand what is meant by a many-body resonance in a system which, ultimately, thermalizes.

We describe the algorithm for the case of a Hermitian operator (the Hamiltonian, H). Generalizations to the unitary case [61,62] are appropriate for the Floquet setting, and are discussed in the Supplemental Material [26].

The algorithm begins by identifying the largest (in absolute value) off-diagonal matrix element of H, H_{ab}, in the z basis. The 2×2 block containing this element is diagonalized by the unitary rotation R_0. (Note that R_0 affects the entire a and b rows and columns of H.) The (a, b) element of H is then set to zero in the rotated matrix $H(\Gamma_1) = R_0^\dagger H R_0$, where Γ_1 is a flow time for the algorithm, defined below in Eq. (6). We say the element H_{ab} of H is decimated, in analogy to the renormalization group.

This process is iterated, so that the weight in the off-diagonal of $H(\Gamma_{j+1}) = R_j^\dagger H(\Gamma_j) R_j$ strictly decreases. This procedure constructs a basis

$$|a(\Gamma_j)\rangle = R_{j-1} \cdots R_1 R_0 |a(\Gamma_0)\rangle, \tag{5}$$

(where $\{|a(\Gamma_0)\rangle\}$ is the bare product state basis) which is dressed by the fast degrees of freedom in H.

The flow time Γ is defined in terms of a physical timescale associated with the basis $\{|a(\Gamma)\rangle\}$ ($\hbar = 1$),

$$\frac{2\pi}{\Gamma} = \sqrt{\frac{1}{L 2^L} \sum_{a \neq b} |\langle b(\Gamma)|H|a(\Gamma)\rangle|^2}. \tag{6}$$

and strictly increases throughout the course of the algorithm [54]. Henceforth, we neglect all subexponential factors of L, as in the denominator of Eq. (6).

The Jacobi algorithm diagonalizes H within $O(4^L)$ steps. Only $O(W 2^L)$ steps are necessary to construct the *dressed basis* useful for computing autocorrelators. The dressed states only have large overlap with $O(1)$ bare states on average, as discussed in the next section.

Dynamics of successive resonance.—Expressing the autocorrelator $C(t)$ in the dressed basis relates it to the statistics of the Jacobi algorithm (11). With two natural assumptions—that dynamics are dominated by *sparse resonances* (9), and that the timescales associated with these resonances are power law distributed (13)—stretched exponential decay follows.

When calculating autocorrelators of some operator Z (assumed to be diagonal in the $\{|a(\Gamma_0)\rangle\}$ basis) for $t \ll \Gamma$, we can treat the Hamiltonian as being diagonal in the basis $\{|a(\Gamma)\rangle\}$ at the cost of introducing a well-controlled error:

$$\begin{aligned} C_Z(t) &= \frac{1}{2^L} [\mathrm{Tr}(Z(t) Z(0))] \\ &= \frac{1}{2^L} \left[\sum_{a,b} |Z_{ab}(\Gamma)|^2 e^{-i\omega_{ab}(\Gamma) t} \right] + O((t/\Gamma)^2), \end{aligned} \tag{7}$$

where $Z_{ab}(\Gamma) = \langle a(\Gamma)|Z|b(\Gamma)\rangle$, $\omega_{ab}(\Gamma) = \langle b(\Gamma)|H|b(\Gamma)\rangle - \langle a(\Gamma)|H|a(\Gamma)\rangle$, and square brackets are again used to denote a disorder average.

The joint distribution function of $|Z_{ab}(\Gamma)|^2$ and $\omega_{ab}(\Gamma)$, $p(Z^2, \omega; \Gamma)$, determines $C_Z(t)$ through

$$C_Z(t) = 2^L [Z^2 e^{-i\omega t}]_{p(\Gamma)} + O[(t/\Gamma)^2]. \tag{8}$$

The subscript on the square brackets indicates the distribution over which the average is performed.

We can deduce properties of p, and hence $C_Z(t)$, from the Jacobi algorithm. Namely, that large matrix elements in the distribution only arise due to occasional large rotations in the Jacobi algorithm.

