

M-CONVEXITY OF VEXILLARY GROTHENDIECK POLYNOMIALS VIA BUBBLING*

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Abstract. We introduce bubbling diagrams and show that they compute the support of the Grothendieck polynomial of any vexillary permutation. Using these diagrams, we show that the support of the top homogeneous component of such a Grothendieck polynomial coincides with the support of the dual character of an explicit flagged Weyl module. We also show that the homogenized Grothendieck polynomial of a vexillary permutation has M-convex support.

Key words. Grothendieck polynomial, M-convex, flagged Weyl module

MSC codes. 05E05, 05E10, 14M15

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1. Introduction. Grothendieck polynomials $\mathfrak{G}_w$ are multivariate polynomials associated to permutations $w \in S_n$. Grothendieck polynomials were introduced by Lascoux and Schützenberger [LS82] as representatives of the classes of Schubert varieties in the K -theory of the flag manifold. They generalize Schubert polynomials, which, in turn, generalize the classical Schur polynomials, a well-known basis of the ring of symmetric functions.

There has been a flurry of research on the support of Grothendieck polynomials as well as the distribution of their coefficients within their support [HMMSD22, Wei21, PSW21, Haf22, MSS22, PS22, CCRMM22]. With Huh and Matherne, the second and fourth author conjectured that homogenized Grothendieck polynomials are Lorentzian (up to appropriate normalization). In particular, this conjecture would imply that their support is *M-convex*, equivalently the set of integer points in a generalized permutahedron. That the support is the set of integer points of a convex polytope was previously conjectured by Monical, Tokcan, and Yong in [MTY19].

To date, it is known that homogenized Grothendieck polynomials are M-convex for several families of permutations. These include permutations of the form 1π with π dominant on $\{2, 3, \dots, n\}$ [MSD20], Grassmannian permutations [EY17], and permutations whose Schubert polynomial has all nonzero coefficients equal to 1 [CCRMM22]. In the present paper, we prove M-convexity for homogenized Grothendieck polynomials of vexillary permutations.

Our inspiration is the proof of the analogous result for all Schubert polynomials [FMS18]. The latter relies heavily on the theory of dual Weyl characters which has no K-theoretic counterpart. Mimicking the dual Weyl character approach, we introduce

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bubbling diagrams, which are diagrams (subsets of the $n \times k$ grid) endowed with additional data affecting the legality of certain local transformations (Definitions 3.3 and 3.5). These diagrams also bear strong similarities with *ghost diagrams* [RY15]. We show that bubbling diagrams compute the support of any vexillary Grothendieck polynomial.

THEOREM 1.1. *If $w \in S_n$ is a vexillary permutation, then $\text{supp}(\mathfrak{G}_w) = \{\text{wt}(\mathcal{D}) : \mathcal{D} \in \mathcal{BD}(w)\}$.*

We also provide a much simpler subset $\mathcal{SBD}(w) \subseteq \mathcal{BD}(w)$ in Definition 4.11 which still realizes the conclusion of Theorem 1.1.

From the characterization of $\text{supp}(\mathfrak{G}_w)$ afforded by Theorem 1.1, we derive two interesting consequences. For a diagram D , let χ_D denote the dual character of the flagged Weyl module of D (see section 2 for definitions). Denote the top degree component of $\mathfrak{G}_w$ by $\mathfrak{G}_w^{\text{top}}$.

THEOREM 1.2. *Let $w \in S_n$ be a vexillary permutation. There is a diagram $D^{\text{top}}(w)$ such that $\text{supp}(\mathfrak{G}_w^{\text{top}}) = \text{supp}(\chi_{D^{\text{top}}(w)})$.*

For vexillary permutations, Theorem 1.2 implies that the *Rajchgot polynomials* of [PSW21] are dual characters of flagged Weyl modules. Consequently, their Newton polytopes are *Schubitoes*, a subclass of generalized permutahedra introduced in [MTY19] whose defining inequalities are derived from a diagram.

In a recent work, Pan and Yu [PY23] also construct a diagram whose weight is the leading monomial of $\mathfrak{G}_w^{\text{top}}$. In general, their diagram is distinct from $D^{\text{top}}(w)$; in particular, the dual character of their diagram does not have the same support as $\mathfrak{G}_w^{\text{top}}$ for vexillary permutations. Tianyi Yu communicated to us that the results of [PY23] along with those in [Yu23] can be used to show that $\text{supp}(\mathfrak{G}_w^{\text{top}})$ is the set of integer points of a Schubitope.

THEOREM 1.3. *Let $w \in S_n$ be a vexillary permutation. Then the homogenized Grothendieck polynomial $\tilde{\mathfrak{G}}_w$ has M -convex support. In particular, each homogeneous component of $\mathfrak{G}_w$ has M -convex support.*

The lowest-degree homogeneous component of $\mathfrak{G}_w$, the Schubert polynomial, equals an integer multiple of some χ_D for any permutation. As a consequence of Theorem 1.2, one might wonder whether this is the case for $\mathfrak{G}_w^{\text{top}}$ or for other homogeneous components of $\mathfrak{G}_w$.

We can use the following result to verify whether or not the Newton polytopes of the homogeneous components of $\mathfrak{G}_w$ are Schubitoes, the Newton polytopes of the polynomials χ_D .

THEOREM 1.4. *Fix $n \geq 1$. The rank functions $r_{\text{SM}_n(I)}$ of Schubert matroids form a basis of the vector space of functions $f: 2^{[n]} \rightarrow \mathbb{R}$ satisfying $f(\emptyset) = 0$. In particular, we have the following:*

- *A generalized permutahedron is a Schubitope if and only if its associated submodular function is a $\mathbb{Z}_{\geq 0}$ -linear combination of rank functions of Schubert matroids, and*
- *two Schubitoes S_D and $S_{D'}$ are equal if and only if D can be obtained from D' by a permutation of columns.*

Using Theorem 1.4, we exhibit two interesting counterexamples:

(1) Example 6.11 provides a vexillary permutation w and a (not top-degree) homogeneous component of $\mathfrak{G}_w$ whose Newton polytope is not a Schubitope. This

suggests focusing attention only on $\mathfrak{S}_w^{\text{top}}$ when looking for Schubitopes among the homogeneous components of $\mathfrak{S}_w$.

(2) Example 6.10 gives a nonvexillary permutation w where $\text{Newton}(\mathfrak{S}_w^{\text{top}})$ is not a Schubitope and so is not a multiple of any χ_D . This suggests restricting attention to vexillary permutations when relating $\mathfrak{S}_w^{\text{top}}$ to χ_D . We conjecture the following strengthening of Theorem 1.2 (tested for all vexillary $w \in S_n$, $n \leq 9$).

CONJECTURE 1.5. *If $w \in S_n$ is vexillary, then $\mathfrak{S}_w^{\text{top}}$ is an integer multiple of $\chi_{D^{\text{top}}(w)}$.*

Outline of this paper. In section 2, we recall some relevant background. In section 3, we establish basic properties of bubbling diagrams, including a nonrecursive characterization of the set of bubbling diagrams (Lemma 3.19) and prove Theorem 1.1 by constructing weight preserving maps between the set of bubbling diagrams and the set of marked bumpless pipe dreams. In section 4, we prove Theorem 1.2 by showing that bubbling diagrams can be systematically padded to obtain a top-degree diagram which is necessarily in $\text{supp}(\chi_{D^{\text{top}}(w)})$ (Theorem 4.6), and we also show that divisibility relations among monomials in $\mathfrak{S}_w$ can be realized by inclusion relations among bubbling diagrams in a strong sense (Theorem 4.10). In section 5, we deduce Theorem 1.3 from a “one-column version” of the result (Proposition 5.6). In section 6, we prove Theorem 1.4 and use it to show that our results are sharp.

2. Background.

Conventions. We will write permutations $w \in S_n$ in one-line notation as words with the letters $\{1, 2, \dots, n\}$. For example, $w = 312 \in S_3$ is the permutation that sends $1 \mapsto 3$, $2 \mapsto 1$, and $3 \mapsto 2$. Throughout, permutations will act on the right (switching positions, not values). For $j \in [n-1]$, let s_j denote the adjacent transposition swapping positions j and $j+1$, so, for example, ws_1 is the permutation w with the numbers $w(1)$ and $w(2)$ swapped. We write $\ell(w)$ for the number of inversions of w .

Grothendieck polynomials.

DEFINITION 2.1. Fix $n \geq 1$ and $j \in [n-1]$. The divided difference operators ∂_j are operators on the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ defined by

$$\begin{aligned} \partial_j(f) &\stackrel{\text{def}}{=} \frac{f - s_j \cdot f}{x_j - x_{j+1}} \\ &= \frac{f(x_1, \dots, x_n) - f(x_1, \dots, x_{j-1}, x_{j+1}, x_j, x_{j+2}, \dots, x_n)}{x_j - x_{j+1}}. \end{aligned}$$

The isobaric divided difference operators $\bar{\partial}_j$ are defined on $\mathbb{Z}[x_1, \dots, x_n]$ by

$$\bar{\partial}_j(f) \stackrel{\text{def}}{=} \partial_j(f - x_{j+1}f).$$

DEFINITION 2.2. The Grothendieck polynomial $\mathfrak{S}_w$ of $w \in S_n$ is defined recursively on the weak Bruhat order. Let w_0 denote the longest permutation in S_n . If $w \neq w_0$, then there is $j \in [n-1]$ with $w(j) < w(j+1)$. The polynomial $\mathfrak{S}_w$ is defined by

$$\mathfrak{S}_w \stackrel{\text{def}}{=} \begin{cases} x_1^{n-1} x_2^{n-2} \dots x_{n-1} & \text{if } w = w_0, \\ \bar{\partial}_j \mathfrak{S}_{ws_j} & \text{if } w(j) < w(j+1). \end{cases}$$

Recall that a permutation $w \in S_n$ is vexillary if it is 2143-avoiding, that is, if there do not exist $i < j < k < \ell$ with $w(j) < w(i) < w(\ell) < w(k)$.

THEOREM 2.3 (see [Haf22, Theorem 3.4]). *Let $w \in S_n$ be a vexillary permutation, and let $\alpha, \gamma \in \text{supp}(\mathfrak{S}_w)$ be such that $x^\alpha \mid x^\gamma$. Then, any $\beta \in \mathbb{Z}^n$ such that $x^\alpha \mid x^\beta \mid x^\gamma$ is also in $\text{supp}(\mathfrak{S}_w)$.*

Marked bumpless pipe dreams. A *bumpless pipe dream* (BPD) is a tiling of the $n \times n$ grid with the tiles



that form a network of n pipes running from the bottom edge of the grid to the right edge [LLS21, Wei21]. A BPD is *reduced* if each pair of pipes crosses at most once.

Given a BPD, one can define a permutation given by labeling the pipes 1 through n along the bottom edge and then reading off the labels on the right edge, ignoring any crossings after the first, i.e., replacing redundant crossing tiles $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ with bump tiles $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$. The set of all BPDs associated to a permutation w is denoted $\text{BPD}(w)$ and the set of all reduced BPDs associated to w is denoted $\text{RBPD}(w)$.

Given a reduced BPD, we label the pipes 1 through n along the bottom edge.

For any permutation $w \in S_n$, the *Rothe bumpless pipe dream* is the unique BPD $P(w) \in \text{BPD}(w)$ which has no up-elbow tiles $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$; each pipe has one down-elbow tile $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ at $(i, w(i))$.

Given $P \in \text{BPD}(w)$, let $D(P)$ denote the set of blank tiles and $U(P)$ denote the up-elbow tiles. A *marked bumpless pipe dream* (MBPD) is a pair (P, S) where $P \in \text{BPD}(w)$ and $S \subseteq U(P)$. The set of MBPDs is denoted $\text{MBPD}(w)$.

PROPOSITION 2.4 (see [Wei21, Corollary 1.5]). *We have*

$$\mathfrak{S}_w = \sum_{(P,S) \in \text{MBPD}(w)} (-1)^{|D(P)|+|S|-\ell(w)} \left(\prod_{(i,j) \in D(P) \cup S} x_i \right).$$

Given an MBPD (P, S) , the *weight* of (P, S) is the vector $\text{wt}(P, S) \in \mathbb{Z}^n$ whose i th component is the number of tiles in the i th row that are blank or up-elbows.

COROLLARY 2.5. *We have*

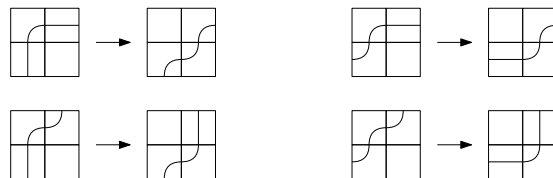
$$\text{supp}(\mathfrak{S}_w) = \{\text{wt}(P, S) : (P, S) \in \text{MBPD}(w)\}.$$

The *rank* of a tile $(i, j) \in D(P)$ is the number of pipes northwest of (i, j) .

LEMMA 2.6 (see [Wei21]). *A permutation $w \in S_n$ is vexillary if and only if every $P \in \text{BPD}(w)$ is reduced.*

Lemma 2.6 guarantees that, for vexillary w , the pipes in any $P \in \text{BPD}(w)$ are labelled.

A *local move* is any of the following local transformations of BPDs:



LEMMA 2.7 (see [Wei21, Lemma 7.4], [Haf22, Lemma 2.3]). *Let $w \in S_n$ be a vexillary permutation, and let $P \in \text{BPD}(w)$. Then P can be obtained from the Rothe*

BPD by using local moves to position the pipe labelled $w(n)$, then to position the pipe labelled $w(n-1)$, and so on through the pipe labelled $w(1)$. Alternatively, P can be obtained from the Rothe BPD by using local moves to position the pipe labelled n , then to position the pipe labelled $n-1$, and so on through the pipe labelled 1.

Supports and Newton polytopes of polynomials. Let $f = \sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha} \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial. The *support* of f is the set

$$\text{supp}(f) = \{\alpha : c_{\alpha} \neq 0\} \subset \mathbb{Z}^n.$$

The *Newton polytope* of f is the convex hull of $\text{supp}(f)$. We denote the Newton polytope by $\text{Newton}(f)$, and we say f has *saturated Newton polytope* (abbreviated SNP) if

$$\text{supp}(f) = \text{Newton}(f) \cap \mathbb{Z}^n.$$

Diagrams. As stated in the introduction, a diagram is a subset $D \subseteq [n] \times [k]$. When we draw diagrams, we read the indices as in a matrix.

Associated to any permutation $w \in S_n$ is the *Rothe diagram* $D(w) \subseteq [n] \times [n]$, defined by

$$D(w) \stackrel{\text{def}}{=} \{(i, j) \in [n] \times [n] : i < w^{-1}(j) \text{ and } j < w(i)\}.$$

The Rothe diagram comes equipped with a *rank function* $r_{D(w)} : D(w) \rightarrow \mathbb{Z}_{\geq 0}$ defined by

$$r_{D(w)}(i, j) = |\{(k, w(k)) : k < i \text{ and } w(k) < j\}|.$$

For $R, S \subseteq [n]$, we say $R \leq S$ if $\#R = \#S$ and the k th smallest element of R does not exceed the k th smallest element of S for every k . For diagrams $C = (C_1, \dots, C_k)$ and $D = (D_1, \dots, D_k)$, we say $C \leq D$ if $C_j \leq D_j$ for every $j \in [k]$.

Flagged Weyl modules. Let Y denote a matrix with indeterminates y_{ij} in the upper triangular positions $i \leq j$ and zeroes elsewhere. Given a matrix $M \in M_n(\mathbb{C})$ and $R, S \subseteq [n]$, let M_R^S denote the submatrix of M obtained by restricting to rows S and columns R .

Let B denote the set of upper triangular matrices in $\text{GL}_n(\mathbb{C})$, and let $\mathfrak{b}$ denote the set of upper triangular matrices in $M_n(\mathbb{C})$. The coordinate ring $\mathbb{C}[\mathfrak{b}]$ is a polynomial ring in the variables $\{y_{ij} : i \leq j\}$. The action of B on $\mathfrak{b}$ by left multiplication induces an action of B on $\mathbb{C}[\mathfrak{b}]$ on the right via $f(Y) \cdot b := f(b^{-1}Y)$.

