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Uniform decomposition of velocity gradient tensor

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Abstract: In this paper, the principal decomposition of the velocity gradient tensor $[\nabla \boldsymbol{v}]$ is discussed in 3 cases based on the discriminant Δ : $\Delta < 0$ with 1 real eigen value and a pair of conjugate complex eigen values, $\Delta > 0$ with 3 distinct real eigen values, and $\Delta = 0$ with 1 or 2 distinct real eigen values. The velocity gradient tensor can also be classified as rotation point, which can be decomposed into three parts, i.e., rotation $[R]$, shear $[S]$ and stretching/compression $[SC]$, and non-rotation point, we defined a new resistance term $[L]$, and the tensor can be decomposed into three parts, i.e., resistance $[L]$, shear $[S]$ and stretching/compression $[SC]$. Example matric are also displayed to demonstrate the new decomposition. Connections of principal decomposition between 3 different cases, and between Resistance and Liutex will also be discussed.

Key words: Velocity gradient, Liutex, resistance, fluid kinematics

0. Introduction

The velocity gradient tensor, or velocity gradient matrix, $(\nabla \boldsymbol{v})$, is related to the convection term in the acceleration of a particle, which can be derived from material derivative formula $\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x}u + \frac{\partial p}{\partial y}v + \frac{\partial p}{\partial z}w$, by applying it on each component of the velocity $\boldsymbol{v} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ as follows

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \begin{bmatrix} \frac{\mathrm{D}u}{\mathrm{D}t} \\ \frac{\mathrm{D}v}{\mathrm{D}t} \\ \frac{\mathrm{D}w}{\mathrm{D}t} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w \\ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w \\ \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w \\ \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w \\ \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial t} \end{bmatrix} +$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v})\boldsymbol{v} \tag{1}$$

where $[\nabla \boldsymbol{v}] = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$ is the velocity gradient tensor, and the acceleration $\boldsymbol{a}$ can be expressed as: $\boldsymbol{a} = \mathrm{D}\boldsymbol{v} / \mathrm{D}t = \partial \boldsymbol{v} / \partial t + (\nabla \boldsymbol{v})\boldsymbol{v}$.

The traditional method for the decomposition of

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velocity gradient tensor $\nabla \boldsymbol{v}$ is first given by Stocks[1] as Cauchy-Stocks decomposition in 1845, which says the velocity gradient matrix can be decomposed uniquely into a symmetric matrix $\boldsymbol{A}$ as strain, and an anti-symmetric matrix $\boldsymbol{B}$ as vorticity, which is used to represent rotation by Helmholtz[2]

$$\nabla \boldsymbol{v} = \boldsymbol{A} + \boldsymbol{B} \tag{2}$$

where $\boldsymbol{A} = (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}})/2$, $\boldsymbol{B} = (\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^{\mathrm{T}})/2$.

Since then, people start to interpret the vorticity tensor $\boldsymbol{B}$ as rotation, until Liutex was discovered in 2018 by Liu et al.[3-6], which points out that vorticity is not rotation, but the combination of rotation and shear, and since then, the rotation part is also called and defined as Liutex. Under the Liutex based principal decomposition, the velocity gradient tensor for rotational points can be decomposed into 3 parts: Stretching/compression, rotation, and shear, which also satisfies the requirements made by Kolář[7]. However, Liu's work is only valid and defined for rotational points but did not consider the non-rotational points. For non-rotational points, the most popular method is Shur's decomposition[8-10], but unfortunately, Shur's decomposition is not unique, for 3×3 matrix, we can find up to 6 decompositions.

Here we give several requirements for decomposition of velocity gradient tensor. First, the velocity gradient decomposition should be unique and defined for all cases. Second, the principal matrix of all cases should be related to and be able to smoothly transfer to each other under certain conditions. In this paper, the method we provide satisfied both requirements, and non-rotational points can be transferred into rotational points when imposed by a disturbance of a rotational antisymmetric matrix if it is strong enough.

To create the smooth transition between rotational and nonrotational points, and to explain why particles can bear an anti-symmetric shear but not having rotation. we defined a new fluid sub-tensor for non-rotational points, which is a symmetric matrix called resistance, $[L]$. Once the non-rotational point is imposed by a rotation term, an anti-symmetric matrix, the resistance will be consumed and be combined into shear. Once the resistance is depleted, it will be transferred from case 3 into case 2. Therefore, the resistance is also the minimum amount of rotation we need to transfer it into case 2. We will choose the eigenvector, along which the resistance is the weakest, as our principal axis $\boldsymbol{r}$, because in that direction, the rotation is most likely to take place, due to the bucket effect.

In this paper, we continue to use the Liutex based principal decomposition method for rotational points and expand it into non-rotational points by using the newly defined resistance. The resistance tensor can be treated as the extension of rotation tensor, and the magnitude of the resistance can also be viewed as an extension from rotational points to non-rotational points.

1. Principal decomposition[11]

The principal decomposition will be classified and discussed in 3 cases according to its discriminant Δ, which determines the types of eigenvalues of the matrix. Case 1: When $\Delta < 0$, we have 1 real eigenvalue λ_r and 2 complex conjugate eigenvalues $\lambda_{cr} \pm \mathrm{i}\lambda_{ci}$. Case 2: When $\Delta = 0$, we have repeated real eigenvalues λ_1, λ_2 or just λ. Case 3: When $\Delta > 0$, we have 3 distinct real eigenvalues $\lambda_1, \lambda_2, \lambda_3$. For $\nabla \boldsymbol{v}$, it will be referred to as non-rotational points if $\Delta \geq 0$ (Cases 2, 3), and it will be referred to as rotational points if $\Delta < 0$ (Case 1).

