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ORIGINAL PAPER

Actualizing the virtuality of the cauchy-riemann equations

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Abstract

In this case study we explored how a mathematician's teaching of the Cauchy-Riemann (CR) equations actualized the virtual aspects of the equations. Using videotaped classroom data, we found that in a three-day period, this mathematician used embodiment to animate and bind formal aspects of the CR equations (including conformality), metaphors, himself, and his students. We found that the mathematician's creative introduction of matrices led to a discussion of transformations which made the CR equations mobile and hence gave him a space to virtualize the CR equations. In our results we summarize how the mathematician's assemblage of *unusual*, *unexpected, and unscripted*, and *without given content* creative acts with materials embedded in algebraic and geometric inscriptions and metaphors *introduced or catalyzed the new*—the virtuality of the CR equations. In our discussion, we highlight how the mathematician bridged the virtual with the abstract via his conceptualizations of the CR equations. Didactic implications include adopting the mathematician's conceptualizations and asking students to bind them. This could stress the mobility of conformal maps which are generally not taught in an undergraduate class. We propose offering professional development for educators focused on learning how to engineer didactic practices that showcase mobility, support binding, and exhibit animation of mathematical concepts.

Keywords Animate · Bind · Cauchy-Riemann equations · Embodiment · Mathematician

1 Introduction

Nathan (2021) summarizes embodiment as "body-based resources to make meaning and to connect new ideas and representations to prior experiences" (p. 4). In his work, he summarizes four types of embodiments, including gestures, defined as "movements of our hands and arms, as well as other body parts, that we make spontaneously as we talk with others and think to ourselves, including pointing and tracing actions" (p. 89). Most gesture research focuses on grades K-12 (e.g., Alibali & Nathan, 2012; Alibali et al., 2019) or mathematicians' gestures in research settings as they explain their understanding of abstract concepts, prove a statement, or solve a mathematical task (e.g., Marghetis et al., 2014; Oehrtman et al., 2019; Soto-Johnson et al., 2016). The purpose of our research is to contribute to the literature on embodiment in a way that rethinks the "body *in* and *of* mathematics" (de Freitas & Sinclair, 2014, p. 200) specifically in the didactic practice of undergraduate mathematics because such research is sparse (e.g., Keene et al., 2012; Stewart et al., 2019; Weinberg et al., 2015). In teaching and learning, de Freitas and Sinclair argue that

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mathematical concepts, humans, the visible such as boards and technology, and the non-visible such as metaphors serve as material agents that become body-assemblages defined as "a set of material relations that … structures other material relations around it" (p. 34).

In their work de Freitas & Sinclair (2014) adopt Châtelet's (1993/2000) notion of the *virtual* which "binds the mathematical and physical together" (p. 201) because the "virtual dimension … animates the mathematical concept" (p. 202). Perceiving the virtual as both physical and mathematical and thus, movable, allows researchers "to study … a [mathematical] concept in terms of the kind of work it can do" (p. 200). Thus, mathematical concepts are material objects with "*virtual* and *actual* dimensions" (p. 202) which one can engage with via gestures, diagrams, materials, and metaphors. For example, we commonly showcase ideas in the Euclidean plane with our hands when we pick up a point, extend it to form a ray, or take two points to form their vector. This is how "mathematical concepts engage in a process of becoming, a process that binds them to" (p. 202) mathematicians' actions and how body-assemblages come to be. Châtelet's examples for virtual and actual components of mathematical concepts tend to omit concepts that are theoretical or relate to areas of mathematics such as algebra and analysis. In this descriptive case study (Merriam, 1998) we attempt to contribute to this gap and explore how a mathematician virtualizes the Cauchy-Riemann (CR) equations as he teaches an undergraduate complex analysis course. Our research question is: *How does a mathematician use embodiment to animate and bind the CR equations and illustrate the virtual dimensions of the material world as they relate to the CR equations*? Recall the theorem: "Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivative of the function u and v exist and satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$" (Zill & Shanahan, 2015, p. 131). Our results indicate that this mathematician's introduction of matrices to teach the CR equations allowed him to discuss transformations which made the equations mobile and gave him space to virtualize the CR equations by binding and animating their formal aspects (including conformality), metaphors, himself, and his class. Didactic implications of this work include adopting the mathematicians' virtuality of the CR equations that avoids the definition-theorem-proof (DTP) model (e.g., Alcock, 2009).

