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On the existence, uniqueness and calculation of Wiener path integral most probable paths for determining the stochastic response of nonlinear dynamical systems

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ABSTRACT

Various techniques are developed for addressing the existence, uniqueness and numerical calculation of Wiener path integral (WPI) most probable path solutions. Specifically, the WPI technique for determining the stochastic response of diverse nonlinear dynamical systems treats the system response joint transition probability density function as a functional integral over the space of all possible paths connecting the initial and the final states of the response vector. This functional integral is evaluated, ordinarily, by resorting to an approximate approach that considers the contribution only of the most probable path. The most probable path corresponds to an extremum of the functional integrand and is determined by solving a functional minimization problem that takes the form of a deterministic boundary value problem (BVP).

In this paper, first, it is shown that for the commonly considered case of the system nonlinearity being of polynomial form, there exist globally optimal solutions corresponding to the most probable path BVP. Further, relying on algebraic geometry concepts and tools, a condition is derived for determining if the BVP for the most probable path exhibits a unique solution over a specific region. Furthermore, a novel solution approach is developed for the BVP by relying on Sylvester's dialytic method of elimination. Notably, the method reduces the complexity of the BVP system of coupled multivariate polynomial equations by eliminating one or more variables. Various numerical examples pertaining to diverse nonlinear oscillators are included for demonstrating the capabilities of the developed techniques.

1. Introduction

Monte Carlo simulation (MCS) constitutes a versatile technique for determining the stochastic response of diverse nonlinear dynamical systems and structures (e.g., [1–3]). Nevertheless, the associated computational cost becomes prohibitive when the objective relates to estimating quite low probability events (e.g., failures). In this regard, various alternative, semi-analytical or purely numerical, techniques have been developed in the field of stochastic engineering dynamics over the past six decades for treating complex structural systems and for computing response and reliability statistics. The interested reader is directed to some standard books and review papers, such as in Refs. [4–8], for a broad perspective.

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Notably, one of the promising techniques, pioneered in stochastic engineering dynamics by Kougioumtzoglou and co-workers [9,10], relies on the mathematical concept of Wiener path integral (WPI) that was originally developed by Wiener [11,12]. Remarkably, the technique exhibits both high accuracy [13–15] and low computational cost [16], and is capable of treating stochastically excited systems exhibiting diverse complex nonlinear/hysteretic behaviors (e.g., [17–19]).

According to the WPI technique (e.g., [20]), the system response joint transition probability density function (PDF) is expressed as a functional integral over the space of all possible paths connecting the initial and the final states of the response vector. Further, the functional integral is evaluated, ordinarily, by resorting to an approximate approach that considers the contribution only of the most probable path. This corresponds to an extremum of the functional integrand and is determined by solving a functional minimization problem that takes the form of a deterministic boundary value problem (BVP).

In general, a plethora of well-established numerical optimization schemes (e.g., [21]) can be employed for solving the resulting deterministic BVP for the most probable path. In this regard, a Newton's iterative optimization scheme was developed in [22]. However, there is generally no guarantee that the optimization algorithm converges to a global minimum. To address this issue, a conceptually different solution approach was also pursued in [22] that relied on computational algebraic geometry concepts and tools. In fact, a Gröbner basis approach was utilized, based on which the entire set of solutions corresponding to the BVP can be computed. Thus, the global minimum (or minima) can be determined. Nevertheless, the associated computational cost becomes non-trivial with increasing system dimensionality.

In this paper, the focus is directed to the existence, uniqueness and calculation of WPI most probable path solutions. In this regard, first, it is shown that for the commonly considered case of the system nonlinearity being of polynomial form, there exist globally optimal solutions corresponding to the most probable path optimization problem. Further, relying on algebraic geometry concepts and tools, a condition is derived for determining if the BVP for the most probable path exhibits a unique solution over a specific region. Furthermore, a novel solution approach is developed for the BVP by relying on Sylvester's dialytic method of elimination [23,24]. The rationale of the method relates to reducing the complexity of the BVP system of coupled multivariate polynomial equations by eliminating one or more variables. In fact, the method yields a univariate polynomial equation to be solved for the suppressed variable. Various numerical examples pertaining to diverse nonlinear oscillators are included for demonstrating the capabilities of the developed techniques.

2. Preliminaries

2.1. Wiener path integral and most probable path approximation

In this section, the salient aspects of the WPI technique, pioneered in the field of engineering mechanics by Kougioumtzoglou and co-workers [9,10] for determining the stochastic response of diverse dynamical systems, are reviewed for completeness. The interested reader is also directed to Refs. [15,16,20] for more details and some more recent developments.

Specifically, consider a nonlinear multi-degree-of-freedom (multi-DOF) system whose dynamics is governed by the second-order stochastic differential equation

$$\mathbf{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \boldsymbol{w}(t). \tag{1}$$

In Eq. (1), $\boldsymbol{x} = \left[x_j(t)\right]_{n\times1}$ is the n-dimensional response displacement vector, $\mathbf{M}$ denotes the $n \times n$ mass matrix, $\boldsymbol{g} = \left[g_j(\boldsymbol{x}, \dot{\boldsymbol{x}})\right]_{n\times1}$ is an arbitrary n-dimensional nonlinear vector-valued function, and $\boldsymbol{w}$ represents a Gaussian white noise stochastic excitation vector process with $\mathbb{E}[\boldsymbol{w}(t)] = \mathbf{0}$ and $\mathbb{E}\left[\boldsymbol{w}(t)\boldsymbol{w}^{\mathrm{T}}(t-\tau)\right] = \mathbf{D}\delta(\tau)$, where $\mathbf{D} \in \mathbb{R}^{n\times n}$ is a deterministic coefficient matrix.

Next, according to the WPI technique (e.g., [12,20]), the joint response transition PDF $p\left(\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\right)$ corresponding to the system of Eq. (1) can be expressed as a functional integral over the space of all possible paths $\mathcal{C}\left\{\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\right\}$ that the response process can follow; that is,

$$p\left(\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\right) = \int_{\mathcal{C}\left\{\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\right\}} \exp\left(-\mathcal{S}[\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}]\right) \mathcal{D}[\boldsymbol{x}(t)], \tag{2}$$

where the stochastic action $\mathcal{S}[\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}]$ is expressed as

$$\mathcal{S}[\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}] = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) dt \tag{3}$$

and

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}) = \frac{1}{2}[\mathbf{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})]^{\mathrm{T}} \mathbf{D}^{-1} [\mathbf{M}\ddot{\boldsymbol{x}} + \boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})] \tag{4}$$

denotes the Lagrangian functional of the system. Further, $\mathcal{D}[\boldsymbol{x}(t)]$ in Eq. (2) represents a functional measure.

Nevertheless, calculating analytically the functional integral of Eq. (2) is, in general, an impossible task. In this regard, the most probable path approximation is routinely employed in the literature for evaluating Eq. (2). This is done by considering the contribution only of the path with the maximum probability of occurrence (e.g., [12,22]). Specifically, the largest contribution to the functional integral of Eq. (2) relates to the trajectory $\boldsymbol{x}_c(t)$ for which the stochastic action in Eq. (3) becomes as small as possible. This leads to the variational (functional minimization) problem

$$\underset{\mathcal{C}\left\{\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\right\}}{\text{minimize}} \ \mathcal{S}[\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}] \tag{5}$$

with the set of boundary conditions, for $j = 1, \dots, n$,

$$x_j\left(t_i\right) = x_{j,i}\ \dot{x}_j\left(t_i\right) = \dot{x}_{j,i}, \quad x_j\left(t_f\right) = x_{j,f}\ \dot{x}_j\left(t_f\right) = \dot{x}_{j,f}. \tag{6}$$

Further, solving Eq. (5) and obtaining $\boldsymbol{x}_c(t)$, the functional integral of Eq. (2) is evaluated approximately as

$$p\left(\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\right) \approx C \exp\left(-\mathcal{S}\left[\boldsymbol{x}_c, \dot{\boldsymbol{x}}_c, \ddot{\boldsymbol{x}}_c\right]\right), \tag{7}$$

where C is a constant to be determined by the normalization condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p\left(\boldsymbol{x}_f, \dot{\boldsymbol{x}}_f, t_f | \boldsymbol{x}_i, \dot{\boldsymbol{x}}_i, t_i\right) d\boldsymbol{x}_f d\dot{\boldsymbol{x}}_f = 1. \tag{8}$$

2.2. Rayleigh–Ritz solution scheme for the most probale path

In general, various methodologies can be employed for treating the optimization problem of Eq. (5) and for determining $\boldsymbol{x}_c(t)$. These range from standard Rayleigh–Ritz type numerical solution schemes (e.g, [10]) to more recently developed techniques relying on computational algebraic geometry concepts and tools [22]. Alternatively, considering Eq. (5) and resorting to calculus of variations (e.g., [25]) yields the corresponding Euler–Lagrange equations, which take the form of a BVP to be solved for obtaining the most probable path $\boldsymbol{x}_c(t)$ (e.g., [26]).

In this section, the basic elements of a Rayleigh–Ritz solution scheme for determining the most probable path $\boldsymbol{x}_c(t)$ are concisely reviewed for completeness; see also [10,19,22] for more details. Specifically, $\boldsymbol{x}(t)$ is approximated by

$$\boldsymbol{x}(t) \approx \hat{\boldsymbol{x}}(t) = \boldsymbol{\psi}(t) + \boldsymbol{Z}\boldsymbol{h}(t), \tag{9}$$

where $\boldsymbol{\psi}(t)$ is selected to satisfy the boundary conditions of Eq. (6), and the trial functions $\boldsymbol{h}(t) = \left[h_l(t)\right]_{L\times 1}$ vanish at the boundaries, i.e., $\boldsymbol{h}\left(t_i\right) = \boldsymbol{h}\left(t_f\right) = \boldsymbol{0}$; $\boldsymbol{Z} \in \mathbb{R}^{n\times L}$ is a coefficient matrix and L is the number of trial functions. Further, utilizing a vectorized form of $\boldsymbol{Z}$, Eq. (9) becomes, equivalently,

$$\hat{\boldsymbol{x}}(t) = \boldsymbol{\psi}(t) + \mathbf{H}(t)\boldsymbol{z} \tag{10}$$

with

$$\boldsymbol{z} = \begin{bmatrix} Z_1^{\mathrm{T}} \\ Z_2^{\mathrm{T}} \\ \vdots \\ Z_n^{\mathrm{T}} \end{bmatrix} \text{ and } \mathbf{H}(t) = \begin{bmatrix} \boldsymbol{h}^{\mathrm{T}}(t) & 0 & \cdots & 0 \\ 0 & \boldsymbol{h}^{\mathrm{T}}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{h}^{\mathrm{T}}(t) \end{bmatrix}, \tag{11}$$

where Z_l denotes the l^{th} row of matrix $\boldsymbol{Z}$ and $\mathbf{H}(t)$ represents an $n\times nL$ matrix. In the ensuing analysis, and without loss of generality, the Hermite interpolating polynomials

$$\psi_j(t) = \sum_{k=0}^{3} a_{j,k} t^k, \tag{12}$$

are used, i.e., $\boldsymbol{\psi}(t) = \left[\psi_j(t)\right]_{n\times 1}$, where the $n \times 4$ coefficients $a_{j,k}$ are determined based on the boundary conditions of Eq. (6). For the trial functions, the shifted Legendre polynomials given by the recursive formula

$$\ell_{q+1}(t) = \frac{2q+1}{q+1}\left(\frac{2t - t_i - t_f}{t_f - t_i}\right)\ell_q(t) - \frac{q}{q+1}\ell_{q-1}(t), \ q = 1, \dots, L-1, \tag{13}$$

are employed, which are orthogonal in the interval $\left[t_i, t_f\right]$ with $\ell_0(t) = 1$ and $\ell_1(t) = \left(2t - t_i - t_f\right)/\left(t_f - t_i\right)$. Ultimately, the trial functions take the form

$$h_l(t) = \left(t - t_i\right)^2\left(t - t_f\right)^2\ell_l(t). \tag{14}$$

Note that $h_l(t)$ is a polynomial of order $l+4$ that vanishes at the boundaries. Further, each component $\hat{x}_j(t)$ of $\hat{\boldsymbol{x}}(t)$ in Eq. (9) is a polynomial of order up to $L+4$ in t.

Overall, the variational problem of Eq. (5) degenerates to an ordinary minimization problem of a function that depends on a finite number of variables. Specifically, the functional $\mathcal{S}$, dependent on the n functions $\boldsymbol{x}(t)$ (and their time derivatives), is cast in the form

$$S(\boldsymbol{z}) := \mathcal{S}(\hat{\boldsymbol{x}}, \dot{\hat{\boldsymbol{x}}}, \ddot{\hat{\boldsymbol{x}}}), \tag{15}$$

that depends on a finite number of nL coefficients $\boldsymbol{z}$. Thus, the optimization problem corresponding to Eq. (5) becomes

$$\min_{\boldsymbol{z}} S(\boldsymbol{z}). \tag{16}$$

Further, the solution $\boldsymbol{z}^*$ of Eq. (16) satisfies the first-order optimality condition

$$\nabla S(\boldsymbol{z}) = \boldsymbol{0}, \tag{17}$$

which represents a system of nL nonlinear algebraic equations to be solved numerically. Once the solution $\boldsymbol{z}^*$ of the optimization problem in Eq. (16) is obtained, the most probable path $\boldsymbol{x}_c$ is determined by Eq. (9).

3. Optimization problem for the Wiener path integral most probable path: Existence, uniqueness and calculation of solutions

3.1. Existence of globally optimal solutions for the Wiener path integral most probable path

In this section, it is shown that for the commonly considered case of the nonlinear function $\mathbf{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})$ being of polynomial form, there exist globally optimal solutions corresponding to the optimization problem of Eq. (16).

Specifically, for a polynomial nonlinear function $\mathbf{g}(\boldsymbol{x}, \dot{\boldsymbol{x}})$ of degree d, the objective function $S(\boldsymbol{z})$ in Eq. (16) becomes a multivariate polynomial of degree $2d$ in $p := nL$ variables; see also [22] for more details. Further, the first-order optimality condition of Eq. (17) yields a polynomial system of p equations of the form

$$\begin{aligned} f_1\left(z_1, z_2, \dots, z_p\right) &= 0, \\ &\vdots \\ f_p\left(z_1, z_2, \dots, z_p\right) &= 0, \end{aligned} \tag{18}$$

where each f_i is a polynomial of degree at most $2d-1$ with real coefficients.