As Z is diagonal in the initial basis $\{|a(\Gamma_0)\rangle\}$, its off-diagonal elements only become large when some rotation R_k affecting that element is also large. This happens when the decimated off-diagonal matrix element is much larger than the difference in diagonal elements $\omega_{ab}(\Gamma_k)$—that is, when the states $|a(\Gamma_k)\rangle$ and $|b(\Gamma_k)\rangle$ are *resonant*. Then, in the next round of iteration,

$$\begin{aligned} |Z_{ab}(\Gamma_{k+1})|^2 &= O(1), \\ H_{aa}(\Gamma_{k+1}) &\approx H_{aa}(\Gamma_k) \pm |H_{ab}(\Gamma_k)|, \end{aligned} \tag{9}$$

and similarly $H_{bb}(\Gamma_{k+1}) \approx H_{bb}(\Gamma_k) \mp |H_{ab}(\Gamma_k)|$.

We make the approximation that rotations are either trivial or cause resonances (9) [22]. Only the resonances produce dynamics.

Before thermalization, resonances are *sparse*. The probability of a resonance occurring in a given rotation is small, $P(|\omega_{ab}| < |H_{ab}|) = O(W^{-1})$. Further, the prefactor hidden in this scaling expression is also small: between 0.1% and 1% of rotations are resonances in the studied parameter regimes. Thus, after $O(W)$ Jacobi steps per state (as in Fig. 3), every dressed state $|a(\Gamma)\rangle$ is involved in $O(1)$ resonances on average.

Technically, the sparse resonance assumption is that $|Z_{ab}|^2 = O(1)$ only for resonant states. This ignores the effects of successive resonances which may reduce $|Z_{ab}|^2$, and the possibility of many small rotations producing a large $|Z_{ab}|^2$. This assumption is valid provided that the number of resonances per state is $O(1)$. This provides a large intermediate window, a few multiples of τ, in which we can make predictions.

The resonance assumption splits p into a part due to resonances, which contributes to $C_Z(t)$, and a part where matrix elements are all close to zero:

$$p(Z^2, \omega; \Gamma) \approx \delta(Z^2) p_0(\omega; \Gamma) + p_{\mathrm{res}}(Z^2, \omega; \Gamma). \tag{10}$$

Equation (9) leads to two conclusions regarding p_{res}. First, the matrix element $|Z_{ab}(\Gamma_k)|^2$, being $O(1)$, does not depend strongly on $\omega_{ab}(\Gamma_k)$. Consequently, the expectation

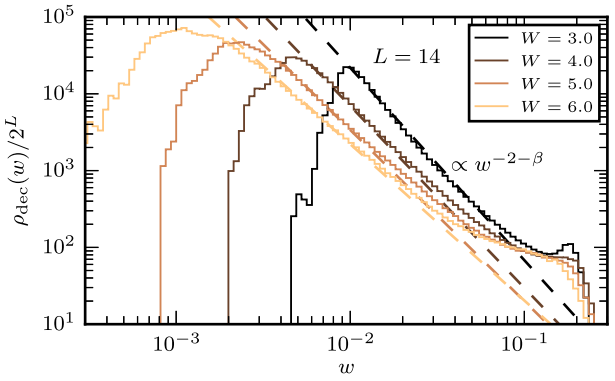

FIG. 3. The Jacobi decimated elements for the Floquet circuit model (the main text discusses the Hamiltonian case) for $200\,2^L$ iterations are approximately power law distributed for intermediate decimated weights w_{ab}, which generalize the decimated matrix elements $|H_{ab}|$ to the unitary case [26]. Furthermore, the power law is in good agreement with the predicted $-2-\beta$ from the successive resonance model (dashed lines). For this number of iterations, the average number of resonances per state is $\lesssim 1$. Matrix element distributions are averaged over 100 disorder realizations, $\rho_{\rm dec}$ is normalized as a number density, and β is fit from Fig. 2.

of Z^2 at fixed ω in $p_{\rm res}$ can be factorized out of Eq. (8). This gives a key intermediate result:

$$C_Z(t) \approx 2^L [Z^2]_{p_{\rm res}(\Gamma)} \mathcal{F}\{p_{\rm res}(\omega;\Gamma)\}(t), \tag{11}$$

where $p_{\rm res}(\omega;\Gamma)$ is the marginal distribution function of the resonance frequencies and $\mathcal{F}\{\cdot\}$ is the Fourier transform.