2 Literature review

In this section we summarize Lakoff & Núñez' (2000) work as it relates to metaphors because metaphors can be agents of body-assemblages. Similarly, because gesture plays a large part in the virtuality of mathematics, we summarize relevant gesture literature. Finally, we summarize the literature on mathematicians' understanding of complex analysis concepts as it is our setting.

2.1 Metaphors

In their seminal work on embodiment and mathematics, Lakoff & Núñez (2000) describe four conceptual mechanisms that humans portray as they engage with mathematics: (a) *image schemas*, (b) *aspectual schemas*, (c) *conceptual blends*, and (d) *conceptual metaphors*. *Image schemas* are perceptual and conceptual in nature, link language and spatial perception, and are frequently used when one reasons spatially. *Aspect schemas* refer to how our bodily movements convey the structure of mathematical objects. A *conceptual blend* combines two distinct structures with a fixed correspondence between them (e.g., relating algebraic and geometric structures). Finally, *conceptual metaphors* are metaphors whose structure maps from one entity to another but in different domains. Lakoff and Núñez use the language of "source" and "target" where the source (the concrete) implies the target (the abstract). Conceptual metaphors are "used unconsciously, effortlessly, and automatically … [and] arise naturally from … our commonplace experience," (p. 41) especially childhood experiences and mirror the structure of image schemas. Thus, conceptual metaphors resemble the animating aspect of the *virtual* but they fail to account for the binding as described in the introduction. As such, a significant difference between conceptual mechanisms and virtuality is that conceptual mechanisms "reinforce the divide between the mathematically abstract and the physically concrete" (de Freitas & Sinclair, 2014, p. 200). In this study we describe how body-assemblage emerged as the professor used metaphors to animate and bind the mathematical and the physical of the CR equations.

2.2 Teachers' gestures

McNeill (2005) categorized gestures as *beat*, *deictic*, *metaphoric*, and *iconic*. *Beat gestures* coincide with the rhythm of speech and *metaphoric gestures* indicate images of an abstract concept such as holding a function in one's hand. *Deictic* or *pointing gestures* occur when one points to something physically present or imagined and *iconic gestures* "present images of concrete entities and/or actions" (p. 39). Alibali & Nathan (2012) merged iconic and metaphoric gestures to coin the term representational gestures, dichotomizing gestures into pointing and representational gestures. Research on teachers' gestures in instructional settings shows that gestures are an integral part of pedagogical communication (Alibali et al., 2014). Teachers produce gestures to help convey mathematical ideas (Alibali

& Nathan, 2012; Font et al., 2010), to scaffold material (Alibali & Nathan, 2007), to maintain or shift focus (Alibali et al., 2019), to establish and maintain common ground (Alibali, et al., 2013), to confirm the correct interpretation of verbal utterances from students, or to develop a shared understanding in the classroom (Alibali et al., 2019). Such gestures are important when an instructor introduces new concepts with specialized language because gestures link verbiage to the physical world (Alibali & Nathan, 2007, 2012; Alibali, et al., 2013; Valenzeno et al., 2003). Interestingly, Châtelet (1993/2000) did not attend to speech in his work; he was only interested in gesture/diagram interplay. de Freitas & Sinclair's (2014) notions of body-assemblages view speech like gestures and diagrams as mathematical material.

While research on the use of teachers' gestures in the K-12 classroom has thrived, research on instructors' use of gestures in undergraduate mathematics classrooms is sparse (Keene et al., 2012; Stewart et al., 2019; Weinberg et al., 2015), but the findings are similar. For example, Weinberg et al. highlighted how an abstract algebra instructor's gestures contribute to their opportunities to communicate abstract mathematical ideas. Others (Oehrtman et al., 2019; Soto-Johnson et al., 2016) documented how mathematicians integrate gestures as they explain geometric interpretations of complex analysis concepts in research settings. Our current work extends such research to a classroom setting, which provides a space to explore how assemblages emerge organically, especially when students are part of the material.

2.3 Mathematicians and complex analysis

Researchers have explored undergraduates' and mathematicians' geometric interpretations of the continuity of complex functions, complex contour integrals, and complex differentiability (Hanke, 2020; Oehrtman et al., 2019; Soto & Oehrtman, 2022; Soto-Johnson et al., 2016; Troup, 2019; Troup et al., 2023). Troup and colleagues showed how dynamic geometric environments help students discover the *amplitwist* geometric interpretation (Needham, 1997) of the complex derivative. Characteristics of the *amplitwist* are small discs (a) dilate by $|f\prime(z)|$, (b) rotate by $Arg(f\prime(z))$, and (c) map to small discs. In their most recent work, Troup et al. (2023) found that technology did not fully aid students in discovering the *amplitwist* concept with the CR equations. Students struggled to coordinate the linear algebra and multivariable calculus notions embedded in the geometry of the CR equations, but they recognized conformal maps. In other words, the students struggled to bind the mathematical with the physical.