Next, consider p-tuples of nonnegative integers $\alpha = \left(\alpha_1, \alpha_2, \dots, \alpha_p\right)$. The monomial in variables $z_1, z_2, \dots, z_p$, i.e., ${z_1}^{\alpha_1} {z_2}^{\alpha_2} \cdots {z_p}^{\alpha_p}$, is represented as z^α and $|\alpha|$ denotes the sum of powers $\sum_{i=1}^{p} \alpha_i$. The degree of the polynomial $S(\boldsymbol{z})$ is defined as the maximum among the sums of powers of all monomials in $S(\boldsymbol{z})$. Thus, a general polynomial of degree d in the variables $z_1, z_2, \dots, z_p$ can be written as [27],

$$S(\boldsymbol{z}) = \sum_{|\alpha| \le d} c_\alpha z^\alpha, \tag{19}$$

where c_α are real coefficients. The following theorem provides a sufficient condition for the existence of global minimizers of Eq. (19).

Theorem 1. *Let $S(\boldsymbol{z})$ be a multivariate polynomial of even degree $2d$, expressed as*

$$S(\boldsymbol{z}) = \sum_{|\alpha| = 2d} c_\alpha z^\alpha + \sum_{|\alpha| < 2d} c_\alpha z^\alpha. \tag{20}$$

If

$$\lim_{\|\boldsymbol{z}\| \to +\infty} \sum_{|\alpha| = 2d} c_\alpha z^\alpha = +\infty, \tag{21}$$

then $S(\boldsymbol{z})$ has at least one global minimizer.

Proof. Assuming that Eq. (21) holds true, it follows from Eq. (20) that

$$\lim_{\|\boldsymbol{z}\| \to +\infty} S(\boldsymbol{z}) = +\infty. \tag{22}$$

Thus, there exists $r > 0$ such that for all $\|\boldsymbol{z}\| > r$,

$$S(\boldsymbol{z}) > S(\mathbf{0}). \tag{23}$$

Further, let $\bar{B}(\mathbf{0}, r)$ denote the set $\{\boldsymbol{z} : \|\boldsymbol{z}\| \le r\}$ that is closed and bounded. Since the function $S(\boldsymbol{z})$ is continuous on $\bar{B}(\mathbf{0}, r)$, $S(\boldsymbol{z})$ has a global minimizer $\boldsymbol{z}^*$ on $\bar{B}(\mathbf{0}, r)$, i.e.,

$$S(\boldsymbol{z}) \ge S\left(\boldsymbol{z}^*\right), \ \boldsymbol{z} \in \bar{B}(\mathbf{0}, r). \tag{24}$$

In particular $S(\mathbf{0}) \ge S\left(\boldsymbol{z}^*\right)$. For $\boldsymbol{z} \notin \bar{B}(\mathbf{0}, r)$,

$$S(\boldsymbol{z}) > S(\mathbf{0}) \ge S\left(\boldsymbol{z}^*\right). \tag{25}$$

Hence, $\boldsymbol{z}^*$ is a global minimizer of $S(\boldsymbol{z})$ on $\mathbb{R}^p$. □

The following Lemma relates to a special form of the polynomial of Eq. (20) that satisfies the condition of Eq. (21), and thus, the corresponding $S(\boldsymbol{z})$ has at least one global minimizer.

Lemma 1. *In Eq. (20), for the monomials corresponding to $|\alpha| = 2d$, if the associated coefficients c_α are positive and the respective powers are even numbers, i.e., $\alpha_i = 2k$ for $i = 1, 2, \dots, p$ and $k \in \mathbb{N}$ and c_α are zero otherwise, then $S(\boldsymbol{z})$ has at least one global minimizer.*

Proof. It is readily seen that if c_α are positive and $\alpha_i = 2k$ for $i = 1, 2, \dots, p$, $k \in \mathbb{N}$, and c_α are zero otherwise, then the first term on the right hand side of Eq. (20) becomes a summation of monomials of even powers with positive coefficients, and thus Eq. (21) holds true. Hence, Theorem 1 is satisfied and $S(\boldsymbol{z})$ has at least one global minimizer. □

3.2. Uniqueness of solution for the Wiener path integral most probable path

In this section, based on algebraic geometry concepts and tools, a condition is derived for determining if Eq. (18) exhibits a unique solution over a region in the domain of $\boldsymbol{z}$. The interested reader is also directed to Refs. [27–32] for a comprehensive exposition to the topic of (computational) algebraic geometry.

In this regard, consider next the polynomial system of Eq. (18) whose entire set of solutions is referred to as the affine variety $V(I)$, where I is the ideal generated by the polynomials $f_1, f_2, \ldots, f_p$. Let $V_{\mathbb{R}}(I)$ denote the real points of the variety $V(I)$. It is assumed that $V(I)$ is a finite set, or equivalently, that the quotient ring $A = \mathbb{R}\left[z_1, z_2, \ldots, z_p\right]/I$ is a finite dimensional $\mathbb{R}$-vector space.

Further, the vector space $A = \mathbb{R}\left[z_1, z_2, \ldots, z_p\right]/I$ is also an algebra, thus for any $f \in A$, multiplication by f induces a vector space endomorphism, denoted by $L_f \in End_{\mathbb{R}}(A)$. This defines a homomorphism $L : A \rightarrow End_{\mathbb{R}}(A)$, such that $L_f L_g = L_{fg}$. Since A is a finite-dimensional algebra, multiplication by f on A can be represented by a matrix m_f. In this regard, the symmetric bilinear form B is defined as

$$B(f,g) = Tr\left(m_f \cdot m_g\right) = Tr\left(m_{fg}\right), \tag{26}$$

where Tr denotes the trace, i.e., the sum of diagonal entries of a square matrix. Furthermore, the matrix of B on the vector space A with basis $\left\{v_1, v_2, \ldots, v_d\right\}$ is given by

$$M_{i,j} = \left(Tr\left(m_{v_i v_j}\right)\right). \tag{27}$$

Note that for a given symmetric bilinear form B with the matrix M, the signature $\sigma(B)$ is equal to the difference between the number of positive eigenvalues and the number of negative eigenvalues of M, and the rank $\rho(B)$ is equal to the rank of matrix M. Also, for a given polynomial $h \in \mathbb{R}\left[z_1, z_2, \ldots, z_p\right]$, the associated bilinear form is defined as

$$B_h(f,g) = B(hf,g) = Tr\left(m_{hfg}\right), \tag{28}$$

and the associated quadratic form Q_h as

$$Q_h(f) = B(hf,f) = Tr\left(m_{hf^2}\right). \tag{29}$$

Lemma 2 (*32*). *Let $V(I)$ be a finite affine variety defined by the ideal I generated by $\left(f_1, f_2, \ldots, f_s\right)$, where $f_i \in \mathbb{R}\left[z_1, z_2, \ldots, z_p\right]$, and $h \in \mathbb{R}\left[z_1, z_2, \ldots, z_p\right]$ be a given polynomial. Then,*

$$\sigma\left(Q_h\right) = \#\left\{\bar{z} \in V_{\mathbb{R}}(I) : \; h(\bar{z}) > 0\right\} - \#\left\{\bar{z} \in V_{\mathbb{R}}(I) : \; h(\bar{z}) < 0\right\}, \tag{30}$$

and

$$\rho\left(Q_h\right) = \#\left\{\bar{z} \in V_{\mathbb{C}}(I) : \; h(\bar{z}) \neq 0\right\}, \tag{31}$$

where σ denotes the signature and ρ denotes the rank of the quadratic form Q_h and $\#A$ denotes the number of elements of the set A.

The uniqueness of solution of Eq. (18) in a region R can be demonstrated by utilizing the following theorem.

Theorem 2. *Given the system in Eq. (18), and a region*

$$R = \left\{\left(z_1, z_2, \ldots, z_p\right) \; : z_i \in \left(a_i, b_i\right) \; , \; i = 1, 2, \ldots, p\right\}, \tag{32}$$

the uniqueness of solution in R is implied if

$$\sigma\left(Q_{h_i}\right) = -1 \text{ and } \sigma\left(Q_{h_i^2}\right) = 1, \tag{33}$$

where $h_i\left(z_1, z_2, \ldots, z_p\right) = \left(z_i - a_i\right)\left(z_i - b_i\right)$, for $i = 1, 2, \ldots, p$.

Proof. Let

$$H_i = \left\{\left(z_1, z_2, \ldots, z_p\right) \; : h_i\left(z_1, z_2, \ldots, z_p\right) < 0\right\}, \tag{34}$$

then $H_i = \left\{\left(z_1, z_2, \ldots, z_p\right) \; : z_i \in \left(a_i, b_i\right)\right\}$ and

$$R = H_1 \cap H_2 \cap \ldots \cap H_p = \bigcap_{i=1}^{p} H_i. \tag{35}$$

Next, employing Lemma 2, Eq. (30) yields

$$\sigma\left(Q_{h_i}\right) = \#\left\{\bar{z} \in V_{\mathbb{R}}(I) \; : h_i(\bar{z}) > 0\right\} - \#\left\{\bar{z} \in V_{\mathbb{R}}(I) \; : h_i(\bar{z}) < 0\right\} = -1. \tag{36}$$

Further, since $h_i^2(\bar{z}) > 0$ at every point $\bar{z}$ such that $h_i(\bar{z}) > 0$ and $h_i(\bar{z}) < 0$, Eq. (30) leads to

$$\sigma\left(Q_{h_i^2}\right) = \#\left\{\bar{z} \in V_{\mathbb{R}}(I) \; : h_i(\bar{z}) > 0\right\} + \#\left\{\bar{z} \in V_{\mathbb{R}}(I) \; : h_i(\bar{z}) < 0\right\} = 1, \tag{37}$$

for $i = 1, 2, \dots, p$. Taking into account Eqs. (36) and (37) yields

$$\#\left\{\bar{z} \in V_{\mathbb{R}}(I) \ : h_i\left(\bar{z}\right) > 0\right\} = 0, \tag{38}$$

and

$$\#\left\{\bar{z} \in V_{\mathbb{R}}(I) \ : h_i\left(\bar{z}\right) < 0\right\} = 1. \tag{39}$$

Thus, $\#V_{\mathbb{R}}(I) \cap H_i = 1$ and $\#V_{\mathbb{R}}(I) \cap H_i^c = 0$ for $i = 1, 2, \dots, p$. Therefore,

$$\#V_{\mathbb{R}}(I) \cap R = \#V_{\mathbb{R}}(I) \cap \left(\bigcap_{i=1}^{p} H_i\right) = 1, \tag{40}$$

and

$$\#V_{\mathbb{R}}(I) \cap R^C = \#V_{\mathbb{R}}(I) \cap \left(\bigcap_{i=1}^{p} H_i\right)^C = 0, \tag{41}$$

where R^C denotes the counterpart of R. In other words, there exists a unique solution of Eq. (18) in R. □

Note that for the special case $h = 1$, the signature of Q_1 is equal to the number of elements in $V_{\mathbb{R}}(I)$. Therefore, solution uniqueness for Eq. (18) is implied if

$$\sigma\left(Q_1\right) = 1. \tag{42}$$

Clearly, the application of Theorem 2 entails the evaluation of the signature of the quadratic form Q_h. To this aim, the following proposition can be employed.

Proposition 1 (*32*)**.** *Let M_h be the matrix of Q_h and*

$$p_h(\lambda) = \det\left(M_h - \lambda I\right) \tag{43}$$

be the characteristic polynomial of M_h. Then, the number of positive eigenvalues of M_h is equal to the number of sign changes in the sequence of coefficients of M_h.

Further, the steps for determining the matrix M_h of quadratic form Q_h of an arbitrary $h \in \mathbb{R}\left[z_1, z_2, \dots, z_p\right]$ corresponding to the system of polynomials in Eq. (18) are presented in Algorithm 1, which is based on the following three main subroutines that can be found in most computer algebra systems; see also [22,33] for more details and some recent applications of the algorithm in engineering dynamics.

- `Groebner` $\left(f_1, f_2, \dots, f_p\right)$: This subroutine computes a Gröbner basis G for the ideal generated by $f_1, f_2, \dots, f_p$ (see for instance `Basis` (.) command in Maple).
- `StandardBasis` (G): This subroutine computes a monomial basis B corresponding to the Gröbner basis G (see for instance `NormalSet` (.) command in Maple).
- `MulMatrix` (f, B, G): This subroutine computes the multiplication matrix for a polynomial f based on the Gröbner basis G and basis B (see for instance `MultiplicationMatrix` (.) command in Maple).

Algorithm 1: Computation of matrix M_h of the quadratic form Q_h

Input: $h, f_1, f_2, \dots, f_p \in \mathbb{R}\left[z_1, z_2, \dots, z_p\right]$
Output: M_h

```
G = Groebner(f_1, f_2, ..., f_p)
B = StandardBasis(G)
n = length(B)
Initialize M_h as an empty n × n matrix
for i = 1 to n do
    for j = 1 to n do
        M_h(i, j) = Tr(MulMatrix(h · B(i) · B(j), B, G))
    end for
end for
return M_h
```

3.3. Calculation of solutions for the Wiener path integral most probable path

In general, a wide range of numerical solution methodologies can be employed for treating the optimization problem described by Eq. (16); see, for instance, Ref. [21] for a broad perspective on various numerical optimization algorithms. In fact, a Newton's iterative optimization scheme was developed in [22] for solving Eq. (16). Nevertheless, although the Newton's scheme exhibits some desirable properties such as a quadratic convergence rate under certain conditions, the associated computational cost becomes non-trivial for an increasing number of $p = nL$ variables in Eq. (16).

Further, a conceptually different solution approach was also pursued in [22] that relies on computational algebraic geometry concepts and employs Gröbner bases. Remarkably, the approach is capable of determining the entire set of solutions corresponding to the first-order optimality conditions of Eq. (17). Note that the convexity of $S(\mathbf{z})$ is implied if the approach yields only one solution. Specifically, following computation of the multiplication matrix M_h via Algorithm 1, the entire set of solutions of Eq. (18) can be determined by calculating the eigenvalues of M_h. In fact, the globally minimum value $S^* = \min S(\mathbf{z})$ is equal to the smallest real eigenvalue of M_S. Nevertheless, the associated computational cost is significant, particularly for higher-dimensional systems.

In this section, an alternative solution approach is developed by relying on Sylvester's dialytic method of elimination [23,24]. The rationale of the method relates to reducing the complexity of the system of coupled multivariate polynomial equations by eliminating one or more variables. In fact, the method yields a univariate polynomial equation to be solved for the suppressed variable. The interested reader is also directed to the review paper [34] for some indicative applications of the approach in robot dynamics.