We use $\nabla \boldsymbol{V}$ to denote the velocity gradient under principal coordinate system while $\nabla \boldsymbol{v}$ is defined as the velocity gradient in the original xyz-coordinate system, and we can transform between 2 coordinate systems as follows

$$\nabla \boldsymbol{V} = \mathcal{M}(\nabla \boldsymbol{v}, \{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}\}) = \boldsymbol{U}^{\mathrm{T}} \nabla \boldsymbol{v} \boldsymbol{U}, \quad \nabla \boldsymbol{v} = \boldsymbol{U} \nabla \boldsymbol{V} \boldsymbol{U}^{\mathrm{T}} \tag{3}$$

where $\boldsymbol{U}$ is an orthogonal coordinate transform matrix, composed of 3 orthonormal vectors, $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$, $\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}}$ forms the identity matrix $\boldsymbol{I}$, and its determinant is 1.

$$\boldsymbol{U} = [\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}] = \begin{bmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ p_z & q_z & r_z \end{bmatrix}, \quad \boldsymbol{U}^{-1} = \boldsymbol{U}^{\mathrm{T}}, \quad \det(\boldsymbol{U}) = 1 \tag{4}$$

After we transform $\nabla \boldsymbol{v}$ into pqr-principal system, we can decompose it easily into 3 terms: Stretching/compression $[SC]$, rotation/resistance $[R/L]$ and shear $[S]$. Then we can bring each term back into the original xyz-system as follows:

$$\nabla \boldsymbol{V} = [SC] + \begin{bmatrix} R \\ L \end{bmatrix} + [S] \Rightarrow [SC]_{xyz} = \boldsymbol{U}[SC]\boldsymbol{U}^{\mathrm{T}} \Rightarrow$$

$$\nabla \boldsymbol{v} = [SC]_{xyz} + \begin{bmatrix} R \\ L \end{bmatrix}_{xyz} + [S]_{xyz} \tag{5a}$$

$$\nabla \boldsymbol{V} = [SC] + \begin{bmatrix} R \\ L \end{bmatrix} + [S] \Rightarrow \begin{bmatrix} R \\ L \end{bmatrix}_{xyz} = \boldsymbol{U}\begin{bmatrix} R \\ L \end{bmatrix}\boldsymbol{U}^{\mathrm{T}} \Rightarrow$$

$$\nabla \boldsymbol{v} = [SC]_{xyz} + \begin{bmatrix} R \\ L \end{bmatrix}_{xyz} + [S]_{xyz} \tag{5b}$$

$$\nabla \boldsymbol{V} = [SC] + \begin{bmatrix} R \\ L \end{bmatrix} + [S] \Rightarrow [S]_{xyz} = \boldsymbol{U}[S]\boldsymbol{U}^{\mathrm{T}} \Rightarrow$$

$$\nabla \boldsymbol{v} = [SC]_{xyz} + \begin{bmatrix} R \\ L \end{bmatrix}_{xyz} + [S]_{xyz} \tag{5c}$$

1.1 *Case 1*

In principal coordinate system, we have principal matrix

$$\nabla \boldsymbol{V} = \begin{bmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ p_z & q_z & r_z \end{bmatrix} =$$

$$\begin{bmatrix} \lambda_{cr} & -\frac{R}{2} & 0 \\ \frac{R}{2} + \varepsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix} \tag{6}$$

where $\boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$ is the unit eigenvector of $\nabla \boldsymbol{v}$ of λ_r, such that $\boldsymbol{\omega} \cdot \boldsymbol{r} > 0$, and $R = \boldsymbol{\omega} \cdot \boldsymbol{r} - \sqrt{(\boldsymbol{\omega} \cdot \boldsymbol{r})^2 - 4\lambda_{ci}^2}$ is the magnitude of Liutex[12]. Though this formula has been proved before, here we provide a proof slightly different, which will be used again later.

First, we can show that $\nabla \boldsymbol{V} - \nabla \boldsymbol{V}^{\mathrm{T}} = \boldsymbol{U}^{\mathrm{T}}(\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^{\mathrm{T}})\boldsymbol{U}$ as follows

$$\nabla \boldsymbol{V} - \nabla \boldsymbol{V}^{\mathrm{T}} = \boldsymbol{U}^{\mathrm{T}} \nabla \boldsymbol{v} \boldsymbol{U} - (\boldsymbol{U}^{\mathrm{T}} \nabla \boldsymbol{v} \boldsymbol{U})^{\mathrm{T}} =$$

$$\boldsymbol{U}^{\mathrm{T}} \nabla \boldsymbol{v} \boldsymbol{U} - \boldsymbol{U}^{\mathrm{T}} \nabla \boldsymbol{v}^{\mathrm{T}} \boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}} (\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^{\mathrm{T}}) \boldsymbol{U} \tag{7}$$

Then we plug in $\nabla \boldsymbol{v}$ and $\nabla \boldsymbol{V}$, and get the following:

$$\begin{bmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{bmatrix} \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ p_z & q_z & r_z \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -\varepsilon - R & -\xi \\ R + \varepsilon & 0 & -\eta \\ \xi & \eta & 0 \end{bmatrix} \tag{8}$$

$$R + \varepsilon = p_x q_y \omega_z - p_x q_z \omega_y - p_y q_x \omega_z + p_y q_z \omega_x +$$

$$p_z q_x \omega_y - p_z q_y \omega_x = \omega_x \begin{vmatrix} p_y & p_z \\ q_y & q_z \end{vmatrix} - \omega_y \begin{vmatrix} p_x & p_z \\ q_x & q_z \end{vmatrix} +$$

$$\omega_z \begin{vmatrix} p_x & p_y \\ q_x & q_y \end{vmatrix} = \omega_x r_x + \omega_y r_y + \omega_z r_z = \boldsymbol{\omega} \cdot \boldsymbol{r} \tag{9}$$

$$\xi = p_x r_y \omega_z - p_x r_z \omega_y - p_y r_x \omega_z + p_y r_z \omega_x + p_z r_x \omega_y -$$

$$p_z r_y \omega_x = -\boldsymbol{\omega} \cdot \boldsymbol{q} \tag{10}$$

$$\eta = q_x r_y \omega_z - q_x r_z \omega_y - q_y r_x \omega_z + q_y r_z \omega_x + q_z r_x \omega_y -$$