The works of Oehrtman et al. (2019) and Soto-Johnson et al. (2016) include the same five mathematicians who regularly teach complex analysis and conduct research in this field. For complex contour integrals, Oehrtman et al. found the mathematicians drew rich parallels from line integrals of real-valued functions as an accumulation of Riemann sums but struggled to conceptually interpret what was accumulated in the complex case, except for one of the mathematicians—Rafael (pseudonym). On the other hand, the mathematicians easily expressed the *amplitwist* concept with gestures. In terms of the geometric interpretation of the CR equations, three mathematicians expressed not thinking about such an interpretation. The fourth mathematician admitted to not having thought about the matter but correctly demonstrated and explained the conformal aspect of the CR equations. The last mathematician, Rafael, conveyed his geometric understanding of the CR equations via gesture/diagram interplay and exhibited rich binding and animating of the physical with the actual world. His virtuality of the CR equations led to this case study in a classroom setting.

3 Theoretical perspective

Châtelet (1993/2000) argues that the materiality of mathematical invention is born from a mathematician's hand when they make a diagram before introducing formal mathematics. Châtelet's interest is not in static diagrams; rather, his interest is in mathematician's diagrams that bring new spaces into being. Moreover, he focuses on the relationship between the actual and virtual because mathematical concepts can be actualizations of the virtuality of the world. Adopting these notions, de Freitas & Sinclair (2014) introduced the *inclusive materialism* framework which (a) views aesthetics and affect as instigators of mathematical activity, (b) attends to how material practices shape and are shaped by socio-political concerns, (c) pursues descriptions of phenomena that privilege differences, and (d) acknowledges that surprise and creativity "are linked to the virtuality and vitality of matter" (p. 43). We elaborate on the last two elements as they are most related to our research. Given that mathematical concepts tend to be perceived as inert, inanimate, and immaterial, the authors seek a different perspective where mathematical concepts are animated. Hence, interaction between a human and mathematics becomes intra-action through embodiment. In discussing creativity, de Freitas and Sinclair describe learning as the creation of something new and clarify that "it is not that individuals are creative or not creative, but rather that creativity flows across the … assemblage" (p. 86). They adopt Châtelet's notions of inventiveness and creativity that can be viewed as a collective process of gesture/diagram/speech/technology/human interplay and argue that material intra-actions in the classroom shift the actual/virtual relationship because they unveil new and unexpected mathematics. As such inventiveness is a “dance between the gesturing and drawing hand, which expresses and captures the temporal and dynamic moment when the new or the original comes into the world at hand” (p. 88). With this in mind de Freitas and Sinclair interpret a creative act as a reassembling and reconfiguring of the world portrayed through four characteristics: (a) *introduces or catalyzes the new*, (b) *is unusual*, (c) *is unexpected or unscripted*, and (d) *is without given content.*

The first characteristic refers to the transformation of the actual/virtual relationship which emerges through body movement intra-actions with material. The second characteristic refers to actions that do not align with the norms of a particular context. The unexpected or unscripted characteristic refers to acts that occur unintentionally and the last characteristic signals how actions change the way language, signs, and meanings are used in a situation; one might interpret this as creating shared meaning. de Freitas and Sinclair stress that mathematical inventiveness is not restricted to an individual body but rather is “a relationship between learners/teachers and the material world” (2014, p. 89), which includes material interplay. We note that geometric or algebraic symbols, as those found in the CR equations, develop from mathematical operations where one acts on inscriptions such as graphs, equations, and manipulatives and hence, constitute a material. In the analysis section, we describe how we adopted the elements of animating mathematical objects and creativity from the framework of *inclusive materialism* for our research.

