J. Math. Anal. Appl. 540 (2024) 128621

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Arbitrarily slow decay in the logarithmically averaged Sarnak conjecture

Amir Algom[a,*], Zhiren Wang[b]

[a] *Department of Mathematics, The University of Haifa at Oranim, Tivon 36006, Israel*
[b] *Department of Mathematics, the Pennsylvania State University, University Park, PA 16802, USA*

ARTICLE INFO

Article history:
Received 7 February 2024
Available online 12 June 2024
Submitted by Y. Huang

Keywords:
Möbius disjointness
Zero topological entropy
Topological dynamics

ABSTRACT

In 2017 Tao proposed a variant Sarnak's Möbius disjointness conjecture with logarithmic averaging: For any zero entropy dynamical system (X,T), $\frac{1}{\log N}\sum_{n=1}^{N}\frac{f(T^n x)\mu(n)}{n} = o(1)$ for every $f \in \mathcal{C}(X)$ and every $x \in X$. We construct examples showing that this $o(1)$ can go to zero arbitrarily slowly. Nonetheless, all of our examples satisfy the conjecture.

1. Introduction

A topological dynamical system is a pair (X,T) where X is compact metric space and $T \in \mathcal{C}(X,X)$. If the system (X,T) has zero topological entropy, then Sarnak's Möbius disjointness conjecture [16, Main Conjecture] asserts that

$$\frac{1}{N}\sum_{n=1}^{N}\mu(n)f(T^n x) = o(1), \qquad \text{for every } f \in \mathcal{C}(X) \text{ and every } x \in X. \tag{1}$$

Here $\mu : \mathbb{N} \to \{-1,0,1\}$ denotes the Möbius function. We refer to the recent comprehensive surveys [6,13,3] for references and reports on progress on the conjecture and related topics.

One strong piece of evidence towards the validity of Conjecture (1) is that it is implied by the Chowla conjecture [18, Conjecture 1.1], see [17]. In 2017, Tao [18] introduced logarithmically averaged versions of both Sarnak's and Chowla's conjectures, that were shown to be logically equivalent. Pertinent to this paper, if (X,T) has zero entropy, then the logarithmically averaged Möbius disjointness conjecture [18, Conjecture 1.5] predicts that

* Corresponding author.
E-mail addresses: amir.algom@math.haifa.ac.il (A. Algom), zhirenw@psu.edu (Z. Wang).

https://doi.org/10.1016/j.jmaa.2024.128621
0022-247X/

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{\mu(n)f(T^n x)}{n} = o(1), \qquad \text{for every } f\in\mathcal{C}(X) \text{ and every } x\in X. \tag{2}$$

The surveys [6,13,3] contain information on progress on this conjecture as well. We also mention the work of Frantzikinakis and Host [7] on it.

Recently, we studied the rate of decay in Sarnak's conjecture, showing that there are systems satisfying Sarnak's conjecture for which the $o(1)$ as in (1) decays to zero arbitrarily slowly [1]. The purpose of this paper is to study this problem for the logarithmically averaged Möbius disjointness conjecture (2). Here is our main result:

Theorem 1.1. *For every decreasing and strictly positive sequence $\tau(n)\to 0$ there is a zero entropy dynamical system (X,T) that satisfies:*

1. *There exist $x\in X$ and $f\in\mathcal{C}(X)$ such that $|f(x)|\equiv 1$ and*

$$\liminf_{N\to\infty}\frac{1}{\log(N)\cdot\tau(N)}\sum_{n=1}^{N}\frac{\mu(n)f(T^n x)}{n} > 0.$$

2. *The system (X,T) satisfies conjecture (2).*

A few remarks are in order:

1. Using the summation by parts identity,

$$\sum_{n=1}^{N}\frac{\mu(n)f(T^n x)}{n} \le \sum_{M=1}^{N-1}\frac{1}{M+1}\left(\frac{1}{M}\sum_{n=1}^{M}\mu(n)f(T^n x)\right) + \frac{1}{N}\sum_{n=1}^{N}\mu(n)f(T^n x),$$

   for any $x\in X$ and $f\in\mathcal{C}(X,\mathbb{R}_+)$ in a given dynamical system. So, a decay rate in conjecture (1) would imply a decay rate in conjecture (2). Thus, as a corollary of Theorem 1.1 we see that there can be no decay rate in conjecture (1). Hence, Theorem 1.1 generalizes our previous result [1, Theorem 1.1] about the (lack of a) decay rate in conjecture (1).
2. Theorem 1.1 part (1) is also formally stronger than [1, Theorem 1.1 part (1)] since it is an assertion about the corresponding $\liminf$ rather than $\limsup$.
3. By [4, Corollary 10], if Conjecture (1) holds true then for every zero entropy system (X,T) and $f\in\mathcal{C}(X)$, (1) holds uniformly in $x\in X$. We remark that we expect a similar result to hold also for the logarithmically averaged version (2), following e.g. the arguments of Gomilko-Lemańczyk-de la Rue [9]. We do not know, however, if this has appeared in print.

Our results and methods are also related to [6,12,8,14,2]. We refer to [1, Section 1] for more discussion about this.

