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String topology in three flavors

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Abstract. We describe two major string topology operations, the Chas–Sullivan product and the Goresky–Hingston coproduct, from geometric and algebraic perspectives. The geometric construction uses Thom–Pontrjagin intersection theory while the algebraic construction is phrased in terms of Hochschild homology. We give computations of products and coproducts on lens spaces via geometric intersection, and deduce that the coproduct distinguishes 3-dimensional lens spaces. Algebraically, we describe the structure these operations define together on the Tate–Hochschild complex. We use rational homotopy theory methods to sketch the equivalence between the geometric and algebraic definitions for simply-connected manifolds and real coefficients, emphasizing the role of configuration spaces. Finally, we study invariance properties of the operations, both algebraically and geometrically.

Dedicated to Dennis Sullivan on the occasion of his 80th birthday

Contents

1. Introduction

String topology is concerned with algebraic structures defined by intersecting, concatenating, and cutting families of paths and loops in a manifold M. It began with Chas and Sullivan's construction of an intersection type product on $H_*(LM)$, the homology of the space $LM = \mathrm{Map}(S^1, M)$ of all loops in M, also known as the free loop space of M [13]. The loop product induces a Lie bracket on $H_*^{S^1}(LM)$, the S^1-equivariant homology of LM, generalizing an earlier construction of Goldman for loops on surfaces [37].

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Over the last twenty years, string topology has branched out to many corners of mathematics:

- It has an algebraic counterpart in Hochschild homology through the Jones [48] and Goodwillie [38] isomorphisms

$$H^*(LM;\mathbb{F}) \cong \mathrm{HH}_*(C^*(M;\mathbb{F}), C^*(M;\mathbb{F}))$$

and

$$H_*(LM) \cong \mathrm{HH}_*(C_*(\Omega M), C_*(\Omega M))$$

for $\mathbb{F}$ a field, ΩM the based loop space of M, and where M is assumed to be simply connected for the first isomorphism, see, e.g., [25, 31, 63, 68, 71];

- It has a symplectic interpretation through the Viterbo [84] isomorphism (with appropriate coefficients)

$$H_*(LM) \cong FH_*(T^*M)$$

with target the Floer homology of the cotangent bundle of M, see, e.g., [3, 4, 22, 78];

- Rich families of string operations have been defined, in particular, using algebraic models for string topology, including, for instance, BV structures, Lie bialgebras, 2-dimensional field theories of various flavors, and more, see, e.g., [28, 36, 50, 51, 82, 86];
- String topology has been used to study closed geodesics on Riemannian manifolds through Morse theory on the energy functional, see, e.g., [39, 43];
- String operations can be defined instead on the loop space LBG for G a Lie group, or more generally on the loop space of stacks, see [8, 14, 42] and see, e.g., [40, 58] for applications to group homology.

We will not be able to cover all aspects of string topology in this note and will instead focus on a few highlights that, we hope, illustrate the richness of the subject. We will restrict our attention to the original loop product of Chas and Sullivan and its "dual," the Goresky–Hingston coproduct. We will describe these two operations geometrically as well as algebraically, and use methods from rational homotopy theory to compare the two descriptions, where the role of configuration spaces will be emphasized. The geometric aspect of string topology will be illustrated through computations of loop products and coproducts via intersections of geometric cycles in examples from lens spaces. Algebraically, we will see that the two operations together define a single product on the Tate–Hochschild complex, defined in Section 3.4, and are encoded by the data of a Manin triple. Finally, we will address the question of homotopy invariance for the product and coproduct.

We describe now in more detail the content of this text. Throughout, M will be a closed oriented manifold of dimension n, and homology is with $\mathbb{Z}$-coefficients unless otherwise stated.

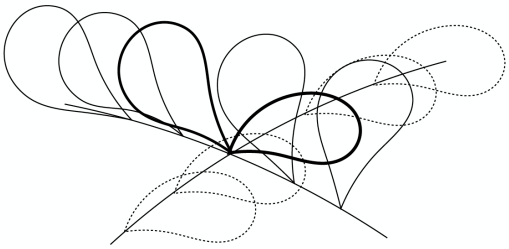

Figure 1. The loop product of two families of loops concatenates the loops that share the same basepoints.

Intersection products

Recall that the classical intersection product

$$\bullet\colon H_p(M) \otimes H_q(M) \to H_{p+q-n}(M)$$

can be computed by geometric intersection for transverse cycles: if $A, B \in H_*(M)$ are homology classes represented by smooth transversally embedded submanifolds, then their product $A \bullet B$ is given by the geometric intersection $A \cap B$ of the cycles. The original idea behind the Chas–Sullivan product is to define a product on $H_*(LM)$ by likewise transversally intersecting two families of loops in M at their basepoints, which is an intersection in M, and concatenating loops at the locus of intersection. This results in a graded commutative and associative product

$$\wedge\colon H_p(LM) \otimes H_q(LM) \to H_{p+q-n}(LM),$$

that is, by construction, compatible with the intersection product under the evaluation map $\mathrm{ev}_0\colon LM \to M$. We will refer to the Chas–Sullivan product as the *loop product* (see Figure 1).

Following ideas going back to Cohen–Jones [25], we give in Section 2.2 a formal definition of this product by lifting the definition of the classical intersection product phrased in terms of a Thom–Pontrjagin construction for the diagonal embedding $\Delta\colon M \to M \times M$.

The Goresky–Hingston coproduct [39], also considered by Sullivan [80] and referred to as the *loop coproduct* here, has the form

$$\vee\colon H_p(LM, M) \to H_{p-n+1}(LM \times LM, LM \times M \cup M \times LM).$$

The idea of the coproduct is, given a family of loops, to look for all the self-intersections in the family of the form $\gamma(0) = \gamma(t)$, for γ a loop and $t \in I$ is a time coordinate, and then cut. Following Hingston–Wahl [44], we show that it can be defined using a simple variant of the definition of the loop product. The operation is most naturally a relative operation

because the interval I has non-trivial boundary; see Remark 2.3 for non-relative versions of the coproduct.

The loop product and coproduct can be diagrammatically described as

$$\begin{array}{ccccc} LM \times LM & \dashleftarrow & \mathrm{Fig}(8) & \xrightarrow{\text{concat}} & LM \\ \downarrow{\scriptstyle \mathrm{ev}_0 \times \mathrm{ev}_0} & & \downarrow{\scriptstyle \mathrm{ev}_0} & & \\ M \times M & \xleftarrow{\Delta} & M & & \end{array} \qquad \begin{array}{ccccc} LM \times I & \dashleftarrow & \mathcal{F} & \xrightarrow{\text{cut}} & LM \times LM \\ \downarrow{\scriptstyle \mathrm{ev}_0 \times \mathrm{ev}_t} & & \downarrow{\scriptstyle \mathrm{ev}_0} & & \\ M \times M & \xleftarrow{\Delta} & M & & \end{array}$$

where the middle spaces $\mathrm{Fig}(8) \cong LM \times_M LM$ and $\mathcal{F} \subset LM \times I$ are the subspaces where the desired intersection holds, and where the dashed arrows are "intersection products" that are only defined on homology (or on chains). In Sections 4.1 and 4.2, we will formulate the data used from M to define these intersection products in terms of an *intersection context* (see Definition 4.7). Our preferred intersection context associated to a manifold M will be

$$\begin{array}{ccc} UTM & \longrightarrow & FM_2 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\Delta} & M \times M, \end{array}$$

where FM_2 is the configuration space of two points in M and UTM the unit tangent bundle of M.

Geometric computations

Just like the intersection product $\bullet$ can be computed by geometric intersection for nice enough cycles, the loop product and coproduct can be computed by a direct intersection for cycles that are appropriately transverse. This is made precise in Proposition 2.4, following [44], and illustrated through the computation of the loop product and coproduct of a family of classes generating $H_3(L\mathcal{L}_{p,q})$, for $\mathcal{L}_{p,q}$ a 3-dimensional lens space; see Propositions 2.5 and 2.8. As an application of the computation, we prove the following:

Theorem A (Theorem 2.11). *The loop coproduct distinguishes non-homeomorphic* 3*-dimensional lens spaces.*

This result is an extension of a computation of the first author in [70], used in that paper to show that the loop coproduct is not homotopy invariant; see below for more details about the invariance properties of the loop product and coproduct.

String topology algebraically

Assume now that M is a simply-connected closed manifold. The isomorphism $\mathrm{HH}_*(C^*(M;\mathbb{F}), C^*(M;\mathbb{F})) \cong H^*(LM;\mathbb{F})$ mentioned above, actually holds independently of the fact that M is a manifold. However, the algebraic structure of the Hochschild

complex becomes much richer once one inputs that $H^*(M)$ satisfies Poincaré duality, or in other words that it is a *Frobenius algebra* (see Definition 3.2). In the above isomorphism, we can replace $C^*(M;\mathbb{F})$ by any algebra A quasi-isomorphic to it in the category of dg-algebras. By a theorem of Lambrechts–Stanley, it is possible to find a model A for the rational cochains $C^*(M;\mathbb{Q})$ that has the structure of a (strict) commutative dg-Frobenius algebra compatible with the Frobenius structure on $H^*(M;\mathbb{Q})$ (see Theorem 3.5 and Example 3.6). The relevant consequence for us is that:

The algebraic structure of the Hochschild chains or cochains of dg-Frobenius algebras reflects rational string topology.

For Frobenius algebras, we indeed have an isomorphism between the linear dual of the Hochschild chain complex $C_*(A,A)$ and the Hochschild cochain complex $C^*(A,A)$, so both complexes are relevant (see Remark 3.11).

There is a wealth of literature on the algebraic structure of the Hochschild chains and cochains of Frobenius algebras, including algebraic versions of the product and coproduct just described, see, e.g., [2, 25, 32, 68] for the loop product and [2, 55] for the loop coproduct, or e.g., [50–52, 82, 86] for larger structures encompassing both, or [55, 85] for a prop of universal operations on the Hochschild complex of symmetric or commutative Frobenius algebras. (See also [9] in the present volume.)

It turns out that the loop product identifies with the classical cup product on Hochschild cochains [31], while the loop coproduct becomes the following product on *relative* Hochschild chains (see Definition 4.1):

Theorem B ([71]). *Let A be a dg-Frobenius algebra model for $C^*(M;\mathbb{R})$. Under a relative version of the Jones isomorphism $H^*(LM;\mathbb{R}) \cong \mathrm{HH}_*(C^*(M;\mathbb{R}), C^*(M;\mathbb{R})) \cong \mathrm{HH}_*(A,A)$, the linear dual of the loop coproduct is given on cochains by the formula*

$$(\overline{a_1}\otimes\cdots\otimes\overline{a_p}\otimes a_{p+1}) * \left(\overline{b_1}\otimes\cdots\otimes\overline{b_q}\otimes b_{q+1}\right) = \sum_i \pm\overline{b_1}\otimes\cdots\otimes\overline{b_{q+1}e_i}\otimes\overline{a_1}\otimes\cdots\otimes\overline{a_p}\otimes a_{p+1}f_i,$$

where $\Delta(1)=\sum_i e_i\otimes f_i \in A\otimes A$ represents the Thom class of the diagonal in $M\times M$. (See Example 3.4 and Definition 3.16).

This result is stated as Theorem 4.2 in the present paper, and we give a sketch proof of the result in Section 4.4.

In Section 3.6, we will focus on the following aspect of the algebraic structure defined by the algebraic product and coproduct:

Theorem C ([76]). *The algebraic product and coproduct extend to define together a single A_∞-structure on the Tate–Hochschild complex*

$$\mathcal{D}^{*,*}(A,A) = \cdots \xrightarrow{\partial_h} s^{1-k}C_{-1,*}(A,A) \xrightarrow{\partial_h} s^{1-k}C_{0,*}(A,A) \xrightarrow{\gamma} C^{0,*}(A,A) \xrightarrow{\delta_h} C^{1,*}(A,A) \xrightarrow{\delta_h} \cdots$$

that is compatible with the natural pairing between Hochschild chains and cochains and with an extension of Connes' operator B to the Tate–Hochschild complex. On cohomology, the product is graded commutative, and $H^*(\mathcal{D}^*(A,A))$ identifies, as an algebra, with the endomorphism algebra of A in the singularity category *of A-A-bimodules (see Remark 3.14).*

Here the Tate–Hochschild complex "glues together" the Hochschild chains and cochains along the map γ that can be thought of as an Euler characteristic, constructed using the Frobenius structure of A, see Section 3.4 for a complete definition of this complex. In Section 3.5, we give a description of this structure in terms of Manin triples, and this implies a form of infinitesimal bialgebra compatibility between the Goresky–Hingston coproduct and the Chas–Sullivan loop product. Note that Cieliebak–Hingston–Oancea have given a geometric version of the above Tate construction, including its algebra structure, using Rabinowitz–Floer homology, a theory that combines symplectic homology and cohomology via a "V-shaped" Hamiltonian [18, 20, 21, 23]. Theorem C is stated as Theorem 3.18 in the text.

The Tate–Hochschild complex satisfies the following strong invariance property, that is a consequence of the interpretation in terms of the singularity category:

Theorem D ([76]). *If two simply-connected symmetric dg-Frobenius algebras are quasi-isomorphic as dg associative algebras, then their Tate–Hochschild cohomologies are isomorphic as algebras.*

This result is stated as Theorem 3.20 in the text. A direct consequence of the result is that the algebraic version of the loop coproduct is a homotopy invariant in the simply-connected setting over the rationals (see Corollary 3.21).

Naturality and invariance

One of the original motivations of Chas and Sullivan in studying free loop spaces was to understand what characterizes the algebraic topology of manifolds and to construct algebraic invariants that could detect beyond the homotopy type; in Sullivan's own words to us

> *"…it is the question that has fascinated me since grad school: What is the algebraic chain level meaning of a space being a combinatorial or smooth manifold?"*

The particular instance of this question we will address here is the following: a homotopy equivalence $M \xrightarrow{\simeq} N$ induces an isomorphism $H_*(LM) \xrightarrow{\cong} H_*(LN)$, and likewise on homology relative to constant loops, and one can ask whether this induced map respects the loop product or coproduct. We summarize in the following result what is known about the question:

Theorem E. *The loop product and coproduct satisfy the following:*

(1) [26] *The loop product on* $H_*(LM)$ *is invariant under homotopy equivalences of manifolds* $M \xrightarrow{\sim} N$.

(2) *(*[76] *and* [71]*) The loop coproduct on* $H_*(LM, M; \mathbb{R})$ *is invariant under homotopy equivalences of simply-connected manifolds* $M \xrightarrow{\sim} N$.

(3) [70] *The loop coproduct on* $H_*(LM, M)$ *is not homotopy invariant in general.*

Alternative proofs of part (1) of the theorem were given by [27, 31, 41]. We give here a sketch proof of this result, in Theorem 4.11, stated in terms of homotopy invariance of general *intersection products*. Part (2) of the theorem is a direct consequence of combining Theorems B and D, while part (3) is a consequence of Theorem A.

The essential difference between the loop product and coproduct is that the loop coproduct uses a *relative intersection product*, and the proof of homotopy invariance of intersection product does not extend to proving the relative result. The article [70] suggests that the failure of invariance of the loop coproduct is related to Reidemeister torsion, which is compatible with Theorem A. See also [45] for a different description of the obstruction to homotopy invariance.

A non-invariance result was earlier obtained by Basu for a modified version of the coproduct [7]. Naef used the lens spaces $\mathcal{L}_{1,7}$ and $\mathcal{L}_{2,7}$ in [70] to show non-homotopy invariance of the coproduct on homology. The very same lens spaces were used by Longoni–Salvatore in [62] to show that the configuration space of two points in a manifold is likewise not a homotopy invariant of the manifold. Although we do not directly relate these two computations of non-homotopy invariance, we have already seen above that the configuration space of two points is an important ingredient in the definition of the loop coproduct, being part of the data needed to define the corresponding (relative) intersection product, see Sections 4.1 and 4.2.

The Lie bialgebra structure at the level of S^1-equivariant homology is a homotopy invariant for simply-connected manifolds by [70]. The recent paper [17] proves that homotopy invariance over the reals is also satisfied for a chain level version of the Lie bialgebra structure (also known as IBL_∞-algebra) in the case of 2-connected manifolds. It is so far unknown whether the chain level Lie bialgebra structure on S^1-equivariant chains (or a chain level version of the coalgebra structure in the non-equivariant case) is a homotopy invariant for simply-connected manifolds.

Organisation of the paper. In Section 2, after recalling the Thom–Pontrjagin definition of the intersection product, we give a chain level definition of the loop product and coproduct. Section 2.3 gives the computations of the loop products and coproducts on $H_3(L\mathcal{L}_{p,q})$ for 3-dimensional lens spaces $\mathcal{L}_{p,q}$. The coproduct computation is used in Section 2.4 to show that the loop coproduct is not homotopy invariant. Then Section 2.5 gives an alternative definition of the loop coproduct as a relative version of the so-called "trivial coproduct," the coproduct on the loop space that only looks for basepoint

self-intersections at time $t = \frac{1}{2}$. This definition will be used in Section 4 to show the equivalence between the algebraic and geometric descriptions of the coproduct.

Section 3 is concerned with the algebraic version of string topology. It starts with recalling and setting in context the concepts of Frobenius algebras, Hochschild chains and cochains. Section 3.4 then gives the definition of the Tate–Hochschild complex of a dg-Frobenius algebra. The loop product and coproduct are defined algebraically in Section 3.5 as products on the Hochschild cochains and chains respectively. These two products are assembled to a single product on the Tate–Hochschild complex in Section 3.6, where it is also interpreted in the language of Manin triples. The invariance of the product on the Tate–Hochschild complex is stated at the end of the section.

Section 4 takes a closer look at the "intersection products" that appear in the definition of the loop product and coproduct. After revisiting the definitions of the loop product and coproduct in Section 4.1, the notion of *intersection context* is defined in Section 4.2, a data one can construct intersection and relative intersection products from. The naturality and invariance properties of such intersection products are discussed in Section 4.3. Finally, Section 4.4 gives a sketch proof of the equivalence between the algebraic and geometric coproduct (Theorem 4.2) using an intersection context featuring the configuration space of two points in M and its real model [11, 46].

2. String topology via geometric intersection

Let M be a closed oriented manifold of dimension n, and pick a Riemannian metric on M. The loop space $LM = \mathrm{Map}(S^1, M)$ is homotopy equivalent to the space ΛM of H^1-loops on which the energy functional is defined:

$$LM \simeq \Lambda M \xrightarrow{E} \mathbb{R}, \quad \text{where } E(\gamma) = \int_{S^1} |\gamma'(t)|^2 dt.$$

The critical points of the energy are precisely the closed geodesics. Given that the energy is nice enough to do Morse theory, it follows that the homology $H_*(LM) \cong H_*(\Lambda M)$ "knows," or even "is built out of" closed geodesics. (See, e.g., [73] for a survey of Morse theory on the free loop space.)

As a graded abelian group, $H_*(LM)$ depends only on the homotopy type of M, whereas the closed geodesics depend on M as a Riemannian manifold. This naturally leads to the question whether there is some additional structure on $H_*(LM)$ that depends on a more refined structure than just the homotopy type of M. When M is a closed manifold, its homology satisfies Poincaré duality, and this duality takes the cup product of $H^*(M)$ to the *intersection product*:

$$H_p(M) \otimes H_q(M) \xrightarrow{\bullet} H_{p+q-n}(M).$$

The lifts of the intersection product given by the Chas–Sullivan product

$$H_p(LM) \otimes H_q(LM) \xrightarrow{\wedge} H_{p+q-n}(LM)$$

and Goresky–Hingston coproduct

$$H_p(LM, M) \xrightarrow{\vee} H_{p+1-n}(LM \times LM, M \times LM \cup LM \times M)$$

briefly described in the introduction, give a potential answer to the above question. Following ideas of Cohen–Jones [25] as implemented in [44], we explain here how both operations can be defined on chains as direct lifts of the intersection product, by using a chain-level definition of the intersection product in terms of a Thom–Pontrjagin construction, lifting along appropriate evaluation maps. Section 2.3 will give example computations, obtained from intersecting geometric cycles, from which we will be able to deduce in Section 2.4 that the coproduct does detect more than the homotopy type. Finally, Section 2.5 will give an alternative definition of the coproduct.

Note that homology in this section will always mean homology with integral coefficients: $H_*(\_) := H_*(\_; \mathbb{Z})$, and the same for cohomology.

2.1. The intersection product as a Thom–Pontrjagin construction

The normal bundle of the diagonal embedding $\Delta\colon M \hookrightarrow M \times M$ is isomorphic to the tangent bundle TM. Identifying $TM \equiv TM_\varepsilon$ with its subbundle of *small vectors*, i.e., vectors of length at most $\varepsilon \ll \rho$ for ρ the injectivity radius of M, the map

$$\nu_M\colon TM \hookrightarrow M \times M \quad \text{defined by} \quad \nu_M(x, V) = (x, \exp_x V)$$

is an explicit tubular neighborhood for Δ, with image the ε-neighborhood of the diagonal

$$\nu_M\colon TM \xrightarrow{\cong} U_M = \{(x, y) \in M \times M \mid |x - y| < \varepsilon\}.$$

Under this identification, the bundle projection map $TM \to M$ becomes the retraction $r\colon U_M \to M$ defined by $r(x, y) = x$. We let

$$\tau_M \in C^n(M \times M, M \times M \setminus M) \xleftarrow{\sim} C^n(TM, TM \setminus M)$$

denote the image of a cochain representative for the Thom class for TM, where $M \subset M \times M$ is the diagonal, and the arrow is the map ν_M^*, which is a quasi-isomorphism by excision.

Out of this data, we can give the following chain level description of the intersection product on $H_*(M)$:

$$\bullet\colon C_p(M) \otimes C_q(M) \xrightarrow{\times} C_{p+q}(M \times M) \xrightarrow{[\tau_M \cap]} C_{p+q-n}(U_M) \xrightarrow{r} C_{p+q-n}(M), \quad (2.1)$$

where the middle map is the following composition:

$$\begin{aligned}[\tau_M \cap]\colon C_*(M \times M) &\to C_*(M \times M, M \times M \setminus M) \\ &\xrightarrow{\sim} C_*(U_M, U_M \setminus M) \xrightarrow{\tau_M \cap} C_{*-n}(U_M),\end{aligned} \quad (2.2)$$

with the middle map being a homotopy inverse to excision, as can be obtained, for example, by subdividing simplices. (To be precise, this definition differs by a sign from the intersection product defined as the Poincaré dual of the cup product, see, e.g., [44, Proposition B.1].)

An important property of the intersection product, for computational purposes, is that it can indeed be computed by geometric intersection for homology classes that can be represented by transverse embedded submanifolds: if $A, B \subset M$ are embedded transverse submanifolds of M, with $[A] \in H_p(M)$ and $[B] \in H_q(M)$ the corresponding homology classes, then

$$[A] \bullet [B] = [A \cap B] \in H_{p+q-n}(M).$$

See, e.g., [10, Chapter VI, Theorem 11.9].

2.2. Definition of the product and coproduct as lifts of the intersection product

Let $\mathrm{ev}_0 \colon LM \to M$ denote the evaluation at 0. The Chas–Sullivan product $\wedge$ being a lift of the intersection product $\bullet$ means that both products should fit in a commutative diagram of the form

$$\begin{array}{ccc} H_p(LM) \otimes H_q(LM) & \xrightarrow{\wedge} & H_{p+q-n}(LM) \\ \downarrow{\scriptstyle \mathrm{ev}_0 \otimes \mathrm{ev}_0} & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ H_p(M) \otimes H_q(M) & \xrightarrow{\bullet} & H_{p+q-n}(M). \end{array} \tag{2.3}$$

We explain now how this can be achieved simply by "pulling back" all the ingredients of the above definition of the intersection product to the loop space along the evaluation map $\mathrm{ev}_0 \times \mathrm{ev}_0$.

Recall from above the ε-neighborhood U_M of the diagonal in $M \times M$ and define

$$U_{\mathrm{CS}} = (\mathrm{ev}_0 \times \mathrm{ev}_0)^{-1} U_M = \big\{(\gamma, \lambda) \in LM \times LM \mid |\gamma(0) - \lambda(0)| < \varepsilon\big\}.$$

The retraction $r \colon U_M \to M$ lifts to a retraction

$$\begin{aligned} R_{\mathrm{CS}} \colon U_{\mathrm{CS}} \longrightarrow \mathrm{Fig}(8) &= \{(\gamma, \lambda) \in LM \times LM \mid \gamma(0) = \lambda(0)\} \\ &= (\mathrm{ev}_0 \times \mathrm{ev}_0)^{-1}(\Delta M \subset M \times M) \end{aligned}$$

by concatenating with a geodesic stick to connect the loops so that they form a "figure 8":

$$R_{\mathrm{CS}}(\gamma, \lambda) = (\gamma, \lambda') \ \text{ with } \ \lambda' = \overline{\gamma(0)\lambda(0)} \star \lambda \star \overline{\lambda(0)\gamma(0)}$$

where, for $x, y \in M$ with $|x - y| < \rho$, $\overline{xy}$ denotes the unique minimal geodesic path $[0, 1] \to M$ from x to y, which is possible by our choice of ε, and $\star$ is the concatenation of paths.[1] See also Figure 2 (a).