4 Methods

4.1 Participant and setting

This study employed a case study design (Merriam, 1998), where we studied our research participant, Rafael, a Ph.D. mathematician with over 20 years of experience teaching undergraduate complex analysis. He served as a research participant for several of the lead author’s research when investigating how mathematicians reason geometrically about complex analysis concepts. This study set out to explore if Rafael provided rich and novel explanations of the CR equations in a classroom setting, as he did in a research setting. This aligns with Merriam’s philosophy that “a case study might be selected for its very uniqueness, for what it can reveal about a phenomenon” (p. 33) and provide access to new knowledge that we may not have otherwise. For his complex analysis course, Rafael adopted an open resource textbook with few diagrams, which he supplemented with his notes and expertise. His affinity for geometric interpretations inspired a course schedule where Rafael documented the topic of the day, related concepts, homework, and visual explorations that he intended to implement as part of the lesson. For the CR equations, Rafael’s calendar indicated that he would spend one day on this topic, relate it to Laplace’s equation and harmonic conjugates, and introduce gradient vector fields and contour lines as visual explorations. His detailed course schedule with anticipated visual explorations suggests Rafael intentionally illustrated the geometry behind the symbolism. During class Rafael used a document camera to project the notes that he wrote as he lectured and tended to move towards the projected material or the students to emphasize key points.

4.2 Data collection and analysis

A third-party videotaped the first six weeks of Rafael’s course, focusing on Rafael’s teaching, and the lead researcher took field notes. In her field notes, the researcher documented students’ questions and responses to Rafael’s questions, as well as a detailed timeline of each class session. We began the analysis by watching different class sessions in teams of two and documenting where Rafael integrated embodiment via visuals, spatial sense, gesture, intuition, and metaphors. We focused on the CR equations lesson because although Rafael’s calendar indicated that he would dedicate one class period to this topic, he spent three days on it: 20 min on the first day, 50 min on the second day, and 35 min on the third day. We transcribed the video data using transcription software and split the data into 10-min segments for each researcher to provide an overall description of their assigned segment and clean the transcript. Using the revised transcripts, we broke the data into small, refined segments, and coded the refined segments. These segments were introduced if Rafael navigated between embodied, symbolic, or formal interpretations of the CR equations. This allowed us to analyze how Rafael exhibited evidence of animating mathematical objects or engaging in one of the four creative acts (de Freitas & Sinclair, 2014) with formal, symbolic, or embodied elements of the CR equations. Using fine-grained, “rich thick description[s] of the phenomenon” (Merriam, 1998, p. 29) we coded instances of gesture/diagram/material/speech interplay as evidence of animating mathematical objects. Instances when Rafael presented the CR equations in a unique or novel way compared to what is in traditional complex analysis texts were coded as *unusual*. It was impossible to be certain when and if Rafael’s lesson was scripted ahead of time, but we were able to compare his lesson to his plans documented in his calendar and use this as evidence of *unexpected or unscripted*. Unique responses to student questions were also coded as *unexpected or unscripted*. Finally, segments where students or Rafael adopted or changed their language or signs in class were coded as *without given content*. Like de Freitas & Sinclair (2014) we did not code episodes where Rafael *introduces or catalyzes the new*, because this creative act manifests as an assemblage of episodes from the other three categories.

Each researcher performed a first coding round for different parts of the lesson. We coded segments where Rafael implemented pointing or representational gestures (Alibali & Nathan, 2012) along with geometric or symbolic inscriptions as embodied and the gestures served as evidence of animating mathematical objects. For all segments that we coded as embodied, we described Rafael's gestures, inscriptions, or use of tangible materials. Segments where Rafael engaged in computations such as determining if a function satisfied the CR equations were coded as symbolic and such segments were not coded as evidence of a creative act. Similarly, we coded segments where Rafael stated a definition, proved something formally, or stated a theorem, as formal and these segments were not coded as engaging in creative acts. Many segments have multiple codes because Rafael effortlessly changed explanations. A second researcher reviewed the first researcher's coding, and each group presented their coding to the research team. This process ensured that two researchers coded all the video segments and allowed for conversations with the entire team to ensure that we coded consistently. Such coding practices serve as a source of validity, reliability, and trustworthiness of results (Merriam, 1998). Using a narrative analysis with rich thick descriptions we summarized Rafael's verbiage, symbolism, inscriptions, and embodiment and how he animated mathematical objects or engaged in one of the creative acts as described below.