We end this introduction with an outline of our construction. Morally, we take advantage of the logarithmic averaging to run a version of our argument from [1] in short intervals, thus obtaining stronger results. More precisely, we consider subshifts of

$$\left(\{-1,0,1\}^{\mathbb{N}}\times\{-1,1\}^{\mathbb{Z}},\, T\right), \text{ where } T(y,z)=(\sigma y,\,\sigma^{y_1}z) \text{ and } \sigma \text{ is the left shift.}$$

Given a rate function τ we first construct two slowly growing sequences $q_k^{(i)}\to\infty$, $i=0,1$. We then construct two subshifts such that their base comes from concatenating words of length $(k+1)^3-k^3$, that have non-zero entries at distance at least $q_k^{(i)}$ from each other. Our space X is a product of these two spaces and a "switch" system: A subshift of $\{0,1\}^{\mathbb{N}}$ generated by elements x satisfying that $x(i) = x(i+1)$ for $k^3 \leq i < (k+1)^3 - 1$. The function f is taken to be

$$f\left((y^{(0)}, z^{(0)}),\, (y^{(1)}, z^{(1)}),\, s\right) = z_0^{(s_1)}.$$

For Theorem 1.1 part (1), our construction of the point $x \in X$ relies on the following observation: For every $k \gg 1$, $i = 0, 1$ and some polynomially growing sequences $M_k^{(i)}$, one may show that for some $c \in [0, q_k^{(0)}]$ or $d \in [0, q_k^{(1)}]$, either

$$\sum_{b=c}^{q_k^{(0)}-1+c} \sum_{n=M_k^{(0)}}^{M_{k+1}^{(0)}} \lambda(q_k^{(0)}(n - M_k^{(0)}) + c + k^3) \cdot \mu(q_k^{(0)}(n - M_k^{(0)}) + b + k^3)$$

or

$$\sum_{b=d}^{q_k^{(1)}-1+d} \sum_{n=M_k^{(1)}}^{M_{k+1}^{(1)}} \lambda(q_k^{(1)}(n - M_k^{(1)}) + c + k^3) \cdot \mu(q_k^{(1)}(n - M_k^{(1)}) + b + k^3)$$

are larger than $\frac{1}{2q_k^{(0)}}$ (or respectively $\frac{1}{2q_k^{(1)}}$) times their statistically expected values, up to a controllable error. Here $\lambda : \mathbb{N} \to \{\pm 1\}$ denotes the Liouville function.

We then construct our point x via working in one of the subshifts in our space: For every k we pick the i giving the inequality above, specifying some digits of the base point and an arithmetic progression in μ to put in some digits of the fiber. The value of the s in the switch coordinate between k^3 and $(k+1)^3$ is determined by the i giving this inequality. With some work, we show that x satisfies Theorem 1.1 part (1).

Finally, to derive part (2) of Theorem 1.1, we show that the systems we construct have bounded measure complexity. This relies on the fact that every ergodic measure ν in the systems we construct is supported on a T-fixed point. The theorem then follows by invoking a recent result of Huang, Wang, and Ye [11]. See Section 3 for more details.

2. Proof of Theorem 1.1 part (1)

2.1. Some preliminaries

Let σ denote the left shift on $\{-1,0,1\}^{\mathbb{Z}}$ as well as on any of the following subspaces: $\{-1,0,1\}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}}$, and $\{-1,1\}^{\mathbb{Z}}$. On $\{-1,0,1\}^{\mathbb{Z}}$ we define the metric

$$d(x,y) = 3^{-\min\{|n| :\, x_n \neq y_n\}}.$$

Also, for every $x \in \{-1,0,1\}^{\mathbb{N}}$ and $k > l \in \mathbb{N}$ let $x|_l^k \in \{-1,0,1\}^{k-l}$ be the word

$$x|_l^k := (x_l, x_{l+1}, ..., x_k),$$

and we use similar notation in the space $\{-1,0,1\}^{\mathbb{Z}}$ as well. Next, for every element $x \in \{-1,0,1\}^{\mathbb{N}}$ and $p \in \mathbb{N}_0$ we define $\sigma^{-p}x \in \{-1,0,1\}^{\mathbb{N}}$ as $\sigma^{-p}x = x$ if $p = 0$, and otherwise

$$\left(\sigma^{-p}x\right)|_1^p = (0, ..., 0), \text{ and for all } n > p,\ \sigma^{-p}x(n) = x(n-p).$$

Now, let

$$Z := \{-1, 0, 1\}^{\mathbb{N}} \times \{-1, 1\}^{\mathbb{Z}}.$$

This is a metric space using the sup-metric on both coordinates. Also, we denote by Π_i, $i = 1, 2$, the coordinate projections on Z. We define the skew-product $T : Z \to Z$ via

$$T(y, z) = (\sigma(y), \sigma^{y_1}(z)).$$

We say that $X \subseteq Z$ is a subshift if it is closed and T-invariant. The following Lemma follows directly from our construction:

Lemma 2.1. *The system (Z, T) satisfies that for every $n \in \mathbb{N}$ and $x = (y, z) \in Z$*

$$T^n(y, z) = \left(\sigma^n y, \sigma^{\sum_{i=1}^n y_i} z\right).$$

2.2. Construction of some zero entropy systems to be used in the proof

Fix a sequence $\tau(n) \to 0$ as in Theorem 1.1. Assuming (as we may) that τ tends to zero sufficiently slowly, we construct a sequence $q_k^{(0)} = q_k^{(0)}(\tau) \to \infty$ that satisfies the following properties:

1. $\frac{1}{2q_k^{(0)}} > \tau(k^3)$, and
2. $q_k^{(0)} < k^{\frac{1}{8}}$.

