

RESOLUTION AND ALTERATION WITH AMPLE EXCEPTIONAL DIVISOR

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ABSTRACT. In this note we construct resolutions or regular alterations admitting an ample exceptional divisor, assuming the existence of projective resolutions or regular alterations. In particular, this implies the existence of such resolutions for three-dimensional schemes.

It is frequently advantageous to have resolutions or alterations that have an ample exceptional divisor. While Hironaka-type methods automatically produce a resolution with ample exceptional divisors, neither the resolution of 3-dimensional schemes presented in [CP19], nor the method of alterations in [dJ96] yields ample exceptional divisors right away. The aim of this note is to outline a simple trick that ensures the existence of ample exceptional divisors. This is especially useful when one needs to run a relative minimal model program; see for instance [CS20, BMP⁺20, Kol21, VP21, BK23].

Note that if D is exceptional and ample, then $-D$ is effective by the Negativity lemma; see [KM98, 3.39].

Definition 1. A morphism $\pi : Y \rightarrow X$ is *component-wise dominant* if every irreducible component of Y dominates an irreducible component of X . A proper, component-wise dominant, generically finite morphism $\pi : Y \rightarrow X$ is called an *alteration*. An alteration of reduced schemes is *birational* if there are dense open subsets $Y^\circ \subset Y$ and $X^\circ \subset X$, such that π induces an isomorphism $\pi^\circ : Y^\circ \cong X^\circ$.

Let $\pi : Y \rightarrow X$ be component-wise dominant and locally of finite type. Let $\text{Ex}(\pi) \subset Y$ denote the smallest closed subset such that π is quasi-finite on $Y \setminus \text{Ex}(\pi)$; it exists by the upper-semicontinuity of the fiber dimension, see [Sta22, Tag 02FZ].

A proper morphism $\pi : Y \rightarrow X$ is a *resolution* if Y is regular, and $\pi : Y \rightarrow \text{red}(X)$ is birational. A resolution is a *log resolution* if, in addition, the exceptional locus $\text{Ex}(\pi)$ is a simple normal crossing divisor.

An alteration $\pi : Y \rightarrow X$ is called *regular* if Y is regular, and *Galois* with group $G = \text{Aut}(Y/X)$ if $Y/G \rightarrow X$ is generically purely inseparable.

The main results are the following.

Theorem 2. *Let X be a Noetherian, normal scheme. Assume that projective resolutions (resp. log resolutions) exist for every scheme $X' \rightarrow X$ that is projective and birational over X .*

Then X has a projective resolution (resp. log resolution) $g : \mathcal{R}(X) \rightarrow X$ by a scheme $\mathcal{R}(X)$, such that $\text{Ex}(g)$ supports a g -ample divisor.

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Theorem 3. *Let X be a Noetherian, normal scheme. Assume that regular, projective, Galois alterations exist for every scheme $X' \rightarrow X$ that is projective and generically purely inseparable over X .*

Then X has a regular, projective, Galois alteration $g : \mathcal{A}(X) \rightarrow X$ by a scheme $\mathcal{A}(X)$, such that $\text{Ex}(g)$ supports a g -ample divisor.

Note that Theorems 2–3 are also valid for algebraic spaces and stacks; see Remark 12 for details. Although both theorems are stated for arbitrary Noetherian schemes, the assumed existence of resolutions implies that the schemes are quasi-excellent, see [Gro60, IV₂, §7.9].

Combining these theorems with [CP19] and [dJ96] gives the following consequences.

Corollary 4. *Let X be a scheme (or algebraic space) of dimension at most three, that is separated and of finite type over an affine, quasi-excellent scheme S . Then X admits a projective log resolution $g : \mathcal{R}(X) \rightarrow X$ by a scheme $\mathcal{R}(X)$, such that $\text{Ex}(g)$ supports a g -ample divisor.*

Corollary 5. *Let X be a scheme (or algebraic space), that is separated and of finite type over a Noetherian, excellent scheme S with $\dim S \leq 2$. Then X admits a regular, projective, Galois alteration $g : \mathcal{A}(X) \rightarrow X$ by a scheme $\mathcal{A}(X)$, such that $\text{Ex}(g)$ supports a g -ample divisor.*

Remark 6. It is clear from the proof that one can find $g : \mathcal{R}(X) \rightarrow X$ and $g : \mathcal{A}(X) \rightarrow X$ with other useful properties. For example, we can choose $\mathcal{R}(X)$ (resp. $\mathcal{A}(X)$) to dominate any finite number of resolutions (resp. alterations).

Also, if $Z_i \subset X$ are finitely many closed subschemes, and embedded resolutions (resp. regular, Galois alterations) exist over X , then we can choose $\mathcal{R}(X)$ (resp. $\mathcal{A}(X)$) to be an embedded resolution (resp. regular, Galois alteration) for the Z_i .

The log version of alterations does not seem to be treated in the literature.

To fix our notation, recall that a normal scheme X is $\mathbb{Q}$ -factorial if, for every Weil divisor D on X , mD is a Cartier divisor for some $m > 0$. Equivalently, if L is a rank 1, reflexive sheaf, then there is an $m > 0$ such that $L^{[m]}$ (the reflexive hull of $L^{\otimes m}$) is invertible.