$$q_z r_y \omega_x = \boldsymbol{\omega} \cdot \boldsymbol{p} \tag{11}$$

Since $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ are coordinate vectors, we can get one of the most important equations in Liutex theory, $\boldsymbol{\omega} = \boldsymbol{S} + \boldsymbol{R}$

$$\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{p})\boldsymbol{p} + (\boldsymbol{\omega} \cdot \boldsymbol{q})\boldsymbol{q} + (\boldsymbol{\omega} \cdot \boldsymbol{r})\boldsymbol{r} = \eta \boldsymbol{p} -$$

$$\xi \boldsymbol{q} + \varepsilon \boldsymbol{r} + R\boldsymbol{r} = \boldsymbol{S} + \boldsymbol{R} \tag{12}$$

By $R + \varepsilon = \boldsymbol{\omega} \cdot \boldsymbol{r}$, we can also get $\varepsilon + R/2 = \boldsymbol{\omega} \cdot \boldsymbol{r} - R/2$, and thus $\nabla \boldsymbol{V} = \begin{bmatrix} \lambda_{cr} & -\frac{R}{2} & 0 \\ \boldsymbol{\omega} \cdot \boldsymbol{r} - \frac{R}{2} & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix}$.

Since orthogonal transformation does not affect the eigenvalues, we thus have

$$\det(\nabla \boldsymbol{V}) = \det(\nabla \boldsymbol{v}) \Rightarrow \lambda_r \left[\lambda_{cr}^2 - \left(-\frac{R}{2} \right) \left(\boldsymbol{\omega} \cdot \boldsymbol{r} - \frac{R}{2} \right) \right] =$$

$$\lambda_r (\lambda_{cr} + \mathrm{i}\lambda_{ci})(\lambda_{cr} - \mathrm{i}\lambda_{ci}) \tag{13}$$

and after simplification, we have

$$\left(\frac{R}{2}\right)^2-(\boldsymbol{\omega}\cdot\boldsymbol{r})\frac{R}{2}+\lambda_{ci}^2=0\Rightarrow$$

$$\frac{R}{2}=\frac{(\boldsymbol{\omega}\cdot\boldsymbol{r})\pm\sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{r})^2-4\lambda_{ci}^2}}{2} \tag{14}$$

If there is no rotation when $\lambda_{ci}=0$, we have

$$R=\boldsymbol{\omega}\cdot\boldsymbol{r}-\sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{r})^2-4\lambda_{ci}^2} \tag{15}$$

The principal decomposition can be written as

$$\begin{bmatrix}\lambda_{cr} & -\frac{R}{2} & 0\\ \frac{R}{2}+\varepsilon & \lambda_{cr} & 0\\ \xi & \eta & \lambda_r\end{bmatrix}=[SC]+[R]+[S]=$$

$$\begin{bmatrix}\lambda_{cr} & 0 & 0\\ 0 & \lambda_{cr} & 0\\ 0 & 0 & \lambda_r\end{bmatrix}+\begin{bmatrix}0 & -\frac{R}{2} & 0\\ \frac{R}{2} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ \varepsilon & 0 & 0\\ \xi & \eta & 0\end{bmatrix} \tag{16}$$

where $[SC]=\begin{bmatrix}\lambda_{cr} & 0 & 0\\ 0 & \lambda_{cr} & 0\\ 0 & 0 & \lambda_r\end{bmatrix}$, $[R]=\begin{bmatrix}0 & -\frac{R}{2} & 0\\ \frac{R}{2} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$,

$[R]=\begin{bmatrix}0 & -\frac{R}{2} & 0\\ \frac{R}{2} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ represents stretching/compression, rotation and shear, and they can be converted back into xyz-system as follows:

$$[SC]_{xyz}=\boldsymbol{U}\left\{\begin{bmatrix}\lambda_{cr} & 0 & 0\\ 0 & \lambda_{cr} & 0\\ 0 & 0 & \lambda_r\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \lambda_r-\lambda_{cr}\end{bmatrix}\right\}\boldsymbol{U}^{\mathrm{T}}=$$

$$\lambda_{cr}\boldsymbol{I}+(\lambda_r-\lambda_{cr})[\boldsymbol{r}\boldsymbol{r}^{\mathrm{T}}] \tag{17}$$

$$[R]_{xyz}=\boldsymbol{U}\begin{bmatrix}0 & -\frac{R}{2} & 0\\ \frac{R}{2} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\boldsymbol{U}^{\mathrm{T}}=\frac{1}{2}\begin{bmatrix}0 & -R_z & R_y\\ R_z & 0 & -R_x\\ -R_y & R_x & 0\end{bmatrix} \tag{18}$$

where $\begin{bmatrix}R_x\\ R_y\\ R_z\end{bmatrix}=\boldsymbol{R}$ is the Liutex vector.

$$[S]_{xyz}=\nabla\boldsymbol{v}-[R]_{xyz}-[SC]_{xyz}=\nabla\boldsymbol{v}-\lambda_{cr}\boldsymbol{I}-$$

$$(\lambda_r-\lambda_{cr})[\boldsymbol{r}\boldsymbol{r}^{\mathrm{T}}]-\frac{1}{2}\begin{bmatrix}0 & -R_z & R_y\\ R_z & 0 & -R_x\\ -R_y & R_x & 0\end{bmatrix} \tag{19}$$