[1] See, e.g., [44, Section 1.2] for a definition of an associative concatenation.

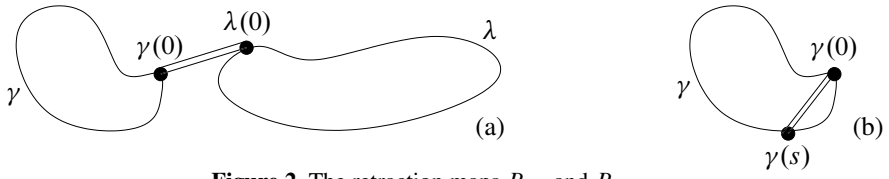

Figure 2. The retraction maps R_{CS} and R_{GH}.

Pulling back our representative of the Thom class τ_M along the evaluation map gives a cochain

$$\tau_{\mathrm{CS}} := (\mathrm{ev}_0 \times \mathrm{ev}_0)^* \tau_M \in C^*(LM \times LM, \mathrm{Fig}(8)^c).$$

Together, U_{CS}, R_{CS} and τ_{CS} are all the ingredients we need to define the desired product:

Definition 2.1. The following sequence of chain maps is a chain model for the Chas–Sullivan product:

$$\begin{aligned} \wedge\colon C_p(LM) \otimes C_q(LM) \xrightarrow{\times} C_{p+q}(LM \times LM) \xrightarrow{[\tau_{\mathrm{CS}} \cap]} C_{p+q-n}(U_{\mathrm{CS}}) \\ \xrightarrow{R_{\mathrm{CS}}} C_{p+q-n}(\mathrm{Fig}(8)) \xrightarrow{\mathrm{concat}} C_{p+q-n}(LM), \end{aligned} \tag{2.4}$$

where, just as in (2.1), the middle map is the composition of a homotopy inverse to excision followed by the capping map.

Naturality of the maps gives that the resulting homology product on the homology $H_*(LM)$ makes diagram (2.3) commute. And it is shown in [44, Proposition 2.4] that this simple-minded chain description of the Chas–Sullivan product agrees in homology with the definition of Cohen–Jones [25] given in terms of a tubular neighborhood of the figure 8 space Fig(8) inside $LM \times LM$.

The coproduct can be defined completely analogously, replacing the evaluation map $\mathrm{ev}_0 \times \mathrm{ev}_0\colon LM \times LM \to M \times M$ by the evaluation map

$$e_I\colon LM \times I \to M \times M \quad \text{defined by } e_I(\gamma, s) = (\gamma(0), \gamma(s)).$$

Indeed, setting

$$U_{\mathrm{GH}} = e_I^{-1} U_M = \big\{(\gamma, s) \in LM \times I \mid |\gamma(0) - \gamma(s)| < \varepsilon\big\},$$

we again have a retraction map

$$\begin{array}{ccccc} U_{\mathrm{GH}} & \xrightarrow{R_{\mathrm{GH}}} & \mathcal{F} = \big\{(\gamma, s) \in LM \times I \mid \gamma(0) = \gamma(s)\big\} & \hookrightarrow & LM \times I \\ \downarrow{\scriptstyle e_I} & & \downarrow{\scriptstyle e_I} & & \downarrow{\scriptstyle e_I} \\ U_M & \xrightarrow{r} & \Delta M & \hookrightarrow & M \times M \end{array}$$

by concatenating with a geodesic stick to force a self-intersection:

$$R_{\mathrm{GH}}(\gamma, s) = (\gamma', s) \text{ with } \gamma' = \gamma[0,s] \star \overline{\gamma(s)\gamma(0)} \star_s \overline{\gamma(0)\gamma(s)} \star \gamma[s,0]$$

where we choose the parametrization of the concatenated loop so that it exactly passes through $\gamma(0)$ at time s; this is possible even if $s = 0$ or 1 as in that case $\gamma(0) = \gamma(s)$ to begin with, and the geodesic sticks are thus length 0. See also Figure 2 (b). Note that the above diagram commutes as $e_I \circ R_{\mathrm{GH}}(\gamma, s) = (\gamma'(0), \gamma'(s)) = (\gamma(0), \gamma(0))$.

We can consider the sequence of maps

$$C_p(LM) \xrightarrow{\times I} C_{p+1}(LM \times I) \xrightarrow{[\tau_{\mathrm{GH}} \cap]} C_{p+1-n}(U_{\mathrm{GH}}) \xrightarrow{R_{\mathrm{GH}}} C_{p+1-n}(\mathcal{F}) \xrightarrow{\mathrm{cut}} C_{p+q-n}(LM \times LM)$$

totally analogous to the maps (2.4) defining the product above. The only new feature of the coproduct, compared to the product, is the first map in the sequence, crossing with an interval, which is not a chain map because the interval has non-trivial boundary. This corresponds to the fact that the operation is now parametrized by an interval I. To obtain an induced operation on homology, we need to appropriately kill the resulting "boundary operation" at the endpoints of the interval. The simplest way to do this is to consider the operation as a relative operation, noting that, when $s = 0$ or 1, the above sequence of maps creates a left or right constant loop.

Definition 2.2. The following sequence of chain maps is a chain model for the Goresky–Hingston–Sullivan coproduct:

$$\vee\colon C_p(LM, M) \xrightarrow{\times I} C_{p+1}(LM \times I, LM \times \partial I \cup M \times I) \xrightarrow{[\tau_{\mathrm{GH}} \cap]} C_{p+1-n}(U_{\mathrm{GH}}, LM \times \partial I \cup M \times I) \xrightarrow{R_{\mathrm{GH}}} C_{p+1-n}(\mathcal{F}, LM \times \partial I \cup M \times I) \xrightarrow{\mathrm{cut}} C_{p+q-n}(LM \times LM, M \times LM \cup LM \times M)$$

This sequence of maps now indeed induces a well-defined degree $1 - n$ coproduct on $H_*(LM, M)$:

$$\vee\colon H_p(LM, M) \longrightarrow H_{p+1-n}(LM \times LM, M \times LM \cup LM \times M);$$

if we work with field coefficients, the target is isomorphic to $H_*(LM, M)^{\otimes 2}$. It is shown in [44, Proposition 2.12] that this chain level description of the Goresky–Hingston–Sullivan coproduct agrees with the definition given in [39] using a tubular neighborhood of $\mathcal{F}$ inside $LM \times I$ away from the boundary $LM \times \partial I$, together with a limit argument reach to the boundary.

Applying the evaluation map e_I after crossing with the interval, and before applying the cut map, gives a diagram of the same form as diagram (2.3), but now with intersection

product *relative to* M on the bottom row:

$$\begin{array}{ccccc} H_p(LM, M) & \xrightarrow{\times I} & H_{p+1}(LM \times I, LM \times \partial I \cup M \times I) & \xrightarrow{\wedge} & H_{p+1-n}(\mathcal{F}, LM \times \partial I \cup M \times I) \\ & & \downarrow{\scriptstyle e_I} & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ & & H_{p+1}(M \times M, \Delta M) & \xrightarrow{\quad\bullet\quad} & H_{p+1-n}(M, M) = 0 \end{array}$$

As the bottom row is now a trivial operation, there is no formal way in which the homology loop coproduct is a lift of the homology intersection product. We will however see in Section 2.3 that the coproduct still can be computed by an appropriate geometric intersection, for nice enough geometric cycles, away from the "trivial self-intersections" coming from constant loops or from the intersection times $s = 0$ and $s = 1$.

Remark 2.3 (Lifting the coproduct to a non-relative operation). There exists several ways to lift the coproduct $\vee$ to a non-relative operation.

(1) One such lift is the *extension by zero* of [44, Section 4], that uses the splitting $H_*(LM) \cong H_*(LM, M) \oplus H_*(M)$ coming from the inclusion of the constant loops and the evaluation cst: $M \rightleftarrows LM : \mathrm{ev}_0$, declaring the coproduct to be zero on constant loops.

(2) If the Euler characteristic of the manifold is zero, one can instead use a nowhere vanishing vector field $\overline{v}$ to define such an extension, by replacing the diagonal $\Delta M \subset M \times M$ in the above definition of the coproduct, with the homotopy equivalent subspace $\Delta_{\overline{v}} M = \{(m, \exp_m \overline{v}_m) \in M \times M \mid m \in M\}$. Indeed, if the vector field has no zeros, the coproduct will then automatically be trivial at the special points with $s = 0$ or $s = 1$. See also [71, Section 3.4] for an analogous definition of a lifted coproduct in the $\chi(M) = 0$ case, using instead a lift of the Thom class.
If the Euler characteristic is not zero, one can instead pick a vector field vanishing only in the neighborhood of a single point, which will yield a coproduct in reduced homology of the loop space instead, corresponding to what we will see in the algebraic version of the coproduct, see Definition 3.16.

(3) The following variant of the previous idea has been described for the case of surfaces in [83, Section 18] and [54]. Instead of attaching the non-vanishing vector field to the manifold M one can attach it to the loop. That is one considers loops in the unit tangent bundle of M. In the case of surfaces, such loops can be identified with regular homotopy classes of immersed curves. Moreover, in case the surface has a non-vanishing vector field, the above construction is recovered by using that every homotopy class of a loop in a surface has a unique representative as an immersed loop with rotation number 0 with respect to the vector field. This is the point of view taken in [6].

(4) As we will see in Section 3.6 in the algebraic context, following the paper [76] (see [18, 23] for a geometric version), the loop product and coproduct together

define a single (non-relative) product on the *Tate–Hochschild complex*, a complex that combines both the chains and cochains of the loop space, attached together using the Euler class (see Section 3.4). When the Euler characteristic of the manifold vanishes, the Tate complex splits and this recovers a non-relative cohomology product, dual to the homology coproduct.

2.3. Computation via geometric intersections

Recall that two smooth maps $f: X \to M$ and $g: Y \to M$ are *transverse* if for every x, y such that $f(x) = m = g(y)$, we have $f_* T_x X + g_* T_y Y = T_m M$. Because the product and coproduct are defined as lifts of the intersection product along evaluation maps, they can both be computed by geometric intersection, under appropriate transversality assumptions on the cycles representing the homology classes:

Proposition 2.4 ([44, Propositions 3.1 and 3.7]). *The loop product and coproduct can be computed as follows:*

(1) *If $Z_1: \Sigma_1 \to LM$ and $Z_2: \Sigma_2 \to LM$ are smooth cycles with the property that the maps $\mathrm{ev}_0 \circ Z_1: \Sigma_1 \to M$ and $\mathrm{ev}_0 \circ Z_2: \Sigma_2 \to M$ are transverse, then the loop product*

$$Z_1 \wedge Z_2 = (Z_1 \star Z_2)|_{\Sigma_1 \times_{\mathrm{ev}_0} \Sigma_2} \in H_*(LM)$$

is the concatenation of the loops of Z_1 and Z_2 along the locus of basepoint-intersections $\Sigma_1 \times_{\mathrm{ev}_0} \Sigma_2 \subset \Sigma_1 \times \Sigma_2$, oriented as stated in [44].

(2) *If $Z: (\Sigma, \Sigma_0) \to (LM, M)$ is a smooth relative cycle with the property that the restriction of $e_I \circ (Z \times I): \Sigma \times I \to M \times M$ to $(\Sigma \setminus \Sigma_0) \times (0,1)$ is transverse to the diagonal, then*

$$\vee Z = \mathrm{cut} \circ (Z \times I)|_{\overline{\Sigma_\Delta}} \in H_*(LM \times LM, M \times LM \cup LM \times M)$$

for $\overline{\Sigma_\Delta}$ the closure in $\Sigma \times I$ of the locus of basepoint self-intersecting loops $\Sigma_\Delta \subset (\Sigma \setminus \Sigma_0) \times (0,1)$, oriented as stated in [44].

We illustrate this proposition here through a loop product and coproduct computation for 3-dimensional lens spaces $M = \mathcal{L}_{p,q}$, on 3-dimensional cycles

$$Z_{m,\ell}: \Sigma = \mathcal{L}_{p,q} \longrightarrow L\mathcal{L}_{p,q}$$

parametrized by the lens spaces themselves. For the product computation, the cycles will turn out to already be transverse, so the computation will be straightforward, while for the coproduct we will need to first deform the cycles to make them appropriately transverse to the diagonal. The coproduct computation will be used in Section 2.4 to show that the coproduct is not homotopy invariant, following [70].

We start by recalling the definition of 3-dimensional lens spaces.

Let S^3 be the 3-sphere, considered as the unit sphere in $\mathbb{C}^2$. We will write elements of S^3 in spherical coordinates as tuples $(\underline{r}, \underline{\theta}) = ((r_1, \theta_1), (r_2, \theta_2))$ with $\theta_i \in \mathbb{R}/\mathbb{Z}$ and

$r_i \geq 0$, satisfying $r_1^2 + r_2^2 = 1$. The lens space $\mathcal{L}_{p,q}$, for p, q coprime, is the quotient of S^3 by the relation

$$\big((r_1, \theta_1), (r_2, \theta_2)\big) \sim \Big(\Big(r_1, \theta_1 + \frac{1}{p}\Big), \Big(r_2, \theta_2 + \frac{q}{p}\Big)\Big).$$

This relation comes from the action of the torus $S^1 \times S^1$ on $S^3 \subset \mathbb{C}^2$ rotating each coordinate, where we have picked a particular subgroup $\mathbb{Z}/p$ inside $S^1 \times S^1$. Note that there is a residual torus action on the lens space:

$$\begin{aligned} \alpha\colon (S^1 \times S^1) \times \mathcal{L}_{p,q} &\to \mathcal{L}_{p,q}, \\ ((s,t), (\underline{r}, \underline{\theta})) &\mapsto \Big(\Big(r_1, \theta_1 + \frac{s}{p}\Big), \Big(r_2, \theta_2 + \frac{sq}{p} + t\Big)\Big). \end{aligned}$$

We can use this residual torus action to define cycles $Z_{\ell,m}$ for a pair of integers (ℓ, m) as follows: Let

$$\delta_{\ell,m}\colon S^1 \to S^1 \times S^1$$

be the loop $t \mapsto (\ell t, mt)$ of slope $\frac{\ell}{m}$. We can combine this loop with the action α of the torus on $\mathcal{L}_{p,q}$ to get a family

$$\begin{aligned} Z_{\ell,m}\colon \mathcal{L}_{p,q} &\to L\mathcal{L}_{p,q} \\ (\underline{r}, \underline{\theta}) &\mapsto \big[\gamma^{\ell,m}_{\underline{r},\underline{\theta}}\colon t \mapsto \alpha(\delta_{\ell,m}(t), (\underline{r}, \underline{\theta}))\big] \end{aligned}$$

associating to each point $(\underline{r}, \underline{\theta})$ in the lens space, the loop $\gamma^{\ell,m}_{\underline{r},\underline{\theta}}$ based at that point and following the image of $\delta_{\ell,m}$ along the torus action. Explicitly, the loop $\gamma^{\ell,m}_{\underline{r},\underline{\theta}}\colon S^1 \to \mathcal{L}_{p,q}$ is defined by

$$\gamma^{\ell,m}_{\underline{r},\underline{\theta}}(t) = \Big(\Big(r_1, \theta_1 + \frac{\ell t}{p}\Big), \Big(r_2, \theta_2 + \frac{q\ell t}{p} + mt\Big)\Big).$$

As above, we denote also by

$$Z_{\ell,m} \in H_3(L\mathcal{L}_{p,q})$$

the associated homology class. Note that each class $Z_{\ell,m}$ is non-trivial as it maps to the fundamental class of $\mathcal{L}_{p,q}$ under the evaluation map

$$\begin{aligned} \mathrm{ev}_0\colon H_3(L\mathcal{L}_{p,q}) &\to H_3(\mathcal{L}_{p,q}) \\ Z_{\ell,m} &\mapsto [\mathcal{L}_{p,q}] \end{aligned}$$

as the basepoints of the loops $\gamma^{\ell,m}_{\underline{r},\underline{\theta}}$ precisely trace the lens space.

We will here compute the loop products and coproducts of the classes $Z_{\ell,m}$, starting with their product under the map:

$$\wedge\colon H_3(L\mathcal{L}_{p,q}) \otimes H_3(L\mathcal{L}_{p,q}) \longrightarrow H_{3+3-3}(L\mathcal{L}_{p,q}) = H_3(L\mathcal{L}_{p,q})$$

defined by the loop product.

Strategy for computing the loop product of the classes $Z_{\ell,m}$: Because the classes $Z_{\ell,m}$ all evaluate at the fundamental class, taking $Z_1 = Z_{\ell_1,m_1}$ and $Z_2 = Z_{\ell_2,m_2}$, the transversality condition of Proposition 2.4 (1) will be automatically satisfied as $\mathrm{ev}_0 \circ Z_1$ and $\mathrm{ev}_0 \circ Z_2$ is simply the identity on $M = \mathcal{L}_{p,q}$. Moreover, the intersection locus is immediately computed to be again the lens space itself, and the resulting product is thus the concatenation of loops from each family at each basepoint $(\underline{r}, \underline{\theta})$. In the following proposition, we identify this family of concatenated loops as a known class, and we give after the statement a detailed proof that the outlined strategy works.

Proposition 2.5. *The Chas–Sullivan loop product of the classes $Z_{\ell,m} \in H_3(L\mathcal{L}_{p,q})$ defined above, is given by summing the indices:*

$$Z_{\ell_1,m_1} \wedge Z_{\ell_2,m_2} = Z_{\ell_1+\ell_2,m_1+m_2}.$$

Proof. The cycles $Z_{\ell,m}\colon \mathcal{L}_{p,q} \to L\mathcal{L}_{pq}$ are smooth cycles parametrized $\mathcal{L}_{p,q}$. To apply Proposition 2.4, we need to check that the maps

$$\mathcal{L}_{p,q} \xrightarrow{Z_{\ell_i,m_i}} L\mathcal{L}_{p,q} \xrightarrow{\mathrm{ev}_0} \mathcal{L}_{p,q}$$

are transverse. But for each (ℓ_i, m_i), this composition is the identity on the lens space, so the maps are certainly transverse, and the locus of basepoint-intersections is the diagonal $\Delta\mathcal{L}_{p,q} \subset \mathcal{L}_{p,q} \times \mathcal{L}_{p,q}$. The product is thus explicitly given by

$$Z_{\ell_1,m_1} \wedge Z_{\ell_2,m_2} = (Z_{\ell_1,m_1} \star Z_{\ell_2,m_2})|_{\Delta\mathcal{L}_{p,q}}\colon \mathcal{L}_{p,q} \equiv \Delta\mathcal{L}_{p,q} \longrightarrow L\mathcal{L}_{p,q}$$

for $\star$ the concatenation of the loops in the image at their common basepoint. At each point $(\underline{r}, \underline{\theta})$ in $\mathcal{L}_{p,q}$, we are thus left to compute the concatenation $\gamma^{\ell_1,m_1}_{\underline{r},\underline{\theta}} \star \gamma^{\ell_2,m_2}_{\underline{r},\underline{\theta}}$ which is exactly the image under the torus action of the concatenation of the loops (ℓ_1, m_1) and (ℓ_2, m_2) in the torus. This concatenation in the torus is homotopic to the loop $(\ell_1 + \ell_2, m_1 + m_2)$ (corresponding to the fact that $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$) and hence the above product is homotopic the loop $\gamma^{\ell_1+\ell_2,m_1+m_2}_{\underline{r},\underline{\theta}}$. As this homotopy originates in the torus, it defines a continuous homotopy over the lens space. It follows that the Chas–Sullivan product of such classes is as claimed. ∎

The coproduct of homology classes of degree 3 in $L\mathcal{L}_{p,q}$ is a map

$$\vee\colon H_3(L\mathcal{L}_{p,q}, \mathcal{L}_{p,q}) \longrightarrow H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}, \mathcal{L}_{p,q} \times L\mathcal{L}_{p,q} \cup L\mathcal{L}_{p,q} \times \mathcal{L}_{p,q}).$$

For the classes $Z_{\ell,m}$, it will be given in terms of B-classes in the target, that we describe now.

Let $\lambda\colon S^1 \to \mathcal{L}_{p,q}$ be the loop defined by $\lambda(t) = ((1, \frac{t}{p}), 0)$, tracing the points $(\underline{r}, \underline{\theta}) \in \mathcal{L}_{p,q}$ with $r_2 = 0$. This is a generator of $\pi_1\mathcal{L}_{p,q} \cong \mathbb{Z}/p$. Note that

$$\gamma^{1,0}_{((1,0),0)} = \left[t \mapsto \left(\left(1, \frac{t}{p}\right), 0\right)\right] = \lambda$$

identifies λ with the evaluation of the class $Z_{1,0}$ at $((1,0),0) \in \mathcal{L}_{p,q}$. In particular, λ is freely homotopic to $\gamma^{1,0}_{(0,(1,0))}$, the evaluation of $Z_{1,0}$ at $(0,(1,0))$ instead, where we note that

$$\gamma^{1,0}_{(0,(1,0))} = \left[t \mapsto \left(0, \left(1, \frac{qt}{p}\right)\right)\right] = (\lambda')^{\star q}$$

for $\lambda' \colon S^1 \to \mathcal{L}_{p,q}$ defined by $\lambda'(t) = (0, (1, \frac{t}{p}))$, the loop tracing the points $(\underline{r}, \underline{\theta})$ with $r_1 = 0$.

We will define 1-cycles $B_{k,k'}$ and $B'_{k,k'}$ in $\mathcal{L}_{p,q}$ using the circle action, reparametrizing the loops, on the concatenation of copies of λ and λ' respectively: Let $\lambda_s \colon S^1 \to \mathcal{L}_{p,q}$ be the rotation of λ, based at $\lambda(s)$, i.e., defined by

$$\lambda_s(t) = \lambda(s+t)$$

and likewise for λ'. Define

$$\begin{aligned} B_{k,k'} \colon S^1 &\to L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q} \subset L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}, \\ s &\mapsto \left((\lambda_s)^{\star k}, (\lambda_s)^{\star k'}\right). \end{aligned}$$

We also denote by $B_{k,k'} \in H_1(L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q})$ or $H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q})$ the associated homology class. Note that the evaluation $\mathrm{ev}_0 \colon H_1(L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q}) \to H_1(\mathcal{L}_{p,q})$ takes $B_{k,k'}$ to λ, now considered as a 1-cycle in $\mathcal{L}_{p,q}$, so the class $B_{k,k'}$ is "doubly" made out of λ, as each loop in the family is a concatenation of copies of λ, but also the family of basepoints follows λ! Define $B'_{k,k'}$ in the same way, replacing λ by λ'.

The coproduct of the classes $Z_{\ell,m}$ will be given by applying the cut map to the families of figure eights $B_{k,k'}$. Both families $B_{k,k'}$ and $B'_{k,k'}$ will naturally arise in the computation of the coproduct, so we start by proving that we can express cycles of the type B' in terms of cycles of the type B, coming from the fact already mentioned above that the loop λ, the classes of type B are made of, is freely homotopic to $\lambda'^{\star q}$, with λ' the loop used to define the classes of type B'.

Lemma 2.6. *Let $B'_{k,k'} \colon S^1 \to L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q} \subset L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}$ be the family of figure eights based at the points of λ' defined by $B'_{k,k'}(s) = ((\lambda'_s)^{\star k}, (\lambda'_s)^{\star k'})$. Then*

$$B_{k,k'} = q\, B'_{qk,qk'} \in H_1(L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q})$$

is the sum of q copies of the class $B'_{qk,qk'}$.

