

NATURALITY AND INNERNESS FOR MORPHISMS OF COMPACT GROUPS AND (RESTRICTED) LIE ALGEBRAS

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ABSTRACT. An extended derivation (endomorphism) of a (restricted) Lie algebra L is an assignment of a derivation (respectively) of L' for any (restricted) Lie morphism $f : L \rightarrow L'$, functorial in f in the obvious sense. We show that (a) the only extended endomorphisms of a restricted Lie algebra are the two obvious ones, assigning either the identity or the zero map of L' to every f ; and (b) if L is a Lie algebra in characteristic zero or a restricted Lie algebra in positive characteristic, then L is in canonical bijection with its space of extended derivations (so the latter are all, in a sense, inner). These results answer a number of questions of G. Bergman.

In a similar vein, we show that the individual components of an extended endomorphism of a compact connected group are either all trivial or all inner automorphisms.

INTRODUCTION

This note was prompted by a number of questions posed in [3]. That paper revolves around the notion of *innerness* for automorphisms, endomorphisms, or other classes of maps between algebraic structures.

One intriguing piece of insight is that innerness (whatever it means in any given context) is automatic for *extended* morphisms (auto, endo, etc.): per [3, Definition 4], an extended endomorphism of an object $c \in \mathcal{C}$ of a category is an endomorphism of the forgetful functor

$$(0.1) \quad c \downarrow \mathcal{C} \ni (c \rightarrow d) \xrightarrow{U_{c,c}} d \in \mathcal{C}$$

from the *comma category* $c \downarrow \mathcal{C}$ consisting of morphisms in $\mathcal{C}$ with domain c ([11, §II.6], [1, Exercise 3K]); the same goes for *automorphisms*. The paradigmatic results ([3, Theorems 1 and 2, Corollary 3]) say that for the category $\mathbf{GP}$ of groups and $G \in \mathbf{GP}$:

- The morphism attaching to $g \in G$ the natural automorphism of $U_{G,\mathbf{GP}}$ (notation as in (0.1)) operating as conjugation by $f(g)$ for any

$$(0.2) \quad \left(G \xrightarrow{f} H \right) \in G \downarrow \mathbf{GP}$$

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is an isomorphism $G \cong \text{Aut}(U_{G, \text{GP}})$.

- The only other extended *endomorphism* of $U_{G, \text{GP}}$ is the one operating trivially on H for every (0.2).

The specific questions that motivated this work are concentrated in [3, §8], which retraces some of this in the context of (restricted [9, §V.7 Definition 4]) Lie algebras and their *derivations*. The category $\mathcal{C}$ of (0.1) is now that of Lie algebras over a field (perhaps restricted, when in positive characteristic), and along with endomorphisms of (0.1) one similarly considers *extended derivations* [3, Definition 10] of a Lie algebra L :

- a derivation ∂_f of the Lie algebra L' for every morphism $f : L \rightarrow L'$;
- with the ∂_f satisfying the obvious compatibility conditions, analogous to those required of extended morphisms.

With that in place,

- (a) (A paraphrase of) [3, Question 11] asks whether the extended derivations of a Lie algebra in characteristic 0 are precisely those of the form

$$(0.3) \quad \partial_{a,f}(l') := [f(a), l'], \quad \forall f : L \rightarrow L', \quad \forall l' \in L'$$

for $a \in L$;

- (b) And, similarly, for *restricted* Lie algebras in positive characteristic;
- (c) While [3, Question 13] asks whether restricted positive-characteristic Lie algebras have any nonobvious extended endomorphisms (the obvious assigning the identity or, respectively, the zero map on L' to every $L \rightarrow L'$).

We answer these in Theorems 1.2 and 1.4:

Theorem. *Let L be either a Lie algebra over a field $\mathbb{k}$, or a restricted Lie algebra when $\mathbb{k}$ has positive characteristic.*

- (a) *If $\text{char } \mathbb{k} = 0$, the map sending $a \in L$ to the extended derivation $(\partial_{a,f})_f$ of (0.3) is an isomorphism between L and the linear space of extended derivations of L .*
- (b) *The same holds in positive characteristic for restricted Lie algebras.*
- (c) *For Lie algebras (regardless of characteristic) the only extended endomorphisms are the two obvious ones:*

$$\begin{aligned} (L \rightarrow L') &\longmapsto \text{id}_{L'} \quad \text{and} \\ (L \rightarrow L') &\longmapsto 0. \end{aligned}$$

- (d) *The analogous statement holds for restricted Lie algebras in positive characteristic.*

Part (c) is already settled in [3, Theorem 12] and is included here only for completeness; the other three items answer the various questions of [3, §8] indicated above.