Specifically, the method aims at recasting Eq. (18) in an appropriate form so that the following renowned linear algebra theorem can be applied.

Theorem 3 (*35*). *The necessary and sufficient condition that p linear equations in p unknowns shall have a solution, other than the trivial one in which each unknown is zero, is that the determinant of the coefficients be zero.*

In the ensuing analysis, without loss of generality and for tutorial effectiveness, the polynomial system in Eq. (18) with $p = 2$ is considered; that is,

$$f_1(z_1, z_2) = \sum_{i+j\le 2d-1} c_{i,j} z_1^i z_2^j = 0, \tag{44}$$

$$f_2(z_1, z_2) = \sum_{i+j\le 2d-1} e_{i,j} z_1^i z_2^j = 0, \tag{45}$$

where $c_{i,j}$ and $e_{i,j}$ denote coefficients. In passing, note that the method can be applied for an arbitrary number of equations p (e.g., [34,36]). Next, the variable z_1 is suppressed and Eqs. (44)–(45) are written, equivalently, as

$$c_{0,2d-1} {z_2}^{2d-1} + \left(c_{1,2d-2} z_1 + c_{0,2d-2}\right) {z_2}^{2d-2} + \cdots + \left(c_{2d-1,0} {z_1}^{2d-1} + \cdots + c_{0,0}\right) = 0, \tag{46}$$

$$e_{0,2d-1} {z_2}^{2d-1} + \left(e_{1,2d-2} z_1 + e_{0,2d-2}\right) {z_2}^{2d-2} + \cdots + \left(e_{2d-1,0} {z_1}^{2d-1} + \cdots + e_{0,0}\right) = 0. \tag{47}$$

Further, ignoring the terms with zero coefficients, Eqs. (46)–(47) are written concisely in the form

$$\tilde{f}_1(z_2) = a_l {z_2}^l + a_{l-1} {z_2}^{l-1} + \cdots + a_0 = 0, \tag{48}$$

$$\tilde{f}_2(z_2) = b_m {z_2}^m + b_{m-1} {z_2}^{m-1} + \cdots + b_0 = 0, \tag{49}$$

where l and m denote the indices of the leading terms corresponding to non-zero coefficients in Eqs. (46) and (47), respectively, and $a_l \neq 0, b_m \neq 0$ are non-zero coefficients dependent on $c_{i,j}$ and $e_{i,j}$ as in Eqs. (46)–(47). Furthermore, a system of $l + m$ homogeneous equations in the variables $z_2^{l+m-1}, z_2^{l+m-2}, \dots, z_2, 1$ can be obtained by multiplying Eqs. (48) and (49) by $z_2^{m-1}, z_2^{m-2}, \dots, z_2$ and $z_2^{l-1}, z_2^{l-2}, \dots, z_2$, respectively, to yield $l + m - 2$ additional equations. This yields

$$\begin{aligned}
a_l {z_2}^{l+m-1} + a_{l-1} {z_2}^{l+m-2} + \cdots + a_0 {z_2}^{m-1} &= 0,\\
a_l {z_2}^{l+m-2} + a_{l-1} {z_2}^{l+m-3} + \cdots + a_0 {z_2}^{m-2} &= 0,\\
&\vdots\\
a_l {z_2}^{l} + a_{l-1} {z_2}^{l-1} + \cdots + a_0 &= 0,\\
b_m {z_2}^{l+m-1} + b_{m-1} {z_2}^{l+m-2} + \cdots + b_0 {z_2}^{l-1} &= 0,\\
b_m {z_2}^{l+m-2} + b_{m-1} {z_2}^{l+m-3} + \cdots + b_0 {z_2}^{l-2} &= 0,\\
&\vdots\\
b_m {z_2}^{m} + b_{m-1} {z_2}^{m-1} + \cdots + b_0 &= 0.
\end{aligned} \tag{50}$$

Next, Theorem 3 is applied to Eq. (50), dictating that the determinant of the coefficients is equal to zero, i.e.,

$$\begin{vmatrix}
a_l & a_{l-1} & \dots & \dots & a_0 & 0 & \dots & \dots & \dots & 0\\
0 & a_l & a_{l-1} & \dots & \dots & a_0 & 0 & \dots & \dots & 0\\
 & & & & \vdots & \vdots & & & & \\
0 & \dots & \dots & \dots & 0 & a_l & a_{l-1} & \dots & \dots & a_0\\
b_m & b_{m-1} & \dots & \dots & \dots & b_0 & 0 & \dots & \dots & 0\\
0 & b_m & b_{m-1} & \dots & \dots & \dots & b_0 & 0 & \dots & 0\\
 & & & & \vdots & \vdots & & & & \\
0 & \dots & \dots & 0 & b_m & b_{m-1} & \dots & \dots & \dots & b_0
\end{vmatrix} = 0. \tag{51}$$

Note that the coefficient matrix of the system of equations in Eq. (50) is known as the Sylvester matrix of $\tilde{f}_1$ and $\tilde{f}_2$. The resultant of $\tilde{f}_1$ and $\tilde{f}_2$, denoted $\mathrm{Res}(\tilde{f}_1, \tilde{f}_2)$, is defined as the determinant of the Sylvester matrix.

It is readily seen that the resultant in Eq. (51) yields a polynomial equation in the suppressed variable z_1 of degree q less than or equal to $(2d-1)^2$; that is,

$$r_q {z_1}^q + r_{q-1} {z_1}^{q-1} + \cdots + r_1 + r_0 = 0, \tag{52}$$

where, clearly, the coefficients $r_i,\ i = 0, \dots, q$, depend on the coefficients $c_{i,j}$ and $e_{i,j}$ of Eqs. (44)–(45).

Remarkably, the complexity of the original system of multivariate polynomial equations has been reduced. In fact, Eqs. (44)–(45) have been recast into Eq. (52) that constitutes a univariate polynomial equation for z_1. This can be readily solved by resorting to a plethora of standard numerical optimization schemes (e.g., [21]). Note that the remaining variable z_2 can be determined in a similar manner by considering z_2 as suppressed and repeating the above steps. Lastly, the solutions satisfying the first-order optimality conditions of Eqs. (44)–(45), i.e., $\nabla S(z) = \mathbf{0}$, are substituted into the objective function $S(z)$ to determine which solution yields the global minimum.

Further, it is shown next that both the Gröbner basis approach employed in [22] and the herein proposed dialytic method yield the same set of solutions for Eqs. (44)–(45). In this regard, the following lemma is utilized in Theorem 4 for showing the equivalence between the two approaches, and, in particular, that the characteristic polynomial of the multiplication matrix M_{z_1} calculated via Algorithm 1 is equal to the resultant of Eq. (51).

Lemma 3 ([27]). *Let $\mathbf{V}(f_1, f_2)$ denote the entire set of solutions of Eqs. (44)–(45). Next, assume that the system*

$$\begin{aligned} \overline{F}_1(z_1, z_2) &= \sum_{i+j=2d-1} c_{i,j} z_1^i z_2^j = 0, \\ \overline{F}_2(z_1, z_2) &= \sum_{i+j=2d-1} d_{i,j} z_1^i z_2^j = 0, \end{aligned} \tag{53}$$

has no nontrivial solutions. Then,

$$\mathrm{Res}\left(\tilde{f}_1, \tilde{f}_2\right) = k \prod_{p \in \mathbf{V}(f_1, f_2)} \left(z_1 - p_1\right)^{m(p)}, \tag{54}$$

for some nonzero scalar $k \in \mathbb{R}$ where $\tilde{f}_1, \tilde{f}_2$ are given by Eqs. (48) and (49), respectively, and $m(p)$ denotes the multiplicity of $p = (p_1, p_2) \in \mathbf{V}(f_1, f_2)$.

Theorem 4. *Let p_{z_1} be the characteristic polynomial of the multiplication matrix M_{z_1} representing the linear map $m_{z_1} : A \to A$ given by multiplication by z_1 on the quotient ring $A = \mathbb{R}[z_1, z_2]/\langle f_1, f_2\rangle$. Then,*

$$\mathrm{Res}\left(\tilde{f}_1, \tilde{f}_2\right) = k p_{z_1} \tag{55}$$

for some nonzero scalar $k \in \mathbb{R}$.

Proof. Let $f \in \mathbb{R}[z_1, z_2]$. The characteristic polynomial p_f of M_f is given by [27]

$$\det\left(\lambda I - M_f\right) = \prod_{p \in \mathbf{V}(f_1, f_2)} (\lambda - f(p))^{m(p)}, \tag{56}$$

where M_f is the multiplication matrix corresponding to the linear map $m_f : A \to A$ given by multiplication by f and $m(p)$ denotes the multiplicity of $p = (p_1, p_2) \in \mathbf{V}(f_1, f_2)$. Next, setting $f(z_1, z_2) = z_1$ leads to

$$\det\left(\lambda I - M_{z_1}\right) = \prod_{p \in \mathbf{V}(f_1, f_2)} \left(\lambda - p_1\right)^{m(p)}. \tag{57}$$

Substituting $\lambda = z_1$ yields

$$p_{z_1} = \prod_{p \in \mathbf{V}(f_1, f_2)} \left(z_1 - p_1\right)^{m(p)}. \tag{58}$$

Then, from Lemma 3,

$$\mathrm{Res}\left(\tilde{f}_1, \tilde{f}_2\right) = k p_{z_1}, \tag{59}$$

where k is a nonzero constant. Therefore, $\mathrm{Res}\left(\tilde{f}_1, \tilde{f}_2\right)$ and p_{z_1} yield the same solutions for z_1. □

3.4. Mechanization of the techniques

Succinctly stated, the developed techniques pertaining to the existence, uniqueness and numerical calculation of WPI most probable paths comprise the following steps:

(a) For a given nonlinear system under consideration governed by Eq. (1), cast the corresponding stochastic action $S(\boldsymbol{z})$ in the form of Eq. (20). Next, apply Theorem 1 to prove the existence of a globally optimal solution.
(b) Use Algorithm 1 to calculate the characteristic polynomial of Eq. (43). Apply Theorem 2 in conjunction with Proposition 1 to prove the solution uniqueness.
(c) Cast the polynomial equations of Eq. (18) into the form of Eqs. (48)–(49). Suppress a specific variable and apply Eq. (51) to yield a polynomial equation that depends on this variable only. Solve Eq. (52) via an appropriate numerical optimization algorithm, e.g., Newton's scheme [21]. Repeat the process for the rest of the variables. Note that the method can be applied, at least in principle, for an arbitrary number of equations $p = nL$ (e.g., [34]).

4. Numerical examples

Various oscillators exhibiting diverse nonlinear behaviors are considered in this section for demonstrating the capabilities of the developed techniques. In fact, to perform direct comparisons between the herein proposed solution approach based on Sylvester's dialytic method and an alternative Gröebner basis approach employed in [22], the same numerical examples used in [22] are considered next.

4.1. Linear oscillator

For the special case of a linear system, i.e., $\boldsymbol{g}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \mathbf{C}\dot{\boldsymbol{x}} + \mathbf{K}\boldsymbol{x}$ in Eq. (1) where $\mathbf{C}$ and $\mathbf{K}$ denote the damping and stiffness matrices, respectively, it has been shown in [26] that the Euler–Lagrange equations corresponding to the minimization problem of Eq. (5) can be solved analytically for the most probable path. Remarkably, substituting the most probable path into Eq. (7) yields the exact system response joint transition PDF that takes a Gaussian form. In other words, the response PDF obtained by the most probable path approach is exact and approximation-free for the case of linear systems.

In the following, a single-DOF linear oscillator is considered whose equation of motion is given by

$$m\ddot{x} + c\dot{x} + kx = w(t), \tag{60}$$

where the parameter values $m = 5,\ c = 0.2,\ k = 1$, and $E(w(t)w(t+\tau)) = 2\pi S_0 \delta(\tau)$ with $S_0 = 0.5$ are used. Next, a normalized version of Eq. (60) is considered, where $\zeta_0 = c/2m\omega_0$ is the damping ratio and $\omega_0 = \sqrt{k/m}$ is the natural frequency of the system. Note that, using $L = 2$ in Eq. (10), it was shown in [22] that the objective function $S(\boldsymbol{z})$ of Eq. (15) is convex. Thus, the Newton's scheme proposed in [22] converges to the exact optimal solution $\boldsymbol{z}^* = (0.0173, 0.0001)$ in a single iteration starting from the arbitrarily chosen point $(50, 50)$. Further, employing the Gröebner basis approach in [22] yielded a single solution corresponding to the objective function value $S(\boldsymbol{z}^*) = 4.4204$, which coincided practically with the estimate obtained by the Newton's scheme.

Next, to apply the techniques developed herein, considering $L = 2$ in Eq. (10) and arbitrary initial and final time instants (t_i and t_f), the objective function $S(\boldsymbol{z})$ cast in the form of Eq. (20) becomes

$$S(\boldsymbol{z}) = c_{2,0}{z_1}^2 + c_{0,2}{z_2}^2 + \sum_{|\alpha|<2} c_\alpha \boldsymbol{z}^\alpha, \tag{61}$$

where

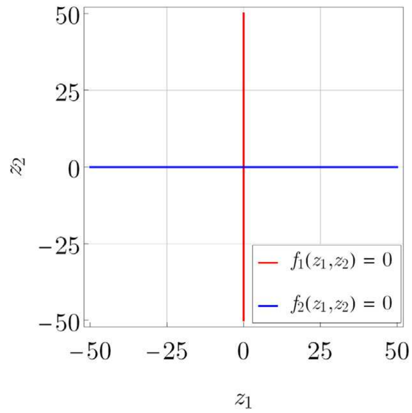
$$c_{2,0} = \frac{1}{4\pi S_0}\frac{(t_f - t_i)^5}{630}\left[\left(\omega_0^2(t_f - t_i)^2 + 24\zeta_0^2 - 12\right)^2 + 576\zeta_0^2\left(1 - \zeta_0^2\right) + 360\right], \tag{62}$$

$$c_{0,2} = \frac{1}{4\pi S_0}\frac{(t_f - t_i)^5}{6930}\left[\left(\omega_0^2(t_f - t_i)^2 + 88\zeta_0^2 - 44\right)^2 + 7744\zeta_0^2\left(1 - \zeta_0^2\right) + 2024\right]. \tag{63}$$

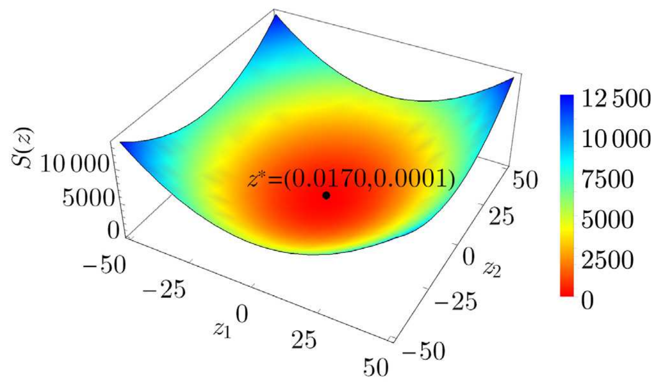
To elaborate further, the stochastic action of Eq. (61) corresponding to the linear oscillator of Eq. (60) with $d = 1$ is a two-variable polynomial of degree 2 (i.e., $2d = 2$). Note that the first two terms in Eq. (61) refer to the monomials whose sums of powers are equal to $2d = 2$. Moreover, it is readily seen that since $\zeta_0^2\left(1 - \zeta_0^2\right) > 0$ for $0 < \zeta_0 < 1$, the coefficients $c_{2,0}$ and $c_{0,2}$ in Eqs. (62) and (63), respectively, are positive for arbitrary values of $\omega_0 > 0$ and $0 < \zeta_0 < 1$. Therefore, according to Lemma 1, there exists a global minimizer for $S(\boldsymbol{z})$.

