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Extremal singular values of random matrix products and Brownian motion on $\mathbf{GL}(N, \mathbb{C})$

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Abstract
We establish universality for the largest singular values of products of random matrices with right unitarily invariant distributions, in a regime where the number of matrix factors and size of the matrices tend to infinity simultaneously. The behavior of the largest log singular values coincides with the large N limit of Dyson Brownian motion with a characteristic drift vector consisting of equally spaced coordinates, which matches the large N limit of the largest log singular values of Brownian motion on $\mathsf{GL}(N, \mathbb{C})$. Our method utilizes the formalism of multivariate Bessel generating functions, also known as spherical transforms, to obtain and analyze combinatorial expressions for observables of these processes.

Mathematics Subject Classification 15B52 · 60B20 · 33D52

1 Introduction

Suppose $X(1), X(2), \ldots$ is a sequence of $N \times N$ independent random matrices, and let

$$Y(M) := X(M) \cdots X(1).$$

As a natural model for systems exhibiting progressive scattering, the study of random matrix products has motivations from a variety of contexts including chaotic dynamical systems [19, 23], deep neural network [30, 32, 53], and wireless communications [56]. If the matrices $X(m)$ are complex and nonsingular, as in this paper, the discrete time (in M) stochastic process $\{Y(M)\}_{M \in \mathbb{Z}_{>0}}$ is a random walk on $\mathsf{GL}(N, \mathbb{C})$.

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In this paper, we establish universality of the largest singular values of $Y(M)$ in the limit as the number of matrix factors M and matrix size N tend to infinity with $M \asymp N$. We focus on random complex matrices $X(m)$ which are *right unitarily invariant*, i.e. the distribution of $X(m)U$ matches the distribution of $X(m)$ for any matrix U in the unitary group $\mathsf{U}(N)$. Under assumptions imposing weak concentration of the support in $\mathbb{R}_{>0}$ and nonvanishing of the average (over m) variance of the empirical measures of $X(m)$, we show that the fluctuations of the largest singular values match those of the $N \to \infty$ limit of *Brownian motion* $\mathsf{Y}^{(N)}(t)$ *on* $\mathsf{GL}(N, \mathbb{C})$.

The approach used in this work is rooted in tools and ideas from integrable probability (summarized in Sect. 1.5). This departs from previous local universality results for random matrices, such as for Wigner matrices (e.g. [21, 57]), which typically employ non-integrable methods.

1.1 Brownian motion on GL(N, $\mathbb{C}$) and Dyson Brownian motion

We introduce Brownian motion on $\mathsf{GL}(N, \mathbb{C})$ and describe the $N \to \infty$ limit of its singular values. Let Δ denote the Laplace-Beltrami operator on $\mathsf{GL}(N, \mathbb{C})$ with respect to the metric induced by the Hilbert-Schmidt inner product

$$\langle X, Y \rangle := \mathrm{Tr}(XY^*)$$

on the Lie algebra $\mathfrak{gl}(N, \mathbb{C})$.

Definition 1.1 *Brownian motion on* $\mathsf{GL}(N, \mathbb{C})$ is a diffusion on $\mathsf{GL}(N, \mathbb{C})$ with infinitesimal generator given by $\frac{1}{2}\Delta$ where $\mathsf{Y}^{(N)}(0)$ is the identity.

Equivalently (see [40, §2.1]), $\mathsf{Y}^{(N)}(t)$ is the $\mathsf{GL}(N, \mathbb{C})$-valued stochastic process $\{\mathsf{Y}^{(N)}(t)\}_{t \geq 0}$ satisfying the Stratonovich equation

$$d\mathsf{Y}^{(N)}(t) = \mathsf{Y}^{(N)}(t) \circ dW^{(N)}(t), \qquad \mathsf{Y}^{(N)}(0) = 1_N$$

where $1_N \in \mathsf{GL}(N, \mathbb{C})$ is the identity and

$$W^{(N)}(t) := \sum_{b \in \beta} W_b(t) b$$

is additive Brownian motion on $\mathfrak{gl}(N, \mathbb{C})$. In the notation above, β is any orthonormal basis of $\mathfrak{gl}(N, \mathbb{C})$ as a real vector space with respect to the Hilbert–Schmidt inner product and $\{W_b\}_{b \in \beta}$ is a family of independent standard real Brownian motions.

The large N limit of $\mathsf{Y}^{(N)}(t)$, in the sense of *-distribution, is the free multiplicative Brownian motion [40]. Additional aspects of the $N \to \infty$ limit are known, including global fluctuations [17] and the limit of the Brown measure (a candidate for the limit of the eigenvalue empirical distribution). In this paper, we consider the $N \to \infty$ limit of the singular values of $\mathsf{Y}^{(N)}(t)$.

Let $\boldsymbol{\xi}^{(N)}(t) = (\xi_1^{(N)}(t) \geq \cdots \geq \xi_N^{(N)}(t))$ denote the logarithms of the squared singular values of $\mathsf{Y}^{(N)}(t)$. It is a remarkable fact (see [37, Corollary 3.3], [14], and

[34]) that the evolution of $\boldsymbol{\xi}^{(N)}(t/4)$ coincides with that of Dyson Brownian motion with drift:

Theorem 1.2 ([37, Corollary 3.3]) *The process $\boldsymbol{\xi}^{(N)}(t/4)$ evolves as Brownian motion on $\mathbb{R}^N$ with drift*

$$\left(\frac{N-1}{2}, \frac{N-3}{2}, \ldots, \frac{-N+3}{2}, \frac{-N+1}{2}\right),$$

started at the origin, and conditioned to remain in the set $\{\boldsymbol{x} = (x_1, \ldots, x_N) : x_1 \geq \cdots \geq x_N\}$.

The original statement from [37] was in terms of the singular values of $\mathsf{Y}^{(N)}(t)$. We note that [37] also provided analogous descriptions for Brownian motions on symmetric spaces G/K where G is a complex semi-simple non-compact connected Lie group with finite center and K is a maximal compact subgroup. In this framework, the drift vector is the sum of the associated positive roots, where our setting[1] corresponds to Type A.

From the identification with Dyson Brownian motion with drift, the process $\boldsymbol{\xi}^{(N)}(t)$ admits determinantal spacetime correlation functions with exact formulas for the correlation kernel. In [39], the $N \to \infty$ limit of $\xi^{(N)}(t)$ was studied via these kernels, where it was shown to exhibit number variance saturation. In this paper, we prove a stronger form of convergence using machinery from [16]. To describe the limiting object, we require the notion of a *line ensemble*. While the formal definition is deferred to Definition 3.1, we may simply view line ensembles as an indexed (possibly infinite) sequence of random paths $(\eta_i(t))_{i \in \Sigma}$.

Theorem 1.3 *There exists a limiting infinite line ensemble (in the sense of* [16]*, see Definition 3.1)*

$$\boldsymbol{\xi}(t) := (\xi_1(t), \xi_2(t), \ldots) = \lim_{N \to \infty} \boldsymbol{\xi}^{(N)}(\frac{t}{4}) - \frac{Nt}{2} - \log N$$

where the convergence holds in the following sense. For any $T > 0$ and any positive integer k, the random continuous function $\left(\xi_1^{(N)}(\frac{t}{4}) - \frac{Nt}{2} - \log N, \ldots, \xi_k^{(N)}(\frac{t}{4}) - \frac{Nt}{2} - \log N\right)$ converges to $(\xi_1(t), \ldots, \xi_k(t))$ as $N \to \infty$ in the weak- topology of probability measures on $C([\frac{1}{T}, T])^k$*

In view of Theorem 1.3, the limiting process $\boldsymbol{\xi}(t)$ may be interpreted as a $\mathbb{Z}_{\geq 1}$-tuple of non-intersecting Brownian motions with drift where the ith Brownian motion (from the top) has drift $-i + \frac{1}{2}$. The joint Laplace transform and correlation kernel of $\boldsymbol{\xi}(t)$ can be explicitly computed, and are given in Theorem 3.2.

[1] Since $\mathsf{GL}(N, \mathbb{C})$ does not have a finite center, Theorem 1.2 follows from the application of this framework to $\mathsf{SL}(N, \mathbb{C})/\mathsf{SU}(N)$ and viewing $\mathsf{GL}(N, \mathbb{C})/\mathsf{U}(N)$ as $\mathsf{SL}(N, \mathbb{C})/\mathsf{SU}(N) \times \mathbb{R}_{>0}$.

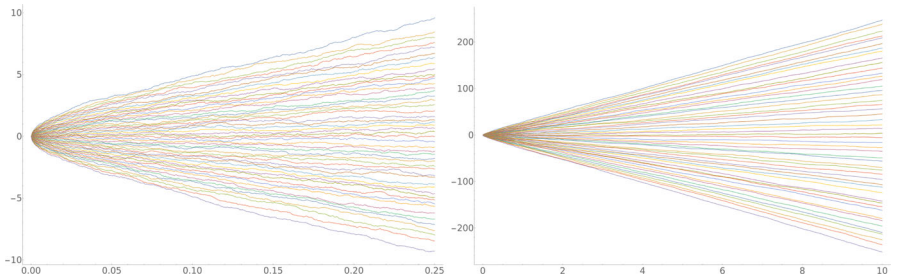

Fig. 1 Samples of $\xi^{(N)}(\frac{t}{4})$ where $N = 50$, with $0 \le t \le 0.25$ (left) and $0 \le t \le 10$ (right)

1.2 Main result

We establish the asymptotic notation used throughout this paper. Given sequences A_N, B_N we write $A_N = O(B_N)$ if there exists a constant $C > 0$ such that $|A_N| \le C|B_N|$ for N sufficiently large. We write $A_N \ll B_N$ or $A_N = o(B_N)$ if $A_N/B_N \to 0$ as $N \to \infty$. We write $A_N \asymp B_N$ if there exist constants $C_1, C_2 > 0$ such that $C_1|B_N| \le |A_N| \le C_2|B_N|$ for N sufficiently large.

Our main result is that the largest log squared singular values of random matrix products $Y^{(N)}(M)$ in the limit $N, M \to \infty$ with $N \asymp M$ converges to the infinite line ensemble $\{\xi(t)\}_{t>0}$ in finite dimensional distribution, under mild assumptions. Given an $N \times N$ matrix X, let

$$|X|^2 := X^* X$$

and $\mathrm{tr}\,(X) = \frac{1}{N}\mathrm{Tr}(X) = \frac{1}{N}\sum_{i=1}^N X_{ii}$ denote the normalized trace.

Theorem 1.4 *Suppose $X^{(N)}(1), X^{(N)}(2), \ldots$ are random complex $N \times N$ matrices with right unitarily invariant distributions and denote by*

$$y_1^{(N)}(M) \ge \cdots \ge y_N^{(N)}(M)$$

the squared singular values of $X^{(N)}(M) \cdots X^{(N)}(1)$. Assume that

(i) there exists $C > 0$ such that

$$1 - \mathbb{P}\left(\textit{all squared singular values of } X^{(N)}(m) \textit{ are contained in } [C^{-1}, C]\right) = o(1/N) \tag{1.1}$$

uniformly over $m = 1, 2, \ldots$, and

(ii) there exists a continuous $\gamma : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\lim_{t \searrow 0} \gamma(t) = 0$ and

$$\frac{1}{N}\sum_{m=1}^{\lfloor tN \rfloor} \frac{\mathrm{tr}\,(|X^{(N)}(m)|^4) - \mathrm{tr}\,(|X^{(N)}(m)|^2)^2}{\mathrm{tr}\,(|X^{(N)}(m)|^2)^2} \to \gamma(t) \tag{1.2}$$

in probability as $N \to \infty$, for each $t > 0$.

Then for each positive integer h, the process (in time t)

$$\log y_j^{(N)}(\lfloor tN \rfloor) - \sum_{m=1}^{\lfloor tN \rfloor} \log\left(\text{tr } |X^{(N)}(m)|^2\right) - \log N, \qquad j = 1, \ldots, h$$

converges in finite dimensional distributions to the top h paths $\xi_1(\gamma(t)), \ldots, \xi_h(\gamma(t))$ *of* $\boldsymbol{\xi}(\gamma(t))$.

The key assumptions, besides unitary invariance, are given by (1.1) and (1.2). Condition (1.1) ensures that the largest singular values are sufficiently unlikely to escape to infinity. While this assumption may be relaxed, it is clear that there must be *some* control over the behavior of the largest singular values. For example, consider unitarily invariant random matrices $X^{(N)}(1), \ldots, X^{(N)}(M-1), X^{(N)}(M) + \lambda_N \boldsymbol{u}\boldsymbol{u}^T$ where $\boldsymbol{u}$ is a uniformly random vector from the sphere in $\mathbb{C}^N$. For $1 \leq j \leq M$, suppose $X^{(N)}(j)$ has singular values between $a, b > 0$ fixed and let $\lambda_N \gg (b/a)^N$. In other words, the first $M-1$ matrices are well-behaved, but the Mth matrix is a well-behaved matrix perturbed by a large rank-one spike. It can be shown that the largest singular value of

$$X^{(N)}(1) \cdots X^{(N)}(M-1)\left(X^{(N)}(M) + \lambda_N \boldsymbol{u}\boldsymbol{u}^T\right)$$

will escape to infinity faster than the second largest, so that our theorem can no longer hold in this setting. Thus, even if a single matrix factor violates Condition (1.1), albeit severely, the conclusion of the theorem no longer holds.

Note that Condition (1.1) also demands that the smallest singular values do not approach 0. However, we expect our condition can be relaxed to include possibly singular matrices, i.e. replace $[C^{-1}, C]$ with $[0, C]$, as long as we require that a nonzero fraction of the singular values are contained in $[C^{-1}, C]$ in the limit. To admit singular matrices, an additional condition of this type is vital to avoid multiplying by zero matrices, or matrices close to the zero matrix.

Condition (1.2) ensures that the time parameter of the process converges to a non-trivial deterministic limit, where the individual summand

$$\frac{1}{N} \frac{\text{tr}\,(|X^{(N)}(m)|^4) - \text{tr}\,(|X^{(N)}(m)|^2)^2}{\text{tr}\,(|X^{(N)}(m)|^2)^2}$$

is the increment of time that the matrix factor $X^{(N)}(m)$ contributes.

An immediate corollary for the case where the matrix factors are i.i.d. is given below.

Corollary 1.5 *Suppose* $X^{(N)}(1), X^{(N)}(2), \ldots$ *is an i.i.d. sequence of random complex* $N \times N$ *matrices with right unitarily invariant distributions and denote by*

$$y_1^{(N)}(M) \geq \cdots \geq y_N^{(N)}(M)$$

the squared singular values of $X^{(N)}(M)\cdots X^{(N)}(1)$. *Suppose* $X^{(N)}(1), X^{(N)}(2), \ldots$ *satisfy* (1.1), *and*

$$\operatorname{tr}(|X^{(N)}(m)|^2) \quad \text{and} \quad \frac{\operatorname{tr}(|X^{(N)}(m)|^4) - \operatorname{tr}(|X^{(N)}(m)|^2)^2}{\operatorname{tr}(|X^{(N)}(m)|^2)^2}$$

converge in probability as $N \to \infty$, *where the latter has a positive limit* $\mathfrak{a}$. *For each positive integer h, the process (in time t)*

$$\log y_j^{(N)}(\lfloor tN \rfloor) - \sum_{m=1}^{\lfloor tN \rfloor} \log\left(\operatorname{tr}\,|X^{(N)}(m)|^2\right) - \log N, \qquad j = 1, \ldots, h$$

converges in finite dimensional distributions to the top h paths $\xi_1(\mathfrak{a}t), \ldots, \xi_h(\mathfrak{a}t)$ *of* $\boldsymbol{\xi}(\mathfrak{a}t)$

Prior to this work, convergence of the largest log singular values to $\boldsymbol{\xi}(t)$ was known for products of Ginibre and truncated unitary matrices [1, 3, 45]. Recall that a *complex Ginibre matrix* is a rectangular matrix of i.i.d. standard complex Gaussian entries, and a *truncated unitary matrix* is a rectangular submatrix of a Haar distributed unitary matrix. The distribution of the former is parametrized by the matrix dimensions and the latter is parametrized by the matrix dimensions and the size of the ambient Haar unitary matrix.

For products of square Ginibre matrices, the convergence of the largest log singular values to $\boldsymbol{\xi}(t)$ for fixed time was shown by [1, 2, 45], with generalizations to products of rectangular Ginibre matrices indicated in [45]. Extensions to joint time convergence and for products of truncated unitary matrices were established in [3]. The accessibility of these examples are due to determinantal and related structures available in those cases [3, 5, 6, 13, 42]. In our setting, this structure is not available in general, thus we appeal to alternative methods which we detail later.

To illustrate the relation between Theorem 1.4 and these previously established results, we briefly review the case for products of Ginibre matrices. This is simpler to state than analogous results for truncated unitary products as it involves fewer parameters (see [3, Theorem 1.7] for details).

Given a sequence of positive integers $\{N_i := N_i(N)\}_{i\geq 0}$ depending on N, where $N_0 = N$ and $N_i \geq N$, let

$$X^{(N)}(m) = ((G^{(N)}(m))^*(G^{(N)}(m))^{1/2}$$

where $G^{(N)}(m)$ is an $N_m \times N$ complex Ginibre matrix ($m \geq 1$), and consider the associated process $\{Y^{(N)}(M)\}_{M\in\mathbb{Z}_{\geq 0}}$ of matrix products. Under mild conditions on the parameters $\{N_m(N)\}_{m\in\mathbb{Z}_{\geq 0}}$ (see [45, Theorem 3.4])[2] which correspond to condition (1.2) in Theorem 1.4, the largest log singular values of $Y^{(N)}(M)$ converge to those

[2] The setup from [45] was in terms of an equivalent, more elegant setup where $X^{(N)}(m)$ is a *rectangular* $N_m \times N_{m-1}$ complex Ginibre, see [3, Appendix A] for details on this equivalence. We stick to products of *square* matrices to remain consistent with the setting of this paper.

of $\boldsymbol{\xi}(t)$ under an appropriate time change and translation, in the regime $M \asymp N$. Theorem 1.4 does not imply this result in general because the singular values of $X^{(N)}(m)$ can get arbitrarily close to 0 if N_m/N approaches 1, violating (1.1). If we include the additional hypothesis that

$$\liminf_{N\to\infty} \inf_{i\geq 1} \frac{N_i(N)}{N} > 1, \tag{1.3}$$

i.e. the ratios N_i/N remain separated from 1, then the result for products of Ginibre matrices now follows from Theorem 1.4.

Although Theorem 1.4 requires the additional assumption (1.3) to reach the full strength of previous results on Ginibre products, the methods in this paper can recover these previous results (without (1.3)) by the integrability of Ginibre and truncated unitary matrices. However, further development is required to deal with general matrices with some singular values approaching 0.