Section 2 focuses on another instance of this same phenomenon, whereby functoriality begets innerness, but this time working in the category CGP_0 of compact connected topological groups (here always assumed Hausdorff). The partial analogue of [3, Corollary 3] is Theorem 2.1, and reads

Theorem. *For a compact connected group G , the individual components of a natural endomorphism of*

$$G \downarrow \text{CGP}_0 \xrightarrow{U_{G, \text{CGP}_0}} \text{CGP}_0$$

are either all trivial or all inner automorphisms.

While very similar in character to the results discussed above, the proofs are by necessity quite different. Bergmann's paper [3] and the citing literature (e.g., [8, 14, 15]) tend to adopt universal-algebra-flavored approaches: the idea is to study the effect of (say) a natural endomorphism of (0.1) on the morphism

$$c \rightarrow \langle c, x \rangle$$

into the structure (set, group, etc.) that *freely adjoins* an element x . Such universal constructions do exist in the category CGP of compact groups; in category-theoretic language, CGP is, for instance, *cocomplete* [1, Definition 12.2]. To construct the *coproduct* [1, §10.63] (or *free product*) $G_1 * G_2$ of two compact groups one must

- form the usual group coproduct $G *_{\text{discrete}} G_2$;
- equip it with the coarsest group topology making the canonical embeddings

$$\iota_i : G_i \rightarrow G_1 *_{\text{discrete}} G_2$$

continuous (e.g., [6, introductory remarks]);

- and then take the *Bohr compactification* ([2, §III.9] or [10, §2.10]) thereof.

It is this last step that disturbs the usual procedure: freely appending an element x to a compact group G amounts to the above with

$$G_1 = G \quad \text{and} \quad G_2 = \text{Bohr compactification of } \mathbb{Z} \cong \langle x \rangle.$$

Words in $x^{\pm 1}$ and elements of G no longer constitute all of $\langle G, x \rangle$, but rather only a dense subgroup thereof; for that reason, arguments such as those in the proof of [3, Theorem 1] are no longer available.

It is perhaps also worth mentioning at this point that for groups (plain, as opposed to topological) there is literature on adjacent problems, seeking to characterize innerness for automorphisms *without* assuming functoriality. An example: an automorphism of a group G is inner precisely if it extends to any group H containing an isomorphic copy of G . This appears as [17, Theorem] and also [16, Corollary to Theorem 3, p.422]; there too, generator-and-relation constructions feature prominently.

1. LIE ALGEBRAS

We work with Lie algebras over fields, for which [9] is an excellent source. Having fixed a field k , L will typically denote

- a Lie algebra unless specified otherwise, or
- a *restricted Lie algebra* in characteristic p , in the sense of [9, §V.7 Definition 4] or [13, Definition 2.3.2].

We further write

- $U(L)$ for the universal enveloping algebra of a Lie k -algebra ([9, §V.1, Definition 1]), and
- $U_p(L)$ for the *restricted enveloping algebra* of the restricted Lie algebra L in characteristic p ($\overline{U}_L$ in [9, §V.7 Theorem 12]).
- $Q(L)$ to denote either $U(L)$ or $U_p(L)$, as appropriate (so as to have uniform notation to refer to both cases).

Note that both $U(L)$ and $U_p(L)$ are naturally Hopf algebras over k [13, Example 1.5.4, Definition 2.3.2] and hence come equipped with counits ε , comultiplications Δ , etc. We will assume basic background on Hopf algebras as covered, for instance, in [13]. The Hopf algebra structure is uniquely determined by the requirement that the elements $x \in L$ be *primitive* [13, Definition 1.3.4], i.e.

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

1.1. Derivations. To streamline the statement of Theorem 1.2, we introduce some terminology.

Definition 1.1. Let k be a field, and L either a Lie algebra (in characteristic zero) or a restricted Lie algebra in characteristic p , and $\langle L, x \rangle$ the (restricted) Lie algebra freely generated by L and a formal variable x .