Further, regarding the uniqueness of the solution, using the values $m = 5,\ c = 0.2$ and $k = 1$, and the boundary conditions $\left(x\left(t_i = 0\right), \dot{x}\left(t_i = 0\right), x\left(t_f = 1\right), \dot{x}\left(t_f = 1\right)\right) = (0, 0, -0.5, -1.0)$, the first-order optimality condition of Eq. (17) yields the polynomial system of Eq. (18) that becomes

$$\begin{aligned} f_1\left(z_1, z_2\right) &= \frac{1}{4\pi S_0}\left(39.6z_1 - 0.674\right) &= 0, \\ f_2\left(z_1, z_2\right) &= \frac{1}{4\pi S_0}\left(28.4z_2 - 2.38\times 10^{-3}\right) &= 0. \end{aligned} \tag{64}$$



(a) First-order optimality condition equations



(b) Objective function

Fig. 1. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a linear oscillator under white noise $\left(x\left(t_f=1\right)=-0.5, \dot{x}\left(t_f=1\right)=-1.0\right)$.

Next, the characteristic polynomial of M_1 (for $h=1$) of the quadratic form Q_1 corresponding to Eq. (64) is calculated via Algorithm 1 yielding

$$p_h(\lambda) = -\lambda + 1. \tag{65}$$

Eq. (65) exhibits a single sign change in the sequence of the polynomial coefficients. According to Proposition 1, M_1 has one positive eigenvalue and, since the polynomial is of degree one, Q_1 has signature $\sigma\left(Q_1\right) = 1 - 0 = 1$. Therefore, relying on Theorem 2, there exists a unique global minimizer for $S(\boldsymbol{z})$.

Furthermore, clearly, the system of Eq. (64) is already in the form of Eqs. (48)–(49), and Sylvester's dialytic method degenerates to analytically solving directly Eq. (64). This yields $\boldsymbol{z}^* = (0.0170, 0.0001)$ and the corresponding objective function value $S\left(\boldsymbol{z}^*\right)$ becomes $S\left(\boldsymbol{z}^*\right) = 4.4204$, which coincides with the estimate based on the Gröbner basis approach in [22], as expected by Theorem 4.

Lastly, to provide further insight, the objective function $S(\boldsymbol{z})$ of the most probable path optimization problem is shown in Fig. 1 by using $L = 2$ trial functions. The first-order optimality condition equations are included as well demonstrating the uniqueness of the solution.

4.2. Duffing nonlinear oscillator

Next, a single-DOF Duffing nonlinear oscillator is considered, whose governing equation is a scalar version of Eq. (1); that is,

$$m\ddot{x} + c\dot{x} + kx + \varepsilon g_{nl}(x, \dot{x}) = w(t), \tag{66}$$

where the parameter ε denotes the nonlinearity magnitude, and the nonlinear function $g_{nl}(x, \dot{x})$ is given by

$$g_{nl}(x, \dot{x}) = kx^3. \tag{67}$$

Considering two trial functions (i.e., $L = 2$) and arbitrary initial and final time instants (t_i and t_f) in Eq. (16), the objective function $S(\boldsymbol{z})$ is expressed in the form of Eq. (20) as

$$S(\boldsymbol{z}) = c_{6,0}{z_1}^6 + c_{4,2}{z_1}^4{z_2}^2 + c_{2,4}{z_1}^2{z_2}^4 + c_{0,6}{z_2}^6 + \sum_{|\alpha|<6} c_\alpha \boldsymbol{z}^\alpha, \tag{68}$$

where

$$c_{6,0} = \frac{1}{4\pi S_0}\frac{(t_f - t_i)^{25}\varepsilon^2 k^2}{67603900} \tag{69}$$

$$c_{4,2} = \frac{1}{4\pi S_0}\frac{(t_f - t_i)^{25}\varepsilon^2 k^2}{121687020} \tag{70}$$

$$c_{2,4} = \frac{1}{4\pi S_0}\frac{(t_f - t_i)^{25}\varepsilon^2 k^2}{1176307860} \tag{71}$$

$$c_{0,6} = \frac{1}{4\pi S_0} \frac{(t_f - t_i)^{25} \varepsilon^2 k^2}{109396630980} \tag{72}$$

According to Lemma 1, since the coefficients $c_{6,0}, c_{4,2}, c_{2,4}$ and $c_{0,6}$ in Eqs. (69)–(72) are positive for arbitrary values $\varepsilon > 0$ and $k \in \mathbb{R}$, there exists a global minimizer for $S(\boldsymbol{z})$.

Next, using the same parameter values for m, c, k and S_0 as in Section 4.1 in Eqs. (66) and (67), and considering the boundary conditions $\left(x\left(t_i = 0\right), \dot{x}\left(t_i = 0\right), x\left(t_f = 1\right), \dot{x}\left(t_f = 1\right)\right) = (0, 0, -0.5, -1.0)$, the characteristic polynomial of M_1 of the quadratic form Q_1 corresponding to Eq. (18) is calculated via Algorithm 1. This yields a 25-th degree polynomial that exhibits 13 sign changes in the sequence of coefficients. According to Proposition 1, M_1 has 13 positive eigenvalues and, since the polynomial is of degree 25, the signature of Q_1 is evaluated as $\sigma\left(Q_1\right) = 13 - 12 = 1$. Therefore, based on Theorem 2, there exists a unique global minimizer for $S(\boldsymbol{z})$.

Further, the Sylvester's dialytic method is employed for solving Eq. (18), which takes the form

$$\begin{aligned} f_1\left(z_1, z_2\right) = \frac{1}{4\pi S_0} [&c_{5,0} z_1^5 + c_{3,2} z_1^3 z_2^2 + c_{1,4} z_1 z_2^4 + c_{4,0} z_1^4 + c_{3,1} z_1^3 z_2 + c_{2,2} z_1^2 z_2^2 + c_{1,3} z_1 z_2^3 + c_{0,4} z_2^4 + c_{3,0} z_1^3 \\ &+ c_{2,1} z_1^2 z_2 + c_{1,2} z_1 z_2^2 + c_{0,3} z_2^3 + c_{2,0} z_1^2 + c_{1,1} z_1 z_2 + c_{0,2} z_2^2 + c_{1,0} z_1 + c_{0,1} z_2 + c_{0,0}], \\ f_2\left(z_1, z_2\right) = \frac{1}{4\pi S_0} [&e_{4,1} z_1^4 z_2 + e_{2,3} z_1^2 z_2^3 + e_{0,5} z_2^5 + e_{4,0} z_1^4 + e_{3,1} z_1^3 z_2 + e_{2,2} z_1^2 z_2^2 + e_{1,3} z_1 z_2^3 + e_{0,4} z_2^4 + e_{3,0} z_1^3 \\ &+ e_{2,1} z_1^2 z_2 + e_{1,2} z_1 z_2^2 + e_{0,3} z_2^3 + e_{2,0} z_1^2 + e_{1,1} z_1 z_2 + e_{0,2} z_2^2 + e_{1,0} z_1 + e_{0,1} z_2 + e_{0,0}], \end{aligned} \tag{73}$$

where

$$\begin{aligned} c_{5,0} &= 8.88 \times 10^{-8} \epsilon^2, & e_{4,1} &= 1.64 \times 10^{-8} \epsilon^2, \\ c_{3,2} &= 3.29 \times 10^{-8} \epsilon^2, & e_{2,3} &= 3.4 \times 10^{-9} \epsilon^2, \\ c_{1,4} &= 1.70 \times 10^{-9} \epsilon^2, & e_{0,5} &= 5.48 \times 10^{-11} \epsilon^2, \\ c_{4,0} &= -1.01 \times 10^{-6} \epsilon^2, & e_{4,0} &= -8.41 \times 10^{-8} \epsilon^2, \\ c_{3,1} &= -3.36 \times 10^{-7} \epsilon^2, & e_{3,1} &= -1.88 \times 10^{-7} \epsilon^2, \\ c_{2,2} &= -2.82 \times 10^{-7} \epsilon^2, & e_{2,2} &= -6.05 \times 10^{-8} \epsilon^2, \\ c_{1,3} &= -4.03 \times 10^{-8} \epsilon^2, & e_{1,3} &= -2.39 \times 10^{-8} \epsilon^2, \\ c_{0,4} &= -5.98 \times 10^{-9} \epsilon^2, & e_{0,4} &= -1.86 \times 10^{-9} \epsilon^2, \\ c_{3,0} &= 5.67 \times 10^{-6} \epsilon^2 \quad -2.63 \times 10^{-3} \epsilon, & e_{3,0} &= 1.03 \times 10^{-6} \epsilon^2, \\ c_{2,1} &= 3.09 \times 10^{-6} \epsilon^2, & e_{2,1} &= 1.28 \times 10^{-6} \epsilon^2 \quad -8.56 \times 10^{-4} \epsilon, \\ c_{1,2} &= 1.28 \times 10^{-6} \epsilon^2 \quad -8.56 \times 10^{-4} \epsilon, & e_{1,2} &= 4.71 \times 10^{-7} \epsilon^2, \\ c_{0,3} &= 1.57 \times 10^{-7} \epsilon^2, & e_{0,3} &= 7.62 \times 10^{-8} \epsilon^2 \quad -6.57 \times 10^{-5} \epsilon, \\ c_{2,0} &= -2.13 \times 10^{-5} \epsilon^2 \quad +1.07 \times 10^{-2} \epsilon, & e_{2,0} &= -6.38 \times 10^{-6} \epsilon^2 \quad +2.63 \times 10^{-3} \epsilon, \\ c_{1,1} &= 1.28 \times 10^{-6} \epsilon^2 \quad -8.56 \times 10^{-4} \epsilon, & e_{1,1} &= -5.67 \times 10^{-6} \epsilon^2 \quad +4.62 \times 10^{-3} \epsilon, \\ c_{0,2} &= -2.84 \times 10^{-6} \epsilon^2 \quad +2.31 \times 10^{-3} \epsilon, & e_{0,2} &= -1.35 \times 10^{-6} \epsilon^2 \quad +9.93 \times 10^{-4} \epsilon, \\ c_{1,0} &= 6.06 \times 10^{-5} \epsilon^2 \quad -3.4 \times 10^{-3} \epsilon + 39.6, & e_{1,0} &= 2.69 \times 10^{-5} \epsilon^2 \quad -1.06 \times 10^{-2} \epsilon, \\ c_{0,1} &= 2.69 \times 10^{-5} \epsilon^2 \quad -1.06 \times 10^{-2} \epsilon, & e_{0,1} &= 1.45 \times 10^{-5} \epsilon^2 \quad -1.12 \times 10^{-2} \epsilon + 28.4, \\ c_{0,0} &= -1.37 \times 10^{-4} \epsilon^2 \quad -0.18 \epsilon - 0.674, & e_{0,0} &= -8.59 \times 10^{-5} \epsilon^2 \quad -7.24 \times 10^{-2} \epsilon - 2.38 \times 10^{-3}. \end{aligned} \tag{74}$$

Next, Eq. (73) is cast in the form of Eqs. (48)–(49). This yields

$$\tilde{f}_1\left(z_2\right) = a_4 z_2^4 + a_3 z_2^3 + a_2 z_2^2 + a_1 z_2 + a_0, \tag{75}$$

$$\tilde{f}_2\left(z_2\right) = b_5 z_2^5 + b_4 z_2^4 + b_3 z_2^3 + b_2 z_2^2 + b_1 z_2 + b_0, \tag{76}$$

where

$$\begin{aligned}
a_4 &= \frac{1}{4\pi S_0}\left(c_{1,4}z_1 + c_{0,4}\right),\\
a_3 &= \frac{1}{4\pi S_0}\left(c_{1,3}z_1 + c_{0,3}\right),\\
a_2 &= \frac{1}{4\pi S_0}\left(c_{3,2}z_1^3 + c_{2,2}z_1^2 + c_{1,2}z_1 + c_{0,2}\right),\\
a_1 &= \frac{1}{4\pi S_0}\left(c_{3,1}z_1^3 + c_{2,1}z_1^2 + c_{1,1}z_1 + c_{0,1}\right),\\
a_0 &= \frac{1}{4\pi S_0}\left(c_{5,0}z_1^5 + c_{4,0}z_1^4 + c_{3,0}z_1^3 + c_{2,0}z_1^2 + c_{1,0}z_1 + c_{0,0}\right),\\
b_5 &= \frac{1}{4\pi S_0}e_{0,5},\\
b_4 &= \frac{1}{4\pi S_0}e_{0,4},\\
b_3 &= \frac{1}{4\pi S_0}\left(e_{2,3}z_1^2 + e_{1,3}z_1 + e_{0,3}\right),\\
b_2 &= \frac{1}{4\pi S_0}\left(e_{2,2}z_1^2 + e_{1,2}z_1 + e_{0,2}\right),\\
b_1 &= \frac{1}{4\pi S_0}\left(e_{4,1}z_1^4 + e_{3,1}z_1^3 + e_{2,1}z_1^2 + e_{1,1}z_1 + e_{0,1}\right),\\
b_0 &= \frac{1}{4\pi S_0}\left(e_{4,0}z_1^4 + e_{3,0}z_1^3 + e_{2,0}z_1^2 + e_{1,0}z_1 + e_{0,0}\right).
\end{aligned} \tag{77}$$