5 Results

Most complex analysis textbook authors devote approximately two pages to the CR equations and generally present them by stating the theorem, proving it, and providing two examples of how to use the CR equations (e.g., Brown & Churchill, 2009; Zill & Shanahan, 2015). These presentations are entirely algebraic in nature and do not illustrate characteristics of a creative act. Below we illustrate how Rafael engaged with and animated mathematical objects through *unusual*, *unexpected, and unscripted*, and *without given content* creative acts. Rafael's assemblage of *unusual*, *unexpected, and unscripted*, and *without given content* creative acts with materials embedded in algebraic and geometric inscriptions and metaphors *introduces or catalyzes the new* – the virtuality of the CR equations. We elaborate on this in the Discussion section. In describing Rafael's verbiage and written inscriptions, we also depict his gestures which are sometimes in parenthesis following the bolded verbiage to indicate the correspondence between the verbiage and gesture.

5.1 Unusual

Three aspects of our data appear *unusual* because they are atypical of how the CR equations are presented in traditional textbooks or because it does not depict the DTP model. We illustrate the *unusual* way that Rafael taught the CR equations by (a) distinguishing between complex and real differentiation, (b) unifying three mathematical concepts, and (c) stressing conformal mappings through rich and diverse metaphors. Moreover, he engaged in this creative act by activating both algebraic and geometric inscriptions.

Rafael introduced complex differentiation by discussing the differentiability of functions that map from R^2 to R^2 using the equation $\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$. The fact that he immediately used matrices to represent functions was in itself *unusual* from what is found in textbooks. Rafael explained that values of such functions at reference points yield "**numerical constants**" (pointed to the corresponding entries in the matrix) and that those constants are the "partial derivatives of y with respect to x" as he pointed to the vector matrices in order. Rafael explained that he started with this dialogue because "R^2 and C are planes" and can be viewed as the "same thing." He reinforced this idea by stating that one could "strip down" a complex-valued function into its real and imaginary parts and "write u as a function of x and y and write v as a function of x and y" and then ask if the function is differentiable as he pointed to Δy_1 and Δy_2 in his original equation. Rafael emphasized that real differentiation is "more inclusive" than complex differentiation as he drew a Venn diagram with complex differentiable functions as a subset of real differentiable functions and stated "if a function is complex differentiable then it is automatically real differentiable" as he pointed to each region of the diagram.

We also found it *unusual* that Rafael elaborated that the complex derivative unites three ideas: the real derivative of a single real variable, the complex derivative of a single complex variable, and the multivariable real derivative. To explain this unification, Rafael conveyed that they wanted to determine if the familiar difference quotient, $\frac{f(z)-f(a)}{z-a}$, "could be expressed as some function $\widetilde{Q}(z)$ which is continuous at $z = a$." He stated that Δf could be expressed as the product $\widetilde{Q}(z)\Delta z$ and that they had a function in terms of Δz, z, and a. To better explain the product $\widetilde{Q}(z)\Delta z$, Rafael expanded the product $\Delta u + i\Delta v = (A + iB)(\Delta x + i\Delta y)$ and clarified that "complex multiplication by a complex quantity could be modeled by a matrix of linear transformations" as he pointed to each of the factors. He then connected the various inscriptions to a matrix inscription and asked students to fill in the matrix shown in Fig. 1. The students provided the correct entries for the matrix; Rafael wrote $(A, -B, B, A)$ in the appropriate matrix entries and reminded the students that the entries are continuous functions. Rafael appeared to bind the symbolism $\Delta u + i\Delta v = (A + iB)(\Delta x + i\Delta y)$, with the words "multiplication," matrix, and linear transformation, in order to view $\Delta u + i\Delta v$ as movement. Moreover, he bound his students with the material by engaging them with the material.

Interpreting the behavior of the symbolism, Rafael said, "complex differentiability is … equivalent to the ability to factor changes in the **output vector** (slid fingers over the vector $\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$) through changes in the **input vector** (slid fingers over the vector $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$) using a **linear [transformation] or matrix** (pointed to each of the matrix entries)." Rafael explained that the entries are not random and exhibit a symmetry where the "diagonal elements are equal, and the off-diagonal elements are opposites." He added that a random continuous matrix with entries without such symmetry meant the function is not complex-differentiable as he pointed to the matrix entries, but that it is real-differentiable. Continuing with symbolic formal interpretations, Rafael uttered and wrote that $w = f(z)$ is complex-differentiable at the point $z = a$ if and only if $\begin{pmatrix} u \\ v \end{pmatrix}$ is real differentiable at $z_0 = a$ and following equalities hold: $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}, A = u_x = v_y$, and $B = v_x = -u_y$. He concluded with "these equations, the symmetries of this matrix, … are called the Cauchy-Riemann equations." As such, Rafael used matrices to distinguish between real and complex-differentiable and to illustrate the unification of the three mathematical concepts found in the CR equations. He explicitly compared the term complex differentiable with real differentiable and concluded class by determining if the functions $f(z) = z$ and $f(z) = \bar{z}$ are complex-differentiable using the *usual* "Cauchy-Riemann way."