1.2 *Case 2*

For $\Delta=0$ cases, we only have 1 or 2 distinct eigenvalues. First, suppose we have 2 distinct eigenvalues λ_1 and λ_2, the dimension of their generalized eigen space is 2 and 1, $\boldsymbol{r}$ is a unit eigenvector of $\nabla\boldsymbol{v}$ with eigenvalue λ_2, such that $\boldsymbol{\omega}\cdot\boldsymbol{r}>0$, then we have the principal matrix:

$$\nabla\boldsymbol{V}=\begin{bmatrix}U_{11} & U_{12} & U_{13}\\ U_{21} & U_{22} & U_{23}\\ r_x & r_y & r_z\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{bmatrix}.$$

$$\begin{bmatrix}U_{11} & U_{21} & r_x\\ U_{12} & U_{22} & r_y\\ U_{13} & U_{23} & r_z\end{bmatrix}=\begin{bmatrix}\lambda_1 & 0 & 0\\ \varepsilon & \lambda_1 & 0\\ \xi & \eta & \lambda_2\end{bmatrix}=$$

$$\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ \varepsilon & 0 & 0\\ \xi & \eta & 0\end{bmatrix}=[SC]+[S] \tag{20}$$

where $[SC]=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{bmatrix}$ is stretching/compression,

$[S]=\begin{bmatrix}0&0&0\\ \varepsilon&0&0\\ \xi&\eta&0\end{bmatrix}$ is shear. And we can change it back to the original coordinates as:

$$[SC]_{xyz}=\boldsymbol{U}\begin{bmatrix}\lambda_1&0&0\\0&\lambda_1&0\\0&0&\lambda_2\end{bmatrix}\boldsymbol{U}^{\mathrm{T}}=\lambda_1\boldsymbol{I}+(\lambda_2-\lambda_1)\cdot$$

$$\begin{bmatrix}r_x^2&r_xr_y&r_xr_z\\ r_yr_x&r_y^2&r_yr_z\\ r_zr_x&r_zr_y&r_z^2\end{bmatrix} \tag{21}$$

$$[S]_{xyz}=[\nabla\boldsymbol{v}]-[SC]_{xyz}=[\nabla\boldsymbol{v}]-\lambda_1\boldsymbol{I}-(\lambda_2-\lambda_1)\cdot$$

$$\begin{bmatrix}r_x^2&r_xr_y&r_xr_z\\ r_yr_x&r_y^2&r_yr_z\\ r_zr_x&r_zr_y&r_z^2\end{bmatrix} \tag{22}$$

Secondly, suppose we only have 1 eigenvalue λ, then $[SC]_{xyz}$ and $[S]_{xyz}$ can be found as follows:

$$[SC]_{xyz}=\boldsymbol{U}\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\boldsymbol{U}^{\mathrm{T}}=\lambda\boldsymbol{I}\,,$$

$$[S]_{xyz}=\nabla\boldsymbol{v}-[SC]_{xyz}=\nabla\boldsymbol{v}-\lambda\boldsymbol{I} \tag{23}$$

1.3 *Case 3*

For $\Delta>0$ cases, we have 3 distinct real eigenvalues. Let $\boldsymbol{r}$ is a unit eigenvector of $\nabla\boldsymbol{v}$ with eigenvalue λ_3, such that $\boldsymbol{\omega}\cdot\boldsymbol{r}>0$, then we have the principal matrix

$$\nabla\boldsymbol{V}=\begin{bmatrix}p_x&p_y&p_z\\q_x&q_y&q_z\\r_x&r_y&r_z\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}&\frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}&\frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x}&\frac{\partial w}{\partial y}&\frac{\partial w}{\partial z}\end{bmatrix}\begin{bmatrix}p_x&q_x&r_x\\p_y&q_y&r_y\\p_z&q_z&r_z\end{bmatrix}=$$

$$\begin{bmatrix}\frac{\lambda_1+\lambda_2}{2}&\frac{L}{2}&0\\ \varepsilon+\frac{L}{2}&\frac{\lambda_1+\lambda_2}{2}&0\\ \xi&\eta&\lambda_3\end{bmatrix}=\begin{bmatrix}\frac{\lambda_1+\lambda_2}{2}&0&0\\0&\frac{\lambda_1+\lambda_2}{2}&0\\0&0&\lambda_3\end{bmatrix}+$$

$$\begin{bmatrix}0&\frac{L}{2}&0\\ \frac{L}{2}&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\ \varepsilon&0&0\\ \xi&\eta&0\end{bmatrix} \tag{24}$$

where $L=\sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{r})^2+(\lambda_1-\lambda_2)^2}-\boldsymbol{\omega}\cdot\boldsymbol{r}$, which can be proved in the same way as follows:

$$\begin{bmatrix}p_x&p_y&p_z\\q_x&q_y&q_z\\r_x&r_y&r_z\end{bmatrix}\begin{bmatrix}0&-\omega_z&\omega_y\\ \omega_z&0&-\omega_x\\ -\omega_y&\omega_x&0\end{bmatrix}\begin{bmatrix}p_x&q_x&r_x\\p_y&q_y&r_y\\p_z&q_z&r_z\end{bmatrix}=$$

$$\begin{bmatrix}0&-\varepsilon&-\xi\\ \varepsilon&0&-\eta\\ \xi&\eta&0\end{bmatrix}\Rightarrow\varepsilon=\boldsymbol{\omega}\cdot\boldsymbol{r}\,,\quad\xi=-\boldsymbol{\omega}\cdot\boldsymbol{q}\,,\quad\eta=\boldsymbol{\omega}\cdot\boldsymbol{p} \tag{25}$$

$$\lambda_1\lambda_2\lambda_3=\det(\nabla\boldsymbol{V})=\lambda_3\left[\left(\frac{\lambda_1+\lambda_2}{2}\right)^2-\frac{L}{2}\left(\varepsilon+\frac{L}{2}\right)\right]\Rightarrow$$

$$\left(\frac{L}{2}\right)^2+\varepsilon\frac{L}{2}-\left(\frac{\lambda_1-\lambda_2}{2}\right)^2=0\Rightarrow$$

$$\frac{L}{2}=\frac{-\varepsilon\pm\sqrt{\varepsilon^2+(\lambda_1-\lambda_2)^2}}{2}\Rightarrow$$

$$L=\sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{r})^2+(\lambda_1-\lambda_2)^2}-\boldsymbol{\omega}\cdot\boldsymbol{r} \tag{26}$$

To calculate each term, first we find the principal axis, which is defined by the lowest resistance, where $\boldsymbol{v}_i$ is the eigenvector of λ_i, λ_j, λ_k are the other two eigenvalues.

$$L_i=\sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{v}_i)^2+(\lambda_j-\lambda_k)^2}-\boldsymbol{\omega}\cdot\boldsymbol{v}_i\,,\quad\lambda_3=\lambda_i\,,\quad\boldsymbol{r}=\boldsymbol{v}_i \tag{27}$$

where $L_i=\min\{L_i,L_j,L_k\}$.