We also define a sequence $q_k^{(1)}$ via

$$q_k^{(1)} := q_k^{(0)} - 1.$$

Note that we also have $\lim_{k\to\infty} q_k^{(1)} = \infty$.

Next, for every k and $i \in \{0, 1\}$ let

$$A_k^{(i)} := \{k^3 + j \cdot q_k^{(i)} : j \in \mathbb{Z}_+, k^3 \le k^3 + j \cdot q_k^{(i)} < (k+1)^3\} \subseteq \mathbb{N}.$$

For every $k \in \mathbb{N}$ and $i \in \{0, 1\}$ we construct elements $s^{(k,i)} \in \{0, 1\}^{\mathbb{N}}$ such that:

1. For every $k^3 + j \cdot q_k^{(i)} \in A_k^{(i)}$,

$$s^{(k,i)}(k^3 + j \cdot q_k^{(i)}) = 1 \text{ if } j \le \left[\frac{(k+1)^3 - k^3}{q_k^{(i)}}\right] - 1.$$

2. $s^{(k,i)}(n) = 0$ for every integer $n \notin A_k^{(i)}$, or if $n \in A_k^{(i)}$ but $n = k^3 + j \cdot q_k^{(i)}$ with $j > \left[\frac{(k+1)^3 - k^3}{q_k^{(i)}}\right] - 1$.

The following Lemma is an immediate consequence of our construction. Recall the definition of $\sigma^{-p}x$ from Section 2.1.

Lemma 2.2. *For every $k \in \mathbb{N}$ large enough, $i \in \{0, 1\}$ and $p = 0, \ldots, q_k^{(i)}$ we have*

$$\sum_{j \in [k^3, (k+1)^3) \cap \mathbb{Z}} \left(\sigma^{-p} s^{(k,i)}\right)(j) = \left[\frac{(k+1)^3 - k^3}{q_k^{(i)}}\right] - 1.$$

Next, for every $k \in \mathbb{N}$ and $i \in \{0,1\}$ define the truncations

$$R_k^{(i)} = \left\{ \left(\sigma^{-p} s^{(k,i)}\right) |_{k^3}^{(k+1)^3-1} : p = 0, ..., q_k^{(i)} \right\} \subseteq \{-1,0,1\}^{(k+1)^3-k^3}.$$

We now define, for every $i \in \{0,1\}$, the space $P^{(i)}$ of all infinite sequences such that

$$P^{(i)} = \{y \in \{-1,0,1\}^{\mathbb{N}} : y|_{k^3}^{(k+1)^3-1} \in R_k^{(i)} \text{ for all } k \in \mathbb{N}\}.$$

The following Lemma is an immediate consequence of Lemma 2.2, summation by parts, and the fact that the Cesàro mean of the sequences $\frac{1}{q_k^{(i)}}$ tends to 0 for both $i = 0, 1$:

Lemma 2.3. *For every* $i \in \{0,1\}$ *and* $y \in P^{(i)}$,

$$\sum_{j=1}^{k^3-1} y(j) = \sum_{j \le k} \left(\left\lceil \frac{j^3 - (j-1)^3}{q_{j-1}^{(i)}} \right\rceil - 1 \right) = o(k^3)$$

Finally, recalling the definition of the system (Z,T) from Section 2.1, for every $i \in \{0,1\}$ we define the subshift of (Z,T)

$$X_i = \mathrm{cl}\left(\bigcup_{n \in \mathbb{N}_0} T^n \left(P^{(i)} \times \{-1,0,1\}^{\mathbb{Z}} \right) \right).$$

For $j \in \{-1,0,1\}$ we denote by $\bar{j} \in \{-1,0,1\}^{\mathbb{N}}$ the constant element $\bar{j}(k) = i$ for every k.

Claim 2.4. *For every* $i \in \{0,1\}$ *we have* $h(X_i, T) = 0$.

Proof. Fix i. We aim to prove the following statement:

$$\text{For every sequence } n_k \to \infty \text{ and } y \in P^{(i)}, \text{ if } \sigma^{n_k} y \to y' \text{ then } \exists p \text{ such that } \sigma^p y' = \bar{0}. \quad (3)$$

Note that (3) implies the Claim: Indeed, let ν be a T ergodic invariant measure. Let ν_1 be its projection to the first coordinate. Then, by (3) and the ergodic Theorem,[1] ν_1 is the Dirac measure on $\{\bar{0}\}$. It follows that for ν-a.e. (y,z), $T(y,z) = (y,z)$. This shows that ν has zero metric entropy, and the Claim follows from the variational principle [19, Chapter 8].