1.3 Discussion

Theorem 1.4 appears closely related to the functional central limit theorem for $\mathsf{GL}(N, \mathbb{C})$. Much like Donsker's invariance theorem for random walks on $\mathbb{R}$, a random walk on a connected Lie group G converges to an appropriate diffusion when the increments approach the identity and the number of steps is properly rescaled [55]. Indeed, making right unitary symmetry and mean zero (of log of the increments) assumptions, random walks on $\mathsf{GL}(N, \mathbb{C})$ converge to $\mathsf{Y}^{(N)}$ with time parametrization dictated by the increments of the original random walk. However, a key distinction which separates Theorem 1.4 from the $\mathsf{GL}(N, \mathbb{C})$ functional limit theorem, besides the fact that $N \to \infty$ in the former, is that the increments are not approaching the identity. The connection with $\mathsf{Y}^{(N)}$ is even more striking when comparing to previous results on global limit shapes [33, 47, 48] and fluctuations [28] of products of right unitarily invariant random matrices, where the behavior was shown to be *non-universal* and independent of the relative growth between N and M.

The connection between $\mathsf{Y}^{(N)}$ and products of random matrices in our regime was hypothesized in [7], based on the main result of that work which established that the drifts of log singular values matched the drifts of the log singular values of $\mathsf{Y}^{(N)}$. Our results bolsters this hypothesis by demonstrating this connection holds at the level of fluctuations, not just in terms of the large time behavior of the processes.

This paper focuses on the regime $N, M \to \infty$ where $N \asymp M$. However, for products of Ginibre matrices, the regimes $N \gg M$ and $N \ll M$ are also known [1, 2, 45]. For $N \ll M$, the so-called picket fence statistics appear, where the ith largest log singular value concentrates near $-i + \frac{1}{2}$ after suitable rescaling. For $N \gg M$, GUE statistics appear. Thus the process $\boldsymbol{\xi}(t)$ may also be viewed as an interpolating process between these two regimes. The appearance of the picket fence statistics is directly related to separation of the curves in $\boldsymbol{\xi}(t)$ as $t \to \infty$ according the drift sequence $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$. The appearance of GUE statistics corresponds to forgetting the drift as $t \to 0$.

In view of this description for Ginibre matrices, Theorem 1.4 suggests (though does not directly imply) that for $N \ll M$, the log squared singular values of the random walk $Y^{(N)}(M)$ converge to the picket fence. This can be recast into a statement about universality of Lyapunov exponents, by interpreting the limit $N \ll M$ as taking limits $M \to \infty$ and $N \to \infty$ in that order. More precisely, Oseledets' multiplicative ergodic theorem [52] asserts the existence of Lyapunov exponents

$$\lambda_i^{(N)} = \lim_{M \to \infty} \frac{1}{2M} \log y_i^{(N)}(M), \qquad 1 \le i \le N$$

for fixed N. While these Lyapunov exponents are not universal, we expect that under general assumptions as $N \to \infty$ the largest Lyapunov exponents should converge to picket fence statistics upon properly rescaling and translating. Results of this type were established for products of truncated unitary and Ginibre matrices in [7].

Likewise, Theorem 1.4 also suggests that for $N \gg M$, the largest singular values of $Y^{(N)}(M)$ should converge to GUE statistics given by the Airy point process. In the extreme case where the number of matrix factors M is fixed and $N \to \infty$, the global limit shape can be understood in terms of free probability [58]. However, local statistics are far less understood, though significant progress [20, 36] has been made in the form of regularity and local law results at the edge. The belief is that the Airy point process should appear as long as the empirical distribution of the matrix factors are suitably regular, and that the regularity can be relaxed as the number of matrix factors M increases. A similar phenomenon was confirmed for local statistics of sums of random Hermitian matrices [4].

1.4 Further directions

A natural question is whether the large N limit of the statistics of $\Upsilon^{(N)}(t)$ appear for products of random matrices beyond the edge. Returning to the case of products of complex Ginibre matrices, it is known [2] that the bulk statistics in the regime $M \asymp N$ continue to the match that of $\Upsilon^{(N)}(t)$, suggesting that this universality holds in the bulk. It is worth noting that for products of square Ginibre matrices, there is a hard edge which is absent for $\Upsilon^{(N)}(t)$.

Let us remark on several related models which do not exhibit right unitary symmetry. For products of complex Wishart matrices (matrices with centered, variance $1/n$, i.i.d. random variables not necessarily Gaussian), we conjecture that the large N limit of $\Upsilon^{(N)}(t)$ statistics continue to appear at the edge and bulk as well. In another direction, one can consider other symmetry classes, such as products of real matrices with right orthogonal invariance or of quaternionic matrices with right symplectic invariance. Natural examples are products of real Ginibre matrices and truncated Haar orthogonal matrices for the former, and products of quaternionic Ginibre matrices and truncated Haar symplectic matrices for the latter. These models, and one-parameter deformations of these models in the Dyson index β in the spirit of β-ensembles, were considered in [3] where tightness results were obtained at the edge in the regime $M \asymp N$. While convergence results beyond $\beta = 2$ are unavailable, we expect $N \to \infty$ limits of β-deformations of $\Upsilon^{(N)}(t)$ to appear, where for $\beta = 1$ and 4 this should be the

corresponding diffusion on $\mathsf{GL}(N, \mathbb{R})$ and $\mathsf{GL}(N, \mathbb{H})$. respectively (where $\mathbb{H}$ is the skew field of real quaternions).

Although convergence results for products of right orthogonally invariant real matrices are not available, there is a similar model which may be accessible via existing methods. The eigenvalues of $X_M^T \cdots X_1^T A X_1 \cdots X_M$ where $X_1, \ldots, X_M$ are real Ginibre matrices and A is some real antisymmetric matrix are determinantal [22, 41], structure which arises from the orthogonal Harish-Chandra-Itzykson-Zuber integral. The behavior of the eigenvalues in the regime $M \asymp N$ have not been studied, though it may be accessible by analysis of correlation kernels. It would be interesting to see what behavior arises in this regime.

There is also recent progress on singular numbers of products of p-adic matrices [59] which exhibit some parallels to our setting. In particular, there is a universality phenomenon where objects corresponding to Lyapunov exponents converge in the large N limit to a geometric progression. It is possible that there may be analogues of our result in this setting.

1.5 Method

The methods in this paper rely on an analogue of the Mellin transform for distributions of right unitarily invariant matrices. Given a random X in $\mathsf{GL}(N, \mathbb{C})$, define

$$\Phi_X(z) = \mathbb{E}\left[\int_{\mathsf{U}(N)} |UX^*XU^{-1}|^z \, dU\right], \qquad z \in \mathbb{C}^N$$

abusing notation by using the random X as a subscript of Φ, where

$$|Y|^z := (\det Y)^{z_N} \prod_{j=1}^{N-1} \left(\det Y_{j\times j}\right)^{z_j - z_{j+1} - 1}$$

is a generalization of the power function for positive definition matrices; $Y_{j\times j}$ is the $j \times j$ top left corner submatrix of Y which itself is positive definite by Sylvester's criterion. The expectation satisfies the factorization property

$$\Phi_{XY}(z) = \Phi_X(z)\Phi_Y(z)$$

where X, Y are independent random right unitarily invariant complex $N \times N$ matrices. These ideas, and their additive analogue, have been used to compute exact density formulas for a variety of matrix ensembles as in [24, 43, 44, 60] under a common framework.

The integral within the expectation of $\Phi_X(z)$ is known as the Gelfand-Naĭmark integral. It can be explicitly evaluated [26]

$$\int_{\mathrm{U}(N)} |U \operatorname{diag}(x_1, \ldots, x_N) U^{-1}|^{\boldsymbol{z}} \, dU = \Delta(\rho_N) \frac{\det x_j^{z_i}}{\Delta(\boldsymbol{z})\Delta(\boldsymbol{x})},$$
$$\Delta(u_1, \ldots, u_N) = \prod_{1 \le i < j \le N} (u_i - u_j)$$

and may be viewed as a multiplicative analogue of the Harish-Chandra-Itzykson-Zuber integral [31, 35]. The right hand side is a (normalized) multivariate Bessel function, a continuous analogue of a Schur function. This connection with symmetric functions yields a collection of tools for the study of right unitarily invariant random matrix products. In particular, we can act on the spherical transforms by certain operators diagonalized by the multivariate Bessel functions to obtain observables for singular values of matrix products. Similar ideas were used to study other random processes with connections to symmetric functions, including polymers [8], measures arising from representation theory [10–12], the β-Jacobi corners process from β-ensembles [9, 29], and many more. This idea was used by [28] to study global fluctuations of random matrix products, where the spherical transform was referred to as the multivariate Bessel generating function, as we will in the body of this paper due to methodological connections with their work.

Our method relies on the extraction of joint observables for singular values of $Y^{(N)}(m)$ via the appropriate family of operators. The observables and corresponding operators used in [28] are not amenable for the analysis of edge statistics in our setting. Thus, we consider a family of operators suitable for our regime corresponding to observables which give the joint Laplace transform of the log singular values of $Y^{(N)}(M)$ (over varying M). This leads to considerable differences from the analyses of [28]. Our observables allow us to access the edge in a similar manner as the method of high moments in [54] probed the edge for Wigner matrices. In short, joint Laplace transforms of the singular values are dominated by the largest singular values in our limit. Using exact formulas, these observables have expressions in terms of large combinatorial sums which can be asymptotically identified with expressions which correspond to the large N limit of observables of $\mathsf{Y}^{(N)}(t)$.

1.6 Organization

The remainder of this paper is organized as follows. In Sect. 2 we introduce the main tools to access the observables of singular values of random matrix products: the multivariate Bessel generating function and associated operators. Section 3 introduces the formalism of line ensembles and is devoted to proving Theorem 1.3, along with auxiliary results for later parts of the paper. Section 4 introduces the S-transform and the ψ function of a measure on $\mathbb{R}_{>0}$, and similarly proves auxiliary results for later parts of the paper. Section 5 obtains asymptotics of multivariate Bessel functions, bootstrapping off of a result of [28], a key input for the asymptotics of the joint Laplace transforms that we want to compute. Finally, Sect. 6 proves the main result Theorem 1.4, containing the core asymptotic analysis of this paper.

2 Joint Laplace transforms and multivariate Bessel functions

In this section, we introduce the multivariate Bessel generating function, also known as the spherical transform, which are expectations of random matrices over the multivariate Bessel function, see e.g. [25] and references therein. The multivariate Bessel generating functions cohere well with matrix products, a fact which furnishes us with expressions for observables of squared singular values of random matrix products.

Definition 2.1 The *multivariate Bessel function* indexed by $\boldsymbol{a} = (a_1 \geq \cdots \geq a_N) \in \mathbb{R}^N$ is the function

$$\mathcal{B}_{\boldsymbol{a}}(z) = \frac{\det\left[e^{z_i a_j}\right]_{i,j=1}^{N}}{\Delta(z)}$$

which is holomorphic for $z \in \mathbb{C}^N$, where

$$\Delta(z) = \prod_{1 \leq i < j \leq N} (z_i - z_j).$$

Definition 2.2 Let X be a random matrix in $\mathsf{GL}(N, \mathbb{C})$ with right unitarily invariant distribution, Denote by $\boldsymbol{x} = (x_1, \ldots, x_N)$ the squared singular values of X. The *multivariate Bessel generating function* of X is defined by

$$\Phi_X(z_1, \ldots, z_N) = \mathbb{E}\left[\frac{\mathcal{B}_{\log \boldsymbol{x}}(z_1, \ldots, z_N)}{\mathcal{B}_{\log \boldsymbol{x}}(\rho_N)}\right],$$

where $\rho_N := (N-1, N-2, \ldots, 0)$, given that this expectation exists in a neighborhood of $(N-1, N-2, \ldots, 0)$.

The normalized multivariate Bessel function within the expectation is the zonal spherical function for the Gelfand pair $\mathsf{GL}(N, \mathbb{C}), \mathsf{U}(N)$ [46, Chapter VII]. If X is a scalar (i.e. $N = 1$), then the multivariate Bessel generating function reduces to

$$\Phi_X(z) = \mathbb{E}[|X|^{2z}]$$

which is the Mellin transform for the distribution of $|X|^2$. Thus, for general N the function Φ_X is an extension of the Mellin transform for positive definite matrices X^*X. Moreover, we can define a generalized power function on $N \times N$ positive definite matrices Y:

$$|Y|^z := (\det Y)^{z_N} \prod_{j=1}^{N-1} \left(\det Y_{j \times j}\right)^{z_j - z_{j+1} - 1}$$

where $Y_{j \times j}$ is the $j \times j$ top left corner submatrix of Y. Indeed, by Sylvester's criterion, the determinants are positive, thus the complex exponentials are well-defined. Then

$$\frac{\mathcal{B}_{\log \boldsymbol{x}}(\boldsymbol{z})}{\mathcal{B}_{\log \boldsymbol{x}}(\rho_N)} = \Delta(\rho_N)\frac{\det[x_j^{z_i}]_{i,j=1}^N}{\Delta(\boldsymbol{z})\Delta(\boldsymbol{x})} = \int_{\mathsf{U}(N)} |U \operatorname{diag}(\boldsymbol{x})U^{-1}|^{\boldsymbol{z}}\, dU$$

where the integral is over the normalized (with volume 1) Haar measure on $\mathsf{U}(N)$. The latter integral is known as the Gelfand–Naĭmark integral [26]. From this perspective, the multivariate Bessel functions are unitarily invariant (under the conjugation action) generalized power functions on positive definite matrices.

Just as products of independent random variables factor under the Mellin transform, the multivariate Bessel generating functions satisfy the following factorization property:

Lemma 2.3 *If X and Y are independent $N \times N$ random matrices with right unitarily invariant distributions, then*

$$\Phi_{XY} = \Phi_X \cdot \Phi_Y.$$

Proof If X and Y have deterministic squared singular values $\boldsymbol{x} \in \mathbb{R}_{>0}^N$ and $\boldsymbol{y} \in \mathbb{R}_{>0}^N$ respectively, then this follows from the identity

$$\frac{\mathcal{B}_{\log \boldsymbol{x}}(\boldsymbol{z})}{\mathcal{B}_{\log \boldsymbol{x}}(\rho_N)}\frac{\mathcal{B}_{\log \boldsymbol{y}}(\boldsymbol{z})}{\mathcal{B}_{\log \boldsymbol{x}}(\rho_N)} = \mathbb{E}\left[\frac{\mathcal{B}_{\log \boldsymbol{w}}(\boldsymbol{z})}{\mathcal{B}_{\log \boldsymbol{w}}(\rho_N)}\right]$$

for zonal spherical functions, where $\boldsymbol{w}$ are the squared singular values of XY, see [46, Chapter VII]. The general case follows from taking mixtures of the aforementioned case. □

We can iterate Lemma 2.3. Let $X(1), X(2), \ldots$ be independent $N \times N$ right-unitarily invariant complex random matrices. Given that the multivariate Bessel generating functions $\Phi_{X(m)}$ $(m \geq 1)$ exist, the product $Y(M) := X(M) \cdots X(1)$ has multivariate Bessel generating function

$$\prod_{m=1}^{M} \Phi_{X(m)}(z_1, \ldots, z_N).$$

Given $c \in \mathbb{C}$, define

$$\mathcal{D}_c := \mathcal{D}_c^{(N)} := \sum_{i=1}^{N}\left(\prod_{j \neq i}\frac{c + z_i - z_j}{z_i - z_j}\right)\mathcal{T}_{c,z_i}$$

where $\mathcal{T}_{c,z_i} f(z_1, \ldots, z_N) = f(z_1, \ldots, z_i + c, \ldots, z_N)$. With these definitions in place, we claim that

$$\mathcal{D}_c \mathcal{B}_{\log \boldsymbol{x}}(z_1, \ldots, z_N) = \left(\sum_{i=1}^{N} x_i^c\right)\mathcal{B}_{\log \boldsymbol{x}}(z_1, \ldots, z_N). \tag{2.1}$$

To see why this eigenrelation holds, observe that

$$\Delta(z)\mathcal{T}_{c,z_i}\Delta(z)^{-1} = \prod_{j\neq i}\frac{z_i - z_j}{c + z_i - z_j}$$

so that

$$\begin{aligned}
&\mathcal{D}_c\mathcal{B}_{\log \boldsymbol{x}}(z_1,\ldots,z_N)\\
&\quad= \sum_{i=1}^{N}\left(\prod_{j\neq i}\frac{c+z_i-z_j}{z_i-z_j}\right)\mathcal{T}_{c,z_i}\frac{\det[x_k^{z_j}]_{j,k=1}^N}{\Delta(z)}\\
&\quad= \frac{1}{\Delta(z)}\sum_{i=1}^{N}\left[\left(\prod_{j\neq i}\frac{c+z_i-z_j}{z_i-z_j}\right)\Delta(z)\mathcal{T}_{c,z_i}\Delta(z)^{-1}\right]\mathcal{T}_{c,z_i}\det[x_k^{z_j}]_{j,k=1}^N\\
&\quad= \frac{1}{\Delta(z)}\sum_{i=1}^{N}\mathcal{T}_{c,z_i}\sum_{\sigma\in\mathfrak{S}_N}(-1)^{\sigma}\prod_{j=1}^{N}x_{\sigma(j)}^{z_j}\\
&\quad= \left(\sum_{i=1}^{N}x_i^c\right)\frac{1}{\Delta(z)}\det[x_k^{z_j}]_{j,k=1}^N = \left(\sum_{i=1}^{N}x_i^c\right)\mathcal{B}_{\log \boldsymbol{x}}(z_1,\ldots,z_N)
\end{aligned}$$

where $\mathfrak{S}_N$ is the symmetric group of order N and $(-1)^\sigma$ is the sign of the permutation σ.

Proposition 2.4 *Let $X(1), X(2), \ldots$ be independent nonsingular $N\times N$ random matrices with right unitarily invariant distributions and multivariate Bessel generating functions $\varphi_1, \varphi_2, \ldots$ respectively. Assume that the multivariate Bessel generating functions are analytic on $\mathbb{C}^N$. Fix real numbers $c_1,\ldots,c_k > 0$ and integers $M_1 \geq \cdots \geq M_k > M_{k+1} = 0$. Suppose $\boldsymbol{y}(M) \in \mathbb{R}^N_{>0}$ is the vector of squared singular values of $Y(M) = X(M)\cdots X(1)$. Then*

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N}y_j(M_i)^{c_i}\right]\\
&\quad= \mathcal{D}_{c_1}\prod_{m_1=M_2+1}^{M_1}\varphi_{m_1}(z_1\ldots,z_N)\cdots\mathcal{D}_{c_k}\prod_{m_k=M_{k+1}+1}^{M_k}\varphi_{m_k}(z_1,\ldots,z_N)\Big|_{z=\rho_N}
\end{aligned}$$

where the $\mathcal{D}_c$ operators act on everything to their right.