The *flagged Weyl module* of a diagram $D \subseteq [n] \times [k]$ is the subrepresentation

$$\mathcal{M}_D \stackrel{\text{def}}{=} \text{Span}_{\mathbb{C}} \left\{ \prod_{j=1}^k \det(Y_{D_j}^{C_j}) : C \leq D \right\}$$

of $\mathbb{C}[\mathfrak{b}]$.

The *dual character* of a representation M of B is the function $\text{char}_M^* : T \rightarrow \mathbb{C}$ given by

$$\text{char}_M^*(\text{diag}(x_1, \dots, x_n)) = \text{tr}(\text{diag}(x_1^{-1}, \dots, x_n^{-1}) : M \rightarrow M).$$

We will write $\chi_D := \text{char}_{\mathcal{M}_D}^*$ for the dual character of $\mathcal{M}_D$.

PROPOSITION 2.8 (cf. [FMS18, Theorem 7]). *The function χ_D equals a polynomial in $\mathbb{Z}[x_1, \dots, x_n]$ whose support is $\{\text{wt}(C) : C \leq D\}$.*

Proof. The elements $\prod_j \det(Y_{D_j}^{C_j}) \in \mathcal{M}_D$ are simultaneous eigenvectors for the action of T with eigenvalue $x^{-\text{wt}(C)}$. Since these elements span $\mathcal{M}_D$, the dual character is a sum of monomials of the form $x^{\text{wt}(C)}$ for $C \leq D$. $\square$

Generalized permutahedra and M-convexity. A function $z: 2^{[n]} \rightarrow \mathbb{R}$ is called *submodular* if

$$z(I) + z(J) \geq z(I \cup J) + z(I \cap J) \quad \text{for all } I, J \subseteq [n].$$

DEFINITION 2.9. *A polytope $P \subset \mathbb{R}^n$ is a generalized permutahedron if there is a submodular function $z: 2^{[n]} \rightarrow \mathbb{R}$ such that $z(\emptyset) = 0$ and*

$$P = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \leq z(I) \text{ for all } I \subseteq [n] \text{ and } \sum_{i=1}^n t_i = z([n]) \right\}.$$

LEMMA 2.10 (see [Fra11, Theorem 14.2.8]). *Let $P \subseteq \mathbb{R}^n$ be a generalized permutahedron defined by a submodular function z with $z(\emptyset) = 0$. Then*

$$z(I) = \max \left\{ \sum_{i \in I} p_i : p \in P \right\}.$$

A set $S \subseteq \mathbb{Z}^n$ is *M-convex* if for any $x, y \in S$ and any $i \in [n]$ for which $x_i > y_i$, there is an index $j \in [n]$ satisfying $x_j < y_j$ and $x - e_i + e_j \in S$ and $y - e_j + e_i \in S$.

Note that the convex hull of an M-convex set is a generalized permutahedron, and the set of integer points of an integer generalized permutahedron is an M-convex set.

Schubert matroid polytopes. A *matroid* is a pair $(E, \mathcal{B})$ consisting of a finite set E and a nonempty collection of subsets $\mathcal{B}$ of E , called the *bases* of M . The set $\mathcal{B}$ is required to satisfy the *basis exchange axiom*: If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 \setminus B_2$, then there is $b_2 \in B_2 \setminus B_1$ such that $B_1 \cup b_2 \in \mathcal{B}$.

DEFINITION 2.11. *Fix positive integers $1 \leq s_1 < \dots < s_r \leq n$. The Schubert matroid $\text{SM}_n(s_1, \dots, s_r)$ is the matroid whose ground set is $[n]$ and whose bases are the sets $\{a_1, \dots, a_r\}$ with $a_1 < \dots < a_r$ such that $a_1 \leq s_1, \dots, a_r \leq s_r$.*

Given a matroid $M = ([n], \mathcal{B})$ and a basis $B \in \mathcal{B}$, let $\zeta^B = (\zeta_1^B, \dots, \zeta_n^B)$ be the vector with $\zeta_i^B = 1$ if $i \in B$ and $\zeta_i^B = 0$ if $i \notin B$. The *matroid polytope* $P(M)$ of M is the convex hull $\text{conv}\{\zeta^B : B \in \mathcal{B}\}$. The *rank function* of M is the function $r_M: 2^E \rightarrow \mathbb{Z}_{\geq 0}$ defined by $r_M(S) = \max\{\#(S \cap B) : B \in \mathcal{B}\}$. The function r_M is submodular and $r_M(\emptyset) = 0$. The matroid polytope $P(M)$ is a generalized permutahedron, defined by the submodular function r_M .

3. Bubbling and supports of Grothendieck polynomials. We establish basic properties of bubbling diagrams, including a nonrecursive characterization of the set of bubbling diagrams (Lemma 3.19), and we prove Theorem 1.1 by constructing weight preserving maps between the set of bubbling diagrams and the set of marked BPDs.

DEFINITION 3.1. *A bubbling diagram is a triple (D, r, F) where $D \subseteq [n] \times [k]$ is a diagram, $r: D \rightarrow \mathbb{Z}_{\geq 0}$ is a function, and $F \subseteq D$ is a collection of squares in D satisfying the following properties:*

- If $(i, j) \in F$, then there exists $(i', j) \in D \setminus F$ with $i' < i$. For the maximal $i' < i$ with $(i', j) \in D \setminus F$, the equality $r(i, j) - r(i', j) = i - i'$ holds.
- If $(i, j) \in F$ and $(i, k) \in F$, then $r(i, j) \neq r(i, k)$.

We refer to the squares $(i, j) \in D \setminus F$ as *live squares*, to the squares $(i, j) \in F$ as *dead squares*, and to the squares $(i, j) \notin D$ as *empty squares*.

When we draw a bubbling diagram, the live squares $D \setminus F$ will be colored green, the dead squares F will be colored grey, and each square $(i, j) \in D$ will be labelled by the value of $r(i, j)$. See Figure 1.

The rank of a square $(i, j) \in D$ is the value $r(i, j)$. The weight of a bubbling diagram $\mathcal{D} = (D, r, F)$ is $\text{wt}(\mathcal{D}) \stackrel{\text{def}}{=} \text{wt}(D)$.

Example 3.2. The Rothe bubbling diagram of a permutation w is the bubbling diagram $\mathcal{D}(w) := (D(w), r_{D(w)}, \emptyset)$.

For example, the Rothe bubbling diagram for $w = 1423$ is shown in Figure 1.

DEFINITION 3.3 (bubbling move). Let $\mathcal{D} = (D, r, F)$ be a bubbling diagram. Suppose that (i, j) is a live square and that $(i - 1, j)$ is an empty square.

Then, a bubbling move at (i, j) produces the bubbling diagram $\mathcal{D}' = (D', r', F')$ where

$$\begin{aligned} D' &:= D \setminus (i, j) \cup (i - 1, j), \\ r'(x, y) &:= \begin{cases} r(x, y) & \text{if } (x, y) \neq (i - 1, j), \\ r(i, j) - 1 & \text{if } (x, y) = (i - 1, j), \end{cases} \\ F' &:= F. \end{aligned}$$

In other words, we “bubble up” a live square (i, j) to $(i - 1, j)$, decreasing the rank of the square by 1 in the process.

Example 3.4. Let $w = 1423$. The bubbling diagram obtained from $\mathcal{D}(w)$ by applying a bubbling move at $(2, 3) \in D(w)$ is shown in Figure 2.

DEFINITION 3.5 (K-bubbling move). Let $\mathcal{D} = (D, r, F)$ be a bubbling diagram. Suppose that (i, j) is a live square and that $(i - 1, j)$ is an empty square. Assume, furthermore, that there are no dead squares $(i, k) \in F$ for which $r(i, j) = r(i, k)$.

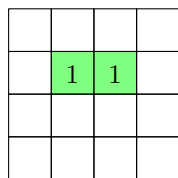


FIG. 1. The Rothe bubbling diagram for $w = 1423$. All squares in $D(w)$ are live.

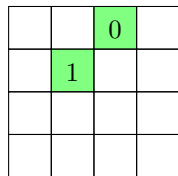


FIG. 2. A bubbling move applied to $(2, 3) \in D(w)$, $w = 1423$.

Then, a K -bubbling move at (i, j) produces the bubbling diagram $\mathcal{D}' = (D', r', F')$ where

$$\begin{aligned} D' &:= D \cup (i-1, j), \\ r'(x, y) &:= \begin{cases} r(x, y) & \text{if } (x, y) \neq (i-1, j), \\ r(i, j) - 1 & \text{if } (x, y) = (i-1, j), \end{cases} \\ F' &:= F \cup (i, j). \end{aligned}$$

In other words, we “bubble up” a live square (i, j) to $(i-1, j)$, decreasing the rank of the square by 1 in the process, while also “leaving behind” a dead copy of the original square (i, j) .

Example 3.6. Let $w = 1423$. The bubbling diagram obtained from $\mathcal{D}(w)$ by applying a K -bubbling move at $(2, 3) \in D(w)$ is shown in Figure 3.

We remark that a dead square cannot be bubbled or K -bubbled, but impacts the K -bubbleability of certain other squares.

DEFINITION 3.7. Let $\mathcal{D}$ be a bubbling diagram. Define $\mathcal{BD}(\mathcal{D})$ to be the set of all bubbling diagrams generated from $\mathcal{D}$ by a series of bubbling moves and K -bubbling moves. For $w \in S_n$, let $\mathcal{BD}(w) := \mathcal{BD}(\mathcal{D}(w))$.

Example 3.8. Let $w = 1423$. Then $\mathcal{BD}(w)$ consists of the bubbling diagrams in Figure 4. Note that the two squares in $D(w)$ cannot both be K -bubbled.

DEFINITION 3.9. Let $\mathcal{D} = (D, r, F)$ be a bubbling diagram where $D \subseteq [n] \times [k]$. For $j \in [k]$, define $\mathcal{D}_j$ to be the j th column of $\mathcal{D}$, i.e., $\mathcal{D}_j := (D_j, r|_{D_j}, F_j)$.

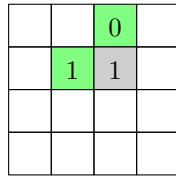


FIG. 3. A K -bubbling move applied to $(2, 3) \in D(w)$, $w = 1423$.

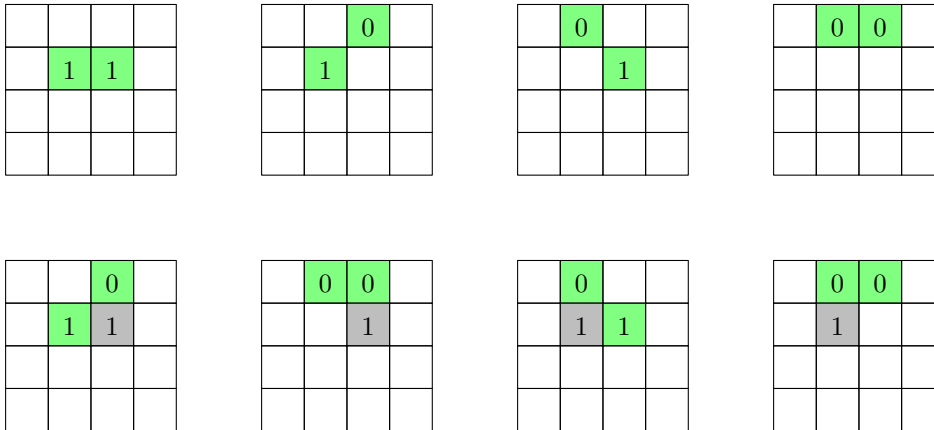


FIG. 4. The set $\mathcal{BD}(w)$ for $w = 1423$.

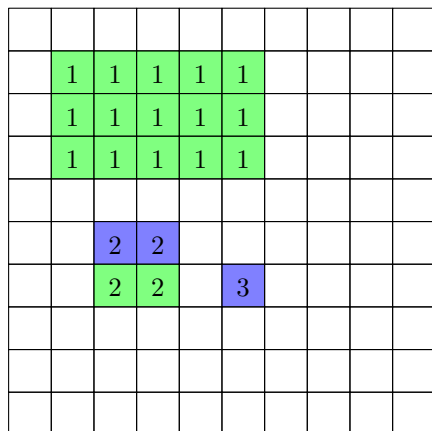


FIG. 5. The squares in the linking class $\{(6,3), (6,4), (7,6)\}$ are shaded blue. (Figure in color online.)

DEFINITION 3.10. Let D be a diagram, and let $r: D \rightarrow \mathbb{Z}_{\geq 0}$ be a function. We say that two squares $(i, j), (i', j') \in D$ are linked if $i - i' = r(i, j) - r(i', j')$. A linking class is an equivalence class of linked squares.

Example 3.11. Let $w = 178925(10)346$. Then, $\{(6,3), (6,4), (7,6)\}$ is a linking class in $\mathcal{D}(w)$. See Figure 5.

LEMMA 3.12. Fix $\mathcal{D} = (D, r, F)$, and let $\mathcal{D}' = (D', r', F') \in \mathcal{BD}(\mathcal{D})$. If the a_j th highest live square in D'_j is linked to the a_k -th highest live square in D'_k , then the a_j th highest live square in D_j is linked to the a_k th highest live square in D_k .

Proof. Let i'_j and i_j denote the a_j th highest live squares in D'_j and D_j , respectively, and let i'_k and i_k denote the a_k th highest live squares in D'_k and D_k , respectively. By assumption,

$$r'(i'_j, j) - r'(i'_k, k) = i'_j - i'_k.$$

Since $\mathcal{D}' \in \mathcal{BD}(\mathcal{D})$, the equalities

$$r'(i'_j, j) - r(i_j, j) = i'_j - i_j \quad \text{and} \quad r'(i'_k, k) - r(i_k, k) = i'_k - i_k$$

hold. It follows that

$$r(i_j, j) - r(i_k, k) = i_j - i_k. \quad \square$$

LEMMA 3.13. Let $w \in S_n$ be a vexillary permutation. Let $(i_1, j_1), (i_2, j_2) \in D(w)$ be two linked squares, and suppose that $j_1 < j_2$. Then,

- (1) $i_1 \leq i_2$, and
- (2) if $(i_1 - 1, j_1), (i_2 - 1, j_2) \notin D(w)$, then the j_1 th and j_2 th columns of $D(w)$ agree above the $(i_1 - 1)$ th row, that is, $(i, j_1) \in D(w)$ if and only if $(i, j_2) \in D(w)$ for all $i \leq i_1 - 1$ and $r_{D(w)}(i, j_1) = r_{D(w)}(i, j_2)$.

Proof. We first show item (1). Suppose that $i_1 > i_2$. Since $j_1 < j_2$, we know that

$$r_{D(w)}(i_2, j_2) \geq |\{(k, w(k)) : k < i_2 \text{ and } w(k) < j_1\}|.$$

Thus

$$\begin{aligned} r_{D(w)}(i_2, j_2) - r_{D(w)}(i_1, j_1) &\geq |\{(k, w(k)) : k < i_2 \text{ and } w(k) < j_1\}| \\ &\quad - |\{(k, w(k)) : k < i_1 \text{ and } w(k) < j_1\}| \\ &= -|\{(k, w(k)) : i_2 \leq k < i_1 \text{ and } w(k) < j_1\}| \\ &\geq i_2 - i_1. \end{aligned}$$

If equality occurs, then $w(k) < j_1$ for all $i_2 \leq k < i_1$; in particular, $w(i_2) < j_1 < j_2$. This contradicts the fact that $(i_2, j_2) \in D(w)$.

We now show item (2). Because $(i_1, j_1) \in D(w)$, we know that $(i, j_2) \in D(w)$ implies $(i, j_1) \in D(w)$ for all $i < i_1$. If $(i, j_1) \in D(w)$ and $(i, j_2) \notin D(w)$, then $i < i_1 - 1 < i_2 < w^{-1}(j_2)$ forms a 2143 pattern. $\square$

Example 3.14. Let $w = 178925(10)346$, as in Figure 5. The live squares $(6, 4)$ and $(7, 6)$ are linked, with $(6, 4)$ to the right of $(7, 6)$. Then, as Lemma 3.13 predicts, (1) the square $(6, 4)$ is above the square $(7, 6)$, and (2) because $(5, 4)$ and $(6, 6)$ are empty, the 4th and 6th columns agree above the 5th row: both columns have live squares in exactly rows 2, 3, 4.

Our next goal is to give a nonrecursive characterization of the elements in $\mathcal{BD}(\mathcal{D})$. In Lemma 3.19, we will show that the axioms below are equivalent to membership in $\mathcal{BD}(\mathcal{D})$.