An element $a \in Q(L)$

- is *constant-less* if it is annihilated by the counit $\varepsilon : Q(L) \rightarrow k$ of the Hopf algebra $Q(L)$.
- *induces a universal derivative* (or is *universally derivative* or a *universal derivative*) if, for a formal variable x , the commutator $[a, x] \in Q(\langle L, x \rangle)$ belongs to $\langle L, x \rangle$.

Note that $L \subset Q(L)$ consists of constant-less universal derivatives. The following result gives the converse, answering [3, Question 11] negatively. It also proves, via the arguments in [3, Section 8], parts (a) and (b) of the first theorem in the Introduction.

Theorem 1.2. *Let k be a field and L either a Lie algebra over k (in characteristic zero) or a restricted Lie algebra (in characteristic p).*

- (a) *If k has characteristic zero then the only constant-less universally derivative elements of $U(L)$ are those of L .*
- (b) *Similarly, if k has characteristic p then the only constant-less universally derivative elements of $U_p(L)$ are those of L .*

Proof. The two statements (and proofs) are very similar, so we treat only the first in detail. The phenomenon driving both arguments is the fact that L can be recovered as precisely the space of primitive elements in the Hopf algebra $Q(L)$.

(a) As hinted above, note first that in characteristic zero, the primitive elements $P(U(G))$ of $U(G)$ (for an arbitrary Lie algebra G) are precisely those of $G \subset U(G)$ [13, Proposition 5.5.3, part 2)]. Take $G = \langle L, x \rangle$. The hypothesis

$$[a, x] \in \langle L, x \rangle$$

(for some constant-less universal derivative $a \in U(L)$) implies that the commutator $[a, x]$ is primitive. This, in turn, implies that a is primitive. To see this, assume otherwise and write

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \sum_i a_{i,1} \otimes a_{i,2},$$

where $a_{i,j}$ are

- constant-less elements of $U(L)$,
- with at least one non-zero term $a_{i,1} \otimes a_{i,2}$,
- and linearly independent $a_{i,2}$ (since we can always group the tensors so as to arrange for this).

Expanding $\Delta([a, x])$, the resulting term $a_{i,1} \otimes a_{i,2}x$ appears only once, and hence will not cancel. This contradicts the primitivity of $[a, x]$, concluding that indeed a must be primitive. But then, by the already-cited [13, Proposition 5.5.3, Part 2)], $a \in L$.

(b) The argument goes through almost verbatim, the only difference being that this time around we use the fact that in characteristic p the primitive elements $P(U_p(L))$ of $U_p(L)$ are those of L . This is not quite what [13, Proposition 5.5.3, Part 3)] says, but that proof can be adapted. The claim (that $P(U_p(L)) = L$) also follows from [18, Proposition 13.2.3]. $\square$

1.2. Endomorphisms. There is an endomorphism (as opposed to derivation) version of Theorem 1.2, which in turn answers [3, Question 13]. Before stating it, some more terminology.

Definition 1.3. Let L be an object of a category $\mathcal{C}$. A *universal endomorphism* of L in $\mathcal{C}$ is an endomorphism of the forgetful functor

$$L \downarrow \mathcal{C} \rightarrow \mathcal{C}.$$

These are the *extended inner endomorphisms* of [3, Definition 4]. The announced universal-endomorphism version of Theorem 1.2 now reads

Theorem 1.4. *Let k be a field and L either a Lie algebra over k (in characteristic zero) or a restricted Lie algebra (in characteristic p).*

- (a) *The only universal endomorphisms of L in the category Lie_k of Lie k -algebras are 0 and id .*
- (b) *Furthermore, if k has characteristic p then the only universal endomorphisms of L in the category $\text{Lie}_{k,p}$ of restricted Lie k -algebras are 0 and id .*

Proof. As explained in [3, discussion preceding Theorem 12], a universal endomorphism as in the statement is determined by elements $a, b \in Q(L)$, acquiring the expression

$$(1.1) \quad M \ni m \mapsto \varphi(a)m\varphi(b),$$

where

- $\varphi : L \rightarrow M$ is a (restricted) Lie algebra morphism as well as the corresponding morphism $Q(L) \rightarrow Q(M)$ it induces, and
- it is understood that for all such φ , the right-hand side of (1.1) belongs to M .