With Theorem 1.2, we can compute the multivariate Bessel generating function for Brownian motion on $\mathsf{GL}(N,\mathbb{C})$ using the following general result:

Proposition 2.5 *Suppose that $\boldsymbol{\eta}(t)$ is the vector of N non-intersecting Brownian motions with drift $\boldsymbol{\mu} = (\mu_1 \geq \cdots \geq \mu_N)$ and $\boldsymbol{\eta}(0) = \mathbf{0}$. If*

$$\boldsymbol{\eta}(t) + a := (\eta_1(t) + a, \ldots, \eta_N(t) + a)$$

for some $a \in \mathbb{R}$, then

$$\mathbb{E}\left[\frac{\mathcal{B}_{\eta(t)+a}(z)}{\mathcal{B}_{\eta(t)+a}(\mu)}\right] = \prod_{i=1}^{N} \frac{e^{\frac{t}{2}(z_i+\frac{a}{t})^2}}{e^{\frac{t}{2}(\mu_i+\frac{a}{t})^2}}.$$

Recall our notation $(\xi_1^{(N)} \geq \cdots \geq \xi_N^{(N)})$ for the log squared singular values of Brownian motion $\mathsf{Y}^{(N)}(t)$ on $\mathsf{GL}(N, \mathbb{C})$. Using our description from Theorem 1.2 for this process as Dyson Brownian motion with drift, Propositions 2.4 and 2.5 imply

Corollary 2.6 *Given $c_1, \ldots, c_k > 0$ and $t_1 > \cdots > t_k > t_{k+1} = 0$, we have*

$$\begin{aligned}&\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\xi_j^{(N)}\left(\frac{t_i}{4}\right)-\frac{Nt_i}{2}\right)}\right] \\ &= \mathcal{D}_{c_1}\prod_{i=1}^{N}\frac{e^{\frac{1}{2}(t_1-t_2)(z_i-N+\frac{1}{2})^2}}{e^{\frac{1}{2}(t_1-t_2)\left(-i+\frac{1}{2}\right)^2}}\cdots\mathcal{D}_{c_k}\prod_{i=1}^{N}\frac{e^{\frac{1}{2}(t_k-t_{k+1})\left(z_i-N+\frac{1}{2}\right)^2}}{e^{\frac{1}{2}(t_k-t_{k+1})\left(-i+\frac{1}{2}\right)^2}}\Bigg|_{z=\rho_N}\end{aligned} \tag{2.2}$$

where the $\mathcal{D}_c$ operators act on everything to their right. The right hand side can be expressed as a multiple contour intergral:

$$\begin{aligned}&\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\xi_j\left(\frac{t_i}{4}\right)-\frac{Nt_i}{2}\right)}\right] \\ &= \frac{1}{(2\pi \boldsymbol{i})^k}\oint\cdots\oint\prod_{1\leq i<j\leq k}\frac{(z_i-z_j)(z_i+c_i-z_j-c_j)}{(z_i-z_j-c_j)(z_i+c_i-z_j)}\prod_{i=1}^{k} \\ &\quad\frac{e^{\frac{t_i}{2}\left(z_i+c_i-\frac{1}{2}\right)^2}}{e^{\frac{t_i}{2}\left(z_i-\frac{1}{2}\right)^2}}\frac{\Gamma(z_i+c_i+N)\Gamma(z_i)}{\Gamma(z_i+c_i)\Gamma(z_i+N)}\frac{dz_i}{c_i}\end{aligned} \tag{2.3}$$

where the z_i contour is positively oriented around $0, -1, \ldots, -N+1$ for $1 \leq i \leq k$ and the z_j contour contains $z_i + c_i$ and $z_i - c_j$ for each $1 \leq i < j \leq k$.

Remark 1 The contour integral formula (2.3) can be viewed as a special case of [4, Propositions 2.8 and 2.9] and is closely related to the formula [3, Theorem B.2] for observables of Schur processes. These ideas go further back to the work of [8] where Macdonald processes, generalizations of the Schur processes [51], were introduced to study directed polymers. In this work, a family of contour integral formulas for observables of Macdonald processes were used to access these polymer models.

We now provide the proofs of these results.

Proof of Proposition 2.4 We show that

$$\begin{aligned}&\mathbb{E}\left[\left(\prod_{i=1}^{k}\sum_{j=1}^{N} y_j(M_i)^{c_i}\right)\frac{\mathcal{B}_{\log \boldsymbol{y}(M_1)}(z_1,\ldots,z_N)}{\mathcal{B}_{\log \boldsymbol{y}(M_1)}(\rho_N)}\right]\\&\quad=\mathcal{D}_{c_1}\prod_{m_1=M_2+1}^{M_1}\varphi_{m_1}(z_1,\ldots,z_N)\cdots\mathcal{D}_{c_k}\prod_{m_k=M_{k+1}+1}^{M_k}\varphi_{m_k}(z_1,\ldots,z_N)\end{aligned}\tag{2.4}$$

by induction on k. The result follows from evaluating the expression above at $z=\rho_N$. Indeed, (2.1) and Lemma 2.3 imply that

$$\mathbb{E}\left[\left(\sum_{j=1}^{N} y_j(M)^{c}\right)\frac{\mathcal{B}_{\log \boldsymbol{y}(M)}(z_1,\ldots,z_N)}{\mathcal{B}_{\log \boldsymbol{y}(M)}(\rho_N)}\right]=\mathcal{D}_c\prod_{m=1}^{M}\varphi_m(z_1,\ldots,z_N)$$

which is the $k=1$ base step in the induction. Next, suppose we know that

$$\begin{aligned}&\mathbb{E}\left[\left(\prod_{i=2}^{k}\sum_{j=1}^{N} y_j(M_i)^{c_i}\right)\frac{\mathcal{B}_{\log \boldsymbol{y}(M_2)}(z_1,\ldots,z_N)}{\mathcal{B}_{\log \boldsymbol{y}(M_2)}(\rho_N)}\right]\\&\quad=\mathcal{D}_{c_2}\prod_{m_2=M_3+1}^{M_2}\varphi_{m_2}(z_1,\ldots,z_N)\cdots\mathcal{D}_{c_k}\prod_{m_k=M_{k+1}+1}^{M_k}\varphi_{m_k}(z_1,\ldots,z_N)\end{aligned}\tag{2.5}$$

which is equivalent to assuming the induction hypothesis for $k-1$. Multiply both sides by

$$\prod_{m_1=M_2+1}^{M_1}\varphi_{m_1}(z_1,\ldots,z_N)$$

and apply $\mathcal{D}_{c_1}$. Then the right hand side of (2.5) becomes the right hand side of (2.4). The left hand side of (2.5) becomes

$$\begin{aligned}&\mathcal{D}_{c_1}\prod_{m_1=M_2+1}^{M_1}\varphi_{m_1}(z_1,\ldots,z_N)\cdot\mathbb{E}\left[\left(\prod_{i=2}^{k}\sum_{j=1}^{N} y_j(M_i)^{c_i}\right)\frac{\mathcal{B}_{\log \boldsymbol{y}(M_2)}(z_1,\ldots,z_N)}{\mathcal{B}_{\log \boldsymbol{y}(M_2)}(\rho_N)}\right]\\&=\mathbb{E}\left[\left(\prod_{i=2}^{k}\sum_{j=1}^{N} y_j(M_i)^{c_i}\right)\mathcal{D}_{c_1}\frac{\mathcal{B}_{\log \boldsymbol{y}(M_2)}(z_1,\ldots,z_N)}{\mathcal{B}_{\log \boldsymbol{y}(M_2)}(\rho_N)}\prod_{m_1=M_2+1}^{M_1}\varphi_{m_1}(z_1,\ldots,z_N)\right]\end{aligned}\tag{2.6}$$

which we want to match with the left hand side of (2.4). Observe that

$$\frac{\mathcal{B}_{\log \boldsymbol{y}(M_2)}(z_1,\dots,z_N)}{\mathcal{B}_{\log \boldsymbol{y}(M_2)}(\rho_N)} \prod_{m_1=M_2+1}^{M_1} \varphi_{m_1}(z_1,\dots,z_N) = \mathbb{E}\left[\frac{\mathcal{B}_{\log \boldsymbol{y}(M_1)}(z_1,\dots,z_N)}{\mathcal{B}_{\log \boldsymbol{y}(M_1)}(\rho_N)} \middle| \boldsymbol{y}(M_2) \right]$$

by Lemma 2.3. In words, the left hand side is the multivariate Bessel generating function for the matrix product $X(M_1)X(M_1-1)\cdots X(M_2+1)Y(M_2) = Y(M_1)$ where we condition $Y(M_2)$ to have squared singular values given by $\boldsymbol{y}(M_2)$. Using the identity above, (2.6) becomes

$$\begin{aligned}
&\mathbb{E}\left[\left(\prod_{i=2}^{k} \sum_{j=1}^{N} y_j(M_i)^{c_i} \right) \mathcal{D}_{c_1} \mathbb{E}\left[\frac{\mathcal{B}_{\log \boldsymbol{y}(M_1)}(z_1,\dots,z_N)}{\mathcal{B}_{\log \boldsymbol{y}(M_1)}(\rho_N)} \middle| \boldsymbol{y}(M_2) \right] \right] \\
&\quad = \mathbb{E}\left[\left(\prod_{i=2}^{k} \sum_{j=1}^{N} y_j(M_i)^{c_i} \right) \mathbb{E}\left[\sum_{j=1}^{N} y_j(M_1)^{c_1} \frac{\mathcal{B}_{\log \boldsymbol{y}(M_1)}(z_1,\dots,z_N)}{\mathcal{B}_{\log \boldsymbol{y}(M_1)}(\rho_N)} \middle| \boldsymbol{y}(M_2) \right] \right]
\end{aligned}$$

by commuting $\mathcal{D}_{c_1}$ with the conditional expectation. Thus we obtain the right hand side of (2.4) by consolidating the expectations. □

Proof of Proposition 2.5 We prove the statement for $a = 0$, the general case follows from the identity

$$\mathcal{B}_{\boldsymbol{\eta}+a}(z_1,\dots,z_N) = \left(\prod_{i=1}^{N} e^{az_i} \right) \mathcal{B}_{\boldsymbol{\eta}}(z_1,\dots,z_N).$$

The density at time t of N Brownian bridges starting at $\boldsymbol{a}$ (at time $t=0$), ending at $\boldsymbol{b}$ (at time T), and conditioned to never intersect is given by

$$\frac{1}{N!\det\left[p_T(a_i,b_j)\right]_{i,j=1}^{N}} \det\left[p_t(a_i,\eta_j)\right]_{i,j=1}^{N} \det\left[p_{T-t}(\eta_i,b_j)\right]_{i,j=1}^{N},$$

$$p_t(x,y) = \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}}$$

which expands out to

$$\frac{1}{(2\pi t(1-\frac{t}{T}))^{N/2}} \left(\prod_{i=1}^{N} e^{-\frac{a_i^2}{2t}+\frac{a_i^2}{2T}-\frac{b_i^2}{2(T-t)}+\frac{b_i^2}{2T}} e^{-\frac{T\eta_i^2}{2t(T-t)}} \right) \frac{\det\left[e^{\frac{a_i\eta_j}{t}} \right]_{i,j=1}^{N} \det\left[e^{\frac{b_i\eta_j}{T-t}} \right]_{i,j=1}^{N}}{N!\det\left[e^{\frac{a_ib_j}{T}} \right]_{i,j=1}^{N}}$$

supported on $\boldsymbol{\eta} \in \mathbb{R}^N$, by e.g. [38]. Here, the density is on the unordered positions of the Brownian motions. Take $a_i = \varepsilon(N-i)$ and $b_i = T\mu_i$. Then the density becomes

$$\frac{1}{N!\left(2\pi t\left(1-\frac{t}{T}\right)\right)^{N/2}}\left(\prod_{i=1}^{N} e^{-\frac{\varepsilon^2(T-t)(N-i)^2}{2tT}-\frac{tT\mu_i^2}{2(T-t)}} e^{-\frac{T\eta_i^2}{2t(T-t)}}\right)\det\left[e^{\frac{T\mu_i\eta_j}{T-t}}\right]_{i,j=1}^{N}$$
$$\prod_{1\le i<j\le N}\frac{e^{\frac{\varepsilon\eta_i}{t}}-e^{\frac{\varepsilon\eta_j}{t}}}{e^{\varepsilon\mu_i}-e^{\varepsilon\mu_j}}$$

where we use the Vandermonde determinant identity

$$\Delta(z)=\prod_{1\le i<j\le N}(z_i-z_j)=\det\left[z_i^{N-j}\right]_{1\le i,j\le N}.$$

Sending $\varepsilon\to 0$, then $T\to\infty$, we obtain

$$\frac{1}{N!(2\pi t)^{N/2}}\left(\prod_{i=1}^{N} e^{-\frac{t\mu_i^2}{2}-\frac{\eta_i^2}{2t}}\right)\det\left[e^{\mu_i\eta_j}\right]_{i,j=1}^{N}\frac{\Delta(\boldsymbol{\eta}/t)}{\Delta(\boldsymbol{\mu})}.$$

This is the time t marginal density for Brownian motion on $\mathbb{R}^N$ starting at $\mathbf{0}$ with drift vector $\boldsymbol{\mu}$, more specifically this density corresponds to the unordered coordinates of this Brownian motion (so the density corresponds to a measure on $\mathbb{R}^N$ rather than on the Weyl chamber $\{x_1\ge\cdots\ge x_N\}$). We have

$$\frac{\mathcal{B}_{\boldsymbol{\eta}}(z)}{\mathcal{B}_{\boldsymbol{\eta}}(\boldsymbol{\mu})}=\frac{\det\left[e^{z_i\eta_j}\right]_{i,j=1}^{N}}{\Delta(z)}\frac{\Delta(\boldsymbol{\mu})}{\det\left[e^{\mu_i\eta_j}\right]_{i,j=1}^{N}}.$$

Then

$$\begin{aligned}\mathbb{E}\left[\frac{\mathcal{B}_{\boldsymbol{\eta}(t)}(z)}{\mathcal{B}_{\boldsymbol{\eta}(t)}(\boldsymbol{\mu})}\right]&=\frac{1}{N!(2\pi t)^{N/2}}\left(\prod_{i=1}^{N} e^{-\frac{t\mu_i^2}{2}}\right)\int_{\mathbb{R}^N}\det\left[e^{z_i\eta_j}\right]_{i,j=1}^{N}\frac{\Delta(\boldsymbol{\eta}/t)}{\Delta(z)}\prod_{i=1}^{N} e^{-\frac{\eta_i^2}{2t}}\,d\boldsymbol{\eta}\\&=\frac{1}{N!(2\pi t)^{N/2}}\left(\prod_{i=1}^{N} e^{-\frac{t\mu_i^2}{2}}\right)\frac{1}{\Delta(z)}\int_{\mathbb{R}^N}\det\left[e^{z_i\eta_j}\right]_{i,j=1}^{N}\det\\&\quad\left[\left(\frac{\eta_i}{t}\right)^{N-j} e^{-\frac{\eta_i^2}{2t}}\right]_{i,j=1}^{N}d\boldsymbol{\eta}\end{aligned}$$

By Andréief's identity, we obtain

$$\begin{aligned}\mathbb{E}\left[\frac{\mathcal{B}_{\boldsymbol{\eta}(t)}(z)}{\mathcal{B}_{\boldsymbol{\eta}(t)}(\boldsymbol{\mu})}\right]&=\frac{1}{(2\pi t)^{N/2}}\left(\prod_{i=1}^{N} e^{-\frac{t\mu_i^2}{2}}\right)\frac{1}{\Delta(z)}\det\left[\int_{\mathbb{R}}\left(\frac{x}{t}\right)^{N-j} e^{xz_i-\frac{x^2}{2t}}\,dx\right]_{i,j=1}^{N}\\&=\frac{1}{(2\pi)^{N/2}}\left(\prod_{i=1}^{N} e^{\frac{t(z_i^2-\mu_i^2)}{2}}\right)\frac{1}{\Delta(z)}\det\left[M_{N-j}(z_i)\right]_{i,j=1}^{N}\end{aligned}\tag{2.7}$$

where

$$M_n(z) := \int_{\mathbb{R}} \left(\frac{x}{t}\right)^n e^{-\frac{(x-tz)^2}{2t}} \frac{dx}{\sqrt{t}} = \sqrt{t} \int_{\mathbb{R}} x^n e^{-\frac{t(x-z)^2}{2}} dx.$$

We claim that $M_n(z)$ is a degree n polynomial in z with leading coefficient $\sqrt{2\pi}$. We proceed by induction on n. Clearly, $M_0(z) = \sqrt{2\pi}$. Observe that

$$M_n(0) = \sqrt{t} \int_{\mathbb{R}} x^n e^{-\frac{tx^2}{2}} dx = \sqrt{t} \int_{\mathbb{R}} (x-z)^n e^{-\frac{t(x-z)^2}{2}} dx = \sqrt{t} \sum_{k=0}^{n} \binom{n}{k} (-z)^{n-k} M_k(z).$$

Rearranging, we get

$$M_n(z) = t^{-1/2} M_n(0) - \sum_{k=0}^{n-1} \binom{n}{k} (-z)^{n-k} M_k(z)$$

By our induction hypothesis, we have

$$M_n(z) = -\left(\sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k}\right) \sqrt{2\pi} z^n + \text{lower degree terms}$$

Thus the top degree term is $\sqrt{2\pi} z^n$, completing the induction. Applying row operations, we have

$$\det[M_{N-j}(z_i)]_{i,j=1}^{N} = (2\pi)^{N/2} \Delta(\boldsymbol{z}).$$

Plugging this into (2.7) completes the proof. □

Proof of Corollary 2.6 Set

$$\Phi_t(z_1, \ldots, z_N) := \mathbb{E}\left[\frac{\mathcal{B}_{\boldsymbol{\xi}^{(N)}(\frac{t}{4}) - \frac{Nt}{2}}(z_1, \ldots, z_N)}{\mathcal{B}_{\boldsymbol{\xi}^{(N)}(\frac{t}{4}) - \frac{Nt}{2}}(\rho_N)}\right].$$

The joint distribution of $\boldsymbol{\xi}^{(N)}(\frac{t_k}{4}) - \frac{Nt_k}{2}, \ldots, \boldsymbol{\xi}^{(N)}(\frac{t_1}{4}) - \frac{Nt_1}{2}$ is given by the joint distribution of the log squared singular values of

$$\begin{aligned} &\mathsf{X}^{(N,k)}(t_k), \\ &\quad \mathsf{X}^{(N,k-1)}(t_{k-1} - t_k)\mathsf{X}^{(N,k)}(t_k), \\ &\quad \vdots \\ &\quad \mathsf{X}^{(N,1)}(t_1 - t_2) \cdots \mathsf{X}^{(N,k-1)}(t_{k-1} - t_k)\mathsf{X}^{(N,k)}(t_k), \end{aligned}$$

where $\mathsf{X}^{(N,1)}(t), \dots, \mathsf{X}^{(N,k)}(t)$ are independent copies of $e^{-\frac{Nt}{4}}\mathsf{Y}^{(N)}(\frac{t}{4})$. Then Proposition 2.4 implies

$$\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\xi_j^{(N)}(\frac{t_i}{4})-\frac{Nt_i}{2}\right)}\right] = \mathcal{D}_{c_1}\Phi_{t_1-t_2}(z_1,\dots,z_N)\cdots\mathcal{D}_{c_k}\Phi_{t_k-t_{k+1}}(z_1,\dots,z_N)\big|_{z=\rho_N}.$$

We compute Φ_t. By Theorem 1.2, $\boldsymbol{\xi}^{(N)}(\frac{t}{4}) - \frac{Nt}{2}$ evolves as

$$\boldsymbol{\eta}(t) - \left(N-\tfrac{1}{2}\right)t$$

where $\boldsymbol{\eta}(s)$ is N non-intersecting Brownian motions with drift $\rho_N = (N-1, N-2, \dots, 0)$, started at the origin. Proposition 2.5 implies

$$\begin{aligned}\Phi_t(z_1,\dots,z_N) &= \mathbb{E}\left[\frac{\mathcal{B}_{\boldsymbol{\eta}^{(N)}(t)-\left(N-\frac{1}{2}\right)t}(z_1,\dots,z_N)}{\mathcal{B}_{\boldsymbol{\eta}^{(N)}(t)-\left(N-\frac{1}{2}\right)t}(\rho_N)}\right]\\ &= \prod_{i=1}^{N}\frac{e^{\frac{t}{2}\left(z_i-N+\frac{1}{2}\right)^2}}{e^{\frac{t}{2}\left(-i+\frac{1}{2}\right)^2}}.\end{aligned}$$

Thus we have shown (2.2).