DEFINITION 3.15. Fix a bubbling diagram $\mathcal{D} = (D, r, F)$. Let $F' \subseteq D'$ be diagrams. We say that (D', F') is $\mathcal{D}$ -admissible if the following occur:

- (1) $F' \supseteq F$.
- (2) $D' \setminus F' \leq D \setminus F$.
- (3) For any $(i, j) \in F' \setminus F$, there is $(i', j) \in D' \setminus F'$ with $i' < i$. Furthermore, suppose that there are m live squares above row i in D'_j . Then the m th highest live square in $D_j \setminus F_j$ is below row i ,
- (4) Suppose that $(i, j) \in F$ and that there are a live squares in D_j above row i and b live squares in D_j below row i . Then there are a live squares in D'_j above row i and b live squares in D'_j below row i ,
- (5) Let $(i, j), (i, k) \in F'$ be dead squares in the same row. Suppose that the m_j th highest square in $D'_j \setminus F'_j$ is the live square immediately above (i, j) and that the m_k th highest square in $D'_k \setminus F'_k$ is the live square immediately above (i, k) . Then the m_j th highest square in D_j and the m_k th highest square in D_k are not linked.

To prove Lemma 3.19, we describe a systematic way to generate a given bubbling diagram $\mathcal{D}' \in \mathcal{BD}(\mathcal{D})$ (Definition 3.17); the legality of the construction is the content of Lemma 3.18.

DEFINITION 3.16. Let $\mathcal{D} = (D, r, F)$ be a bubbling diagram, and let $F' \subseteq D'$ be a pair of diagrams. Let $\mathcal{D}' = (D', F')$ and $\mathcal{D}'_j = (D'_j, F'_j)$. We say that $\mathcal{D}_j$ and $\mathcal{D}'_j$ weakly agree below row s if the following occur:

- $D_j \cap \{s, \dots, n\} = D'_j \cap \{s, \dots, n\}$, and
- $F_j \cap \{s, \dots, n\} = F'_j \cap \{s, \dots, n\}$.

We write $s(\mathcal{D}_j, \mathcal{D}'_j)$ to mean the minimal integer s so that $\mathcal{D}_j$ and $\mathcal{D}'_j$ weakly agree below row s . If no such row exists, then we set $s(\mathcal{D}_j, \mathcal{D}'_j) := n + 1$.

DEFINITION 3.17. Let $\mathcal{D} = (D, r, F)$ be a bubbling diagram. Suppose that $F' \subseteq D'$ are $\mathcal{D}$ -admissible. The canonical bubbling sequence of (D', F') with respect to $\mathcal{D}$ is the sequence $(\mathcal{D}^n, \dots, \mathcal{D}^0)$ defined by the following:

- $\mathcal{D}^n := \mathcal{D}$.
- For $m \geq 1$, $\mathcal{D}^{m-1}$ is obtained from $\mathcal{D}^m = (D^m, r^m, F^m)$ by applying the following bubbling and K -bubbling moves. For each column j with an empty square in row m directly above a live square in row $m+1$, let $k_j := s(\mathcal{D}_j^m, \mathcal{D}'_j)$. If $k_j > m+1$ and $k_j - 1 \notin D'_j$, then apply bubbling moves at $m+1, m+2, \dots, k_j - 1$; if $k_j > m+1$ and $k_j - 1 \in F'_j$, then apply bubbling moves at $m+1, m+2, \dots, k_j - 2$ and then a K -bubbling move at $k_j - 1$.

See Figure 6 for an example.

To ensure the legality of the bubbling moves in Definition 3.17, we use the following lemma.

LEMMA 3.18. Let $\mathcal{D} = (D, r, F)$ be a bubbling diagram. Suppose that $F' \subseteq D'$ is $\mathcal{D}$ -admissible, and let $(\mathcal{D}^n, \dots, \mathcal{D}^0)$ denote the canonical bubbling sequence. Write $\mathcal{D}^m = (D^m, r^m, F^m)$. Suppose that for some m and j , we have $(m, j) \notin D^m$ and $(m+1, j) \in D^m \setminus F^m$. Let $\ell_{j,m} \geq m+1$ be maximal so that $(m+1, j), \dots, (\ell_{j,m}, j) \in D^m \setminus F^m$. Let $k_{j,m} := s(\mathcal{D}_j^m, \mathcal{D}'_j)$. Then,

- (1) $\ell_{j,m} \geq k_{j,m} - 1$.
- (2) Suppose that $k_{j,m} - 1 \in F'_j$ and that for some $h \neq j$, either $k_{j,m} - 1 \in F_h^m$ or $m \notin D_h^m$, $m+1 \in D_h^m$, $k_{h,m} = k_{j,m}$, and $k_{h,m} - 1 \in F'_h$. Then $r^m(k_{j,m} - 1, j) \neq r^m(k_{h,m} - 1, h)$.

Furthermore, $D^0 = D'$ and $F^0 = F'$.

Proof. We first show item (1) using induction. Let $m_1 > \dots > m_h$ be the integers for which $\mathcal{D}_j^{m_i} \neq \mathcal{D}_j^{m_i-1}$. Thus, m_1 is maximal so that $m_1 \notin D_j$ and $m_1 + 1 \in D_j \setminus F_j$. It follows that if $i > \ell_{j,m_1}$ and $i \in D_j \setminus F_j$, then there is $m' < i$ with $m' \in F_j$ and $m' + 1, m' + 2, \dots, i \in D_j \setminus F_j$. Condition (1) implies that $m' \in F'_j$, and conditions (2) and (4) together imply

$$(D_j \setminus F_j) \cap \{\ell_{j,m_1} + 1, \dots, n\} = (D'_j \setminus F'_j) \cap \{\ell_{j,m_1} + 1, \dots, n\}.$$

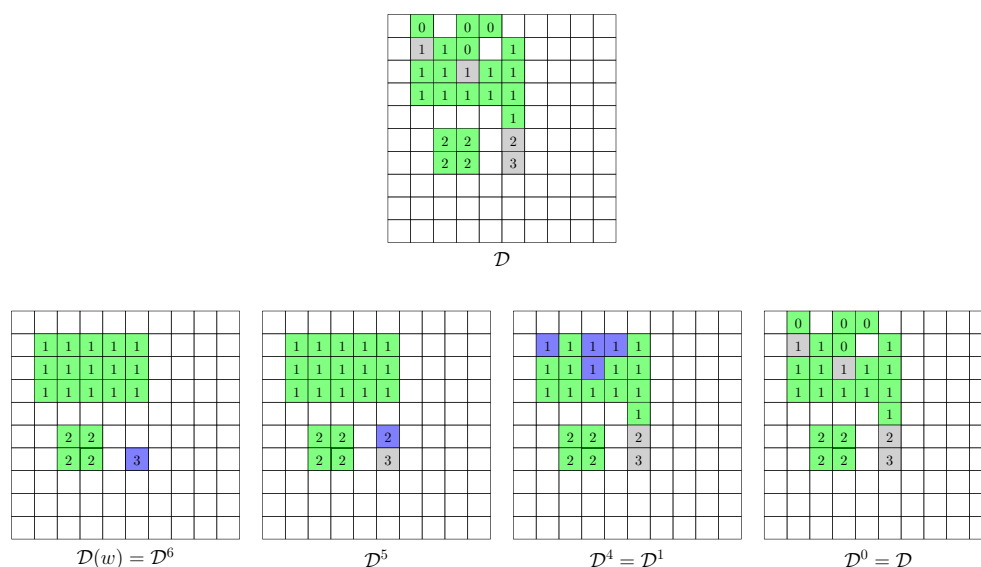


FIG. 6. The canonical bubbling sequence for a bubbling diagram $\mathcal{D} \in \mathcal{BD}(w)$ with $w = 178925(10)346$. The squares which are bubbled or K -bubbled are shaded blue. (Figure in color online.)

Then condition (3) implies that $(F' \setminus F) \cap \{\ell_{j,m_1} + 1, \dots, n\} = \emptyset$. We conclude that $\ell_{j,m_1} \geq k_{j,m_1} - 1$.

Now assume that $\ell_{j,m_i} \geq k_{j,m_i} - 1$. By construction of the canonical bubbling sequence, $\mathcal{D}^{m_{i+1}} = \mathcal{D}_j^{m_i-1}$ weakly agrees with $\mathcal{D}'_j$ below row $k_{j,m_i} - 1$. Furthermore, $k_{j,m_i} - 1 \notin D'_j \setminus F'_j$, so $\ell_{j,m_{i+1}} \leq k_{j,m_i} - 2$. Thus

$$k_{j,m_{i+1}} - 1 \leq k_{j,m_i} - 2 \leq \ell_{j,m_{i+1}}.$$

We now show item (2). By induction, we may assume that $\mathcal{D}^m \in \mathcal{BD}(\mathcal{D})$. Suppose that $k_{j,m} \in D_j^m \setminus F_j^m$ is the a_j th highest live square in its column. Define a_h as follows: if $(k_{j,m} - 1, h) \in F^m$, then suppose that the a_h th highest square in $D_h^m \setminus F_h^m$ is the live square immediately above $(k_{j,m} - 1, h)$; if $k_{j,m} - 1 \in D_h^m \setminus F_h^m$, then suppose that $k_{j,m} - 1$ is the a_h th highest live square in its column. If $r^m(k_{j,m} - 1, j) \neq r^m(k_{h,m} - 1, h)$, then the a_j th highest live square in D_j^m and the a_h th highest live square in D_h^m are linked. Lemma 3.12 implies that the a_j -th highest live square in D_j and the a_h th highest live square in D_h are linked, contrary to condition (5) in Lemma 3.19.

We now show that $D^0 = D'$ and $F^0 = F'$. We claim that $F' \subseteq D'$ is $\mathcal{D}_j^0$ -admissible, that is,

- (1) The canonical bubbling sequence introduces a dead square $(i, j) \in F^0 \setminus F$ only if $(i, j) \in F' \setminus F$, so $F' \supseteq F^0$.
- (2) The canonical bubbling sequence bubbles the m th highest live square $i \in D_j^m$ only if the m th highest live square in D'_j is above row i , so $D' \setminus F' \leq D^0 \setminus F^0$.
- (3) Since $F' \supseteq F^0$, for any $(i, j) \in F' \setminus F^0$, there is $(i', j) \in D' \setminus F'$ with $i' < i$. If the m th and $(m+1)$ th highest live squares in $D_j^0 \setminus F_j^0$ are in row k and ℓ , respectively, then $F_j^0 \cap \{k+1, \dots, \ell-1\} = F'_j \cap \{k+1, \dots, \ell-1\}$; thus, if the m th highest live square in $D'_j \setminus F'_j$ is the live square that is immediately above (i, j) , then the m -th highest live square in $D_j^0 \setminus F_j^0$ is below row i .
- (4) Suppose that $(i, j) \in F^0$ and that there are a live squares in D_j^0 above row i and b live squares in D'_j below row i . If $(i, j) \in F^m \setminus F^{m+1}$ for some m , then $i+1 = k_{j,m+1}$ and there are a and b live squares in D_j^m above and below row i , respectively. It follows that there are a and b live squares in D'_j above and below row i , respectively.
- (5) Let $(i, j), (i, k) \in F'$ be dead squares in the same row. Suppose that the m_j th highest square in $D'_j \setminus F'_j$ is the live square immediately above (i, j) and that the m_k th highest square in $D'_k \setminus F'_k$ is the live square immediately above (i, k) . Then the m_j th highest square in D_j and the m_k th highest square in D_k are not linked; thus Lemma 3.12 guarantees that the m_j th highest square in D_j^0 and the m_k th highest square in D_k^0 are not linked.

Let m be minimal so that $m \notin D_j^0$ and $m+1 \in D_j^0 \setminus F_j^0$; if no such m exists, then set $m := n+1$. It follows that if $i < m$ and $i \in D_j^0 \setminus F_j^0$, then there is $m' < i$ with $m' \in F_j^0$ and $m' + 1, \dots, i \in D_j^0 \setminus F_j^0$. Condition (1) implies that $m' \in F'_j$, and conditions (2) and (4) together imply $(D_j^0 \setminus F_j^0) \cap \{1, \dots, m-1\} = (D'_j \setminus F'_j) \cap \{1, \dots, m-1\}$. Then condition (3) implies that $(F' \setminus F) \cap \{1, \dots, m-1\} = \emptyset$, so it suffices to show that $\mathcal{D}_j^0$ and $\mathcal{D}'_j$ agree below row $m+1$.

Either $m \notin D_j^m$, $m+1 \in D_j^m$, and $s(\mathcal{D}_j^m, \mathcal{D}'_j) \leq m+1$ or $m, m+1 \in D_j^m$ and there exists $\ell < m$ with $k_{j,\ell} = m+1$ so that $s(\mathcal{D}_j^\ell, \mathcal{D}'_j) \leq m$. Furthermore, in both cases, the canonical bubbling sequence then leaves rows $\{m+1, \dots, n\}$ invariant. It follows that $s(\mathcal{D}_j^0, \mathcal{D}'_j) \leq m+1$, and we conclude that $D_j^0 = D'_j$ and $F_j^0 = F'_j$. $\square$

LEMMA 3.19. *Let $\mathcal{D} = (D, r, F)$ be a bubbling diagram, and let $F' \subseteq D'$ be diagrams. Then there exists $r': D' \rightarrow \mathbb{Z}_{\geq 0}$, necessarily unique, so that $(D', r', F') \in \mathcal{BD}(\mathcal{D})$ if and only if (D', F') is $\mathcal{D}$ -admissible.*

Proof of Lemma 3.19. A straightforward check confirms that if $\mathcal{D}' \in \mathcal{BD}(\mathcal{D})$ is $\mathcal{D}$ -admissible, then any bubbling diagram $\mathcal{D}''$ obtained from $\mathcal{D}'$ via a bubbling or K-bubbling move is $\mathcal{D}$ -admissible. Thus, the forward implication follows.

Conversely, take any $\mathcal{D}$ -admissible $F' \subseteq D'$. The canonical bubbling sequence of (D', F') with respect to $\mathcal{D}$ gives a bubbling diagram $\mathcal{D}^0 = (D^0, r^0, F^0) \in \mathcal{BD}(\mathcal{D})$, and Lemma 3.18 guarantees that $D^0 = D'$ and $F^0 = F'$. $\square$

THEOREM 1.1. *Let $w \in S_n$ be a vexillary permutation. Then $\text{supp}(\mathfrak{G}_w) = \{\text{wt}(\mathcal{D}) : \mathcal{D} \in \mathcal{BD}(w)\}$.*

Proof of Theorem 1.1. We first show that $\text{supp}(\mathfrak{G}_w) \subseteq \{\text{wt}(\mathcal{D}) : \mathcal{D} \in \mathcal{BD}(w)\}$. By Corollary 2.5, it suffices to show that for every marked bumpless pipe dream $(P, S) \in \text{MBPD}(w)$, there exists a bubbling diagram $\mathcal{D} \in \mathcal{BD}(w)$ so that $\text{wt}(\mathcal{D}) = \text{wt}(P, S)$. We will construct such a bubbling diagram $\mathcal{D}$ as follows; see Example 3.20 for an example.

Fix $(P, S) \in \text{MBPD}(w)$. Let $(P_m, S_m) \in \text{MBPD}(w)$ denote the MBPD whose $w(m+1), w(m+2), \dots, w(n)$ th marked pipes agree with those of (P, S) and whose $w(1), w(2), \dots, w(m)$ th marked pipes agree with those of the Rothe BPD $P(w)$. Let B_m denote the set of blank tiles of P_m which are not southeast of any of the pipes $w(m+1), \dots, w(n)$.

We will use induction to construct diagrams $\mathcal{D}^m = (D^m, r^m, F^m) \in \mathcal{BD}(w)$ and bijections $f_m : [n] \rightarrow [n]$ so that we have the following:

- The diagram $\mathcal{D}^m$ agrees with $\mathcal{D}(w)$ above row m .
- $\text{wt}(\mathcal{D}^m) = \text{wt}(P_m, S_m)$.
- $(i, j) \in B_m$ if and only if $(i, f_m(j)) \in D^m \setminus F^m$ for every $1 \leq i \leq \max\{k : (k, j) \in B_m\}$, and
- the rank of any square $(i, j) \in B_m$ is equal to $r^m(i, f_m(j))$.