Proof. An explicit homotopy $\lambda \simeq_h (\lambda')^{\star q} \colon S^1 \to \mathcal{L}_{p,q}$ is given by picking a "straight line" in $\mathcal{L}_{p,q}$ from $((1,0),0)$ to $(0,(1,0))$ and evaluating $Z_{1,0}$: we let $h \colon S^1 \times I \to \mathcal{L}_{p,q}$ be defined by the evaluation of $Z_{1,0}$ along the line $((\sqrt{1-\tau^2}, 0), (\tau, 0))$, giving the formula

$$h(s,\tau) = \left(\left(\sqrt{1-\tau^2}, \frac{s}{p}\right), \left(\tau, \frac{qs}{p}\right)\right) \in \mathcal{L}_{p,q}.$$

This lifts to a homotopy $H\colon S^1 \times I \to L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q}$ of loops based at h, defined by

$$H(s,\tau) = \Big[t \mapsto \Big(\Big(\sqrt{1-\tau^2}, \frac{s+kt}{p}\Big), \Big(\tau, \frac{q(s+kt)}{p}\Big)\Big)\Big] \\ \star \Big[t \mapsto \Big(\Big(\sqrt{1-\tau^2}, \frac{s+k't}{p}\Big), \Big(\tau, \frac{q(s+k't)}{p}\Big)\Big)\Big]$$

that starts at

$$H(s,0) = \Big[t \mapsto \Big(\Big(1, \frac{s+kt}{p}\Big), 0\Big)\Big] \star \Big[t \mapsto \Big(\Big(1, \frac{s+k't}{p}\Big), 0\Big)\Big] = (\lambda_s)^{\star k} \star (\lambda_s)^{\star k'},$$

that identifies precisely with the family $B_{k,k'}$, and ends at

$$H(s,1) = \Big[t \mapsto \Big(0, \Big(1, \frac{qs+qkt}{p}\Big)\Big)\Big] \star \Big[t \mapsto \Big(0, \Big(1, \frac{qs+qk't}{p}\Big)\Big)\Big],$$

that exactly runs q times, as s runs along S^1, the family $B'_{qk,qk'}$. In particular, as a homology cycle, it represents $q\, B'_{qk,qk'}$. ∎

Lemma 2.7. *We have that:*

(1) $B_{k,k'} = B_{h,h'} \in H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q})$ *if and only if* $k = h \mod p$ *and* $k' = h' \mod p$*;*

(2) *the relative classes*

$$\{\mathrm{B}_{k,k'}\}_{\substack{0<k<p \\ 0<k'<p}} \in H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}, \mathcal{L}_{p,q} \times L\mathcal{L}_{p,q} \cup L\mathcal{L}_{p,q} \times \mathcal{L}_{p,q})$$

are linearly independent over $\mathbb{Z}_p$.

Proof. The evaluation at 0 takes the family of figure eights $B_{k,k'}$ to the generator λ of $\pi_1(\mathcal{L}_{p,q}) \cong \mathbb{Z}/p$. Hence, the map $H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}) \to H_1(\mathcal{L}_{p,q})$ projecting on the first component and evaluating at 0, takes $B_{k,k'}$ to the generator of $H_1(\mathcal{L}_{p,q})$. In particular, each class $B_{k,k'} \in H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q})$ is non-trivial.

Note now that $B_{k,k'}$ has image in the component ($k \mod p, k' \mod p$) of the space $L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}$, as each loop $[s \mapsto ((1, \frac{t+ks}{p}), 0)]$ is a rotated version of $\lambda^{\star k}$. Given that the classes are non-zero, $B_{k,k'} = B_{h,h'}$ thus necessarily requires that $k = h \mod p$ and $k' = h' \mod p$, just to be in the same component. The converse follows from the fact that any homotopy $\lambda^{\star p} \simeq *$ extends continuously over such a family of loops $B_{k,k'}$, using the S^1-action to push it along its parametrizing family of basepoints λ, proving that $B_{k,k'} = B_{k+np,k'+mp}$ in homology for any $n, m \in \mathbb{N}$, which proves (1).

Finally, by the above, $B_{k,k'}$ is non-zero in relative homology precisely when k and k' are not equal to 0 mod p, as $B_{0,k'}$ and $B_{k,0}$ are trivial in relative homology. And the classes are linearly independent as they live in different components. ∎

We are now ready to compute the coproduct of ρ-classes, where we will assume that ℓ and m are positive for simplicity. We start by explaining the general idea of the computation.

Strategy for computing the loop coproduct of the classes $Z_{\ell,m}$: Recall that $Z_{\ell,m}\colon \mathcal{L}_{p,q} \to L\mathcal{L}_{p,q}$ is a family of loops coming from, at each point of the lens space, applying to that point the restriction of the torus action to the loop of slope $\frac{\ell}{m}$ in the torus. Such families of loops, specially when ℓ and m are big, will a priori have many self-intersections, whose exact pattern depend on the Z/p action defining the lens space. To compute the coproduct, we though first need to make this family *transverse* in the sense of Proposition 2.4. This can be thought of as "pushing the loops in the family to avoid as many of these self-intersections as possible." Now for $(\underline{r}, \underline{\theta})$ in $\mathcal{L}_{p,q}$, the loop $Z_{\ell,m}(\underline{r}, \underline{\theta})$ is based at $(\underline{r}, \underline{\theta}) = ((r_1, \theta_1), (r_2, \theta_2))$ and runs along points of the form $((r_1, \theta_1(t)), (r_2, \theta_2(t)))$, with only the angle coordinates varying, and with self-intersections coming from the fact that, sometimes, both $\theta_1(t) = \theta_1$ and $\theta_2(t) = \theta_2$ for some $0 < t < 1$. To avoid such self-intersections, we could simply try to make sure that, at such points of intersection, the radii do not match. To achieve this, we can deform the family of loops so that it takes the form $((\widetilde{r}_1(t), \theta_1(t)), (\widetilde{r}_2(t), \theta_2(t)))$ with $\widetilde{r}_1(t), \widetilde{r}_2(t)$ never equal to r_1, r_2 when $0 < t < 1$. (Note that $\widetilde{r}_1(t)^2 + \widetilde{r}_2(t)^2 = 1$, so we only really have one variable to play with here.) This simple idea can be used *as long as* $(r_1, r_2) \notin \{(1, 0), (0, 1)\}$. Indeed, for these special choice of radii, deforming, e.g., $r_1 = 1$ not to be 1 anymore, forces one to pick r_2 not 0, but when r_2 is zero, there is no angle θ_2 attached to it, and we cannot, continuously over the lens space, suddenly choose $\theta_2(t)$'s to associate to newly non-zero $\widetilde{r}_2(t)$'s. This is how the subfamilies parametrized by λ (the loop of points with $r_1 = 1$) and λ' (those with $r_2 = 1$) will enter as the parametrizing families for the loci of self-intersections. The actual intersections will then be given by classes $B_{k,k'}$ with $k + k' = \ell$, which is the total length of loops over the points of λ, or $k + k' = q\ell + pm = q\ell \mod p$, for the points over λ'. Finally, the latter cycles can be rewritten in terms of classes $B_{k,k'}$ (with $k + k' = \ell$) using Lemmas 2.6 and 2.7. We prove below that this strategy works, and yields the following formula, where the first group of terms counts the intersections along λ and the second groups the intersections along λ':

Proposition 2.8. *The coproduct of the class $Z_{\ell,m} \in H_3(L\mathcal{L}_{p,q}, \mathcal{L}_{p,q})$ with $\ell, m \geq 0$ is given by the formula*

$$\vee Z_{\ell,m} = \sum_{\substack{0<k<\ell \\ k,(\ell-k)\neq 0 \bmod p}} B_{k,\ell-k} \; + \; q' \sum_{\substack{0<k<q\ell+pm \\ k,(\ell-kq')\neq 0 \bmod p}} B_{kq',\ell-kq'}$$

where q' is the multiplicative inverse of q mod p.

Using the previous lemma, one deduces that the coproduct of Z-classes is non-trivial most of the time.

Proof. We make precise the sketch of proof given above before the statement.

To compute the coproduct $\vee Z_{\ell,m}$ by geometric intersection applying Proposition 2.4, we need the map

$$\mathcal{L}_{p,q} \times (0,1) \xrightarrow{Z_{\ell,m}\times \mathrm{id}} L\mathcal{L}_{p,q} \times (0,1) \xrightarrow{e_I} \mathcal{L}_{p,q} \times \mathcal{L}_{p,q},$$

where e_I evaluates the loops at 0 and $s \in (0,1) \subset I$, to be transverse to the diagonal embedding $\Delta\colon \mathcal{L}_{p,q} \to \mathcal{L}_{p,q} \times \mathcal{L}_{p,q}$ after removing the locus of constant loops. In the present case, either $(\ell, m) = (0,0)$ in which case all loops are constant, with $Z_{(0,0)} = 0$ in homology relative to the constant loops, or $(\ell, m) \neq (0,0)$ and the cycle has no constant loop in its image. So we can assume $(\ell, m) \neq (0,0)$ and work with the parametrizing pair $(\Sigma, \Sigma_0) = (\mathcal{L}_{p,q}, \emptyset)$ for our relative cycle (using the notation of Proposition 2.4).

As explained above, to achieve transversality, we will represent the homology class of $Z_{\ell,m}$ by the homotopic family $\widetilde{Z}_{\ell,m}\colon \mathcal{L}_{p,q} \to L\mathcal{L}_{p,q}$ defined by $\widetilde{Z}_{\ell,m}(\underline{r}, \underline{\theta}) = \widetilde{\gamma}^{\ell,m}_{\underline{r},\underline{\theta}}$ for $\widetilde{\gamma}^{\ell,m}_{\underline{r},\underline{\theta}}\colon S^1 \to \mathcal{L}_{p,q}$ the loop based at $(\underline{r}, \underline{\theta})$ given by

$$\widetilde{\gamma}^{\ell,m}_{\underline{r},\underline{\theta}}(t) = \left(\left(\widetilde{r}_1(t), \theta_1 + \frac{\ell t}{p}\right), \left(\widetilde{r}_2(t), \theta_2 + \frac{q\ell + pm}{p} t\right)\right),$$

where $(\widetilde{r}_1(t), \widetilde{r}_2(t))$ is a deformation of (r_1, r_2) with $(\widetilde{r}_1(t), \widetilde{r}_2(t)) = (r_1, r_2)$ only when r_1 or $r_2 = 0$, or when $t = 0$ or 1. Such a deformation can be obtained by, e.g., interpolating back and forth between the identity on r_1 at times $t = 0$ and 1 and r_1^2 at $t = \frac{1}{2}$, with $\widetilde{r}_2(t) = \sqrt{1 - \widetilde{r}_1(t)^2}$.

The map $e_I \circ (\widetilde{Z}_{\ell,m} \times \mathrm{id})|_{\mathcal{L}_{p,q} \times (0,1)}$ intersects the diagonal whenever a loop $\widetilde{\gamma}^{\ell,m}_{\underline{r},\underline{\theta}}$ has a self-intersection $\widetilde{\gamma}^{\ell,m}_{\underline{r},\underline{\theta}}(0) = \widetilde{\gamma}^{\ell,m}_{\underline{r},\underline{\theta}}(t)$ for some $t \in (0,1)$. Such self-intersections can only happen when $r_1 = 0$ or $r_2 = 0$, as otherwise $\widetilde{r}_i(t) \neq \widetilde{r}_i(0) = r_i$, making the equality impossible. When $r_2 = 0$, the equality happens exactly if the first angle coordinate at time t agrees with $\theta_1 \mod \frac{1}{p}\mathbb{Z}$, and when $r_1 = 0$, if the second angle coordinate agrees with $\theta_2 \mod \frac{1}{p}\mathbb{Z}$. (Note that in the lens space, we indeed have $(0, (1, \theta_2)) \sim (0, (1, \theta_2 + \frac{k}{p}))$ for any $k \in \mathbb{Z}$ as p and q are assumed to be coprime.) This yields the following condition on the parameters:

$$\begin{cases} 0 < t = \frac{a}{\ell} < 1 \ \text{ for some } a \in \mathbb{N}, & \text{if } r_2 = 0; \\ 0 < t = \frac{b}{q\ell + pm} < 1 \ \text{ for some } b \in \mathbb{N}, & \text{if } r_1 = 0. \end{cases}$$

That is the locus of self-intersections of $\widetilde{Z}_{\ell,m} \times \mathrm{id}\,|_{\mathcal{L}_{p,q} \times (0,1)}$ is

$$\Sigma_\Delta = (\lambda \times I_1) \cup (\lambda' \times I_2) \subset \mathcal{L}_{p,q} \times (0,1)$$

for $I_1 = \{\frac{1}{\ell}, \ldots, \frac{\ell-1}{\ell}\}$ and $I_2 = \{\frac{1}{q\ell+pm}, \ldots, \frac{q\ell+pm-1}{q\ell+pm}\}$, and λ, λ' the loops parametrizing the points with $r_2 = 0$ and $r_1 = 0$, respectively, as above.

We need to check that these self-intersections are transverse to the diagonal. This is to be expected as we have "pushed away self-intersections as much as we could," but needs to be checked, which can only be done by actually computing the maps

$$\mathcal{L}_{p,q} \times (0,1) \xrightarrow{e_I \circ (\widetilde{Z}_{\ell,m} \times \mathrm{id})} \mathcal{L}_{p,q} \times \mathcal{L}_{p,q} \hookleftarrow \Delta\mathcal{L}_{p,q}$$

at the points of the intersection locus $\Sigma_\Delta \subset \mathcal{L}_{p,q} \times (0,1)$. Now Σ_Δ consists of two components: the component $\lambda \times I_1$ of points with coordinate $r_2 = 0$ in the lens space,

and the component $\lambda' \times I_2$ of points with coordinate $r_1 = 0$ in the lens space. Note that the map $e_I \circ (\widetilde{Z}_{\ell,m} \times \mathrm{id})$ takes points with r_i coordinate 0 to points of the same form in the diagonal. We will do the transversality computation in local coordinates $(r_1, \theta_1, \theta_2, t) = (z, \theta_2, t) \in \mathbb{C} \times \mathbb{R}^2/(\mathbb{Z}/p)$ around points with $r_1 = 0$ in $\mathcal{L}_{p,q} \times (0,1)$, with the coordinates $((z,\theta),(z',\theta'))$ in the target $\mathcal{L}_{p,q} \times \mathcal{L}_{p,q}$, and similarly with coordinates $(\theta_1, r_2, \theta_2, t) = (\theta_1, z, t)$ when $r_2 = 0$. In those coordinates, the function $e_I \circ (\widetilde{Z}_{\ell,m} \times \mathrm{id})$ has the form

$$(z,\theta,t) \mapsto \big((z,\theta), \big(e^{2\pi i\alpha t} r(t) z, \theta + \beta t\big)\big)$$

where $r(t)$ is a function so that $r(t) = 1$ only for $t = 0, 1$, while the diagonal is the set of points $\Delta = \cup_k \Delta_k$ for

$$\Delta_k = \begin{cases} \big(z, \theta, e^{2\pi i k/p} z, \theta + \frac{kq}{p}\big), & \text{when } r_2 = 0; \\ \big(z, \theta, e^{2\pi i k/p} z, \theta + \frac{kq'}{p}\big), & \text{when } r_1 = 0. \end{cases}$$

Now transversality holds because the zeros of the functions

$$f_k(z,\theta,t) = \Big(\big(e^{2\pi i\alpha t} r(t) - e^{2\pi i k/p}\big) z, \beta t - \frac{kq^{(\prime)}}{p}\Big)$$

are transversal. Indeed, away from $t = 0, 1$ the factor $(e^{2\pi i\alpha t} r(t) - e^{2\pi i k/p})$ is never zero, so, up to translation, f_k has the form $f_k(z,\theta,t) = (a(t)z, \beta t)$ for $0 \neq a(t) \in \mathbb{C}$ and $\beta > 0$, either equal to $\frac{\ell}{p}$ or to $\frac{q\ell + pm}{p}$.

Applying Proposition 2.4, it now follows that the coproduct

$$\vee \widetilde{Z}_{\ell,m} = [\mathrm{cut} \circ (\widetilde{Z}_{\ell,m} \times I)|_{\overline{\Sigma_\Delta}}]$$

where $\overline{\Sigma_\Delta}$ is the closure inside $\mathcal{L}_{p,q} \times I$ of Σ_Δ, with Σ_Δ oriented so that the isomorphism

$$T_{(\underline{r},\underline{\theta},t)}(\mathcal{L}_{p,q} \times I) \cong N\Delta\mathcal{L}_{p,q} \oplus T_{(\underline{r},\underline{\theta},t)}\Sigma_\Delta,$$

coming from transversality, is orientation preserving.[2] Our computation above shows that $\overline{\Sigma_\Delta} = \Sigma_\Delta$ is the disjoint union of circles $\lambda \times I_1 \cup \lambda' \times I_2 \subset \mathcal{L}_{p,q} \times (0,1)$. Given that the sign depends on choices and conventions, we only give here the important part of the sign computation for us, namely that it is independent of $t \in I_1 \cup I_2$, and independent of ℓ, m.

Orient $T_{(\underline{r},\underline{\theta},t)}(\mathcal{L}_{p,q} \times I)$ around $r_1 = 0$ as $\mathbb{R}^4\langle r_1, \theta_1, \theta_2, t\rangle$. Then we have $T_{(\underline{r},\underline{\theta},t)}(\mathcal{L}_{p,q} \times I) \cong -\mathbb{R}^3\langle r_1, \theta_1, t\rangle \oplus T\Sigma_\Delta\langle\theta_2\rangle$ at the intersections with $r_1 = 0$. Around $r_2 = 0$, we then have $T_{(\underline{r},\underline{\theta},t)}(\mathcal{L}_{p,q} \times I) \cong \mathbb{R}^4\langle r_2, \theta_2, \theta_1, t\rangle$ as $r_2 = \sqrt{1 - r_1^2}$ is orientation preserving, and hence likewise $T_{(\underline{r},\underline{\theta},t)}(\mathcal{L}_{p,q} \times I) \cong -\mathbb{R}^3\langle r_2, \theta_2, t\rangle \oplus T\Sigma_\Delta\langle\theta_1\rangle$. And in local coordinate (z,θ,t), the map considered has the form $(z,\theta,t) \mapsto ((z,\theta),(c(t)z, \theta + \beta))$, independently of the point of Σ_Δ.

[2] In our conventions, $N\Delta M$ is oriented so that $\tau_M \cap [M \times M] = [M]$, for τ_M the corresponding Thom class.

Finally, we have that

$$\mathrm{cut} \circ (\widetilde{Z}_{\ell,m} \times I)|_{\lambda \times I_1 \cup \lambda' \times I_2} = \Big(\sum_{k=1}^{\ell-1} B_{k,\ell-k} + \sum_{k=1}^{q\ell+pm-1} B'_{k,q\ell+pm-k} \Big)$$

as a family of pairs of loops. The result thus follows from Lemmas 2.6 and 2.7. ∎

2.4. Homotopy invariance

A diffeomorphism $f\colon M \xrightarrow{\cong} N$ induces an isomorphism $Lf_*\colon H_*(LM) \xrightarrow{\cong} H_*(LN)$, and likewise for relative homology, that preserves both the loop product and coproduct, as all their defining ingredients are identified by diffeomorphisms. It is natural to ask whether only assuming that f is a homotopy equivalence could be enough for the induced isomorphism Lf_* to preserve the loop product and coproduct. Note that if f satisfies the even weaker assumption of being a degree 1 map, then $f_*\colon H_*(M) \to H_*(N)$ already preserves the intersection product, see, e.g., [10, Chapter VI, Proposition 14.2].

The following two results show that the answer to the above question is yes for the product, and no for the coproduct.

Theorem 2.9 ([26] (see also [27, 31, 41])). *Let $f\colon M \to N$ be a degree 1 homotopy equivalence between two closed oriented manifolds. Then $Lf_*\colon H_*(LM) \to H_*(LN)$ is an isomorphism of algebras with respect to the Chas–Sullivan product.*

The main ingredient of the proof of this theorem is sketched in Section 4.3 (see Theorem 4.11), where we will revisit the question of invariance of the loop product and coproduct after going through a deeper analysis of their defining ingredients.

In the meanwhile, as noted by the first author in [70], the computations presented in Section 2.3 can already be used to show that the loop coproduct is not homotopy invariant:

Theorem 2.10 ([70]). *Let $f\colon \mathscr{L}_{7,1} \to \mathscr{L}_{7,2}$ be a homotopy equivalence and $Z_{1,0} \in H_3(L\mathscr{L}_{7,1})$ be as in Section 2.3. Then*

$$\begin{aligned} 0 &= (Lf_* \otimes Lf_*)(\vee(Z_{1,0})) \\ &\neq \vee(Lf_*(Z_{1,0})) \in H_1(L\mathscr{L}_{7,2} \times L\mathscr{L}_{7,2}, \mathscr{L}_{7,2} \times L\mathscr{L}_{7,2} \cup L\mathscr{L}_{7,2} \times \mathscr{L}_{7,2}). \end{aligned}$$

In particular, the loop coproduct $\vee$ is not preserved by f.

The manifolds $\mathscr{L}_{7,1}$ and $\mathscr{L}_{7,2}$ are the simplest examples of lens spaces that are homotopy equivalent, but not simple homotopy equivalent. They were also used in [62] to prove that the configuration space of two points in a manifold is not a homotopy invariant of the manifold. In Section 4.2, we will see that the same configuration of two points plays an important role in the definition of the loop coproduct.

Proof. The class $Z_{1,0} \in H_3(L\mathscr{L}_{7,1})$ has trivial coproduct by Proposition 2.8 as $\ell = q = 1$ and $m = 0$. (This also follows, using [44, Theorem 3.10], from the fact that $Z_{1,0}$ is a family of simple loops whenever $q = 1$.).

We need to compute the coproduct of the image $f_*(Z_{1,0})$. The free loop space $L\mathcal{L}_{7,q}$ has 7 components, and each component $L_\ell\mathcal{L}_{7,q}$ has $H_3(L_\ell\mathcal{L}_{7,q}) \cong \mathbb{Z} \oplus \mathbb{Z}/7$ (see [70, Section 2.1]). From Lemma 2.7 and Proposition 2.8, one can deduce that, e.g., the classes $Z_{\ell,0}$ and $Z_{\ell,1}$ generate $H_3(L_\ell\mathcal{L}_{7,q})$ since both results together show that their images under the coproduct are linearly independent, which implies that they are themselves linearly independent and hence must generate $H_3(L_\ell\mathcal{L}_{7,q})$. Now [69, Lemma 6.9] tells us that, because f is a homotopy equivalence, $Z_{1,0}$ has image in $L_\ell\mathcal{L}_{7,2}$ for $\ell = 2$ or 5, depending on whether f has degree 1 or -1. If f has degree 1, then $f(Z_{0,1}) = aZ_{\ell,0} + (1-a)Z_{\ell,1}$ for some $a \in \{0,\dots,6\}$, with $\ell = 2$, where $a \in Z/7$ and the coefficients sum to 1 because all the classes $Z_{\ell,m}$ evaluate to the fundamental class of the lens space, differing only in their $Z/7$ component, and the property of evaluating to the fundamental class is preserved by f. Now Proposition 2.8 for $\ell = 2$ shows that $\vee Z_{2,0} = 5B_{1,1} + 4(B_{4,5} + B_{5,4})$ while $\vee Z_{2,1} = 2B_{1,1} + B_{4,5} + B_{5,4} + 4(B_{3,6} + B_{6,3})$. And one checks readily that there is no a such that $\vee(aZ_{\ell,0} + (1-a)Z_{\ell,1}) = 0$. A similar computation rules out the possibility in the case $\ell = 5$ with f of degree -1. ∎

Combining the invariance of the corresponding (co)product in algebra (see Theorem 3.20), with the fact that the algebraic model indeed models the loop coproduct (see Theorem 4.2), it follows that, when working over real coefficients and with simply-connected manifolds, the coproduct is homotopy invariant, as stated in Theorem E. By contrast, in the non-simply-connected case and with integer coefficients, the above computation can be extended to show the following:

Theorem 2.11. *A degree 1 homotopy equivalence $f\colon \mathcal{L}_{p,q_1} \to \mathcal{L}_{p,q_2}$ between two 3-dimensional spaces such that $Lf_*\colon H_*(L\mathcal{L}_{p,q_1}, \mathcal{L}_{p,q_1}) \to H_*(L\mathcal{L}_{p,q_2}, \mathcal{L}_{p,q_2})$ preserves the loop coproduct of degree 3 classes is homotopic to a homeomorphism.*

The idea of the proof is the same as that of the previous theorem: we take the class with the simplest coproduct in the source, namely $Z_{1,0}$, and show that the equality $\vee Lf_*(\mathbb{Z}_{1,0}) = (Lf_* \otimes Lf_*)(\vee Z_{1,0})$ is only possible under some number theoretic conditions that, in all cases, force known conditions for the lens spaces to be homeomorphic. We only do the computation in the case of degree 1 maps because it is involved enough, and because it is the most interesting case.

Proof. Suppose f is such a homotopy equivalence. Let $Z_{1,0} \in H_3(L\mathcal{L}_{p_1,q_1})$ be as above. We will compare $(Lf_* \otimes Lf_*)(\vee(Z_{1,0}))$ with $\vee(Lf_*(Z_{1,0}))$.