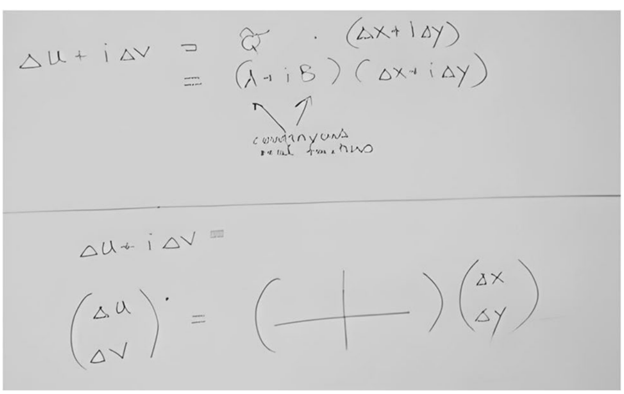


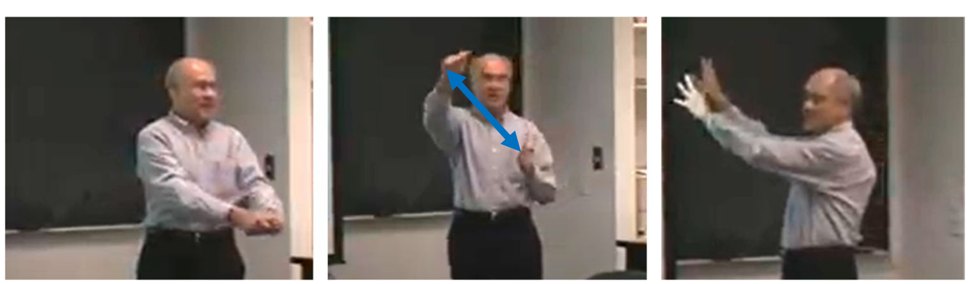
Fig. 1 Matrix of Linear Transformations

Readers might think that this is the end of the story of Rafael teaching the CR equations, but on the second day he described the equations via metaphors which he depicted through diagrams, imagination, and animation. This is *unusual* because conformal mappings are usually introduced in a graduate-level complex analysis course. This could also be considered unexpected and unscripted because Rafael did not have it listed as a geometric exploration on his calendar, but we are not sure if he changed his plans. Below we describe how Rafael animated his *unusual* metaphors consisting of elastic fabric, silly putty, rubber patch, and dough and how *unusually* effortlessly he navigated between metaphors and hence, bound his metaphors.

Rafael introduced the geometry of complex functions and their derivatives by drawing and projecting two copies of the complex plane and using a function, $w = f(z)$, to go between them. Rafael described the planes as an "elastic fabric" and the function as something that "warped" the plane onto the other plane. He further animated this metaphor by walking to the front of the class, positioning his hands in front of himself to make a warping motion, and "tossing" it towards the output plane while saying, "Pick it up off the plane. Stretch it out, do whatever we want to it. And then slap it down over on the other plane" (see Fig. 2).

After this enactment, Rafael symbolized his "elastic fabric" metaphor by drawing a horizontal and vertical coordinate system on the input plane about some point z_0 and depicting the function as warping the coordinate system in the output plane. He described the horizontal lines in the input plane as "threads on the elastic fabric" that map to

Fig. 2 Pick it Up, Stretch it Out, and Slap it Down



curves in the output plane. As he uttered this description, he ran his fingers along the horizontal line (see Fig. 3) and conveyed the notion of conformality by saying that the perpendicular parts of the input coordinate system will be perpendicular in the output system. Here Rafael appeared to bind the notions of the elastic fabric metaphor, coordinate system, and conformality.

To illustrate the geometry behind complex differentiable mappings, Rafael drew a curved grid which represented the image around the image point $w_0 = f(z_0)$ (see Fig. 4). Rafael used his finger to trace the horizontal lines in the domain, which he referred to as "threads" and emphasized that only these horizontal lines mapped to the curved vertical lines in the image. Similarly, he explained that the family of vertical curves in the domain all map to the curved horizontal lines. He further described how the curved lines in the image are "perfectly perpendicular" and inscribed a square at their intersection. Using tick marks, he indicated that the length of the square from the domain was preserved in the warped square of the image. Rafael warned the students that a formal description of Fig. 4 would only be "literally true under infinite magnification" and waved his hand to represent a physical manifestation of a dynamic function. He also made a sweeping movement with his hand whilst pinching together his index finger and thumb as he said, "infinite magnification" to indicate the notion of looking at infinitesimally small neighborhoods around the image point. He then offered a silly putty metaphor for Fig. 4.