Since (P_n, S_n) is the Rothe BPD, we set $\mathcal{D}^n := \mathcal{D}(w)$ and $f_n := \text{id}$; the three items above hold because $B_n = D(w)$ (as subsets of $[n] \times [n]$) and $S_n = F_n = \emptyset$.

If $(P_{m-1}, S_{m-1}) = (P_m, S_m)$, then we define $\mathcal{D}^{m-1} := \mathcal{D}^m$ and $f_{m-1} := f_m$.

Now suppose $(P_{m-1}, S_{m-1}) \neq (P_m, S_m)$. Assume we are given $\mathcal{D}^m$ and f_m satisfying the items above. Let Q_m denote the set of blank tiles in B_m which are displaced upon replacing the $w(m)$ th pipe in P_m with the $w(m)$ th (marked) pipe of P to obtain (P_{m-1}, S_{m-1}) .

Any square $(i, j) \in Q_m$ that is northernmost in its column satisfies $i = m+1$; it follows by construction of f_m that $(m+1, f_m(j)) \in D_m \setminus F_m$ and that $(m, f_m(j)) \notin D_m$. Furthermore, any square in Q_m has the same rank, $Q_m \cap S_{m-1}$ has at most one square in each row, and any square in $Q_m \cap S_{m-1}$ is southernmost in its column. Thus, we may apply bubbling moves to $\mathcal{D}_m$ at the squares $\{(i, f_m(j)) : (i, j) \in Q_m \setminus S_{m-1}\}$ followed by K-bubbling moves at the squares $\{(i, f_m(j)) : (i, j) \in Q_m \cap S_{m-1}\}$ to produce a bubbling diagram $\mathcal{D}_{m-1}$. This bubbling diagram agrees with $\mathcal{D}(w)$ above row $m-1$ and satisfies $\text{wt}(\mathcal{D}_{m-1}) = \text{wt}(P_{m-1}, S_{m-1})$.

It remains to define the bijection f_{m-1} . Let $j_1 < \dots < j_k$ denote the columns which have squares in Q_m , and set $j_0 := j_1 - 1$. Let $\sigma_m : \{j_0, \dots, j_k\} \rightarrow \{j_0, \dots, j_k\}$ be the map $\sigma_m(j_\ell) = j_{\ell+1}$ (with $j_{k+1} := j_0$), and let $f_{m-1} := f_m \circ \sigma_m$. Since $(m, j_0), \dots, (m, j_k) \notin D(w)$, Lemma 3.13 guarantees that the $j_0, \dots, j_k$ th columns of $D(w)$ all agree above row m . Combined with the fact that $(i, j) \in B_m$ if and only if $(i, f_m(j)) \in D^m \setminus F^m$ for all $1 \leq i \leq \max\{k : (k, j) \in B_m\}$, it follows that $(i, j) \in B_{m-1}$ if and only if $(i, f_{m-1}(j)) \in D^{m-1} \setminus F^{m-1}$ for every $1 \leq i \leq \max\{k : (k, j) \in B_{m-1}\}$ and, furthermore, that the rank of any square $(i, j) \in B_{m-1}$ is equal to $r^{m-1}(i, f_{m-1}(j))$.

We now show that $\text{supp}(\mathfrak{G}_w) \supseteq \{\text{wt}(\mathcal{D}) : \mathcal{D} \in \mathcal{BD}(w)\}$. By Corollary 2.5, it suffices to show that for every diagram $\mathcal{D} \in \mathcal{BD}(w)$, there exists a MBPD $(P, S) \in \text{MBPD}(w)$ such that $\text{wt}(\mathcal{D}) = \text{wt}(P, S)$. We accomplish this using the following construction; see Example 3.21 for an example.

Fix $\mathcal{D} = (D, r, F) \in \mathcal{BD}(w)$. Let $\mathcal{D}^m = (D^m, r^m, F^m)$ denote the canonical bubbling sequence. We will construct MBPDs (P_m, S_m) and bijections $g_m : [n] \rightarrow [n]$ so that

- The $w(1), \dots, w(m)$ th pipes of P_m have no up-elbow tiles.
- $\text{wt}(\mathcal{D}^m) = \text{wt}(P_m, S_m)$.
- $(i, j) \in B_m$ if and only if $(i, g_m(j)) \in D^m \setminus F^m$ for every $1 \leq i \leq \max\{k : (k, j) \in B_m\}$.
- The rank of any square $(i, j) \in B_m$ is equal to $r_m(i, g_m(j))$.

Since $\mathcal{D}^n$ is the Rothe bubbling diagram, we set (P_n, S_n) to be the Rothe BPD and $g_n := \text{id}$. If $\mathcal{D}^{m-1} = \mathcal{D}^m$, then we define $(P_{m-1}, S_{m-1}) := (P_m, S_m)$ and $f_{m-1} := f_m$.

Now suppose $\mathcal{D}^{m-1} \neq \mathcal{D}^m$. Assume we are given (P_m, S_m) and f_m satisfying the items above. Let $j_1, \dots, j_\ell$ be the columns which are bubbled when constructing $\mathcal{D}^{m-1}$, and let $k_{j_i} = s(\mathcal{D}_{j_i}^m, \mathcal{D}_{j_i}^{m-1})$, indexed so that $k_{j_1} \geq \dots \geq k_{j_\ell}$ and so that if $(k_{j_i}, j_i) \in F$, then $k_{j_i} > k_{j_{i+1}}$.

By assumption on g_m and by Lemma 3.18, the squares $(x, g_m^{-1}(j_i))$ are in B_m for $m+1 \leq x \leq k_{j_i} - 1$ while the squares $(m, g_m^{-1}(j_i))$ are not in B_m . It follows that the squares $(i, g_m^{-1}(j_i))$ are southeast of pipe $w(m)$.

Let

$$R_m = \{(x, w^{-1}(m) + y) : m+1 \leq x \leq k_{j_y} - 1, 1 \leq y \leq \ell\}.$$

We define (P_{m-1}, S_{m-1}) to be the BPD obtained from P_m by replacing pipe $w(m)$ with the pipe that traces the southeasternmost squares of R_m and marking the up-elbow tiles at $(k_{j_i} - 1, w^{-1}(m))$ whenever $(k_{j_i} - 1, j_i) \in F$. The $w(1), \dots, w(m-1)$ th pipes of P_{m-1} have no up-elbow tiles, and this BPD satisfies $\text{wt}(\mathcal{D}_{m-1}) = \text{wt}(P_{m-1}, S_{m-1})$.

It remains to define the bijection g_{m-1} . Let h_m be the maximal integer such that $(m+1, h_m) \in B_m$. Let $\psi_m : \{w^{-1}(m), w^{-1}(m) + 1, w^{-1}(m) + 2, \dots, h_m\} \rightarrow \{w^{-1}(m), w^{-1}(m) + 1, w^{-1}(m) + 2, \dots, h_m\}$ denote a fixed bijection which sends $w^{-1}(m) + i - 1$ to $g_m^{-1}(j_i)$. Then we define $g_{m-1} := g_m \circ \psi_m$. Since $(m, w^{-1}(m) + 1), \dots, (m, h_m) \notin D(w)$, Lemma 3.13 guarantees that the $w^{-1}(m) + 1, \dots, h_m$ th rows of $D(w)$ all agree above row m . Combined with the fact that $(i, j) \in B_m$ if and only if $(i, g_m(j)) \in D^m \setminus F^m$ for all $1 \leq i \leq \max\{k : (k, j) \in B_m\}$, it follows that $(i, j) \in B_{m-1}$ if and only if $(i, g_{m-1}(j)) \in D^{m-1} \setminus F^{m-1}$ for every $1 \leq i \leq \max\{k : (k, j) \in B_{m-1}\}$ and, furthermore, that the rank of any square $(i, j) \in B_{m-1}$ is equal to $r^{m-1}(i, g_{m-1}(j))$. $\square$

Example 3.20. Let $(P, S) \in \text{MBPD}(w)$ be as in Figure 7.

The MBPDs (P_m, S_m) are equal to $(P(w), \emptyset)$ for $m \geq 5$, so the bubbling diagrams $\mathcal{D}^m$ are equal to $\mathcal{D}(w)$ for $m \geq 5$. Furthermore, f_m is the identity for $m \geq 5$. The set

$$Q_5 = \{(6, 3), (7, 3), (6, 4)\}$$

is, however, nonempty, so $(P_4, S_4) \neq (P_5, S_5)$. The set Q_5 contains squares in columns 3 and 4, and the map $f_4 = \sigma_5$ cyclically permutes the set $(2, 3, 4)$.

Then $(P_m, S_m) = (P_4, S_4)$ for $1 \leq m \leq 4$, so $\mathcal{D}^m = \mathcal{D}^4$ for $1 \leq m \leq 4$; however,

$$Q_1 = \{(2, 2), (3, 2), (4, 2), (2, 3), (3, 3), (4, 3), (2, 4)\}$$

is nonempty, so $(P_0, S_0) \neq (P_1, S_1)$. The set Q_1 contains squares in columns 2, 3, and 4, and the map σ_1 cyclically permutes the set $(1, 2, 3, 4)$. Thus, the map $f_0 = f_4 \circ \sigma_1$ cyclically permutes the set $(1, 3, 2, 4)$. See Figure 8.

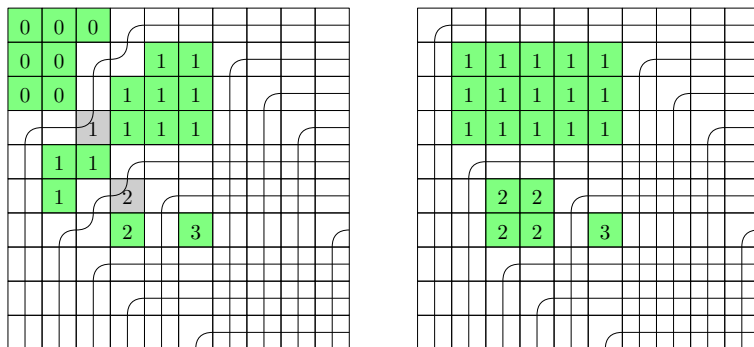


FIG. 7. Left: A marked pipe dream $(P, S) \in \text{MBPD}(w)$ for $w = 178925(10)346$ with blank tiles shaded green and marked up-elbows shaded grey; each blank and marked tile is labeled by its rank. Right: The Rothe MBPD $(P(w), \emptyset)$ with blank and marked tiles labeled by their ranks. (Figure in color online.)

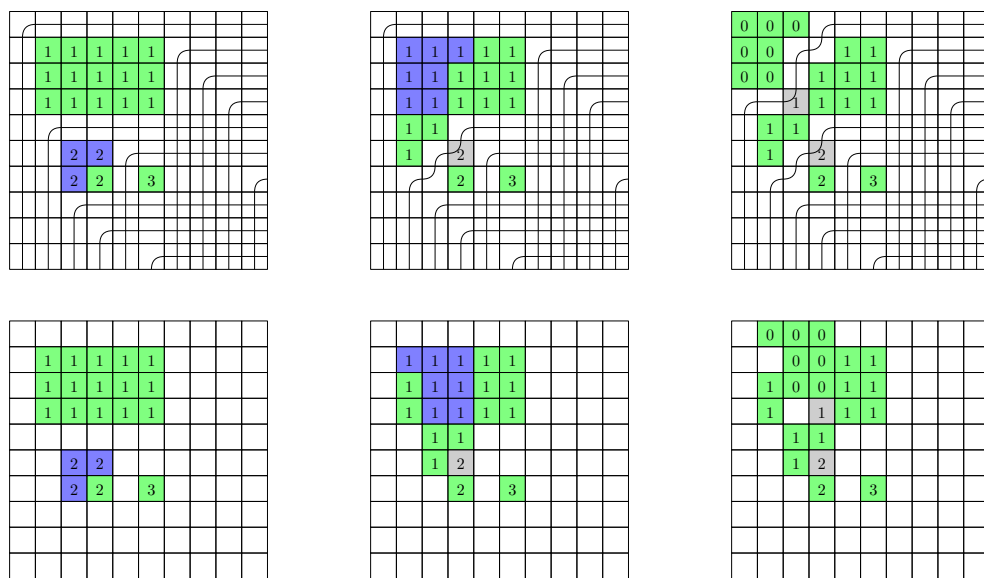


FIG. 8. Left column: The pipe dream $(P_5, S_5) = (P(w), \emptyset)$ for P as in Figure 7 along with the diagram $\mathcal{D}^5 = \mathcal{D}(w)$. The squares in Q_5 and its image under f_5 are shaded blue. Middle column: The pipe dream (P_1, S_1) along with $\mathcal{D}^1$. The squares in Q_1 and its image under f_1 are shaded blue. Right column: The pipe dream $(P_0, S_0) = P$ along with the diagram $\mathcal{D}^0 = \mathcal{D}$. (Figure in color online.)

Example 3.21. Let $\mathcal{D} \in \mathcal{BD}(w)$ be as in Figure 9. The bubbling diagrams $\mathcal{D}^m$ are equal to $\mathcal{D}(w)$ for $m \geq 6$, so the MBPDs (P_m, S_m) are equal to $(P(w), \emptyset)$ for $m \geq 6$. Furthermore, g_m is the identity for $m \geq 6$. Then, $\mathcal{D}^5$ is obtained from $\mathcal{D}^6$ by applying a K-bubbling move to $(7, 6) \in D^6 \setminus F^6$. We have

$$R_6 = \{(7, 6)\},$$

so $(P_5, S_5) \neq (P_6, S_6)$. The map $\psi_6: \{5, 6\} \rightarrow \{5, 6\}$ is a fixed bijection which sends $w^{-1}(6) + 1 - 1 = 5$ to $g_6^{-1}(6) = 6$, so ψ_6 will cyclically permute $(5, 6)$. Thus, $g_5 = \psi_6$ cyclically permutes $(5, 6)$.

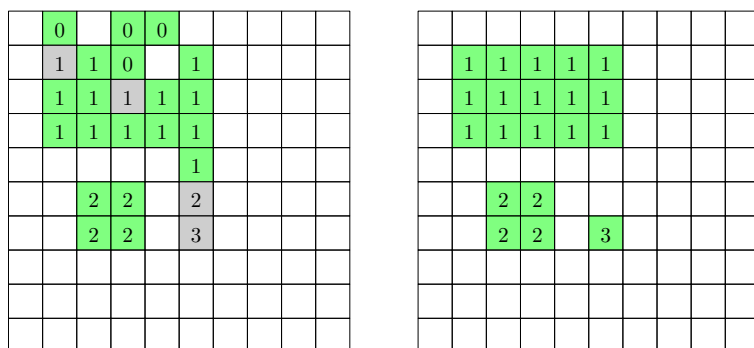


FIG. 9. Left: A bubbling diagram $\mathcal{D} \in \mathcal{BD}(w)$ for $w = 178925(10)346$. Right: The Rothe bubbling diagram $\mathcal{D}(w)$.

Next, $\mathcal{D}^4$ is obtained from $\mathcal{D}^5$ by applying a K-bubbling move to $(6, 6) \in D^5 \setminus F^5$. We have

$$R_5 = \{(6, 3)\},$$

so $(P_4, S_4) \neq (P_5, S_5)$. The map $\psi_5: \{2, 3, 4, 5\} \rightarrow \{2, 3, 4, 5\}$ is a fixed bijection which sends $w^{-1}(5) + 1 - 1 = 2$ to $g_5^{-1}(6) = 5$; suppose that ψ_5 cyclically permutes $(2, 5)$. Then, $g_4 = g_5 \circ \psi_5$ cyclically permutes $(2, 6, 5)$.

Then, $\mathcal{D}^m = \mathcal{D}^4$ and $g_m = g_4$ for $1 \leq m \leq 4$. The bubbling diagram $\mathcal{D}^0 = \mathcal{D}$ is obtained from $\mathcal{D}^1$ by applying bubbling moves to $\{(2, 4), (2, 5)\} \in D^1 \setminus F^1$ and then K-bubbling moves to $\{(2, 1), (3, 4)\} \in D^1 \setminus F^1$. We have

$$R_1 = \{(2, 2), (3, 2), (2, 3), (2, 4)\}.$$

The map $\psi_1: \{1, 2, 3, 4, 5, 6\} \rightarrow \{1, 2, 3, 4, 5, 6\}$ is a fixed bijection which sends $w^{-1}(1) + 1 - 1 = 1$ to $g_1^{-1}(4) = 4$, $w^{-1}(1) + 2 - 1 = 2$ to $g_1^{-1}(5) = 6$, and $w^{-1}(1) + 3 - 1 = 3$ to $g_1^{-1}(2) = 5$; suppose that ψ_1 cyclically permutes $(1, 4)$ and $(2, 6)$ and $(3, 5)$. Then, $g_0 = g_1 \circ \psi_1$ cyclically permutes $(1, 4)$ and $(2, 5, 3)$.