Multiplying Eqs. (75) and (76) by z_2^4, z_2^3, z_2^2, z_2 and z_2^3, z_2^2, z_2, respectively, yields the equations

$$\begin{aligned}
a_4z_2^8 + a_3z_2^7 + a_2z_2^6 + a_1z_2^5 + a_0z_2^4 &= 0,\\
a_4z_2^7 + a_3z_2^6 + a_2z_2^5 + a_1z_2^4 + a_0z_2^3 &= 0,\\
a_4z_2^6 + a_3z_2^5 + a_2z_2^4 + a_1z_2^3 + a_0z_2^2 &= 0,\\
a_4z_2^5 + a_3z_2^4 + a_2z_2^3 + a_1z_2^2 + a_0z_2 &= 0,\\
a_4z_2^4 + a_3z_2^3 + a_2z_2^2 + a_1z_2 + a_0 &= 0,\\
b_5z_2^8 + b_4z_2^7 + b_3z_2^6 + b_2z_2^5 + b_1z_2^4 + b_0z_2^3 &= 0,\\
b_5z_2^7 + b_4z_2^6 + b_3z_2^5 + b_2z_2^4 + b_1z_2^3 + b_0z_2^2 &= 0,\\
b_5z_2^6 + b_4z_2^5 + b_3z_2^4 + b_2z_2^3 + b_1z_2^2 + b_0z_2 &= 0,\\
b_5z_2^5 + b_4z_2^4 + b_3z_2^3 + b_2z_2^2 + b_1z_2 + b_0 &= 0.
\end{aligned} \tag{78}$$

Furthermore, according to Theorem 3, the determinant of the coefficients of Eq. (78) is set equal to zero as in Eq. (51). This yields a polynomial equation in the form of Eq. (52) with $q = 25$ that depends only on the suppressed variable z_1. Next, considering $\varepsilon = 1$ and the initial condition $z_1^{(0)} = 0$, Eq. (52) is solved numerically by employing a standard Newton's iterative optimization scheme [21]. Similarly, the remaining variable z_2 is obtained by considering z_2 as suppressed and repeating the above steps. Numerical results related to the iterations of the Newton's scheme are summarized in Table 1. The objective functions $S(\boldsymbol{z})$ of the most probable path optimization problem and the first-order optimality condition equations for $\varepsilon = 1, 10$ and 20 are shown in Figs. 2–4, respectively. The solutions $\boldsymbol{z}^*$ and $S(\boldsymbol{z}^*)$ for various values of ε are summarized in Table 2. As anticipated based on Theorem 4, the solutions coincide practically with the estimates obtained by the Gröbner basis approach applied in [22].

4.3. Nonlinear oscillator with an asymmetric response PDF

Further, a single-DOF nonlinear oscillator with an asymmetric response PDF is considered, whose governing equation takes the form

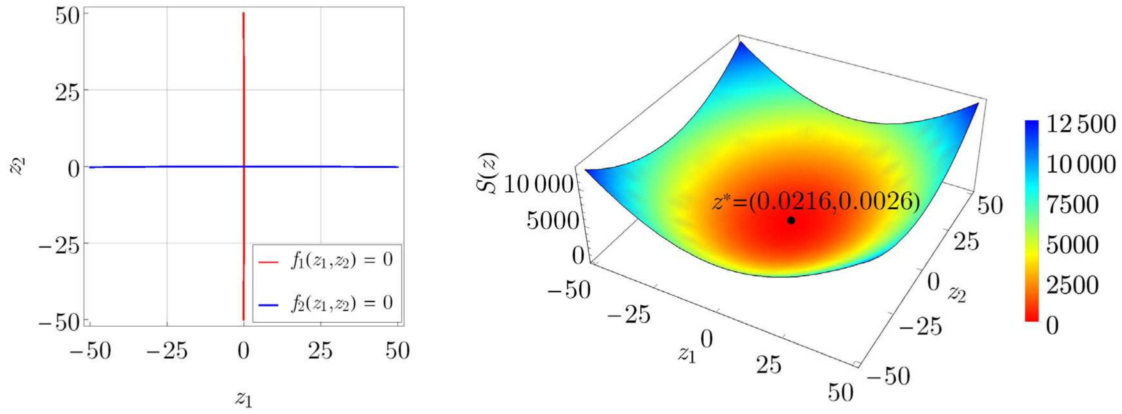
$$m\ddot{x} + c\dot{x} + kx + \varepsilon g_{nl}(x,\dot{x}) = w(t), \tag{79}$$

where the parameter ε denotes the nonlinearity magnitude, and the nonlinear function $g_{nl}(x,\dot{x})$ is given by

$$g_{nl}(x,\dot{x}) = ax^2 + x^3. \tag{80}$$

Considering two trial functions (i.e., $L = 2$) and arbitrary initial and final time instants (t_i and t_f) in Eq. (16), the objective function $S(\boldsymbol{z})$ is expressed in the form of Eq. (20) as

$$S(\boldsymbol{z}) = c_{6,0}{z_1}^6 + c_{4,2}{z_1}^4{z_2}^2 + c_{2,4}{z_1}^2{z_2}^4 + c_{0,6}{z_2}^6 + \sum_{|\alpha|<6} c_\alpha \boldsymbol{z}^\alpha, \tag{81}$$



(a) First-order optimality condition equations

(b) Objective function

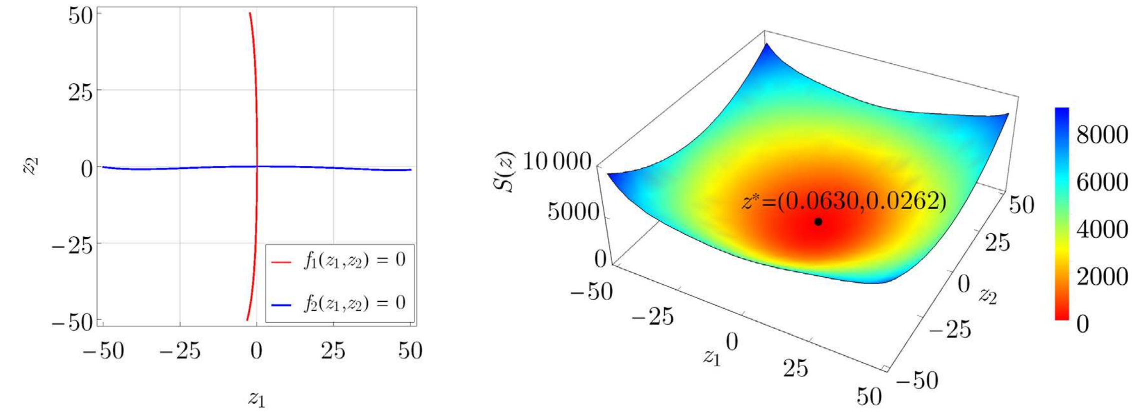
Fig. 2. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a Duffing nonlinear oscillator with $\varepsilon = 1.0$ and using $L = 2$ trial functions $\left(x\left(t_f = 1\right) = -0.5, \dot{x}\left(t_f = 1\right) = -1.0\right)$.

Table 1
Numerical optimization iterations for a Duffing nonlinear oscillator.

$\varepsilon = 1.0$		$\varepsilon = 10$		$\varepsilon = 20$		$\varepsilon = 100$	
$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$	$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$	$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$	$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$
0	2.11E−2	0	5.93E−2	0	9.89E−2	0	0.329
0.0211	4.52E−4	0.0593	3.72E−3	0.0989	1.08E−2	0.3286	0.146
0.0216	2.04E−7	0.0630	1.41E−5	0.1097	1.22E−4	0.4743	2.81E−2
0.0216	4.14E−14	0.0630	2.02E−10	0.1098	1.54E−8	0.5023	9.61E−4
0.0216	3.34E−18	0.0630	1.82E−19	0.1098	2.48E−16	0.5033	1.1E−6
						0.5033	1.44E−12
						0.5033	4.76E−17
$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$	$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$	$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$	$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$
0	2.65E−3	0	2.73E−2	0	5.84E−2	0	0.749
0.0027	1.08E−5	0.0273	1.12E−3	0.0584	4.98E−3	0.7491	0.674
0.0026	1.82E−10	0.0262	1.98E−6	0.0534	3.95E−5	0.07491	0.407
0.0026	1.12E−19	0.0262	6.12E−12	0.0534	2.46E−9	0.4816	0.288
		0.0262	1.17E−19	0.0534	1.24E−17	0.1939	0.139
						0.333	2.66E−2
						0.3064	1.26E−3
						0.3052	2.71E−6
						0.3052	1.27E−11
						0.3052	5.74E−18

Table 2
Optimal solution and objective function values for a Duffing nonlinear oscillator under white noise.

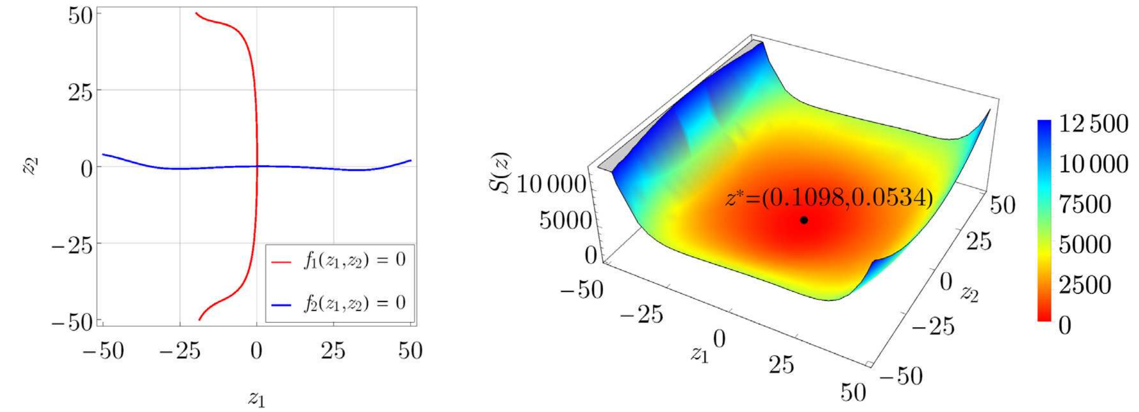
ε	$\boldsymbol{z}^*$	$S(\boldsymbol{z}^*)$
$\varepsilon = 1.0$	(0.0216, 0.0026)	4.4517
$\varepsilon = 10$	(0.0630, 0.0262)	4.7427
$\varepsilon = 20$	(0.1098, 0.0534)	5.0861
$\varepsilon = 100$	(0.5033, 0.3052)	8.4904



(a) First-order optimality condition equations

(b) Objective function

Fig. 3. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a Duffing nonlinear oscillator with $\varepsilon = 10$ and using $L = 2$ trial functions $\left(x\left(t_f = 1\right) = -0.5, \dot{x}\left(t_f = 1\right) = -1.0\right)$.



(a) First-order optimality condition equations

(b) Objective function

Fig. 4. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a Duffing nonlinear oscillator with $\varepsilon = 20$ and using $L = 2$ trial functions $\left(x\left(t_f = 1\right) = -0.5, \dot{x}\left(t_f = 1\right) = -1.0\right)$.

where

$$c_{6,0} = \frac{1}{4\pi S_0} \frac{(t_f - t_i)^{25}\varepsilon^2}{67603900}, \tag{82}$$

$$c_{4,2} = \frac{1}{4\pi S_0} \frac{(t_f - t_i)^{25}\varepsilon^2}{121687020}, \tag{83}$$

$$c_{2,4} = \frac{1}{4\pi S_0} \frac{(t_f - t_i)^{25}\varepsilon^2}{1176307860}, \tag{84}$$

$$c_{0,6} = \frac{1}{4\pi S_0} \frac{(t_f - t_i)^{25}\varepsilon^2}{109396630980}. \tag{85}$$

According to Lemma 1, since the coefficients $c_{6,0}, c_{4,2}, c_{2,4}$ and $c_{0,6}$ in Eqs. (82)–(85), are positive for arbitrary values $\varepsilon > 0$, there exists a global minimizer for $S(\boldsymbol{z})$.

Next, utilizing the parameter values, $m = 1$, $c = 0.2$, $k = 1$, and $E(w(t)w(t+\tau)) = 2\pi S_0 \delta(\tau)$ with $S_0 = 0.5$ in Eqs. (79) and (80), and considering the boundary conditions $(x(t_i = 0), \dot{x}(t_i = 0), x(t_f = 1), \dot{x}(t_f = 1)) = (0, 0, -0.3, -0.8)$, the characteristic polynomial of M_1 of the quadratic form Q_1 corresponding to Eq. (18) is calculated via Algorithm 1. This yields a 25-th degree polynomial exhibiting 13 sign changes in the sequence of coefficients. According to Proposition 1, M_1 has 13 positive eigenvalues and, since the polynomial is of degree 25, the signature of Q_1 takes the value $\sigma(Q_1) = 13 - 12 = 1$. Therefore, based on Theorem 2, there exists a unique global minimizer for $S(\boldsymbol{z})$.