To prove (3), let $y \in P^{(i)}$. Suppose $\sigma^{n_k} y \to y'$. Let $m \in \mathbb{N}$. Then there is some k_0 such that for all $k > k_0$ we have that

$$y|_{n_k}^{n_k+m} = y'|_1^m.$$

Note that, assuming k is large enough (depending on m), there can be at most 2 non-zero digit in $y|_{n_k}^{n_k+m}$: Indeed, such entries appear in places of the form $\ell^3 + jq_\ell^{(i)} + p$ for some ℓ and $p \in \{0, ..., q_\ell^{(i)}\}$. We make the following two observations:

1. If there is some ℓ and $j_1 < j_2$ such that

$$n_k \le \ell^3 + j_i q_\ell^{(i)} + p \le n_k + m, \quad i = 1, 2,$$

[1] For example, one can apply [5, Exercise 2.3.7] with (3) to see that there exists a ν_1 generic point y that admits a p with $\sigma^p y = \bar{0}$.

then

$$(j_2 - j_1) \cdot q_\ell^{(i)} \leq m.$$

So, assuming $\ell = \ell(k)$ is large enough, we see that $j_2 - j_1 < 1$, a contradiction.

2. If there is some ℓ and j_1, j_2, p_1, p_2 such that

$$n_k \leq \ell^3 + j_1 q_\ell^{(i)} + p_1 \leq n_k + m, \text{ and } n_k \leq (\ell+2)^3 + j_2 q_{\ell+2}^{(i)} + p_2 \leq n_k + m$$

then, since $\ell^3 + j_1 q_\ell^{(i)} + p_1 \leq (\ell+1)^3$ and $(\ell+2)^3 + j_2 q_{\ell+2}^{(i)} + p_2 \geq (\ell+2)^3$, we have that

$$(\ell+2)^3 - (\ell+1)^3 \leq (\ell+2)^3 + j_2 q_{\ell+2}^{(i)} + p_2 - \left(\ell^3 + j_1 q_\ell^{(i)} + p_1\right) \leq m.$$

Assuming ℓ is large enough, this is impossible. Note that the same argument works for with $\ell + 2$ swapped for $\ell + a$ for any $a \geq 2$.

We conclude that for every m the word $y'|_1^m$ consists of 0's, with the exception of at most two non-zero entries (note that these non-zero entries must be the same regardless of m). So, there exists some $j \in \mathbb{N}, j = j(y')$, such that $\sigma^j y' = \bar{0}$, proving (3). □

Finally, let

$$A := \{w \in \{0,1\}^{\mathbb{N}} : w(i) = w(i+1),\, k^3 \leq i < (k+1)^3 - 1\},$$

and define

$$\Sigma := \mathrm{cl}\left(\bigcup_{l \in \mathbb{N}_0} \sigma^l A\right).$$

We require the following Lemma:

Lemma 2.5. $h(\Sigma, \sigma) = 0$.

Proof. As in the proof of Claim 2.4, suppose we show that

$$\text{For every sequence } n_k \to \infty \text{ and } w \in A, \text{ if } \sigma^{n_k} w \to w' \text{ then } \exists p \text{ such that } \sigma^p w' \text{ is a fixed point.} \tag{4}$$

Then the Lemma will follow from the variational principle, since (4) implies that every ergodic measure for (Σ, σ) is a Dirac mass on a σ-fixed point of the form $\bar{i}$ for $i = 0, 1$.

To prove (4), suppose $\sigma^{n_k} w \to w'$ for some $w \in \Sigma$. Let $d(\cdot, \cdot)$ denote the usual distance between a point and a set in $\mathbb{R}$. Let $\mathcal{C}$ denote the set of cubic positive integers. Fix $m \in \mathbb{N}$. Then there are two options:

1. If $\lim_{k\to\infty} d(n_k, \mathcal{C}) = \infty$ then there is some k_0 large enough such that for all $k > k_0$, $[n_k, n_k + m]$ does not contain a cubic number.
2. Otherwise, there is some $k_0 = k_0(m)$ such that for every $k > k_0$ the interval $[n_k, n_k + m]$ may contain at most 1 cubic number.

Now, if (1) happens then for every k large enough all the digits of $w|_{n_k}^{n_k+m} = w'|_1^m$ are the same. If (2) happens then still this might occur. Otherwise, there is some $j = j(w')$ where all the digits $w'|_1^j$ are equal, and then all the digits $w'|_{j+1}^{m}$ are equal, but perhaps the constant digit occurring after j differs from that occurring before j. Note that j must be unique, and does not depend on m. So, either $w' = \bar{i}$ for $i = 0, 1$ or $\sigma^j w' = \bar{i}$, proving (4). □

2.3. Correlations along arithmetic progressions in the Möbius function

Recall the definition of Z from Section 2.1 and let $g : Z \to \{-1, 1\}$ be the function

$$g(y, z) = z_0.$$

For every k, $i = 0, 1$, and r, c such that $r, c \in [0, q_k^{(i)}]$, writing

$$M_k^{(i)} := \sum_{j \le k-1} \left(\left\lceil \frac{j^3 - (j-1)^3}{q_{j-1}^{(i)}} \right\rceil - 1 \right)$$

let

$$S_{r,c}^{k,i} := \sum_{b=r}^{q_k^{(i)}-1+r} \sum_{n=M_k^{(i)}}^{M_{k+1}^{(i)}} \lambda(q_k^{(i)}(n - M_k^{(i)}) + c + k^3) \cdot \mu(q_k^{(i)}(n - M_k^{(i)}) + b + k^3).$$

In the following Lemma we use the construction from Section 2.2.

