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On the Support of Grothendieck Polynomials

Karola Mészáros, Linus Setiabrata and Avery St. Dizier

Abstract. Grothendieck polynomials $\mathfrak{G}_w$ of permutations $w \in S_n$ were introduced by Lascoux and Schützenberger (C R Acad Sci Paris Sér I Math 295(11):629–633, 1982) as a set of distinguished representatives for the K-theoretic classes of Schubert cycles in the K-theory of the flag variety of $\mathbb{C}^n$. We conjecture that the exponents of nonzero terms of the Grothendieck polynomial $\mathfrak{G}_w$ form a poset under componentwise comparison that is isomorphic to an induced subposet of $\mathbb{Z}^n$. When $w \in S_n$ avoids a certain set of patterns, we conjecturally connect the coefficients of $\mathfrak{G}_w$ with the Möbius function values of the aforementioned poset with $\hat{0}$ appended. We prove special cases of our conjectures for Grassmannian and fireworks permutations

Mathematics Subject Classification. Primary 05E05.

1. Introduction

Grothendieck polynomials $\mathfrak{G}_w$ are multivariate polynomials associated to permutations $w \in S_n$. Grothendieck polynomials were introduced by Lascoux and Schützenberger [24] as a set of distinguished representatives for the K-theoretic classes of Schubert cycles in the K-theory of the flag variety of $\mathbb{C}^n$. The lowest degree component of $\mathfrak{G}_w$ is the Schubert polynomial $\mathfrak{S}_w$. Schubert polynomials have many combinatorial constructions and are well-understood [1,2,10,11,13,15,16,20,23,25,26,31,39]. However, there is not nearly as much known combinatorially or discrete-geometrically about Grothendieck polynomials. A recent flurry of work [4,5,9,29,32,34,38] has uncovered novel formulas and perspectives on Grothendieck polynomials.

The main objective of this paper is to shed light on the combinatorial structure of the support of Grothendieck polynomials. While there have been

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recent breakthroughs in the degree of Grothendieck polynomials [29,34,35], much less is known about the structure of the support. The support has previously been conjecturally connected to generalized permutahedra via flow polytopes [30, Conjecture 5.1], and via the Lorentzian property [20, Conjecture 22]. In this paper, we give a new poset theoretic perspective on the support of any Grothendieck polynomial.

1.1. Supports of Grothendieck Polynomials

For a permutation $w \in S_n$, the support of $\mathfrak{G}_w$ is the set of exponents of terms in $\mathfrak{G}_w$ with nonzero coefficient. We endow the support $\mathrm{supp}(\mathfrak{G}_w)$ with the following poset structure. For $\alpha, \beta \in \mathbb{Z}^n$, define the *componentwise comparison* $\leq$ by $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i \in [n]$.

For $w = 15324$, we have

$$\begin{aligned}\mathfrak{G}_w = &(x_1^3x_2 + x_1^3x_3 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^3 + x_1x_2^2x_3 + x_2^3x_3)\\ &- (x_1^3x_2^2 + 2x_1^3x_2x_3 + x_1^2x_2^3 + 2x_1^2x_2^2x_3 + 2x_1x_2^3x_3)\\ &+ (x_1^3x_2^2x_3 + x_1^2x_2^3x_3).\end{aligned}$$

The Hasse diagram of $\mathrm{supp}(\mathfrak{G}_w)$ as a poset under componentwise comparison is shown in Fig. 1.

Conjecture 1.1. If $\alpha \in \mathrm{supp}(\mathfrak{G}_w)$ and $|\alpha| < \deg \mathfrak{G}_w$, then there exists $\beta \in \mathrm{supp}(\mathfrak{G}_w)$ with $\alpha < \beta$.

We prove Conjecture 1.1 for fireworks permutations in Theorem 3.16. A natural strengthening of Conjecture 1.1 is:

Conjecture 1.2. If $\alpha \in \mathrm{supp}(\mathfrak{G}_w)$ and $|\alpha| < \deg \mathfrak{G}_w$, then there exists $\beta \in \mathrm{supp}(\mathfrak{G}_w)$ with $\alpha < \beta$ and $|\beta| = |\alpha| + 1$.

We also conjecture that $\mathrm{supp}(\mathfrak{G}_w)$ is closed under taking intervals in componentwise comparison.

Conjecture 1.3. Fix any $w \in S_n$. If $\alpha, \gamma \in \mathrm{supp}(\mathfrak{G}_w)$, then

$$\{\beta \in \mathbb{Z}^n \mid \alpha \leq \beta \leq \gamma\} \subseteq \mathrm{supp}(\mathfrak{G}_w).$$

A discrete-geometric strengthening of Conjecture 1.3 is:

Conjecture 1.4. For all $w \in S_n$, $\mathfrak{G}_w$ has saturated Newton polytope and $\mathrm{Newton}(\mathfrak{G}_w)$ is a generalized polymatroid.

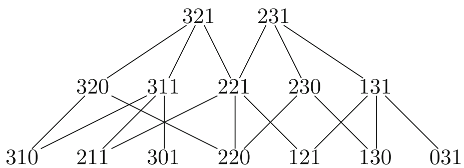


FIGURE 1. The Hasse diagram of $\mathrm{supp}(\mathfrak{G}_{15324})$ under componentwise comparison (writing exponents $(a, b, c, 0, 0) \in \mathbb{Z}^5$ as abc)

See Sects. 2.3 and 2.4 for definitions and background on saturated Newton polytopes and generalized polymatroids respectively. The assertion that $\mathfrak{G}_w$ always has saturated Newton polytope appeared in [32, Conjecture 5.5].

We note that Conjectures 1.1–1.4 make sense for double Grothendieck polynomials and Lascoux polynomials, and appear to hold there as well.

Conjectures 1.1–1.4 are related to each other in three ways:

- Conjecture 1.2 implies Conjecture 1.1.
- Conjecture 1.4 implies Conjecture 1.3 via the defining inequalities of generalized polymatroids (Definition 2.12).
- Conjectures 1.1 and 1.3 together imply Conjecture 1.2.

We also note that via a property of generalized polymatroids ([17, Theorem 14.2.5]), Conjecture 1.4 is implied by [20, Conjecture 22]. Conjecture 1.4 is a strengthening of [30, Conjecture 5.1].

We prove Conjectures 1.1–1.4 for Grassmannian permutations in Theorem 3.19. In Theorem 3.6, we provide a polytope containing the Newton polytope $\mathrm{Newton}(\mathfrak{G}_w)$. Assuming Conjectures 1.1 and 1.4, we characterize when equality with the Newton polytope occurs in Proposition 3.20.

1.2. Coefficients of Grothendieck Polynomials

In Definition 4.1, we define a poset $P_w \subseteq \mathbb{Z}^n$ (under componentwise comparison) containing $\mathrm{supp}(\mathfrak{G}_w)$. It appears for permutations whose Schubert polynomial $\mathfrak{S}_w$ has all nonzero coefficients equal 1, that the coefficients of $\mathfrak{G}_w$ agree with the Möbius function of P_w:

Conjecture 1.5. Let w be a permutation such that all nonzero coefficients of $\mathfrak{S}_w$ equal 1. If μ_w is the Möbius function of P_w, then

$$\mathfrak{G}_w = -\sum_{\alpha \in P_w - \hat{0}} \mu_w(\hat{0}, \alpha) x^\alpha.$$

Conjecture 1.5 has now been proved by Pechenik and Satriano [33].

It is known that all nonzero coefficients of $\mathfrak{S}_w$ equal 1 exactly when w avoids the patterns 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, and 315642 ([12, Theorem 4.8]). We conjecture one final property connecting the poset structure of $\mathrm{supp}(\mathfrak{G}_w)$ to the coefficients of $\mathfrak{G}_w$. Let $\mathfrak{G}_w^{\mathrm{top}}$ denote the top degree homogeneous component of $\mathfrak{G}_w$.