Further, the Sylvester's dialytic method is employed for solving Eq. (18), which takes the form

$$\begin{aligned} f_1(z_1, z_2) = \frac{1}{4\pi S_0}[&c_{5,0}z_1^5 + c_{3,2}z_1^3z_2^2 + c_{1,4}z_1z_2^4 + c_{4,0}z_1^4 + c_{3,1}z_1^3z_2 + c_{2,2}z_1^2z_2^2 + c_{1,3}z_1z_2^3 + c_{0,4}z_2^4 + c_{3,0}z_1^3 \\ &+ c_{2,1}z_1^2z_2 + c_{1,2}z_1z_2^2 + c_{0,3}z_2^3 + c_{2,0}z_1^2 + c_{1,1}z_1z_2 + c_{0,2}z_2^2 + c_{1,0}z_1 + c_{0,1}z_2 + c_{0,0}], \\ f_2(z_1, z_2) = \frac{1}{4\pi S_0}[&d_{4,1}z_1^4z_2 + d_{2,3}z_1^2z_2^3 + d_{0,5}z_2^5 + d_{4,0}z_1^4 + d_{3,1}z_1^3z_2 + d_{2,2}z_1^2z_2^2 + d_{1,3}z_1z_2^3 + d_{0,4}z_2^4 + d_{3,0}z_1^3 \\ &+ d_{2,1}z_1^2z_2 + d_{1,2}z_1z_2^2 + d_{0,3}z_2^3 + d_{2,0}z_1^2 + d_{1,1}z_1z_2 + d_{0,2}z_2^2 + d_{1,0}z_1 + d_{0,1}z_2 + d_{0,0}], \end{aligned} \tag{86}$$

where

$$\begin{aligned}
c_{5,0} &= 8.88\times10^{-8}\epsilon^2, \\
c_{3,2} &= 3.28\times10^{-8}\epsilon^2, \\
c_{1,4} &= 1.70\times10^{-9}\epsilon^2, \\
c_{4,0} &= 2.58\times10^{-6}a\epsilon^2 \ -4.20\times10^{-7}\epsilon^2, \\
c_{3,1} &= -\ 1.72\times10^{-7}\epsilon^2, \\
c_{2,2} &= 6.72\times10^{-7}a\epsilon^2 \ -1.25\times10^{-7}\epsilon^2, \\
c_{1,3} &= -\ 2.09\times10^{-8}\epsilon^2, \\
c_{0,4} &= 1.34\times10^{-8}a\epsilon^2 \ -2.76\times10^{-9}\epsilon^2, \\
c_{3,0} &= 1.83\times10^{-5}a^2\epsilon^2 \ -1.01\times10^{-5}a\epsilon^2 \ +1.09\times10^{-6}\epsilon^2 \ -4.96\times10^{-4}\epsilon, \\
c_{2,1} &= -\ 3.71\times10^{-6}a\epsilon^2 \ +7.18\times10^{-7}\epsilon^2, \\
c_{1,2} &= 2.89\times10^{-6}a^2\epsilon^2 \ -1.86\times10^{-6}a\epsilon^2 \ +2.89\times10^{-7}\epsilon^2 \ -1.67\times10^{-4}\epsilon, \\
c_{0,3} &= -\ 1.79\times10^{-7}a\epsilon^2 \ +3.87\times10^{-8}\epsilon^2, \\
c_{2,0} &= -\ 5.66\times10^{-5}a^2\epsilon^2 \ +2.20\times10^{-5}a\epsilon^2 \ -2.12\times10^{-6}\epsilon^2 \ -6.43\times10^{-3}a\epsilon \ +6.06\times10^{-4}\epsilon, \\
c_{1,1} &= -\ 1.72\times10^{-5}a^2\epsilon^2 \ +1.17\times10^{-5}a\epsilon^2 \ -1.50\times10^{-6}\epsilon^2 \ +4.72\times10^{-4}\epsilon, \\
c_{0,2} &= -\ 4.51\times10^{-6}a^2\epsilon^2 \ +2.59\times10^{-6}a\epsilon^2 \ -3.49\times10^{-7}\epsilon^2 \ -1.12\times10^{-3}a\epsilon \ +1.91\times10^{-4}\epsilon, \\
c_{1,0} &= 9.86\times10^{-5}a^2\epsilon^2 \ -3.66\times10^{-5}a\epsilon^2 \ +3.61\times10^{-6}\epsilon^2 \ +1.41\times10^{-3}a\epsilon \ +7.78\times10^{-4}\epsilon + 1.53, \\
c_{0,1} &= 3.33\times10^{-5}a^2\epsilon^2 \ -1.60\times10^{-5}a\epsilon^2 \ +1.83\times10^{-6}\epsilon^2 \ +2.66\times10^{-3}a\epsilon \ -2.41\times10^{-4}\epsilon, \\
c_{0,0} &= -\ 1.30\times10^{-4}a^2\epsilon^2 \ +5.30\times10^{-5}a\epsilon^2 \ -5.75\times10^{-6}\epsilon^2 \ +3.11\times10^{-2}a\epsilon \ -7.85\times10^{-3}\epsilon - 0.109,
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
d_{4,1} &= 1.64\times10^{-8}\epsilon^2, \\
d_{2,3} &= 3.40\times10^{-9}\epsilon^2, \\
d_{0,5} &= 5.48\times10^{-11}\epsilon^2 \\
d_{4,0} &= -\ 4.30\times10^{-8}\epsilon^2, \\
d_{3,1} &= 4.48\times10^{-7}a\epsilon^2 \ -8.34\times10^{-8}\epsilon^2, \\
d_{2,2} &= -\ 3.14\times10^{-8}\epsilon^2, \\
d_{1,3} &= 5.38\times10^{-8}a\epsilon^2 \ -1.1\times10^{-8}\epsilon^2, \\
d_{0,4} &= -\ 9.79\times10^{-10}\epsilon^2, \\
d_{3,0} &= -\ 1.24\times10^{-6}a\epsilon^2 \ +2.39\times10^{-7}\epsilon^2, \\
d_{2,1} &= 2.89\times10^{-6}a^2\epsilon^2 \ -1.86\times10^{-6}a\epsilon^2 \ +2.89\times10^{-7}\epsilon^2 \ -1.67\times10^{-4}\epsilon,
\end{aligned}$$

$$
\begin{aligned}
d_{1,2} &= -\ 5.38\times10^{-7}a\epsilon^2\ +1.16\times10^{-7}\epsilon^2,\\
d_{3,0} &= \ \ 1.37\times10^{-7}a^2\epsilon^2\ -9.86\times10^{-8}a\epsilon^2\ +1.87\times10^{-8}\epsilon^2\ -1.29\times10^{-5}\epsilon,\\
d_{2,0} &= -\ 8.62\times10^{-6}a^2\epsilon^2\ +5.86\times10^{-6}a\epsilon^2\ -7.51\times10^{-7}\epsilon^2\ +2.36\times10^{-4}\epsilon,\\
d_{1,1} &= -\ 9.01\times10^{-6}a^2\epsilon^2\ +5.19\times10^{-6}a\epsilon^2\ -6.99\times10^{-7}\epsilon^2\ -2.24\times10^{-3}a\epsilon\ +3.82\times10^{-4}\epsilon,\\
d_{0,2} &= -\ 1.55\times10^{-6}a^2\epsilon^2\ +1.19\times10^{-6}a\epsilon^2\ -1.77\times10^{-7}\epsilon^2\ +9.8\times10^{-5}\epsilon,\\
d_{1,0} &= \ \ 3.33\times10^{-5}a^2\epsilon^2\ -1.6\times10^{-5}a\epsilon^2\ +1.83\times10^{-6}\epsilon^2\ +2.66\times10^{-3}a\epsilon\ -2.41\times10^{-4}\epsilon,\\
d_{0,1} &= \ \ 1.68\times10^{-5}a^2\epsilon^2\ -8.56\times10^{-6}a\epsilon^2\ +1.06\times10^{-6}\epsilon^2\ +2.87\times10^{-3}a\epsilon\ -4.22\times10^{-4}\epsilon+1.12,\\
d_{0,0} &= -\ 7.32\times10^{-5}a^2\epsilon^2\ +3.36\times10^{-5}a\epsilon^2\ -3.94\times10^{-6}\epsilon^2\ +1.12\times10^{-2}a\epsilon\ -3.92\times10^{-3}\epsilon-1.25\times10^{-2}.
\end{aligned}
\tag{88}
$$

Next, Eq. (86) is cast in the form of Eqs. (48)–(49). This yields

$$\tilde{f}_1\left(z_2\right) = a_4 z_2^4 + a_3 z_2^3 + a_2 z_2^2 + a_1 z_2 + a_0, \tag{89}$$

$$\tilde{f}_2\left(z_2\right) = b_5 z_2^5 + b_4 z_2^4 + b_3 z_2^3 + b_2 z_2^2 + b_1 z_2 + b_0, \tag{90}$$

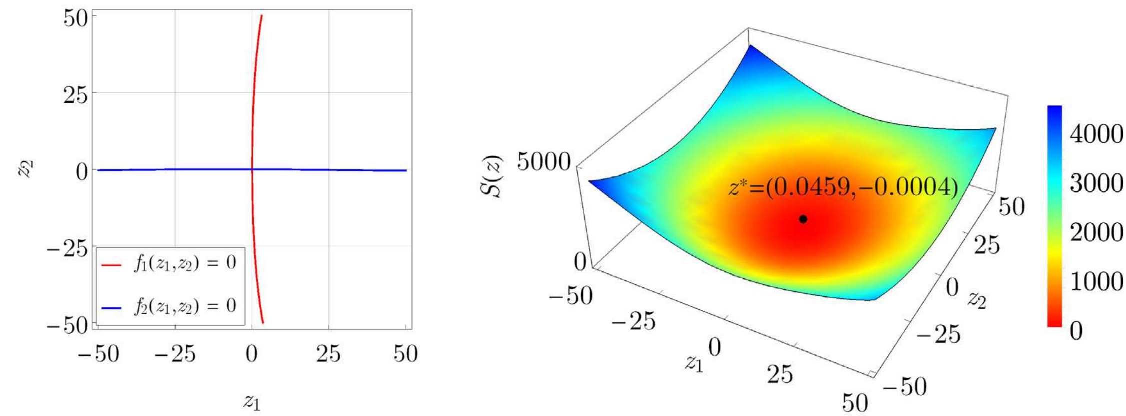
where

$$
\begin{aligned}
a_4 &= \frac{1}{4\pi S_0}\left(c_{1,4}z_1 + c_{0,4}\right),\\
a_3 &= \frac{1}{4\pi S_0}\left(c_{1,3}z_1 + c_{0,3}\right),\\
a_2 &= \frac{1}{4\pi S_0}\left(c_{3,2}z_1^3 + c_{2,2}z_1^2 + c_{1,2}z_1 + c_{0,2}\right),\\
a_1 &= \frac{1}{4\pi S_0}\left(c_{3,1}z_1^3 + c_{2,1}z_1^2 + c_{1,1}z_1 + c_{0,1}\right),\\
a_0 &= \frac{1}{4\pi S_0}\left(c_{5,0}z_1^5 + c_{4,0}z_1^4 + c_{3,0}z_1^3 + c_{2,0}z_1^2 + c_{1,0}z_1 + c_{0,0}\right),\\
b_5 &= \frac{1}{4\pi S_0}d_{0,5},\\
b_4 &= \frac{1}{4\pi S_0}d_{0,4},\\
b_3 &= \frac{1}{4\pi S_0}\left(d_{2,3}z_1^2 + d_{1,3}z_1 + d_{0,3}\right),\\
b_2 &= \frac{1}{4\pi S_0}\left(d_{2,2}z_1^2 + d_{1,2}z_1 + d_{0,2}\right),\\
b_1 &= \frac{1}{4\pi S_0}\left(d_{4,1}z_1^4 + d_{3,1}z_1^3 + d_{2,1}z_1^2 + d_{1,1}z_1 + d_{0,1}\right),\\
b_0 &= \frac{1}{4\pi S_0}\left(d_{4,0}z_1^4 + d_{3,0}z_1^3 + d_{2,0}z_1^2 + d_{1,0}z_1 + d_{0,0}\right).
\end{aligned}
\tag{91}
$$

Multiplying Eqs. (89) and (90) by z_2^4, z_2^3, z_2^2, z_2 and z_2^3, z_2^2, z_2, respectively, yields the equations

$$
\begin{aligned}
a_4 z_2^8 + a_3 z_2^7 + a_2 z_2^6 + a_1 z_2^5 + a_0 z_2^4 &= 0,\\
a_4 z_2^7 + a_3 z_2^6 + a_2 z_2^5 + a_1 z_2^4 + a_0 z_2^3 &= 0,\\
a_4 z_2^6 + a_3 z_2^5 + a_2 z_2^4 + a_1 z_2^3 + a_0 z_2^2 &= 0,\\
a_4 z_2^5 + a_3 z_2^4 + a_2 z_2^3 + a_1 z_2^2 + a_0 z_2 &= 0,\\
a_4 z_2^4 + a_3 z_2^3 + a_2 z_2^2 + a_1 z_2 + a_0 &= 0,\\
b_5 z_2^8 + b_4 z_2^7 + b_3 z_2^6 + b_2 z_2^5 + b_1 z_2^4 + b_0 z_2^3 &= 0,\\
b_5 z_2^7 + b_4 z_2^6 + b_3 z_2^5 + b_2 z_2^4 + b_1 z_2^3 + b_0 z_2^2 &= 0,\\
b_5 z_2^6 + b_4 z_2^5 + b_3 z_2^4 + b_2 z_2^3 + b_1 z_2^2 + b_0 z_2 &= 0,\\
b_5 z_2^5 + b_4 z_2^4 + b_3 z_2^3 + b_2 z_2^2 + b_1 z_2 + b_0 &= 0.
\end{aligned}
\tag{92}
$$

Furthermore, according to Theorem 3, the determinant of the coefficients of Eq. (92) is set equal to zero as in Eq. (51). This yields a polynomial equation in the form of Eq. (52) with $q = 25$ that depends only on the suppressed variable z_1. Next, considering $\varepsilon = 1$, $a = 1.5$ and the initial condition $z_1^{(0)} = 0$, Eq. (52) is solved numerically by employing a standard Newton's iterative optimization scheme [21]. Similarly, the remaining variable z_2 is obtained by considering z_2 as suppressed and repeating the above steps. Numerical results related to the iterations of the Newton's scheme are summarized in Table 3. The objective functions $S(\boldsymbol{z})$ of the most probable path optimization problem for $\varepsilon = 1$ and $a = 1.5$, $\varepsilon = 10$ and $a = \frac{3\sqrt{10}}{20}$, and $\varepsilon = 50$ and $a = \frac{3\sqrt{2}}{20}$ are shown in Figs. 5–7, respectively. The solutions $\boldsymbol{z}^*$ and $S(\boldsymbol{z}^*)$ for various values of ε and a are summarized in Table 4. As anticipated based on Theorem 4, the solutions coincide practically with the estimates obtained by the Gröbner basis approach applied in [22].



(a) First-order optimality condition equations (b) Objective function

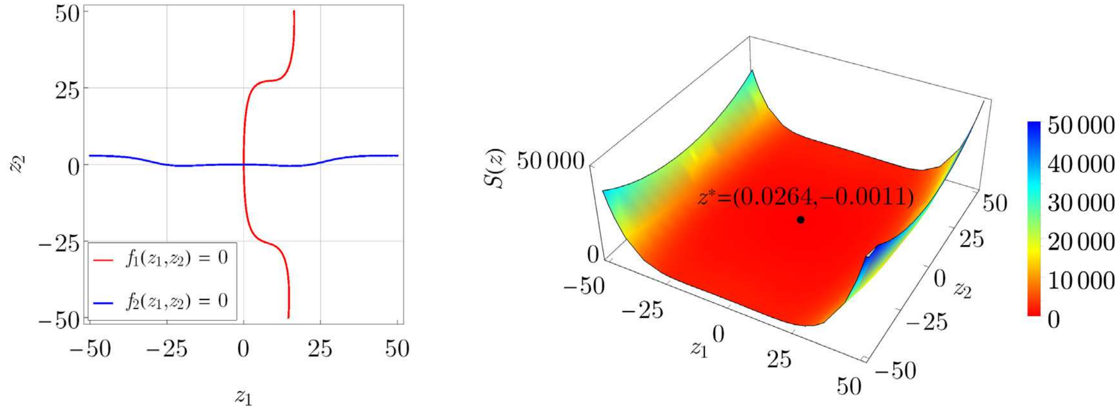
Fig. 5. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a nonlinear oscillator with an asymmetric response PDF with $\varepsilon = 1$ and $a = 1.5$ using $L = 2$ trial functions $\left(x\left(t_f = 1\right) = -0.3,\ \dot{x}\left(t_f = 1\right) = -0.8\right)$.