Secondly, we can find $\boldsymbol{p}$, $\boldsymbol{q}$ as follows: Let $N(\boldsymbol{v})$ be a normalizing function $N(\boldsymbol{v})=\boldsymbol{v}/\|\boldsymbol{v}\|$, and the projection of the other 2 eigenvectors on $\boldsymbol{r}^{\perp}$ plain can be calculated as $\boldsymbol{v}_i'=\boldsymbol{v}_i-(\boldsymbol{v}_i\cdot\boldsymbol{r})\boldsymbol{r}$, then $\boldsymbol{p}$, $\boldsymbol{q}$ can be found as their bisector

$$\boldsymbol{q}_0 = N[N(\boldsymbol{v}_1') + N(\boldsymbol{v}_2')], \quad \boldsymbol{p}_0 = \boldsymbol{q}_0 \times \boldsymbol{r}$$

If $\nabla \boldsymbol{v}(\boldsymbol{q}_0) \cdot \boldsymbol{p}_0 \neq \frac{L}{2}$ then $\boldsymbol{p} = -\boldsymbol{q}_0$, $\boldsymbol{q} = \boldsymbol{p}_0$,

otherwise $\boldsymbol{p} = \boldsymbol{p}_0$, $\boldsymbol{q} = \boldsymbol{q}_0$ (28)

Finally, we can obtain $[SC]_{xyz}$, $[L]_{xyz}$, $[S]_{xyz}$ as follows:

$$[SC]_{xyz} = \boldsymbol{U}\left\{\frac{\lambda_1+\lambda_2}{2}\boldsymbol{I} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3 - \frac{\lambda_1+\lambda_2}{2} \end{bmatrix}\right\}\boldsymbol{U}^{\mathrm{T}} =$$

$$\frac{\lambda_1+\lambda_2}{2}\boldsymbol{I} + \left(\lambda_3 - \frac{\lambda_1+\lambda_2}{2}\right)[\boldsymbol{r}\boldsymbol{r}^{\mathrm{T}}] \quad (29)$$

$$[L]_{xyz} = \boldsymbol{U}\begin{bmatrix} 0 & \frac{L}{2} & 0 \\ \frac{L}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{U}^{\mathrm{T}} = \frac{L}{2}([\boldsymbol{q}\boldsymbol{p}^{\mathrm{T}}] + [\boldsymbol{p}\boldsymbol{q}^{\mathrm{T}}]) \quad (30)$$

$$[S]_{xyz} = \nabla \boldsymbol{v} - [SC] - [L] \quad (31)$$

1.4 *Examples*

Here we provide an example matrix in case 3 and its decomposition:

Let $\nabla \boldsymbol{v} = \begin{bmatrix} 2.5 & 1.0 & 0.5 \\ -1.1 & 0.4 & -1.3 \\ 0.7 & 0.2 & 3.1 \end{bmatrix}$, then vorticity $\boldsymbol{\omega} = \begin{bmatrix} 1.5 \\ -0.2 \\ -2.1 \end{bmatrix}$, eigenvalues are 1, 2, 3, and the eigenvectors are $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-7}{5\sqrt{3}} \\ \frac{-1}{5\sqrt{3}} \end{bmatrix}$, $\begin{bmatrix} \frac{5}{\sqrt{35}} \\ \frac{-1}{\sqrt{35}} \\ \frac{-3}{\sqrt{35}} \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$, the resistance are 0.34641016151377513, 0.7319547637260868 and 0.26053577119208793, thus we take $\lambda_3 = 3$, and $\boldsymbol{r} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$. Then

$$\boldsymbol{v}_1' = \boldsymbol{v}_1 - (\boldsymbol{v}_1 \cdot \boldsymbol{r})\boldsymbol{r} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-6}{\sqrt{75}} \\ \frac{-3}{\sqrt{75}} \end{bmatrix},$$

$$\boldsymbol{v}_2' = \boldsymbol{v}_2 - (\boldsymbol{v}_2 \cdot \boldsymbol{r})\boldsymbol{r} = \begin{bmatrix} \frac{5}{\sqrt{35}} \\ \frac{-2}{\sqrt{35}} \\ \frac{-1}{\sqrt{35}} \end{bmatrix} \quad (32)$$

and thus

$$\boldsymbol{q}_0 = N[N(\boldsymbol{v}_1') + N(\boldsymbol{v}_2')] = \begin{bmatrix} 0.78045432 \\ -0.55920734 \\ -0.27960367 \end{bmatrix},$$

$$\boldsymbol{p}_0 = \boldsymbol{q}_0 \times \boldsymbol{r} = \begin{bmatrix} 0.62521281 \\ 0.69805956 \\ 0.34902978 \end{bmatrix} \quad (33)$$