Conjecture 1.6. Fix $w \in S_n$ and let $\mathfrak{G}_w = \sum_{\alpha \in \mathbb{Z}^n} C_{w\alpha} x^\alpha$. For any $\beta \in \mathrm{supp}(\mathfrak{G}_w^{\mathrm{top}})$,

$$\sum_{\alpha \leq \beta} C_{w\alpha} = 1.$$

When the poset $\mathrm{supp}(\mathfrak{G}_w)$ has a unique maximum element, Conjecture 1.6 coincides with the principal specialization $\mathfrak{G}_w(1, \ldots, 1)$. While we could not locate the original source in the literature, it is well-known that $\mathfrak{G}_w(1, \ldots, 1) = 1$ (see for instance [21, Comment 3.2], and the recent [37, Corollary 3.15]). We provide a proof in Proposition 4.4 for completeness, concluding a special case of Conjecture 1.6. We have tested Conjectures 1.1–1.6 for all $w \in S_8$.

1.3. Current Status of the Conjectures

Proofs of Conjectures 1.1–1.3 for vexillary permutations are given in [19, Proposition 3.1, Theorem 3.4]. Conjecture 1.2 was proven for inverse fireworks permutations in [8]. Conjecture 1.4 was proven for permutations w whose Schubert polynomial has all nonzero coefficients equal to 1 in [6, Theorem B]. Conjectures 1.1, 1.2, and 1.6 was proven for permutations w whose Grothendieck polynomial has all nonzero coefficients equal to 1 in [7, Theorem 1.10].

Finally, as Conjecture 1.1 and 1.3 together imply Conjecture 1.2, combining Theorem 3.16 and [6, Theorem B] gives Conjecture 1.2 for fireworks permutations w whose Schubert polynomial has all nonzero coefficients equal to 1.

The above discussion is summarized in the following table.

Conjecture 1.1	Fireworks (Theorem 3.16)
	Everything for which Conjecture 1.2 holds
Conjecture 1.2	Grassmannian (Theorem 3.19)
	Vexillary [19, Proposition 3.1]
	Inverse fireworks [8]
	Zero-one Grothendieck [7, Theorem 1.10]
	Fireworks zero-one Schubert (Theorem 3.16 and [6, Theorem B])
Conjecture 1.3	Vexillary [19, Theorem 3.4]
	Everything for which Conjecture 1.4 holds
Conjecture 1.4	Zero-one Schubert [6, Theorem B]
	Grassmannian (Theorem 3.19)
Conjecture 1.5	Fully proven [33]
Conjecture 1.6	Zero-one Grothendieck [7, Thm 1.10]

1.4. Outline of the Paper

Section 2 covers the necessary background for the paper. In Sect. 3 we elaborate on Conjectures 1.1–1.4 and prove related results on the support of Grothendieck polynomials. In Sect. 4 we conjecturally connect the coefficients of certain Grothendieck polynomials to the Möbius function of a poset. We conclude by considering the principal specialization of Grothendieck polynomials.

2. Background

2.1. Conventions

For $n \in \mathbb{N}$, we use the notation $[n]$ to mean the set $\{1, 2, \ldots, n\}$. We reserve lowercase Greek letters $\alpha, \beta, \gamma, \delta, \epsilon$ for nonnegative integer vectors in $\mathbb{R}^n$; we opt for t to denote arbitrary vectors in $\mathbb{R}^n$. We write $|\alpha|$ for $\alpha_1 + \cdots + \alpha_n$. We use x to represent the collection of variables $x_1, x_2, \ldots, x_n$, so x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

For $j \in [n-1]$, s_j will denote the adjacent transposition in the symmetric group S_n swapping j and $j+1$. We otherwise represent permutations $w \in S_n$ in one-line notation as a word $w(1)w(2)\cdots w(n)$, so $w = 312 \in S_3$ is the permutation that sends $1 \mapsto 3$, $2 \mapsto 1$, and $3 \mapsto 2$. Throughout, we will take permutations as acting on the right (switching positions, not values). For example ws_1 equals w with the numbers $w(1)$ and $w(2)$ swapped. We write $\ell(w)$ for the number of inversions of w.

2.2. Schubert and Grothendieck Polynomials

Definition 2.1. Fix any $n \geq 1$. The *divided difference operators* ∂_j for $j \in [n-1]$ are operators on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ defined by

$$\partial_j(f) = \frac{f - (s_j \cdot f)}{x_j - x_{j+1}} = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{j-1}, x_{j+1}, x_j, x_{j+2}, \ldots, x_n)}{x_j - x_{j+1}}.$$

The *isobaric divided difference operators* $\overline{\partial}_j$ are defined on $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$\overline{\partial}_j(f) = \partial_j((1 - x_{j+1})f).$$

Definition 2.2. The *Schubert polynomial* $\mathfrak{S}_w$ of $w \in S_n$ is defined recursively on the weak Bruhat order. Let $w_0 = n\ n{-}1\ \cdots\ 2\ 1 \in S_n$, the longest permutation in S_n. If $w \neq w_0$ then there is $j \in [n-1]$ with $w(j) < w(j+1)$ (called an *ascent* of w). The polynomial $\mathfrak{S}_w$ is defined by

$$\mathfrak{S}_w = \begin{cases} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} & \text{if } w = w_0, \\ \partial_j \mathfrak{S}_{ws_j} & \text{if } w(j) < w(j+1). \end{cases}$$

Definition 2.3. The *Grothendieck polynomial* $\mathfrak{G}_w$ of $w \in S_n$ is defined analogously to the Schubert polynomial, with

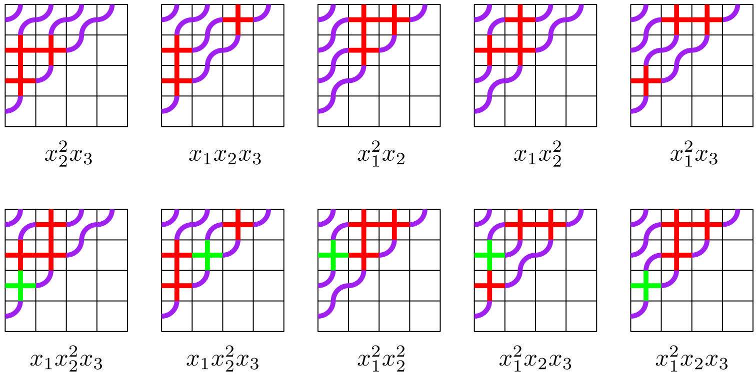
$$\mathfrak{G}_w = \begin{cases} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} & \text{if } w = w_0, \\ \overline{\partial}_j \mathfrak{G}_{ws_j} & \text{if } w(j) < w(j+1). \end{cases}$$

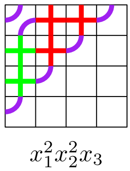
It can be seen from the recursive definitions that $\mathfrak{S}_w$ is homogeneous of degree equal to $\ell(w)$, and equals the lowest-degree nonzero homogeneous component of $\mathfrak{G}_w$. See [27] for a deeper introduction to Schubert polynomials. We now recall pipe dreams, one of many combinatorial constructions of Schubert and Grothendieck polynomials.

Definition 2.4. A *pipe dream* for $w \in S_n$ is a tiling of an $n \times n$ matrix with crosses $+$ and elbows ⌟⌜ such that

- All tiles in the weak south-east triangle are elbows, and
- If you write $1, 2, \ldots, n$ on the top left-to-right and follow the strands (treating second crossings among the same strands as elbows instead), they come out on the left edge and read w from top to bottom.