Lemma 2.6. *For every k and $i \in \{0, 1\}$ and for every two integers $c, r \in [0, q_k^{(i)}]$, let $x \in P^{(i)} \times \{-1, 0, 1\}^{\mathbb{Z}} \subseteq X_i$ be any element such that:*

1. *For $k^3 \le n < (k+1)^3$, $\Pi_1 x(n) = s_k^{(i)}(n - r)$.*
2. *For $M_k^{(i)} \le n < M_{k+1}^{(i)}$, $\Pi_2 x(n) = \lambda\left(q_k^{(i)}(n - M_k^{(i)}) + c + k^3\right)$.*

Then

$$\sum_{n=k^3}^{(k+1)^3} g(T^n x)\mu(n) = S_{r,c}^{k,i} + O(q_k^{(0)}).$$

Note that by the construction of $P^{(i)} \times \{-1, 0, 1\}^{\mathbb{Z}}$ in Section 2.2, there exists an element x as in the statement of the Lemma in that space.

Proof. In this proof we suppress the i in our notation and simply write q, M_k. First, for every two integers $j \in [M_k, M_{k+1}]$ and $b \in [r, q + r - 1]$,

$$\begin{aligned}
\sum_{d=1}^{q(j-M_k)+b+k^3} (\Pi_1 x)(d) &= \sum_{d=1}^{k^3-1} (\Pi_1 x)(d) + \sum_{d=k^3}^{q(j-M_k)+b+k^3-1} (\Pi_1 x)(d) \\
&= M_k + \sum_{d=k^3}^{q(j-M_k)+b+k^3-1} s_k^{(i)}(d - r) \\
&= M_k + \sum_{d=k^3-r}^{q(j-M_k)+b+k^3-r-1} s_k^{(i)}(d) = M_k + j - M_k = j.
\end{aligned}$$

Note the use of Lemma 2.3 in the second equality, and the use of the definition of $s_k^{(i)}$ together with the fact that $M_{k+1} - M_k = \left[\frac{(k+1)^3 - k^3}{q}\right] - 1$ in the last one. Therefore,

$$\begin{aligned}
&\sum_{n=k^3}^{(k+1)^3} g(T^n x)\mu(n) \\
&= \sum_{j=M_k}^{M_{k+1}} \sum_{b=r}^{q+r-1} g(T^{q\cdot(j-M_k)+b+k^3} x)\mu(q\cdot(j-M_k)+b+k^3) + O(q) \\
&= \sum_{j=M_k}^{M_{k+1}} \sum_{b=r}^{q+r-1} g\left(\sigma^{q\cdot(j-M_k)+b+k^3}\Pi_1 x,\, \sigma^{\sum_{d=1}^{q\cdot(j-M_k)+b+k^3}(\Pi_1 x)(d)}\Pi_2 x\right)\mu(q\cdot(j-M_k)+b+k^3) \\
&\quad +O(q) \\
&= \sum_{b=r}^{q+r-1} g\left(\sigma^{q\cdot(j-M_k)+b-r+k^3} s_k^{(i)},\, \sigma^j \Pi_2 x\right)\mu(q(j-M_k)+b+k^3) + O(q) \\
&= \sum_{j=M_k}^{M_{k+1}} \sum_{b=r}^{q+r-1} \lambda(q\cdot(j-M_k)+c+k^3)\cdot\mu(q\cdot(j-M_k)+b+k^3) + O(q)
\end{aligned}$$

Indeed: The first equality follows since $g(T^n x)$ and μ are both bounded sequences, in the second equality we use Lemma 2.1, and in the third equality we are using the previous equation array and the definition of x. This definition along with the definition of $s_k^{(i)}$ justifies the last equality, where we simply get the definition of $S_{r,c}^{k,i}$. □

Remark 2.7. In the setup of Lemma 2.6, we may similarly find another $x \in P^{(i)} \times \{-1, 0, 1\}^{\mathbb{Z}}$ that satisfies the conclusion of Lemma 2.6, but for $-S_{r,c}^{k,i}$. Indeed, this follows from the very same proof by picking $x \in P^{(i)} \times \{-1, 0, 1\}^{\mathbb{Z}}$ to be any element such that for every $k^3 \le n < (k+1)^3$ we have $\Pi_1 x(n) = s_k^{(i)}(n-r)$, and for $M_k^{(i)} \le n < M_{k+1}^{(i)}$ we put $\Pi_2 x(n) = -\lambda\left(q_k^{(i)}(j - M_k^{(i)}) + c + k^3\right)$.