Since $\nabla \boldsymbol{v}(\boldsymbol{q}_0) \cdot \boldsymbol{p}_0 = L/2 = 0.13026788559604374$, then $\boldsymbol{p} = \boldsymbol{p}_0$, $\boldsymbol{q} = \boldsymbol{q}_0$, and we have the following:

$$[SC]_{xyz} = \frac{\lambda_1+\lambda_2}{2}\boldsymbol{I} + \left(\lambda_3 - \frac{\lambda_1+\lambda_2}{2}\right)[\boldsymbol{r}\boldsymbol{r}^{\mathrm{T}}] =$$

$$\begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.8 & -0.6 \\ 0 & -0.6 & 2.7 \end{bmatrix} \quad (34)$$

$$[L]_{xyz} = \frac{L}{2}([\boldsymbol{q}\boldsymbol{p}^{\mathrm{T}}] + [\boldsymbol{p}\boldsymbol{q}^{\mathrm{T}}]) =$$

$$\begin{bmatrix} 0.12712844 & 0.02542569 & 0.01271284 \\ 0.02542569 & -0.10170275 & -0.05085138 \\ 0.01271284 & -0.05085138 & -0.02542569 \end{bmatrix} \quad (35)$$

$$[S]_{xyz} = \nabla \boldsymbol{v} - [SC] - [L] =$$

$$\begin{bmatrix} 0.87287156 & 0.97457431 & 0.48728716 \\ -1.12542569 & -1.29829725 & -0.64914862 \\ 0.68728716 & 0.85085138 & 0.42542569 \end{bmatrix} \tag{36}$$

$$([S]_{xyz})^3 =$$

$$\begin{bmatrix} -1.2373898\times10^{-16} & 1.14343214\times10^{-17} & -1.11890087\times10^{-17} \\ 1.70240278\times10^{-16} & -1.94917722\times10^{-19} & 1.54855144\times10^{-17} \\ -1.16261382\times10^{-16} & -4.14264310\times10^{-18} & -9.89234621\times10^{-18} \end{bmatrix} \tag{37}$$

Since $([S]_{xyz})^3 = [0]$, thus the shear is indeed nilpotent.

2. Uniform decomposition

2.1 *The resistance to rotation*

We can use the resistance to explain why non-rotational points can remain non-rotational once imposed by an anti-symmetric matrix $\begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, for some $a \in (0, L/2)$

$$\begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{L}{2} & 0 \\ \varepsilon+\frac{L}{2} & \frac{\lambda_1+\lambda_2}{2} & 0 \\ \xi & \eta & \lambda_3 \end{bmatrix} + \begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{L}{2}-a & 0 \\ \varepsilon+\frac{L}{2}+a & \frac{\lambda_1+\lambda_2}{2} & 0 \\ \xi & \eta & \lambda_3 \end{bmatrix} \tag{38}$$

Then the resistance will be consumed into $L' = L - 2a$, and the shear on $\boldsymbol{r}$ will increases to $\varepsilon' = \varepsilon + 2a$, hence the principal matrix become

$$\begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{L}{2}-a & 0 \\ \varepsilon+\frac{L}{2}+a & \frac{\lambda_1+\lambda_2}{2} & 0 \\ \xi & \eta & \lambda_3 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{L'}{2} & 0 \\ \varepsilon'+\frac{L'}{2} & \frac{\lambda_1+\lambda_2}{2} & 0 \\ \xi & \eta & \lambda_3 \end{bmatrix} \tag{39}$$

2.2 *From non-rotation to rotation*

Now suppose we impose a larger anti-symmetric matrix $\begin{bmatrix} 0 & -\frac{L}{2} & 0 \\ \frac{L}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ on it

$$\begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{L}{2} & 0 \\ \varepsilon+\frac{L}{2} & \frac{\lambda_1+\lambda_2}{2} & 0 \\ \xi & \eta & \lambda_3 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{L}{2} & 0 \\ \frac{L}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & 0 & 0 \\ \varepsilon+L & \frac{\lambda_1+\lambda_2}{2} & 0 \\ \xi & \eta & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_2' & 0 & 0 \\ \varepsilon' & \lambda_2' & 0 \\ \xi & \eta & \lambda_1 \end{bmatrix} \tag{40}$$

Then the resistance is depleted, it will be transferred from case 3 into case 2, with $\lambda_2' = (\lambda_1+\lambda_2)/2$, $\varepsilon' = \varepsilon + L$.

Now suppose we impose any anti-symmetric matrix $\begin{bmatrix} 0 & -\frac{R}{2} & 0 \\ \frac{R}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ on it, then the resistance is depleted, it will be converted from case 2 into case 1, with $\lambda_r = \lambda_1$, $\lambda_{cr} = \lambda_2$

$$\begin{bmatrix} \lambda_2 & 0 & 0 \\ \varepsilon & \lambda_2 & 0 \\ \xi & \eta & \lambda_1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{R}{2} & 0 \\ \frac{R}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_2 & -\frac{R}{2} & 0 \\ \varepsilon+\frac{R}{2} & \lambda_2 & 0 \\ \xi & \eta & \lambda_1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} \lambda_{cr} & -\frac{R}{2} & 0 \\ \frac{R}{2}+\varepsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix} \tag{41}$$

2.3 *Resistance and Liutex*

We can use the formula for negative resistance $-L$ for Liutex magnitude. Let $\lambda_{1,2} = \lambda_{cr} \pm \mathrm{i}\lambda_{ci}$, then we have

$$(\lambda_1 - \lambda_2)^2 = -4\lambda_{ci}^2 \Rightarrow -L = \boldsymbol{\omega}\cdot\boldsymbol{r} - \sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{r})^2 + (\lambda_1 - \lambda_2)^2} = \boldsymbol{\omega}\cdot\boldsymbol{r} - \sqrt{(\boldsymbol{\omega}\cdot\boldsymbol{r})^2 - 4\lambda_{ci}^2} = R \tag{42}$$

Mathematically, we can directly use the formula to demonstrate the relation, and physically, since the resistance is the minimum amount of rotation we need, to transfer it into case 2, a stage about to rotate, and thus we can use negative Liutex to represent it. Therefore, $-L$ can be used as a uniform parameter for both rotation and non-rotation points, and it is Liutex for rotation points, and negative resistance for non-rotational points.