A pipe dream is *reduced* if no two strands cross twice. Let $\mathrm{RPD}(w)$ and $\mathrm{PD}(w)$ denote respectively the sets of reduced pipe dreams of w and all pipe dreams of w.





$$\mathfrak{S}_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1^2x_2 + x_1x_2^2 + x_1^2x_3$$

$$\mathfrak{G}_{1432} = \mathfrak{S}_{1432} - (2x_1x_2^2x_3 + x_1^2x_2^2 + 2x_1^2x_2x_3) + x_1^2x_2^2x_3$$

FIGURE 2. The set PD(1432). Second crossing of strands are shown in green

Theorem 2.5 ([1,14,15]). *For any permutation $w \in S_n$,*

$$\mathfrak{S}_w = \sum_{P \in \mathrm{RPD}(w)} x^{\mathrm{wt}(P)} \quad \textit{and} \quad \mathfrak{G}_w = \sum_{P \in \mathrm{PD}(w)} (-1)^{\#\mathrm{crosses}(P) - \ell(w)} x^{\mathrm{wt}(P)}$$

where $\mathrm{wt}(P)_i = \#$crosses in row i of P.

Example 2.6. The pipe dreams of $w = 1432$ are shown in Fig. 2.

Pipe dreams also carry additional geometric structure, which we utilize in Sect. 4.

Theorem 2.7 ([22]). *For any $w \in S_n$, there is a simplicial complex Δ_w whose faces correspond to pipe dreams $\{\mathrm{PD}(v) : v \geq w\}$ and whose face inclusions correspond to reverse inclusion of cross tiles. The dimension of Δ_w is $\binom{n}{2} - \ell(w) - 1$. The facets of Δ_w correspond to reduced pipe dreams $\mathrm{RPD}(w)$ of w. The interior faces of Δ_w correspond to $\mathrm{PD}(w)$. The boundary of Δ_w is the union of all complexes Δ_v where $v > w$.*

Above, $\geq$ and $>$ denotes the strong Bruhat order; we refer to [3] for background.

Theorem 2.8 ([22, Corollary 3.8]). *The simplicial complex Δ_w of $w \in S_n$ is a ball whenever $w \neq w_0$, and Δ_{w_0} is empty.*

Example 2.9. When $w = 1432$, Δ_w is shown in Fig. 3 (with faces labeled by their pipe dream).

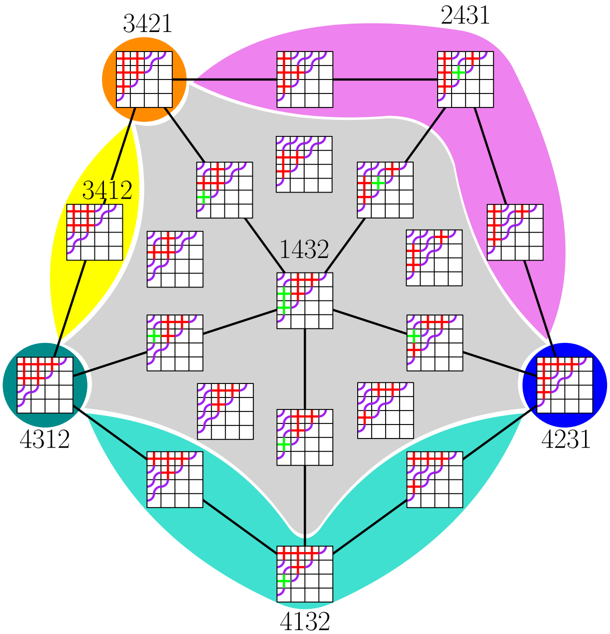


FIGURE 3. The simplicial complex Δ_{1432}. The colored regions indicate pipe dreams corresponding to a particular permutation

2.3. Support and Newton Polytopes of Polynomials

The *support* of a polynomial $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_n]$ is the set

$$\mathrm{supp}(f) = \{\alpha \mid c_\alpha \neq 0\} \subset \mathbb{Z}^n.$$

The *Newton polytope* of f, denoted $\mathrm{Newton}(f)$, is the convex hull of $\mathrm{supp}(f)$. When

$$\mathrm{Newton}(f) \cap \mathbb{Z}^n = \mathrm{supp}(f),$$

f is said to have *saturated Newton polytope*, abbreviated SNP.

2.4. Generalized Permutahedra and Generalized Polymatroids

A function $z : 2^{[n]} \to \mathbb{R}$ is called *submodular* if

$$z(I) + z(J) \geq z(I \cup J) + z(I \cap J) \text{ for all } I, J \subseteq [n].$$

Similarly, z is called *supermodular* if $-z$ is submodular.

Definition 2.10. A polytope $P \subset \mathbb{R}^n$ is a *generalized permutahedron* if there is a submodular function z such that

$$P = \left\{ t \in \mathbb{R}^n \;\middle|\; \sum_{i \in I} t_i \leq z(I) \text{ for all } I \subseteq [n] \text{ and } \sum_{i=1}^{n} t_i = z([n]) \right\}.$$

Definition 2.11. A pair (y, z) of functions $2^{[n]} \to \mathbb{R}$ is called a *paramodular pair* if y is supermodular, z is submodular, and

$$z(I) - y(J) \geq z(I \setminus J) - y(J \setminus I) \text{ for all } I, J \subseteq [n].$$

Definition 2.12. A polytope $Q \subset \mathbb{R}^n$ is called a *generalized polymatroid* if there is a paramodular pair (y, z) such that

$$Q = \left\{ t \in \mathbb{R}^n \;\middle|\; y(I) \leq \sum_{i \in I} t_i \leq z(I) \text{ for all } I \subseteq [n] \right\}.$$

Generalized permutahedra are special cases of generalized polymatroids.

Lemma 2.13 ([17, Theorem 14.2.8]). *Let $Q \subseteq \mathbb{R}^n$ be a generalized polymatroid with paramodular pair (y, z). Then y and z are uniquely determined from Q as*

$$y(I) = \min \left\{ \sum_{i \in I} q_i \;\middle|\; q \in Q \right\} \quad \textit{and} \quad z(I) = \max \left\{ \sum_{i \in I} q_i \;\middle|\; q \in Q \right\}.$$

Lemma 2.14 ([17, Theorem 14.2.10]). *If Q is a generalized polymatroid defined by an integral paramodular pair (y, z), then Q is an integral polyhedron. Furthermore, there are always integral optimizers for*

$$\min \left\{ \sum_{i \in I} q_i \;\middle|\; q \in Q \right\} \quad \textit{and} \quad \max \left\{ \sum_{i \in I} q_i \;\middle|\; q \in Q \right\}.$$

The following proposition is immediate from [18, Theorem 1].

Proposition 2.15. *If $Q, Q' \subset \mathbb{R}^n$ are generalized polymatroids, then so is $Q + Q'$.*

2.5. Diagrams

By a *diagram*, we mean a sequence

$$D = (D_1, D_2, \ldots, D_n)$$

of finite subsets of $[n]$, called the *columns* of D. We interchangeably think of D as a collection of boxes (i, j) in a grid, viewing an element $i \in D_j$ as a box in row i and column j of the grid. When we draw diagrams, we read the indices as in a matrix: i increases top-to-bottom and j increases left-to-right. Associated to any permutation $w \in S_m$ is the *Rothe diagram* $D(w)$, defined by

$$D(w) = \{(i, j) \in [n] \times [n] \mid i < w^{-1}(j) \text{ and } j < w(i)\}.$$

For $R, S \subseteq [n]$, we write $R \preccurlyeq S$ if $\#R = \#S$ and the kth least element of R does not exceed the kth least element of S for each k. For any diagrams $C = (C_1, \ldots, C_n)$ and $D = (D_1, \ldots, D_n)$, we say $C \preccurlyeq D$ if $C_j \preccurlyeq D_j$ for all $j \in [n]$.

2.6. Matroids and Polytopes

A *matroid* M is a pair $(E, \mathcal{B})$ consisting of a finite set E and a nonempty collection of subsets $\mathcal{B}$ of E, called the *bases* of M. $\mathcal{B}$ is required to satisfy the *basis exchange axiom*: If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $B_1 - b_1 \cup b_2 \in \mathcal{B}$. By choosing a labeling of the elements of E, we always assume $E = [n]$ for some n.