The class $Lf_*(Z_{1,0})$ lies in $H_3(L_\ell\mathcal{L}_{p,q_1})$ for some ℓ satisfying $q_1 = \ell^2 q_2 \mod p$, because f is a degree 1 homotopy equivalence, with f inducing multiplication by ℓ on π_1, see, e.g., [69, Theorem 6.11], where $0 < q_1, q_2, \ell < p$. We want to show that f is homotopic to a homeomorphism. By [69, Lemma 6.8], it is enough to check that the two lens spaces are homeomorphic, which happens precisely if either $q_1 q_2 = \pm 1 \mod p$ or $q_1 = \pm q_2 \mod p$, see, e.g., [24, Section 31] or [69, Theorem 1.3]. We may assume without loss of generality that $q_2 \neq 1$.

To avoid confusion, denote by $\bar{Z}_{\ell,0}, \bar{Z}_{\ell,1} \in H_3(L\mathcal{L}_{p,q_2})$ the classes in the second lens space, and likewise for the B-classes. As argued above for $\mathcal{L}_{7,2}$, we have that $\bar{Z}_{\ell,0}, \bar{Z}_{\ell,1}$ generate $H_3(L_\ell\mathcal{L}_{p,q_2})$, so we know that $(Lf_* \otimes Lf_*)(Z_{1,0}) = a\bar{Z}_{\ell,0} + (1-a)\bar{Z}_{\ell,1}$ for some $a \in \{0, \dots, p-1\}$, with the coefficients summing to 1 again since f has degree 1 and the classes $Z_{\ell,m}$ evaluate to the fundamental class.

The coproducts $(Lf_* \otimes Lf_*)(\vee(Z_{1,0}))$ and $\vee(Lf_*(Z_{1,0}))$ will be given in terms of the classes $\bar{B}_{k,\ell-k} \in H_1(L\mathcal{L}_{p,q_2} \times L\mathcal{L}_{p,q_2})$ (or the corresponding relative homology group). As these classes only depends on the parameter k, we will denote them by $[k]$ below, for better readability. Note also that $(Lf_* \otimes Lf_*)(B_{k,1-k}) = \ell\bar{B}_{\ell k,\ell-\ell k}$ as f is multiplication by ℓ on π_1.

From our computation above, we have that

$$\vee Z_{1,0} = \; q_1' \sum_{0<k<q_1} B_{kq_1',1-kq_1'} = \; q_1' \sum_{0<k<q_1} [kq_1']$$

so in the above notation,

$$(Lf_* \otimes Lf_*)(\vee Z_{1,0}) = \; \ell q_1' \sum_{0<k<q_1} [k\ell q_1'] = \; \ell' q_2' \sum_{0<k<q_1} [k\ell' q_2']$$

using that $\ell q_1' = \ell' q_2$ for the second equality. On the other hand,

$$\begin{aligned}
\vee\big((1-a)\bar{Z}_{\ell,0} + a\bar{Z}_{\ell,1}\big) &= \sum_{0<k<\ell} [k] \; + \; q_2' \sum_{\substack{0<k<q_2\ell \\ k,(\ell-kq_2')\neq 0 \bmod p}} [kq_2'] \; + \; aq_2' \sum_{\substack{q_2\ell\le k<q_2\ell+p \\ k,(\ell-kq_2')\neq 0 \bmod p}} [kq_2'] \\
&= \sum_{0<k<\ell} [k] \; + \; \Big(q_2' \sum_{0<k<d} [kq_2'] \; + \; cq_2' \sum_{\substack{d\le k<d+p \\ k,(\ell-kq_2')\neq 0 \bmod p}} [kq_2']\Big) \\
&\qquad + \; aq_2' \sum_{\substack{d\le k<d+p \\ k,(\ell-kq_2')\neq 0 \bmod p}} [kq_2'] \\
&= \sum_{0<k<\ell} [k] \; + \; q_2' \sum_{0<k<d} [kq_2'] \; + \; (a+c)q_2' \sum_{0<k<p} [k],
\end{aligned}$$

where $0 < d < p$ is such that $q_2\ell = cp + d$, used to split the second summation term in the first line and simplify the third, and where, for the last equality, we note that summing $[kq_2']$ from letting k vary between d and $d+p$ runs precisely once through all the possible values of $[kq_2']$ and hence can be more directly written as a sum over $[k]$ from k running between 0 and p instead.

The equality $\vee(Lf_*(Z_{1,0})) - (Lf_* \otimes Lf_*)(\vee(Z_{1,0})) = 0$ holds precisely if all possible terms $[s]$ appear with coefficient a multiple of p. A necessary condition for this to hold is that the terms $[s]$ all appear in

$$\sum_{0<k<\ell} [k] \; + \; q_2' \sum_{0<k<d} [kq_2'] \; - \; \ell' q_2' \sum_{0<k<q_1} [k\ell' q_2']$$

with the same total coefficient. Consider the sets

$$\begin{aligned} A &= \{k \mid 0 < k < \ell\}, \\ B &= \{kq_2' \mid 0 < k < d\}, \\ C &= \{k\ell' q_2' \mid 0 < k < q_1\}. \end{aligned}$$

Case 1: $A = \emptyset$, or equivalently $\ell = 1$. Then $q_1 = \ell^2 q_2 = q_2 \mod p$ and f is homotopic to a homeomorphism.

Case 2: $B = \emptyset$. Then $\ell q_2 = 1 \mod p$ (as $d = 1$), so that $q_1 = \ell^2 q_2 = \ell$. But then $q_1 q_2 = 1 \mod p$, which also gives that f is homotopic to a homeomorphism.

Case 3: $C = \emptyset$ with $A, B \neq \emptyset$. So $q_1 = 1$, and either $A = B$ or $A = B^c$. If $A = B$, then $\ell = d = \ell q_2 \mod p$, giving $q_2 = 1$. If $A = B^c$, we would need $q_2' = 1$ for the coefficients to agree. In both cases, this contradicts our assumption that $q_2 \neq 1$.

Case 4: $A, B, C \neq \emptyset$. If the sets are disjoint, we need the three coefficients to be equal, giving in particular $q_2' = 1$ which contradicting again $q_2 \neq 1$. The case $A = B$ is ruled out above. If $A = C$, then $\ell = q_1 = \ell^2 q_2 \mod p$, giving $\ell q_2 = 1 \mod p$, i.e., $d = 1$ contradicting that B is non-empty. And if $B = C$, $q_2 \ell = q_1 = \ell^2 q_2 \mod p$, giving $\ell = 1$, contradicting that $A \neq \emptyset$. We are now left with the case when all three sets intersect, but none are equal. In that case, we need all sums of coefficients to agree: $1 + q_2' = 1 - \ell' q_2' = q_2' - \ell' q_2'$ modulo p, implying in particular $q_2' = 1 \mod p$, again a contradiction. ∎

2.5. The good and the bad coproduct

The coproduct we have described looks for self-intersections of the form $\gamma(t) = \gamma(0)$ in families of loops γ where $t \in I$ is any time along the interval. One could instead define a coproduct $\vee_{\frac{1}{2}}$ that only looks for self-intersections at time $t = \frac{1}{2}$, i.e., defined just like $\vee$ but without crossing with I and replacing the evaluation e_I by the map $\mathrm{ev}_{0,\frac{1}{2}} = (\mathrm{ev}_0, \mathrm{ev}_{\frac{1}{2}}) \colon LM \to M \times M$. Denoting $\mathrm{Fig}(8) = \mathrm{ev}_{0,\frac{1}{2}}^{-1}(\Delta M) \subset LM$ the space of "figure eights," i.e., loops γ with a self-intersection $\gamma(0) = \gamma(\frac{1}{2})$, and $U_\varepsilon(\mathrm{Fig}(8)) = \mathrm{ev}_{0,\frac{1}{2}}^{-1}(U_M)$ its ε-neighborhood, we have

$$\vee_{\frac{1}{2}} \colon H_p(LM) \xrightarrow{(\mathrm{ev}_{0,1/2})^* \tau_M \cap} H_{p-n}(U_\varepsilon(\mathrm{Fig}(8))) \xrightarrow{R_{1/2}} H_{p-n}(\mathrm{Fig}(8)) \xrightarrow{\mathrm{cut}} H_{p-n}(LM \times LM),$$

for $R_{\frac{1}{2}}$ a retraction map defined just like the retraction map R_{GH} used for $\vee$.

This leads to a rather trivial coproduct though, as first noted by Tamanoi in [81]. Indeed, the coproduct $\vee_{\frac{1}{2}}$ is homotopic to the coproduct $\vee_0$ that looks for ("left-trivial") self-intersections at $t = 0$, i.e., of the form $\gamma(0) = \gamma(0)$, or likewise to the coproduct $\vee_1$ looking for ("right-trivial") self-intersection at $t = 1$ only. Whether we set $t = \frac{1}{2}$, 0 or 1,

we again have a commutative diagram:

$$\begin{array}{lccc} \vee_t\colon & H_p(LM) & \xrightarrow{R\circ(\mathrm{ev}_{0,t}^*\,\tau_M\cap)} & H_{p-n}(\mathrm{Fig}(8)) \xrightarrow{\mathrm{cut}} H_{p-n}(LM\times LM) \\ & \downarrow{\scriptstyle \mathrm{ev}_{0,t}} & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ & H_p(M\times M) & \xrightarrow{\quad\bullet\quad} & H_{p-n}(M) \end{array}$$

Setting $t = 0$ or 1, the left vertical map has image inside the diagonal. Note that the intersection product takes the diagonal $[\Delta M] \in H_n(M \times M)$ to $(-1)^n\chi(M)[\{*\}] \in H_0(M)$, the Euler characteristic $\chi(M)$ being an obstruction to moving the diagonal away from itself. Combining this with the equality $\vee_{\frac{1}{2}} = \vee_0 = \vee_1$ can be used to show that the coproduct $\vee_{\frac{1}{2}}$ is only non-trivial in homology on the fundamental class of $[M]$, considered as a family of constant loops, and only when $\chi(M) \neq 0$, with $\vee_{\frac{1}{2}}[M] = (-1)^n\chi(M)[\{*\} \times \{*\}] \in H_0(LM \times LM)$ (see, e.g., [44, Lemma 4.5]). In fact, the "good" coproduct $\vee$ that we have worked with here can be thought of as a secondary operation, coming from these two reasons that $\vee_{\frac{1}{2}}$ is trivial, homotoping it to its $t = 0$ or $t = 1$ versions.

One way to formulate this relationship between the two coproduct is as follows: the coproduct $\vee$ can be defined as a *relative version* of the coproduct $\vee_{\frac{1}{2}}$, as we explain now. This form of definition first appeared in [39, Section 9], in the definition of the dual cohomology product.

Let $J\colon LM \times I \to LM$ be the reparametrizing map defined by $J(\gamma, s) = \gamma \circ \theta_{\frac{1}{2}\to s}$ where $\theta_{\frac{1}{2}\to s}\colon [0, 1] \to [0, 1]$ is the piecewise linear map that fixes 0 and 1 and takes $\frac{1}{2}$ to s. Note that J restricts on the boundary to a map $J\colon LM \times \partial I \to \mathcal{R}$ for

$$\mathcal{R} := \left\{\gamma \in LM \mid \gamma|_{[0,\frac{1}{2}]} \text{ or } \gamma|_{[\frac{1}{2},1]} \text{ is constant}\right\}$$

the subspace of LM of *half-constant loops*.

Proposition 2.12. *The loop coproduct $\vee$ can equivalently be defined as the composition of the following sequence of maps:*

$$\begin{aligned} H_*(LM, M) &\xrightarrow{\times I} H_{*+1}(LM \times I, LM \times \partial I \cup M \times \partial I) \xrightarrow{J} H_{*+1}(LM, \mathcal{R}) \\ &\xrightarrow{(\mathrm{ev}_{0,1/2})^*\tau_M\cap} H_{*+1-n}(U_\varepsilon(\mathrm{Fig}(8)), \mathcal{R}) \xrightarrow{R_{1/2}} H_{*+1-n}(\mathrm{Fig}(8), \mathcal{R}) \\ &\xrightarrow{\mathrm{cut}} H_{*+1-n}(LM \times LM, M \times LM \cup LM \times M). \quad (2.5) \end{aligned}$$

See [44, Theorem 2.13] for a proof that this new definition is equivalent to the one of Section 2.2. Note that the last three maps in the statement indeed compose to a relative version of the coproduct $\vee_{\frac{1}{2}}$.

3. String topology via Hochschild complexes

In this section we define a product on the Tate–Hochschild complex of any connected dg-Frobenius algebra A. The Tate–Hochschild complex is an amalgam of the Hochschild chains and cochains, two chain complexes that model, by results of Jones and Chen, the cohomology and homology of the free loop space of simply-connected manifolds, respectively. We will see below and in Section 4 that the product on the Tate–Hochschild complex relates to both the Chas–Sullivan product, when restricted to the Hochschild cochains, and the Goresky–Hingston coproduct, when restricted to the Hochschild chains.

3.1. Differential graded algebras

Let $\mathbb{K}$ be a commutative ring with unit. Recall that a *dg $\mathbb{K}$-module*, or *chain complex*, is a graded $\mathbb{K}$-module $V = \bigoplus_{j\in\mathbb{Z}} V^j$ equipped with a differential $d_V\colon V \to V$; in this section, all differentials will have degree $+1$. The *dual* of (V, d_V) is the dg $\mathbb{K}$-module $(V^\vee, d_{V^\vee})$ with $(V^\vee)^{-j} = \mathrm{Hom}_{\mathbb{K}}(V^j, \mathbb{K})$ and the differential defined by $d_{V^\vee}(\alpha)(x) = -(-1)^{|\alpha|}\alpha(d_V(x))$ on homogeneous elements $\alpha \in V^\vee$, where $|\alpha|$ denotes the degree of α.

A *dg $\mathbb{K}$-algebra* $A = (A, d, \mu)$, or *dg-algebra* for short, is a dg $\mathbb{K}$-module (A, d) equipped with an associative product $\mu\colon A \otimes A \to A$ of degree zero satisfying the Leibniz rule

$$\mu \circ (d \otimes \mathrm{id} + \mathrm{id} \otimes d) = d \circ \mu.$$

We write $\mu(a \otimes b) = ab$. The multiplication is (graded) commutative if $ab = (-1)^{|a||b|}ba$, and *unital* if there is a map $u\colon \mathbb{K} \to A$ such that the image of $1 \in \mathbb{K}$ is a unit for the multiplication of A.

The *cohomology* $H^*(A)$ of a dg-algebra $A = (A, d, \mu)$ becomes a graded $\mathbb{K}$-algebra with product $H^*(A) \otimes H^*(A) \to H_*(A)$ induced by $\mu\colon A \otimes A \to A$. A morphism of dg-algebras $f\colon A \to A'$ is a *quasi-isomorphism* if it induces an isomorphism of graded algebras $H^*(f)\colon H^*(A) \xrightarrow{\cong} H^*(A')$.

Example 3.1. The following examples are particularly relevant to our discussion:

(1) The singular cochains on a topological space X equipped with the simplicial differential and cup product define a dg-algebra $(C^*(X;\mathbb{K}), d, \smile)$. The cup product is unital associative and homotopy commutative.

(2) When $\mathbb{K} = \mathbb{Q}$ the dg-algebra $(C^*(X;\mathbb{Q}), d, \smile)$ is quasi-isomorphic to a *commutative* dg-algebra $(\mathcal{A}_{\mathrm{pl}}(X), d, \wedge)$ of $\mathbb{Q}$-polynomial differential forms, as shown by Sullivan.

One of the main theorems discussed in this note, Theorem 3.18, involves the weaker notion of an A_∞-algebra. Recall that an *A_∞-algebra* is a graded $\mathbb{K}$-module A equipped with linear maps $\{m_n\colon A^{\otimes n} \to A\}_{n\in\mathbb{Z}_{>0}}$, where each m_n is of degree $2-n$, satisfying the following relations:

- $m_1 \circ m_1 = 0$, in other words, (A, m_1) is a dg $\mathbb{K}$-module;

- $m_1 \circ m_2 = m_2 \circ (m_1 \otimes \mathrm{id}_A + \mathrm{id}_A \otimes m_1)$, in other words, the product m_2 satisfies Leibniz rule with respect to m_1;
- more generally, for each positive integer n we have

$$\sum (-1)^{p+qr} m_{p+1+r} \circ (\mathrm{id}_A^{\otimes p} \otimes m_q \otimes \mathrm{id}_A^{\otimes r}) = 0,$$

where the sum runs over all triples of positive integers (p, q, r) such that $n = p + q + r$.

In particular, the last equation implies that $m_3 \colon A^{\otimes 3} \to A$ is a chain homotopy for the associativity of m_2. Hence, for any A_∞-algebra A, the cohomology $H^*(A, m_1)$ has an induced graded associative algebra structure.

3.2. Differential graded Frobenius algebras

The notion of a symmetric dg-Frobenius algebra consists of a dg-algebra equipped with a non-degenerate symmetric bilinear pairing compatible with the product structure. Our interest in symmetric dg-Frobenius algebras is motivated by Poincaré duality.

Definition 3.2. A *dg-Frobenius $\mathbb{K}$-algebra of dimension n* is a non-negatively graded unital dg $\mathbb{K}$-algebra (A, d, μ) equipped with a pairing $\langle -, - \rangle \colon A \otimes A \to \mathbb{K}$ such that

(1) $\langle -, - \rangle$ is of degree $-n$, i.e., non-zero only on $A^i \otimes A^{n-i}$ for $i = 0, \ldots, n$;

(2) $\langle -, - \rangle$ is non-degenerate, namely, the induced map

$$\rho \colon A \to A^\vee, \quad a \mapsto (b \mapsto \langle a, b \rangle)$$

is an isomorphism of degree $-n$;

(3) $\langle ab, c \rangle = \langle a, bc \rangle$ for any $a, b, c \in A$;

(4) $\langle d(a), b \rangle = -(-1)^{|a|} \langle a, d(b) \rangle$ for any $a, b \in A$.

Conditions (3) and (4) imply that $\rho \colon A \to A^\vee$ is a map of dg A-A-bimodules of degree $-n$, where the A-A-bimodule structure on $A^\vee$ is given by

$$(a \otimes b) \cdot \beta(c) = (-1)^{|\beta|(|a|+|b|)+|a|(|b|+|c|)} \beta(bca), \quad \text{for any } \beta \in A^\vee \text{ and } a, b, c \in A.$$

A dg-Frobenius algebra A is said to be *symmetric* if $\langle a, b \rangle = (-1)^{|a||b|} \langle b, a \rangle$ for any $a, b \in A$.

Note that the isomorphism $\rho \colon A \to A^\vee$ gives rise to a degree n product on $A^\vee$:

$$A^\vee \otimes A^\vee \xrightarrow{\rho^{-1} \otimes \rho^{-1}} A \otimes A \xrightarrow{\mu} A \xrightarrow{\rho} A^\vee.$$

When A is a finitely generated free $\mathbb{K}$-module, e.g., when $\mathbb{K}$ is a field, the linear dual of that product becomes a coproduct on A:

$$\Delta \colon A \xrightarrow{\rho} A^\vee \xrightarrow{\mu^\vee} (A \otimes A)^\vee \cong A^\vee \otimes A^\vee \xrightarrow{\rho^{-1} \otimes \rho^{-1}} A \otimes A \tag{3.1}$$

This coproduct is a map of A-A-bimodules.

Remark 3.3 (2-dimensional field theories). Assume that $\mathbb{K}$ is a field. While commutative Frobenius algebras classify 2-dimensional (closed) topological field theories, symmetric Frobenius algebras classify open topological field theories, and non-commutative Frobenius algebras classify planar open topological field theories, see [56] and [61, Corollaries 4.5–4.7]. We do not require commutativity for our algebras.

Example 3.4 (Poincaré duality and relationship to the intersection product). Let M be a closed manifold of dimension n. The graded cohomology ring $(H^*(M;\mathbb{K}), \smile)$ with coefficients in the commutative ring $\mathbb{K}$ is an example of a symmetric dg-Frobenius algebra of dimension n with trivial differential $d = 0$ and pairing given by Poincaré duality.

When $\mathbb{K}$ is a field, the corresponding coproduct $\Delta\colon H^*(M;\mathbb{K}) \to H^*(M;\mathbb{K}) \otimes H^*(M;\mathbb{K})$, as given by (3.1), is the composition

$$\begin{array}{ccc} H^k(M;\mathbb{K}) & \xrightarrow{\ \Delta\ } & \bigoplus_{i+j=n-k} H^{n-i}(M;\mathbb{K}) \otimes H^{n-j}(M;\mathbb{K}) \\ {\scriptstyle [M]\cap}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle [M]\cap\otimes[M]\cap} \\ H_{n-k}(M;\mathbb{K}) & \xrightarrow{(\Delta_M)_*} H_{n-k}(M\times M;\mathbb{K}) \xleftarrow{\ \cong\ } & \bigoplus_{i+j=n-k} H_i(M;\mathbb{K}) \otimes H_j(M;\mathbb{K}) \end{array}$$

where the bottom composition is the linear dual of the cup product, induced by the diagonal $\Delta_M\colon M \to M \times M$. This coproduct is actually also the linear dual of the intersection product on homology; see, e.g., [44, Appendix B] for the relationship between that definition of the intersection product and the one given in Section 2.1.

Applying the above composition of maps to $1 \in H^0(M;\mathbb{K}) \cong \mathbb{K}$ we get a class $\Delta(1) \in \bigoplus_{i+j=n} H^{n-i}(M;\mathbb{K}) \otimes H^{n-j}(M;\mathbb{K})$. Writing also $\Delta(1)$ for its image in $H^n(M\times M;\mathbb{K}) \cong \bigoplus_{i+j=n} H^{n-i}(M;\mathbb{K}) \otimes H^{n-j}(M;\mathbb{K})$, we see that it is characterized as the unique class such that $[M\times M] \cap \Delta(1) = (\Delta_M)_*[M]$. Hence, $\Delta(1)$ maps to the Thom class $\tau_M \in H^n(M\times M, M\times M \setminus M)$ of Section 2.1 in relative cohomology, as the Thom class is determined by this very same relation.

A dg-algebra A is *simply connected* if it is non-negatively graded, $A^0 = \mathbb{K}$, and $A^1 = 0$. The following result of Lambrechts and Stanley shows that, when $\mathbb{K}$ is a field and A is commutative and simply connected, a Frobenius structure on $H^*(A)$ can be "lifted" to A.

Theorem 3.5 ([59, Theorem 1.1]). *Let $\mathbb{K}$ be any field and $\mathcal{A}$ be a simply-connected commutative dg $\mathbb{K}$-algebra equipped with a pairing $\langle -,-\rangle_{\mathcal{A}}\colon \mathcal{A}\otimes\mathcal{A} \to \mathbb{K}$ which induces a graded Frobenius algebra structure of dimension k on its cohomology $H^*(\mathcal{A})$. Then there exists a simply-connected commutative symmetric dg-Frobenius $\mathbb{K}$-algebra A and a zigzag of quasi-isomorphisms of commutative dg-algebras between A and $\mathcal{A}$ inducing an isomorphism $H^*(A) \cong H^*(\mathcal{A})$ of graded Frobenius algebras.*

Example 3.6 (Frobenius models of manifolds). Let M be a simply-connected oriented closed manifold and assume $\mathbb{K} = \mathbb{Q}$. Then the polynomial forms $\mathcal{A}_{\mathrm{pl}}(M) \simeq C^*(M, \mathbb{Q})$ are a strictly commutative, simply-connected model of the cochains. The above theorem then yields a commutative dg-Frobenius algebra $A_M \simeq C^*(M, \mathbb{Q})$, that "lifts" the graded Frobenius structure of $H^*(M; \mathbb{Q})$ to the cochain level.

3.3. Hochschild chains and cochains

We recall here the definition of the Hochschild chain and cochain complexes and their relevance in homological algebra and topology. We will work with the normalized version of the Hochschild complex, assuming that the algebra is unital. Let $\overline{A}$ denote the cokernel of the unit map $\mathbb{K} \to A$.

For any dg $\mathbb{K}$-module (V, d) we denote by $(s^i V, s^i d)$ the i-th shifted module given by $(s^i V)^j = V^{i+j}$ and $s^i d(v) = (-1)^i d(s^i v)$ for any $v \in V$. The definition of the Hochschild complex will use the suspension $s\overline{A}$. For simplicity, we write $\overline{a}$ for the element $sa \in s\overline{A}$ where $a \in \overline{A}$.