2.4 *DNS results about the generation of resistance*

Here are Figs. 1-4 of $-L$ from DNS data[13], which is presented in the 22nd international conference on parallel computational fluid dynamics (ParCFD 2010) in Kaohsiung City, Taiwan, China, May 17-22, 2010. As we can see from the first figure, the region of resistance (blue) occurs between vortices (red), which shows that resistance could be related to vortices. From the second figure, we can see that at $X = 398$, the vortices are generated due to bottom friction, and the resistance show up in the middle where 2 vortices are close to each other. As X increases, the flow becomes more turbulent with more vortices show up and the distance between them becomes closer. We can find that under this trend, the resistance also grows larger. So, we can make a hypothesis about the generation of resistance here: the resistance is generated between vortices, and due to the different rotational direction of them. To verify this hypothesis, we also labeled some rotational point

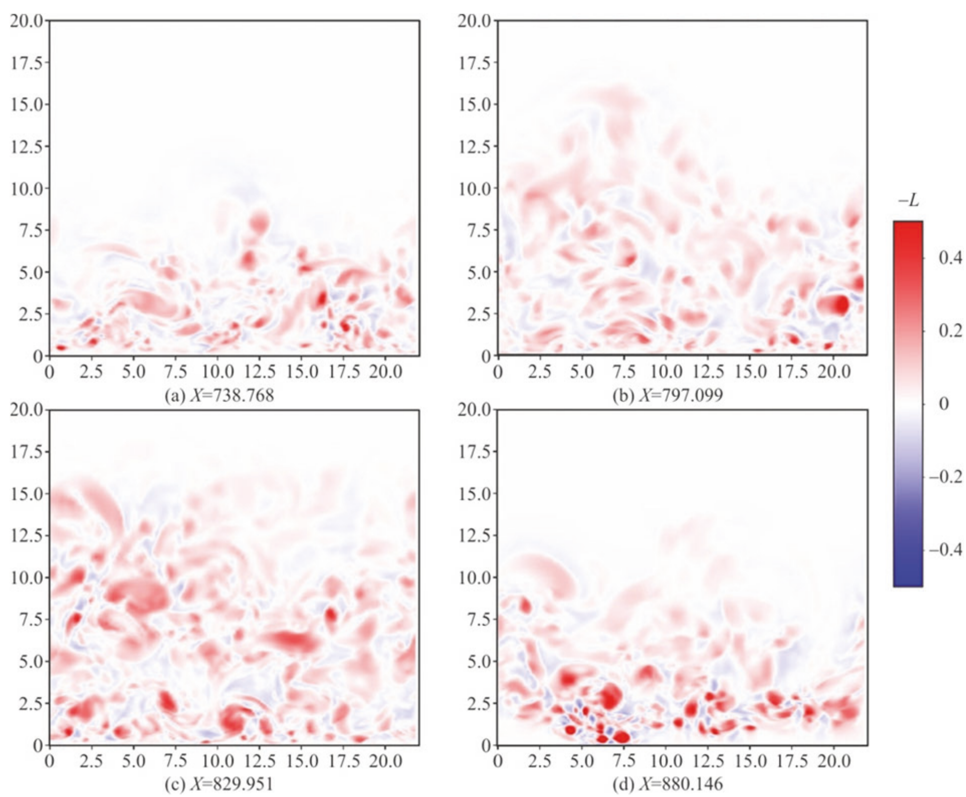

Fig. 1 (Color online) Four plots of $-L$ in turbulent regions, with X (streamwise direction) between 700-900. We only plot from $z=0$ to $z=20$, because $-L$ is near 0 everywhere above $z=20$

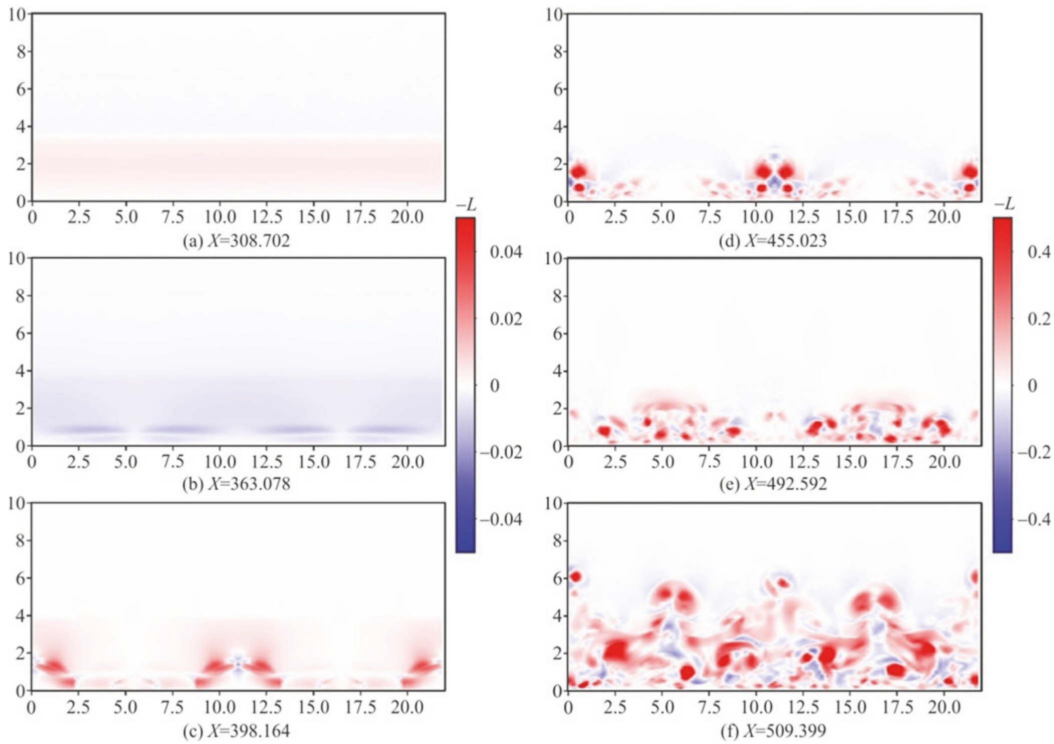

Fig. 2 (Color online) Six plots of $-L$ in the early stage of turbulence development, with X (streamwise direction) between 300-600. We only plot from $z=0$ to $z=10$, because $-L$ is near 0 everywhere above $z=10$