We start with three lemmas; the first two are well known.

Lemma 7. *Let X be a Noetherian, normal, $\mathbb{Q}$ -factorial scheme, $\pi : X' \rightarrow X$ a projective, birational morphism with X' normal. Then there is a π -ample, π -exceptional divisor E on X' .*

Proof. Let H be a π -ample line bundle on X' . Choose $m > 0$ such that $(\pi_* H)^{[m]}$ is invertible. Then $H^m \otimes \pi^*((\pi_* H)^{[-m]})$ is π -ample and trivial on $X' \setminus \text{Ex}(\pi)$. Thus it is linearly equivalent to a π -exceptional divisor E . $\square$

Lemma 8. *Let X be a Noetherian, normal scheme, $\pi_1 : X_1 \rightarrow X$ a projective, generically purely inseparable morphism, and H_1 a line bundle on X_1 . Set $U_1 := X_1 \setminus \text{Ex}(\pi_1)$.*

Then there is a coherent, rank 1, reflexive sheaf L on X and $q > 0$, such that, $\pi_1^ L|_{U_1} \cong H_1^q|_{U_1}$.*

Proof. Consider the Stein factorization $X_1 \xrightarrow{\rho'} X' \xrightarrow{\rho} X$ of π_1 . The images of U_1 give $U' \subset X'$ and $U \subset X$. So $\rho'_* H_1$ is a line bundle on U' . Since $U' \rightarrow U$ is finite

and purely inseparable, it factors through a power of Frobenius; cf. [Sta22, Tag 0CNF]. Hence there is a line bundle L_U on U such that $\rho^*L_U \cong \rho'_*H_1^q|_{U'}$, where we can take $q = \deg \rho$. We can then extend L_U to a coherent, reflexive sheaf L on X . $\square$

Lemma 9. *Let X be a Noetherian, normal scheme and $\pi_1 : X_1 \rightarrow X$ a projective, generically purely inseparable morphism. Assume that X_1 is $\mathbb{Q}$ -factorial and let H_1 be a π_1 -ample line bundle on X_1 . Let L be a coherent, rank 1, reflexive sheaf on X as in Lemma 8. Set $M := \mathcal{H}om_X(L, \mathcal{O}_X)$ and*

$$\pi_2 : X_2 := \text{Proj}_X \oplus_{m \geq 0} (M^{\otimes m} / (\text{torsion})) \rightarrow X.$$

Let $\pi_3 : X_3 \rightarrow X$ be a projective, generically purely inseparable morphism that dominates both X_1 and X_2 , with X_3 normal. Then there is a π_3 -ample, π_3 -exceptional divisor E on X_3 .

Note that we work with $\oplus_{m \geq 0} (M^{\otimes m} / (\text{torsion}))$, not with $\oplus_{m \geq 0} M^{[m]}$, since the latter may not be finitely generated; see [Koll10, Sec.5] for such examples related to flips and flops.

Proof. For $i = 1, 2$, let $\tau_i : X_3 \rightarrow X_i$ be the natural maps and $X_3 \xrightarrow{\tau'} X'_1 \xrightarrow{\tau} X_1$ the Stein factorization of τ_1 . As in Lemma 8, τ is either an isomorphism or factors through a power of Frobenius.

Applying Lemma 8 to $X'_1 \xrightarrow{\tau} X_1$ and $U_1 := X_1$ we see that X'_1 is $\mathbb{Q}$ -factorial since X_1 is $\mathbb{Q}$ -factorial.

By Lemma 7 there is a τ' -ample, τ' -exceptional divisor E_3 on X_3 . Then $\tau_1^*H_1^m(E_3)$ is π_3 -ample for $m \gg 0$.

Set $H_2 := \mathcal{O}_{X_2}(1)$. Since H_2 is π_2 -ample, its pull-back $\tau_2^*H_2$ is π_3 -nef. Therefore $\tau_2^*H_2^m \otimes \tau_1^*H_1^{qm}(qE_3)$ is π_3 -ample for every $q > 0$.

Set $U_3 := X_3 \setminus \text{Ex}(\pi_3)$; its images give open subschemes $U \subset X$ and $U_i \subset X_i$. By Lemma 8 there is a coherent, rank 1, reflexive sheaf L on X and $q > 0$, such that, $\pi_1^*L|_{U_1} \cong H_1^q|_{U_1}$. Then

$$\tau_2^*H_2^m \otimes \tau_1^*H_1^{qm}(qE_3)|_{U_3} \cong \pi_3^*(M^m|_U \otimes L^m|_U) \cong \mathcal{O}_{U_3}.$$

This gives a rational section of $\tau_2^*H_2^m \otimes \tau_1^*H_1^{qm}(qE_3)$ whose divisor is π_3 -ample and π_3 -exceptional. $\square$

10 (Proof of Theorem 2). We may assume that X is integral. Start with a projective (log) resolution $\pi_1 : X_1 \rightarrow X$ and construct $\pi_2 : X_2 \rightarrow X$ as in Lemma 9. Let $X_{12} \subset X_1 \times_X X_2$ be the irreducible component that dominates X , and $X_3 \rightarrow X_{12}$ a projective (log) resolution. By Lemma 9, $\pi_3 : X_3 \rightarrow X$ has a π_3 -ample, π_3 -exceptional divisor. $\square$

11 (Proof of Theorem 3). Start with a regular, projective, Galois alteration $\bar{\pi}_1 : \bar{X}_1 \rightarrow X$. Let $\pi_1 : X_1 \rightarrow X$ be its quotient by the Galois group of $k(\bar{X}_1/X)$. Note that X_1 is $\mathbb{Q}$ -factorial.