Definition 3.7. Let A be a unital dg-algebra. The *Hochschild chain complex of* A is the complex $(C_*(A, A), \partial = \partial_v + \partial_h)$ where

$$C_*(A, A) = \bigoplus_{m \geq 0} (s\overline{A})^{\otimes m} \otimes A$$

and where ∂_v, the *vertical* differential, is given by

$$\begin{aligned}\partial_v(\overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1}) &\\ = -\sum_{i=1}^{m} (-1)^{\varepsilon_{i-1}} \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{d(a_i)} \otimes \overline{a_{i+1}} \otimes \cdots \otimes a_{m+1} &\\ + (-1)^{\varepsilon_m} \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes d(a_{m+1}) &\end{aligned}$$

and ∂_h, the *horizontal* differential, is given by

$$\begin{aligned}\partial_h(\overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1}) &\\ = \sum_{i=1}^{m-1} (-1)^{\varepsilon_i} \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes a_{m+1} &\\ - (-1)^{\varepsilon_{m-1}} \overline{a_1} \otimes \cdots \otimes \overline{a_{m-1}} \otimes a_m a_{m+1} &\\ + (-1)^{(|a_2| + \cdots + |a_{m+1}| - m + 1)|a_1|} \overline{a_2} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} a_1. &\end{aligned}$$

Here we denote $\varepsilon_i = |a_1| + \cdots + |a_i| - i$ and $\varepsilon_0 = 0$.

We will denote by $C_{-m,k}(A, A) = ((s\overline{A})^{\otimes m} \otimes A)^k$ the elements in $(s\overline{A})^{\otimes m} \otimes A$ of total degree k. In particular, $C_k(A, A) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} C_{-m,k}(A, A)$

The Hochschild homology of A is defined to be the homology of $(C_*(A,A), \partial = \partial_v + \partial_h)$, and it is denoted by $\mathrm{HH}_*(A,A)$. Hochschild homology is functorial with respect to maps of unital dg-algebras. Furthermore, a quasi-isomorphism $f\colon A \to A'$ between unital dg-algebras that are flat as $\mathbb{K}$-modules induces an isomorphism

$$\mathrm{HH}_*(f)\colon \mathrm{HH}_*(A,A) \to \mathrm{HH}_*(A',A').$$

Remark 3.8 (The Hochschild complex in algebra and topology). The Hochschild chain complex originates in the context of homological algebra. When A is a dg-algebra which is projective as a $\mathbb{K}$-module, $C_*(A,A)$ is a model for $A \otimes^{\mathbb{L}}_{A\otimes A^{\mathrm{op}}} A$, the derived tensor product of A with itself in the category of A-A-bimodules. Hence, $\mathrm{HH}_*(A,A) = \mathrm{Tor}^{A\otimes A^{\mathrm{op}}}_*(A,A)$.

In topology, when $\mathbb{K} = \mathbb{F}$ is a field, and $A \simeq C^*(X;\mathbb{F})$ is a dg-algebra cochain model for the singular cochains of a simply-connected space X, then there is a quasi-isomorphism $C_*(A,A) \simeq C^*(LX;\mathbb{F})$ between the Hochschild chains of A and the singular cochains of the free loop space of X. This relationship may be deduced over the reals using Chen iterated integrals (as introduced by Chen in [15], see also [34,68]), or over any field using a cosimplicial model for the free loop space (as done by Jones in [48]). A dual version of the result, in terms of the coHochschild complex of the singular chains coalgebra, that works for coefficients in an arbitrary ring $\mathbb{K}$ may be found in [75].

Goodwillie gave in [38] the following "Koszul dual" version of this model of the free loop space that does not assume simple connectivity. Let $\mathbb{K}$ be any commutative ring and assume X is a path-connected space and set instead $A = C_*(\Omega X;\mathbb{K})$, the singular chains on the space of (Moore) loops in X, equipped with the concatenation product. Then there is a quasi-isomorphism $C_*(A,A) \simeq C_*(LX;\mathbb{K})$.

Definition 3.9. Let A be a unital dg-algebra. The *Hochschild cochain complex of A* is the complex $(C^*(A,A), \delta = \delta^v + \delta^h)$ where

$$C^*(A,A) = \prod_{m\geq 0} \mathrm{Hom}_{\mathbb{K}}((s\overline{A})^{\otimes m}, A)$$

and where δ^v is given by

$$\begin{aligned}\delta^v(f)(\overline{a_1}\otimes\cdots\otimes\overline{a_m}) &\\ = d(f(\overline{a_1}\otimes\cdots\otimes\overline{a_m})) &+ \sum_{i=1}^{m}(-1)^{|f|+\varepsilon_{i-1}} f(\overline{a_1}\otimes\cdots\otimes\overline{d(a_i)}\otimes\cdots\otimes\overline{a_m}),\end{aligned}$$

and δ^h by

$$\begin{aligned}\delta^h(f)(\overline{a_1}\otimes\cdots\otimes\overline{a_{m+1}}) &= -(-1)^{(|a_1|-1)|f|}a_1 f(\overline{a_2}\otimes\cdots\otimes\overline{a_{m+1}})\\ &\quad - \sum_{i=1}^{m}(-1)^{|f|+\varepsilon_i} f(\overline{a_1}\otimes\cdots\otimes\overline{a_{i-1}}\otimes\overline{a_ia_{i+1}}\otimes\overline{a_i}\otimes\cdots\otimes\overline{a_{m+1}})\\ &\quad + (-1)^{|f|+\varepsilon_m} f(\overline{a_1}\otimes\cdots\otimes\overline{a_m})a_{m+1},\end{aligned}$$

with $\varepsilon_i = |a_1| + \cdots + |a_i| - i$ and $\varepsilon_0 = 0$ as before.

Denoting by $C^{m,k}(A,A) = \mathrm{Hom}^n_{\mathbb{K}}((s\bar{A})^{\otimes m}, A)$ the submodule of $\mathbb{K}$-linear maps of degree $k \in \mathbb{Z}$, we have $C^k(A,A) = \prod_{m\geq 0} C^{m,k}(A,A)$.

The Hochschild cohomology $\mathrm{HH}^*(A,A)$ of A is defined to be the cohomology of $(C^*(A,A), \delta = \delta_v + \delta_h)$. The Hochschild cochain complex is not as such natural in maps of dg-algebras, but if $f\colon A \to A'$ is a quasi-isomorphism of unital dg-algebras that are flat as $\mathbb{K}$-modules, then there is an isomorphism $\mathrm{HH}^*(A,A) \cong \mathrm{HH}^*(A',A')$. We will see in the next section that the product structure of Hochschild cohomology is also invariant under quasi-isomorphisms.

Remark 3.10 (Gerstenhaber algebra). When A is projective as a $\mathbb{K}$-module, the complex $C^*(A,A)$ is a model for $\mathbb{R}\,\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A,A)$, the derived hom from A to itself in the category of A-A-bimodules. Hence, $\mathrm{HH}^*(A,A) = \mathrm{Ext}^*_{A\otimes A^{\mathrm{op}}}(A,A)$. The Yoneda product on $\mathrm{Ext}^*_{A\otimes A^{\mathrm{op}}}(A,A)$ can be modeled via the chain level cup product $\cup$ on $C^*(A,A)$ of Definition 3.15. The graded algebra $(\mathrm{HH}^*(A,A), \cup)$ may also be equipped with a Lie bracket of degree -1 which is compatible with the cup product. The resulting algebraic structure is known as a Gerstenhaber algebra and was described in [33]. The Gerstenhaber algebra structure on $\mathrm{HH}^*(A,A)$ may be lifted to an E_2-algebra structure at the cochain level on $C^*(A,A)$. This statement is known as the *Deligne conjecture* and was solved in [67].

Remark 3.11 (Duality). For any dg-algebra A the graded hom-tensor adjunction provides an isomorphism

$$C_{-m,*}(A,A)^\vee \cong C^{m,*}(A,A^\vee).$$

If A is a symmetric dg-Frobenius algebra which is a finitely generated free $\mathbb{K}$-module then the isomorphism of A-A-bimodules $A \cong A^\vee$ induces an isomorphism of graded $\mathbb{K}$-modules

$$C_{-m,*}(A,A)^\vee \cong C^{m,*}(A,A^\vee) \cong C^{m,*}(A,A).$$

In particular, if A is a symmetric dg-Frobenius algebra model over a field $\mathbb{F}$ for a simply-connected closed manifold M, e.g., as provided by Theorem 3.5, combining this duality with Remark 3.8 gives an isomorphism $\mathrm{HH}^*(A,A) \cong H_*(LM;\mathbb{F})$. In Section 4 we discuss how the Gerstenhaber algebra structure of $\mathrm{HH}^*(A,A)$ corresponds to the Chas–Sullivan product of Section 2 and a loop bracket that in addition uses the circle action, see also [32].

3.4. Tate–Hochschild complex

In the presence of a Frobenius structure on an algebra A we may combine Hochschild chains and cochains of A into a single unbounded complex through a construction reminiscent of the Tate cohomology of a finite group.

Definition 3.12 ([76]). Let A a symmetric dg-Frobenius $\mathbb{K}$-algebra of dimension $n > 0$. Write $\Delta(1) = \sum_i e_i \otimes f_i \in A \otimes A$. The *Tate–Hochschild complex* $(\mathcal{D}^*(A,A), \delta)$ of A is

the totalization of the double complex

$$\begin{aligned}&\mathcal{D}^{*,*}(A,A)\\&= \cdots \xrightarrow{\partial_h} s^{1-n}C_{-1,*}(A,A) \xrightarrow{\partial_h} s^{1-n}C_{0,*}(A,A) \xrightarrow{\gamma} C^{0,*}(A,A) \xrightarrow{\delta_h} C^{1,*}(A,A) \xrightarrow{\delta_h} \cdots\end{aligned}$$

where $\gamma\colon s^{1-n}C_{0,*}(A,A) \cong s^{1-n}A \to A \cong C^{0,*}(A,A)$ is given by

$$\gamma(s^{1-n}a) = \sum_i (-1)^{|f_i||a|} e_i a f_i, \quad \text{for any } a \in A.$$

The fact that $\partial_h \circ \gamma = 0 = \gamma \circ \delta^h$ follows from (4) Definition 3.2. Here totalization means the direct sum totalization in the Hochschild chains direction and the direct product totalization in the Hochschild cochains direction:

$$\begin{aligned}\mathcal{D}^k(A,A) &= \prod_{p\geq 0} \mathrm{Hom}_{\mathbb{K}}((s\bar{A})^{\otimes p}, A)^k \oplus \bigoplus_{p\in\geq 0} ((s\bar{A})^{\otimes p} \otimes A)^{k-n+1}\\ &= C^k(A,A) \oplus C_{k-n+1}(A,A).\end{aligned}$$

One can equivalently define the Tate–Hochschild complex $\mathcal{D}^*(A,A)$ as the mapping cone of the chain map

$$\widetilde{\gamma}\colon s^{-n}C_*(A,A) \to C^*(A,A) \tag{3.2}$$

defined by $\widetilde{\gamma}(\alpha) = 0$ if $\alpha \in C_{-m,*}(A,A)$ for $m \neq 0$ and $\widetilde{\gamma}(\alpha) = \sum_i (-1)^{|f_i||a|} e_i a f_i$ if $\alpha \in A = C_{0,*}(A,A)$.

Definition 3.13. Let A be a dg-Frobenius algebra with pairing $\langle -,-\rangle_A\colon A \otimes A \to \mathbb{K}$. Define a pairing

$$\langle -,-\rangle_{\mathcal{D}}\colon \mathcal{D}^*(A,A) \otimes \mathcal{D}^*(A,A) \to \mathbb{K}$$

by

$$\langle f, \alpha\rangle_{\mathcal{D}} := \langle f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}), a_{m+1}\rangle_A \qquad \text{and} \qquad \langle \alpha, f\rangle_{\mathcal{D}} := (-1)^{|\alpha||f|}\langle f, \alpha\rangle_{\mathcal{D}}$$

for any $\alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes a_{m+1} \in C_{-m,*}(A,A)$ and $f \in C^{m,*}(A,A)$, and 0 otherwise.

The above pairing is compatible with the Tate–Hochschild differential, i.e., it satisfies

$$\langle \delta x, y\rangle_{\mathcal{D}} = (-1)^{|x|}\langle x, \delta y\rangle_{\mathcal{D}}.$$

Consequently, we obtain an induced pairing $H^*(\mathcal{D}^*(A,A)) \otimes H^*(\mathcal{D}^*(A,A)) \to \mathbb{K}$.

Remark 3.14 (The Tate complex in algebra and topology). Let $\mathbb{K}$ be a field and A a symmetric dg-Frobenius $\mathbb{K}$-algebra A. Then $H^*(\mathcal{D}^*(A,A))$ is isomorphic to the graded $\mathbb{K}$-vector space of morphisms from A to itself in the *singularity category*

$$\mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}}) = \mathcal{D}^b(A \otimes A^{\mathrm{op}})/\mathrm{Perf}(A \otimes A^{\mathrm{op}}),$$

i.e., the Verdier quotient of the bounded derived category of finitely generated dg A-A-bimodules by the full subcategory of perfect dg A-A-bimodules. This statement was originally proven in [87, Proposition 6.9] when A is a (non-graded) symmetric Frobenius algebra and extended in [76, Proposition 3.11] to the case when A is a symmetric dg-Frobenius algebra.

The singularity category was used in [74] to study singularities of algebraic varieties.

In topology, when A is a commutative symmetric dg-Frobenius model for $C^*(M, \mathbb{K})$ for a simply-connected manifold M, using Remarks 3.8 and 3.11, we can think of $\mathcal{D}^*(A, A)$ as a way of connecting the singular chains and cochains on LM into a single unbounded complex via the Euler characteristic of M. Indeed, the map $\gamma\colon A \to A$ in that case takes the product with the element $\sum_i e_i f_i$, that identifies with the Euler class of M. In other words, the map γ is determined by taking a representative of the Poincaré dual of the fundamental class $[M]$ to the Euler characteristic $\chi(M)$ thought of as a top-dimensional cochain on M by using a representative of the volume form. On cohomology this is just multiplication by $\chi(M)$ thought of as a map $\mathbb{K} \cong H^0(A) \to H^n(A) \cong \mathbb{K}$. A symplectic version of the Tate–Hochschild construction has been described and studied in [18, 23] by combining symplectic homology and cohomology via a "V-shaped" Hamiltonian.

3.5. Two operations on Hochschild complexes

We recall the classical cup product on the Hochschild cochains of a dg-algebra, and define afterwards a form of dual operation on the Hochschild chains.

Definition 3.15. Let A be a dg $\mathbb{K}$-algebra. The *cup product*

$$\cup\colon C^{m,*}(A, A) \otimes C^{n,*}(A, A) \to C^{m+n,*}(A, A)$$

is defined on any $f \in C^{m,*}(A, A), g \in C^{n,*}(A, A)$ by the formula

$$f \cup g(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+n}}) = (-1)^{|g|\varepsilon_m} f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) g(\overline{a_{m+1}} \otimes \cdots \otimes \overline{a_{m+n}}),$$

where $\varepsilon_m = \sum_{i=1}^{m} |a_i| - m$.

The cup product gives rise to an associative product of degree 0 on $C^*(A, A)$ that satisfies the graded Leibniz identity with respect to the Hochschild cochains differential δ. Therefore $(C^*(A, A), \delta, \cup)$ is a dg-algebra and, consequently, the induced product on $\mathrm{HH}^*(A, A)$ defines a graded associative algebra structure. This computes the endomorphism graded algebra $\mathrm{Ext}^*_{A \otimes A^{\mathrm{op}}}(A, A)$ with the categorical Yoneda product.

We now describe a product on the Hochschild chains of a symmetric dg-Frobenius algebra that behaves as a "dual" to this cup product, following [76, Section 2.3]. This product has also appeared in a slight variation in, e.g., [1, Section 6] and [55, Example 2.12].

A dg-algebra A is *connected* if it is non-negatively graded and $A^0 = \mathbb{K}$. When A is a Frobenius of dimension n, finitely generated free as a $\mathbb{K}$-module, this implies that also $A^n \cong \mathbb{K}$.

Definition 3.16. Suppose A is a connected symmetric dg-Frobenius $\mathbb{K}$ algebra of dimension $n > 0$. The *algebraic Goresky–Hingston product*

$$*: C_*(A, A) \otimes C_*(A, A) \to C_*(A, A)$$

is defined on any $\alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1}$ and $\beta = \overline{b_1} \otimes \cdots \otimes \overline{b_q} \otimes b_{q+1}$ by the formula

$$\alpha * \beta = \sum_i (-1)^{\eta_i} \overline{b_1} \otimes \cdots \otimes \overline{b_{q+1} e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1} f_i,$$

where $\eta_i = |\alpha||f_i| + |b_{q+1}| + (|\alpha| + n - 1)(|\beta| + n - 1)$. The product $*$ induces a degree zero product on the $(1 - n)$-shifted graded $\mathbb{K}$-module $s^{1-n} C_*(A, A)$.

Note that $*$ does *not* satisfy the Leibniz rule with respect to the Hochschild chains differential ∂. In fact, the product $*$ may be understood as a secondary operation, or a chain homotopy, between two operations. If $p > 0$ and $q > 0$ we do have

$$\partial(\alpha * \beta) - \partial(\alpha) * \beta - (-1)^{|\alpha|+k-1} \alpha * \partial(\beta) = 0.$$

However, if $p = 0$, so that $\alpha = a_1 \in C_{0,*}(A, A) = A$, we may compute

$$\partial(\alpha * \beta) - \partial(\alpha) * \beta - (-1)^{|\alpha|+n-1} \alpha * \partial(\beta) = \sum_i (-1)^{\eta_i + |\beta| - 1 - |b_{q+1}|} \overline{b_1} \otimes \cdots \otimes \overline{b_q} \otimes b_{q+1} e_i a_1 f_i.$$

The case $q = 0$ is analogous.

Note that, for degree reasons, $e_i a_1 f_i$ is only non-zero if $a_1 \in A^0 \cong \mathbb{K}$ and, in such case, $e_i a_1 f_i \in A^n \cong \mathbb{K}$. It follows that $*$ induces a well-defined chain map on the complement of $C_{0,0}(A, A) = A^0 \cong \mathbb{K} \subset C_*(A, A)$, which we call the reduced Hochschild complex.

Definition 3.17. The *reduced Hochschild chain complex* $\overline{C}_*(A, A)$ of a connected dg-algebra A is the subcomplex $\overline{C}_{*,*}(A, A) \subset C_{*,*}(A, A)$ given by $\overline{C}_{0,0}(A, A) = 0$ and $\overline{C}_{i,j}(A, A) = C_{i,j}(A, A)$ for all pairs of integers $(i, j) \neq (0, 0)$. We denote by $\overline{\mathrm{HH}}_*(A, A)$ its homology.

The algebraic Goresky–Hingston product $*$ gives rise to an associative product of degree 0

$$*: s^{1-n} \overline{C}_*(A, A) \otimes s^{1-n} \overline{C}_*(A, A) \to s^{1-n} \overline{C}_*(A, A)$$

that satisfies the graded Leibniz identity with respect to the reduced Hochschild chains differential. The elements that lead to obstructions for the Leibniz rule on $C_*(A, A)$ to be satisfied are now removed in the sub-complex $\overline{C}_*(A, A)$. Consequently, the induced product on $s^{1-n} \overline{\mathrm{HH}}_*(A, A)$ defines a graded associative algebra structure.

3.6. Cyclic A_∞-algebra on the Tate–Hochschild complex

The following natural questions now arise:

(Q_1) In what sense are the products $\cup$ and $*$ dual to each other?

(Q_2) What is the compatibility between $\cup$ and $*$ and what is the general algebraic structure they are part of?

(Q_3) Do $\cup$ and $*$ satisfy a form of homotopy invariance?

(Q_4) Is there a homological interpretation for the product $*$ similar to the interpretation of $\cup$ as the endomorphism algebra in the derived category of A-A-bimodules?

(Q_5) What is the precise relationship between the geometrically defined Chas–Sullivan and Goresky–Hingston operations and $\cup$ and $*$?

Question (Q_5) will be discussed in Section 4, following [71]. The following two statements address the remaining questions (Q_1)–(Q_4), saying in particular that $\cup$ and $*$ naturally combine to a single product on the Tate–Hochschild complex.

Theorem 3.18 ([76, Theorem 6.3, Proposition 6.5]). *Let $\mathbb{K}$ be a field and A be a connected symmetric dg-Frobenius $\mathbb{K}$-algebra of dimension n. There exists a (strictly unital) A_∞-algebra structure $\{m_1, m_2, m_3, \ldots\}$ on $\mathcal{D}^*(A,A) = s^{1-n}C_*(A,A) \oplus C^*(A,A)$ such that*

(1) *$m_1 = \delta$ is the Tate–Hochschild complex differential, m_2 extends both $*$ and $\cup$ (i.e., $m_2|_{s^{1-n}C_*(A,A)} = *$, $m_2|_{C^*(A,A)} = \cup$), and $m_i = 0$ for $i > 3$.*

(2) *The A_∞-algebra is cyclically compatible with the pairing $\langle -,- \rangle_{\mathcal{D}}$:*

$$\langle m_p(\alpha_0 \otimes \cdots \otimes \alpha_{p-1}), \alpha_p \rangle_{\mathcal{D}} = (-1)^{|\alpha_0|(|\alpha_1|+\cdots+|\alpha_p|)} \langle m_p(\alpha_1 \otimes \cdots \otimes \alpha_p), \alpha_0 \rangle_{\mathcal{D}}.$$

(3) *The induced homology product is (graded) commutative, and there is an isomorphism of graded algebras*

$$H^*(\mathcal{D}^*(A,A)) \cong \mathrm{HH}^*_{sg}(A,A),$$

where the latter is the endomorphism algebra from A to itself in the singularity category of A-A-bimodules.

(4) *Connes' operator $B\colon C_*(A,A) \to C_{*-1}(A,A)$ extends to an operator*

$$B_{\mathcal{D}}\colon \mathcal{D}^*(A,A) \to \mathcal{D}^{*-1}(A,A)$$

satisfying $B_{\mathcal{D}} \circ \delta + \delta \circ B_{\mathcal{D}} = 0$, $B_{\mathcal{D}} \circ B_{\mathcal{D}} = 0$, and making $H^*(\mathcal{D}^*(A,A))$ into a BV-algebra.

Statement (3) in Theorem 3.18 provides a homological algebra interpretation for the graded associative algebra structure on $H^*(\mathcal{D}^*(A,A))$, thus giving an answer to (Q_4). We now give answers to questions (Q_1) and (Q_2) by further discussing the kind of algebraic structure on $H^*(\mathcal{D}^*(A,A))$ we obtain from statements (1) and (2) in Theorem 3.18.

The product $m_2\colon \mathcal{D}^*(A,A) \otimes \mathcal{D}^*(A,A) \to \mathcal{D}^*(A,A)$ is associative up to a chain homotopy given by m_3, so it induces an associative product of degree 0 on

$H^*(\mathcal{D}^*(A,A))$, which we denote by

$$\star\colon H^*(\mathcal{D}^*(A,A)) \otimes H^*(\mathcal{D}^*(A,A)) \to H^*(\mathcal{D}^*(A,A)).$$

Furthermore, this product (at the cohomology level) is graded commutative; this is part of statement (4) in Theorem 3.18. Observe that there is an isomorphism

$$H^*(\mathcal{D}^*(A,A)) \cong H^*(\ker(\overline{\gamma})) \oplus H^*(\operatorname{coker}(\overline{\gamma})),$$

where $\overline{\gamma}\colon s^{1-n}C_*(A,A) \to C^*(A,A)$ is the degree $+1$ map defined by $\overline{\gamma} = \widetilde{\gamma} \circ s^{-1}$, where $s^{-1}\colon s^{1-n}C_*(A,A) \to s^nC_*(A,A)$ is the shift map and $\widetilde{\gamma}$ is as defined in (3.2). In this language, the above result implies the existence of a commutative product $\star$ on the direct sum $H^*(\ker(\overline{\gamma})) \oplus H^*(\operatorname{coker}(\overline{\gamma}))$, together with a pairing $\langle -,-\rangle_{\mathcal{D}}$, satisfying the following properties:

Proposition 3.19. *Let A be a connected symmetric dg-Frobenius algebra of dimension n and $(H^*(\mathcal{D}^*(A,A)), \star)$ the Tate–Hochschild cohomology algebra as described above.*

(i) *The pairing $\langle -,-\rangle_{\mathcal{D}}$ of Definition 3.13 is non-degenerate with respect to the "monomial length" chain level filtration on $\mathcal{D}^{*,*}(A,A) = s^{1-n}C_{*,*}(A,A) \oplus C^{*,*}(A,A)$. More precisely, it induces an isomorphism of graded vector spaces*

$$C_{-m,*}(A,A) \xrightarrow{\cong} C^{m,*}(A,A)^\vee.$$

(ii) *For any $x, y, z \in H^*(\ker(\overline{\gamma})) \oplus H^*(\operatorname{coker}(\overline{\gamma}))$ we have $\langle x \star y, z\rangle_{\mathcal{D}} = \langle x, y \star z\rangle_{\mathcal{D}}$.*

(iii) *Both $(H^*(\operatorname{coker}(\overline{\gamma})), \cup)$ and $(H^*(\ker(\overline{\gamma})), *)$ are isotropic sub-algebras of*

$$\left(H^*(\ker(\overline{\gamma})) \oplus H^*(\operatorname{coker}(\overline{\gamma})), \star\right)$$

with respect to the pairing $\langle -,-\rangle_{\mathcal{D}}$.