Given a matroid $M = (E, \mathcal{B})$ with $E = [n]$ and a basis $B \in \mathcal{B}$, let ζ^B be the indicator vector of B. That is, let $\zeta^B = (\zeta_1^B, \ldots, \zeta_n^B) \in \mathbb{R}^n$ with $\zeta_i^B = 1$ if $i \in B$ and $\zeta_i^B = 0$ if $i \notin B$ for each i. The *matroid polytope* of M is the polytope

$$P(M) = \mathrm{Conv}\{\zeta^B : B \in \mathcal{B}\}.$$

Any set $S \subseteq [n]$ with $S \supseteq B$ for $B \in \mathcal{B}$ is called a *spanning set* of M. The *spanning set polytope* $P_{\mathrm{sp}}(M)$ is the polytope

$$P_{\mathrm{sp}}(M) = \mathrm{Conv}\{\zeta^S \mid S \subseteq [n] \text{ is a spanning set of } M\}.$$

The *rank function* of M is the function

$$r : 2^E \to \mathbb{Z}_{\geq 0}$$

defined by $r(S) = \max\{\#(S \cap B) : B \in \mathcal{B}\}$. The sets $S \cap B$ where $S \subseteq [n]$ and $B \in \mathcal{B}$ are called the *independent sets* of M.

The following result is well-known, see for instance [17,36].

Proposition 2.16. *For any matroid M on ground set $[n]$, $P(M)$ is a generalized permutahedron and $P_{\mathrm{sp}}(M)$ is a generalized polymatroid.*

As a generalized permutahedron, a matroid polytope is parameterized by the (submodular) rank function r of the underlying matroid:

$$P(M) = \left\{ t \in \mathbb{R}^n \mid \sum_{i \in I} t_i \leq r(I) \text{ for } I \subseteq E, \text{ and } \sum_{i \in E} t_i = r(E) \right\}.$$

As a generalized polymatroid, a spanning set polytope is parameterized by

$$P_{\mathrm{sp}}(M) = \left\{ t \in \mathbb{R}^n \mid r(E) - r(E \setminus I) \leq \sum_{i \in I} t_i \leq |I| \text{ for all } I \subseteq E \right\}.$$

Definition 2.17. Fix positive integers $1 \leq c_1 < \cdots < c_r \leq n$. The sets $\{a_1, \ldots, a_r\}$ of positive integers with $a_1 < \cdots < a_r$ such that $a_1 \leq c_1, \ldots, a_r \leq c_r$ are the bases of a matroid (with ground set $[n]$), called the *Schubert matroid* $\mathrm{SM}_n(c_1, \ldots, c_r)$.

Theorem 2.18 ([11, Theorem 11]). *For any permutation $w \in S_n$ with Rothe diagram $D(w) = (D_1, \ldots, D_n)$,*

$$\mathrm{Newton}(\mathfrak{S}_w) = \sum_{j=1}^{n} P(\mathrm{SM}_n(D_j)).$$

In particular, each $\alpha \in \mathrm{supp}(\mathfrak{S}_w)$ can be written as a sum

$$\alpha = \alpha^{(1)} + \cdots + \alpha^{(n)}$$

$$\mathrm{Newton}(\mathfrak{G}_{21543}) = P(\mathrm{SM}_5(1)) + P(\mathrm{SM}_5(3,4)) + P(\mathrm{SM}_5(3))$$

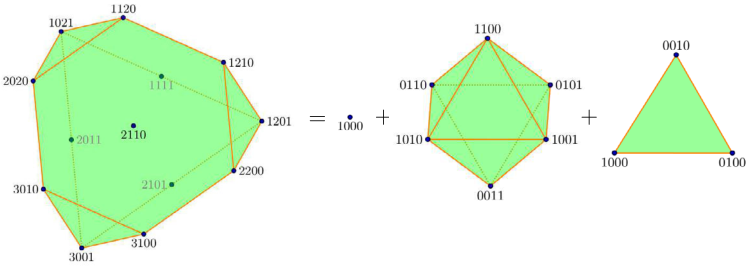
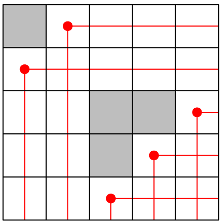


FIGURE 4. The Schubert matroid polytope decomposition of $\mathrm{Newton}(\mathfrak{G}_{21543})$

where $\alpha^{(j)}$ *is the indicator vector of a basis of* $\mathrm{SM}_n(D_j)$*.*

Example 2.19. Consider the permutation $w = 21543$. Then

$D(21543) =$  , with

$$\begin{aligned} \mathrm{SM}_5(1) &= \{\{1\}\}, \\ \mathrm{SM}_5(3,4) &= \{\{1,2\},\{1,3\},\{1,4\}, \\ &\qquad \{2,3\},\{2,4\},\{3,4\}\}, \\ \mathrm{SM}_5(3) &= \{\{1\},\{2\},\{3\}\}. \end{aligned}$$

The Minkowski sum decomposition of $\mathrm{Newton}(\mathfrak{G}_{21543})$ is shown in Fig. 4.

3. Supports of Grothendieck Polynomials for Certain Classes of Permutations

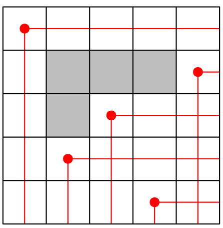
For a permutation $w \in S_n$, recall the support of the Grothendieck polynomial of w is the set $\mathrm{supp}(\mathfrak{G}_w)$ of exponents of terms in $\mathfrak{G}_w$ with nonzero coefficient. We endow $\mathrm{supp}(\mathfrak{G}_w)$ with the following poset structure.

Definition 3.1. For $\alpha, \beta \in \mathbb{Z}^n$, define the *componentwise comparison* $\leq$ by

$$\alpha \leq \beta \text{ if } \alpha_i \leq \beta_i \text{ for all } i \in [n].$$

We study the subsets $\mathrm{supp}(\mathfrak{G}_w) \subset \mathbb{Z}^n$ with the inherited poset structure.

Example 3.2. For $w = 15324$, we have

$D(15324) =$  ,

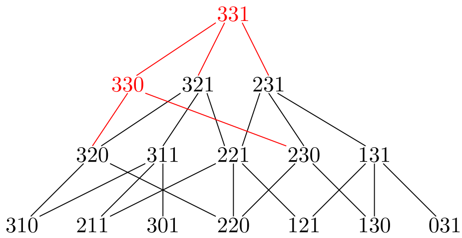


FIGURE 5. Black: The Hasse diagram of $\mathrm{supp}(\mathfrak{G}_{15324})$ (writing exponents $(a, b, c, 0, 0) \in \mathbb{Z}^5$ as abc); cf. Fig. 1. Red: Elements of $(P_{\mathrm{sp}}(\mathrm{SM}_3(\{2,3\})) + P_{\mathrm{sp}}(\mathrm{SM}_2(\{2\})) + P_{\mathrm{sp}}(\mathrm{SM}_2(\{2\}))) \cap \mathbb{Z}^3$ not in $\mathrm{supp}(\mathfrak{G}_{15324})$

$$\begin{aligned}\mathfrak{S}_w &= x_1^3x_2 + x_1^3x_3 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^3 + x_1x_2^2x_3 + x_2^3x_3,\\ \mathfrak{G}_w &= \mathfrak{S}_w - (x_1^3x_2^2 + 2x_1^3x_2x_3 + x_1^2x_2^3 + 2x_1^2x_2^2x_3 + 2x_1x_2^3x_3)\\ &\quad + (x_1^3x_2^2x_3 + x_1^2x_2^3x_3).\end{aligned}$$

The Hasse diagram of $\mathrm{supp}(\mathfrak{G}_w)$ as a poset under componentwise comparison is shown in black in Fig. 5 (cf. also Figure 1).