See Figure 10.

4. Supports of top degree components of Grothendieck polynomials.

We prove Theorem 1.2 by showing that bubbling diagrams can be systematically padded to obtain a top-degree diagram which is necessarily in $\text{supp}(\chi_{D^{\text{top}}(w)})$ (Theorem 4.6), and we show that divisibility relations among monomials in $\mathfrak{S}_w$ can be realized by inclusion relations among bubbling diagrams in a strong sense (Theorem 4.10).

DEFINITION 4.1. Let $w \in S_n$ be a vexillary permutation. We will construct an ordered set $A(w)$ of distinguished live squares using the following procedure:

- (1) Endow the squares in $D(w)$ with the total ordering given by $(i, j) \prec (i', j')$ if the following occur:
 - (a) $i > i'$, or
 - (b) $i = i'$ and (i, j) has fewer squares below it than (i', j') , or
 - (c) $i = i'$, (i, j) has the same number of squares below it as does (i', j') , and $j < j'$.
- (2) Add the first square in this ordering to $A(w)$.

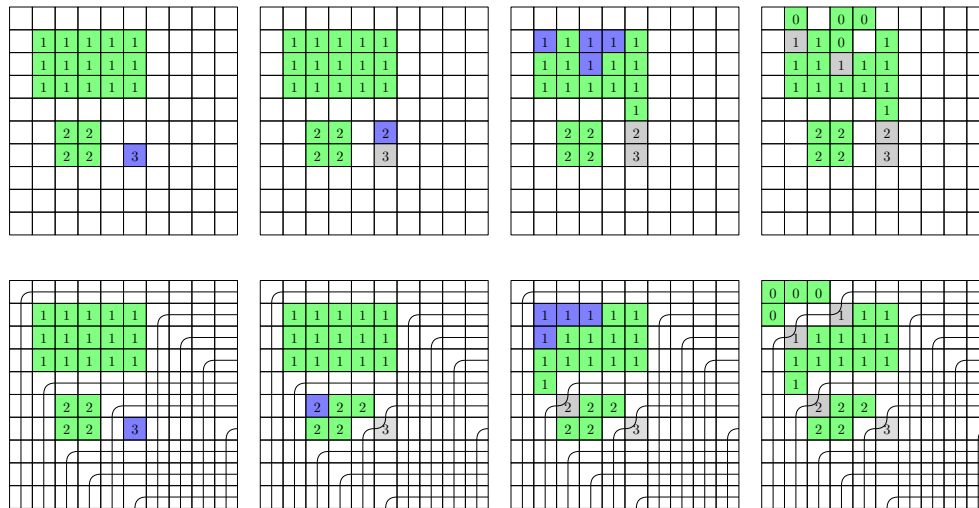


FIG. 10. Left column: The diagram $\mathcal{D}^6 = \mathcal{D}(w)$ for $\mathcal{D}$ as in Figure 9 along with the pipe dream $(P_6, S_6) = (P(w), \emptyset)$; R_6 and its image under g_6 are shaded blue. Second column: The diagram $\mathcal{D}^5$ along with (P_5, S_5) ; R_5 and its image under g_5 are shaded blue. Third column: The diagram $\mathcal{D}^1$ along with (P_1, S_1) ; R_1 and its image under g_1 are shaded blue. Right column: The diagram $\mathcal{D}^0 = \mathcal{D}$ along with $(P_0, S_0) = (P, S)$. (Figure in color online.)

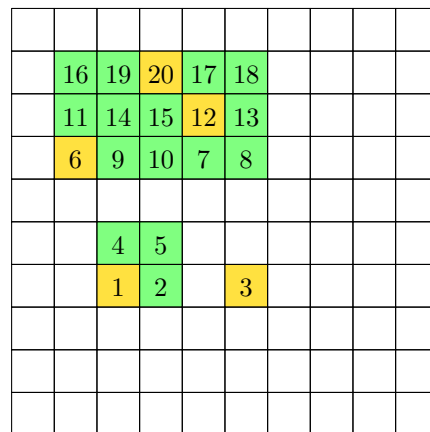


FIG. 11. The $\prec$ ordering on $D(w)$, along with the set $A(w)$, for $w = 178925(10)346$.

- (3) Each subsequent square in the ordering will be appended to $A(w)$ if and only if $A(w)$ does not already contain a square in the same column and $A(w)$ does not already contain a square in the same linking class.

Example 4.2. The diagram $D(w)$ for $w = 178925(10)346$ with squares labeled by their positions in the $\prec$ order is shown in Figure 11. The squares in $A(w)$ are colored gold.

DEFINITION 4.3. Let $w \in S_n$ be a vexillary permutation, and let $\mathcal{D} \in \mathcal{BD}(w)$. Suppose that the k th square in $A(w)$ is the m_k th highest square in column j_k . Define the distinguished live squares $A(\mathcal{D})$ of $\mathcal{D}$ to be the ordered set whose k th element is the m_k th highest live square in D_{j_k} .

LEMMA 4.4. Let $w \in S_n$ be a vexillary permutation, and let $\mathcal{D} = (D, r, F) \in \mathcal{BD}(w)$. Suppose that a dead square $(i, j) \in F$ is linked to the k th highest live square in D_j . Suppose that the ℓ th square in $A(\mathcal{D})$ is the k' th highest live square in D_j for $k' < k$. Then (i, j) is linked to the ℓ' th square in $A(\mathcal{D})$ for some $\ell' < \ell$.

Proof. As the k' th highest live square in D_j is in $A(\mathcal{D})$, the k' th highest square in $D(w)_j$ is in $A(w)$. Similarly, the k th highest square in $D(w)_j$ is not in $A(w)$. As the k th highest square in $D(w)_j$ precedes the k' th highest square in the $\prec$ order, it follows that the k th highest square in $D(w)_j$ is linked to the ℓ' th square in $A(w)$ for $\ell' < \ell$. Thus, the k th highest live square in D_j is linked to the ℓ' th square in $A(\mathcal{D})$. $\square$

DEFINITION 4.5. Let $w \in S_n$ be a vexillary permutation. Construct the bubbling diagram

$$\mathcal{D}^{\text{top}}(w) := (D^{\text{top}}(w), r_{D(w)}^{\text{top}}, F^{\text{top}}(w)) \in \mathcal{BD}(w)$$

from $\mathcal{D}(w)$ by repeatedly applying bubbling moves to every square that is above a distinguished live square until it is no longer possible to do so and then repeatedly applying K -bubbling moves to every distinguished live square until it is no longer possible to do so.

For example, the bubbling diagram $\mathcal{D}^{\text{top}}(w)$ for $w = 178925(10)346$ is shown in Figure 12.

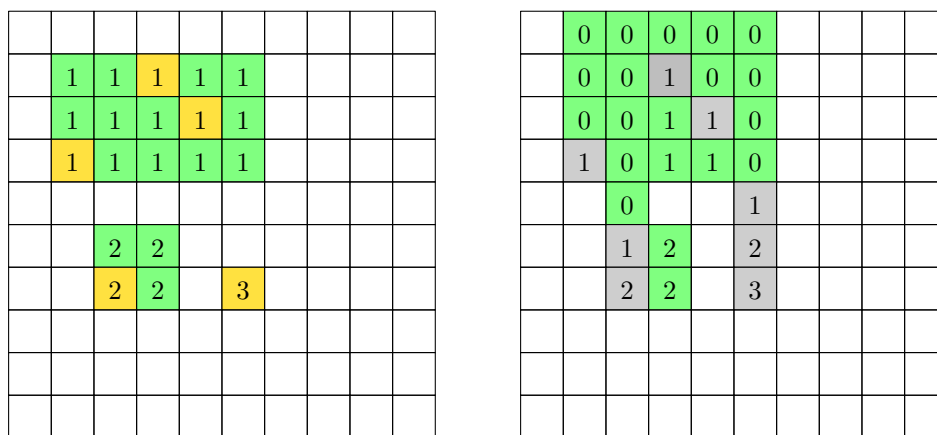
THEOREM 4.6. Let $w \in S_n$ be a vexillary permutation. For any $\mathcal{D} = (D, r, F) \in \mathcal{BD}(w)$, there is $\mathcal{D}' = (D', r', F') \in \mathcal{BD}(w)$ with $F' = F^{\text{top}}(w)$ and $x^{\text{wt}(D)} \mid x^{\text{wt}(D')}$.

The proof of Theorem 4.6 uses the following three lemmas.

LEMMA 4.7. Let $w \in S_n$ be vexillary. Let $(i_1, j_1), (i_2, j_2) \in D(w)$ be two linked squares, and suppose that $i_1 < i_2$. Then $(i', j_2) \in D(w)$ implies $(i', j_1) \in D(w)$ for all $i' > i_2$. In particular, (i', j_1) has at least as many squares below it as (i', j_2) does, and the k th square of $D(w)_{j_2}$ below row i' is in the same row or in a lower row than the k th square of $D(w)_{j_1}$ below row i' .

Proof. Lemma 3.13 implies that $j_1 < j_2$. Observe that if

$$|\{(k, w(k)) : k < i_2 \text{ and } w(k) < j_2\}| - |\{(k, w(k)) : k < i_1 \text{ and } w(k) < j_1\}| = i_2 - i_1,$$



then there exists $(k, w(k)) \in \{(k, w(k)) : k < i_2 \text{ and } w(k) < j_2\}$ with $k \leq i_1$ but $w(k) > j_1$. Since $w(i_1) = j_1$, it follows that $k < i_1$.

If there exists $i' > i_2$ with $(i', j_2) \in D(w)$ and $(i', j_1) \notin D(w)$, then $k < w^{-1}(j_1) < i' < w^{-1}(j_2)$ is a 2143 pattern. $\square$

LEMMA 4.8. *Let $w \in S_n$ be vexillary. Let $(i, j_1), (i, j_2) \in D(w)$, and suppose that (i, j_1) has more squares below it than (i, j_2) does. Then $(i', j_2) \in D(w)$ implies $(i', j_1) \in D(w)$ for all $i' > i$. In particular, the k th square of $D(w)_{j_2}$ below row i is in the same row or in a lower row than the k th square of $D(w)_{j_1}$ below row i .*

Proof. Suppose there exists $i' > i$ such that $(i', j_2) \in D(w)$ but $(i', j_1) \notin D(w)$. Because (i, j_1) has more squares below it than (i, j_2) does, there exists $\ell > i$ with $(\ell, j_1) \in D(w)$ and $(\ell, j_2) \notin D(w)$. If $j_1 < j_2$, then $j_1 < w^{-1}(\ell) < j_2 < w^{-1}(i')$ is a 2143 pattern; if $j_2 < j_1$, then $j_2 < w^{-1}(i') < j_1 < w^{-1}(\ell)$ is a 2143 pattern. $\square$

LEMMA 4.9. *Let $w \in S_n$ be vexillary. Let $(i_1, j_1), (i_2, j_2) \in D(w)$ be two linked squares with $j_1 < j_2$, and let $i'_1 < i_1$ be minimal so that $(i'_1, j_1) \in D(w)$ for all $i'_1 \leq i \leq i_1$. Then there are exactly $i_1 - i'_1$ indices $k_1 < \dots < k_{i_1 - i'_1}$ for which $i'_1 \leq k_m < i_2$ and $(k_m, j_2) \in D(w)$. Furthermore, (k_m, j_2) is linked to $(i'_1 + m - 1, j_1)$.*

Proof. Lemma 3.13 implies that $i_1 < i_2$. Thus,

$$\{(k, w(k)) : k < i_1 \text{ and } w(k) < j_1\} \subseteq \{(k, w(k)) : k < i_2 \text{ and } w(k) < j_2\}.$$

Since $(i, j_1) \in D(w)$ for all $i'_1 \leq i \leq i_1$, we know $w(i) > j_1$ for all $i'_1 \leq i \leq i_1$ and hence,

$$\{(k, w(k)) : k < i'_1 \text{ and } w(k) < j_1\} = \{(k, w(k)) : k < i_1 \text{ and } w(k) < j_1\}.$$

If some $(k, w(k)) \in \{(k, w(k)) : k < i_2 \text{ and } w(k) < j_2\} \setminus \{(k, w(k)) : k < i'_1 \text{ and } w(k) < j_1\}$ satisfies $k < i'_1$, then $w(k) \geq j_1 > w^{-1}(i'_1 - 1)$, and it follows that $w^{-1}(i'_1 - 1) < w(k) < j_2 < w^{-1}(i_2)$ forms a 2143 pattern. Because w is vexillary, it follows that all $i_2 - i_1$ such elements $(k, w(k))$ satisfies $k \geq i'_1$. There are, thus, exactly $i_2 - i_1$ elements b such that $i'_1 \leq b < i_2$ and $(b, j_2) \notin D(w)$; therefore, there are exactly $(i_2 - i'_1) - (i_2 - i_1) = i_1 - i'_1$ elements k such that $i'_1 \leq k < i_2$ and $(k, j_2) \in D(w)$.

The linkedness result follows from the facts that

$$r_{D(w)}(i, j_1) = r_{D(w)}(i_1, j_1) \quad \text{for all } i'_1 \leq i < i_1$$

and

$$r_{D(w)}(k_m, j_2) = r_{D(w)}(i_2, j_2) - i_2 + k_m + (i_1 - i'_1 - m + 1) \quad \text{for all } m \in [i_1 - i'_1]. \quad \square$$

Proof of Theorem 4.6. Denote the k th square in $A(w)$ by (i_k^*, j_k) , and suppose that (i_k^*, j_k) is the m_k th highest square of $D(w)$ in its column. We will construct diagrams $\mathcal{D}^k = (D^k, r^k, F^k)$ satisfying the following properties for all k :

- $\mathcal{D}^k \in \mathcal{BD}(w)$,
- $x^{\text{wt}(\mathcal{D}^{k-1})} \mid x^{\text{wt}(\mathcal{D}^k)}$,
- $(i, j_\ell) \in D^k \setminus F^k$ for all $\ell \in [k]$ and $1 \leq i \leq m_\ell$,
- $(i, j_\ell) \in F^k$ for all $\ell \in [k]$ and $m_\ell + 1 \leq i \leq i_\ell^*$,
- $(i, j_\ell) \notin F^k$ for all $\ell \in [k]$ and $i_\ell^* + 1 \leq i \leq n$,
- $\mathcal{D}_{j_\ell}^k = \mathcal{D}_{j_\ell}^{k-1}$ for all $\ell \in [k-1]$.

Set $\mathcal{D}^0 := \mathcal{D}$. Given $\mathcal{D}^{k-1}$, we construct $\mathcal{D}^k$ according to the following procedure.

Let $i_k \in D_{j_k}^{k-1} \setminus F_{j_k}^{k-1}$ be the m_k th highest live square in its column. Observe that $\mathcal{D}_{j_k}^{k-1}$ contains no dead squares (f, j_k) linked to a live square below row i_k : Lemma 4.4

guarantees that (f, j_k) is linked to (m_ℓ, j_ℓ) for some $\ell < k$, and the definition of $\prec$ guarantees that (f, j_k) is linked to a dead square (f, j_ℓ) in the same row. In particular, $\mathcal{D}_{j_k}^{k-1}$ contains no dead squares below row $i_k^* + 1$.

Let L denote the set of dead squares in F^{k-1} which are below row i_k and are linked to (i_k, j_k) . Let $C = \{c_1, \dots, c_r\}$ be the set of columns that have a square in L ; note that $C \cap \{j_1, \dots, j_{k-1}\}$ is empty as all dead cells in column j_ℓ are linked to (m_ℓ, j_ℓ) and hence not linked to (i_k, j_k) .