We now show (2.3). We first claim that if $f_1(z), \dots, f_k(z)$ are entire functions, then (recalling $\mathcal{D}_c$ acts on everything to its right in an expression)

$$\begin{aligned}\mathcal{D}_{c_1}\left(\prod_{i=1}^{N} f_1(z_i)\right)\cdots\mathcal{D}_{c_k}\left(\prod_{i=1}^{N} f_k(z_i)\right) &= \left(\prod_{i=1}^{N} f_1(z_i)\cdots f_k(z_i)\right)\\ \times\frac{1}{(2\pi\mathbf{i})^k}\oint\cdots\oint\prod_{1\le i<j\le k}\frac{(w_i-w_j)(w_i+c_i-w_j-c_j)}{(w_i-w_j-c_j)(w_i+c_i-w_j)}&\\ \prod_{i=1}^{k}\left(\prod_{\ell=i}^{k}\frac{f_\ell(w_i+c_i)}{f_\ell(w_i)}\right)\left(\prod_{j=1}^{N}\frac{w_i+c_i-z_j}{w_i-z_j}\right)\frac{dw_i}{c_i}&\end{aligned}$$

where the w_i contour is positively oriented around $z_1, \dots, z_N$ for $1 \le i \le k$ and the w_j contour contains $w_i + c_i$ and $w_i - c_j$ for $1 \le i < j \le k$. This can be proved by induction on k using the residue theorem and the definition of $\mathcal{D}_c$, see e.g. [3, Appendix B].

If we set

$$f_\ell(z) = e^{(t_\ell - t_{\ell+1})\left(z-N+\frac{1}{2}\right)^2}$$

for $\ell = 1, \dots, k$, and apply (2.2), we obtain

$$\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\xi_j^{(N)}\left(\frac{t_i}{4}\right)-\frac{Nt_i}{2}\right)}\right] = \frac{1}{(2\pi\mathbf{i})^k}\oint\cdots\oint \prod_{1\le i<j\le k}\frac{(w_i-w_j)(w_i+c_i-w_j-c_j)}{(w_i-w_j-c_j)(w_i+c_i-w_j)} \times \prod_{i=1}^{k}\left(\prod_{\ell=i}^{k}\frac{e^{\frac{(t_\ell-t_{\ell+1})}{2}(w_i+c_i-N+\frac{1}{2})^2}}{e^{\frac{(t_\ell-t_{\ell+1})}{2}(w_i-N+\frac{1}{2})^2}}\right)\left(\prod_{j=1}^{N}\frac{w_i+c_i-N+j}{w_i-N+j}\right)\frac{dw_i}{c_i}.$$

Observe that

$$\prod_{j=1}^{N}\frac{w+c_i-N+j}{w-N+j} = \frac{\Gamma(w_i+c_i+1)\Gamma(w_i-N+1)}{\Gamma(w_i+c_i-N+1)\Gamma(w_i+1)}.$$

By changing variables $w_i = z_i + N - 1$ and consolidating the product over ℓ, (2.3) follows. □

3 Limiting line ensemble

The purpose of this section is to introduce line ensembles introduced in [16] and prove the existence of the limiting line ensemble $\boldsymbol{\xi}(t)$ and the convergence result Theorem 1.3. We prove auxiliary lemmas on the way to the proof of Theorem 1.3 for later usage.

Definition 3.1 Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be intervals. Consider the topological space $C(\Sigma \times \Lambda)$ with the topology of uniform convergence on compact subsets of $\Sigma \times \Lambda$. We may view $C(\Sigma \times \Lambda)$ as the space $\Lambda \times C(\Lambda)$ of sequences $(\eta_i(t))_{i\in\Sigma}$ of continuous functions on Λ by the identification $\eta(i,t) = \eta_i(t)$ for $\boldsymbol{\eta} \in C(\Sigma \times \Lambda)$. A *line ensemble (on Λ)* is a probability measure on $C(\Sigma \times \Lambda)$ with respect to the Borel σ-algebra. For us, the set Σ will always be $\{1, \dots, k\}$ for some k or $\mathbb{Z}_{>0}$. An *infinite line ensemble* will then be a line ensemble with $\Sigma = \mathbb{Z}_{>0}$. A line ensemble $\boldsymbol{\eta}$ is *non-intersecting* if $\eta_i(t) > \eta_j(t)$ for all $i < j$ and $t \in \Lambda$ almost surely.

Theorem 1.3 claims the existence of an infinite line ensemble $\{\boldsymbol{\xi}(t)\}_{t>0}$ which is the limit of $\{\boldsymbol{\xi}^{(N)}(\frac{t}{4}) - \frac{Nt}{2} - \log N\}_{t>0}$. The following theorem gives explicit expressions for certain observables of $\boldsymbol{\xi}(t)$.

Theorem 3.2 *We have:*

(i) For $c_1, \ldots, c_k > 0$,

$$\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{\infty} e^{c_i \xi_j(t_i)}\right] = \int \frac{dz_1}{2\pi \boldsymbol{i} c_1} \cdots \int \frac{dz_k}{2\pi \boldsymbol{i} c_k} \left(\prod_{1\le i<j\le k} \frac{(z_i - z_j)(z_i + c_i - z_j - c_j)}{(z_i + c_i - z_j)(z_i - z_j - c_j)}\right) \tag{3.1}$$

$$\times \prod_{i=1}^{k} \frac{e^{\frac{t_i}{2}\left(z_i + c_i - \frac{1}{2}\right)^2}}{e^{\frac{t_i}{2}\left(z_i - \frac{1}{2}\right)^2}} \frac{\Gamma(z_i)}{\Gamma(z_i + c_i)}$$

where the z_i contour is an infinite contour positively oriented around $0, -1, -2, \ldots$ which starts at $-\infty - \boldsymbol{i}\epsilon$ and ends at $-\infty + \boldsymbol{i}\epsilon$ for $1 \le i \le k$, and the z_j contour encloses $z_i + c_i$ and $z_i - c_j$ whenever $1 \le i < j \le k$.

(ii) The spacetime correlation kernel for $\boldsymbol{\xi}(t)$ is given by

$$\rho_k(t_1, x_1; \ldots; t_k, x_k) = \det\left[K(t_i, x_i; t_j, x_j)\right]_{1\le i,j\le k}$$

where

$$K(s, x; t, y) = -\frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} \boldsymbol{I}[t > s]$$

$$+ \int \frac{dz}{2\pi \boldsymbol{i}} \int_{c-\boldsymbol{i}\infty}^{c+\boldsymbol{i}\infty} \frac{dw}{2\pi \boldsymbol{i}} \frac{e^{\frac{tw^2}{2} - yw}}{e^{\frac{sz^2}{2} - xz}} \frac{1}{w - z} \frac{\Gamma\left(z + \frac{1}{2}\right)}{\Gamma\left(w + \frac{1}{2}\right)}. \tag{3.2}$$

and the z contour is an infinite contour positively oriented around $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$ which starts at $-\infty - \boldsymbol{i}\epsilon$ and ends at $-\infty + \boldsymbol{i}\epsilon$.

Remark 2 The explicit expression for the correlation function will not be used directly in this paper, we only use the fact that it is determinantal.

The remainder of this section is devoted to the proofs of Theorems 1.3 and 3.2. Our first step is to show the convergence of joint Laplace transforms and correlation functions.

Proposition 3.3 *Fix $t_1 \ge \cdots \ge t_k > 0$. Suppose $\tau_1(N) \ge \cdots \ge \tau_k(N) > 0$ such that $t_i := \lim_{N\to\infty} \tau_i(N)$ for $1 \le i \le k$.*

(i) For any $c_1, \ldots, c_k > 0$,

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\xi_j^{(N)}\left(\frac{\tau_i(N)}{4}\right) - \frac{N\tau_i(N)}{2} - \log N\right)}\right]$$

$$= \int \frac{dz_1}{2\pi \boldsymbol{i} c_1} \cdots \int \frac{dz_k}{2\pi \boldsymbol{i} c_k} \left(\prod_{1\le i<j\le k} \frac{(z_i - z_j)(z_i + c_i - z_j - c_j)}{(z_i + c_i - z_j)(z_i - z_j - c_j)}\right)$$

$$\prod_{i=1}^{k} \frac{e^{\frac{t_i}{2}(z_i+c_i-\frac{1}{2})^2}}{e^{\frac{t_i}{2}\left(z_i-\frac{1}{2}\right)^2}} \frac{\Gamma(z_i)}{\Gamma(z_i+c_i)}$$

where $c_1, \ldots, c_k > 0$, the z_i contour is an infinite contour positively oriented around $0, -1, -2, \ldots$ which starts at $-\infty - \boldsymbol{i}\epsilon$ and ends at $-\infty + \boldsymbol{i}\epsilon$ for $1 \le i \le k$, and the z_j contour encloses $z_i + c_i$ and $z_i - c_j$ whenever $1 \le i < j \le k$.

(ii) Let $\rho_k^{(N)}(\tau_1, x_1; \ldots, \tau_k, x_k)$ denote the kth space-time correlation function of

$$\left(\xi_1^{(N)}\left(\frac{\tau}{4}\right) - \frac{N\tau}{2} - \log N, \ldots, \xi_N^{(N)}\left(\frac{\tau}{4}\right) - \frac{N\tau}{2} - \log N\right)_{\tau \ge 0}.$$

Then

$$\lim_{N\to\infty} \rho_k^{(N)}(\tau_1(N), x_1; \ldots, \tau_k(N), x_k) = \det\left[K(t_i, x_i; t_j, x_j)\right]_{1\le i,j\le k}$$

where $K(s, x; t, y)$ is given by (3.2).

Proof of Proposition 3.3 Let $c_1, \ldots, c_k > 0$. By Corollary 2.6, we have

$$\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\xi_j(\frac{\tau_i(N)}{4}) - \frac{N\tau_i(N)}{2} - \log N\right)}\right] = \left(\prod_{i=1}^{k} N^{-c_i}\right)$$
$$\times \oint \frac{dz_1}{2\pi \mathbf{i} c_1} \cdots \oint \frac{dz_k}{2\pi \mathbf{i} c_k} \left(\prod_{1\le i<j\le k} \frac{(z_i - z_j)(z_i + c_i - z_j - c_j)}{(z_i - z_j - c_j)(z_i + c_i - z_j)}\right)$$
$$\prod_{i=1}^{k} \frac{e^{\frac{\tau_i(N)}{2}\left(z_i+c_i-\frac{1}{2}\right)^2}}{e^{\frac{\tau_i(N)}{2}\left(z_i-\frac{1}{2}\right)^2}} \frac{\Gamma(z_i + c_i + N)\Gamma(z_i)}{\Gamma(z_i + c_i)\Gamma(z_i + N)}$$

where the z_i contour is positively oriented around $0, -1, \ldots, -N+1$ for $1 \le i \le k$ and the z_j contour contains $z_i + c_i$ and $z_i - c_j$ for $1 \le i < j \le k$. From Stirling's formula [50, p141] (see also [3, Lemma 6.6]) for the Gamma function, we have

$$\frac{\Gamma(z_i + c_i + N)}{\Gamma(z_i + N)} = \frac{(z_i + c_i + N)^{z_i+c_i+N-\frac{1}{2}}}{(z_i + N)^{z_i+N-\frac{1}{2}}} e^{-c_i}(1 + O(1/N)) = N^{c_i}(1 + O(1/N))$$

which holds uniformly on compact subsets of the z_i contour. Combining this with the decay of the integrand for $\operatorname{Re} z \ll 0$ but $\operatorname{Re} z > -N - 1$ and $|\operatorname{Im} z|$ bounded away from 0, we obtain the desired expression

$$\int \frac{dz_1}{2\pi \mathbf{i} c_1} \cdots \int \frac{dz_k}{2\pi \mathbf{i} c_k} \left(\prod_{1 \le i < j \le k} \frac{(z_i - z_j)(z_i + c_i - z_j - c_j)}{(z_i + c_i - z_j)(z_i - z_j - c_j)} \right) \prod_{i=1}^{k} \frac{e^{\frac{t_i}{2}\left(z_i + c_i - \frac{1}{2}\right)^2}}{e^{\frac{t_i}{2}\left(z_i - \frac{1}{2}\right)^2}} \frac{\Gamma(z_i)}{\Gamma(z_i + c_i)}$$

in the limit as $N \to \infty$. Note that the decay of the exponential terms at infinity along the contour is clear. To see the decay of the gamma quotient, we may use the reflection formula for the Gamma function

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.$$

Recalling Theorem 1.2, we can explicitly write down the spacetime correlation kernel for $\{\boldsymbol{\xi}^{(N)}(\frac{t}{4}) - \frac{Nt}{2}\}_{t>0}$ by [38] (see also [18, Proposition 4.1]). It is given by

$$K_N(s, x; t, y) = -\frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} \mathbf{1}[t > s] + \oint_{\gamma} \frac{dz}{2\pi \mathbf{i}} \int_{\Gamma_c} \frac{dw}{2\pi \mathbf{i}} \frac{e^{\frac{tw^2}{2} - yw}}{e^{\frac{sz^2}{2} - xz}} \frac{1}{w - z} \prod_{i=1}^{N} \frac{w + i - \frac{1}{2}}{z + i - \frac{1}{2}}$$

where γ is a simple closed curve positively oriented around $\{-i + \frac{1}{2}\}_{i=1}^{N}$ and $\Gamma_c : \tau \mapsto c + \mathbf{i}\tau$, $\tau \in \mathbb{R}$ such that γ and Γ_c are disjoint. Thus

$$\rho_k^{(N)}(\tau_1, x_1; \ldots; \tau_k, x_k) = \det\left[K_N(\tau_i, x_i + \log N; \tau_j, x_j + \log N)\right]_{i,j=1}^{k}.$$

We can write

$$K_N(s, x + \log N; t, y + \log N) = -\frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} \mathbf{1}[t > s] + \oint_{\gamma} \frac{dz}{2\pi \mathbf{i}} \int_{\Gamma_c} \frac{dw}{2\pi \mathbf{i}} \frac{e^{\frac{tw^2}{2} - yw}}{e^{\frac{sz^2}{2} - xz}} \frac{N^{z-w}}{w - z} \frac{\Gamma\left(w + N + \frac{1}{2}\right)}{\Gamma\left(z + N + \frac{1}{2}\right)} \frac{\Gamma\left(z + \frac{1}{2}\right)}{\Gamma\left(w + \frac{1}{2}\right)}.$$

From Stirling's formula for the Gamma function as before, we find

$$\frac{\Gamma\left(w + N + \frac{1}{2}\right)}{\Gamma\left(z + N + \frac{1}{2}\right)} = \frac{\left(w + N + \frac{1}{2}\right)^{w+N}}{\left(z + N + \frac{1}{2}\right)^{z+N}} e^{z-w}(1 + O(1/N)) = N^{w-z}(1 + O(1/N)).$$

Thus, we have

$$\lim_{N \to \infty} K_N(s, x + \log N; t, y + \log N) = -\frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} \mathbf{1}[t > s] + \int \frac{dz}{2\pi \mathbf{i}} \int_{\Gamma_c} \frac{dw}{2\pi \mathbf{i}} \frac{e^{\frac{tw^2}{2} - yw}}{e^{\frac{sz^2}{2} - xz}} \frac{1}{w - z} \frac{\Gamma\left(z + \frac{1}{2}\right)}{\Gamma\left(w + \frac{1}{2}\right)}$$

where the z contour is an infinite contour positively oriented around $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$, starting at $-\infty - \mathbf{i}\epsilon$ and ending at $-\infty + \mathbf{i}\epsilon$. For full rigor, we must control the tail of the z-contour for $\mathrm{Re}\, z \ll 0$. This is managed by the reflection formula for the gamma function and the $e^{-\frac{sz^2}{2}}$ term, as before. □

The next two lemmas are the key to proving Theorems 1.3 and 3.2. They are stated in a manner convenient for later usage. The first lemma establishes the existence of a limiting process.

Lemma 3.4 *There exists a process* $\{\boldsymbol{\xi}(t) := (\xi_1(t), \xi_2(t), \ldots)\}_{t>0}$ *with joint Laplace transform given by* (3.1) *and spacetime correlation kernel given by* (3.2).

The next lemma links convergence of Laplace transforms with convergence in finite dimensional distributions.

Lemma 3.5 *Fix* $t_1 \geq \cdots \geq t_k > 0$. *Let* $\tau_1(N) \geq \cdots \geq \tau_k(N) > 0$ *such that* $t_i := \lim_{N\to\infty} \tau_i(N)$ *for* $1 \leq i \leq k$. *Suppose* $\left\{(\mathsf{y}_1^{(N)}(\tau) \geq \cdots \geq \mathsf{y}_N^{(N)}(\tau)\right\}_{\tau>0}$ *is a random* $\mathbb{R}^N$*-valued process such that there exists* $\varepsilon > 0$ *(which may vary with* k*) satisfying*

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i \mathsf{y}_j^{(N)}(\tau_i(N))}\right] = \mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{\infty} e^{c_i \xi_j(t_i)}\right] \tag{3.3}$$

for $0 < c_1, \ldots, c_k \leq \varepsilon$. *Then*

$$\lim_{N\to\infty} \mathbb{P}\left(\mathsf{y}_j(\tau_i(N)) \leq a_{i,j} : 1 \leq i \leq k, 1 \leq j \leq h\right) = \mathbb{P}\left(\xi_j(t_i) \leq a_{i,j} : 1 \leq i \leq k, 1 \leq j \leq h\right)$$

for any real numbers $a_{i,j}$ $(1 \leq i \leq k,\ 1 \leq j \leq h)$ *and any positive integer* k.