Rafael physically grounded the domain by comparing it to silly putty and gestured the function mapping as "pounding down on it," with the silly putty "ooz[ing] out in all directions" as he spread his hands away from the location where he pounded (see Fig. 5) and produced an audible embodiment. Rafael reinforced the conformality of complex differentiable as he said, "anything that was orthogonal, when you pound on it, stretches out in equal amounts," thereby staying orthogonal. He further embodied this geometric description as he crossed his hands on the projector to represent the domain (see Fig. 6) and then gestured the stretching along the same angles by separating his pinched thumb and index fingers from his two hands (see Fig. 7). These animations of the silly putty binded it with the conformal map and the elastic fabric metaphor.

After this discussion, Rafael pointed to the entries of the Jacobian matrix and explained that the vector Δw can be obtained from Δz by multiplication of the Jacobian of the function because $f(x, y) = u(x, y) + iv(x, y)$ is a real differentiable function. He clarified that he was interested

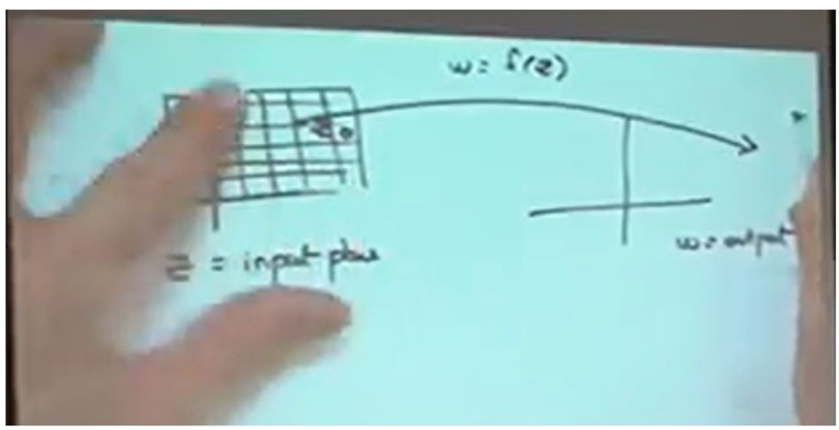


Fig. 3 Threads on Elastic Fabric

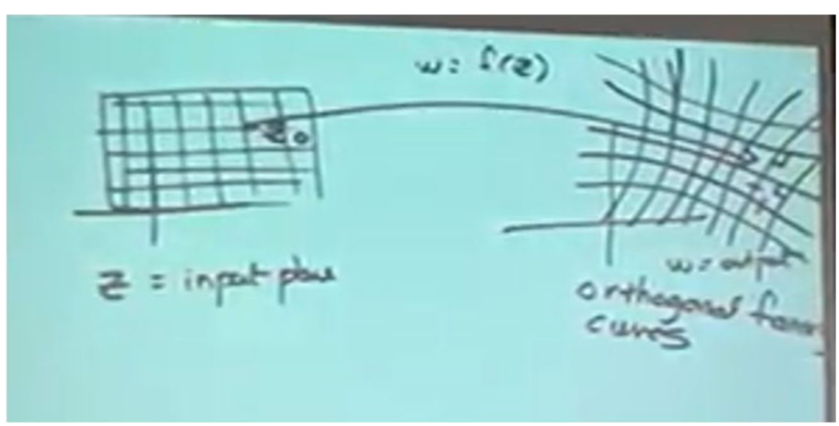


Fig. 4 Geometry of Complex Differentiable Maps

Fig. 5 Pounding Down on It, Oozing Out, In All Directions

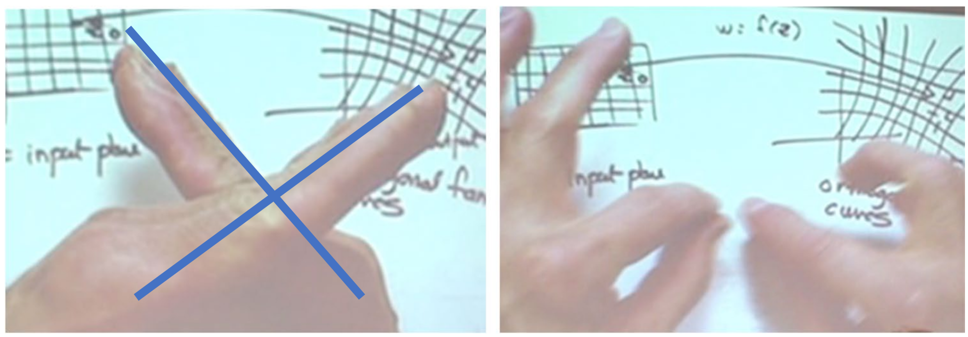


Fig. 6 Lines Intersecting at the Point z_0 and the Image f(z_0) = w_0

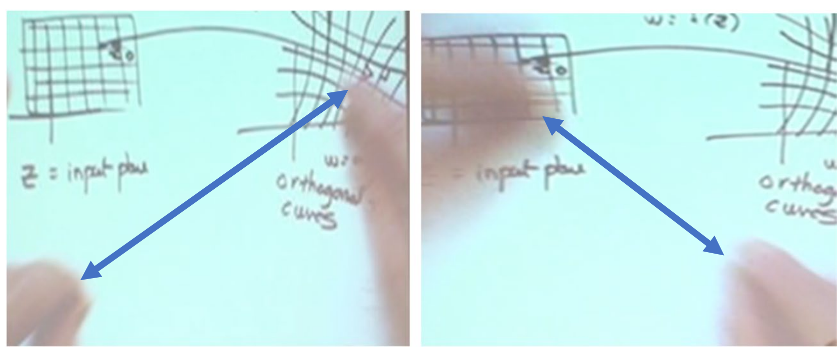


Fig. 7 Stretching the Lines Emanating from w_0 Along Same Angles as Angles in the Domain

in situations where the mapping is described by a “dilation and a rotation” (i.e. an *amplitwist*) as he wrote the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ underneath the Jacobian matrix. Rafael summarized how this gave a “visual, or geometrical, interpretation of what the complex derivative is good for. A function is complex differentiable if and only if, when you look at it at a high magnification, it looks like a conformal map” and projected a 3D visualization of an arbitrary quadratic polynomial created with *Mathematica*. Moving his hands from one side to the other, he explained how he visualized this “as a mapping from the z-plane to the w-plane and what we’re actually doing is taking a little disk, like a little rubber patch in the z-plane and mapping it to the w-plane.” He described “carving out … circular patches” and then switched the screen to the lecture notes to