We shall first move squares in L horizontally between the columns in C and reindex the c_i so that whenever $k < k'$, all squares of L in column c_k are above all squares of L in column $c_{k'}$ using the following process. Let (r_i^-, c_i) denote the live square in column c_i that is linked to (i_k, j_k) , and let $r_i^+ > r_i^-$ be minimal such that (r_i^+, c_i) is live. The squares in (i, c_j) for $r_j^- < i < r_j^+$ are either dead or empty; if they are dead, then they are linked to (i_k, j_k) . Furthermore, if $r_j^- < i < r_j^+$ and $r_k^- < i < r_k^+$, then (i, c_j) and (i, c_k) cannot both be dead. If $[r_i^-, r_i^+] \supseteq [r_j^-, r_j^+]$ for some i, j , then we may move all dead squares in $\{(i, c_j) : r_j^- < i < r_j^+\}$ to column c_i (to break a tie $[r_i^-, r_i^+] = [r_j^-, r_j^+]$, we move all dead squares to the column with the smaller index). Now, reorder the c_i so that $r_1^- > r_2^- > \dots$, and move all dead squares in $\{(i, c_j) : r_j^- < i < r_j^+\}$ to (i, c_k) for k minimal such that $i < r_k^+$.

We now modify each column $c_i \in C$, starting from c_1 and working towards c_r , according to the following procedure.

Let $x_1 < \dots < x_{k_1}$ be the rows below i_k where (x_s, c_i) is dead and linked to (i_k, j_k) and where (x_s, j_k) is live. Also, let $y_1 < \dots < y_{k_2}$ be the rows below i_k where (y_s, c_i) is dead and linked to (i_k, j_k) and where (y_s, j_k) is empty. Let $z_1 < \dots < z_{k_1}$ be the first k_1 rows below $\max\{x_{k_1}, y_{k_2}\}$ where (z_s, c_i) is live and (z_s, j_k) is empty; such rows $z_1, \dots, z_{k_1}$ exist because the live square (a, c_i) immediately above the dead squares (x_s, c_i) and (y_s, c_i) is linked to $(i_k, j_k) \in A(\mathcal{D}^k)$, so Lemma 4.7 implies (a, c_i) has at least as many live squares below it as (i_k, j_k) does.

Let k_3 be the number of rows between i_k and $\max\{x_{k_1}, y_{k_2}\}$ which have an empty space in column c_i and a live square in column j_k . Modify the portions of columns c_i and j_k below row i_k such that

- Column c_i has live squares in rows $x_1, \dots, x_{k_1}$ and any rows below i_k which previously had live squares, except for rows $z_1, \dots, z_{k_1}$,
- column j_k has dead squares in all rows between $i_k + 1$ and $\max\{x_{k_1}, y_{k_2}\}$, inclusive, along with dead squares in any other rows which already has dead squares, and
- column j_k has live squares in all rows below $\max\{x_{k_1}, y_{k_2}\}$ which already had a live square, rows $z_1, \dots, z_{k_1}$, and the first other k_3 rows below $\max\{x_{k_1}, y_{k_2}\}$.

See Figure 13 for an example. Letting $S_{c_i; i}$ and $S_{j_k; i}$ denote the set of live squares in the modified columns c_i and j_k , respectively, Lemmas 4.7 and 4.8 imply that $S_{c_i; i} \leq D(w)_{c_i}$ and $S_{j_k; i} \leq D(w)_{j_k}$. Then, Lemma 3.19 implies that the resulting diagram is in $\mathcal{BD}(w)$.

At this point, every square in L is in column j_k . We now use the following procedure to bubble up squares in column j_k so that (i, j_k) is live for all $\ell \in [k]$ and $1 \leq i \leq m_\ell$:

- If there is a column $j \neq j_k$ such that $(i_k, j) \in F$ is dead and linked to (i_k, j_k) , swap the portions of columns j and j_k in and above row i_k . Then fill in dead squares between (i_k, j_k) and the next lowest live square above it, removing matching dead squares from other columns if necessary. Such a j is necessarily not equal to j_ℓ for $\ell \in [k-1]$, as those columns contain only dead squares

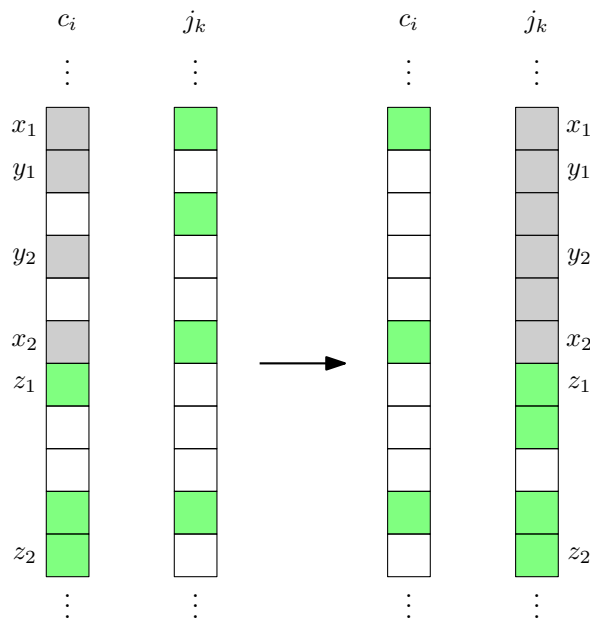


FIG. 13. In this example, $k_3 = 1$ because row $(y_1 + 1, c_i)$ is empty and $(y_1 + 1, j_k)$ is live. After performing the procedure, $(z_2 - 3, j_k)$ is live because it is the first row below $\max\{x_2, y_2\}$ which did not already have a live square and is not row z_1 or z_2 .

linked to (m_ℓ, j_ℓ) . Letting S_j and S_{j_k} denote the set of live squares in the modified columns j and j_k , respectively, Lemmas 3.13 and 4.9 imply that $S_j \leq D(w)_j$ and $S_{j_k} \leq D(w)_{j_k}$. Then Lemma 3.19 implies that the resulting diagram is in $\mathcal{BD}(w)$.

- If there is no such column, and for the maximal $i < i_k$ so that $i \notin D_{j_k} \setminus F_{j_k}$, (i, j_k) is empty, then apply bubbling moves at $i + 1, \dots, i_k - 1$ followed by a K-bubbling move at i_k . The resulting diagram is in $\mathcal{BD}(w)$.
- If there is no such column, and for the maximal $i < i_k$ so that $i \notin D_{j_k} \setminus F_{j_k}$, (i, j_k) is dead, then remove this dead square and apply bubbling moves at $i + 1, \dots, i_k - 1$ followed by a K-bubbling move at i_k . The resulting diagram is in $\mathcal{BD}(w)$.

When this procedure terminates, the square (i, j_k) is live for all $1 \leq i \leq m_k$, is dead for all $m_k + 1 \leq i \leq i_k^*$, and is live or empty otherwise. Columns $j_1, \dots, j_{k-1}$ were left invariant throughout this construction. We may push down any remaining live squares in $\{(i_k + 1, j_k), \dots, (i_k^*, j_k)\}$ so that there are no live squares in this region and then fill in any empty squares in $\{(i_k + 1, j_k), \dots, (i_k^*, j_k)\}$ with dead squares. We set $\mathcal{D}^k$ to be the resulting diagram. $\square$

THEOREM 1.2. *Let $w \in S_n$ be a vexillary permutation. Then $\text{supp}(\mathfrak{G}_w^{\text{top}}) = \text{supp}(\chi_{D^{\text{top}}(w)})$.*

Proof of Theorem 1.2. Theorem 4.6 implies that any monomial appearing in $\mathfrak{G}_w^{\text{top}}$ is equal to $x^{\text{wt}(D)}$ for some $(D, r, F) \in \mathcal{BD}(w)$ with $F = F^{\text{top}}(w)$. Any such diagram D satisfies $D \leq D^{\text{top}}(w)$, so

$$\text{supp}(\mathfrak{G}_w^{\text{top}}) \subseteq \{\text{wt}(D) : D \leq D^{\text{top}}(w)\}.$$

By construction, $D^{\text{top}}(w) \in \mathcal{BD}(w)$. Furthermore, if $(i, j) \in F^{\text{top}}(w)$, then $(i', j) \in D^{\text{top}}(w)$ for all $i' < i$. It follows (e.g., by Lemma 3.19) that

$$\{\text{wt}(D) : D \leq D^{\text{top}}(w)\} \subseteq \{\text{wt}(\mathcal{D}) : \mathcal{D} \in \mathcal{BD}(D^{\text{top}}(w))\} \subseteq \text{supp}(\mathfrak{G}_w^{\text{top}}).$$

We conclude that

$$\text{supp}(\mathfrak{G}_w^{\text{top}}) = \{\text{wt}(D) : D \leq D^{\text{top}}(w)\}.$$

Finally, Proposition 2.8 guarantees that $\{\text{wt}(D) : D \leq D^{\text{top}}(w)\} = \text{supp}(\chi_{D^{\text{top}}(w)})$. $\square$

The next result asserts that if a monomial $x^{\text{wt}(D)}$ appearing in $\mathfrak{G}_w$ is represented by a bubbling diagram $\mathcal{D} = (D, r, F)$, then any monomial x^α which divides $x^{\text{wt}(D)}$ and appears in $\mathfrak{G}_w$ can be represented by a bubbling diagram whose dead squares are contained in F .

THEOREM 4.10. *Let $\mathcal{D} = (D, r, F) \in \mathcal{BD}(w)$. Suppose that there exists $i \in [n]$ so that $x^{\text{wt}(D)}/x_i$ appears with nonzero coefficient in $\mathfrak{G}_w$. Then there is $\mathcal{D}^- = (D^-, r^-, F^-) \in \mathcal{BD}(w)$ so that $\text{wt}(D^-) = \text{wt}(D) - e_i$, $F^- \subsetneq F$, and $r = r^-$ on F^- .*

Proof. Fix a diagram $\mathcal{C} = (C, s, G) \in \mathcal{BD}(w)$ so that $\text{wt}(C) = \text{wt}(D) - e_i$.

If row i of $\mathcal{D}$ contains a dead square, then removing that square gives the desired diagram $\mathcal{D}^-$. Otherwise, there must be a square (i, j) so that $(i, j) \in D \setminus F$ and $(i, j) \notin C$. Suppose that $(i, j) \in D \setminus F$ is the k th uppermost live square in the column D_j . There are two cases:

- (1) Suppose that the k th uppermost live square in the column C_j is above row i . Let $i' < i$ be maximal so that D_j does not have a live square in the i' th row; such a position exists because C_j has its k th uppermost square above row i . Apply a bubbling move to the live squares $(i, j), (i-1, j), \dots, (i'+1, j)$ of D . If $(i', j) \in F$, then simply remove it to make the bubbling move legal. The resulting diagram is in $\mathcal{BD}(w)$.
- (2) Suppose that the k th uppermost live square in the column C_j is below row i . Let $i' > i$ be minimal so that D_j does not have a live square in the i' th row. Because the k th uppermost live square in C_j is below row i , the diagram obtained from D by “pushing down” the live squares $(i, j), (i+1, j), \dots, (i'-1, j)$ of D by one space, removing a dead square at (i', j) if it exists, is again a diagram in $\mathcal{BD}(w)$.

In either case, if a dead square was removed then the resulting diagram has weight $\text{wt}(D) - e_i$, giving our desired bubbling diagram $\mathcal{D}^-$. If no dead square was removed, then the resulting diagram has weight $\text{wt}(D) - e_i + e_{i'}$ and has more squares in row i' than does C , so we may repeat the process using row i' until a dead square is removed.

At each step of the process, the squares in D move closer to their counterparts in C . Because $\text{wt}(C) = \text{wt}(D) - e_i$ and $|D \setminus F| = |C \setminus G|$, there is a row r so that $\#\{j : (r, j) \in C \setminus G\} > \#\{j : (r, j) \in D \setminus F\}$. In particular, there is a column j_* in which a live square $(r, j_*) \in C \setminus G$ is not in the same row as its counterpart in $(r', j_*) \in D \setminus F$; the algorithm will eventually move (r', j_*) to (r, j_*) , so this procedure will terminate. $\square$

DEFINITION 4.11. *Let $\mathcal{SBD}(w)$ denote the set of bubbling diagrams $\mathcal{D} = (D, r, F) \in \mathcal{BD}(w)$ for which every dead square is linked to a distinguished live square in its column.*

THEOREM 4.12. *If $w \in S_n$ is vexillary, then $\text{supp}(\mathfrak{G}_w) = \{\text{wt}(\mathcal{D}) : \mathcal{D} \in \mathcal{SBD}(w)\}$.*

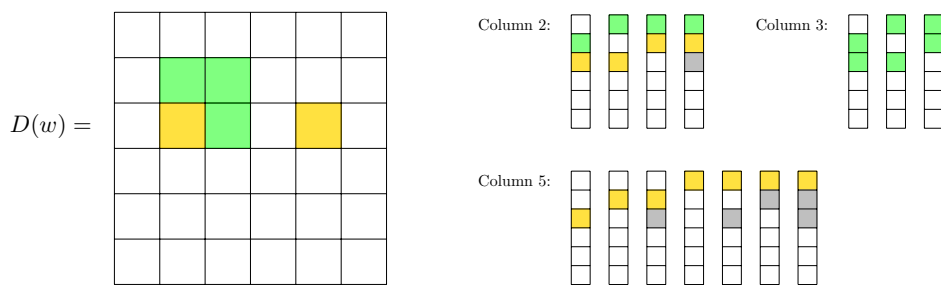


FIG. 14. Construction of $\mathcal{SBD}(w)$ for $w = 146235$. The set $A(w)$ of distinguished live squares is shown in gold. Any combination of the above options for columns 2, 3, and 5 will yield a valid diagram in $\mathcal{SBD}(w)$. (Figure in color online.)

Proof. By Theorem 1.1, any monomial appearing in $\mathfrak{G}_w$ is of the form $x^{\text{wt}(D)}$ for some $\mathcal{D} = (D, r, F) \in \mathcal{BD}(w)$. By Theorem 4.6, there is a bubbling diagram $\mathcal{C} = (C, s, F^{\text{top}}(w)) \in \mathcal{BD}(\mathcal{D}^{\text{top}}(w))$ so that $x^{\text{wt}(D)} \mid x^{\text{wt}(C)}$. Repeated application of Theorems 2.3 and 4.10 give the desired result. $\square$

Observe that $\mathcal{SBD}(w)$ is precisely the set of diagrams which can be generated from $D(w)$ by any series of the following moves:

- (1) Bubble up any live square.
- (2) K-bubble any distinguished live square.

In particular, once the set $A(w)$ of distinguished live squares has been determined, this procedure makes no further reference to the ranks of squares (since no pair of squares in $A(w)$ can be linked). The possible states of each column in $\mathcal{SBD}(w)$ are, thus, independent of the states of the other columns. Figure 14 shows an example of $\mathcal{SBD}(w)$.

5. Supports of homogenized Grothendieck polynomials. We deduce Theorem 1.3 from a “one-column version” of the result (Proposition 5.6).

DEFINITION 5.1 (see [MTY19]). *Let $D \subseteq [n] \times [k]$ be a diagram. The Schubitope $\mathcal{S}_D$ is the Newton polytope of the dual character χ_D of the flagged Weyl module.*

By [FMS18], the Schubitope $\mathcal{S}_D$ is the Minkowski sum

$$\mathcal{S}_D = \sum_{i=1}^k P(\text{SM}_n(D_i))$$

of Schubert matroid polytopes.

We recall the combinatorial interpretation, due to [MTY19], for the rank functions of Schubert matroids. For $I, J \subseteq [n]$, construct a string denoted $\text{word}_I^n(J)$ by setting $k = 1, \dots, n$ and recording

- $_$ if $k \notin I$ and $k \notin J$;
- $($ if $k \notin I$ and $k \in J$;
- $)$ if $k \in I$ and $k \notin J$;
- $\star$ if $k \in I$ and $k \in J$.

Define

$$\theta_I^n(J) \stackrel{\text{def}}{=} \#\{\text{matched } ()'s \text{ in } \text{word}_I^n(J)\} + \#\{\star's \text{ in } \text{word}_I^n(J)\},$$

where parentheses are matched iteratively left-to-right, removing matched pairs.