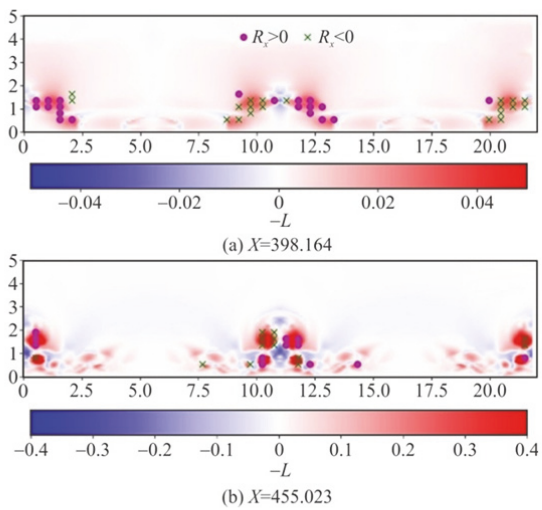

Fig. 3 (Color online) Same plot of $-L$ at $X = 398.164$, 455.023. The rotation direction is marked as magenta dot and green cross, if the R_x is positive or negative

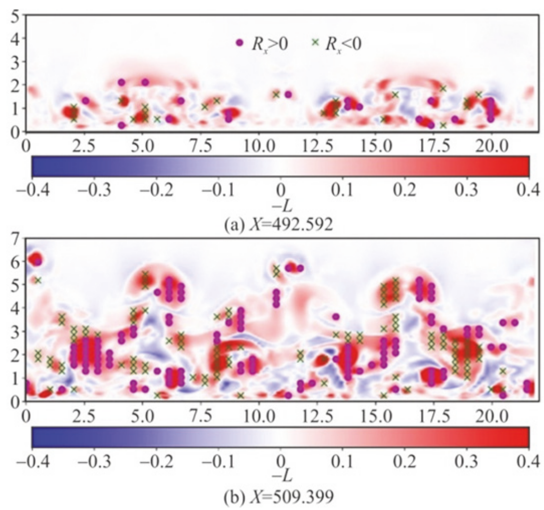

Fig. 4 (Color online) Same plot of $-L$ at $X = 492.592$, 509.399. The rotation direction is marked as magenta dot and green cross, if the R_x is positive or negative

according to the x value of their Liutex vector, R_x. And as we can see from the third figure, the resistance region did occur in the middle, and on both left and right boundaries, between vortices with different rotation directions. However, on the fourth figure, we can find that, as the flow becomes more and more turbulent, this pattern became not that obvious. On one hand, the number of vortices increases, thus some resistance region could be surrounded by 3 or even more vortices, and, on the other hand, the shape of the vortices are not round and perfect anymore.

Because the vortex structure in transitional and turbulent flows are all three-dimensional, the above figures are all two-dimensional and cannot show the accurate full vortex structures, which can just be used to show rough ideas.

3. Conclusions

The velocity gradient tensor in all 3 cases can be decomposed into 3 terms, stretching/compression $[SC]$, rotation/resistance $[R/L]$, and shear $[S]$:

$$\Delta < 0: \begin{bmatrix} \lambda_{cr} & -\frac{R}{2} & 0 \\ \frac{R}{2}+\varepsilon & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix} = [SC]+[R]+[S] =$$

$$\begin{bmatrix} \lambda_{cr} & 0 & 0 \\ 0 & \lambda_{cr} & 0 \\ 0 & 0 & \lambda_r \end{bmatrix} + \begin{bmatrix} 0 & -\frac{R}{2} & 0 \\ \frac{R}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \varepsilon & 0 & 0 \\ \xi & \eta & 0 \end{bmatrix} \tag{43}$$

$$\Delta = 0: \begin{bmatrix} \lambda_1 & 0 & 0 \\ \varepsilon & \lambda_1 & 0 \\ \xi & \eta & \lambda_2 \end{bmatrix} = [SC]+[S] =$$

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \varepsilon & 0 & 0 \\ \xi & \eta & 0 \end{bmatrix} \tag{44}$$

$$\Delta > 0: \begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{L}{2} & 0 \\ \varepsilon+\frac{L}{2} & \frac{\lambda_1+\lambda_2}{2} & 0 \\ \xi & \eta & \lambda_3 \end{bmatrix} = [SC]+[L]+[S] =$$

$$\begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & 0 & 0 \\ 0 & \frac{\lambda_1+\lambda_2}{2} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{L}{2} & 0 \\ \frac{L}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ \varepsilon & 0 & 0 \\ \xi & \eta & 0 \end{bmatrix} \tag{45}$$

For stretching/compression $[SC]$, it is a symmetric matrix with $tr([SC]) = tr(\nabla \boldsymbol{v})$, shear $[S]$ is a nilpotent matrix, and rotation/resistance $[R/L]$ is a symmetric or anti-symmetric matrix with $tr([R/L]) = \det(R/L) = 0$. For rotation points, the $[R/L]$ is an antisymmetric matrix stands for rotation $[R]$, for non-rotation points, the $[R/L]$ is a symmetric matrix stands for resistance $[L]$, and for case 2 points, $[R/L]=[0]$, which is both symmetric and anti-symmetric. Here, $[R/L]$ means R or L.

In this paper, we also defined a uniform parameter $-L$ for all velocity gradient tensor, which has positive values as Liutex magnitude for rotational points and negative values as rotation-resistance for non-rotational points. From the DNS data, the resistance may be generated between vortices by the balance of vortices with different rotation directions.

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Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interest. Chaoqun Liu is editorial board member for the Journal of Hydrodynamics and was not involved in the editorial review, or the decision to publish this article. All authors declare that there are no other competing interests.

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