Statement (i) above follows directly from the fact that the pairing of A is non-degenerate, (ii) follows directly from part (2) of Theorem 3.18, and (iii) from part (1) of Theorem 3.18 together with the way we have defined the pairing $\langle -,-\rangle_{\mathcal{D}}$.

The algebraic structure described in Proposition 3.19 is reminiscent of a *Manin triple*, a notion originally introduced in the context of quantum groups. A Manin triple was originally defined by Drinfeld as a triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ over a field $\mathbb{K}$ such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces and $\mathfrak{g}$ is equipped with a symmetric bilinear pairing $\langle -,-\rangle_{\mathfrak{g}}\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}$ satisfying $\langle [x,y], z\rangle_{\mathfrak{g}} = \langle x, [y,z]\rangle_{\mathfrak{g}}$, inducing an isomorphism $\mathfrak{g}_+ \cong \mathfrak{g}_-^\vee$, and for which $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic Lie sub-algebras. If $\mathfrak{h}$ is a finite-dimensional Lie algebra then there is a 1-1 correspondence between Manin triples with $\mathfrak{g}_+ = \mathfrak{h}$ and Lie bialgebra structures on $\mathfrak{h}$. In particular, if $\mathfrak{g}$ is a Lie bialgebra then one can describe a canonical Lie bialgebra structure on $\mathfrak{g} \oplus \mathfrak{g}^\vee$ called the *Drinfeld double of* $\mathfrak{g}$. Drinfeld showed this construction yields a quasi-triangular Lie bialgebra. A complete reference for these notions and results is [12].

We may interpret the structure of $H^*(\mathcal{D}^*(A,A))$ as a graded commutative version of a Manin triple. More precisely, we may define analogously a *graded commutative Manin triple* to be a triple of graded commutative $\mathbb{K}$-algebras (V, V_+, V_-) over a field $\mathbb{K}$ such that

(i) $V = V_+ \oplus V_-$ as a vector space and V is equipped with a symmetric bilinear pairing $\langle -,-\rangle_V : V \otimes V \to \mathbb{K}$ inducing an isomorphism $V_+ \cong V_-^\vee$,

(ii) for any $a, b, c \in V$, we have $\langle ab, c\rangle_V = \langle a, bc\rangle_V$, and

(iii) both V_+ and V_- are isotropic sub-algebras of V.

As in the Lie case, one can use the duality given by the pairing to reformulate the defining equations of this structure in terms of a type of bialgebra structure on V. More precisely, if W is a finite-dimensional graded commutative algebra, there is a 1-1 correspondence between graded commutative Manin triples with $V_+ = W$ and graded commutative cocommutative *infinitesimal bialgebra* structures on W, as introduced by Joni and Rota in [49]. The data of a graded infinitesimal bialgebra structure on W consists of a product $\cdot : W \otimes W \to W$ of degree 0 and coproduct $\Delta : W \to W \otimes W$ of degree k such that Δ is a derivation of the product, namely

$$\Delta(a \cdot b) = \Delta(a) \cdot b + (-1)^{|a|k} a \cdot \Delta(b),$$

where we define $(a' \otimes a'') \cdot b := a' \otimes (a'' \cdot b)$ and $a \cdot (b' \otimes b'') := (a \cdot b') \otimes b''$. See [5] for more about infinitesimal bialgebras. See [72] for (a non-graded version of) the correspondence between commutative cocommutative infinitesimal bialgebras with Manin triples of commutative algebras and, more generally, between Poisson bialgebras and Manin triples of Poisson algebras.

The following result provides an answer to question (Q_3).

Theorem 3.20 ([77, Theorem 1.1]). *Let $\mathbb{K}$ be a field and $(A, \langle -,-\rangle_A)$ and $(B, \langle -,-\rangle_B)$ be two simply-connected symmetric dg-Frobenius $\mathbb{K}$-algebras of dimension n. Suppose that there is a zigzag of quasi-isomorphisms of dg-algebras*

$$A \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} B.$$

Then there is an isomorphism of algebras

$$(H^*(\mathcal{D}^*(A,A)), \star) \cong (H^*(\mathcal{D}^*(B,B)), \star)$$

restricting to an isomorphism of subalgebras

$$(s^{1-n}\overline{\mathrm{HH}}_*(A,A), *) \cong (s^{1-n}\overline{\mathrm{HH}}_*(B,B), *).$$

The proof the above theorem relies on the homological interpretation of the Tate–Hochschild cohomology algebra as the endomorphism algebra in the singularity category of A-A-bimodules (see Remark 3.14). The isomorphism class of the latter, just like for the Hochschild cohomology algebra with cup product, is an invariant of the quasi-isomorphism type of the underlying dg-algebra. A careful analysis of the relationship

between Tate–Hochschild cohomology and singular Hochschild cohomology allows concluding that the isomorphism $(H^*(\mathcal{D}^*(A,A)),\star) \cong (H^*(\mathcal{D}^*(B,B)),\star)$ restricts to an isomorphism $(s^{1-n}\overline{\mathrm{HH}}_*(A,A),*) \cong (s^{1-n}\overline{\mathrm{HH}}_*(B,B),*)$ in the simply-connected case. We refer to [77] for further details.

A direct consequence of Theorem 3.20 is the following:

Corollary 3.21 ([77, Corollary 1.2]). (1) *Let M be a simply-connected oriented closed manifold of dimension n and A a Poincaré duality model for the cdga of rational polynomial forms $\mathcal{A}_{\mathrm{pl}}(M,\mathbb{Q})$, as provided by Theorem 3.5. The isomorphism class of the graded algebra structure on $s^{1-n}\overline{\mathrm{H}}^*(LM;\mathbb{Q})$ induced by the product $*\colon s^{1-n}\overline{\mathrm{HH}}_*(A,A)^{\otimes 2} \to s^{1-n}\overline{\mathrm{HH}}_*(A,A)$ through the isomorphism $\overline{\mathrm{H}}^*(LM;\mathbb{Q}) \cong \overline{\mathrm{HH}}_*(A,A)$ is independent of the choice of Poincaré duality model $A \simeq \mathcal{A}_{\mathrm{pl}}(M,\mathbb{Q})$.*

(2) *If M and M' are homotopy equivalent simply-connected oriented closed manifolds of dimension n, then the algebra structures on $s^{1-n}\overline{\mathrm{H}}^*(LM;\mathbb{Q})$ and $s^{1-n}\overline{\mathrm{H}}^*(LM';\mathbb{Q})$ are isomorphic.*

3.7. Final remarks

One would like to understand the complete algebraic chain level structure of the Tate–Hochschild complex of a symmetric dg-Frobenius algebra. The type of cyclic A_∞-algebra described in Theorem 3.18 is a finite type version of a notion discussed in [47] and [57] under the name of *pre-Calabi–Yau* algebra. In particular, Theorem 3.18 says that for any symmetric dg-Frobenius algebra there is a pre-Calabi–Yau algebra structure on $C^*(A,A)$ extending the cup product of Hochschild cochains and the algebraic Goresky–Hingston on Hochschild chains. It is explained in [47] how the associator m_3 gives rise to a *double Poisson bracket*. A precise formula for the map m_3 on the Tate–Hochschild complex may be found in [76, Remark 6.4].

This is only the tip of the iceberg of a very rich algebraic structure on the Tate–Hochschild complex. Part (4) of Theorem 3.18 tells us that $B_{\mathcal{D}}$ and the product $\star$ define a *BV-algebra* structure on $H^*(\mathcal{D}^*(A,A)) = H^*(\mathrm{coker}(\gamma)) \oplus H^*(\ker(\gamma))$. By definition, a BV-algebra consists of a triple $(V,\star,B)$ where $(V,\star)$ is a graded commutative algebra, $B\colon V \to V$ is a degree -1 operator satisfying $B\circ B = 0$, and the operation

$$\{x,y\} := B(x\star y) - B(x)\star y - (-1)^{|x|}x\star B(y)$$

is a Lie bracket of degree -1 which is a derivation of $\star$ on each variable, i.e., $\{-,-\}$ is Poisson compatible with $\star$.

The BV-algebra structure on Tate–Hochschild cohomology extends the BV-algebra structure of the Hochschild cohomology of a symmetric dg-Frobenius algebra. Furthermore, in [53] a lift of the BV-algebra structure of Tate–Hochschild cohomology to the chain level is constructed, building upon the framework of [50, 51], solving a cyclic Deligne conjecture for the Tate–Hochschild complex. The Lie bracket associated to the

BV-algebra structure on Tate–Hochschild cohomology gives rise to a compatible (Lie) graded Manin triple structure on $(H^*(\mathcal{D}^*(A,A)), H^*(\mathrm{coker}(\gamma)), H^*(\ker(\gamma)))$ extending the classical Gerstenhaber algebra structure on Hochschild cohomology. This Lie algebra structure on $H^*(\mathcal{D}^*(A,A))$ was also lifted to a cyclic L_∞-algebra structure on $\mathcal{D}^*(A,A)$ in [76]. After dualizing and completing the tensor product appropriately, we obtain on $H^*(\mathcal{D}^*(A,A))$ a graded commutative cocommutative infinitesimal bialgebra equipped with a Gerstenhaber bracket and a Gerstenhaber cobracket that are Lie bialgebra compatible. Furthermore, the Gerstenhaber bracket and the cocommutative coproduct, as well as the Gerstenhaber cobracket and the commutative product, satisfy additional second order compatibility equations. This algebraic structure, which may be called a *Gerstenhaber bialgebra*, is a graded version of a *Poisson bialgebra*, defined and studied in [72].

Gerstenhaber bialgebras are reminiscent of similar structures appearing in the theory of quantum groups, where associated to a Lie bialgebra $\mathfrak{g}$, such as the structure induced on the tangent Lie algebra of a Poisson–Lie group, one may consider the commutative cocommutative Hopf algebra $S(\mathfrak{g})$, the symmetric algebra on the vector space $\mathfrak{g}$, with the Poisson bracket and Poisson cobracket induced by the Lie bialgebra structure on $\mathfrak{g}$. Then one proceeds to deform the product to obtain the non-commutative cocommutative universal enveloping algebra $U(\mathfrak{g})$ and then deforms the coproduct in the Poisson cobracket direction to obtain a non-commutative non-cocommutative Hopf algebra $U_h(\mathfrak{g})$. Motivated by the above discussion and by the question of constructing examples of non-commutative non-cocommutative infinitesimal bialgebras one can replace the notion of Hopf algebra by infinitesimal bialgebra. More precisely, one could ask if given a Poisson bialgebra A there exists a deformation to a (possibly non-commutative non-cocommutative) infinitesimal bialgebra $A[[h]]$ in the direction of the Poisson bracket and cobracket. One may also study analogous questions in the graded setting for Gerstenhaber bialgebras.

Lie bialgebras also appear in S^1-equivariant string topology. In fact, the Chas–Sullivan loop product and the Goresky–Hingston loop coproduct induce a Lie bialgebra structure once we pass to the reduced S^1-equivariant homology of the free loop space of a manifold. This structure generalizes previous constructions of Goldman and Turaev from surfaces to manifolds of arbitrary dimension [37, 80, 83]. In the algebraic context, this construction is modeled by a dg-Lie bialgebra structure on the reduced cyclic chain complex of a dg-Frobenius algebra [16, 19, 71], a construction foreshadowed by Ginzburg's necklace Lie bialgebra [35]. Turaev described the quantization of the Lie bialgebra structure on the zeroth S^1-equivariant homology of the free loop space of a surface in terms of skein invariants of links in 3-manifolds. This quantization has also been studied from an algebraic perspective: in [79] a quantization of Ginzburg's necklace Lie bialgebra of a quiver is constructed and this is generalized in [16] where a quantization of the Lie bialgebra on the cyclic homology of a Frobenius algebra is constructed. We expect that the functorial theory of quantization of Lie bialgebras described by Etingof and Kazhdan in [29] may be adapted to quantize infinitesimal bialgebras in the direction of a compatible bracket and cobracket. This theory should give rise to explicit and interesting examples of non-commutative non-cocommutative infinitesimal bialgebras associated to dg-Frobenius

algebras by quantizing the infinitesimal bialgebra structure of $H^*(\mathcal{D}^*(A,A))$ in the direction of the Gerstenhaber bracket and cobracket.

4. String topology and configuration spaces

In this section we compare the geometrically defined string topology operations of Section 2 with the ones defined algebraically using a dg-Frobenius model as in Section 3, under the assumption that the coefficients $\mathbb{K} = \mathbb{R}$ are the real numbers. The main ingredient is an algebraic model for the Fulton–McPherson compactification of $M \times M \setminus M$, the configuration space of two points in M.

Let M be a simply-connected oriented closed manifold. By a theorem of Lambrechts and Stanley (stated here as Theorem 3.5), applied to the case $\mathbb{K} = \mathbb{R}$, there exists a commutative symmetric dg-Frobenius algebra A quasi-isomorphic to real cochains $C^*(M,\mathbb{R})$. As discussed in Remark 3.8, we have isomorphisms

$$\mathrm{HH}_*(A,A) \cong \mathrm{HH}_*\big(C^*(M;\mathbb{R}), C^*(M;\mathbb{R})\big) \cong H^*(LM;\mathbb{R}). \tag{4.1}$$

Definition 4.1. Define the *relative Hochschild complex* by

$$\underline{C}_*(A,A) = \bigoplus_{m\geq 1} (s\bar{A})^{\otimes m} \otimes A.$$

Because A is commutative, $\underline{C}_*(A,A)$ is a sub-chain complex of $C_*(A,A)$.

The chain complex $\underline{C}_*(A,A)$ may also be regarded as the kernel of the natural chain map $C_*(A,A) \to A$, which models the map cst: $M \to LM$ (see Example 4.15). The isomorphism (4.1) restricts to an isomorphism

$$\underline{\mathrm{HH}}_*(A,A) \cong H^*(LM,M;\mathbb{R}). \tag{4.2}$$

The algebraic Goresky–Hingston product given in Definition 3.16 induces a product on this relative version of the Hochschild chain complex (see also, e.g., [1, Section 6]). The purpose of this section is to sketch a proof of the following result:

Theorem 4.2 ([71, Theorem 1.3]). *Let M be a simply-connected oriented closed manifold with commutative dg-Frobenius algebra model $A \simeq C^*(M;\mathbb{R})$. Then the isomorphism* (4.2)

$$\underline{\mathrm{HH}}_*(A,A) \cong H^*(LM,M;\mathbb{R}),$$

intertwines the algebraic with the topological Goresky–Hingston product of Definitions 2.2 *(dualized) and* 3.16*.*

We will sketch a proof of this theorem following the line of argument of [71]. A similar argument to the one presented here gives the equivalence between the algebraic and topological Chas–Sullivan products of Definitions 2.1 and 3.15 (dualized), giving an

alternative proof of [31, Theorem 11]. Here we focus on the Goresky–Hingston product, the dual of the loop coproduct.

We will use the definition of the coproduct given in Section 2.5. Before embarking into the proof of the theorem in Section 4.4, we will take a closer look at the crucial step in the definition of the coproduct, namely the intersection map, defining a general notion of *intersection products* (see Sections 4.1 and 4.2). Section 4.3 then analyses invariance properties of such intersection products.

Remark 4.3 (Dependence on the manifold M). Note that we can take A *any* commutative dg-Frobenius model of $C^*(M;\mathbb{R})$ in the statement. As the right-hand side in the theorem is model-independent, it follows that the algebraic Goresky–Hingston product on $H_*(A,A)$ does not depend on the particular model A. This partially recovers Corollary 3.21.

We saw in Section 2.3 through a lens space example that the coproduct on $H_*(LM)$ is in general not a homotopy invariant of M, at least with integral coefficients, see Theorem 2.10. In the proof of Theorem 4.2, the topology of M will enter through the homotopy type of the complement of the diagonal $M \times M \setminus M$. This last space identifies with the configuration space of 2 points, a space known to depend in general on more than the homotopy type from the same lens space example, see [62]. We will use a recent result by Campos–Willwacher and Idrissi [11, 46] to obtain an algebraic model for this space over the reals in the case of simply-connected manifolds (together with some compatibility datum).

To simplify presentation and notation, we will show the corresponding statement for the operation

$$H_{*+n-1}(LM;\mathbb{R}) \to H_{*+n-1}(LM,M;\mathbb{R}) \xrightarrow{\vee} H_*(LM,M;\mathbb{R})^{\otimes 2},$$

that is the pre-composition with the canonical map $H_*(LM;\mathbb{R}) \to H_*(LM,M;\mathbb{R})$.

4.1. Intersection products

Recall from Section 2.5 that the loop coproduct can be defined as a relative version of the trivial coproduct $\vee_{\frac{1}{2}}$, intersecting with the figure eights space $\mathrm{Fig}(8) \subset LM$. The crucial step in this definition of the coproduct is the composition

$$R_{\frac{1}{2}} \circ \big(\big(\mathrm{ev}_{0,\frac{1}{2}}\big)^* \tau_M \cap\big)\colon H_*(LM,\mathcal{R}) \to H_{*-n}(\mathrm{Fig}(8),\mathcal{R}), \tag{4.3}$$

see (2.5). Here $\mathcal{R}$ is the subspace of half-constant loops, $\mathrm{ev}_{0,\frac{1}{2}} = (\mathrm{ev}_0, \mathrm{ev}_{\frac{1}{2}})\colon LM \to M \times M$ is the evaluation at 0 and $\frac{1}{2}$, the cochain $\tau_M \in C^n(M \times M, M \times M \setminus M)$ is a representative of the Thom class of the normal bundle of the diagonal $M \to M \times M$, and $R_{\frac{1}{2}}$ is a retraction map. In Sections 4.1–4.3, homology can be taken with integral coefficients.

Note that Fig(8) is the pullback of $\mathrm{ev}_{0,\frac{1}{2}}$ along the diagonal

$$\begin{array}{ccc} \mathrm{Fig}(8) & \longrightarrow & LM \\ \downarrow{\scriptstyle \mathrm{ev}_0} & & \downarrow{\scriptstyle \mathrm{ev}_{0,1/2}} \\ M & \xrightarrow{\Delta} & M \times M, \end{array}$$

and one can show that, just like the evaluation map ev_0, the map $\mathrm{ev}_{0,\frac{1}{2}}$ is a fibration. The map (4.3) is the lift along $\mathrm{ev}_{0,\frac{1}{2}}$ of the intersection product $H_*(M \times M) \xrightarrow{\bullet} H_{*-n}(M)$, taken relative to $\mathcal{R}$. We will think of it as a "relative intersection product" and will now abstract what is needed to define it.

4.1.1. Relative intersection products. The definition of the relative intersection product (4.3) immediately generalizes to the following situation. Suppose $p_{\mathcal{E}}\colon \mathcal{E} \to M \times M$ is a fibration, and $\mathcal{R}$ is a space equipped with maps $p_{\mathcal{R}}\colon \mathcal{R} \to M$ and $f\colon \mathcal{R} \to \mathcal{E}$ such that the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{f} & \mathcal{E} \\ \downarrow{\scriptstyle p_{\mathcal{R}}} & & \downarrow{\scriptstyle p_{\mathcal{E}}} \\ M & \xrightarrow{\Delta} & M \times M \end{array} \tag{4.4}$$

commutes. From this data, we can define the following zigzag of chain maps:

$$C_*(\mathcal{E}) \longrightarrow C_*(\mathcal{E}, \mathcal{E}|_{M\times M\setminus M}) \xleftarrow{\sim} C_*(\mathcal{E}|_{U_M}, \mathcal{E}|_{U_M\setminus M}) \xrightarrow{\cap p_{\mathcal{E}}^*\tau_M} C_{*-n}(\mathcal{E}|_{U_M}) \xleftarrow{\sim} C_{*-n}(\mathcal{E}|_M),$$

where $TM \cong U_M \subset M \times M$ is a tubular neighborhood of the diagonal as in Section 2.1. Both wrong-way maps are quasi-isomorphisms: the first one by excision and the second one since we are pulling back a fibration along the homotopy equivalence $M \xrightarrow{\sim} U_M$. Thus we get a map in homology

$$H_*(\mathcal{E}) \xrightarrow{\mathrm{int}_M} H_{*-n}(\mathcal{E}|_M), \tag{4.5}$$

which we call the *(absolute) intersection product* associated to the fibration $p_{\mathcal{E}}$. To refine this operation to a relative version, we note that the following diagram commutes.

$$\begin{array}{ccccccccc} C_*(\mathcal{E}) & \longrightarrow & C_*(\mathcal{E}, \mathcal{E}|_{M\times M\setminus M}) & \xleftarrow{\sim} & C_*(\mathcal{E}|_{U_M}, \mathcal{E}|_{U_M\setminus M}) & \xrightarrow{\cap p_{\mathcal{E}}^*\tau} & C_{*-n}(\mathcal{E}|_{U_M}) & \xleftarrow{\sim} & C_{*-n}(\mathcal{E}|_M) \\ {\scriptstyle f}\uparrow & & {\scriptstyle f}\uparrow & & {\scriptstyle f}\uparrow & & {\scriptstyle f}\uparrow & & {\scriptstyle f}\uparrow \\ C_*(\mathcal{R}) & = & C_*(\mathcal{R}) & = & C_*(\mathcal{R}) & \xrightarrow{\cap f^* p_{\mathcal{E}}^*\tau} & C_{*-n}(\mathcal{R}) & = & C_{*-n}(\mathcal{R}) \end{array} \tag{4.6}$$

Taking vertical mapping cones, this again defines a zigzag of complexes such that the wrong-way maps are quasi-isomorphisms, and thus we obtain a map in homology

$$H_*(\mathcal{E}, \mathcal{R}) \xrightarrow{\mathrm{int}_M} H_{*-n}(\mathcal{E}|_M, \mathcal{R}) \tag{4.7}$$

which we call the *relative intersection product* associated to the diagram (4.4).

Proposition 4.4. *For $\mathcal{E} = LM$ with $p_{\mathcal{E}} = \mathrm{ev}_{0,\frac{1}{2}} = (\mathrm{ev}_0, \mathrm{ev}_{\frac{1}{2}}) \colon LM \to M \times M$ and $\mathcal{R} \hookrightarrow LM$ the space of half-constant loops, the operation*

$$\mathrm{int}_M \colon H_*(LM, \mathcal{R}) \longrightarrow H_{*-n}(\mathrm{Fig}(8), \mathcal{R})$$

coincides with the corresponding map in the definition (2.5) of the loop coproduct.

Proof. The first three commuting squares in (4.6) are simply spelling out the details in (2.5) (as in (2.2)), with the only difference that a homotopy inverse to excision was chosen in (2.5). The last step follows from the fact that the retraction map $R_{\frac{1}{2}}$ in (2.5) is a homotopy inverse to the inclusion $\mathrm{Fig}(8) \hookrightarrow LM|_{U_M}$ (this is essentially [44, Lemma 2.11]), thus inducing an inverse to the map $H_*(\mathrm{Fig}(8), \mathcal{R}) \to H_*(LM|_{U_M}, \mathcal{R})$ in relative homology. ∎

Similarly, we obtain the loop product as an example of the (non-relative) intersection product:

Proposition 4.5. *For $\mathcal{E} = LM \times LM$ with $p_{\mathcal{E}} = (\mathrm{ev}_0, \mathrm{ev}_0) \colon LM \times LM \to M \times M$ and $\mathcal{R} = \emptyset$, the operation*

$$\mathrm{int}_M \colon H_*(LM \times LM) \to H_{*-n}(\mathrm{Fig}(8))$$

coincides with the corresponding map in the definition (2.4) of the loop product.

The following properties of the relative intersection product follow directly from the definitions.