Proof of Lemmas 3.4 and 3.5 The argument below closely follows the ideas from [54, Section 5] and [49, Section 4.1.3] to show that the convergence of Laplace transforms of the correlation functions implies the desired convergence in finite dimensional distributions. Let $\rho_k^{(N)}(\tau_1, x_1; \ldots; \tau_k, x_k)$ denote the space-time correlation function for the process

$$\left(\mathsf{y}_1^{(N)}(\tau), \ldots, \mathsf{y}_k^{(N)}(\tau)\right).$$

Our assumption (3.3) implies the existence of the limits

$$\lim_{N\to\infty} \int_{\mathbb{R}^k} e^{c_1 x_1 + \cdots + c_k x_k} \rho_k^{(N)}(\tau_1(N), x_1; \ldots; \tau_k(N), x_k) dx_1 \cdots dx_k \tag{3.4}$$

for $0 < c_1, \ldots, c_k < \varepsilon$ where the limit is given by a finite linear combination of the right hand side of (3.1). We want to show that this limit is given by some limiting measure $\rho_k(t_1, x_1; \ldots; t_k, x_k)$. For this, define the measure

$$\varrho_k^{(N)}(x_1,\dots,x_k)dx_1\cdots dx_k := e^{\theta x_1+\cdots+\theta x_k}\rho_k^{(N)}(\tau_1(N),x_1;\dots;\tau_k(N),x_k)dx_1\cdots dx_k$$

where $\theta = \varepsilon/2$. The existence and form of the limits (3.4) implies the weak convergence of $\varrho_k^{(N)}$ to some limiting finite measure ϱ_k as measures on $\mathbb{R}^k$. Define ρ_k by

$$\rho_k(t_1,x_1;\dots;t_k,x_k)dx_1\cdots dx_k := e^{-\theta x_1-\cdots-\theta x_k}\varrho_k(x_1,\dots,x_k)dx_1\cdots dx_k$$

where we note the suppression of the dependence on the τ's and t's in the notation for $\varrho_k^{(N)}$ and ϱ_k. Thus

$$\rho_k^{(N)}(\tau_1(N),x_1;\dots;\tau_k(N),x_k)dx_1\cdots dx_k \to \rho_k(t_1,x_1;\dots;t_k,x_k)dx_1\cdots dx_k$$

weakly on $\mathbb{R}^k$. By Proposition 3.3, this convergence holds in particular for $\mathsf{y}^{(N)}(\tau) = \boldsymbol{\xi}^{(N)}(\frac{\tau}{4}) - \frac{N\tau}{2} - \log N$ so that

$$\rho_k(t_1,x_1;\dots,t_k,x_k) = \det\left[K(t_i,x_i;t_j,x_j)\right]_{1\le i,j\le k}$$

where $K(s,x;t,y)$ is given by (3.2).

The weak convergence of the correlation functions implies that the joint moments of random variables of the form

$$\mathcal{Y}^{(N)}_{\tau_i(N)}(S) := |\{j : \mathsf{y}_j^{(N)}(\tau_i(N)) \in S\}|, \qquad S\subset[c,\infty), \qquad 1\le i\le k, \qquad c>0$$

converge to corresponding joint moments of some limiting random variables

$$\mathcal{Y}_{t_i}(S), \qquad S\subset[c,\infty), \qquad 1\le i\le k, \qquad c>0.$$

Since the limit ρ_k is determinantal, the joint moments of the $\mathcal{Y}_{t_i}(S)$ do not grow faster than factorials so that the convergence of joint moments implies convergence in distribution. Therefore the probabilities

$$\mathbb{P}\left(\mathsf{y}_j^{(N)}(\tau_i(N)) \le a_{i,j} : 1\le i\le k, 1\le j\le h\right)$$

converge as $N\to\infty$ as they can be expressed as a finite linear combination of probabilities of the form

$$\mathbb{P}\Big(\mathcal{Y}^{(N)}_{\tau_1(N)}(S_{1,1}) = n_{1,1},\dots\mathcal{Y}^{(N)}_{\tau_1(N)}(S_{1,r_1}) = n_{1,r_1},\dots,\mathcal{Y}^{(N)}_{\tau_k(N)}(S_{k,1}) = n_{k,1},\dots,\mathcal{Y}^{(N)}_{\tau_k(N)}(S_{k,r_k}) = n_{k,r_k}\Big),$$

where the sets $S_{i,r}$ are among $(a_{i,1},\infty)$, $(a_{i,2},a_{i,1}]$, $\dots$, $(a_{i,h},a_{i,h-1}]$. This proves the existence of the limit (in finite dimensional distributions) process $\{(\xi_1,\xi_2,\dots)\}_{t>0}$,

where the Laplace transform and spacetime correlation kernel are necessarily given by (3.1) and (3.2). Thus Lemmas 3.4 and 3.5 follow. □

Proof of Theorem 1.3 and Theorem 3.2 We want to upgrade the convergence in finite dimensional distributions of

$$\left(\xi_1^{(N)}\left(\tfrac{t}{4}\right) - \tfrac{Nt}{2} - \log N, \ldots, \xi_N^{(N)}\left(\tfrac{t}{4}\right) - \tfrac{Nt}{2} - \log N\right)$$

implied by Proposition 3.3 and Lemma 3.5 to the stronger notion of convergence of line ensembles for Theorem 1.3. The machinery for this is supplied by [16, Proposition 3.6]. We can argue as in [16, Proposition 3.12] to check that our line ensembles satisfy the hypotheses of [16, Proposition 3.6], using the determinantal structure of the line ensembles from Proposition 3.3. The statements in [16] are for line ensembles on $[-T, T]$, so minor modifications in the statement of hypotheses need to be made to obtain the convergence of our line ensembles on $[\frac{1}{T}, T]$. Theorem 3.2 follows from Lemmas 3.4 and 3.5. □

4 The *S*-transform and ψ

Given a probability measure μ on $\mathbb{R}_{\geq 0}$, we can define its ψ-function and S-transform. The former is a generating function for the moments of μ and the latter plays the role of the log characteristic function from classical probability in the context of free probability, where the multiplicative free convolution corresponds to summation of independent random variables, see e.g. [15, 58]. We collect several properties of these functions for the analysis in subsequent sections.

Definition 4.1 Given a probability measure μ supported in $\mathbb{R}_{\geq 0}$, let

$$\psi_\mu(z) := \int \frac{zx}{1-zx}\, d\mu(x), \qquad z \in \mathbb{C} \setminus \operatorname{supp} \mu.$$

Definition 4.2 Let $\mathcal{M}$ denote the set of compactly supported Borel probability measures on $\mathbb{R}_{>0}$, in particular $\inf \operatorname{supp} \mu > 0$ for $\mu \in \mathcal{M}$. We view $\mathcal{M}$ as a topological space under the weak topology. Given a closed interval $I \subset \mathbb{R}_{>0}$, let $\mathcal{M}_I \subset \mathcal{M}$ denote the subset of probability measures supported in I, which is compact under the weak topology.

Assume that $\mu \in \mathcal{M}$. Then ψ_μ is analytic on $(\mathbb{C} \cup \{\infty\})\setminus J$ where J is some bounded interval in $\mathbb{R}_{>0}$ which contains $\{x^{-1} : x \in \operatorname{supp} \mu\}$. Moreover,

$$\psi'_\mu(z) = \int \frac{x}{(1-zx)^2}\, d\mu(x) \tag{4.1}$$

which is positive for $z \leq 0$. Thus there exists a meromorphic inverse ψ_μ^{-1} defined in a neighborhood of $[-1, 0]$ and mapping to the Riemann sphere with a simple pole at -1 and a zero at 0.

Definition 4.3 The *S-transform* of $\mu \in \mathcal{M}$ is given by

$$S_\mu(u) := \frac{1+u}{u}\psi_\mu^{-1}(u).$$

In view of the discussion above, for μ compactly supported in $\mathbb{R}_{>0}$, the S-transform is defined in a neighborhood of $[-1, 0]$.

Proposition 4.4 *Fix a compact subset $I \subset \mathbb{R}_{>0}$. Then there exists a neighborhood $U \subset \mathbb{C}$ of $[-1, 0]$ such that for all $\mu \in \mathcal{M}_I$*

(i) $\psi_\mu^{-1}(z)$ is well-defined, injective, meromorphic function on U with a unique pole at -1 and zero at 0;
(ii) $S_\mu(z)$ is holomorphic with no zeros on U, and
(iii) the map $\mu \mapsto S_\mu$ on $\mathcal{M}_I$ is continuous where the topology of the images are with respect to uniform convergence on compact subsets of U.

Proof Items (i) and (iii) follow from [15, Proposition 3.3]. Given that ψ_μ^{-1} has a simple pole at -1 and a zero at 0, (ii) follows from the definition of S_μ. □

Here are additional properties of the S-transform which follow from [15, Proposition 3.1]:

Proposition 4.5 *(i) $S'_\mu(u) \le 0$ for $u \in [-1, 0]$.*
(ii) $S_\mu(u) > 0$ for $u \in [-1, 0]$.
(iii) $\overline{S_\mu(u)} = S_\mu(\overline{u})$.

We record a lemma which evaluates the S-transform and its first and second derivatives at 0.

Lemma 4.6 *Suppose $\mu \in \mathcal{M}$. Let*

$$\kappa_1(\mu) := \int x\, d\mu(x), \qquad \kappa_2(\mu) := \int x^2\, d\mu(x) - \left(\int x\, d\mu(x)\right)^2$$

denote the mean and variance of μ respectively. Then

$$S_\mu(0) = \frac{1}{\kappa_1(\mu)}, \qquad S'_\mu(0) = -\frac{\kappa_2(\mu)}{\kappa_1(\mu)^3},$$
$$S''_\mu(0) = 4\frac{\left(\int x^2\, d\mu(x)\right)^2}{\left(\int x\, d\mu(x)\right)^5} - 2\frac{\int x^3\, d\mu(x)}{\left(\int x\, d\mu(x)\right)^4} - 2\frac{\int x^2\, d\mu(x)}{\left(\int x\, d\mu(x)\right)^3}.$$

Proof From the expansion

$$\psi_\mu(z) = z\int x\, d\mu(x) + z^2\int x^2\, d\mu(x) + z^3\int x^3\, d\mu(x) + O(|z|^4), \qquad |z| \to 0,$$

we get

$$\psi_\mu^{-1}(u) = u\frac{1}{\int x\,d\mu(x)} - u^2\frac{\int x^2\,d\mu(x)}{\left(\int x\,d\mu(x)\right)^3} + u^3\left(2\frac{\int x^2\,d\mu(x)}{\left(\int x\,d\mu(x)\right)^5} - \frac{\int x^3\,d\mu(x)}{\left(\int x\,d\mu(x)\right)^4}\right) + O(|u|^4), \qquad |u| \to 0$$

so that

$$S_\mu(u) = \frac{1}{\int x\,d\mu(x)} + \left(\frac{1}{\int x\,d\mu(x)} - \frac{\int x^2\,d\mu(x)}{\left(\int x\,d\mu(x)\right)^3}\right)u + \left(2\frac{\left(\int x^2\,d\mu(x)\right)^2}{\left(\int x\,d\mu(x)\right)^5} - \frac{\int x^3\,d\mu(x)}{\left(\int x\,d\mu(x)\right)^4} - \frac{\int x^2\,d\mu(x)}{\left(\int x\,d\mu(x)\right)^3}\right)u^2 + O(|u|^3)$$

as $|u| \to 0$. The result follows. □

We conclude this section with a lemma on ratios of Cauchy determinants involving the ψ-function, for later use.

Lemma 4.7 *Fix a compact subset $I \subset \mathbb{R}_{>0}$ and a positive integer k. Then there exists a neighborhood $U \subset \mathbb{C}$ of $[-1, 0]$ such that for all $\mu \in \mathcal{M}_I$ and $u_1, \ldots, u_k, v_1, \ldots, v_k \in U$, the bound*

$$C^{-1} < \left| \frac{\det\left(\frac{1}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_j)}\right)_{1\le i,j\le k}}{\det\left(\frac{1}{u_i-v_j}\right)_{1\le i,j\le k}} \prod_{i=1}^k \frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u_i))\psi_\mu'(\psi_\mu^{-1}(v_i))}} \right| < C \tag{4.2}$$

holds for some constant $C > 0$ independent of $\mu \in \mathcal{M}_I$. Moreover,

$$\frac{\det\left(\frac{1}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_j)}\right)_{1\le i,j\le k}}{\det\left(\frac{1}{u_i-v_j}\right)_{1\le i,j\le k}} \prod_{i=1}^k \frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u_i))\psi_\mu'(\psi_\mu^{-1}(v_i))}} = 1 + O\left(\max_{1\le i\le k} |u_i - v_i|^2\right) \tag{4.3}$$

uniformly over $\mu \in \mathcal{M}_I$, $u_1, \ldots, u_k, v_1, \ldots, v_k \in U$.

Remark 3 From Proposition 4.4, $\psi_\mu'(\psi_\mu^{-1}(u))$ is nonzero for u in a neighborhood U of $[-1, 0]$ and positive on $[-1, 0]$. Therefore, the square root is well-defined, where we take the standard branch for $u \in [-1, 0]$ and extend by continuity on U.

Proof of Lemma 4.7 Our starting point is a proof of the case $k = 1$, restated in the following claim:

Claim 4.8 *Fix a compact subset $I \subset \mathbb{R}_{>0}$. Then there exists a neighborhood $U \subset \mathbb{C}$ of $[-1, 0]$ such that for all $\mu \in \mathcal{M}_I$ and $u, v \in U$, we have*

$$C^{-1} < \frac{1}{\psi_\mu^{-1}(u) - \psi_\mu^{-1}(v)} \frac{u - v}{\sqrt{\psi'_\mu(\psi_\mu^{-1}(u))\psi'_\mu(\psi_\mu^{-1}(v))}} < C \tag{4.4}$$

for some constant C independent of $\mu \in \mathcal{M}_I$. Moreover,

$$\frac{1}{\psi_\mu^{-1}(u) - \psi_\mu^{-1}(v)} \frac{u - v}{\sqrt{\psi'_\mu(\psi_\mu^{-1}(u))\psi'_\mu(\psi_\mu^{-1}(v))}} = 1 + O(|u - v|^2) \tag{4.5}$$

uniformly over $\mu \in \mathcal{M}_I$ and $u, v \in U$.

Proof of Claim 4.8 Choose $U \supset [-1, 0]$ so that ψ_μ^{-1} is meromorphic, with a unique pole at -1 and zero at 0, on its closure for every $\mu \in \mathcal{M}_I$, where existence is guaranteed by Proposition 4.4. Observe that

$$\mathrm{C}(u, v) := \frac{1}{\psi_\mu^{-1}(u) - \psi_\mu^{-1}(v)} \frac{u - v}{\sqrt{\psi'_\mu(\psi_\mu^{-1}(u))\psi'_\mu(\psi_\mu^{-1}(v))}}$$

and its reciprocal have no poles of codimension 1 and are thus holomorphic on $\mathrm{cl}(U)^2$ by Riemann's second extension theorem [27, Theorem 7.1.2], as in [28, Proof of Lemma 3.5]. Therefore $\mathrm{C}(u, v)$ is bounded and does not vanish on U. This implies (4.4) where the uniformity of C follows from the compactness of $\mathcal{M}_I$ and $\mathrm{cl}(U)$, and the continuity of $\mathrm{C}(u, v)$ as a function of μ, u, and v.

It remains to show (4.5). Assume without loss of generality that $I = [a^{-1}, a]$ for some $a > 1$. Fix $\delta > 0$ small and let $W_\delta := \{w \in U : |w + 1| \geq \delta\}$.

We start by showing (4.5) for $u, v \in W_\delta$. Assuming $u, v \in W_\delta$, since

$$\psi_\mu^{-1}(u) - \psi_\mu^{-1}(v) = \frac{1}{\psi'_\mu(\psi_\mu^{-1}(v))}(u - v) - \frac{1}{2}\frac{\psi''_\mu(\psi_\mu^{-1}(v))}{\psi'_\mu(\psi_\mu^{-1}(v))^3}(u - v)^2 + O(|u - v|^3)$$

we have

$$\begin{aligned}
&\frac{1}{\psi_\mu^{-1}(u) - \psi_\mu^{-1}(v)} \frac{u - v}{\sqrt{\psi'_\mu(\psi_\mu^{-1}(u))\psi'_\mu(\psi_\mu^{-1}(v))}} \\
&= \frac{\sqrt{\psi'_\mu(\psi_\mu^{-1}(v))}}{\sqrt{\psi'_\mu(\psi_\mu^{-1}(u))}} \frac{1}{1 - \frac{1}{2}\frac{\psi''_\mu(\psi_\mu^{-1}(v))}{\psi'_\mu(\psi_\mu^{-1}(v))^2}(u - v) + O(|u - v|^2)} \\
&= \frac{\sqrt{\psi'_\mu(\psi_\mu^{-1}(v))}}{\sqrt{\psi'_\mu(\psi_\mu^{-1}(u))}} \left(1 + \frac{1}{2}\frac{\psi''_\mu(\psi_\mu^{-1}(v))}{\psi'_\mu(\psi_\mu^{-1}(v))^2}(u - v) + O(|u - v|^2)\right).
\end{aligned}$$

Since

$$\log \psi'_\mu(\psi_\mu^{-1}(u)) = \log \psi'_\mu(\psi_\mu^{-1}(v)) + \frac{\psi''_\mu(\psi_\mu^{-1}(v))}{\psi'_\mu(\psi_\mu^{-1}(v))^2}(u-v) + O(|u-v|^2),$$

we have

$$\begin{aligned}\frac{\sqrt{\psi'_\mu(\psi_\mu^{-1}(v))}}{\sqrt{\psi'_\mu(\psi_\mu^{-1}(u))}} &= \exp\left(\frac{1}{2}\log \psi'_\mu(\psi_\mu^{-1}(v)) - \frac{1}{2}\log \psi'_\mu(\psi_\mu^{-1}(u))\right)\\ &= \exp\left(-\frac{1}{2}\frac{\psi''_\mu(\psi_\mu^{-1}(v))}{\psi'_\mu(\psi_\mu^{-1}(v))^2}(u-v) + O(|u-v|^2)\right)\\ &= 1 - \frac{1}{2}\frac{\psi''_\mu(\psi_\mu^{-1}(v))}{\psi'_\mu(\psi_\mu^{-1}(v))^2}(u-v) + O(|u-v|^2).\end{aligned}$$

Combining these estimates proves (4.5) holds for $u, v \in W_\delta$.