We first present two known properties of the posets $\mathrm{supp}(\mathfrak{G}_w)$.

Lemma 3.3. *Fix any permutation $w \in S_n$. For each $\beta \in \mathrm{supp}(\mathfrak{G}_w)$, with $|\beta| > \ell(w)$, there is $\alpha \in \mathrm{supp}(\mathfrak{G}_w)$ with $\alpha \leq \beta$ and $|\alpha| = |\beta| - 1$.*

Proof. Choose any pipe dream P of w with weight β. Since $|\beta| > \ell(w)$, P is not reduced. Removing any single second crossing in P yields a pipe dream P' whose weight α satisfies the conditions of the lemma. □

For any diagram $D \subseteq [n]^2$, define the *weight* of D to be the vector $\mathrm{wt}(D) \in \mathbb{Z}^n$ whose ith component counts the number of boxes in row i of D.

Definition 3.4. For any diagram D, the *upper closure* $\overline{D}$ is the diagram

$$\overline{D} = \{(i, j) \mid j = j' \text{ and } i \leq i' \text{ for some } (i', j') \in D\}.$$

Theorem 3.5. ([29, Theorem 1.2]) *For any permutation $w \in S_n$,*

$$\mathrm{wt}(\overline{D(w)}) \geq \alpha \textit{ for all } \alpha \in \mathrm{supp}(\mathfrak{G}_w).$$

Consequently,

$$\deg \mathfrak{G}_w \leq \#\overline{D(w)}.$$

The following theorem gives a polytopal interpretation for Theorem 3.5. For the Minkowski sum below, we use the natural inclusions $\mathbb{R}^{k-1} \to \mathbb{R}^k$ by appending a zero.

Theorem 3.6. *Let $w \in S_n$ be any permutation and let $D(w)$ have columns $D_1, \ldots, D_n$. Set $d_j = \max(D_j)$, taking $\max(\emptyset) = 0$. Then,*

$$\mathrm{Newton}(\mathfrak{G}_w) \subseteq \sum_{j=1}^{n} P_{\mathrm{sp}}(\mathrm{SM}_{d_j}(D_j)).$$

Proof. Let $\alpha \in \mathrm{supp}(\mathfrak{G}_w)$. By repeated use of Lemma 3.3, we can write $\alpha = \beta + \gamma$ where $\beta \in \mathrm{supp}(\mathfrak{S}_w)$ and $\gamma \geq 0$. By Theorem 2.18, we can find a decomposition

$$\beta = \beta^{(1)} + \cdots + \beta^{(n)}$$

where each $\beta^{(j)}$ is the indicator vector of a basis of $\mathrm{SM}_n(D_j)$.

Let $\delta = \mathrm{wt}(\overline{D(w)})$, and note that

$$\delta = \delta^{(1)} + \cdots + \delta^{(n)}$$

where $\delta^{(j)}$ is the indicator vector in $\mathbb{R}^n$ of $\overline{D_j} = [d_j]$. Let A be the $n \times n$ matrix with columns $\delta^{(j)}$. Equivalently, $A_{i,j} = 1$ if and only if $(i,j) \in \overline{D(w)}$.

Observe that $\beta_i^{(j)} = 1$ means $A_{ij} = 1$. Call the entry (i,j) of A marked if $\beta_i^{(j)} = 1$. Fix any $i \in [n]$. Since $\gamma_i \geq 0$, we have $\beta_i \leq \alpha_i$, so $\delta_i - \beta_i \geq \delta_i - \alpha_i$. Since $\delta_i - \beta_i$ is the number of unmarked entries of A in row i, there are at least $\delta_i - \alpha_i$ unmarked entries in row i of A.

For each $p \in [n]$, pick any $\delta_i - \alpha_i$ unmarked entries in row i of A. Set all these entries to 0 to get a new matrix B. Let $\epsilon^{(j)}$ be the jth column of B for each $j \in [n]$. By construction,

$$\epsilon^{(1)} + \cdots + \epsilon^{(n)} = \delta - (\delta - \alpha) = \alpha.$$

From the use of marked entries, we see that $\delta^{(j)} \geq \epsilon^{(j)} \geq \beta^{(j)}$ for each j, so $\epsilon^{(j)} \in P_{\mathrm{sp}}(\mathrm{SM}_{d_j}(D_j))$. Thus,

$$\alpha = \sum_{j=1}^{n} \epsilon^{(j)} \in \sum_{j=1}^{n} P_{\mathrm{sp}}(\mathrm{SM}_{d_j}(D_j)).$$

□

Example 3.7. For $w = 15324$, the inclusion in Theorem 3.6 is strict, as we now show. The columns $D_1, \ldots, D_5$ of $D(w)$ are equal to $\emptyset, \{2,3\}, \{2\}, \{2\}, \emptyset$ respectively. Compute that

$$P_{\mathrm{sp}}(\mathrm{SM}_3(\{2,3\})) = \mathrm{conv}\{\zeta^{\{2,3\}}, \zeta^{\{1,3\}}, \zeta^{\{1,2\}}, \zeta^{\{1,2,3\}}\}$$
$$P_{\mathrm{sp}}(\mathrm{SM}_2(\{2\})) = \mathrm{conv}\{\zeta^{\{2\}}, \zeta^{\{1\}}, \zeta^{\{1,2\}}\}.$$

The polytope $P_{\mathrm{sp}}(\mathrm{SM}_3(\{2,3\}))+P_{\mathrm{sp}}(\mathrm{SM}_2(\{2\}))+P_{\mathrm{sp}}(\mathrm{SM}_2(\{2\}))$ has 16 integer points, shown in Figure 5, whereas $\mathrm{supp}(\mathfrak{G}_{15324})$ has just 14 elements.

We now make two pairs of conjectures describing the support of Grothendieck polynomials. We provide partial results and describe some implications of the conjectures.

Theorem 3.5 shows the vector $\mathrm{wt}(\overline{D(w)})$ is an upper bound (in $\mathbb{Z}^n$) for $\mathrm{supp}(\mathfrak{G}_w)$. When $\mathrm{wt}(\overline{D(w)}) \in \mathrm{supp}(\mathfrak{G}_w)$, it is the unique maximal element. We conjecture that all maximal elements of $\mathrm{supp}(\mathfrak{G}_w)$ have the same degree.

Conjecture 1.1. *If* $\alpha \in \mathrm{supp}(\mathfrak{G}_w)$ *and* $|\alpha| < \deg \mathfrak{G}_w$*, then there exists* $\beta \in \mathrm{supp}(\mathfrak{G}_w)$ *with* $\alpha < \beta$.

The following is a natural strengthening of Conjecture 1.1, dual to Lemma 3.3.

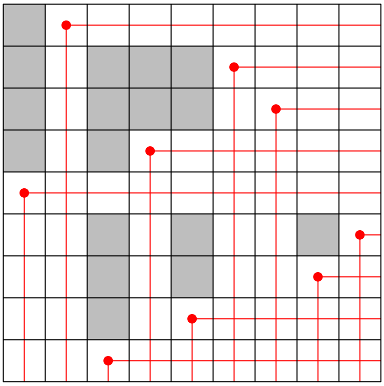


FIGURE 6. The Rothe diagram of the fireworks permutation $w = 267419853$

Conjecture 1.2. *If* $\alpha \in \operatorname{supp}(\mathfrak{G}_w)$ *and* $|\alpha| < \deg \mathfrak{G}_w$*, then there exists* $\beta \in \operatorname{supp}(\mathfrak{G}_w)$ *with* $\alpha < \beta$ *and* $|\beta| = |\alpha| + 1$*.*

We prove Conjecture 1.1 in the special case of fireworks permutations by confirming $\operatorname{wt}(\overline{D(w)}) \in \operatorname{supp}(\mathfrak{G}_w)$. Conjecture 1.2 is shown for vexillary permutations in [19, Proposition 4.1].