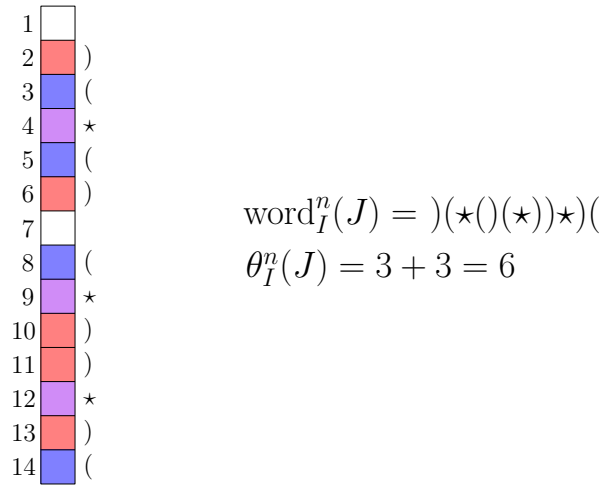


FIG. 15. Left: the sets I , J , and $I \cap J$ from Example 5.2. Right: the word $\text{word}_I^n(J)$ and the number $\theta_I^n(J)$.

Example 5.2. Let $n = 14$, $I = \{2, 4, 6, 9, 10, 11, 12, 13\}$, and $J = \{3, 4, 5, 8, 9, 12, 14\}$. Coloring $I - J$, $J - I$, and $I \cap J$, respectively, red, blue, and purple, we show how to compute $\theta_I^n(J)$ in Figure 15.

THEOREM 5.3 (see [FMS18, Theorem 10]). *For any n and any $I, J \subseteq [n]$,*

$$r_{\text{SM}_n(I)}(J) = \theta_I^n(J).$$

Let $S \subseteq [n]$ be a one-column diagram with a single distinguished square $s \in S$. Set $S^{(0,s)} := S$, and whenever $[s] \not\subseteq S^{(k-1,s)}$, define $S^{(k,s)}$ from $S^{(k-1,s)}$ by

$$S^{(k,s)} = S^{(k-1,s)} \cup \max\{i : i < s \text{ and } i \notin S^{(k-1,s)}\}.$$

Let $d = s - |\{i \in S : i \leq s\}|$ so that $[s] \subseteq S^{(d,s)}$.

LEMMA 5.4. *For all k , we have*

$$r_{\text{SM}_n(S^{(k+1,s)})}(J) - r_{\text{SM}_n(S^{(k,s)})}(J) \in \{0, 1\}.$$

When $r_{\text{SM}_n(S^{(k+1,s)})}(J) = r_{\text{SM}_n(S^{(k,s)})}(J)$, then we have

$$\begin{aligned} r_{\text{SM}_n(S^{(k'+1,s)})}(J) &= r_{\text{SM}_n(S^{(k',s)})}(J) && \text{for all } k' > k, \text{ and} \\ r_{\text{SM}_n(S^{(k+1,s)})}(J') &= r_{\text{SM}_n(S^{(k,s)})}(J') && \text{for all } J' \subset J. \end{aligned}$$

Proof. Suppose that $S^{(k+1,s)} \setminus S^{(k,s)} = \{i\}$. If $i \in J$, then $\text{word}_{S^{(k+1,s)}}^n(J)$ is obtained from $\text{word}_{S^{(k,s)}}^n(J)$ by replacing the $($ in the i th position of $\text{word}_{S^{(k,s)}}^n(J)$ with a $\star$, while if $i \notin J$, then $\text{word}_{S^{(k+1,s)}}^n(J)$ is obtained from $\text{word}_{S^{(k,s)}}^n(J)$ by replacing the $_$ in the i th position of $\text{word}_{S^{(k,s)}}^n(J)$ with a $)$. In either case, $r_{\text{SM}_n(S^{(k+1,s)})}(J) - r_{\text{SM}_n(S^{(k,s)})}(J) \in \{0, 1\}$.

To prove the rest of the lemma, we will use the following observation. Let $a \in [n]$ and suppose that $T, K \subset [n]$ are such that $a \notin T$ and every $($ in the first a positions of $\text{word}_T^n(K)$ is matched to a $)$. In this case, observe that $\text{word}_{T \cup \{a\}}^n(K)$ is obtained from $\text{word}_T^n(K)$ as follows:

- If $a \in K$, then we replace the $($ in the a th position, which is matched by assumption, with a $\star$; then the number of matched $()$'s decreases by one and the number of $\star$'s by one, so that $r_{\text{SM}_n(T \cup \{a\})}(K) = r_{\text{SM}_n(T)}(K)$.
- If $a \notin K$, then we replace the $_$ in the a th position with a $)$; then the number of matched $()$'s does not increase as every $($ to the left of position a was already matched. Thus $r_{\text{SM}_n(T \cup \{a\})}(K) = r_{\text{SM}_n(T)}(K)$.

When $r_{\text{SM}_n(S^{(k+1,s)})}(J) = r_{\text{SM}_n(S^{(k,s)})}(J)$, we know that every $($ in the first i positions of $\text{word}_{S^{(k,s)}}^n(J)$ is matched. Furthermore, for any $k' > k$, observe that $\text{word}_{S^{(k',s)}}^n(J)$ is obtained from $\text{word}_{S^{(k,s)}}^n(J)$ by modifying entries strictly to the right of $a := S^{(k'+1,s)} \setminus S^{(k',s)}$, so every $($ in the first a positions of $\text{word}_{S^{(k',s)}}^n(J)$ is matched. The observation guarantees that $r_{\text{SM}_n(S^{(k'+1,s)})}(J) = r_{\text{SM}_n(S^{(k',s)})}(J)$.

Similarly, let $J' \subseteq J$. Note that for any set T , $\text{word}_T^n(J')$ is obtained from $\text{word}_T^n(J)$ by replacing, for every $j \in (J \setminus J') \cap T$, the $\star$ in the j -th position with a $)$ and, for every $j \in (J \setminus J') \setminus T$, the $($ in the j -th position with a $_$. Because $\text{word}_T^n(J')$ is obtained from $\text{word}_T^n(J)$ by replacing $\star$'s with $)$'s and $($'s with $_$'s, every $($ in the first i positions of $\text{word}_{S^{(k,s)}}^n(J')$ is matched. The observation guarantees that $r_{\text{SM}_n(S^{(k+1,s)})}(J') = r_{\text{SM}_n(S^{(k,s)})}(J')$. $\square$

COROLLARY 5.5. *The Schubert matroid rank function of $S^{(k,s)}$ is given by*

$$r_{\text{SM}_n(S^{(k,s)})}(I) = \min\{r_{\text{SM}_n(S^{(d,s)})}(I), r_{\text{SM}_n(S)}(I) + k\} \quad \text{for all } I \subseteq [n].$$

Furthermore, if $J' \subseteq J$, then

$$r_{\text{SM}_n(S^{(d,s)})}(J) - r_{\text{SM}_n(S^{(d,s)})}(J') \geq r_{\text{SM}_n(S)}(J) - r_{\text{SM}_n(S)}(J').$$

Proof. By Lemma 5.4, we know $r_{\text{SM}_n(S^{(k+1,s)})}(J) - r_{\text{SM}_n(S^{(k,s)})}(J) \in \{0, 1\}$. This implies

$$r_{\text{SM}_n(S^{(k,s)})}(J) \leq \min\{r_{\text{SM}_n(S^{(d,s)})}(I), r_{\text{SM}_n(S)}(I) + k\}.$$

Furthermore, Lemma 5.4 implies the following:

- If $r_{\text{SM}_n(S^{(k+1,s)})}(J) - r_{\text{SM}_n(S^{(k,s)})}(J) = 0$, then $r_{\text{SM}_n(S^{(k'+1,s)})}(J) - r_{\text{SM}_n(S^{(k',s)})}(J) = 0$ for all $k' > k$, so $r_{\text{SM}_n(S^{(k,s)})}(J) = r_{\text{SM}_n(S^{(d,s)})}(J)$.
- If $r_{\text{SM}_n(S^{(k+1,s)})}(J) - r_{\text{SM}_n(S^{(k,s)})}(J) = 1$, then $r_{\text{SM}_n(S^{(k'+1,s)})}(J) - r_{\text{SM}_n(S^{(k',s)})}(J) = 1$ for all $k' < k$, so $r_{\text{SM}_n(S^{(k,s)})}(J) = r_{\text{SM}_n(S)}(J) + k$.

If $J' \subseteq J$, then Lemma 5.4 implies that

$$r_{\text{SM}_n(S^{(d,s)})}(J) - r_{\text{SM}_n(S)}(J) \geq r_{\text{SM}_n(S^{(d,s)})}(J') - r_{\text{SM}_n(S)}(J').$$

Rearranging the inequality gives the desired result. $\square$

For $B \leq S^{(k,s)}$, let $\tilde{\zeta}^B = (\zeta_1^B, \dots, \zeta_n^B, d-k) \in \mathbb{R}^{n+1}$ be the vector with $\zeta_i^B = 1$ if $i \in B$ and $\zeta_i^B = 0$ if $i \notin B$. Define the polytope

$$\mathcal{P}(S^{(s)}) \stackrel{\text{def}}{=} \text{conv}\{\tilde{\zeta}^B : B \leq S^{(k,s)} \text{ for some } k \leq d\}.$$

PROPOSITION 5.6. *The polytope $\mathcal{P}(S^{(s)})$ is a generalized permutahedron, and*

$$\mathcal{P}(S^{(s)}) \cap \mathbb{Z}^{n+1} = \{\tilde{\zeta}^B : B \leq S^{(k,s)} \text{ for some } k \leq d\}.$$

Proof. Consider the function $z: 2^{[n+1]} \rightarrow \mathbb{R}$ defined by

$$z(I) = \begin{cases} r_{\text{SM}_n(S^{(d,s)})}(I) & \text{if } n+1 \notin I, \\ r_{\text{SM}_n(S)}(I \setminus \{n+1\}) + d & \text{if } n+1 \in I. \end{cases}$$

We claim that z is submodular. Indeed,

- If $n+1 \notin I, J$, then $z(I) + z(J) \geq z(I \cup J) + z(I \cap J)$ because $r_{\text{SM}_n(S^{(d,s)})}$ is submodular.
- If $n+1 \in I \setminus J$, then

$$\begin{aligned} z(I) + z(J) &= r_{\text{SM}_n(S)}(I \setminus \{n+1\}) + d + r_{\text{SM}_n(S^{(d,s)})}(J) \\ &\geq r_{\text{SM}_n(S)}(I \setminus \{n+1\}) + d + r_{\text{SM}_n(S)}(J) - r_{\text{SM}_n(S)}(I \cap J) \\ &\quad + r_{\text{SM}_n(S^{(d,s)})}(I \cap J) \\ &\geq d + r_{\text{SM}_n(S)}(I \cup J \setminus \{n+1\}) + r_{\text{SM}_n(S^{(d,s)})}(I \cap J) \\ &= z(I \cup J) + z(I \cap J), \end{aligned}$$

where the first inequality uses Corollary 5.5 applied to $r_{\text{SM}_n(S^{(d,s)})}(J)$ and the second inequality uses the submodular inequality $r_{\text{SM}_n(S)}(J) - r_{\text{SM}_n(S)}(I \cap J) \geq r_{\text{SM}_n(S)}(I \cup J \setminus \{n+1\}) - r_{\text{SM}_n(S)}(I \setminus \{n+1\})$.

- If $n+1 \in I, J$, then $z(I) + z(J) \geq z(I \cup J) + z(I \cap J)$ because $r_{\text{SM}_n(S)}$ is submodular.

Since z is submodular, we have a generalized permutahedron

$$P = \left\{ t \in \mathbb{R}^{n+1} : \sum_{i \in I} t_i \leq z(I) \text{ for all } I \subseteq [n+1] \text{ and } \sum_{i=1}^{n+1} t_i = z([n+1]) \right\}.$$

We now claim that $\mathcal{P}(S^{(s)}) = P$. To prove this, fix any $B \leq S^{(k,s)}$ and $I \subseteq [n+1]$. If $n+1 \notin I$, then

$$\sum_{i \in I} \zeta_i^B \leq r_{\text{SM}_n(S^{(k,s)})}(I) \leq r_{\text{SM}_n(S^{(d,s)})}(I),$$

and if $n+1 \in I$, then

$$\begin{aligned} \left(\sum_{i \in I \setminus \{n+1\}} \zeta_i^B \right) + d - k &\leq r_{\text{SM}_n(S^{(k,s)})}(I \setminus \{n+1\}) + d - k \\ &\leq r_{\text{SM}_n(S)}(I \setminus \{n+1\}) + d, \end{aligned}$$

where we use the inequality $r_{\text{SM}_n(S^{(k,s)})}(I) - k \leq r_{\text{SM}_n(S)}(I)$ from Corollary 5.5. Furthermore,

$$\sum_{i \in [n]} \zeta_i^B + d - k = (|S| + k) + d - k = z([n+1]),$$

so $\tilde{\zeta}^B \in P$. We conclude that $\mathcal{P}(S^{(s)}) \subseteq P$.

Now fix any $t \in P \cap \mathbb{Z}^{n+1}$. Observe that $z([n]) = z([n+1])$, so $t_{n+1} \geq 0$. Furthermore, $z(\{n+1\}) = d$, so $t_{n+1} \leq d$. Write $t_{n+1} = d - k$. Observe that for any $I \subseteq [n]$, we have

$$\begin{aligned} \sum_{i \in I} t_i &\leq z(I) = r_{\text{SM}_n(S^{(d,s)})}(I) \quad \text{and} \\ \sum_{i \in I} t_i &\leq z(I \cup \{n+1\}) - t_{n+1} = r_{\text{SM}_n(S)}(I \setminus \{n+1\}) + k, \end{aligned}$$

so that

$$\sum_{i \in I} t_i \leq \min\{r_{\text{SM}_n(S^{(d,s)})(I)}, r_{\text{SM}_n(S)}(I \setminus \{n+1\}) + k\} = r_{\text{SM}_n(S^{(k,s)})(I)}.$$

In particular, $(t_1, \dots, t_n)$ is an integer point of the Schubitope $\mathcal{S}_{S^{(k,s)}}$. It follows that $(t_1, \dots, t_n) = \zeta^B$ for some $B \leq S^{(k,s)}$; hence, $t = \tilde{\zeta}^B$. Thus, $P \cap \mathbb{Z}^{n+1} = \{\tilde{\zeta}^B : B \leq S^{(k,s)} \text{ for some } k\}$ and $P \supseteq \mathcal{P}(S^{(s)})$. We conclude that $P = \mathcal{P}(S^{(s)})$ and that $\mathcal{P}(S^{(s)}) \cap \mathbb{Z}^{n+1} = P \cap \mathbb{Z}^{n+1} = \{\tilde{\zeta}^B : B \leq S^{(k,s)}\}$. $\square$

Let $f_i^{\text{top}} := \#F^{\text{top}}(w)_i$, and write $f^{\text{top}} = (f_1^{\text{top}}, \dots, f_n^{\text{top}})$. For $f \leq f^{\text{top}}$, let $D^f(w) = (D^f(w)_1, \dots, D^f(w)_n)$ be the diagram with

$$D^f(w)_k = (D(w)_k)^{(f_k, i_k)}, \quad \text{where } (i_k, k) \in A(w).$$

LEMMA 5.7. *We have*

$$\begin{aligned} &\{D : \text{there exist } r, F \text{ so that } (D, r, F) \in \mathcal{SBD}(w)\} \\ &= \{D : D \leq D^f(w) \text{ for some } f \leq f^{\text{top}}\}. \end{aligned}$$

Proof. Let $(D, r, F) \in \mathcal{SBD}(w)$. Define $(f_1, \dots, f_n)$ by $f_k := \#F_k = \#D_k - \#D(w)_k$. Because $F \subseteq F^{\text{top}}(w)$, we know that $f \leq f^{\text{top}}$. Suppose that $(i_k, k) \in A(w)$ is the m_k th highest square in $D(w)_k$. Writing d_i for the i th highest square in D_k , we have $F_k = \{d_{m_k+1} < \dots < d_{m_k+f_k}\}$ with $d_{m_k+f_k} \leq i_k$. Since $D \in \mathcal{BD}(w)$, Lemma 3.19 implies that $D_k \setminus F_k \leq D(w)_k$. It follows that $D_k \leq (D^f(w))_k$, and by varying k , we deduce that $D \leq D^f(w)$.