To complete the proof, we show that (4.5) hold for $u, v \in \{w \in U : |w| \geq \delta\} = -(W_\delta + 1)$. For δ sufficiently small, W_δ and $-(W_\delta + 1)$ cover U. This is sufficient since the estimate (4.5) holds trivially if u, v are separated. We prove this by reduction to the case for W_δ. We may write

$$\begin{aligned}&\frac{1}{\psi_\mu^{-1}(u) - \psi_\mu^{-1}(v)} \frac{u-v}{\sqrt{\psi'_\mu(\psi_\mu^{-1}(u))\psi'_\mu(\psi_\mu^{-1}(v))}}\\ &= \frac{1}{\psi_\nu^{-1}(\widetilde{u}) - \psi_\nu^{-1}(\widetilde{v})} \frac{\widetilde{u}-\widetilde{v}}{\sqrt{\psi'_\nu(\psi_\nu^{-1}(\widetilde{u}))\psi'_\nu(\psi_\nu^{-1}(\widetilde{v}))}}\end{aligned}$$

where $\widetilde{u} = -(u+1)$, $\widetilde{v} = -(v+1)$, and let ν denote the Borel probability measure on $\mathbb{R}_{>0}$ determined by $\nu([c_1, c_2]) = \mu([c_2^{-1}, c_1^{-1}])$ for any $0 < c_1 \leq c_2 < \infty$. Indeed, observe

$$\begin{aligned}\psi_\nu(z) &= -\psi_\mu(z^{-1}) - 1\\ \psi_\nu^{-1}(w) &= \psi_\mu^{-1}(-(w+1))^{-1}\\ \psi'_\nu(z) &= \frac{1}{z^2}\psi'_\mu(z^{-1})\\ \psi'_\nu(\psi_\nu^{-1}(w)) &= \psi_\mu^{-1}(-(w+1))^2\psi'_\mu(\psi_\mu^{-1}(-(w+1))).\end{aligned}$$

Since $\nu \in \mathcal{M}_I$ (recall $I = [a^{-1}, a]$), this completes the proof. □

By the Cauchy determinant formula, which states

$$\det\left(\frac{1}{a_i - b_j}\right)_{1\leq i,j\leq k} = \frac{\prod_{1\leq i<j\leq k}(a_i - a_j)(b_j - b_i)}{\prod_{i,j=1}^k (a_i - b_j)},$$

we have

$$\begin{aligned}
&\frac{\det\left(\frac{1}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_j)}\right)_{1\le i,j\le k}}{\det\left(\frac{1}{u_i-v_j}\right)_{1\le i,j\le k}}\prod_{i=1}^{k}\frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u_i))\psi_\mu'(\psi_\mu^{-1}(v_i))}}\\
&=\prod_{i=1}^{k}\frac{u_i-v_i}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_i)}\frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u_i))\psi_\mu'(\psi_\mu^{-1}(v_i))}}\\
&\times\prod_{1\le i<j\le k}\frac{u_i-v_j}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_j)}\frac{v_i-u_j}{\psi_\mu^{-1}(v_i)-\psi_\mu^{-1}(u_j)}\\
&\frac{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(u_j)}{u_i-u_j}\frac{\psi_\mu^{-1}(v_i)-\psi_\mu^{-1}(v_j)}{v_i-v_j}.
\end{aligned}$$

Setting

$$\mathrm{C}(u,v):=\frac{u-v}{\psi_\mu^{-1}(u)-\psi_\mu^{-1}(v)}\frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u))\psi_\mu'(\psi_\mu^{-1}(v))}},$$

we obtain

$$\begin{aligned}
&\frac{\det\left(\frac{1}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_j)}\right)_{1\le i,j\le k}}{\det\left(\frac{1}{u_i-v_j}\right)_{1\le i,j\le k}}\prod_{i=1}^{k}\frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u_i))\psi_\mu'(\psi_\mu^{-1}(v_i))}}\\
&=\prod_{i=1}^{k}\mathrm{C}(u_i,v_i)\prod_{1\le i<j\le k}\frac{\mathrm{C}(u_i,v_j)\mathrm{C}(v_i,u_j)}{\mathrm{C}(u_i,u_j)\mathrm{C}(v_i,v_j)}.
\end{aligned}$$

Then Claim 4.8 implies the bound (4.2).

For the estimate (4.3), first note that

$$\mathrm{C}(u_i,v_i)=1+O(|u_i-v_i|^2)$$

by Claim 4.8, and

$$\frac{\mathrm{C}(u_i,v_j)\mathrm{C}(v_i,u_j)}{\mathrm{C}(u_i,u_j)\mathrm{C}(v_i,v_j)}=1+O\left(\max(|u_i-v_i|^2,|u_j-v_j|^2)\right),$$

which can be seen by Taylor expanding in u_i near v_i and u_j near v_j. □

5 Asymptotics of multivariate Bessel functions

Given $v_1, \ldots, v_k \in \{N-1, N-2, \ldots, 0\}$, define

$$B_\mu^{(N)}(u_1, \ldots, u_k; v_1, \ldots, v_k) := \frac{\mathcal{B}_{\boldsymbol{a}}(u_1, \ldots, u_k, N-1, \ldots, \widehat{v_1}, \ldots, \widehat{v_k}, \ldots, 0)}{\mathcal{B}_{\boldsymbol{a}}(N-1, \ldots, 0)}$$

where $\mu := \frac{1}{N}\sum_{i=1}^N \delta_{e^{a_i}}$ and the hat notation means that $v_1, \ldots, v_k$ are omitted from $N-1, N-2, \ldots, 0$. In other words, the multivariate Bessel function in the numerator takes as input ρ_N with $v_1, \ldots, v_k$ replaced by $u_1, \ldots, u_k$. In this section, we obtain asymptotics for these normalized multivariate Bessel functions in preparation for proving Theorem 1.4. We note that the asymptotics from this section are refinements of those from [28, Theorem 3.4]. Moreover, we obtain our asymptotics by bootstrapping off the latter.

Definition 5.1 Define

$$H_\mu(u) := -(u+1)\log S_\mu(u) - \int \log\left((u+1)S_\mu(u)^{-1} - ux\right) d\mu(x)$$

where the logarithms are given by the standard branch.

Observe that

$$H_\mu(u) = -(u+1)\log(u+1) + u\log u - u\log\psi_\mu^{-1}(u) - \int \log(1 - x\psi_\mu^{-1}(u))d\mu(x).$$

Using the fact that

$$-u - \int \frac{x\psi_\mu^{-1}(u)}{1 - x\psi_\mu^{-1}(u)} d\mu(x) = u - \psi_\mu(\psi_\mu^{-1}(u)) = 0,$$

we have

$$H_\mu'(u) = -\log(u+1) + \log u - \log\psi_\mu^{-1}(u) = -\log S_\mu(u) \tag{5.1}$$

and

$$H_\mu''(u) = -\frac{S_\mu'(u)}{S_\mu(u)}. \tag{5.2}$$

Definition 5.2 Let $\mathcal{R}^N$ denote the subset of $\mathcal{M}_I$ consisting of probability measures of the form

$$\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$$

where $x_1, \ldots, x_N \in I$.

Theorem 5.3 *Fix a closed interval $I \subset \mathbb{R}_{>0}$. There exists an open neighborhood U of $[-1, 0]$ such that*

$$B_\mu(N(u_1+1), \ldots, N(u_k+1); N(v_1+1), \ldots, N(v_k+1)) = \frac{\det\left(\frac{1}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_j)}\right)_{1\le i,j\le k}}{\det\left(\frac{1}{u_i-v_j}\right)_{1\le i,j\le k}} \prod_{i=1}^{k} \left[\frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u_i))\psi_\mu'(\psi_\mu^{-1}(v_i))}} \frac{\sqrt{S_\mu(v_i)}e^{NH_\mu(u_i)}}{\sqrt{S_\mu(u_i)}e^{NH_\mu(v_i)}}(1+o(|u_i-v_i|))\right]$$

as $N \to \infty$, uniformly over $\mu \in \mathcal{M}_I \cap \mathcal{R}^N$, $u_1, \ldots, u_k \in U$, and $v_1, \ldots, v_k \in \frac{1}{N}\mathbb{Z} \cap [-1, 0]$.

Remark 4 To translate between our notation and that of [28], our H_μ corresponds to their $\widetilde{\Psi}_{\rho_N}$ and our ψ_μ corresponds to their $M_{\widetilde{\rho}_N}$.

Remark 5 We note the peculiarity in Theorem 5.3 that the uniformity $\mu \in \mathcal{M}_I \cap \mathcal{R}^N$ is over a set varying with N.

Proof of Theorem 5.3 Our starting point is [28, Theorem 3.4] which states that there is some neighborhood U of $[-1, 0]$ such that

$$B_\mu^{(N)}(N(u_1+1), \ldots, N(u_k+1); N(v_1+1), \ldots, N(v_k+1)) = \frac{\det\left(\frac{1}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_j)}\right)_{1\le i,j\le k}}{\det\left(\frac{1}{u_i-v_j}\right)_{1\le i,j\le k}} \prod_{i=1}^{k} \left[\frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u_i))\psi_\mu'(\psi_\mu^{-1}(v_i))}} \frac{\sqrt{S_\mu(v_i)}e^{NH_\mu(u_i)}}{\sqrt{S_\mu(u_i)}e^{NH_\mu(v_i)}}(1+o(1))\right]$$

as $N \to \infty$, uniformly for $u_1, \ldots, u_k, v_1, \ldots, v_k \in U$ and $\mu \in \mathcal{M}_I \cap \mathcal{R}^N$. We note that the original statement of [28, Theorem 3.4] is in the regime where $\mu = \mu_N$ converges weakly to a measure in $\mathcal{M}_I$ as $N \to \infty$, but the proof also implies uniform asymptotics for $\mu \in \mathcal{M}_I \cap \mathcal{R}^N$. Thus, it remains to improve the relative $o(1)$ error.

Define

$$\mathfrak{B}_\mu^{(N)}(u_1, \ldots, u_k; v_1, \ldots, v_k) = \frac{\det\left(\frac{1}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_j)}\right)_{1\le i,j\le k}}{\det\left(\frac{1}{u_i-v_j}\right)_{1\le i,j\le k}} \prod_{i=1}^{k} \left[\frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u_i))\psi_\mu'(\psi_\mu^{-1}(v_i))}} \frac{\sqrt{S_\mu(v_i)}e^{NH_\mu(u_i)}}{\sqrt{S_\mu(u_i)}e^{NH_\mu(v_i)}}\right]$$

for $u_1, \ldots, u_k, v_1, \ldots, v_k \in U$. Then for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} &B_\mu^{(N)}(N(u_1+1), \ldots, N(u_k+1); N(v_1+1), \ldots, N(v_k+1)) \\ &\quad = \mathfrak{B}_\mu^{(N)}(u_1, \ldots, u_k; v_1, \ldots, v_k)(1+o(1)) \end{aligned} \tag{5.3}$$

as $N \to \infty$, uniformly over $u_1, \ldots, u_k \in U_\varepsilon$, and $v_1, \ldots, v_k \in \frac{1}{N}\mathbb{Z} \cap [-1, 0]$, where U_ε denotes the ε-neighborhood of $[-1, 0]$.

By Lemma 4.7, the quotient of Cauchy determinants in the definition of $\mathfrak{B}_\mu^{(N)}$ is bounded and bounded away from 0 for $u_1, \ldots, u_k, v_1, \ldots, v_k \in U_\varepsilon$, for $\varepsilon > 0$ sufficiently small. Similarly, since (see Proposition 4.4 and Proposition 4.5 (ii))

$$\psi_\mu'(\psi_\mu^{-1}(0)) = \psi_\mu'(0) \neq 0 \qquad \text{and} \qquad S_\mu(0) \neq 0,$$

we have

$$\frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u))\psi_\mu'(\psi_\mu^{-1}(v))}} \frac{\sqrt{S_\mu(v)}}{\sqrt{S_\mu(u)}}$$

is bounded and bounded away from 0 for $u_1, \ldots, u_k, v_1, \ldots, v_k \in U_\varepsilon$, given that ε is sufficiently small. For each integer $k \geq 1$, define

$$\begin{aligned} &F_k^{(N)}(u_1, \ldots, u_k; v_1, \ldots, v_k) \\ &\quad := \log\left(\frac{B_\mu^{(N)}(N(u_1+1), \ldots, N(u_k+1); N(v_1+1), \ldots, N(v_k+1))}{B_\mu^{(N)}(N(u_1+1), \ldots, N(u_{k-1}+1); N(v_1+1), \ldots, N(v_{k-1}+1))}\right) \\ &\mathfrak{F}_k^{(N)}(u_1, \ldots, u_k; v_1, \ldots, v_k) \\ &\quad := \log\left(\frac{\mathfrak{B}_\mu^{(N)}(Nu_1, \ldots, Nu_k; Nv_1, \ldots, Nv_k)}{\mathfrak{B}_\mu^{(N)}(Nu_1, \ldots, Nu_{k-1}; Nv_1, \ldots, Nv_{k-1})}\right) \end{aligned}$$

where in the case $k = 1$, we take the denominator in the logarithm to be 1. Then $F_k^{(N)}$ and $\mathfrak{F}_k^{(N)}$ are analytic for $u_1, \ldots, u_k \in U_\varepsilon$, where $v_1, \ldots, v_k \in \frac{1}{N}\mathbb{Z} \cap [-1, 0]$ and N is sufficiently large. Moreover, we have the convergence $F_k^{(N)} - \mathfrak{F}_k^{(N)} \to 0$ as $N \to \infty$ on this region by definition and the convergence (5.3). Furthermore, $F_k^{(N)}$ and $\mathfrak{F}_k^{(N)}$ vanish whenever $u_k = v_k$.

Then

$$\begin{aligned} &\mathcal{G}_i^{(N)}(u_1, \ldots, u_i; v_1, \ldots, v_i) \\ &\quad := \frac{1}{u_i - v_i} F_i^{(N)}(u_1, \ldots, u_i; v_1, \ldots, v_i) - \frac{1}{u_i - v_i} \mathfrak{F}_i^{(N)}(u_1, \ldots, u_i; v_1, \ldots, v_i) \end{aligned}$$

is analytic for $u_1, \ldots, u_i \in U_\varepsilon$, where $v_1, \ldots, v_i \in \frac{1}{N}\mathbb{Z} \cap [-1, 0]$ and N is sufficiently large. Since $F_i^{(N)} - \mathfrak{F}_i^{(N)} \to 0$, we have

$$\mathcal{G}_i^{(N)}(u_1, \ldots, u_i; v_1, \ldots, v_i) = o(1)$$

uniformly for $u_1, \ldots, u_i \in \partial U_{2\varepsilon/3}$ and $v_1, \ldots, v_i \in \frac{1}{N}\mathbb{Z} \cap [-1, 0]$, since the restriction of $u_1, \ldots, u_i$ to the boundary of $U_{2\varepsilon/3}$ keeps $1/(u_i - v_i)$ bounded.

By Cauchy integral formula,

$$\begin{aligned}&\mathcal{G}_i^{(N)}(u_1, \ldots, u_i; v_1, \ldots, v_i)\\ &\quad = \frac{1}{(2\pi\mathbf{i})^i} \oint_{\partial U_{2\varepsilon/3}} \cdots \oint_{\partial U_{2\varepsilon/3}} \frac{G(w_1, \ldots, w_i; v_1, \ldots, v_i)}{(w_1 - u_1) \cdots (w_i - u_i)}\, dw_1 \cdots dw_i = o(1)\end{aligned}$$

uniformly for $u_1, \ldots, u_i \in U_{\varepsilon/2}$ and $v_1, \ldots, v_i \in \frac{1}{N}\mathbb{Z} \cap [-1, 0]$. Therefore,

$$\begin{aligned}&B_\mu^{(N)}(N(u_1 + 1), \ldots, N(u_k + 1); N(v_1 + 1), \ldots, N(v_k + 1))\\ &\quad = \exp\left(\sum_{i=1}^k F_i^{(N)}(u_1, \ldots, u_i; v_1, \ldots, v_i)\right)\\ &\quad = \exp\left(\sum_{i=1}^k \left(\mathfrak{F}_i^{(N)}(u_1, \ldots, u_i; v_1, \ldots, v_i) + o(|u_i - v_i|)\right)\right)\\ &\quad = \mathfrak{B}_\mu^{(N)}(u_1, \ldots, u_k; v_1, \ldots, v_k)(1 + o(\max_i |u_i - v_i|))\end{aligned}$$

as $N \to \infty$, uniformly for $u_1, \ldots, u_k \in U_{\varepsilon/2}$ and $v_1, \ldots, v_k \in \frac{1}{N}\mathbb{Z} \cap [-1, 0]$. This completes the proof of Theorem 5.3. □

6 Proof of Theorem 1.4

In this section, we prove our main result Theorem 1.4. Throughout this section, we fix some notation. Given a sequence $X^{(N)}(1)$, $X^{(N)}(2)$, ..., denote by $\mu_N^{(m)}$ the empirical distribution of the squared singular values of $X^{(N)}(m)$. Given a compactly supported probability measure μ, let $\kappa_1(\mu)$ and $\kappa_2(\mu)$ denote the mean (first cumulant) and variance (second cumulant) of μ respectively.

The key step is to establish the following intermediate result.

Theorem 6.1 *Suppose that $X^{(N)}(1)$, $X^{(N)}(2)$, ... have deterministic squared singular values, all contained in a fixed compact interval $I \subset \mathbb{R}_{>0}$, and that the hypotheses of Theorem 1.4 (i.e. conditions (1.1) and (1.2)) are satisfied. Let*

$$y_1^{(N)}(M) \geq \cdots \geq y_N^{(N)}(M)$$

denote the squared singular values of $X^{(N)}(M)\cdots X^{(N)}(1)$. *Then for any* $t_1 \geq \cdots \geq t_k > 0$ *and* $c_1, \ldots, c_k > 0$ *such that* $c_1 + \cdots + c_k \in (0, 1)$, *we have*

$$\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i \log y_j^{(N)}(\lfloor t_i N\rfloor)}\right] = \left(\prod_{i=1}^{k} e^{c_i \mathcal{E}_N(\lfloor t_i N\rfloor)}\right)$$
$$\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\xi_j^{(N)}(\frac{1}{4}\mathcal{V}_N(\lfloor t_i N\rfloor)) - \frac{N}{2}\mathcal{V}_N(\lfloor t_i N\rfloor)\right)}\right](1 + o(1))$$

as $N \to \infty$, *where*

$$\mathcal{E}_N(M) := \sum_{m=1}^{M} \log \kappa_1(\mu_N^{(m)}), \qquad \text{and} \qquad \mathcal{V}_N(M) := \frac{1}{N}\sum_{m=1}^{M} \frac{\kappa_2(\mu_N^{(m)})}{\kappa_1(\mu_N^{(m)})^2}.$$

This convergence holds uniformly over sequences $X_1^{(N)}, X_2^{(N)}, \ldots$ *satisfying* (1.1) *and* (1.2) *such that the squared singular values of* $X^{(N)}(m)$ *lie in* I *for every* $1 \leq i \leq M_1$.

The proof of Theorem 6.1 combines the asymptotics from the previous sections and our formalism of multivariate Bessel functions. Note that Theorem 6.1 makes the assumption that the matrices have *non-random* singular values. The proof of Theorem 1.4 proceeds straightforwardly from Theorem 6.1 by bootstrapping from the deterministic case, see Sect. 6.2.