Construct $\pi_2 : X_2 \rightarrow X$ as in Lemma 9. Let $X_{12} \subset X_1 \times_X X_2$ be the irreducible component that dominates X , and $\bar{X}_3 \rightarrow X_{12}$ a regular, projective, Galois alteration. Let $X_3 \rightarrow X_{12}$ be its quotient by the Galois group of $k(\bar{X}_3/X_{12})$. By Lemma 9, $\pi_3 : X_3 \rightarrow X$ has a π_3 -ample, π_3 -exceptional divisor. Its pull-back to $\bar{X}_3$ is a $\bar{\pi}_3$ -ample, $\bar{\pi}_3$ -exceptional divisor, where $\bar{\pi}_3 : \bar{X}_3 \rightarrow X$ is the natural morphism. $\square$

Remark 12. Theorems 2–3 are valid for every integral, Noetherian algebraic space (resp. stack) X with $\mathcal{R}(X)$ or $\mathcal{A}(X)$ being an algebraic space (resp. stack), assuming the appropriate representable resolutions or regular alterations by algebraic spaces (resp. stacks) exist for every algebraic space (resp. stack) X' admitting a representable projective birational (resp. generically purely inseparable) morphism to X . As for algebraic spaces, we note that all of the above constructions can be performed in the category of algebraic spaces and their validity may be verified étale locally. As for algebraic stacks, we note that every algebraic stack admits a presentation as a quotient of an algebraic space by a smooth groupoid [Sta22, Tag 04T3], and that quotients of algebraic spaces by smooth groupoids always exist [Sta22, Tag 04TK]. We can then conclude, since each step in our constructions is equivariant with respect to a chosen presentation.

If X is an algebraic space and the appropriate resolutions or regular alterations of all algebraic spaces admitting representable, projective, birational or generically purely inseparable morphisms to X exist as schemes, then we can achieve that $\mathcal{R}(X)$ and $\mathcal{A}(X)$ are schemes.

Here, a representable morphism of quasi-compact quasi-separated algebraic spaces (resp. algebraic stacks) is *projective* if it is proper and there exists a relatively ample invertible sheaf (cf. [Ryd16, Definition 8.5 and Theorem 8.6]).

13 (Proof of Corollary 4). We may harmlessly replace X by the irreducible components of its normalization. Thus we may assume that X is normal and integral.

When X is a scheme, the assumptions of Theorem 2 are valid for integral affine quasi-excellent schemes of dimension at most three by [CP19], see [BMP⁺20, Theorem 2.5 and 2.7].

If X is an algebraic space, then by Chow’s lemma [Sta22, Tag 088U] we can find a projective birational morphism $h: Y \rightarrow X$ such that the scheme Y is quasi-projective over S . Temkin extended [CP19] to give a projective resolution for such a scheme Y ; the proof is discussed in [BMP⁺20, Sec.2.3].

Similarly, we obtain projective resolutions of all algebraic spaces admitting a projective birational morphism to X . By Remark 12 we can obtain $\mathcal{R}(X)$ as a scheme.

14 (Proof of Corollary 5). As before, we may assume that X is normal and integral.

When X is a scheme, the assumptions of Theorem 3 are valid for all integral schemes that are separated and of finite type over an excellent scheme S with $\dim S \leq 2$ (see [dJ97, Corollary 5.15] and [Tem17, 4.3.1]).

If X is an algebraic space, then a regular, projective, Galois alteration of X (and of all algebraic spaces admitting a projective generically purely inseparable morphism to X) exists by Chow’s lemma as in the proof of Corollary 4, and so we can conclude by Remark 12 to get $\mathcal{A}(X)$, which is a scheme.

Remark 15. The above proofs of Corollaries 4–5 do not immediately apply to algebraic stacks. Indeed, Chow’s lemma for algebraic stacks only ensures the existence of a proper surjective cover by a quasi-projective scheme. This cover need not be birational. On the other hand, one could try to construct a resolution equivariantly with respect to a presentation, but we do not know whether the algorithms for the existence of resolutions and regular alterations from [CP19] and [dJ96] can be run equivariantly (in contrast to the characteristic zero case). For Deligne-Mumford stacks of finite type over a Noetherian scheme, the proper surjective cover from

Chow’s lemma may be assumed to be generically étale [LMB00, Corollaire 16.6.1]. In particular, they admit regular alterations (and so also regular, Galois alterations) and Corollary 5 holds for them.

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