We will also require the following Lemma:

Lemma 2.8. *For every k large enough there is either some $c \in [0, q_k^{(0)})$ such that*

$$S_{c,c}^{k,0} \ge \frac{1}{2q_k^{(0)}} \sum_{m=k^3}^{(k+1)^3} \mu(m)\mu(m) - O(q_k^{(0)}),$$

or some $d \in [0, q_k^{(1)})$ with

$$-S_{d+1,d}^{k,1} \ge \frac{1}{2q_k^{(1)}} \sum_{m=k^3}^{(k+1)^3} \mu^2(m) - O(q_k^{(0)}).$$

Proof. In this proof we again simply write q, M for $q_k^{(0)}, M_k^{(0)}$ respectively. Now, for every $c, r \in [0, q]$,

$$\sum_{c=0}^{q-1} S_{c+r,c}^{k,0} = \sum_{m=1}^{(k+1)^3-k^3} \lambda(m+k^3)\cdot\left(\mu(m+r+k^3) + \ldots + \mu(m+r+q-1+k^3)\right)$$

$$+O\left(\frac{q^2}{(k+1)^3-k^3}\right)$$

So,

$$\sum_{c=0}^{q-1} S_{c,c}^{k,0} = \sum_{m=1}^{(k+1)^3-k^3} \lambda(m+k^3)\cdot\left(\mu(m+k^3)+...+\mu(m+q-1+k^3)\right)+O\left(q^2\right).$$

Similarly,

$$\sum_{c=1}^{q-1} S_{c+1,c}^{k,1} = \sum_{m=1}^{(k+1)^3-k^3} \lambda(m+k^3)\cdot\left(\mu(m+1+k^3)+...+\mu(m+q-1+k^3)\right)+O\left(q^2\right).$$

Combining these equations,

$$\sum_{c=0}^{q-1} S_{c,c}^{k,0} - \sum_{d=1}^{q-1} S_{d+1,d}^{k,1} = \sum_{m=k^3}^{(k+1)^3} \lambda(m)\mu(m)+O\left(q^2\right) = \sum_{m=k^3}^{(k+1)^3} \mu^2(m)+O\left(q^2\right).$$

This implies the Lemma. □

2.4. Construction of the point and system as in Theorem 1.1

Recall that for every k the inequality from Lemma 2.8 is given by $q_k^{(i)}$ where i is either 0 or 1. Recalling the spaces constructed in Section 2.2, we define

$$X := X_0 \times X_1 \times \Sigma. \tag{5}$$

We now construct a point $x \in X$ as follows: For every $k \in \mathbb{N}$ and $k^3 \le n < (k+1)^3$, let $i = 0, 1$ correspond to the term yielding the inequality from Lemma 2.8. We put $\ell := i$ and then define in the ℓ-th coordinate $x^{(\ell)}(n) := x(n)$ where x is as in Lemma 2.6 if $i = 0$ or Remark 2.7 if $i = 1$, corresponding to k, and either $r = c$ and c (if $i = 0$) or $r = d + 1$ and $c = d$ (if $i = 1$) yielding the inequality from Lemma 2.8. Also, for the indices $k^3 \le n < (k+1)^3$ we put ℓ in the Σ-coordinate of x. For all $\ell = 0, 1$ and digits not covered by the procedure above, we make some choice that ensures $x^{(\ell)} \in P^{(\ell)} \times \{-1, 0, 1\}^{\mathbb{Z}}$. Note that by Lemma 2.6 and the construction of $P^{(\ell)}$, such a choice is readily available.

Finally, we make X a dynamical system via the self-map $\hat{T} \in \mathcal{C}(X)$ defined by

$$\hat{T}(p^{(0)}, p^{(1)}, s) = (Tp^{(0)}, Tp^{(1)}, \sigma(s)).$$

The function $f \in \mathcal{C}(X)$ is taken to be

$$f((y^{(0)}, z^{(0)}), (y^{(1)}, z^{(1)}),\ s) = z_0^{(s_0)}.$$

We now prove part (1) of Theorem 1.1 via the following two claims:

Claim 2.9. *We have $h(X, \hat{T}) = 0$.*

Proof. By Claim 2.4 and Lemma 2.5 each factor in the product space X has zero entropy, which implies the assertion via standard arguments. □

Claim 2.10. *For all N large enough,*

$$\sum_{n=1}^{N} \frac{f(\hat{T}^n x)\mu(n)}{n} \geq \tau(N) \sum_{n=1}^{N} \frac{\mu^2(n)}{n} - O(1),$$

where $O(1)$ does not depend on N. In particular,

$$\liminf_{N\to\infty} \frac{1}{(\log N)\cdot \tau(N)} \sum_{n=1}^{N} \frac{f(\hat{T}^n x)\mu(n)}{n} \geq \frac{6}{\pi^2}.$$

Proof. In this proof whenever we write q_k we mean $q_k^{(0)}$ (this is of little consequence since $q_k^{(1)} = q_k^{(0)} - 1$). First, we claim that for every large enough $k \in \mathbb{N}$,

$$\sum_{n=k^3}^{(k+1)^3} f(\hat{T}^n x)\mu(n) \geq \frac{1}{2q_k} \sum_{n=k^3}^{(k+1)^3} \mu(n)^2 - O(q_k). \tag{6}$$

Indeed, this follows since by our construction,

$$\sum_{n=k^3}^{(k+1)^3} f(\hat{T}^n x)\mu(n) = \sum_{n=k^3}^{(k+1)^3} g(\hat{T}^n x^{(\ell)}(n))\mu(n)$$

where g and $x^{(\ell)}(n)$ are as in Lemma 2.6 (corresponding to the parameters as in the choice of x). Then (6) follows directly from a combination of Lemma 2.6 and Lemma 2.8, together with the construction of x and of q.