Table 3
Numerical optimization iterations for a nonlinear oscillator with an asymmetric response PDF.

$\varepsilon = 1,\ a = 1.50$		$\varepsilon = 10,\ a = \frac{3\sqrt{10}}{20}$		$\varepsilon = 50,\ a = \frac{3\sqrt{2}}{20}$		$\varepsilon = 100,\ a = 0.150$	
$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$	$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$	$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$	$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$
0	4.67E−2	0	2.78E−2	0	9.43E−2	0	1.09E−1
0.0467	7.95E−4	0.0278	1.34E−3	0.09431	1.48E−2	0.1086	8.11E−2
0.0459	2.34E−7	0.0265	3.23E−6	0.1091	7.68E−4	0.1897	5.13E−2
0.0459	2.02E−14	0.0264	1.89E−11	0.1084	2.57E−6	0.241	2.09E−2
0.0459	2.07E−18			0.1084	2.86E−11	0.2619	3.09E−3
						0.265	6.07E−5
						0.2651	2.30E−8
						0.2651	3.41E−15
$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$	$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$	$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$	$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$
0	3.89E−4	0	1.05E−3	0	1.12E−1	0	10
−0.0004	5.22E−7	−0.0011	3.79E−6	0.112	2.99E−2	10	9
−0.0004	9.36E−13	−0.0011	4.91E−11	0.0818	3.42E−3	1	0.5
−0.0004	1.20E−20			0.0783	4.21E−5	0.5	0.25
				0.0783	6.32E−9	0.25	3.47E−2
				0.0783	1.39E−16	0.2153	4.98E−3
						0.2103	9.59E−5
						0.2102	3.51E−8
						0.2102	4.69E−15
						0.2102	1.98E−18

Table 4
Optimal solution and objective function values for a nonlinear oscillator with an asymmetric response PDF under white noise.

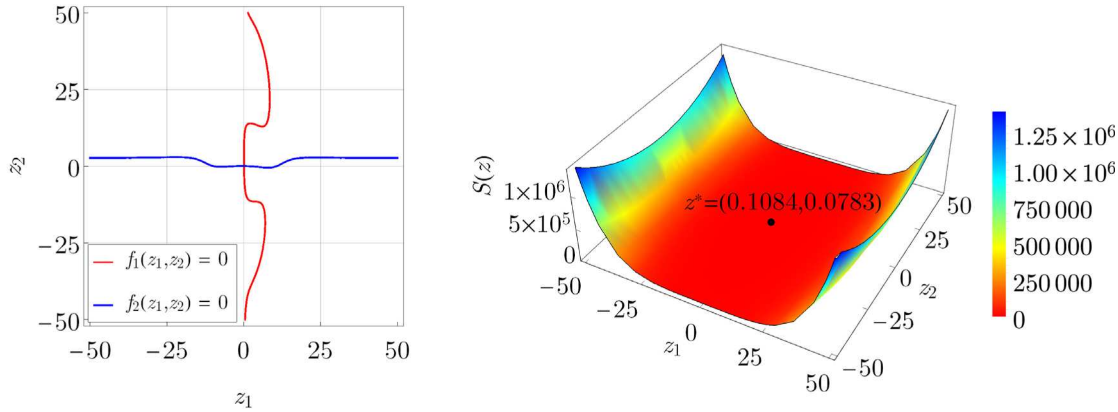
$\varepsilon,\ a$	$\boldsymbol{z}^*$	$S(\boldsymbol{z}^*)$
$\varepsilon = 1,\ a = 1.50$	$(0.0459, -0.0004)$	1.6827
$\varepsilon = 10,\ a = \frac{3\sqrt{10}}{20}$	$(0.0264, -0.0011)$	1.5984
$\varepsilon = 50,\ a = \frac{3\sqrt{2}}{20}$	$(0.1084, 0.0783)$	1.7934
$\varepsilon = 100,\ a = 0.150$	$(0.2651, 0.2102)$	2.2707



(a) First-order optimality condition equations

(b) Objective function

Fig. 6. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a nonlinear oscillator with an asymmetric response PDF with $\varepsilon = 10$ and $a = \frac{3\sqrt{10}}{20}$ using $L = 2$ trial functions $\left(x\left(t_f = 1\right) = -0.3,\ \dot{x}\left(t_f = 1\right) = -0.8\right)$.



(a) First-order optimality condition equations

(b) Objective function

Fig. 7. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a nonlinear oscillator with an asymmetric response PDF with $\varepsilon = 50$ and $a = \frac{3\sqrt{2}}{20}$ using $L = 2$ trial functions $\left(x\left(t_f = 1\right) = -0.3,\ \dot{x}\left(t_f = 1\right) = -0.8\right)$.

4.4. Nonlinear oscillator with a bimodal response PDF

Next, a single-DOF nonlinear oscillator with a bimodal response PDF is considered, whose governing equation is given by

$$m\ddot{x} + c\dot{x} + kx + \varepsilon g_{nl}(x, \dot{x}) = w(t), \tag{93}$$

where

$$g_{nl}(x, \dot{x}) = -ax + x^3. \tag{94}$$

Considering two trial functions (i.e., $L = 2$) and arbitrary initial and final time instants (t_i and t_f) in Eq. (16), the objective function $S(\boldsymbol{z})$ is expressed in the form of Eq. (20) as

$$S(\boldsymbol{z}) = c_{6,0}{z_1}^6 + c_{4,2}{z_1}^4{z_2}^2 + c_{2,4}{z_1}^2{z_2}^4 + c_{0,6}{z_2}^6 + \sum_{|\alpha|<6} c_\alpha z^\alpha, \tag{95}$$

where

$$c_{6,0} = \frac{1}{4\pi S_0}\frac{(t_f - t_i)^{25}\varepsilon^2}{67603900}, \tag{96}$$

$$c_{4,2} = \frac{1}{4\pi S_0}\frac{(t_f - t_i)^{25}\epsilon^2}{121687020}, \tag{97}$$

$$c_{2,4} = \frac{1}{4\pi S_0}\frac{(t_f - t_i)^{25}\epsilon^2}{1176307860}, \tag{98}$$

$$c_{0,6} = \frac{1}{4\pi S_0}\frac{(t_f - t_i)^{25}\epsilon^2}{109396630980}. \tag{99}$$

According to Lemma 1, since the coefficients $c_{6,0}, c_{4,2}, c_{2,4}$ and $c_{0,6}$ in Eqs. (96)–(99) are positive for arbitrary values $\epsilon > 0$, there exists a global minimizer for $S(z)$.

Next, using the parameter values $m = 1,\ c = 1.0,\ k = 1.0$, and $E(w(t)w(t+\tau)) = 2\pi S_0\delta(\tau)$ with $S_0 = 0.0637$ in Eqs. (93) and (94), and considering the boundary conditions $(x(t_i = 0), \dot{x}(t_i = 0), x(t_f = 1), \dot{x}(t_f = 1)) = (0, 0, 0.8, 0.9)$, the characteristic polynomial of M_1 of the quadratic form Q_1 corresponding to Eq. (18) is calculated via Algorithm 1. This yields a 25-th degree polynomial exhibiting 13 sign changes in the sequence of coefficients. According to Proposition 1, M_1 has 13 positive eigenvalues and, since the polynomial is of degree 25, Q_1 has signature $\sigma(Q_1) = 13 - 12 = 1$. Therefore, based on Theorem 2, there exists a unique global minimizer for $S(z)$.

Further, the Sylvester's dialytic method is employed for solving Eq. (18), which takes the form

$$\begin{aligned} f_1(z_1, z_2) = \frac{1}{4\pi S_0}[&c_{5,0}z_1^5 + c_{3,2}z_1^3z_2^2 + c_{1,4}z_1z_2^4 + c_{4,0}z_1^4 + c_{3,1}z_1^3z_2 + c_{2,2}z_1^2z_2^2 + c_{1,3}z_1z_2^3 + c_{0,4}z_2^4 + c_{3,0}z_1^3 \\ &+ c_{2,1}z_1^2z_2 + c_{1,2}z_1z_2^2 + c_{0,3}z_2^3 + c_{2,0}z_1^2 + c_{1,1}z_1z_2 + c_{0,2}z_2^2 + c_{1,0}z_1 + c_{0,1}z_2 + c_{0,0}], \\ f_2(z_1, z_2) = \frac{1}{4\pi S_0}[&d_{4,1}z_1^4z_2 + d_{2,3}z_1^2z_2^3 + d_{0,5}z_2^5 + d_{4,0}z_1^4 + d_{3,1}z_1^3z_2 + d_{2,2}z_1^2z_2^2 + d_{1,3}z_1z_2^3 + d_{0,4}z_2^4 + d_{3,0}z_1^3 \\ &+ d_{2,1}z_1^2z_2 + d_{1,2}z_1z_2^2 + d_{0,3}z_2^3 + d_{2,0}z_1^2 + d_{1,1}z_1z_2 + d_{0,2}z_2^2 + d_{1,0}z_1 + d_{0,1}z_2 + d_{0,0}], \end{aligned} \tag{100}$$

where

$$\begin{aligned} c_{5,0} &= 8.88\times10^{-8}\epsilon^2, \\ c_{3,2} &= 3.28\times10^{-8}\epsilon^2, \\ c_{1,4} &= 1.7\times10^{-9}\epsilon^2, \\ c_{4,0} &= 2.26\times10^{-6}\epsilon^2, \\ c_{3,1} &= 6.41\times10^{-7}\epsilon^2, \\ c_{2,2} &= 6.07\times10^{-7}\epsilon^2, \\ c_{1,3} &= 7.61\times10^{-8}\epsilon^2, \\ c_{0,4} &= 1.24\times10^{-8}\epsilon^2, \\ c_{3,0} &= -3.66\times10^{-5}a\epsilon^2 + 2.69\times10^{-5}\epsilon^2 - 4.96\times10^{-4}\epsilon, \\ c_{2,1} &= 1.25\times10^{-5}\epsilon^2, \\ c_{1,2} &= -5.77\times10^{-6}a\epsilon^2 + 5.35\times10^{-6}\epsilon^2 - 1.67\times10^{-4}\epsilon, \\ c_{0,3} &= 6.02\times10^{-7}\epsilon^2, \\ c_{2,0} &= -5.89\times10^{-4}a\epsilon^2 + 1.97\times10^{-4}\epsilon^2 - 5.02\times10^{-3}\epsilon, \\ c_{1,1} &= -1.26\times10^{-4}a\epsilon^2 + 1.01\times10^{-4}\epsilon^2 - 2.05\times10^{-3}\epsilon, \\ c_{0,2} &= -4.09\times10^{-5}a\epsilon^2 + 2.18\times10^{-5}\epsilon^2 - 9.85\times10^{-4}\epsilon, \\ c_{1,0} &= 3.17\times10^{-3}a^2\epsilon^2 - 4.14\times10^{-3}a\epsilon^2 + 9.79\times10^{-4}\epsilon^2 + 6.98\times10^{-2}a\epsilon - 1.18\times10^{-2}\epsilon + 1.57, \\ c_{0,1} &= -1.01\times10^{-3}a\epsilon^2 + 3.75\times10^{-4}\epsilon^2 - 1.03\times10^{-2}\epsilon, \\ c_{0,0} &= 2.02\times10^{-2}a^2\epsilon^2 - 1.53\times10^{-2}a\epsilon^2 + 3.24\times10^{-3}\epsilon^2 - 1.60\times10^{-1}a\epsilon + 1.45\times10^{-1}\epsilon + 8.02\times10^{-2}, \end{aligned} \tag{101}$$

and

$$\begin{aligned}
d_{4,1} &= 1.64\times10^{-8}\epsilon^2,\\
d_{2,3} &= 3.40\times10^{-9}\epsilon^2,\\
d_{0,5} &= 5.48\times10^{-11}\epsilon^2,\\
d_{4,0} &= 1.60\times10^{-7}\epsilon^2,\\
d_{3,1} &= 4.05\times10^{-7}\epsilon^2,\\
d_{2,2} &= 1.14\times10^{-7}\epsilon^2,\\
d_{1,3} &= 4.98\times10^{-8}\epsilon^2,\\
d_{0,4} &= 3.48\times10^{-9}\epsilon^2,\\
d_{3,0} &= 4.16\times10^{-6}\epsilon^2,\\
d_{2,1} &= -5.77\times10^{-6}a\epsilon^2 +5.35\times10^{-6}\epsilon^2 -1.67\times10^{-4}\epsilon,\\
d_{1,2} &= 1.81\times10^{-6}\epsilon^2,\\
d_{0,3} &= -2.75\times10^{-7}a\epsilon^2 +2.97\times10^{-7}\epsilon^2 -1.29\times10^{-5}\epsilon,\\
d_{2,0} &= -6.29\times10^{-5}a\epsilon^2 +5.04\times10^{-5}\epsilon^2 -1.02\times10^{-3}\epsilon,\\
d_{1,1} &= -8.19\times10^{-5}a\epsilon^2 +4.35\times10^{-5}\epsilon^2 -1.97\times10^{-3}\epsilon,\\
d_{0,2} &= -1.09\times10^{-5}a\epsilon^2 +9.72\times10^{-6}\epsilon^2 -3.75\times10^{-4}\epsilon,\\
d_{1,0} &= -1.01\times10^{-3}a\epsilon^2 +3.75\times10^{-4}\epsilon^2 -1.03\times10^{-2}\epsilon,\\
d_{0,1} &= 2.89\times10^{-4}a^2\epsilon^2 -5.09\times10^{-4}a\epsilon^2 +1.89\times10^{-4}\epsilon^2 +2.48\times10^{-2}a\epsilon -9.68\times10^{-3}\epsilon +1.13,\\
d_{0,0} &= 4.37\times10^{-3}a^2\epsilon^2 -6.67\times10^{-3}a\epsilon^2 +1.79\times10^{-3}\epsilon^2 +3.13\times10^{-2}a\epsilon +3.34\times10^{-2}\epsilon -1.56\times10^{-2}.
\end{aligned} \tag{102}$$