The second consequence is found by repeatedly applying Eq. (9) to find the energy differences ω. They are of the form

$$\omega_{ab}(\Gamma) = \sum_{\Gamma_k<\Gamma} \mu_k |H(\Gamma_k)|, \tag{12}$$

where $\mu_k = \pm 1$, and the sum runs over matrix elements $|H(\Gamma)|$ responsible for a resonance in either state $|a(\Gamma)\rangle$ or $|b(\Gamma)\rangle$ at flow time Γ. We have neglected the initial value $\omega_{ab}(\Gamma_0)$, which must be small if the states are to become resonant. Equation (12) encodes the effect of many resonances, each contributing to dynamics at progressively longer timescales $2\pi/|H(\Gamma_k)|$.

Equation (9) relates the frequencies $\omega_{ab}(\Gamma)$ to the resonance timescales, and hence the distribution of decimated elements. Our central assumption, verified numerically in Fig. 3, is that the distribution of decimated elements is a *power law* (cf. Ref. [42]). This is natural if the dressed basis is quasilocal, as the distribution of matrix elements of a quasilocal operator in a quasilocal basis is a power law in one dimension [19,22,50,63]. (Matrix elements decrease exponentially with spatial range, but there are exponentially many of them).

As $\omega_{ab}(\Gamma)$ is the sum of many independent variables, the central limit theorem may be invoked. The limit distribution for a sum of power-law distributed variables is not normal, but is rather a *Lévy stable distribution* [64]. The Fourier transform of a Lévy distribution is a stretched exponential, which leads to the observed form of the decay (3) through Eq. (11).

The distribution of decimated elements is parametrized as a power law ansatz with an exponent θ [22]:

$$\rho_{\rm dec}(|H|) = 2^L C |H|^{-2+\theta}. \tag{13}$$

Explicitly reinserting a local energy scale $J = O(\Gamma_0^{-1})$, dimensional analysis gives $C = O(J^{1-\theta})$. The distribution of $|H(\Gamma_k)|$ involved in resonances (treating $\omega_{ab}(\Gamma)$ and $|H_{ab}(\Gamma)|$ as uncorrelated) is

$$\begin{aligned}\rho_{\rm res}(|H|) &= P(|\omega| < |H|)\rho_{\rm dec}(|H|) \\ &\approx 2^{L+1} p(\omega = 0) C |H|^{-1+\theta},\end{aligned} \tag{14}$$

where we assumed $|H|$ is small, so that $P(|\omega| < |H|) \approx 2p(\omega = 0)|H|$, and $p(\omega) = O(W^{-1})$ is the ω marginal of $p(Z^2, \omega)$.

The distribution of resonances (14) is also a power law, but with a larger exponent, $-1+\theta$, than $\rho_{\rm dec}$.

The exponent θ appearing in Eqs. (13) and (14) is the central parameter of the *single* resonance model introduced in Ref. [22] (see also Ref. [43]). With the chosen parametrization, $\theta < 0$ implies thermalization. Successive resonances may cause a drift of θ with Γ. However, for sufficiently negative θ, the system thermalizes before any significant drift, and θ may be treated as a constant. Figure 3 shows this is a reasonable approximation in accessible parameter regimes.

The exponent θ also controls the distribution of matrix elements of generic local operators in the $\{|a(\Gamma)\rangle\}$ basis, not just the decimated elements [26,63].

The Supplemental Material [26] computes the Fourier transform (11) and shows that, for $\theta < 0$,

$$C_Z(t) \sim A e^{-(t/\tau)^{-\theta}} \quad \text{for } J^{-1} \ll t \ll \omega_c^{-1}, \tag{15}$$

where A is a constant, $\omega_c = O(\tau^{-1})$ is a small frequency cutoff, and J gives the large frequency cutoff.