Definition 3.8 ([34, Definition 3.5]). A permutation $w \in S_n$ is called *fireworks* if the initial elements of its decreasing runs occur in increasing order.

Example 3.9. Consider $w = 267419853$. The decreasing runs of w are $2|6|741|9853$, so w is fireworks since $2 < 6 < 7 < 9$. The Rothe diagram of w is shown in Fig. 6.

Observe in the previous example that there is a dot directly below the southmost box in each column of $D(w)$. We show this property characterizes fireworks permutations.

Proposition 3.10. *Let* $w \in S_n$ *and* $D(w)$ *have columns* $D_1, \dots, D_n$*. Then* w *is fireworks if and only if* $D_{w(j)} \neq \emptyset$ *implies* $\max(D_{w(j)}) = j - 1$*.*

Proof. First, note that $D_{w(j)} \neq \emptyset$ if and only if there is $i < j$ with $w(i) > w(j)$. In particular when $w(j-1) > w(j)$, the box $(j-1, w(j))$ is the southmost box in column $w(j)$ of $D(w)$.

Suppose w is fireworks. If $w(j)$ is initial in a decreasing run of w, then $w(j) > w(i)$ for all $i < j$, so $D_{w(j)} = \emptyset$. If $w(j)$ is not initial in a decreasing run of w, then $w(j-1) > w(j)$ and we are done.

Conversely, suppose w is not fireworks. Then, we can find a decreasing run $w(i) > w(i+1) > \cdots > w(j-1)$ with $w(j-1) < w(j) < w(i)$. Then $D_{w(j)} \neq \emptyset$, but $(j-1, w(j)) \notin D_{w(j)}$, so $\max(D_{w(j)}) < j - 1$. □

Definition 3.11 ([34]). The *Rajchgot code* of $w \in S_n$ is the vector $\operatorname{rajcode}(w) = (r_1, \dots, r_n)$, where r_j is defined as follows. Choose an increasing subsequence of $w(j), w(j+1), \dots, w(n)$ containing $w(j)$ which has greatest length among all such subsequences. Let r_j be the number of terms from $w(j), w(j+1), \ldots, w(n)$ omitted to form the chosen subsequence.

Theorem 3.12 ([34, Theorem 1.1]). *Let $w \in S_n$ and $\mathrm{rajcode}(w) = (r_1, \ldots, r_n)$. Then $\deg \mathfrak{G}_w = r_1 + \cdots + r_n$, and in any term order satisfying $x_1 < x_2 < \cdots < x_n$, the leading term of $\mathfrak{G}_w$ is a scalar multiple of $x^{\mathrm{rajcode}(w)}$.*

The following two lemmas describe $\mathrm{rajcode}(w)$ when w is a fireworks permutation.

Lemma 3.13. *Let $w \in S_n$ be fireworks. The Rajchgot code $\mathrm{rajcode}(w) = (r_1, \ldots, r_n)$ can be read off from w as follows:*

$$r_i = \begin{cases} 0 & \text{if } i = n \\ r_{i+1} & \text{if } w(i) < w(i+1) \\ r_{i+1} + 1 & \text{if } w(i) > w(i+1) \end{cases}$$

Proof. Since w is fireworks, one greatest length increasing subsequence of $w(j), w(j+1), \ldots, w(n)$ (starting with $w(j)$) consists of $w(j)$ together with the initial elements of all subsequent decreasing runs. □

Lemma 3.14. *When $w \in S_n$ is fireworks,* $\mathrm{rajcode}(w) = \mathrm{wt}(\overline{D(w)})$

Proof. Let $\mathrm{rajcode}(w) = (r_1, \ldots, r_n)$. From Proposition 3.10 and its proof together with Lemma 3.13 we see that

$$\begin{aligned} \mathrm{wt}(\overline{D(w)})_i &= \#\text{ of columns of } D(w) \text{with a box in any row } j \geq i \\ &= \#\{j \in [n] \mid w(j) \text{ is not initial in a decreasing run of } w, \\ \text{and } j \geq i+1 &= r_i. \end{aligned}$$

□

Denote by $\mathfrak{G}_w^{\mathrm{top}}$ the highest-degree nonzero homogeneous component of $\mathfrak{G}_w$.

Theorem 3.15. *Let $w \in S_n$ be fireworks. Then* $\mathrm{supp}(\mathfrak{G}_w^{\mathrm{top}}) = \{\mathrm{wt}(\overline{D(w)})\}$.

Proof. Theorem 3.5 establishes $\mathrm{wt}(\overline{D(w)})$ as an upper bound for $\mathrm{supp}(\mathfrak{G}_w)$ (under componentwise comparison). Theorem 3.12 together with Lemma 3.14 show that when w is fireworks,

$$\mathrm{wt}(\overline{D(w)}) = \mathrm{rajcode}(w) \in \mathrm{supp}(\mathfrak{G}_w^{\mathrm{top}}).$$

Since no two elements in the support of a homogeneous polynomial can be componentwise comparable, the theorem follows. □

Theorem 3.16. *Conjecture 1.1 holds for fireworks permutations.*

We now make one other pair of conjectures, further specifying the poset structure of $\mathrm{supp}(\mathfrak{G}_w)$.

Conjecture 1.3. *Fix any $w \in S_n$. If $\alpha, \gamma \in \mathrm{supp}(\mathfrak{G}_w)$, then*

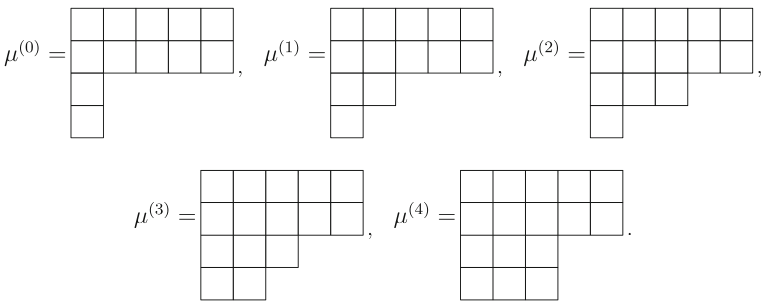
$$\{\beta \in \mathbb{Z}^n \mid \alpha \leq \beta \leq \gamma\} \subseteq \mathrm{supp}(\mathfrak{G}_w).$$

Conjecture 1.4. *For all* $w \in S_n$, $\mathfrak{G}_w$ *has SNP and* $\mathrm{Newton}(\mathfrak{G}_w)$ *is a generalized polymatroid.*

We prove Conjectures 1.2 and 1.4 for Grassmannian permutations. We begin by reviewing the main result of [9]. Recall a permutation w is *Grassmannian* if w has exactly one descent. It is well-known that Grassmannian permutations are in bijection with partitions: when w has a descent at position r, the corresponding partition λ is given by $\lambda = (w(r) - r, \cdots, w(2) - 2, w(1) - 1)$.

Let $\lambda \in \mathbb{Z}^n$ be a partition and consider the Young diagram of λ in English notation. Set $\mu^{(0)} = \lambda$. For $j \geq 1$, define $\mu^{(j)}$ to be $\mu^{(j-1)}$ with a box added to the northmost row r such that the addition still yields a partition, and $\mu_r^{(j-1)} - \mu_r^{(0)} < r - 1$. Stop when no such box exists. Let the resulting partitions be $\mu^{(0)}, \ldots, \mu^{(N)}$. For Grassmannian $w \in S_n$ corresponding to λ, define $\mathrm{Par}(w) = \{\mu^{(0)}, \ldots, \mu^{(N)}\}$, the partitions constructed from λ. Recall that *dominance order* on partitions is defined by $\rho \trianglelefteq \nu$ if $\rho_1 + \cdots + \rho_i \leq \nu_1 + \cdots + \nu_i$ for all i and $|\rho| = |\nu|$.