One can package all of this into its universal (or generic) instance: take $\varphi : L \rightarrow M$ to be the inclusion

$$L \subseteq \langle L, x \rangle$$

for a formal variable x , and require that $axb \in \langle L, x \rangle$. Denote by ε the counit of the Hopf algebra $Q(L)$, and decompose

$$a = \varepsilon(a) + \bar{a}, \quad b = \varepsilon(b) + \bar{b}$$

for constant-less $\bar{a}$ and $\bar{b}$; the goal is to show that these two latter elements must vanish.

The argument is now similar to that in the proof of Theorem 1.2

$$\Delta(\bar{a}) = \bar{a} \otimes 1 + 1 \otimes \bar{a} + \dots$$

and similarly for $\bar{b}$, where the missing summands indicated by ‘ $\dots$ ’ are simple tensors with constant-less tensorands.

Because $Q(\langle L, x \rangle)$ is the coproduct (over k) of $Q(L)$ and $k[x]$, if $\bar{a}$ and $\bar{b}$ are both non-vanishing then the term $\bar{a} \otimes x\bar{b}$ of $\Delta(axb) \in Q(\langle L, x \rangle)^{\otimes 2}$ will not cancel out, contradicting the fact that

$$axb \in \langle L, x \rangle \subset Q(\langle L, x \rangle)$$

is primitive, i.e.,

$$\Delta(axb) = axb \otimes 1 + 1 \otimes axb.$$

It follows that at least one of a and b is scalar. Suppose it is b , so that we can absorb the constant into a and work with ax in place of the original axb . Now repeat the argument: if $\bar{a} \neq 0$ then the term $\bar{a} \otimes x$ will be present in $\Delta(ax)$, again contradicting the primitivity of ax .

In conclusion a and b are both scalar, as claimed. $\square$

Remark 1.5. Alternatively, once we find that one of a and b is scalar we can conclude using the fact that, according to [3, proof of Theorem 12], $ba = 1$; this was not used above.

Part (a) of Theorem 1.4 recovers [3, Theorem 12], while part (b) answers [3, Question 13] negatively.

Remark 1.6. *Finite-dimensional* (rather than arbitrary) Lie algebras are much more interesting, as extended endomorphisms or automorphisms go.

By [4, §III.6.1, Theorem 1] the category $\mathbf{LALG}_{f, \mathbb{k}}$ of finite-dimensional Lie algebras over the real or complex field $\mathbb{k}$ is equivalent to that of *simply-connected* Lie groups over $\mathbb{k}$. Consequently, for any finite-dimensional Lie algebra $L \in \mathbf{LALG}_{f, \mathbb{k}}$, the corresponding simply-connected Lie group G_L with Lie algebra L gives a wealth of “inner” automorphisms of the forgetful functor

$$L \downarrow \mathbf{LALG}_{f, \mathbb{k}} \xrightarrow{U_L} \mathbf{LALG}_{f, \mathbb{k}} :$$

an element $g \in G_L$ operates on the codomain L' of a morphism

$$(1.2) \quad L \xrightarrow{f} L' \text{ in } \mathbf{LALG}_{f, \mathbb{k}}$$

via the adjoint action of $\tilde{f}(g)$, where

$$G_L \xrightarrow{\tilde{f}} G_{L'}$$

is the lift of (1.2) to simply connected Lie groups (once more, [4, §III.6.1, Theorem 1]).

2. COMPACT GROUPS

Some notation:

- $\mathbf{CGP}$ the category of compact (Hausdorff) topological groups;
- $\mathbf{CGP}_0$ that of *connected* compact groups;
- And in general, for an object $c \in \mathcal{C}$ and a full subcategory $\mathcal{C}' \subseteq \mathcal{C}$, we write

$$(2.1) \quad c \downarrow \mathcal{C}' \xrightarrow{U_{c, \mathcal{C}'}} \mathcal{C}'$$

for the respective forgetful functor (as in (0.1), with $\mathcal{C}'$ in place of $\mathcal{C}$).

The present section is concerned with the following (almost) “automatic innerness” for extended automorphisms of compact connected groups.

Theorem 2.1. *Let $G \in \mathbf{CGP}_0$ be a compact connected group and $\alpha \in \text{End}(U_{G, \mathbf{CGP}_0})$ a natural endomorphism of the forgetful functor defined as in (2.1), assigning an endomorphism $\alpha_f \in \text{End}(H)$ to*

$$(2.2) \quad f : G \rightarrow H, \quad H \text{ compact connected.}$$

One of the two following possibilities obtains:

- α is trivial, in the sense that α_f is the trivial endomorphism of H for every (2.2);
- or every α_f is an inner automorphism of H .