Proposition 4.6. *The relative intersection product (4.7) is natural in diagrams (4.4) over a fixed manifold M and refines the absolute intersection product (4.5) in the sense that*

$$\begin{array}{ccc} H_*(\mathcal{E}, \mathcal{R}) & \xrightarrow{\mathrm{int}_M} & H_{*-n}(\mathcal{E}|_M, \mathcal{R}) \\ \uparrow & & \uparrow \\ H_*(\mathcal{E}) & \xrightarrow{\mathrm{int}_M} & H_{*-n}(\mathcal{E}|_M) \end{array}$$

commutes. The absolute intersection product is natural in fibrations $p_{\mathcal{E}}$ over a fixed manifold M, and identifies with the classical intersection product of Section 2.1 in the case $\mathcal{E} = M \times M$ with $p_{\mathcal{E}} = \mathrm{id}$:

$$H_*(M \times M) \xrightarrow{\mathrm{int}_M = \bullet} H_{*-n}(M).$$

4.2. Intersection contexts

The definition of the relative intersection product uses the following data from the manifold: the diagram

$$\begin{array}{ccccc} & & U_M \setminus M & \hookrightarrow & M \times M \setminus M \\ & & \downarrow & & \downarrow \\ M & \xrightarrow{\sim} & U_M & \hookrightarrow & M \times M \end{array} \tag{4.8}$$

and the class $\tau_M \in H^n(M \times M, M \times M \setminus M) \cong H^n(U_M, U_M \setminus M)$. This is also the data used to define the classical intersection product. We note that the spaces U_M, $U_M \setminus M$ and $M \times M \setminus M$ only appear in the intermediate steps of the definition.

We now describe a slight generalization of a relative intersection product

$$H_*(\mathcal{E}, \mathcal{R}) \to H_{*-n}(\mathcal{E}|_M, \mathcal{R}).$$

Such a construction may be defined from the data of a diagram (4.4) as before together with the "manifold data" recorded by any homotopy pushout diagram of the shape

$$\begin{array}{ccccc} & & A & \longrightarrow & B \\ & & \downarrow & & \downarrow \\ M & \xrightarrow{\sim} & C & \longrightarrow & M \times M \end{array} \tag{4.9}$$

playing the role of (4.8), equipped with a class $\tau \in H^n(C, A)$. Indeed, if we denote $\mathcal{E}|_A, \mathcal{E}|_B, \mathcal{E}|_C$ the pullback of $\mathcal{E}$ along the maps $A, B, C \to M \times M$, to construct the relative intersection using the corresponding zigzag (4.6), all we need is that the maps

$$C_*(\mathcal{E}, \mathcal{E}|_B) \longleftarrow C_*(\mathcal{E}|_C, \mathcal{E}|_A)$$

and

$$C_*(\mathcal{E}|_C) \longleftarrow C_*(\mathcal{E}|_M)$$

are quasi-isomorphisms. For the second one, this follows as before from our assumption that $M \to C$ is a homotopy equivalence, given that $p_{\mathcal{E}}$ is a fibration. For the first, it follows from the assumption that (4.9) is a homotopy pushout, using Mather's second cube theorem [64, Theorem 25] applied to the pullback of the square along the fibration $p_{\mathcal{E}}$, as a replacement of excision.

Note that the construction goes through for any diagram of topological spaces (not necessarily manifolds) of shape (4.9) satisfying the homotopy pushout condition.

Definition 4.7. We call a homotopy pushout diagram of the shape (4.9) an *intersection context*, and a cohomology class $\tau \in H^n(C, A)$ an *n-orientation.*

Let us define a map between oriented intersection contexts to be a map between the corresponding diagrams (4.9) that is compatible with the orientations. We say that such a

map is an equivalence, if it induces a weak equivalence for each of the spaces A, B and C. Finally, we say that two oriented intersection contexts are equivalent if they can be related by a zigzag of equivalences. A diagram chase gives the following.

Proposition 4.8. *Two equivalent oriented intersection contexts associate the same relative intersection map*

$$\mathrm{int}_M \colon H_*(\mathcal{E}, \mathcal{R}) \to H_{*-n}(\mathcal{E}|_M, \mathcal{R})$$

to a tuple $(\mathcal{E}, \mathcal{R}, p_{\mathcal{E}}, p_{\mathcal{R}}, f)$ *as in diagram* (4.4).

The intersection context we will be using in our proof of Theorem 4.2 is the following. Let FM_2 denote the Fulton–McPherson compactification of the configurations space of two points. It is obtained as the real oriented blowup of $M \times M$ along the diagonal. That is FM_2 is a manifold with boundary whose interior is $M \times M \setminus M$ and with boundary the unit tangent bundle UTM of M. In particular, it fits into the following commuting square

$$\begin{array}{ccccc} & & UTM & \longrightarrow & FM_2 \\ & & \downarrow & & \downarrow \\ M & = & M & \longrightarrow & M \times M. \end{array} \tag{4.10}$$

Proposition 4.9. *Together with the class* $\tau_M \in H^n(M, UTM) \cong H^n(U_M, U_M \setminus M)$, *diagram* (4.10) *defines an oriented intersection context equivalent to* (4.8).

Proof. There is a zigzag of equivalences between the two diagrams coming from the pair of zigzag $UTM \to \bar{U}_M \setminus M \leftarrow U_M \setminus M$ and $FM_2 = FM_2 \leftarrow M \times M \setminus M$, for $\bar{U}_M \setminus M$ an ε-neighborhood of UTM in FM_2. ∎

4.3. Invariance of intersection products

Suppose $f \colon M \to N$ is a smooth map, and that M comes equipped with an intersection context, for example, one of the form (4.10). Composing with f, we obtain an intersection context for N from that of M. We denote the corresponding relative intersection product by $f_* \,\mathrm{int}_M$. By construction, we have the following naturality property:

Lemma 4.10. *For*

$$\begin{array}{ccc} \mathcal{R} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ N & \longrightarrow & N \times N \end{array}$$

as in (4.4), *the square*

$$\begin{array}{ccc} H_*(f^*\mathcal{E}, f^*\mathcal{R}) & \xrightarrow{\mathrm{int}_M} & H_{*-n}(f^*\mathcal{E}|_M, f^*\mathcal{R}) \\ \downarrow & & \downarrow \\ H_*(\mathcal{E}, \mathcal{R}) & \xrightarrow{f_* \,\mathrm{int}_M} & H_{*-n}(\mathcal{E}|_N, \mathcal{R}) \end{array}$$

commutes, where $f^*\mathcal{E}$ *and* $f^*\mathcal{R}$ *are the homotopy pullback of* $\mathcal{E}$ *and* $\mathcal{R}$ *along* $f \times f : M \times M \to N \times N$ *and* $f : M \to N$. *Note that the vertical maps are isomorphisms if* f *is a homotopy equivalence.*

We are interested in the case $\mathcal{E} = LN \to N \times N$ with $\mathcal{R}_N \to N$ the space of half-constant loops, as defined in Section 2.5. In that case, f also induces compatible natural maps $LM \to LN$ and $\mathcal{R}_M \to \mathcal{R}_N$ giving a commuting diagram

$$\begin{array}{ccc} H_*(LM, \mathcal{R}_M) & \xrightarrow{\text{int}_M} & H_{*-n}(\text{Fig}(8)_M, \mathcal{R}_M) \\ \downarrow & & \downarrow \\ H_*(f^*LN, f^*\mathcal{R}_N) & \xrightarrow{\text{int}_M} & H_{*-n}(f^*LN|_M, f^*\mathcal{R}_N) \\ \downarrow & & \downarrow \\ H_*(LN, \mathcal{R}_N) & \xrightarrow{f_* \text{int}_M} & H_{*-n}(\text{Fig}(8)_N, \mathcal{R}_N), \end{array}$$

where again the vertical arrows are all isomorphisms if f is a homotopy equivalence. Hence, comparing the loop coproduct for two manifolds M and N is equivalent to comparing the relative intersection products $f_* \text{int}_M$ and int_N on the pair $(LN, \mathcal{R}_N)$. In general, these are not equal. Otherwise, since the loop coproduct may be described in terms of the above intersection products (as in Proposition 2.12), this would yield a proof for homotopy invariance of the loop coproduct, contradicting Theorem 2.10.

In contrast, the loop product is known to satisfy homotopy invariance (see Theorem 2.9), and the (failed) line of argument suggested above for the coproduct does go through for the product. The essential difference is that the loop product only uses the non-relative intersection product (see Proposition 4.5). Its homotopy invariance follows from the following result.

Theorem 4.11. *Let* $f : M \to N$ *be an orientation-preserving homotopy equivalence of manifolds, each equipped with its intersection context of the form* (4.10). *Then for any fibration* $\mathcal{E} \to N \times N$ *the intersection product*

$$\text{int}_N : H_*(\mathcal{E}) \longrightarrow H_{*-n}(\mathcal{E}|_N)$$

coincides with the transferred intersection product $f_* \text{int}_M$.

Sketch proof. The above theorem is proved in the papers [26, 27, 31, 41] in the context of string topology, i.e., in the special case when $\mathcal{E} = LN \times LN \to N \times N$, as the crucial ingredient in the homotopy invariance of the loop product, and the proofs generalize to our context. The proof of Gruher–Salvatore in [41] is closest to our language, so we follow that paper. Translating to our notation, Theorem 8 in that paper defines a product preserving map θ_f in homology from $(\mathcal{E}, \text{int}_N)$ to $(f^*\mathcal{E}, \text{int}_M)$. This map can be composed by the product-preserving map $(f^*\mathcal{E}, \text{int}_M) \to (\mathcal{E}, f_* \text{int}_M)$ given by the non-relative version of Lemma 4.10. As both maps preserve the product, it is enough to show that they compose to the identity on $\mathcal{E}$. This statement corresponds to [41, last display in the proof

of Proposition 23]. This last computation is only stated in the case of the loop space in that paper, but it comes from an analysis of the maps using Thom isomorphisms that only use what the maps do on the underlying manifolds. ∎

An alternative approach to the above statement is to use parametrized homotopy theory as in [65], identifying the intersection product considered here with the evaluation map of the Costenoble–Waner duality for M.

Remark 4.12. As the example of lens spaces shows (Theorem 2.10), the above theorem does not generalize to the relative intersection product. The above argument fails in that the composition $\iota \circ \theta_f$ may fail to be equal to the identity in relative homology. This is equivalent to the lack of a Thom isomorphism type map in the computation to be an isomorphism in relative homology, relating to the issue discussed in [45, Section 4.10].

4.4. Equivalence between algebraic and geometric models for the loop coproduct

We will now give a sketch of the proof of Theorem 4.2. We first describe real models (in the sense of rational homotopy theory) for each of the steps in the definition of the loop coproduct and compare the final result with the description in Definition 3.16. More precisely, up to crossing with an interval, we can write the geometric coproduct (2.5) as the composition of the following three maps:

$$C_{*+n}(LM \times I, LM \times \partial I) \xrightarrow{J} C_{*+n}(LM, \mathcal{R}) \xrightarrow{\text{int}} C_*(\text{Fig}(8), \mathcal{R}) \xrightarrow{\text{cut}} C_*(LM, M)^{\otimes 2}, \tag{4.11}$$

where the middle map is the intersection product discussed in Sections 4.1 and 4.2. We will give models for each of these three maps. Most of what we do in this section can be done with rational coefficients; real coefficients will only be needed at the very end of the section, when picking a particular model of the configuration space FM_2. For simplicity, we will ignore sign issues in this section.

A major ingredient will be the Eilenberg–Moore theorem, that we will use to give rational models of homotopy pullbacks. We will apply it to the functorial rational model of polynomial forms $\mathcal{A}_{\text{pl}}$, with $\mathcal{A}_{\text{pl}}(X) \simeq C^*(X; \mathbb{Q})$:

Theorem 4.13 (Eilenberg–Moore; see, for instance, [66, Theorem 7.14]). *Suppose that*

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is a homotopy pullback of spaces, such that Z is simply connected and either X or Y are connected. Then the natural map

$$\mathcal{A}_{\text{pl}}(X) \otimes^L_{\mathcal{A}_{\text{pl}}(Z)} \mathcal{A}_{\text{pl}}(Y) \longrightarrow \mathcal{A}_{\text{pl}}(X) \otimes_{\mathcal{A}_{\text{pl}}(Z)} \mathcal{A}_{\text{pl}}(Y) \longrightarrow \mathcal{A}_{\text{pl}}(W)$$

induced by $f^*\colon \mathcal{A}_{\mathrm{pl}}(X) \to \mathcal{A}_{\mathrm{pl}}(W)$ *and* $g^*\colon \mathcal{A}_{\mathrm{pl}}(Y) \to \mathcal{A}_{\mathrm{pl}}(W)$ *is a quasi-isomorphism. Here* $\mathcal{A}_{\mathrm{pl}}(X) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y)$ *denotes the derived tensor product.*

In the following we use the bar construction model for the derived tensor product:

$$\mathcal{A}_{\mathrm{pl}}(X) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y) \;=\; \bigoplus_{p\geq 0} \mathcal{A}_{\mathrm{pl}}(X) \otimes s\overline{\mathcal{A}_{\mathrm{pl}}(Z)}^{\otimes p} \otimes \mathcal{A}_{\mathrm{pl}}(Y),$$

with differential analogous to that of the Hochschild complex of Definition 3.7 (see, for example, [1, (2.5)]). Note that with this definition $\mathcal{A}_{\mathrm{pl}}(X) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y)$ is a quasi-free (i.e., free after forgetting the differential) $\mathcal{A}_{\mathrm{pl}}(X) \otimes \mathcal{A}_{\mathrm{pl}}(Y)$-module. Moreover, there is an $\mathcal{A}_{\mathrm{pl}}(X) \otimes \mathcal{A}_{\mathrm{pl}}(Y)$-module map

$$\mathcal{A}_{\mathrm{pl}}(X) \otimes \mathcal{A}_{\mathrm{pl}}(Y) \longrightarrow \mathcal{A}_{\mathrm{pl}}(X) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y)$$

given by inclusion of the $(p = 0)$-summand.

The map

$$\mathcal{A}_{\mathrm{pl}}(X) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y) \longrightarrow \mathcal{A}_{\mathrm{pl}}(W)$$

in the theorem is then given by projecting onto the $\mathcal{A}_{\mathrm{pl}}(X) \otimes_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y)$ summand on which the map is $f^* \cup g^*$. We obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{\mathrm{pl}}(X) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y) & \xrightarrow{\sim} & \mathcal{A}_{\mathrm{pl}}(W) \\ \uparrow & \nearrow & \\ \mathcal{A}_{\mathrm{pl}}(X) \otimes \mathcal{A}_{\mathrm{pl}}(Y) & & \end{array} \tag{4.12}$$

of $\mathcal{A}_{\mathrm{pl}}(X) \otimes \mathcal{A}_{\mathrm{pl}}(Y)$-modules. Note that $\mathcal{A}_{\mathrm{pl}}(X) \otimes \mathcal{A}_{\mathrm{pl}}(Y)$ is the free $\mathcal{A}_{\mathrm{pl}}(X) \otimes \mathcal{A}_{\mathrm{pl}}(Y)$-module on one generator, and hence a module map out of it is given by a single element in the target. With that in mind (4.12) is saying that both $\mathcal{A}_{\mathrm{pl}}(X) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y)$ and $\mathcal{A}_{\mathrm{pl}}(W)$ come with a distinguished element, which we will call the *pointing* and the equivalence respects that distinguished element. That is, we have the following

Corollary 4.14. *The map* $\mathcal{A}_{\mathrm{pl}}(X) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y) \xrightarrow{\sim} \mathcal{A}_{\mathrm{pl}}(W)$ *of Theorem* 4.13 *is a quasi-isomorphism of pointed* $\mathcal{A}_{\mathrm{pl}}(X) \otimes \mathcal{A}_{\mathrm{pl}}(Y)$*-modules.*

Example 4.15 (The Hochschild complex as a model for LM). The loop space LM can be defined as a pullback

$$\begin{array}{ccc} LM & \xrightarrow{\mathrm{ev}_0} & M \\ \downarrow & & \downarrow{\scriptstyle \Delta_M} \\ PM & \xrightarrow{\mathrm{ev}_0 \times \mathrm{ev}_1} & M \times M. \end{array}$$

We then obtain the following zigzag of pointed $\mathcal{A}_{\mathrm{pl}}(M) \otimes \mathcal{A}_{\mathrm{pl}}(M)$-modules

$$\mathcal{A}_{\mathrm{pl}}(M) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(M)\otimes\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M) \xleftarrow{\sim} \mathcal{A}_{\mathrm{pl}}(PM) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(M)\otimes\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M) \xrightarrow{\sim} \mathcal{A}_{\mathrm{pl}}(LM),$$

where the first arrow is the quasi-isomorphism induced by $\mathcal{A}_{\mathrm{pl}}(PM) \simeq \mathcal{A}_{\mathrm{pl}}(M)$ and the second one comes from Theorem 4.13. The above zigzag thus exhibits $\mathcal{A}_{\mathrm{pl}}(M) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(M)\otimes\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M)$ as a model for $C^*(LM;\mathbb{Q})$. Additionally, we obtain a model for the map $\mathrm{ev}_0\colon LM \to M$ as follows:

$$\begin{array}{ccccc}
\mathcal{A}_{\mathrm{pl}}(M) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(M)\otimes\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M) & \xleftarrow{\sim} & \mathcal{A}_{\mathrm{pl}}(PM) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(M)\otimes\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M) & \xrightarrow{\sim} & \mathcal{A}_{\mathrm{pl}}(LM) \\
{\scriptstyle 1\otimes\mathrm{id}}\uparrow & & {\scriptstyle 1\otimes\mathrm{id}}\uparrow & & {\scriptstyle \mathrm{ev}_0^*}\uparrow \\
\mathcal{A}_{\mathrm{pl}}(M) & = & \mathcal{A}_{\mathrm{pl}}(M) & = & \mathcal{A}_{\mathrm{pl}}(M).
\end{array}$$

Finally, let

$$B\mathcal{A}_{\mathrm{pl}}(M) = \bigoplus_{p\geq 0} \mathcal{A}_{\mathrm{pl}}(M) \otimes \overline{s\mathcal{A}_{\mathrm{pl}}(M)}^{\otimes p} \otimes \mathcal{A}_{\mathrm{pl}}(M)$$

denote the two-sided bar construction computing $\mathcal{A}_{\mathrm{pl}}(M) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M)$. There is a quasi-isomorphism of pointed $\mathcal{A}_{\mathrm{pl}}(M) \otimes \mathcal{A}_{\mathrm{pl}}(M)$-modules

$$B\mathcal{A}_{\mathrm{pl}}(M) \xrightarrow{\sim} \mathcal{A}_{\mathrm{pl}}(M).$$

Since $B\mathcal{A}_{\mathrm{pl}}(M)$ is a quasi-free $\mathcal{A}_{\mathrm{pl}}(M) \otimes \mathcal{A}_{\mathrm{pl}}(M)$-module we obtain quasi-isomorphisms

$$\begin{array}{ccc}
B\mathcal{A}_{\mathrm{pl}}(M) \otimes_{\mathcal{A}_{\mathrm{pl}}(M)\otimes\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M) & \xleftarrow{\sim} & B\mathcal{A}_{\mathrm{pl}}(M) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(M)\otimes\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M) \\
{\scriptstyle 1\otimes\mathrm{id}}\uparrow & \nearrow^{1\otimes\mathrm{id}} & \downarrow{\scriptstyle \sim} \\
\mathcal{A}_{\mathrm{pl}}(M) & \xrightarrow{1\otimes\mathrm{id}} & \mathcal{A}_{\mathrm{pl}}(M) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(M)\otimes\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M).
\end{array}$$

The left-hand side is now exactly the definition of the Hochschild complex:

$$C_*(\mathcal{A}_{\mathrm{pl}}(M), \mathcal{A}_{\mathrm{pl}}(M)) \equiv B\mathcal{A}_{\mathrm{pl}}(M) \otimes_{\mathcal{A}_{\mathrm{pl}}(M)\otimes\mathcal{A}_{\mathrm{pl}}(M)} \mathcal{A}_{\mathrm{pl}}(M).$$

This shows that the Hochschild complex, as a pointed $\mathcal{A}_{\mathrm{pl}}(M)$-module, is a model for $\mathrm{ev}_0\colon LM \to M$, giving a proof of the isomorphism (4.1) in the rational case. (See also [30, Proposition 1].)

Note that the above computations also show that the map $\mathcal{A}_{\mathrm{pl}}(M) \otimes \mathcal{A}_{\mathrm{pl}}(M) \to B\mathcal{A}_{\mathrm{pl}}(M)$ is a model for the fibration $PM \to M \times M$.

Remark 4.16. Given a pair of spaces (X, A) we will use the formula

$$C^*(X, A;\mathbb{Q}) := \mathrm{cone}(C^*(X;\mathbb{Q}) \to C^*(A;\mathbb{Q})),$$

as the definition of relative cochains in the following. Here, as we are working with cochain complexes, by "cone" we mean the following construction:

$$\mathrm{cone}(A \xrightarrow{f} B) = (A \oplus sB, d_A + d_B + f).$$

By naturality of $\mathcal{A}_{\mathrm{pl}}(-)$ we obtain an equivalence

$$C^*(X, A;\mathbb{Q}) \simeq \mathrm{cone}(\mathcal{A}_{\mathrm{pl}}(X) \to \mathcal{A}_{\mathrm{pl}}(A)).$$

Example 4.17. Let $p\colon \mathcal{E} \to Z$ be a fibration and suppose we are given a map $f\colon Y \to Z$ and a class $T \in H^k(Z, Y)$. From this we obtain the homology operation

$$- \cap p^*(T)\colon H_*(\mathcal{E}, \mathcal{E}|_Y) \to H_{*-k}(\mathcal{E}),$$

or dually

$$- \cup p^*(T)\colon H^*(\mathcal{E}) \to H^{*+k}(\mathcal{E}, \mathcal{E}|_Y).$$

More precisely, let $p^*(T) = (u, v) \in \operatorname{cone}(\mathcal{A}_{\mathrm{pl}}^*(\mathcal{E}) \to \mathcal{A}_{\mathrm{pl}}^*(\mathcal{E}|_Y))$, then one obtains the chain map

$$\begin{aligned} \mathcal{A}_{\mathrm{pl}}^*(\mathcal{E}) &\longrightarrow \mathcal{A}_{\mathrm{pl}}^{*+k}(\mathcal{E}, \mathcal{E}|_Y), \\ x &\longmapsto (x \cup u, f_{\mathcal{E}}^*(x) \cup v), \end{aligned}$$

where $f_{\mathcal{E}}\colon \mathcal{E}|_Y \to \mathcal{E}$ is the pullback of the map f. Note that in the above case the pair (u, v) is pulled back from $\operatorname{cone}(\mathcal{A}_{\mathrm{pl}}^*(Z) \to \mathcal{A}_{\mathrm{pl}}^*(Y))$ and thus the formula only uses $\mathcal{A}_{\mathrm{pl}}^*(\mathcal{E})$ and $\mathcal{A}_{\mathrm{pl}}^*(\mathcal{E}|_Y)$ as $C^*(Z)$ and $C^*(Y)$-modules, respectively. We thus obtain that under the equivalence

$$\mathcal{A}_{\mathrm{pl}}(\mathcal{E}) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y) \xrightarrow{\sim} \mathcal{A}_{\mathrm{pl}}(\mathcal{E}|_Y),$$

the above operation is given by

$$\begin{aligned} - \cup p^*(T)\colon \mathcal{A}_{\mathrm{pl}}(\mathcal{E}) &\longrightarrow \operatorname{cone}\big(\mathcal{A}_{\mathrm{pl}}(\mathcal{E}) \to \mathcal{A}_{\mathrm{pl}}(\mathcal{E}) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Y)\big), \\ x &\longmapsto (x \cup u, x \otimes v). \end{aligned}$$

By writing $\mathcal{A}_{\mathrm{pl}}(\mathcal{E}) \simeq \mathcal{A}_{\mathrm{pl}}(\mathcal{E}) \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} \mathcal{A}_{\mathrm{pl}}(Z)$ we can understand this map as being

$$\mathrm{id}_{\mathcal{A}_{\mathrm{pl}}(E)} \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} (- \cup T)$$

where

$$\begin{aligned} - \cup T\colon \mathcal{A}_{\mathrm{pl}}(Z) &\longrightarrow \operatorname{cone}(\mathcal{A}_{\mathrm{pl}}(Z) \to \mathcal{A}_{\mathrm{pl}}(Y)), \\ x &\longmapsto (x \cup u, f^*(x) \cup v). \end{aligned}$$

More succinctly,

$$- \cup p^*(T) = \mathrm{id}_{\mathcal{A}_{\mathrm{pl}}(E)} \otimes^L_{\mathcal{A}_{\mathrm{pl}}(Z)} (- \cup T). \tag{4.13}$$

Let $A \simeq \mathcal{A}_{\mathrm{pl}}(M)$ be any commutative dg-algebra model of M. We can now replace $\mathcal{A}_{\mathrm{pl}}(M)$ by A in the models for LM and PM that we have obtained. As above, let

$$BA = \bigoplus_{p \geq 0} A \otimes (s\bar{A})^{\otimes p} \otimes A$$

be the two-sided bar-resolution, considered as a pointed $A \otimes A$-module. We then have models

$$A \otimes A \longrightarrow BA \quad \text{and} \quad A \longrightarrow C_*(A, A)$$

for the fibrations $\mathrm{ev}_0 \times \mathrm{ev}_1 \colon PM \to M \times M$ and $\mathrm{ev}_0 \colon LM \to M$. The latter map admits a section cst: $M \to LM$, which, under the identification $LM \cong PM \times_{M\times M} M$, is given by the diagonal embedding. Analysing the zigzags in Example 4.15 we find that it is modeled by

$$C_*(A,A) = BA \otimes_{A^{\otimes 2}} A \longrightarrow A \otimes_{A^{\otimes 2}} A \xrightarrow{m} A$$

where $m \colon A \otimes A \to A$ is the multiplication map of A.