6.1 Proof of Theorem 6.1

Let $M_i := M_i(N) := \lfloor t_i N\rfloor$ for $1 \leq i \leq k$ and $M_{k+1} := 0$. Let $\boldsymbol{x}(m) = (x_1^{(m)}, \ldots, x_N^{(m)})$ denote the squared singular values of $X^{(N)}(m)$. By Proposition 2.4,

$$\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i \log y_j^{(N)}(M_i)}\right]$$
$$= \mathcal{D}_{c_1}^{(N)} \prod_{m_1=M_2+1}^{M_1} \frac{\mathcal{B}_{\log \boldsymbol{x}(m_1)}(z_1, \ldots, z_N)}{\mathcal{B}_{\log \boldsymbol{x}(m_1)}(\rho_N)} \cdots \mathcal{D}_{c_k}^{(N)} \prod_{m_k=M_{k+1}+1}^{M_k} \frac{\mathcal{B}_{\log \boldsymbol{x}(m_k)}(z_1, \ldots, z_N)}{\mathcal{B}_{\log \boldsymbol{x}(m_k)}(\rho_N)}\Bigg|_{z=\rho_N}.$$

Recall our convention that $\mathcal{D}_c = \mathcal{D}_c^{(N)}$ acts on everything to its right (see Sect. 2). Expanding out the $\mathcal{D}_c$ terms, we obtain

$$\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i \log y_j^{(N)}(M_i)}\right]$$
$$= \sum_{i_1,\ldots,i_k=1}^{N}\left(\prod_{j_1\neq i_1}\frac{c_1 + z_{i_1} - z_{j_1}}{z_{i_1} - z_{j_1}}\right)\mathcal{T}_{c_1, z_{i_1}} \prod_{m_1=M_2+1}^{M_1} \frac{\mathcal{B}_{\log \boldsymbol{x}(m_1)}(z_1, \ldots, z_N)}{\mathcal{B}_{\log \boldsymbol{x}(m_1)}(\rho_N)}$$

$$\cdots \left(\prod_{j_k \neq i_k} \frac{c_k + z_{i_k} - z_{j_k}}{z_{i_k} - z_{j_k}} \right) \mathcal{T}_{c_k, z_{i_k}} \prod_{m_k = M_{k+1}+1}^{M_k} \frac{\mathcal{B}_{\log \boldsymbol{x}(m_k)}(z_1, \ldots, z_N)}{\mathcal{B}_{\log \boldsymbol{x}(m_k)}(\rho_N)} \Bigg|_{z=\rho_N}.$$

The products over $j_\ell \neq i_\ell$ are understood to range over $1 \leq j_\ell \leq N$ (for $1 \leq \ell \leq k$). Like $\mathcal{D}_c$, the shift operators $\mathcal{T}_{c,z_i}$ act on everything to their right. If the $\mathcal{T}_{c,z_i}$ is contained between parentheses, its action is confined within those parentheses.

Since $\mathcal{T}_c fg = (\mathcal{T}_c f)(\mathcal{T}_c g)$, we get

$$\mathbb{E}\left[\prod_{i=1}^{k} \sum_{j=1}^{N} e^{c_i \log y_j^{(N)}(M_i)} \right] = \sum_{i_1, \ldots, i_k = 1}^{N} \sigma_{i_1, \ldots, i_k}$$

where

$$\sigma_{i_1,\ldots,i_k} = \prod_{\ell=1}^{k} \left(\mathcal{T}_{c_1,z_{i_1}} \cdots \mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}} \prod_{j_\ell \neq i_\ell} \frac{c_\ell + z_{i_\ell} - z_{j_\ell}}{z_{i_\ell} - z_{j_\ell}} \Bigg|_{z=\rho_N} \right) \times \left(\prod_{m_\ell = M_{\ell+1}+1}^{M_\ell} \mathcal{T}_{c_1,z_{i_1}} \cdots \mathcal{T}_{c_\ell,z_{i_\ell}} \frac{\mathcal{B}_{\log \boldsymbol{x}(m_\ell)}(z_1, \ldots, z_N)}{\mathcal{B}_{\log \boldsymbol{x}(m_\ell)}(\rho_N)} \Bigg|_{z=\rho_N} \right).$$

Set

$$\begin{aligned}
\tau_{i_1,\ldots,i_k} &:= \left(\prod_{i=1}^{k} e^{c_i \mathcal{E}_N(M_i)} \right) \left(\prod_{j_1 \neq i_1} \frac{c_1 + z_{i_1} - z_{j_1}}{z_{i_1} - z_{j_1}} \right) \mathcal{T}_{c_1,z_{i_1}} \left(\prod_{a_1=1}^{N} \frac{\exp\left[\Delta_1\left(z_{a_1} - N + \frac{1}{2}\right)^2\right]}{\exp\left[\Delta_1\left(-a_1 + \frac{1}{2}\right)^2\right]} \right) \\
&\quad \times \cdots \times \left(\prod_{j_k \neq i_k} \frac{c_k + z_{i_k} - z_{j_k}}{z_{i_k} - z_{j_k}} \right) \mathcal{T}_{c_k,z_{i_k}} \left(\prod_{a_k=1}^{N} \frac{\exp\left[\Delta_k\left(z_{a_k} - N + \frac{1}{2}\right)^2\right]}{\exp\left[\Delta_k\left(-a_k + \frac{1}{2}\right)^2\right]} \right) \Bigg|_{z=\rho_N} \\
&= \left(\prod_{i=1}^{k} e^{c_i \mathcal{E}_N(M_i)} \right) \mathcal{D}_{c_1} \left(\prod_{a_1=1}^{N} \frac{\exp\left[\Delta_1\left(z_{a_1} - N + \frac{1}{2}\right)^2\right]}{\exp\left[\Delta_1\left(-a_1 + \frac{1}{2}\right)^2\right]} \right) \cdots \\
&\quad \mathcal{D}_{c_k} \left(\prod_{a_k=1}^{N} \frac{\exp\left[\Delta_k\left(z_{a_k} - N + \frac{1}{2}\right)^2\right]}{\exp\left[\Delta_k\left(-a_k + \frac{1}{2}\right)^2\right]} \right) \Bigg|_{z=\rho_N}
\end{aligned}$$

where

$$\Delta_\ell := \frac{1}{2N} \sum_{m=M_{\ell+1}+1}^{M_\ell} \frac{\kappa_2(\mu_N^{(m)})}{\kappa_1(\mu_N^{(m)})^2} = \frac{1}{2} \left(\mathcal{V}_N(M_\ell) - \mathcal{V}_N(M_{\ell+1}) \right), \qquad 1 \leq \ell \leq k.$$

The equality following the definition of $\tau_{i_1,\dots,i_k}$ follows from the definition for $\mathcal{D}_c$, as in the calculation (though in reverse) at the start of this proof.

We prove that

$$\sigma_{i_1,\dots,i_k} = \tau_{i_1,\dots,i_k}(1+o(1)) \tag{6.1}$$

as $N \to \infty$, uniformly over $1 \le i_1, \dots, i_k \le N^{1/3}$. Furthermore, we prove that if N is sufficiently large then

$$\sigma_{i_1,\dots,i_k} > 0, \tag{6.2}$$
$$\tau_{i_1,\dots,i_k} > 0, \tag{6.3}$$

for $1 \le i_1, \dots, i_k \le N$, and there exists $c > 0$ such that

$$\sigma_{i_1,\dots,i_k} \le \sigma_{1,\dots,1} e^{-cN^{1/3}}, \tag{6.4}$$
$$\tau_{i_1,\dots,i_k} \le \sigma_{1,\dots,1} e^{-cN^{1/3}} \tag{6.5}$$

for $1 \le i_1, \dots, i_k \le N$ such that $i_j > N^{1/3}$ for some $1 \le j \le k$. Indeed, Theorem 6.1 would follow because

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i \log y_j^{(N)}(M_i)}\right] &= \sum_{i_1,\dots,i_k=1}^{N} \sigma_{i_1,\dots,i_k} = (1+o(1)) \sum_{i_1,\dots,i_k=1}^{N} \tau_{i_1,\dots,i_k} \\
&= (1+o(1)) \left(\prod_{i=1}^{k} e^{c_i \mathcal{E}_N(M_i)}\right) \mathcal{D}_{c_1} \left(\prod_{a_1=1}^{N} \frac{\exp\left[\Delta_1\left(z_{a_1} - N + \frac{1}{2}\right)^2\right]}{\exp\left[\Delta_1\left(-a_1+\frac{1}{2}\right)^2\right]}\right) \\
&\quad \cdots \mathcal{D}_{c_k} \left(\prod_{a_k=1}^{N} \frac{\exp\left[\Delta_k\left(z_{a_k} - N + \frac{1}{2}\right)^2\right]}{\exp\left[\Delta_k\left(-a_k+\frac{1}{2}\right)^2\right]}\right)\Bigg|_{z=\rho_N} \\
&= (1+o(1)) \left(\prod_{i=1}^{k} e^{c_i \mathcal{E}_N(M_i)}\right) \mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\xi_j^{(N)}\left(\frac{1}{4}\mathcal{V}_N(M_i)\right) - \frac{N}{2}\mathcal{V}_N(M_i)\right)}\right].
\end{aligned}$$

The second equality comes from (6.1) applied to the terms with $i_1, \dots, i_k \le N^{1/3}$, and the remaining terms are tail terms which can be replaced by (6.2)–(6.5). The third equality follows from the definition of $\tau_{i_1,\dots,i_k}$ and the definition of $\mathcal{D}_c$. The fourth equality uses Corollary 2.6.

Therefore, our goal is to prove (6.1)–(6.5). For this, we rely on the following claims:

Claim 6.2 *For any $1 \le i_1, \dots, i_\ell \le N$, we have*

$$\mathcal{T}_{c_1,z_{i_1}} \cdots \mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}} \prod_{j \ne i_\ell} \frac{c_\ell + z_{i_\ell} - z_j}{z_{i_\ell} - z_j}\Bigg|_{z=\rho_N} > 0 \tag{6.6}$$

$$\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}}\prod_{j\neq i_\ell}\frac{c_\ell+z_{i_\ell}-z_j}{z_{i_\ell}-z_j}\bigg|_{z=\rho_N}\leq C\,\mathcal{T}_{c_1,z_1}\cdots\mathcal{T}_{c_{\ell-1},z_1}\prod_{j\neq 1}\frac{c_\ell+z_1-z_j}{z_1-z_j}\bigg|_{z=\rho_N} \tag{6.7}$$

for some constant $C>1$ uniform in the $i_1,\ldots,i_\ell$ but depending on $c_1,\ldots,c_\ell>0$ satisfying $c_1+\cdots+c_\ell<1$.

Claim 6.3 *Let $\mu_N=\frac{1}{N}\sum_{i=1}^N\delta_{x_i}$ and $\boldsymbol{x}=(x_1,\ldots,x_N)$. Then*

$$\begin{aligned}&\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_\ell,z_{i_\ell}}\frac{\mathcal{B}_{\log\boldsymbol{x}}(z_1,\ldots,z_N)}{\mathcal{B}_{\log\boldsymbol{x}}(\rho_N)}\bigg|_{z=\rho_N}\\&=\prod_{j=1}^{\ell}\exp\left(-c_j\log S_{\mu_N}\left(-\tfrac{i_j}{N}\right)-\frac{c_j(c_j+1)}{2N}\frac{S'_{\mu_N}\left(-\frac{i_j}{N}\right)}{S_{\mu_N}\left(-\frac{i_j}{N}\right)}\right)(1+o(N^{-1}))\end{aligned} \tag{6.8}$$

as $N\to\infty$, uniformly over $\mu_N\in\mathcal{M}_I\cap\mathcal{R}^N$ and $i_1,\ldots,i_\ell\in\frac{1}{N}\mathbb{Z}\cap[0,1]$. In particular, if $i_1,\ldots,i_\ell\leq N^{1/3}$, then

$$\begin{aligned}&\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_\ell,z_{i_\ell}}\frac{\mathcal{B}_{\log\boldsymbol{x}}(z_1,\ldots,z_N)}{\mathcal{B}_{\log\boldsymbol{x}}(\rho_N)}\bigg|_{z=\rho_N}\\&=\prod_{j=1}^{\ell}\kappa_1(\mu_N)^{c_j}\exp\left[\frac{1}{N}\frac{\kappa_2(\mu_N)}{\kappa_1(\mu_N)^2}\left(-c_ji_j+\frac{c_j(c_j+1)}{2}\right)\right](1+o(N^{-1})).\end{aligned} \tag{6.9}$$

Before providing the proofs of Claims 6.2 and 6.3, we explain how (6.1)-(6.5) follow from the claims. We can rewrite

$$\begin{aligned}\tau_{i_1,\ldots,i_k}&=\left(\prod_{i=1}^k e^{c_i\mathcal{E}_N(M_i)}\right)\prod_{\ell=1}^k\left(\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}}\prod_{j_\ell\neq i_\ell}\frac{c_\ell+z_{i_\ell}-z_{j_\ell}}{z_{i_\ell}-z_{j_\ell}}\bigg|_{z=\rho_N}\right)\\&\quad\times\prod_{m_\ell=M_{\ell+1}+1}^{M_\ell}\left(\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_\ell,z_{i_\ell}}\prod_{a=1}^N\frac{\exp\left[\frac{1}{2N}\frac{\kappa_2(\mu_N^{(m_\ell)})}{\kappa_1(\mu_N^{(m_\ell)})^2}\left(z_a-N+\frac{1}{2}\right)^2\right]}{\exp\left[\frac{1}{2N}\frac{\kappa_2(\mu_N^{(m_\ell)})}{\kappa_1(\mu_N^{(m_\ell)})^2}\left(-a+\frac{1}{2}\right)^2\right]}\bigg|_{z=\rho_N}\right)\\&=\prod_{\ell=1}^k\left(\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}}\prod_{j_\ell\neq i_\ell}\frac{c_\ell+z_{i_\ell}-z_{j_\ell}}{z_{i_\ell}-z_{j_\ell}}\bigg|_{z=\rho_N}\right)\\&\quad\times\left(\prod_{m_\ell=M_{\ell+1}+1}^{M_\ell}\prod_{j=1}^{\ell}\kappa_1(\mu_N^{(m_\ell)})^{c_j}\exp\left[\frac{1}{N}\frac{\kappa_2(\mu_N^{(m_\ell)})}{\kappa_1(\mu_N^{(m_\ell)})^2}\left(-c_ji_j+\frac{c_j(c_j+1)}{2}\right)\right]\right)\end{aligned}$$

Then (6.9) and the definition of $\sigma_{i_1,\ldots,i_k}$ immediately imply (6.1). The positivity statements (6.2) and (6.3) also follow from (6.6) and (6.8).

We must still prove (6.4) and (6.5). To prove (6.5), let us rewrite $\tau_{i_1,\dots,i_k}$ as

$$\left[\prod_{\ell=1}^{k}\left(\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}}\prod_{j_\ell\neq i_\ell}\frac{c_\ell+z_{i_\ell}-z_{j_\ell}}{z_{i_\ell}-z_{j_\ell}}\bigg|_{z=\rho_N}\right)\right]\left[\prod_{\ell=1}^{k}\prod_{m_\ell=M_{\ell+1}+1}^{M_\ell}\prod_{j=1}^{\ell}\kappa_1\left(\mu_N^{(m_\ell)}\right)^{c_j}\right]$$
$$\times\exp\left[\sum_{j=1}^{k}\left(\frac{1}{N}\sum_{m=1}^{M_j}\frac{\kappa_2\left(\mu_N^{(m)}\right)}{\kappa_1\left(\mu_N^{(m)}\right)^2}\right)\left(-c_ji_j+\frac{c_j(c_j+1)}{2}\right)\right]$$

where we swap the sum over ℓ and the sum over j in the summation inside the exponential and recall that $M_{k+1}=0$. We see that the last line in the expression above for $\tau_{i_1,\dots,i_k}$ is strictly decreasing in each of $i_1,\dots,i_k$, and the decay is exponential. Moreover, by (1.2), each of the terms $\frac{1}{N}\sum_{m=1}^{M_j}\frac{\kappa_2\left(\mu_N^{(m)}\right)}{\kappa_1\left(\mu_N^{(m)}\right)^2}$ is converging as $N\to\infty$ to some positive value. Thus if N is sufficiently large, then there exists $c>0$ such that

$$\tau_{i_1,\dots,i_k}\le\tau_{1,\dots,1}e^{-cN^{1/3}}$$

for $1\le i_1,\dots,i_k\le N$ such that $i_j>N^{1/3}$ for some $1\le j\le k$. Note that we use Claim 6.2 to compare the first line of the latest expression for $\tau_{i_1,\dots,i_k}$ with that of the special case of $\tau_{1,\dots,1}$. By (6.1), we obtain (6.5).

To prove (6.4), observe that by Claim 6.3,

$$\begin{aligned}\sigma_{i_1,\dots,i_k}&=\prod_{\ell=1}^{k}\left(\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}}\prod_{j_\ell\neq i_\ell}\frac{c_\ell+z_{i_\ell}-z_{j_\ell}}{z_{i_\ell}-z_{j_\ell}}\bigg|_{z=\rho_N}\right)\\&\quad\times\left(\prod_{m_\ell=M_{\ell+1}+1}^{M_\ell}\prod_{j=1}^{\ell}\exp\left[-c_j\log S_{\mu_N^{(m_\ell)}}\left(-\tfrac{i_j}{N}\right)-\frac{c_j(c_j+1)}{2N}\frac{S'_{\mu_N^{(m_\ell)}}\left(-\frac{i_j}{N}\right)}{S_{\mu_N^{(m_\ell)}}\left(-\frac{i_j}{N}\right)}\right]\right)(1+o(1))\\&=\prod_{\ell=1}^{k}\left(\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}}\prod_{j_\ell\neq i_\ell}\frac{c_\ell+z_{i_\ell}-z_{j_\ell}}{z_{i_\ell}-z_{j_\ell}}\bigg|_{z=\rho_N}\right)\\&\quad\times\left(\prod_{j=1}^{\ell}\exp\left[\sum_{m=1}^{M_\ell}\left(-c_j\log S_{\mu_N^{(m)}}\left(-\tfrac{i_j}{N}\right)-\frac{c_j(c_j+1)}{2N}\frac{S'_{\mu_N^{(m)}}\left(-\frac{i_j}{N}\right)}{S_{\mu_N^{(m)}}\left(-\frac{i_j}{N}\right)}\right)\right]\right)(1+o(1))\end{aligned}$$

as $N\to\infty$, uniformly over $1\le i_1,\dots,i_k\le N$. It suffices to show that

$$\begin{aligned}&\sum_{m=1}^{M_\ell}\left(-c_j\log S_{\mu_N^{(m)}}\left(-\tfrac{i}{N}\right)-\frac{c_j(c_j+1)}{2N}\frac{S'_{\mu_N^{(m)}}\left(-\frac{i}{N}\right)}{S_{\mu_N^{(m)}}\left(-\frac{i}{N}\right)}\right)\\&\le-CN^{1/3}\cdot\mathbf{1}[N^{1/3}<i\le N]+\sum_{m=1}^{M_\ell}\left(-c_j\log S_{\mu_N^{(m)}}\left(-\tfrac{1}{N}\right)-\frac{c_j(c_j+1)}{2N}\frac{S'_{\mu_N^{(m)}}\left(-\frac{1}{N}\right)}{S_{\mu_N^{(m)}}\left(-\frac{1}{N}\right)}\right)\end{aligned}\tag{6.10}$$

for $1 \le i \le N$. Indeed, by Claim 6.2, we may disregard the first line in the expression for $\sigma_{i_1,\ldots,i_k}$ above, and focus on comparing the terms inside the exponential.