The proof requires some preparation.

Remarks 2.2.

- (1) Consider a full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ (as in (2.1)). The application

$$\mathcal{C} \ni c \longmapsto \text{End}(U_{c, \mathcal{C}'})$$

is functorial (albeit taking values, in principle, in the category of set-theoretically large monoids).

In the sequel we use this repeatedly, and mostly tacitly. This functoriality will usually make an appearance (with, say, $\mathcal{C} = \mathcal{C}' = \mathbf{CGP}$ or $\mathcal{C} = \mathcal{C}' = \mathbf{CGP}_0$) in the following form: given an endomorphism

$$\alpha \in \text{End}(U_{c, \mathcal{C}'})$$

and a morphism $f : c \rightarrow d$ with codomain $d \in \mathcal{C}'$, the component α_f is itself the identity component

$$\beta_{\text{id}} = \alpha_f \text{ of some } \beta \in \text{End}(U_{d, \mathcal{C}'})$$

induced by α .

- (2) Ideally, one would expect a stronger version of Theorem 2.1, along the lines of [3, Theorems 1 and 2]: every extended automorphism of G is presumably of the form $\alpha_f = \text{conjugation by } f(g)$ for a unique $g \in G$.

I do not know whether this is the case; the problem appears to require a more careful analysis of commutators in pushouts in the category $\mathbf{CGP}$ than could be carried out here, and is the subject of future work.

Proposition 2.3.

- (a) *For a compact connected group $G \in \mathbf{CGP}_0$, a non-trivial natural endomorphism of the forgetful functor*

$$G \downarrow \mathbf{CGP}_0 \xrightarrow{U_{G, \mathbf{CGP}_0}} \mathbf{CGP}_0$$

is automatically a natural automorphism.

- (b) *The same goes for the forgetful functor $U_{G, \mathbf{CGP}}$.*

Proof. We focus on (a) to fix ideas; the other proof is largely parallel.

- (1) **Each α_f is either trivial or injective.** Per Remark 2.2 (1), we may as well take $f = \text{id}$.

Let $g \in G$ be an element not annihilated by α_{id} , and consider a morphism

$$(2.3) \quad G \xrightarrow{f} H \text{ in } \mathbf{CGP},$$

chosen judiciously (more on this momentarily). We have a commutative diagram.

$$(2.4) \quad \begin{array}{ccccc} & & G & & \\ & \nearrow^{\alpha_{\text{id}}} & & \searrow^f & \\ G & & & & H \\ & \searrow_f & & \nearrow_{\alpha_f} & \\ & & H & & \end{array}$$

Suppose now that

- $f(\alpha_{\text{id}}(g))$ is nontrivial;
- as is $f(g')$ for an arbitrary $1 \neq g' \in \ker \alpha_{\text{id}}$;
- and H is a compact, connected, *simple* Lie group: one with no non-trivial proper normal subgroups or equivalently [7, Theorem 9.90], no such subgroups that are *closed*.

That such f exist is easily seen, and relegated to Lemma 2.4. The upper path in (2.4) fails to annihilate $g \in G$, hence so does the lower. H being simple, α_f must be one-to-one (because it cannot be trivial). But this means that the lower composition in (2.4) fails to annihilate g' , contradicting the fact that the upper path does.

The contradiction stems from our assumption that there are non-trivial $g' \in \ker \alpha_{\text{id}}$, so that map must in fact be injective.

- (2) **Each α_f is either trivial or surjective.** Once more, take $f = \text{id}$. Assume, this time around, that α_{id} is neither trivial nor onto.

The subgroup $\alpha_{\text{id}}(G) \leq G$ is then proper but nontrivial. Because compact groups are inverse limits of their Lie quotients [7, Corollary 2.36], we can find such a quotient (2.3) so that the image

$$(2.5) \quad \alpha_f(H) = f(\alpha_{\text{id}}(G)) \leq H = f(G)$$

is both proper and nontrivial (the first equality follows from the commutativity of (2.4)). But then

- α_f cannot be trivial, since its image is not;
- hence must be injective by part (1);
- and thus also surjective, because it is a one-to-one map of compact *Lie* groups: it restricts to an isomorphism on every connected component for dimension reasons, and a compact Lie group has finitely many connected components.