The linear scaling of $\log(J\tau)$ with W/J follows from our previous assumptions and a linearization of $-\theta^{-1}$ in W/J [22,26]. The power law form of $\rho_{\rm res}$ is appropriate while each state is involved in few resonances. It breaks down when

$$\int_{\omega_c}^{J} \rho_{\rm res}(H) dH = O(2^L), \tag{16}$$

which immediately provides

$$\log(J\tau) = O[-\theta^{-1} \log(-\theta W/J)] \tag{17}$$

[using $p(\omega = 0) = O(W^{-1})$, $CJ^{\theta} = O(J)$, and that the ratio J/ω_c is proportional to $J\tau$]. The exact dependence of $-\theta^{-1}$ on W/J is unknown. Nevertheless, as $-\theta^{-1}$ is slowly varying in the numerically accessible W/J range [inset of Fig. 2(b)], we linearize it. This gives $\log(J\tau) = O(W/J)$.

The dependence of $\log(J\tau)$ may be different for larger W/J. The purely linear form for $\log(J\tau)$ obtained numerically [Fig. 2(b)] thus remain unexpected in the successive resonance model.

Equation (15) accounts for both of our main observations in Fig. 2. Furthermore, we have arrived at a falsifiable prediction: the stretch exponent β of the decay should be given by $\beta = -\theta$ (1). An independent numerical measurement of θ can test this prediction. Indeed, fitting a power law $-2 + \theta$ to the distribution of decimated elements in Fig. 3 produces exponents $-\theta$ which are in broad agreement with β as fit to $C(t)$ [Fig. 1(b)]. This agreement extends to several models with prethermal MBL regimes [23–26]. Note that, as ρ_{dec} is not a pure power law, agreement should not be exact.

Discussion.—Theory forbids a stable MBL phase in many settings in which experiment and numerics observe MBL phenomenology [65–76]. Even in one dimension, MBL may only exist at much larger disorder strengths than previously anticipated [10,17–19,77]. These settings are instances of prethermal MBL. We have begun the study of such prethermal dynamics in one dimension. The successive resonance model predicts stretched exponential decay of autocorrelators, an exponentially long thermalization time, and the value of the stretch exponent, all of which are supported by numerics in several models at intermediate values of W. These values of W lie in the regime that diagonalization studies previously identified as critical.

At larger values of W in the prethermal regime (previously identified as many-body localized), the decimated elements can be controlled by a power law exponent θ that is positive for small Jacobi flow times. Here, the flow of θ cannot be ignored, as thermalization should be signaled by a negative θ. Testing resonance model predictions then requires longer flow times and larger system sizes, or an analytic theory of the flow of θ.

The successive resonance model does not use any properties of the random potential. As such, its predictions are identical for correlated potentials in one dimension.

More generally, successive resonance model predictions are identical in any setting where ρ_{dec} is a power law with $\theta < 0$. Future work should check this in higher dimensions [65–67], translationally invariant MBL [68,69], and with long range interactions [73–76]. Whenever ρ_{dec} has a strong separation of scales, the model may still be predictive, although decay need not be stretched exponential. An interesting open question is the applicability to Floquet prethermalization [78], in which there is only one long-lived global conservation law, but fidelities still show slow decay [79].

In Anderson models on random regular graphs and related random matrix models, return probabilities exhibit stretched exponential decay [11,80–85]. With the assumption of sparse resonances, the formal calculations in these models are very similar to ours. The application of the Jacobi method to random regular graphs is worth exploring.

The Jacobi algorithm provides an effective off-diagonal matrix element distribution at different time scales. Its applications to quantum dynamics, the emergence of hydrodynamics, and connections to other techniques [45,86–88] deserve further investigation. Indeed, the most rigorous analysis of MBL uses the Jacobi algorithm [9,89].

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*dmlong@bu.edu

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