4.4.1. Reparametrization map J. In this section, we will give a model of the reparametrization map

$$J \colon sC_*(LM) \cong C_*(LM \times I, LM \times \partial I) \longrightarrow C_*(LM, \mathcal{R}).$$

We have so far seen that the Hochschild complex $C_*(A,A)$ can be used to model the loop space together with the evaluation and inclusion maps $LM \leftrightarrows M$. This model however does not come with a convenient description of the map $\mathrm{ev}_{0,\frac{1}{2}} = (\mathrm{ev}_0, \mathrm{ev}_{\frac{1}{2}}) \colon LM \to M \times M$. We start the section by giving a model of LM that is more convenient to describe that map.

Lemma 4.18. *The fibration* $\mathrm{ev}_{0,\frac{1}{2}} \colon LM \to M \times M$ *admits the following pointed* $A \otimes A$*-module model:*

$$A \otimes A \longrightarrow A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$$

where $A^{\otimes 2}$ *is an* $A^{\otimes 4}$ *module via the map* $(x,y,z,w) \to (xz, yw)$*. As a vector space*

$$A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} = \bigoplus_{p,q\geq 0} (s\bar{A})^{\otimes p} \otimes A \otimes (s\bar{A})^{\otimes q} \otimes A$$

and the map is the inclusion into the summand with $p, q = 0$*.*

Proof. The map $\mathrm{ev}_{0,\frac{1}{2}} = (\mathrm{ev}_0, \mathrm{ev}_{\frac{1}{2}}) \colon LM \to M \times M$ is the product over $M \times M$ of two copies of the path fibration, i.e., we have a homotopy pullback square

$$\begin{array}{ccc} LM & \longrightarrow & PM \times PM \\ \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_{1/2})} & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)\times(\mathrm{ev}_0, \mathrm{ev}_1)} \\ M \times M & \xrightarrow{\Delta_M \times \Delta_M} & (M \times M) \times (M \times M), \end{array}$$

where the unlabeled map $LM \to PM \times PM$ is given by restricting a loop in LM to the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. As in the Example 4.15 we use $A \otimes A \to BA$ as a model for $PM \to M \times M$. Applying Theorem 4.13, we get a model for $\mathrm{ev}_{0,\frac{1}{2}} \colon LM \to M \times M$ as

$$\mathcal{A}_{\mathrm{pl}}(LM) \xleftarrow{\sim} (A \otimes A) \otimes^L_{A^{\otimes 4}} (BA \otimes BA) \simeq A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$$

where one checks that $a \otimes b \in A \otimes A$ is mapped to the right-hand side as claimed in the statement. ∎

Lemma 4.19. *The fibration* $\mathrm{Fig}(8) \to M$ *admits the following pointed* A*-module model*

$$A \to C_*(A,A) \otimes_A C_*(A,A) \cong A \otimes_{A^{\otimes 4}} (BA \otimes BA)$$

with the cut map and inclusions $LM \times LM \xleftarrow{\text{cut}} \mathrm{Fig}(8) \hookrightarrow LM$ *given by the quotient maps*

$$C_*(A,A) \otimes C_*(A,A) \to C_*(A,A) \otimes_A C_*(A,A) \cong A \otimes_{A^{\otimes 4}} (BA \otimes BA) \leftarrow BA \otimes_{A^{\otimes 2}} BA.$$

Proof. The space Fig(8) can be seen to be the pullback of $PM \times PM \to M^2 \times M^2 = M^4$ along the diagonal $M \to M^4$, which gives us a model for $\mathrm{Fig}(8) \to M$ as $A \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$. Consider the two factorizations of the diagonal as $M \to M^2 \to M^4$, where the second map $M^2 \to M^4$ is either $(x,y) \mapsto (x,x,y,y)$ or $(x,y) \mapsto (x,y,x,y)$. The first version exhibits Fig(8) as the pullback of $LM \times LM \to M \times M$ along the diagonal and gives the description of the cut map. The second version exhibits Fig(8) as the pullback of $\mathrm{ev}_{0,\frac{1}{2}} : LM \to M \times M$ giving the description of the inclusion $\mathrm{Fig}(8) \to LM$. ∎

The above description of the figure eight space, allows us now to give a model for the map $\mathcal{R} \to LM$. Let

$$C_*(A,A) \oplus_A C_*(A,A) = \mathrm{cone}\big(C_*(A,A) \oplus C_*(A,A) \to A\big)$$

where the map is the composition $C_*(A,A) \oplus C_*(A,A) \to A \oplus A \to A$, with the second map being the difference.

Lemma 4.20. *The map* $\mathcal{R} \to \mathrm{Fig}(8)$ *is modeled by the map*

$$\begin{aligned} C_*(A,A) \otimes_A C_*(A,A) &\longrightarrow C_*(A,A) \oplus_A C_*(A,A), \\ \overline{\mathbf{a}} \otimes \overline{\mathbf{b}} \otimes c &\longmapsto \varepsilon(\overline{\mathbf{a}})(\overline{\mathbf{b}} \otimes c) \ \oplus \ \varepsilon(\overline{\mathbf{b}})(\overline{\mathbf{a}} \otimes c), \end{aligned}$$

where $\varepsilon(\overline{a_1} \otimes \cdots \otimes \overline{a_p}) = 0$ *if* $p \geq 1$ *and* 1 *if* $p = 0$.

Proof. Consider the commuting diagram

$$\begin{array}{ccc} LM \times LM & \longleftarrow & LM \times M \\ \uparrow & & \uparrow \\ M \times LM & \longleftarrow & M \times M \end{array}$$

of spaces over $M \times M$. By pulling back along the diagonal we obtain

$$\begin{array}{ccc} \mathrm{Fig}(8) & \longleftarrow & LM \\ \uparrow & & \uparrow \\ LM & \longleftarrow & M \end{array}$$

and $\mathcal{R}$ is the pushout of the lower-right triangle; the diagram thus encodes the inclusion map $\mathcal{R} \to \mathrm{Fig}(8)$. Hence, we can get a model for that commuting square in algebra,

by pulling back in the same way the previous square and using the natural part of Theorem 4.13. This becomes

$$\begin{array}{ccc} C_*(A,A) \otimes_A C_*(A,A) & \xrightarrow{\mathrm{id}\otimes\varepsilon} & C_*(A,A) \\ \downarrow{\scriptstyle \varepsilon\otimes\mathrm{id}} & & \downarrow{\scriptstyle \varepsilon} \\ C_*(A,A) & \xrightarrow{\quad\varepsilon\quad} & A, \end{array}$$

from which one can read off the map given in the statement. ∎

We now assemble the models of LM, Fig(8) and $\mathcal{R}$ just obtained to give a model of the reparametrization map:

Proposition 4.21. *In our models, the reparametrization map* $J^*\colon \mathcal{A}_{\mathrm{pl}}(LM, \mathcal{R}) \to s\mathcal{A}_{\mathrm{pl}}(LM)$

$$\mathrm{cone}\big(A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} \to C_*(A,A) \oplus_A C_*(A,A)\big) \xrightarrow{J^*} sC_*(A,A)$$

takes $\alpha = (\overline{a_1} \otimes \cdots \otimes \overline{a_p}) \otimes c \otimes (\overline{b_1} \otimes \cdots \otimes \overline{b_q}) \otimes d$ *of the subcomplex* $A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$ *of the source to*

$$B(\alpha) := \pm\big(\overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes \overline{c} \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_q}\big) \otimes d$$

in the target and maps $\beta \oplus \gamma \in C_*(A,A) \oplus_A C_*(A,A)$ *to* $\beta - \gamma$.

One can give a proof of the above proposition using Chen's iterated integrals, see [71, Section 4.2]. We give here an alternative proof.

Proof. We split the reparametrization into two maps

$$(LM \times I, LM \times \partial I) \xrightarrow{\simeq} (LM, LM \sqcup LM) \to (LM, \mathcal{R})$$

where $LM \sqcup LM \to LM$ maps the two copies of LM to the left (resp. right) half-constant loops. Now there is an equivalence of pairs (an equivalence of the corresponding cones, to be precise)

$$(LM \times I, LM \times \partial I) \xrightarrow{\simeq} (\mathrm{pt}, LM \sqcup \mathrm{pt})$$

via the map that sends one of the LM factors to $\{\mathrm{pt}\}$. We can thus think of the reparametrization map as the zigzag

$$(\mathrm{pt}, LM \sqcup \mathrm{pt}) \xleftarrow{\simeq} (LM, LM \sqcup LM) \to (LM, \mathcal{R}).$$

In our rational model, this becomes a map

$$sC_*(A,A) \xrightarrow{\sim} \mathrm{cone}\begin{pmatrix} A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} \\ \downarrow \\ C_*(A,A) \oplus C_*(A,A) \end{pmatrix} \longleftarrow \mathrm{cone}\begin{pmatrix} A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} \\ \downarrow \\ C_*(A,A) \oplus_A C_*(A,A) \end{pmatrix}$$

where the first map is the inclusion $C_*(A,A) \to C_*(A,A) \oplus C_*(A,A)$ in the first summand, and the second map is the natural projection. It remains to give a left-inverse to the first map. One can check that sending $(\alpha, \beta \oplus \gamma) \to B(\alpha) + \beta - \gamma$ defines such a chain model for such a homotopy inverse. The result follows. ∎

Remark 4.22. Note that the map $B\colon A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} \to sC_*(A,A)$ is not by itself a chain map. Instead, it is a homotopy between the two maps $A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2} \to C_*(A,A)$ given by

$$\overline{\mathbf{a}} \otimes c \otimes \overline{\mathbf{b}} \otimes d \mapsto \pm\varepsilon(\overline{\mathbf{b}})(\overline{\mathbf{a}} \otimes bc),$$

and

$$\overline{\mathbf{a}} \otimes c \otimes \overline{\mathbf{b}} \otimes d \mapsto \pm\varepsilon(\overline{\mathbf{a}})(\overline{\mathbf{b}} \otimes bc),$$

respectively, that model the inclusions $LM \to LM$ of left and right half-constant loops.

4.4.2. Cut map. We give now a model for the cut map used in the definition of the coproduct. Its target is $C^*(LM \times LM, M \times LM \cup LM \times M) \simeq C^*(LM,M)^{\otimes 2}$. Recall that the relative Hochschild chain complex $\underline{C}_*(A,A)$ is the kernel of the (surjective) map $C_*(A,A) \to A$ and hence a model for $C^*(LM,M)$.

Proposition 4.23. *The cut map* $(\mathrm{Fig}(8), \mathcal{R}) \to (LM \times LM, M \times LM \cup LM \times M)$ *can be modeled as the map*

$$\mathrm{cone}\begin{pmatrix} C_*(A,A) \otimes_A C_*(A,A) \\ \downarrow \\ C_*(A,A) \oplus_A C_*(A,A) \end{pmatrix} \xleftarrow{\text{cut}} \mathrm{cone}\begin{pmatrix} C_*(A,A) \\ \downarrow \\ A \end{pmatrix}^{\otimes 2} \xleftarrow{\sim} \underline{C}_*(A,A)^{\otimes 2}$$

defined by

$$\begin{aligned}\mathrm{cut}\big((\overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1}) \otimes \big(\overline{b_1} \otimes \cdots \otimes \overline{b_q} \otimes b_{q+1}\big)\big) \\ = \pm(\overline{a_1} \otimes \cdots \otimes \overline{a_p}) \otimes \big(\overline{b_1} \otimes \cdots \otimes \overline{b_q}\big) \otimes a_{p+1}b_{q+1}\end{aligned}$$

sitting in the subcomplex $C_*(A,A) \otimes_A C_*(A,A)$ *of the target.*

Proof. We have already seen in Lemma 4.19 that the cut map $\mathrm{Fig}(8) \to LM \times LM$ can be described as the quotient map

$$C_*(A,A) \otimes_A C_*(A,A) \longleftarrow C_*(A,A) \otimes C_*(A,A).$$

To see that this map descends to a relative map, we use the same diagrams of spaces as in the proof of Lemma 4.20. More precisely, we note that the second diagram in the proof of Lemma 4.20 maps into the first one. This gives us a map between the pairs consisting of upper-left corner and pushout of lower-right triangle, which models the cut map

$$(\mathrm{Fig}(8), R) \to (LM \times LM, M \times LM \cup LM \times M).$$

The result now follows from naturality of the proof of Lemma 4.20 in the diagram. ∎

4.4.3. Model for the relative intersection. We are left to find a model for the relative intersection step of (4.11). We will use the decomposition of this map given by the relative intersection product using the oriented intersection context (4.10):

$$\begin{array}{ccccccc} C_*(LM) & \to & C_*(LM, LM|_{FM_2}) & \xleftarrow{\sim} & C_*(LM|_M, LM|_{UTM}) & \xrightarrow{\cap p_{\mathcal{E}}^*\tau} & C_{*-n}(LM|_M) \\ {\scriptstyle f}\uparrow & & {\scriptstyle f}\uparrow & & {\scriptstyle f}\uparrow & & \\ C_*(\mathcal{R}) & = & C_*(\mathcal{R}) & = & C_*(\mathcal{R}) & \xrightarrow{\cap f^* p_{\mathcal{E}}^*\tau} & C_{*-n}(\mathcal{R}). \end{array} \tag{4.14}$$

The middle map is the "excision" map

$$C_*(\mathrm{Fig}(8), LM|_{UTM}) \cong C_*(LM|_M, LM|_{UTM}) \xrightarrow{\sim} C_*(LM, LM|_{FM_2})$$

induced by the pullback diagram (4.10), and we need a cochain model for a homotopy inverse of that map. We start by giving a model of the spaces involved, starting from appropriate models of UTM and FM_2.

Suppose now that $A = A_M$ is a Poincare duality model for M, as given by Theorem 3.5. Then A has a coproduct map $\Delta\colon s^{-n}A \to A \otimes A$ (dual to the intersection product of M, see Example 3.4). Lambrechts–Stanley conjectured in [60] explicit commutative dg-algebra models for configuration spaces. This conjecture was shown to hold over the reals by Idrissi and Campos–Willwacher, see [46], [11, Appendix A].

For FM_2, this model is the quotient of the truncated polynomial algebra

$$\mathcal{F}_A = \Big(\frac{A \otimes A[\omega_{1,2}]}{(\omega_{1,2}^2 = 0, (a \otimes 1)\omega_{1,2} = (1 \otimes a)\omega_{1,2})}, d\omega_{1,2} = \Delta(1) \Big),$$

where $\omega_{1,2}$ is a degree $n-1$ class. The spherical fibration UTM more classically admits a model

$$\mathcal{U}_A = \Big(\frac{A[\theta]}{(\theta^2 = 0)}, d\theta = e \Big),$$

where θ has degree $n-1$, representing the fiber, and $e = (m \circ \Delta)(1) \in A$ is the Euler class of M.

These algebras fit into the commutative diagram

$$\begin{array}{ccc} \mathcal{U}_A & \longleftarrow & \mathcal{F}_A \\ \uparrow & & \uparrow \\ A & \xleftarrow[m]{} & A \otimes A, \end{array} \tag{4.15}$$

where the vertical maps are the natural inclusions and the top map takes θ to $\omega_{1,2}$.

Theorem 4.24. *Let A be a Poincare duality model for a simply-connected manifold M. Then the following hold:*

(1) *The diagram (4.15) is a real model for (4.10), i.e., there exists a zigzag of quasi-isomorphisms of squares of commutative dg $\mathbb{R}$-algebras connecting (4.15) to the diagram obtained from (4.10) by applying $\mathcal{A}_{\mathrm{pl}}(-)$.*

(2) *The map* $\phi\colon \mathrm{cone}(A \to \mathcal{U}_A) \to \mathrm{cone}(A \otimes A \to \mathcal{F}_A)$ *taking* $(x, y + z\theta) \in A \oplus s\mathcal{U}_A$ *to* $(\Delta(z), (z \otimes 1)\omega_{1,2}) \in A \otimes A \oplus s\mathcal{F}_A$*, is a model for the homotopy inverse of the map of pairs* $\iota\colon (M, UTM) \xrightarrow{\sim} (M \times M, FM_2)$*, and is a map of* $A \otimes A$*-modules.*

(3) *A representative of the Thom class* $\tau \in \mathrm{cone}(A \to \mathcal{U}_A)$ *is given by*

$$\tau = (e, \theta),$$

where $e = m \circ \Delta(1) \in A$ *is the Euler class as above.*

Proof sketch. Part (1) follows from the works [11] and [46]: the model of FM_2 given here is that of Lambrechts–Stanley, and it is a commutative dg-algebra model of FM_2 over the reals by these two papers. Analysing the models, we see that the maps in diagram (4.10) are modeled as stated, as the multiplication of A models the diagonal, and the class $\omega_{1,2}$ corresponds to the class of the sphere in UTM. Going through the proof in [11] or [46] that $\mathcal{F}_A$ is quasi-isomorphic to $\mathcal{A}_{\mathrm{pl}}(FM_2, \mathbb{R})$, one can strengthen the statements to obtain a zigzag of squares of commutative dg-algebras, as claimed. See also [71, Proposition 8.3].

For part (2), note that part (1) implies that the map $\iota\colon (M, UTM) \to (M \times M, FM_2)$ is modeled by the map $\hat{m}\colon \mathrm{cone}(A \to \mathcal{U}_A) \to \mathrm{cone}(A \otimes A \to \mathcal{F}_A)$ obtained by taking vertical cones of the diagram (4.15). Since diagram (4.15) is a model for diagram (4.10) which in turn is a homotopy pushout, it follows that $\hat{m}$ is a homotopy equivalence. So it is enough to check that ϕ is a one-sided homotopy inverse to $\hat{m}$. The composite $\hat{m} \circ \phi$ takes $(x, y + z\theta)$ to $(ze, z\theta)$ in $\mathrm{cone}(A \to \mathcal{U}_A)$. To see that this is homotopic to the identity note that the quotient map $q\colon \mathrm{cone}(A \to \mathcal{U}_A) \to (s\mathcal{U}_A)/A = As\theta$ given by $q(x, y + z\theta) = zs\theta$ is an equivalence and that $q \circ \hat{m} \circ \phi = q$. One checks that ϕ is a map of $A \otimes A$-modules.

Part (3) follows from the analysis of the models in (1). Alternatively, using the above equivalence $\mathrm{cone}(A \to \mathcal{U}_A) \sim As\theta = s^n A$ and thus there is only one candidate up to a scalar for the Thom class. The scalar is determined by the condition that the image of the Thom class under the isomorphism $H^n(M, UTM) \cong H^n(M \times M, M \times M \setminus M) \to H^n(M \times M)$ is the diagonal class. By (2) this image is $\Delta(1) \in A \times A$, which is the diagonal class. ∎

4.4.4. Proof of Theorem 4.2. We now assemble the results of the previous sections to give a sketch proof of Theorem 4.2. Let

$$\overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1},\ \overline{b_1} \otimes \cdots \otimes \overline{b_q} \otimes b_{q+1} \ \in\ \underline{C}_*(A, A)$$

be two Hochschild chains. By Proposition 4.23, applying the cut map to their tensor product we get

$$\pm(\overline{a_1} \otimes \cdots \otimes \overline{a_p}) \otimes \left(\overline{b_1} \otimes \cdots \otimes \overline{b_q}\right) \otimes a_{p+1}b_{q+1} \in C_*(A, A) \otimes_A C_*(A, A).$$

Next we apply the relative intersection product as given by our algebraic model of diagram (4.14): Let us write $\mathcal{L}_A := A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$ and $\mathcal{R}_A := C_*(A, A) \oplus_A C_*(A, A)$ for our models of LM (as a fibration over $M \times M$) and $\mathcal{R}$ of Section 4.4.1. We then

apply Eilenberg–Moore theorem (Theorem 4.13) to the homotopy pullbacks $LM|_M = LM \times_{M\times M} M$, $LM|_{FM_2} = LM \times_{M\times M} FM_2$ and $LM|_{UTM} = LM \times_{M\times M} UTM$ to obtain that

$$\begin{array}{ccc} \mathcal{L}_A \otimes_{A^{\otimes 2}} \mathcal{U}_A & \longleftarrow & \mathcal{L}_A \otimes_{A^{\otimes 2}} \mathcal{F}_A \\ \uparrow & & \uparrow \\ \mathcal{L}_A \otimes_{A^{\otimes 2}} A & \longleftarrow & \mathcal{L}_A \otimes_{A^{\otimes 2}} A^{\otimes 2} \end{array}$$

is a model for

$$\begin{array}{ccc} LM|_{UTM} & \longrightarrow & LM|_{FM_2} \\ \downarrow & & \downarrow \\ LM|_M & \longrightarrow & LM. \end{array}$$

With this we obtain a model for the diagram (4.14) defining the relative intersection product is equivalent to the diagram

$$\begin{array}{ccccccccc} \mathcal{L}_A & \leftarrow & \mathcal{L}_A \otimes_{A^{\otimes 2}} \mathrm{cone}(A^{\otimes 2} \to \mathcal{F}_A) & \xrightarrow{\sim} & \mathcal{L}_A \otimes_{A^{\otimes 2}} \mathrm{cone}(A \to \mathcal{U}_A) & \xleftarrow{\cup\tau_M} & \mathcal{L}_A \otimes_{A^{\otimes 2}} A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_A & = & \mathcal{R}_A & = & \mathcal{R}_A & \xleftarrow{\cup\tau_M} & \mathcal{R}_A, \end{array}$$

where the first map has degree n, with source

$$\mathcal{L}_A \otimes_{A^{\otimes 2}} A = \left(A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}\right) \otimes_{A^{\otimes 2}} A \cong C_*(A,A) \otimes_A C_*(A,A).$$

Note that the right most commuting square is given by the presentation (4.13) of the relative cup product. Recall from Theorem 4.24 (3) that the Thom class is given by $\tau_M = (e, \theta)$ in our model, so applying the first map to our element gives

$$\pm(\overline{a_1} \otimes \cdots \otimes \overline{a_p}) \otimes \left(\overline{b_1} \otimes \cdots \otimes \overline{b_q}\right) \otimes (a_{p+1}b_{q+1}e, a_{p+1}b_{q+1}\theta)$$

in $C_*(A,A) \otimes_A C_*(A,A) \otimes_A \mathrm{cone}(A \to \mathcal{U}_A) \cong \mathcal{L}_A \otimes_{A^{\otimes 2}} \mathrm{cone}(A \to \mathcal{U}_A)$. Now we apply the explicit inverse of $\mathrm{cone}(A \otimes A \to \mathcal{F}_A) \to \mathrm{cone}(A \to \mathcal{U}_A)$ given in Theorem 4.24 which yields

$$\pm(\overline{a_1} \otimes \cdots \otimes \overline{a_p}) \otimes \left(\overline{b_1} \otimes \cdots \otimes \overline{b_q}\right) \otimes \left(\Delta(a_{p+1}b_{q+1}), (a_{p+1}b_{q+1} \otimes 1)\omega_{1,2}\right)$$

in $\mathcal{L}_A \otimes_{A^{\otimes 2}} \mathrm{cone}(A \otimes A \to \mathcal{F}_A)$. Next applying $\mathrm{cone}(A^{\otimes 2} \to \mathcal{F}_A) \to A^{\otimes 2}$, we obtain

$$\pm(\overline{a_1} \otimes \cdots \otimes \overline{a_p}) \otimes \left(\overline{b_1} \otimes \cdots \otimes \overline{b_q}\right) \otimes \Delta(a_{p+1}b_{q+1})$$

in $\mathcal{L}_A \otimes_{A^{\otimes 2}} A^{\otimes 2} \cong A^{\otimes 2} \otimes_{A^{\otimes 4}} (BA)^{\otimes 2}$. Finally, the reparametrization map J is given by Proposition 4.21 after applying the last identification and yields the formula for the coproduct as

$$\sum \pm\left(\overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes \overline{a_{p+1}e_i} \otimes \overline{b_1} \otimes \cdots \otimes \overline{b_q}\right) \otimes b_{q+1}f_i \in sC_*(A,A)$$

matching the formula for the algebraic Goresky–Hingston product of Definition 3.16 (up to switching the factors, which does not make a difference on cohomology by the graded commutativity of the product, see Theorem 3.18).

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