This latter remark contradicts the properness of (2.5), and the contradiction concludes the proof of part (2).

This, so far, shows that the *individual* components α_f of α are each either trivial or bijective. It remains to argue that we cannot have a mixture of these: if α_{id} is trivial (bijective) then so, respectively, is every α_f .

- (3) **If**

$$\alpha_{\text{id}} \in \text{End}(G) \quad \text{and} \quad \alpha_{\text{triv}} \in \text{End } H$$

are not both trivial, then the former is injective and the latter surjective. We will make repeated use of the diagram

$$(2.6) \quad \begin{array}{ccccc} & & G \times H & & \\ & \nearrow \iota & \downarrow \alpha_\iota & \nwarrow \pi & \\ G & & & & H \\ \alpha_{\text{id}} \downarrow & & & & \downarrow \alpha_{\text{TRIV}} \\ G & & G \times H & & H \\ & \nwarrow \iota & \nearrow \pi & & \end{array}$$

(with ι and π the obvious inclusion and projection). By claims (1) and (2), if at least one of α_{id} and α_{TRIV} is nontrivial, then α_ι must be bijective. This, in turn, entails the claimed injectivity and surjectivity.

- (4) α_{id} and α_f are simultaneously (non)trivial. Assuming not, there are two cases to consider:

- (I) α_{id} is trivial, while α_f isn't. By parts (1) and (2) α_f is bijective, so the commutativity of (2.4) shows that $f : G \rightarrow H$ itself is trivial. It follows that α_f is the α_{TRIV} of (2.6), and hence

$$\alpha_{\text{id}} = \text{TRIV is injective by part (3).}$$

But then G itself is trivial, so that $\alpha_{\text{id}} = \text{TRIV}$ is itself bijective.

- (II) α_f is trivial, while α_{id} isn't. Similar to the preceding: (2.4) and the bijectivity (by (1) and (2)) of α_{id} force $f = \text{TRIV}$, so that

$$\alpha_f = \alpha_{\text{TRIV}} = \text{TRIV is surjective by (3).}$$

But this means that H itself is trivial, so $\alpha_f = \text{TRIV}$ is also bijective. This settles claim (4), and hence also the proposition as a whole. $\square$

The following result is presumably well known, but we isolate it here for reference in the proof of Proposition 2.3.

Lemma 2.4. *For any finite set $F \subseteq G$ of nontrivial elements of a compact group there is a morphism $f : G \rightarrow PSU(n)$ to some projective special unitary group with $f(g) \neq 1$ for $g \in F$.*

If G is Lie then f can be chosen injective.

Proof. Because G is the inverse limit of its Lie quotient groups [7, Corollary 2.36], we certainly have a morphism f to a unitary group with these properties; it is thus enough to assume that $G = U(m)$.

Next, $U(m)$ further embeds into $SU(2m)$ via

$$U(m) \ni x \mapsto \begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix} \in SU(2m)$$

(with the overline denoting complex conjugation), so that we can in fact set $G = S(N)$.

Denoting by ρ the N -dimensional defining representation of $SU(N)$, the representation $\rho \oplus \rho^{\otimes 2}$ gives an embedding

$$SU(N) \longrightarrow SU(n), \quad n := N + N^2$$

with the property that the center $\mathbb{Z}/N \subset SU(N)$ intersects that of $SU(n)$ trivially, so that further surjecting onto $PSU(n)$ finally gives the desired embedding (of G , now assumed Lie, into $PSU(n)$). $\square$

We will take it for granted that the automorphism groups of the unitary $U(n)$ are

$$(2.7) \quad \text{Aut}(U(n)) \cong PSU(n) \rtimes \mathbb{Z}/2,$$

where

- the *projective* special unitary group $PSU(n)$ is

$$PSU(n) = U(n)/(\text{central circle } \mathbb{S}^1) \cong SU(n)/(\text{central } \mathbb{Z}/n)$$

is the *inner* automorphism group, acting on $U(n)$ by conjugation;

- and the generator of $\mathbb{Z}/2$ is complex conjugation.