Example 3.17. Let $\lambda = (5, 5, 1, 1)$. Then we obtain the sequence of partitions



Theorem 3.18 ([9]). *Suppose* w *is a Grassmannian permutation with* $\mathrm{Par}(w) = \{\mu^{(0)}, \ldots, \mu^{(N)}\}$. *Then* $\deg \mathfrak{G}_w = |\mu^{(N)}|$, *and for* $0 \leq j \leq N$, *the support of the* $(\ell(w) + j)$*th degree homogeneous component of* $\mathfrak{G}_w$ *is exactly*

$$\{\alpha \in \mathbb{Z}^n \mid \alpha \geq 0 \text{ and } \alpha \trianglelefteq \mu^{(j)}\}.$$

In particular, $\mathfrak{G}_w$ *has SNP.*

Theorem 3.19. *Conjectures 1.2 and 1.4 hold when* w *is a Grassmannian permutation.*

Proof. Let $\mathrm{Par}(w) = \{\mu^{(0)}, \ldots, \mu^{(N)}\}$ with $\lambda = \mu^{(0)}$. We first confirm Conjecture 1.2. Let $\alpha \in \mathrm{supp}(\mathfrak{G}_w)$ with $|\alpha| = \ell(w) + j < \deg \mathfrak{G}_w$. By Theorem 3.18, $\alpha \trianglelefteq \mu^{(j)}$. Let e_i be the standard basis vector such that $\mu^{(j+1)} - \mu^{(j)} = e_i$. Then $\alpha + e_i \trianglelefteq \mu^{(j+1)}$, so $\alpha + e_i \in \mathrm{supp}(\mathfrak{G}_w)$. Hence $\beta = \alpha + e_i$ confirms Conjecture 1.2.

We now confirm Conjecture 1.4. Define functions $y, z : 2^{[n]} \to \mathbb{R}$ by

$$y(I) = \lambda_n + \cdots + \lambda_{n-\#I+1} \quad \text{and} \quad z(I) = \mu_1^{(N)} + \cdots \mu_{\#I}^{(N)}.$$

It is straightforward to check that (y, z) is a paramodular pair. Let Q be the corresponding generalized polymatroid

$$Q = \left\{ t \in \mathbb{R}^n \;\middle|\; y(I) \le \sum_{i \in I} t_i \le z(I) \text{ for all } I \subseteq [n] \right\}.$$

Observe that

$$\min_{t \in \text{Newton}(\mathfrak{G}_w)} \sum_{i \in I} t_i = \lambda_n + \cdots + \lambda_{n-\#I+1} \quad \text{and}$$
$$\max_{t \in \text{Newton}(\mathfrak{G}_w)} \sum_{i \in I} t_i = \mu_1^{(N)} + \cdots + \mu_{\#I}^{(N)},$$

so $\text{Newton}(\mathfrak{G}_w) \subseteq Q$. We prove the opposite inclusion.

Let $q \in Q \cap \mathbb{Z}^n$ and $|q| = |\lambda| + j$. We must show $q \trianglelefteq \mu^{(j)}$. We have

$$y(I) \le \sum_{i \in I} q_i = \sum_{i=1}^{n} q_i - \sum_{i \notin I} q_i = |\lambda| + j - \sum_{i \notin I} q_i,$$

so that

$$\sum_{i \notin I} q_i \le \lambda_1 + \cdots + \lambda_{n-\#I} + j = \lambda_1 + \cdots + \lambda_{\#([n] \setminus I)} + j.$$

Replacing $[n] \setminus I$ with I, we see

$$\sum_{i \in I} q_i \le \lambda_1 + \cdots + \lambda_{\#I} + j \text{ for all } I \subseteq [n]. \tag{1}$$

Set α to be the vector $\alpha = \mu^{(N)} - \lambda$, so

$$\sum_{i \in I} q_i \le z(I) = (\lambda_1 + \alpha_1) + \cdots + (\lambda_{\#I} + \alpha_{\#I}) \text{ for all } I \subseteq [n]. \tag{2}$$

Fix any $k \in [n]$. Note that $j \le \alpha_1 + \cdots + \alpha_k$. Then (1) shows

$$\mu_1^{(j)} + \cdots + \mu_k^{(j)} = \lambda_1 + \cdots + \lambda_k + j \ge q_1 + \cdots + q_k.$$

Hence, we have shown $q \trianglelefteq \mu^{(j)}$. □

The following characterization of equality in Theorem 3.6 would follow from Conjectures 1.1 and 1.4.

Proposition 3.20. *Let $w \in S_n$ be any permutation and let $D(w)$ have columns $D_1, \ldots, D_n$. Set $d_j = \max(D_j)$, taking $\max(\emptyset) = 0$. Assuming Conjectures 1.1 and 1.4 hold, it follows that*

$$\deg \mathfrak{G}_w = \#\overline{D(w)} \textit{ if and only if } \text{Newton}(\mathfrak{G}_w) = \sum_{j=1}^{n} P_{\text{sp}}(\text{SM}_{d_j}(D_j)).$$

Proof. Since, Conjectures 1.1 and 1.3 together imply Conjecture 1.2, we are assuming Conjecture 1.2 holds as well. Suppose first that the polyhedral equality holds. By linearity,

$$\mathrm{Newton}(\mathfrak{G}_w^{\mathrm{top}}) = \{\mathrm{wt}(\overline{D(w)})\}.$$

Thus $\deg \mathfrak{G}_w = \#\overline{D(w)}$.

Conversely, assume $\deg \mathfrak{G}_w = \#\overline{D(w)}$. By Theorem 3.5, we have Newton $(\mathfrak{G}_w^{\mathrm{top}}) = \{\mathrm{wt}(\overline{D(w)})\}$. Since we are assuming Conjecture 1.4, $\mathfrak{G}_w$ has SNP and $\mathrm{Newton}(\mathfrak{G}_w)$ is a generalized polymatroid.

Set $Q = \sum_{j=1}^{n} P_{\mathrm{sp}}(\mathrm{SM}_{d_j}(D_j))$. By Propositions 2.15 and 2.16, Q is a generalized polymatroid. Denote its associated paramodular pair by (y, z). Observe that the integer points of $P_{\mathrm{sp}}(\mathrm{SM}_{d_j}(D_j))$ satisfy the conclusions of Lemma 3.3 and Conjecture 1.2. Then, Lemmas 2.13 and 2.14 imply

$$\begin{aligned} y(I) &= \min\left\{\sum_{i\in I} q_i \;\;\middle|\; q \in Q \cap \mathbb{Z}^n\right\} = \min\left\{\sum_{i\in I} q_i \;\;\middle|\; q \in \mathbb{Z}^n \cap \sum_{j=1}^{n} P(\mathrm{SM}_n(D_j))\right\} \\ &= \min\left\{\sum_{i\in I} q_i \;\;\middle|\; q \in \mathbb{Z}^n \cap \mathrm{Newton}(\mathfrak{G}_w)\right\} = \min\left\{\sum_{i\in I} q_i \;\;\middle|\; q \in \mathrm{Newton}(\mathfrak{G}_w)\right\}. \end{aligned}$$

Similarly, Lemmas 2.13 and 2.14 together with Conjecture 1.2 imply

$$\begin{aligned} z(I) &= \max\left\{\sum_{i\in I} q_i \;\;\middle|\; q \in Q \cap \mathbb{Z}^n\right\} = \max\left\{\sum_{i\in I} q_i \;\;\middle|\; q = \mathrm{wt}(\overline{D(w)})\right\} \\ &= \max\left\{\sum_{i\in I} q_i \;\;\middle|\; q \in \mathbb{Z}^n \cap \mathrm{Newton}(\mathfrak{G}_w^{\mathrm{top}})\right\} = \max\left\{\sum_{i\in I} q_i \;\;\middle|\; q \in \mathrm{Newton}(\mathfrak{G}_w)\right\}. \end{aligned}$$