Now suppose that $D \leq D^f(w)$. As above, suppose that $(i_k, k) \in A(w)$ is the m_k th highest square in $D(w)_k$, and write d_i for the i th highest square in D_k . Let $F_k := \{d_{m_k+1} < \dots < d_{m_k+f_k}\}$. Then $D \setminus F \leq D(w)$, and furthermore, $d_{m_k+f_k} \leq i_k$. The live square immediately above any square in F_k is $d_{m_k} \in D_k \setminus F_k$, and no two squares in $A(w)$ are linked. Thus, Lemma 3.19 implies that there exists r so that $(D, r, F) \in \mathcal{BD}(w)$. $\square$

For $D \leq D^f(w)$, let $\widetilde{\text{wt}}(D) \in \mathbb{Z}^{n+1}$ denote the vector whose i th coordinate counts the number of squares in the i th row of D for $i \leq n$ and whose $(n+1)$ th coordinate is $\deg(\mathfrak{G}_w) - |D|$.

Recall that $\mathfrak{G}_w$ denotes the *homogenized Grothendieck polynomial*

$$\widetilde{\mathfrak{G}}_w(x_1, \dots, x_n, z) := \sum_{k=\ell(w)}^{\deg(\mathfrak{G}_w)} \mathfrak{G}_w^{(k)}(x_1, \dots, x_n) z^{\deg(\mathfrak{G}_w)-k}.$$

THEOREM 1.3. *Let $w \in S_n$ be a vexillary permutation. Then, the homogenized Grothendieck polynomial $\widetilde{\mathfrak{G}}_w$ has M -convex support. In particular, each degree component $\mathfrak{G}_w^{(k)}$ has M -convex support.*

Proof of Theorem 1.3. By Theorem 4.12 and Lemma 5.7, we know that

$$\begin{aligned} \text{supp}(\widetilde{\mathfrak{G}}_w) &= \left\{ \widetilde{\text{wt}}(D) : D \leq D^f(w) \text{ for some } f \leq f^{\text{top}} \right\} \\ &= \sum_{k=1}^n \left\{ \tilde{\zeta}^{D_k} : D_k \leq (D(w)_k)^{(f_k, i_k)} \text{ for some } f_k \leq f_k^{\text{top}} \right\}. \end{aligned}$$

Thus

$$\text{Newton}(\widetilde{\mathfrak{G}}_w) = \sum_{k=1}^n \mathcal{P}((D(w)_k)^{(i_k)})$$

is a generalized permutahedron. Furthermore, [Sch03, Corollary 46.2c] implies that any $t \in \text{Newton}(\widetilde{\mathfrak{G}}_w) \cap \mathbb{Z}^{n+1}$ can be written as a sum

$$t = t_1 + \cdots + t_n, \quad t_i \in \mathcal{P}((D(w)_k)^{(i_k)}) \cap \mathbb{Z}^{n+1}.$$

Since $t_i = \widetilde{\zeta}^{D_i}$, we conclude that $t = \widetilde{\text{wt}}(D) \in \text{supp}(\widetilde{\mathfrak{G}}_w)$ for $D = (D_1, \dots, D_n)$, so $\widetilde{\mathfrak{G}}_w$ has SNP.

M-convexity of $\text{supp}(\mathfrak{G}_w^{(k)})$ follows from the equality

$$\text{supp}(\mathfrak{G}_w^{(k)}) = \text{supp}(\widetilde{\mathfrak{G}}_w) \cap \{t \in \mathbb{R}^{n+1} : t_{n+1} = \deg(\mathfrak{G}_w) - k\}. \quad \square$$

6. Linear independence of Schubert matroid rank functions. We prove Theorem 1.4 and use it to show that our results are sharp.

DEFINITION 6.1. For each n , denote by V_n the set of all nonempty subsets of $[n]$ with the following total order: if $I, J \in V_n$, then $I \prec J$ if

$$\max(I \setminus J) \leq \max(J \setminus I),$$

where we take $\max(\emptyset) := 0$.

Example 6.2. V_4 is the chain:

$$\begin{aligned} \{1\} \prec \{2\} \prec \{1, 2\} \prec \{3\} \prec \{1, 3\} \prec \{2, 3\} \prec \{1, 2, 3\} \\ \prec \{4\} \prec \{1, 4\} \prec \{2, 4\} \prec \{1, 2, 4\} \prec \{3, 4\} \prec \{1, 3, 4\} \prec \{2, 3, 4\} \prec \{1, 2, 3, 4\}. \end{aligned}$$

Note that V_{n-1} is an initial segment of V_n .

DEFINITION 6.3. For each $n \geq 1$, define A_n to be the $(2^n - 1) \times (2^n - 1)$ matrix

$$A_n = (r_{\text{SM}_n(I)}(J))_{I, J \in V_n}.$$

Example 6.4. For $n = 3$ and $n = 4$, we have

$$A_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 2 & 1 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 \\ 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 2 & 3 & 2 & 3 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 2 & 3 & 3 & 3 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 2 & 3 & 3 & 4 \end{pmatrix}.$$

Because V_{n-1} is an initial segment of V_n , the upper left justified $(2^{n-1}-1) \times (2^{n-1}-1)$ submatrix of A_n is equal to A_{n-1} .

We would like to show that the columns of A_n are linearly independent. To do this, we will use symmetries of A_n which relate blocks of A_n with A_{n-1} . We first give a motivating example.

Example 6.5. Take A_4 as above. For each $I \in V_3$, subtract row I from row $I \cup \{n\}$ to get

$$\left(\begin{array}{cccccccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 2 & 1 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 0 \\ 1 & 1 & 2 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 0 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right) = \left(\begin{array}{cccc|c|cccc} & & & & 0 & & & & \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ & & & & 0 & & & & \\ \hline & & & & 1 & & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right).$$

More generally, Lemmas 6.6 and 6.7 imply that the matrix A'_n obtained from A_n by subtracting row I from row $I \cup \{n\}$ for each $I \in V_{n-1}$ has a block decomposition as in Figure 16.

LEMMA 6.6. *The row and column of A_n indexed by $\{n\}$ are given by*

$$r_{\text{SM}_n(\{n\})}(J) = 1 \quad \text{and} \quad r_{\text{SM}_n(I)}(\{n\}) = \begin{cases} 1 & \text{if } n \in I, \\ 0 & \text{if } n \notin I, \end{cases}$$

respectively.

Proof. It is straightforward to check that

$$\text{word}_{\{n\}}^n(J) = \begin{cases} (\dots(\star) & \text{if } n \in J, \\ (\dots()) & \text{if } n \notin J \end{cases} \quad \text{and} \quad \text{word}_I^n(\{n\}) = \begin{cases} (\dots)\star & \text{if } n \in I, \\ (\dots)() & \text{if } n \notin I. \end{cases} \quad \square$$

$$\left(\begin{array}{cccc|c|cccc} & & & & 0 & & & & \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ & & & & 0 & & & & \\ \hline 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ & 0 & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ & & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ & & & 1 & 1 & 1 & \dots & 1 \\ & & & 0 & 1 & 1 & \dots & 1 \end{array} \right)$$

FIG. 16. The matrix A'_n .

LEMMA 6.7. Let $I, J \in V_n \setminus \{\{n\}\}$. The rank functions $r_{\text{SM}_n(I)}$ satisfy the following properties:

- (1) If $n \notin I$ and $n \notin J$, then $r_{\text{SM}_n(I)}(J) = r_{\text{SM}_{n-1}(I)}(J)$.
- (2) If $n \notin I$ and $n \in J$, then $r_{\text{SM}_n(I)}(J) = r_{\text{SM}_n(I)}(J \setminus \{n\})$.
- (3) If $n \in I$ and $n \in J$, then $r_{\text{SM}_n(I)}(J) = r_{\text{SM}_n(I \setminus \{n\})}(J \setminus \{n\}) + 1$.
- (4) If $n \in I$, $n \notin J$, and $I \setminus \{n\} \prec J$, then $r_{\text{SM}_n(I)}(J) = r_{\text{SM}_n(I \setminus \{n\})}(J) + 1$.
- (5) If $n \in I$, then $r_{\text{SM}_n(I)}(I \setminus \{n\}) = r_{\text{SM}_n(I \setminus \{n\})}(I \setminus \{n\})$.

Proof. If $n \notin I$ and $n \notin J$, then $\text{word}_I^n(J) = \text{word}_I^{n-1}(J)$. Thus, $r_{\text{SM}_n(I)}(J) = r_{\text{SM}_{n-1}(I)}(J)$.

If $n \notin I$ and $n \in J$, then $\text{word}_I^n(J)$ is obtained by appending a (to the end of $\text{word}_I^n(J \setminus \{n\})$. Doing so does not change the number of $\star$ s or paired $()$ s, so $r_{\text{SM}_n(I)}(J) = r_{\text{SM}_n(I)}(J \setminus \{n\})$.

If $n \in I$ and $n \in J$, then $\text{word}_I^n(J)$ is obtained by appending a $\star$ to the end of $\text{word}_{I \setminus \{n\}}^n(J \setminus \{n\})$, so $r_{\text{SM}_n(I)}(J) = r_{\text{SM}_n(I \setminus \{n\})}(J \setminus \{n\}) + 1$.

If $n \in I$ and $n \notin J$, then $\text{word}_I^n(J)$ is obtained by appending a) to the end of $\text{word}_{I \setminus \{n\}}^n(J)$. On the other hand, if $I \setminus \{n\} \prec J$, then $\max(I \setminus \{n\} \setminus J) < \max(J \setminus I)$; thus, $\text{word}_{I \setminus \{n\}}^n(J)$ contains an unmatched left parenthesis to the right of all closed parentheses. Combined, we deduce $r_{\text{SM}_n(I)}(J) = r_{\text{SM}_n(I \setminus \{n\})}(J) + 1$.

If $n \in I$, then $\text{word}_I^n(I \setminus \{n\})$ is obtained by appending a) to the end of $\text{word}_{I \setminus \{n\}}^n(I \setminus \{n\})$. Thus, $r_{\text{SM}_n(I)}(I \setminus \{n\}) = r_{\text{SM}_n(I \setminus \{n\})}(I \setminus \{n\})$. $\square$

PROPOSITION 6.8. The Schubert matroid rank functions $r_{\text{SM}_n(I)}$ are linearly independent.

Proof. We will show that the columns of A_n are linearly independent. First, let A'_n denote the matrix obtained from A_n by subtracting row I from row $I \cup \{n\}$ for each $I \in V_{n-1}$, as in Figure 16. Let A''_n denote the matrix obtained from A'_n by subtracting each row in $V_n \setminus (V_{n-1} \cup \{n\})$ from the row above it, working from the top row to the bottom row. Then A''_n has a block decomposition.

$$\left(\begin{array}{cc|cc} & & 0 & \\ & & \vdots & \\ & & 0 & \\ & A_{n-1} & & A_{n-1} \\ & & 0 & \\ \hline 1 & 0 & \cdots & 0 \\ & 1 & \cdots & 0 \\ & & \ddots & \\ & & & 1 & 0 \\ & \star & & 1 & 1 \end{array} \right)$$

Let v_J denote the column vector of A''_n indexed by $J \in V_n$, and suppose that

$$(\diamond) \quad \sum_{J \in V_n} c_J v_J = 0$$

is a linear dependence between the vectors v_J . The columns of A_{n-1} are linearly independent, so $c_J + c_{J \cup \{n\}} = 0$ for all $J \in V_{n-1}$. Furthermore, comparing coordinates of the dependence $(\diamond)$ corresponding to $V_n \setminus (V_{n-1} \cup \{n\})$, working from the smallest element to the largest element, gives that $c_J = 0$ for all $J \in V_{n-1}$. It follows that $c_{J \cup \{n\}} = 0$ for all $J \in V_{n-1}$ as well. Thus, the linear dependence $(\diamond)$ reads $c_{\{n\}}v_{\{n\}} = 0$, and since $v_{\{n\}} \neq 0$, it follows that $c_{\{n\}} = 0$.

We conclude that the columns of A_n'' , and hence the columns of A_n , are independent. $\square$

THEOREM 1.4. *Fix $n \geq 1$. The rank functions $r_{\text{SM}_n(I)}$ of Schubert matroids form a basis of the vector space of functions $f: 2^{[n]} \rightarrow \mathbb{R}$ satisfying $f(\emptyset) = 0$. In particular, we have the following:*

- *A generalized permutahedron is a Schubitope if and only if its associated submodular function is a $\mathbb{Z}_{\geq 0}$ -linear combination of rank functions of Schubert matroids, and*
- *two Schubitopes $\mathcal{S}_D$ and $\mathcal{S}_{D'}$ are equal if and only if D can be obtained from D' by a permutation of columns.*

Proof of Theorem 1.4. The vector space of functions $f: 2^{[n]} \rightarrow \mathbb{R}$ satisfying $f(\emptyset) = 0$ is $(2^n - 1)$ -dimensional and contains the $2^n - 1$ functions $r_{\text{SM}_n(I)}$. Proposition 6.8 guarantees that these functions are linearly independent, so they form a basis.

Let $D = (D_1, \dots, D_k)$ be a collection of columns. The submodular function of the Schubitope $\mathcal{S}_D$ is given by $r_{\text{SM}_n(D_1)} + \dots + r_{\text{SM}_n(D_k)}$; in particular, it is a $\mathbb{Z}_{\geq 0}$ -linear combination of Schubert matroid rank functions.

Lemma 2.10 guarantees that a generalized permutahedron P is uniquely determined by its submodular function z . Because Schubitopes are generalized permutahedra, an arbitrary generalized permutahedron P is equal to a Schubitope $\mathcal{S}_D$ if and only if the submodular function z defining P is a $\mathbb{Z}_{\geq 0}$ -linear combination of rank functions of Schubert matroid polytopes.

Combined with the linear independence of rank functions of Schubert matroid polytopes, it also follows that two Schubitopes $\mathcal{S}_D$ and $\mathcal{S}_{D'}$ are equal if and only if D can be obtained from D' by a permutation of columns. $\square$

Remark 6.9. One can show that $\det(A_n) = 1$, so the Schubert matroid rank functions in fact form a $\mathbb{Z}$ -basis for the space of functions $f: 2^{[n]} \rightarrow \mathbb{Z}$ with $f(\emptyset) = 0$.

The following examples provide counterexamples to natural generalizations of Theorems 1.2 and 1.3.

Example 6.10. Consider the nonvexillary permutation $w = 2168534(10)79 \in S_{10}$. We show that the Newton polytope of $\mathfrak{S}_w^{\text{top}}$ is not a Schubitope. The defining inequalities of $\text{Newton}(\mathfrak{S}_w^{\text{top}})$ show it is a generalized permutahedron. Its submodular function z expands in the basis of Schubert matroid rank functions as

$$\begin{aligned} z = & r_{\text{SM}_n(\{1\})} - r_{\text{SM}_n(\{2,3,4\})} + 2r_{\text{SM}_n(\{1,2,3,4\})} + r_{\text{SM}_n(\{3,4,5\})} \\ & + r_{\text{SM}_n(\{1,2,3,4,5\})} + r_{\text{SM}_n(\{2,3,4,8\})} + r_{\text{SM}_n(\{1,2,3,4,5,6,7,8\})}. \end{aligned}$$

Because there is a negative coefficient in this expansion, Theorem 1.4 implies that $\text{Newton}(\mathfrak{S}_w^{\text{top}})$ is not a Schubitope.

Example 6.11. Let $w = 14253 \in S_5$. We show that the Newton polytope of $\mathfrak{S}_w^{(\ell(w)+1)}$ is not a Schubitope. Since w is vexillary, Theorem 1.3 implies $\text{Newton}(\mathfrak{S}_w^{(\ell(w)+1)})$ is a generalized permutahedron. Its submodular function z expands in the basis of Schubert matroid rank functions as

$$z = r_{\text{SM}_n(\{1,2\})} + r_{\text{SM}_n(\{2,4\})} - r_{\text{SM}_n(\{1,2,4\})} + r_{\text{SM}_n(\{2,3,4\})}.$$

Because there is a negative coefficient in this expansion, Theorem 1.4 implies that $\text{Newton}(\mathfrak{S}_w^{(\ell(w)+1)})$ is not a Schubitope.

Based on the previous two examples, we conclude with the following conjecture, a generalization of Theorem 1.2.

CONJECTURE 1.5. *If $w \in S_n$ is vexillary, then $\mathfrak{S}_w^{\text{top}}$ is an integer multiple of $\chi_{D^{\text{top}}(w)}$.*

We tested Conjecture 1.5 for all vexillary $w \in S_n$, $n \leq 9$.

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