We can make a further reduction. The continuity of $\mu \mapsto S_\mu$ on $\mathcal{M}_I$, via Proposition 4.4, and the compactness of $\mathcal{M}_I$ implies that

$$\frac{d}{du} \log S_\mu(u) = \frac{S'_\mu(u)}{S_\mu(u)}$$

is bounded, uniformly over $u \in [-1, 0]$ and $\mu \in \mathcal{M}_I$. Thus

$$-\frac{1}{2N} \sum_{m=1}^{M_\ell} \frac{S'_{\mu_N^{(m)}}\left(-\frac{i}{N}\right)}{S_{\mu_N^{(m)}}\left(-\frac{i}{N}\right)}$$

is bounded, uniformly over $1 \le i \le N$ and in N. We see then that to prove (6.10) (and therefore prove (6.4)), it is enough to show that

$$-\sum_{m=1}^{M_\ell} c_j \log S_{\mu_N^{(m)}}\left(-\tfrac{i}{N}\right) \le -CN^{1/3} \cdot \mathbf{1}[N^{1/3} < i \le N] - \sum_{m=1}^{M_\ell} c_j \log S_{\mu_N^{(m)}}\left(-\tfrac{1}{N}\right) \tag{6.11}$$

for $1 \le i \le N$.

We know that $-\log S_\mu(u)$ is an increasing function on $[-1, 0]$ with

$$-\frac{d}{du} \log S_\mu(u)\Big|_{u=0} = -\frac{S'_\mu(0)}{S_\mu(0)} = \frac{\kappa_2(\mu)}{\kappa_1(\mu)^2}$$

where we use Lemma 4.6. Condition (1.2) then implies

$$-\sum_{m=1}^{M_\ell} \frac{d}{du} \log S_{\mu_N^{(m)}}(u)\Big|_{u=0} = \sum_{m=1}^{M_\ell} \frac{\kappa_2(\mu_N^{(m)})}{\kappa_1(\mu_N^{(m)})^2} = N\gamma(t_\ell)(1 + o(1)) \tag{6.12}$$

as $N \to \infty$, where we recall $\gamma(t_\ell) > 0$. Arguing as we did earlier using the compactness of $\mathcal{M}_I$ and the continuity of $\mu \mapsto S_\mu$, we can show that

$$-\sum_{m=1}^{M_\ell} \frac{d^2}{du^2} \log S_{\mu_N^{(m)}}(u)\Big|_{u=0} = O(N). \tag{6.13}$$

Then for N sufficiently large

$$-\sum_{m=1}^{M_\ell} \log S_{\mu_N^{(m)}}\left(-\tfrac{i}{N}\right) \le -\sum_{m=1}^{M_\ell} \log S_{\mu_N^{(m)}}\left(-\tfrac{1}{N}\right) \qquad \text{for } 1 \le i \le N,$$

$$-\sum_{m=1}^{M_\ell} \log S_{\mu_N^{(m)}}\left(-\tfrac{i}{N}\right) \le -\sum_{m=1}^{M_\ell} \log S_{\mu_N^{(m)}}\left(-\tfrac{1}{N}\right) - CN^{1/3} \qquad \text{for } N^{1/3} < i \le N$$

for some constant $C > 0$ (since $M_\ell \asymp N$). The latter follows from expanding the function of $-\frac{i}{N}$ around $-\frac{1}{N}$. In this expansion, the behavior is dominated by the linear term, due to the bound (6.13), and since the derivative is positive and of order N by (6.12), we can glean the sign $-CN^{1/3}$. These inequalities prove (6.11) and therefore (6.4).

Having justified that Claims 6.2 and 6.3 imply (6.1)-(6.5), it remains to prove these claims.

Proof of Claim 6.2 To work with our expression, it will be convenient to account for the repetition of variables z_i among our operators $\mathcal{T}_{c_1,z_{i_1}}, \dots, \mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}}$ with distinguished tracking of repetitions of z_{i_ℓ}. Thus we introduce the following. Let $i'_1, \dots, i'_{r-1}$ be the distinct elements of $\{i_1, \dots, i_{\ell-1}\} \setminus \{i_\ell\}$ and set $i'_r = i_\ell$. Define

$$c'_1 := \sum_{\substack{1 \le a < r: \\ i_a = i'_1}} c_a, \quad \dots, \quad c'_r := \sum_{\substack{1 \le a < r: \\ i_a = i'_r}} c_a,$$

where we note that $c'_1, \dots, c'_{r-1} > 0$, but $c'_r > 0$ if and only if $i'_r = i_\ell$ is among $i_1, \dots, i_{\ell-1}$, i.e. $i_\ell \in \{i_1, \dots, i_{\ell-1}\}$. Then

$$\begin{aligned}\mathcal{T}_{c_1,z_{i_1}} \cdots \mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}} \prod_{j \ne i_\ell} \frac{c_\ell + z_{i_\ell} - z_j}{z_{i_\ell} - z_j} &= \mathcal{T}_{c'_1,z_{i'_1}} \cdots \mathcal{T}_{c'_r,z_{i'_r}} \prod_{j \ne i'_r} \frac{c_\ell + z_{i'_r} - z_j}{z_{i'_r} - z_j} \\ &= \left(\prod_{a=1}^{r-1} \frac{(z_{i'_r} - z_{i'_a})(c_\ell + c'_r + z_{i'_r} - c'_a - z_{i'_a})}{(c_\ell + c'_r + z_{i'_r} - z_{i'_a})(z_{i'_r} - c'_a - z_{i'_a})}\right) \left(\prod_{j \ne i'_r} \frac{c_\ell + c'_r + z_{i'_r} - z_j}{z_{i'_r} - z_j}\right).\end{aligned}$$

Since $0 < c_1 + \cdots + c_\ell < 1$, upon evaluating at $\boldsymbol{z} = \rho_N = (N-1, N-2, \dots, 0)$, we see that the expression above is positive. This proves (6.6). Furthermore, we have a bound

$$\prod_{a=1}^{r-1} \frac{(z_{i'_r} - z_{i'_a})(c_\ell + c'_r + z_{i'_r} - c'_a - z_{i'_a})}{(c_\ell + c'_r + z_{i'_r} - z_{i'_a})(z_{i'_r} - c'_a - z_{i'_a})}\bigg|_{z=\rho_N} \le C$$

where we may make C uniform over $1 \le i_1, \dots, i_\ell \le N$ by virtue of $0 < c_1 + \cdots + c_\ell < 1$ and the positivity of $c_1, \dots, c_\ell$. Thus

$$\mathcal{T}_{c_1,z_{i_1}} \cdots \mathcal{T}_{c_{\ell-1},z_{i_{\ell-1}}} \prod_{j \ne i_\ell} \frac{c_\ell + z_{i_\ell} - z_j}{z_{i_\ell} - z_j} \le C \prod_{j \ne i'_r} \frac{c_\ell + c'_r + z_{i'_r} - z_j}{z_{i'_r} - z_j}\bigg|_{z=\rho_N}.$$

Next, observe that

$$\begin{aligned}\prod_{j\neq i}\frac{c_\ell+c_r'+z_i-z_j}{z_i-z_j}\bigg|_{z=\rho_N} &= \prod_{j\neq i}\frac{c_\ell+c_r'+j-i}{j-i}\\ &\leq \prod_{j\neq 1}\frac{c_\ell+c_r'+j-1}{j-1}\\ &\leq \prod_{j\neq 1}\frac{c_1+\cdots+c_\ell+j-1}{j-1}\\ &= \mathcal{T}_{c_1,z_1}\cdots\mathcal{T}_{c_{\ell-1},z_1}\prod_{j\neq 1}\frac{c_\ell+z_1-z_j}{z_1-z_j}\bigg|_{z=\rho_N}\end{aligned}$$

where the third line uses the fact that $c_\ell + c_r' \leq c_1 + \cdots + c_\ell$, recalling that products are restricted over $1 \leq j \leq N$. Combining these inequalities, the claim follows. □

Proof of Claim 6.3 Applying Theorem 5.3 with $u_j = \frac{1}{N}(N - i_j + c_j)$ and $v_j = \frac{1}{N}(N - i_j)$, so that $|u_i - v_i| = O(1/N)$, we have

$$\begin{aligned}&\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_\ell,z_{i_\ell}}\frac{\mathcal{B}_{\log \boldsymbol{x}}(z_1,\ldots,z_N)}{\mathcal{B}_{\log \boldsymbol{x}}(\rho_N)}\bigg|_{z=\rho_N}\\ &= B_{\mu_N}(N-i_1+c_1,\ldots,N-i_\ell+c_\ell;N-i_1,\ldots,N-i_\ell)\\ &= \frac{\det\left(\frac{1}{\psi_\mu^{-1}(u_i)-\psi_\mu^{-1}(v_j)}\right)_{1\leq i,j\leq k}}{\det\left(\frac{1}{u_i-v_j}\right)_{1\leq i,j\leq k}}\\ &\quad\prod_{j=1}^{\ell}\left[\frac{1}{\sqrt{\psi_\mu'(\psi_\mu^{-1}(u_i))\psi_\mu'(\psi_\mu^{-1}(v_i))}}\frac{\sqrt{S_{\mu_N}\left(\frac{-i_j}{N}\right)}e^{NH_{\mu_N}\left(\frac{-i_j+c_j}{N}\right)}}{\sqrt{S_{\mu_N}\left(\frac{-i_j+c_j}{N}\right)}e^{NH_{\mu_N}\left(\frac{-i_j}{N}\right)}}(1+o(N^{-1}))\right]\end{aligned}$$

uniformly over $\mu_N \in \mathcal{M}_I \cap \mathcal{R}^N$ and $i_1, \ldots, i_\ell \in \frac{1}{N}\mathbb{Z} \cap [0, 1]$. Applying (4.3) from Lemma 4.7, we may drop the Cauchy determinant and the terms expressed via ψ_μ so that

$$\begin{aligned}&\mathcal{T}_{c_1,z_{i_1}}\cdots\mathcal{T}_{c_\ell,z_{i_\ell}}\frac{\mathcal{B}_{\log \boldsymbol{x}}(z_1,\ldots,z_N)}{\mathcal{B}_{\log \boldsymbol{x}}(\rho_N)}\bigg|_{z=\rho_N}\\ &= \prod_{j=1}^{\ell}\frac{\sqrt{S_{\mu_N}\left(\frac{-i_j}{N}\right)}e^{NH_{\mu_N}\left(\frac{-i_j+c_j}{N}\right)}}{\sqrt{S_{\mu_N}\left(\frac{-i_j+c_j}{N}\right)}e^{NH_{\mu_N}\left(\frac{-i_j}{N}\right)}}(1+o(N^{-1}))\end{aligned}$$

$$= \prod_{j=1}^{\ell} \exp\left(-c_j \log S_{\mu_N}\left(-\tfrac{i_j}{N}\right) - \frac{c_j(c_j+1)}{2N} \frac{S'_{\mu_N}\left(-\tfrac{i_j}{N}\right)}{S_{\mu_N}\left(-\tfrac{i_j}{N}\right)} \right) (1 + o(N^{-1})).$$

uniformly over $\mu_N \in \mathcal{M}_I \cap \mathcal{R}^N$ and $i_1, \ldots, i_\ell \in \frac{1}{N}\mathbb{Z} \cap [0,1]$. Note that the final equality follows from the estimates

$$N\left(H_\mu\left(\tfrac{u+c}{N}\right) - H_\mu\left(\tfrac{u}{N}\right)\right) = cH'_\mu\left(\tfrac{u}{N}\right) + \frac{c^2}{2N} H''_\mu\left(\tfrac{u}{N}\right) + O(N^{-2})$$

$$\frac{\sqrt{S_{\mu_N}\left(\tfrac{u}{N}\right)}}{\sqrt{S_{\mu_N}\left(\tfrac{u+c}{N}\right)}} = \exp\left(\frac{1}{2}\left(\log S_{\mu_N}\left(\tfrac{u}{N}\right) - \log S_{\mu_N}\left(\tfrac{u+c}{N}\right)\right)\right)$$

$$= \exp\left(-\frac{c}{2N} \frac{S'_{\mu_N}\left(\tfrac{u}{N}\right)}{S_{\mu_N}\left(\tfrac{u}{N}\right)} + O(N^{-2})\right)$$

which hold for fixed $c > 0$, uniformly over $u \in [-1, 0]$ and $\mu \in \mathcal{M}_I$ as $N \to \infty$, and from expressing H'_μ and H''_μ in terms of the S-transform as in (5.1) and (5.2). This proves (6.8).

If $|u| \le N^{1/3}$, then

$$\begin{aligned} &- c \log S_\mu\left(\tfrac{u}{N}\right) - \frac{c(c+1)}{2N} \frac{S'_\mu\left(\tfrac{u}{N}\right)}{S_\mu\left(\tfrac{u}{N}\right)} \\ &= -c \log S_\mu(0) - \frac{1}{N} \frac{S'_\mu(0)}{S_\mu(0)} \left(u + \frac{c(c+1)}{2}\right) + O(N^{-4/3}) \\ &= c \log \kappa_1(\mu) + \frac{1}{N} \frac{\kappa_2(\mu)}{\kappa_1(\mu)^2} \left(u + \frac{c(c+1)}{2}\right) + O(N^{-4/3}) \end{aligned}$$

uniformly in u and $\mu \in \mathcal{M}_I$, where the second equality uses the evaluations of $S_\mu(0)$ and $S'_\mu(0)$ from Lemma 4.6. This proves (6.9). □

6.2 Proof of Theorem 1.4

Let

$$\mathcal{E}_N(M) := \sum_{m=1}^{M} \log \kappa_1(\mu_N^{(m)}) \qquad \text{and} \qquad \mathcal{V}_N(M) := \frac{1}{N} \sum_{m=1}^{M} \frac{\kappa_2(\mu_N^{(m)})}{\kappa_1(\mu_N^{(m)})^2}$$

as in Theorem 6.1. In contrast with the setting of Theorem 6.1, $\mathcal{E}_N(M)$ and $\mathcal{V}_N(M)$ are not deterministic in general.

Fix $t_1 \geq \cdots \geq t_k > 0$. Our goal is to show that for any positive integers k and h,

$$\lim_{N\to\infty} \mathbb{P}\left(\log y_j^{(N)}(\lfloor t_i N\rfloor) - \mathcal{E}_N(\lfloor t_i N\rfloor) - \log N \leq a_{i,j} : 1 \leq i \leq k, 1 \leq j \leq h\right) = \mathbb{P}(\xi_j(\gamma(t_i)) \leq a_{i,j} : 1 \leq i \leq k, 1 \leq j \leq h) \tag{6.14}$$

for every $a_{i,j} \in \mathbb{R}$ such that $a_{i,1} \geq \cdots \geq a_{i,h}$.

We may assume that there exists a compact interval $I \subset \mathbb{R}_{>0}$ such that

$$\operatorname{supp} \mu_N(m) \subset I, \qquad \text{for all } 1 \leq m \leq \lfloor t_1 N\rfloor \text{ and } N \geq 1. \tag{6.15}$$

We show why this reduction is valid. Indeed, by (1.1)

$$1 - \mathbb{P}\left(\operatorname{supp} \mu_N(m) \subset I\right) = o(1/N)$$

uniformly over $m = 1, 2, \ldots$. Then if $\mathcal{I}_N$ is the event that $\operatorname{supp} \mu_N(m) \subset I$ for every $1 \leq m \leq \lfloor t_1 N\rfloor$, we have

$$1 - \mathbb{P}\left(\mathcal{I}_N\right) \leq \lfloor t_1 N\rfloor \cdot o(1/N) = o(1).$$

Thus, to prove (6.14), we may assume without loss of generality that $\mathbb{P}(\mathcal{I}_N) = 1$ and that the complement of $\mathcal{I}_N$ is empty, which is the desired reduction.

We make one more reduction. By (1.2), we have that

$$\varepsilon_{i,N} := \mathcal{V}_N(\lfloor t_i N\rfloor) - \gamma(t_i) \to 0$$

in probability as $N \to \infty$. Thus there exists a sequence of events $\widetilde{\mathcal{I}}_N$ and a sequence $a_N > 0$ such that

$$|\varepsilon_{i,N}| \leq a_N$$

on $\widetilde{\mathcal{I}}_N$, and $\mathbb{P}(\widetilde{\mathcal{I}}_N) \to 1$ and $a_N \to 0$ as $N \to \infty$. Therefore, we may assume without loss of generality that $\mathbb{P}(\widetilde{\mathcal{I}}_N) = 1$.

Let

$$\mathcal{X}_N := \left\{x_j^{(N)}(m) : 1 \leq m \leq M_1, 1 \leq j \leq N\right\}$$

denote the collection of squared singular values of $X^{(N)}(1), \ldots, X^{(N)}(M_1)$. We have $\mathcal{X}_N \subset I$ from (6.15). Under the assumption $\mathbb{P}(\widetilde{\mathcal{I}}_N) = 1$, we have

$$\mathcal{V}_N(\lfloor t_i N\rfloor) = \gamma(t_i) + o(1) \tag{6.16}$$

as $N \to \infty$ for $1 \leq i \leq k$, where the $o(1)$ is uniform over all realization of $\mathcal{X}_N$.

Suppose $c_1, \ldots, c_k > 0$ such that $c_1 + \cdots + c_k < 1$. Then

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\log y_j^{(N)}(\lfloor t_i N\rfloor)-\mathcal{E}_N(\lfloor t_i N\rfloor)-\log N\right)}\,\middle|\,\mathcal{X}_N\right] \\
&\quad= \mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\xi_j^{(N)}(\frac{1}{4}\mathcal{V}_N(\lfloor t_i N\rfloor))-\frac{N}{2}\mathcal{V}_N(\lfloor t_i N\rfloor)-\log N\right)}\,\middle|\,\mathcal{X}_N\right](1+o(1)) \\
&\quad= \mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{\infty} e^{c_i\xi_j(\mathcal{V}_N(\lfloor t_i N\rfloor))}\,\middle|\,\mathcal{X}_N\right](1+o(1)) \\
&\quad= \mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{\infty} e^{c_i\xi_j(\gamma(t_i))}\right](1+o(1))
\end{aligned}$$

as $N \to \infty$, uniformly over all realizations of $\mathcal{X}_N$. The first equality follows from Theorem 6.1, using the uniformity statement in that theorem along with fact that $\mathcal{X}_N \subset I$. The second equality uses Proposition 3.3 and Lemma 3.4. The third equality follows from (6.16). Taking an overall expectation then yields

$$\lim_{N\to\infty}\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{N} e^{c_i\left(\log y_j^{(N)}(\lfloor t_i N\rfloor)-\mathcal{E}_N(\lfloor t_i N\rfloor)-\log N\right)}\right] = \mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{\infty} e^{c_i\xi_j(\gamma(t_i))}\right] \tag{6.17}$$

where the uniformity of the $o(1)$ error permits the commutation of the limit with the expectation. The result now follows from Lemma 3.5.

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