Thus, the paramodular pairs of Q and $\mathrm{Newton}(\mathfrak{G}_w)$ coincide. Consequently, $Q = \mathrm{Newton}(\mathfrak{G}_w)$. ☐

4. Coefficients and Principal Specialization of Grothendieck Polynomials

For each permutation w, we describe a poset P_w. For certain permutations, we connect the Möbius function of P_w to the coefficients of the Grothendieck polynomial $\mathfrak{G}_w$. Recall the *Möbius function* μ of a finite poset P is the unique function $P \times P \to \mathbb{Z}$ defined by

$$\mu(p,q) = \begin{cases} 0 & \text{if } p \not\leq q, \\ 1 & \text{if } p = q, \\ -\displaystyle\sum_{p\leq r<q} \mu(p,r) & \text{if } p < q. \end{cases}$$

Definition 4.1. For each $w \in S_n$, define P_w to be the poset

$$\{\beta \in \mathbb{Z}^n \mid \alpha \leq \beta \leq \mathrm{wt}(\overline{D(w)}) \text{ for some } \alpha \in \mathrm{supp}(\mathfrak{G}_w)\},$$

together with a minimum element denoted $\hat{0}$.

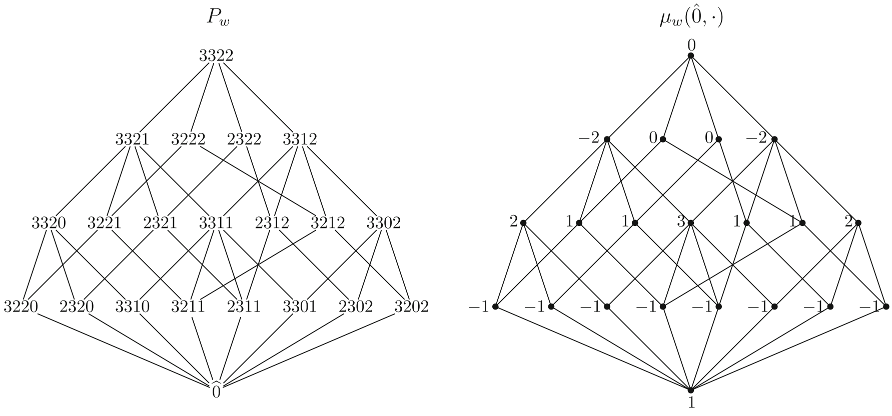


FIGURE 7. The poset P_w and its Möbius function μ_w

Conjecture 1.5. *Let w be a permutation such that all nonzero coefficients of $\mathfrak{S}_w$ equal 1. If μ_w is the Möbius function of P_w, then*

$$\mathfrak{G}_w = -\sum_{\beta \in P_w - \hat{0}} \mu_w(\hat{0}, \beta) x^\beta.$$

Conjecture 1.5 has now been proved by Pechenik and Satriano [33].

Example 4.2. Set $w = 351624$. Then $\mathrm{wt}(\overline{D(w)}) = (3,3,2,2)$, and

$$\begin{aligned}\mathfrak{G}_w = \ & (x_1^3x_2^2x_3^2 + x_1^2x_2^3x_3^2 + x_1^3x_2^3x_3 + x_1^3x_2^2x_3x_4 + x_1^2x_2^3x_3x_4 + x_1^3x_2^3x_4 \\ & + x_1^2x_2^3x_4^2 + x_1^3x_2^2x_4^2) - (2x_1^3x_2^3x_3^2 + x_1^3x_2^2x_3^2x_4 + x_1^2x_2^3x_3^2x_4 + 3x_1^3x_2^3x_3x_4 \\ & + x_1^2x_2^3x_3x_4^2 + x_1^3x_2^2x_3x_4^2 + 2x_1^3x_2^3x_4^2) + (2x_1^3x_2^3x_3^2x_4 + 2x_1^3x_2^3x_3x_4^2).\end{aligned}$$

The poset P_w and its Möbius function μ_w are shown in Fig. 7.

The class of permutations covered by Conjecture 1.5 is characterized by the following result.

Theorem 4.3 ([12, Theorem 4.8]). *A permutation w avoids the patterns 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, and 315642 if and only if all nonzero coefficients of $\mathfrak{S}_w$ equal 1.*

We conjecture one last property connecting the poset structure of $\mathrm{supp}(\mathfrak{G}_w)$ to the coefficients of $\mathfrak{G}_w$.

Conjecture 1.6. *Fix $w \in S_n$ and let $\mathfrak{G}_w = \sum_{\alpha \in \mathbb{Z}^n} C_{w\alpha} x^\alpha$. For any $\beta \in \mathrm{supp}(\mathfrak{G}_w^{\mathrm{top}})$,*

$$\sum_{\alpha \leq \beta} C_{w\alpha} = 1.$$

When $\mathrm{supp}(\mathfrak{G}_w)$ has a unique maximum element (such as for fireworks permutations), Conjecture 1.6 specializes to

$$\mathfrak{G}_w(1, \ldots, 1) = 1.$$

The principal specialization of β-Grothendieck polynomials has been previously considered, for example by Kirillov [21, Comment 3.2] as well as by the first author of this paper [28]. While we assume that $\mathfrak{G}_w(1,\ldots,1) = 1$ for all $w \in S_n$ is well-known to experts, we could not find the original source of this fact. We include a short proof below based on pipe dream complexes, essentially a restatement of the proof given in [37, Corollary 3.5].

We recall basic facts about the Euler characteristic of simplicial complexes. Let Δ be any simplicial complex. Denote the interior faces of Δ by $\mathrm{int}(\Delta)$, and the boundary faces by $\mathrm{bd}(\Delta)$. Suppose Δ has f_i faces of dimension i for each $i \geq 0$. Recall the *Euler characteristic* $\chi(\Delta)$ is the alternating sum

$$\chi(\Delta) = f_0 - f_1 + f_2 - f_3 + \cdots$$

It is well-known that $\chi(\Delta) = 1$ when Δ is a ball, and that $\chi(\Delta) = 1 + (-1)^{d-1}$ when Δ is the boundary of a d-dimensional ball. Recall the pipe dream complex Δ_w of $w \in S_n$ (see Theorem 2.7 and Example 2.9).

Proposition 4.4. ([37, Corollary 3.5]) *For any permutation $w \in S_n$,*

$$\mathfrak{G}_w(1,\ldots,1) = 1.$$

Proof. Let $w \in S_n$. If $w = w_0$, then clearly the result holds. Otherwise, Theorem 2.8 implies Δ_w is a ball of dimension $d = \binom{n}{2} - \ell(w) - 1$. Thus

$$\chi(\mathrm{int}(\Delta_w)) = \chi(\Delta_w) - \chi(\mathrm{bd}(\Delta_w)) = 1 - (1 + (-1)^{d-1}) = (-1)^d.$$

From Theorems 2.5 and 2.7, one observes $\mathfrak{G}_w(1,\ldots,1) = (-1)^d \chi(\mathrm{int}(\Delta_w))$. Hence $\mathfrak{G}_w(1,\ldots,1) = 1$. □

Alternatively, one can also deduce the preceding result from [28, Lemma 2.3]. Theorem 3.15 yields the following.

Corollary 4.5. *Conjecture 1.6 holds for fireworks permutations.*

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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Karola Mészáros
Department of Mathematics
Cornell University
Ithaca
NY 14853
USA
e-mail: karola@math.cornell.edu

Linus Setiabrata
Department of Mathematics
University of Chicago
Chicago
IL 60637
USA
e-mail: linus@math.uchicago.edu

Avery St. Dizier
Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana
IL 61801
USA
e-mail: stdizie2@illinois.edu

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