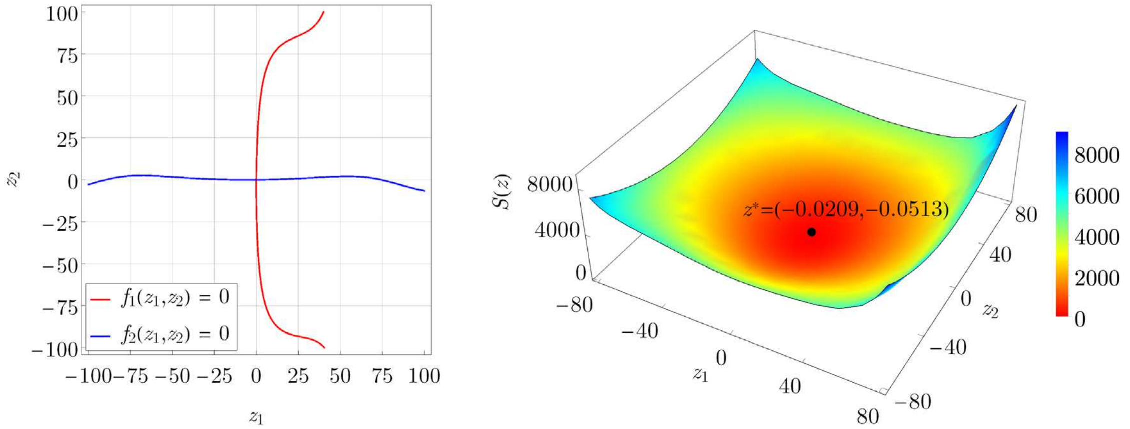
Next, Eq. (100) is cast in the form of Eqs. (48)–(49). This yields

$$\tilde{f}_1\left(z_2\right) = a_4 z_2^4 + a_3 z_2^3 + a_2 z_2^2 + a_1 z_2 + a_0, \tag{103}$$

$$\tilde{f}_2\left(z_2\right) = b_5 z_2^5 + b_4 z_2^4 + b_3 z_2^3 + b_2 z_2^2 + b_1 z_2 + b_0, \tag{104}$$

where

$$\begin{aligned}
a_4 &= \frac{1}{4\pi S_0}\left(c_{1,4}z_1 + c_{0,4}\right),\\
a_3 &= \frac{1}{4\pi S_0}\left(c_{1,3}z_1 + c_{0,3}\right),\\
a_2 &= \frac{1}{4\pi S_0}\left(c_{3,2}z_1^3 + c_{2,2}z_1^2 + c_{1,2}z_1 + c_{0,2}\right),\\
a_1 &= \frac{1}{4\pi S_0}\left(c_{3,1}z_1^3 + c_{2,1}z_1^2 + c_{1,1}z_1 + c_{0,1}\right),\\
a_0 &= \frac{1}{4\pi S_0}\left(c_{5,0}z_1^5 + c_{4,0}z_1^4 + c_{3,0}z_1^3 + c_{2,0}z_1^2 + c_{1,0}z_1 + c_{0,0}\right),\\
b_5 &= \frac{1}{4\pi S_0}d_{0,5},\\
b_4 &= \frac{1}{4\pi S_0}d_{0,4},\\
b_3 &= \frac{1}{4\pi S_0}\left(d_{2,3}z_1^2 + d_{1,3}z_1 + d_{0,3}\right),\\
b_2 &= \frac{1}{4\pi S_0}\left(d_{2,2}z_1^2 + d_{1,2}z_1 + d_{0,2}\right),\\
b_1 &= \frac{1}{4\pi S_0}\left(d_{4,1}z_1^4 + d_{3,1}z_1^3 + d_{2,1}z_1^2 + d_{1,1}z_1 + d_{0,1}\right),\\
b_0 &= \frac{1}{4\pi S_0}\left(d_{4,0}z_1^4 + d_{3,0}z_1^3 + d_{2,0}z_1^2 + d_{1,0}z_1 + d_{0,0}\right).
\end{aligned} \tag{105}$$



(a) First-order optimality condition equations (b) Objective function

Fig. 8. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a nonlinear oscillator with a bimodal response PDF with $a = 1.3$ and $\varepsilon = 1$ using $L = 2$ trial functions $\left(x\left(t_f = 1\right) = 0.8, \dot{x}\left(t_f = 1\right) = 0.9\right)$.

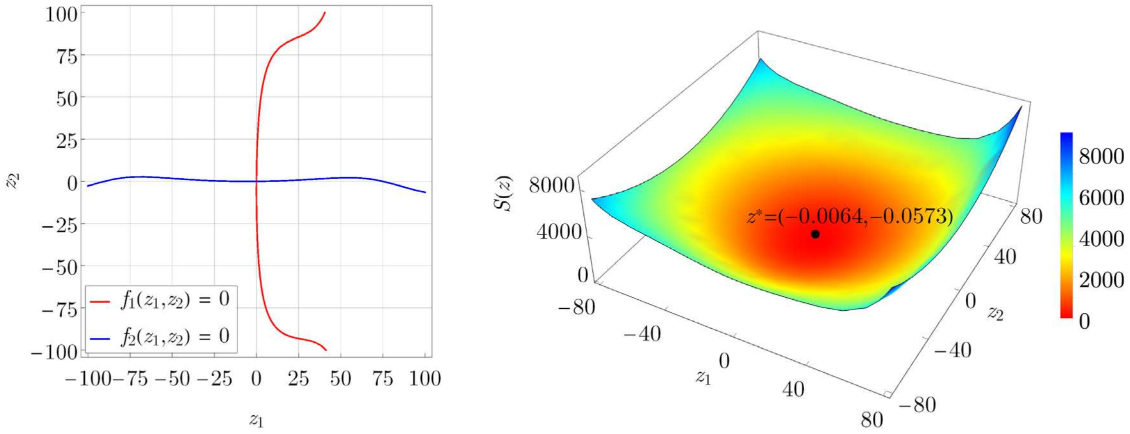
Table 5
Numerical optimization iterations for a nonlinear oscillator with a bimodal response PDF.

$a = 1.3$		$a = 1.5$		$a = 1.8$	
$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$	$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$	$z_1^{(k)}$	$\lvert z_1^{(k+1)} - z_1^{(k)}\rvert$
0	2.07E−02	0	5.98E−02	0	1.31E−02
−0.0207	2.16E−04	−0.0598	2.48E−03	0.0131	8.50E−05
−0.0209	2.33E−08	−0.0573	4.44E−06	0.013	3.60E−09
−0.0209	2.71E−16	−0.0573	1.42E−11	0.013	6.52E−18
$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$	$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$	$z_2^{(k)}$	$\lvert z_2^{(k+1)} - z_2^{(k)}\rvert$
0	5.33E−02	0	5.98E−02	0	7.01E−02
−0.0533	1.99E−03	−0.0598	2.48E−03	−0.0701	3.36E−03
−0.0513	2.88E−06	−0.0573	4.44E−06	−0.0668	8.07E−06
−0.0513	6.00E−12	−0.0573	1.42E−11	−0.0667	4.64E−11
−0.0513	4.58E−18	−0.0573	4.40E−18	−0.0667	4.19E−18

Multiplying Eqs. (103) and (104) by z_2^4, z_2^3, z_2^2, z_2 and z_2^3, z_2^2, z_2, respectively, yields the equations

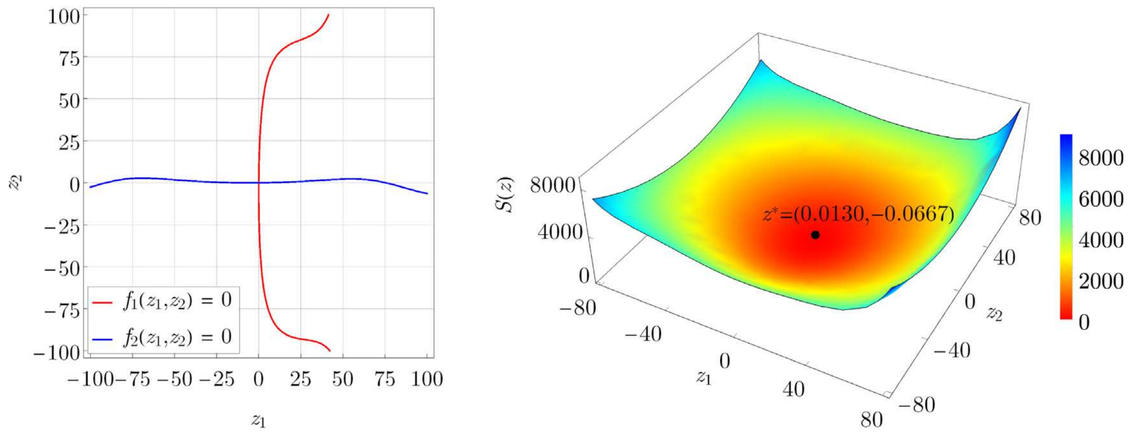
$$
\begin{aligned}
a_4 z_2^8 + a_3 z_2^7 + a_2 z_2^6 + a_1 z_2^5 + a_0 z_2^4 &= 0,\\
a_4 z_2^7 + a_3 z_2^6 + a_2 z_2^5 + a_1 z_2^4 + a_0 z_2^3 &= 0,\\
a_4 z_2^6 + a_3 z_2^5 + a_2 z_2^4 + a_1 z_2^3 + a_0 z_2^2 &= 0,\\
a_4 z_2^5 + a_3 z_2^4 + a_2 z_2^3 + a_1 z_2^2 + a_0 z_2 &= 0,\\
a_4 z_2^4 + a_3 z_2^3 + a_2 z_2^2 + a_1 z_2 + a_0 &= 0,\\
b_5 z_2^8 + b_4 z_2^7 + b_3 z_2^6 + b_2 z_2^5 + b_1 z_2^4 + b_0 z_2^3 &= 0,\\
b_5 z_2^7 + b_4 z_2^6 + b_3 z_2^5 + b_2 z_2^4 + b_1 z_2^3 + b_0 z_2^2 &= 0,\\
b_5 z_2^6 + b_4 z_2^5 + b_3 z_2^4 + b_2 z_2^3 + b_1 z_2^2 + b_0 z_2 &= 0,\\
b_5 z_2^5 + b_4 z_2^4 + b_3 z_2^3 + b_2 z_2^2 + b_1 z_2 + b_0 &= 0.
\end{aligned}
\tag{106}
$$

Furthermore, according to Theorem 3, the determinant of the coefficients of Eq. (106) is set equal to zero as in Eq. (51). This yields a polynomial equation in the form of Eq. (52) with $q = 25$ that depends only on the suppressed variable z_1. Next, considering $a = 1.3$, $\varepsilon = 1$ and the initial condition $z_1^{(0)} = 0$, Eq. (52) is solved numerically by employing a standard Newton's iterative optimization scheme [21]. Similarly, the remaining variable z_2 is obtained by considering z_2 as suppressed and repeating the above steps. Numerical results related to the iterations of the Newton's scheme are summarized in Table 5. The objective functions $S(\boldsymbol{z})$ of the most probable path optimization problem and the first-order optimality condition equations for $a = 1.3,\ 1.5$ and 1.8, considering $\varepsilon = 1$ are shown in Figs. 8–10, respectively. The solutions $\boldsymbol{z}^*$ and $S(\boldsymbol{z}^*)$ for various values of a are summarized in Table 6. As anticipated based on Theorem 4, the solutions coincide practically with the estimates obtained by the Gröbner basis approach applied in [22].



(a) First-order optimality condition equations (b) Objective function

Fig. 9. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a nonlinear oscillator with a bimodal response PDF with $a = 1.5$ and $\varepsilon = 1$ using $L = 2$ trial functions $\left(x\left(t_f = 1\right) = 0.8, \dot{x}\left(t_f = 1\right) = 0.9\right)$.



(a) First-order optimality condition equations (b) Objective function

Fig. 10. First-order optimality condition equations and objective function of the most probable path optimization problem corresponding to a nonlinear oscillator with a bimodal response PDF with $a = 1.8$ and $\varepsilon = 1$ using $L = 2$ trial functions $\left(x\left(t_f = 1\right) = 0.8, \dot{x}\left(t_f = 1\right) = 0.9\right)$.

Table 6
Optimal solution and objective function values for a nonlinear oscillator with a bimodal response PDF under white noise.

a	$\boldsymbol{z}^*$	$S(\boldsymbol{z}^*)$
$a = 1.3$	$(-0.0209, -0.0513)$	4.6669
$a = 1.5$	$(-0.0064, -0.0573)$	4.5140
$a = 1.8$	$(\ 0.0130, -0.0667)$	4.3156

5. Concluding remarks

Various techniques have been developed in this paper for addressing the existence, uniqueness and numerical calculation of WPI most probable path solutions. Specifically, for the first time in the literature, results have been obtained regarding the existence and uniqueness of solutions pertaining to the most probable path BVP described by the coupled system of multivariate polynomial equations shown in Eq. (18). To elaborate further, first, it has been shown that for the commonly considered case of the system nonlinearity being of polynomial form, there exist globally optimal solutions corresponding to the most probable path optimization problem. Second, relying on algebraic geometry concepts and tools, a condition has been derived for determining if the BVP for the most probable path exhibits a unique solution over a specific region.

Furthermore, a novel approach based on Sylvester's dialytic method of elimination has been developed for calculating numerically the most probable paths. The rationale of the method relates to reducing the complexity of the nL system of coupled multivariate polynomial equations described by Eq. (18). In fact, the computational cost associated with solving numerically Eq. (18), by applying, indicatively, the Newton's iterative scheme developed in [22], becomes non-trivial for an increasing number of unknowns ($p = nL$) when higher-dimensional n-DOF systems are considered. Remarkably, it has been shown that the proposed method circumvents the above challenge by eliminating one or more variables successively, and thus, yielding nL univariate polynomial equations to be solved independently. Notably, it has also been proved that both the Gröbner basis approach employed in [22] and the herein proposed dialytic method yield the same set of solutions for the BVP. Various numerical examples pertaining to diverse nonlinear oscillators have been considered for demonstrating the capabilities of the developed techniques.

CRediT authorship contribution statement

Asela Nawagamuwage: Conceptualization, Methodology, Software, Writing – original draft, Visualization, Validation. **Ioannis A. Kougioumtzoglou:** Conceptualization, Methodology, Writing – review & editing, Supervision, Project administration, Funding acquisition. **Athanasios A. Pantelous:** Conceptualization, Methodology, Writing – review & editing, Supervision, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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