Given N let N' be such that $(N')^3$ is the largest cube satisfying $(N')^3 \leq N$. Then

$$N^{\frac{1}{3}} - 1 \leq N' \leq N^{\frac{1}{3}}.$$

And,

$$\begin{aligned}
&\sum_{k=1}^{N} \frac{f(\hat{T}^n x)\mu(n)}{n} \\
&= \sum_{k=1}^{N'-1} \sum_{n=k^3}^{(k+1)^3} \frac{f(\hat{T}^n x)\mu(n)}{n} + \sum_{n=(N')^3}^{N} \frac{f(\hat{T}^n x)\mu(n)}{n} \\
&= \sum_{k=1}^{N'-1} \sum_{n=k^3}^{(k+1)^3} \frac{f(\hat{T}^n x)\mu(n)}{k^3} - O\left(\sum_{k=1}^{N'-1} ((k+1)^3 - k^3)\left(\frac{1}{k^3} - \frac{1}{(k+1)^3}\right)\right) - O\left(\sum_{n=(N')^3}^{N} \frac{1}{n}\right)
\end{aligned}$$

We now make use of (6) and get

$$\begin{aligned}
&\sum_{k=1}^{N} \frac{f(\hat{T}^n x)\mu(n)}{n} \\
&\geq \sum_{k=1}^{N'-1} \left(\sum_{n=k^3}^{(k+1)^3} \frac{\mu^2(n)}{2q_k k^3} - O\left(\frac{q_k^2}{k^3}\right)\right) + O\left(\sum_{k=1}^{N'-1} \frac{1}{k^2}\right) - O\left(\log N - \log (N')^3\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2q_{N'-1}} \sum_{k=1}^{N'-1} \sum_{n=k^3}^{(k+1)^3} \frac{\mu^2(n)}{k^3} - O\left(\sum_{k=1}^{N'-1} \frac{q_k^2}{k^3}\right) - O\left(\sum_{k=1}^{N'-1} \frac{1}{k^2}\right) - O\left(\log \frac{N}{(N')^3}\right) \\
&\geq \frac{1}{2q_{N'-1}} \sum_{k=1}^{N'-1} \sum_{n=k^3}^{(k+1)^3} \frac{\mu^2(n)}{n} - O(1) \\
&\geq \tau(N) \sum_{k=1}^{N'-1} \sum_{n=k^3}^{(k+1)^3} \frac{\mu^2(n)}{n} - O(1) \\
&\geq \tau(N) \sum_{n=1}^{N} \frac{\mu^2(n)}{n} - O\left(\tau(N) \sum_{n=(N')^3+1}^{N} \frac{1}{n}\right) - O(1) \\
&\geq \tau(N) \sum_{n=1}^{N} \frac{\mu^2(n)}{n} - O(1).
\end{aligned}
$$

Note that we made of the facts that $q_k \leq k^{\frac{1}{8}}$ and $\frac{1}{2q_k} > \tau(k^3)$ in the computations. The proof of the Claim, and thus of Theorem 1.1, follows immediately by the standard fact that $\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{\mu^2(n)}{n} = \frac{6}{\pi^2}$. $\square$

3. Proof of Theorem 1.1 part (2)

In this Section we prove Part (2) of Theorem 1.1. That is, we show that the system $(X, \hat{T})$ given in (5) satisfies the logarithmically averaged Möbius disjointness conjecture (2). In fact, we will prove a stronger claim, that $(X, \hat{T})$ satisfies the "usual" Möbius disjointness conjecture (1).

To this end, we will invoke the following (special case of a) Theorem of Huang, Wang, and Ye [11]: Let ρ be an invariant measure for $(X, \hat{T})$. Letting d be the sup metric on $X = X_0 \times X_1 \times \Sigma$, for every n define a metric on X via

$$\bar{d}(x, y) = \frac{1}{n} \sum_{i=0}^{n-1} d(\hat{T}^i x, \hat{T}^i y).$$

Let $\epsilon > 0$ and let

$$S_n(d, \rho, \epsilon) = \left\{ \min m : \exists x_1, ..., x_m \text{ s.t. } \rho\left(\bigcup_{i=1}^{m} B_{\bar{d}}(x_i, \epsilon)\right) > 1 - \epsilon \right\}.$$

We say that ρ has bounded measure complexity if for every $\epsilon > 0$ we have that $S_n(d, \rho, \epsilon) = O_{\epsilon,\rho}(1)$.

Theorem 3.1. *[11, Theorem 1.1] If every invariant measure ρ has bounded measure complexity then $(X, \hat{T})$ satisfies the Möbius disjointness conjecture (1).*

Thus, via Theorem 3.1, if we prove the following Claim then Theorem 1.1 part (2) will follow:

Claim 3.2. *The system $(X, \hat{T})$ has bounded measure complexity.*

Proof. Recall that

$$X = X_0 \times X_1 \times \Sigma$$

where all these spaces were constructed in Section 2.2. We now show that (3) and (4), that were already proved in Section 2.2, imply the Claim: Indeed, let ν be a $\hat{T}$ ergodic invariant measure. Let $\tilde{\nu}$ be its projection to $\Pi_1 X_1 \times \Pi_1 X_2 \times \Sigma$, where Π_1 is the projection to the first coordinate. Then, by (3) and (4) and the ergodic Theorem, $\tilde{\nu}$ is the Dirac measure on $\{\overline{0}\} \times \{\overline{0}\} \times \{\overline{i}\}$ for some $i \in \{0,1\}$. It follows that for ν-a.e. x we have that $\hat{T}x = x$. Therefore, for any invariant measure ρ we have that for ρ-a.e. x, $\hat{T}x = x$. This clearly implies that ρ has bounded measure complexity, as claimed. □

Thus, Theorem 1.1 part (2) is proved.