This is fairly standard, though a statement specifically to this effect seems difficult to locate in the literature. The proof is certainly no more difficult than that of the analogous result for $SU(n)$, which in turn follows from the classification of the automorphisms of the complexified Lie algebra

$$\mathfrak{sl}(n) = \mathfrak{su}(n) \otimes_{\mathbb{R}} \mathbb{C},$$

described, say, in [9, §IX.5].

Proposition 2.5. *Let $G \in \text{CGP}_0$ be a compact connected group and α an automorphism of the forgetful functor U_{G, CGP_0}*

For $f : G \rightarrow H$ the automorphism $\alpha_f \in \text{Aut}(H)$ is inner.

Proof. Note that it is enough to prove α_{id} itself inner: every α_f as in the statement is the identity component of an automorphism of U_{H, CGP_0} induced by α .

The proof runs through a number of intermediate steps.

- (1) **The circle:** $G = \mathbb{S}^1$. In this case α_{id} is either the identity (which we claim is the case) or the other automorphism: $z \mapsto z^{-1}$. Consider, now, the central embedding

$$\mathbb{S}^1 \xrightarrow[\cong]{f} (\text{diagonal matrices}) \subset U(n)$$

into a unitary group and the corresponding element α_f of $\text{Aut}(U(n))$.

Because the image of f is central in $U(n)$, further composition with an arbitrary conjugation

$$Ad_u := u(-)u^* \in \text{Aut}(U(n)), \quad u \in U(n)$$

will leave f invariant, so the naturality of α gives a commutative diagram

$$\begin{array}{ccccc} & \alpha_f & & Ad_u & \\ & \nearrow & & \searrow & \\ U(n) & & U(n) & & U(n) \\ & \searrow & & \nearrow & \\ & Ad_u & & \alpha_f & \end{array}$$

In other words, α_f lies in the centralizer of $PSU(n)$ in (2.7). That centralizer is easily seen to be trivial, so that $\alpha_f = \text{id}$ and $\alpha_{\text{id}} = \text{id}$ by the commutativity of (2.4).

- (2) **Unitary groups.** We are assuming now that $G = U(n)$. The preceding point shows that the automorphism $\alpha_{\det}$ associated to the determinant morphism

$$U(n) \xrightarrow{\det} \mathbb{S}^1$$

is trivial. But then, once more by the commutativity of (2.4) (with $G = U(n)$, $H = \mathbb{S}^1$ and $f = \det$), this means that $\alpha_{\text{id}} \in \text{Aut}(U(n))$ fixes determinants (rather than inverting them), and hence must be inner by (2.7).

- (3) **Arbitrary connected compact G .** We know from the preceding discussion that every

$$\alpha_f \in \text{Aut}(U(n)), \quad f : G \rightarrow U(n)$$

induced by α is inner, i.e. conjugation by some $u_f \in U(n)$. But then the commutativity of

$$\begin{array}{ccccc} & & \alpha_{\text{id}} & \rightarrow & G & \xrightarrow{f} & U(n) \\ G & \searrow & & & & & \\ & f & \rightarrow & U(n) & \xrightarrow{\alpha_f} & & \end{array} ,$$

means that u_f , regarded as an automorphism of the carrier space $V \cong \mathbb{C}^n$ of the G -representation f , implements an isomorphism between f and the α_{id} -twisted representation $f \circ \alpha_{\text{id}}$.

This holds for arbitrary representations f of G , so α_{id} is an automorphism of the latter leaving invariant (the isomorphism classes of) all of its irreducible representations. For *connected* G this implies that α_{id} must be inner [12, Corollary 2].

This concludes the proof. $\square$

Remark 2.6. While I do not know whether Theorem 2.1 holds for disconnected compact groups, the argument in the proof of Proposition 2.5 certainly does not go through: it uses the connectedness of G crucially, in concluding via [12, Corollary 2] that automorphisms that preserve the isomorphism classes of all (or equivalently, all irreducible) representations are inner.

For *finite* groups, for instance, the automorphisms with this preservation property are precisely the ones termed *class-preserving* in the rich literature on the topic: an equivalent characterization is that they leave every conjugacy class invariant.

The reader can consult, for instance, [5] and their references ([19, 20], and so on) for extensive discussions and examples of finite groups which admit *outer* class-preserving automorphisms.

At this point, not much is left to do.

Proof of Theorem 2.1. By Proposition 2.3, the statement reduces to the already-proven Proposition 2.5. $\square$

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