Finally, we make two more remarks. First, in [10] it is shown that systems with bounded measure complexity have zero entropy. So, for an alternative proof of Claims 2.4 and 2.5 we could have argued (as we do above) that (3) and (4) imply bounded measure complexity, and then appeal directly to [10]. Also, via Claims 2.4 and 2.5 and their proof, an alternative proof of Theorem 1.1 part (2) may be derived from [15] using the arguments presented in [6, Section 3.4.1].

Acknowledgments

This research was supported by Grant No. 2022034 from the United States - Israel Binational Science Foundation (BSF), Jerusalem, Israel. Z.W. was also supported by the NSF grant DMS-1753042 and a von Neumann Fellowship at the IAS.

References

[1] Amir Algom, Zhiren Wang, Arbitrarily slow decay in the Möbius disjointness conjecture, in: Ergodic Theory and Dynamical Systems, 2022.
[2] Dmitry Dolgopyat, Changguang Dong, Adam Kanigowski, Péter Nándori, Flexibility of statistical properties for smooth systems satisfying the central limit theorem, Invent. Math. 230 (1) (2022) 31–120.
[3] Michael Drmota, Mariusz Lemańczyk, Clemens Müllner, Joël Rivat, Some recent developments on the Sarnak conjecture.
[4] E.H. el Abdalaoui, Joanna Kułaga-Przymus, Mariusz Lemańczyk, Thierry de la Rue, Möbius disjointness for models of an ergodic system and beyond, Isr. J. Math. 228 (2) (2018) 707–751.
[5] Manfred Einsiedler, Thomas Ward, Ergodic Theory with a View Towards Number Theory, Graduate Texts in Mathematics, vol. 259, Springer-Verlag, London, 2011.
[6] Sébastien Ferenczi, Joanna Kułaga-Przymus, Mariusz Lemańczyk, Sarnak's conjecture: what's new, in: Ergodic Theory and Dynamical Systems in Their Interactions with Arithmetics and Combinatorics, in: Lecture Notes in Math., vol. 2213, Springer, Cham, 2018, pp. 163–235.
[7] Nikos Frantzikinakis, Bernard Host, The logarithmic Sarnak conjecture for ergodic weights, Ann. Math. (2) 187 (3) (2018) 869–931.
[8] Krzysztof Frączek, Adam Kanigowski, Mariusz Lemańczyk, Prime number theorem for regular Toeplitz subshifts, Ergod. Theory Dyn. Syst. 42 (4) (2022) 1446–1473.
[9] Alexander Gomilko, Mariusz Lemańczyk, Thierry de la Rue, Möbius orthogonality in density for zero entropy dynamical systems, Pure Appl. Funct. Anal. 5 (6) (2020) 1357–1376.
[10] Wen Huang, Jian Li, Jean-Paul Thouvenot, Leiye Xu, Xiangdong Ye, Bounded complexity, mean equicontinuity and discrete spectrum, Ergod. Theory Dyn. Syst. 41 (2) (2021) 494–533.
[11] Wen Huang, Zhiren Wang, Xiangdong Ye, Measure complexity and Möbius disjointness, Adv. Math. 347 (2019) 827–858.
[12] Adam Kanigowski, Mariusz Lemańczyk, Maksym Radziwiłł, Prime number theorem for analytic skew products, preprint, arXiv:2004.01125, 2020.
[13] Joanna Kułaga-Przymus, Mariusz Lemańczyk, Sarnak's conjecture from the ergodic theory point of view, Encyclopedia Complexity Systems Sci. (2024), in press, https://link-springer-com.ezproxy.haifa.ac.il/referenceworkentry/10.1007/978-1-0716-2388-6_735.
[14] Zhengxing Lian, Ruxi Shi, A counter-example for polynomial version of Sarnak's conjecture, Adv. Math. 384 (2021) 107765.
[15] Kaisa Matomäki, Maksym Radziwiłł, Multiplicative functions in short intervals, Ann. Math. (2) 183 (3) (2016) 1015–1056.
[16] Peter Sarnak, Mobius randomness and dynamics, Not. S. Afr. Math. Soc. 43 (2) (2012) 89–97.
[17] Terence Tao, The Chowla conjecture and the Sarnak conjecture, https://terrytao.wordpress.com/2012/10/14/the-chowla-conjecture-and-the-sarnak-conjecture/.
[18] Terence Tao, Equivalence of the logarithmically averaged Chowla and Sarnak conjectures, in: Number Theory—Diophantine Problems, Uniform Distribution and Applications, Springer, Cham, 2017, pp. 391–421.
[19] Peter Walters, An Introduction to Ergodic Theory, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982.