show how one takes circular patches in the z-plane and zoomed in on them in the w-plane. Again, he made use of Fig. 4 as he pointed from the z-plane to the w-plane. Rafael conveyed that when he makes the “patch bigger and bigger, [he is] carving out bigger and bigger pieces.” He gestured a patch by making a circle with his hands and explained that it was “like dough and [he’s] pulling it out and also curving it around and then making a pirouette, and laying it back on top of itself, and so some of the dough is overlapping other parts.” While describing this, Rafael gestured pulling out the dough by expanding his hands, curving them to represent the curving of the dough, and slapping his hands together while turning them to show the dough laying on top of itself. He then returned to the *Mathematica* visual and pointed to the part where the layers overlap in the image and proceeded to play with the sliders to dynamically illustrate different views of the overlap.

Again, we observed evidence of how Rafael's animations bound his various metaphors, conformal map notion, and the matrix, $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ which was *unusual*.

5.2 Unexpected or unscripted

Two aspects of our data were *unexpected or unscripted*. The first was the amount of time that Rafael dedicated to teaching the CR equations. Based on Rafael's calendar, we expected him to spend one day on the material and from experience and conversations with others we know that mathematicians spend at most one day on the equations. Furthermore, Rafael did not deviate from his calendar for other topics. As previously mentioned, the *unexpected* amount of time that he spent on this topic was one of the reasons for focusing our research on the CR equations. A second aspect that we coded as *unexpected or unscripted* was an episode where a student asked Rafael a question following Rafael's explanation of conformal maps. The student asked whether Fig. 4 could be compared to an elastic fishing net, "where all the places where the nets tie together stay the same." Rafael confirmed and elaborated on the student's theory through embodiment with tangible material. He took the cap off his marker and used it and the marker itself to represent "tubes" that are perpendicular. Rafael explained that one could run ropes through these tubes as he gestured left and right and then up and down along the direction of the tubes. He explained how the intersection of the tubes, represented by the marker and its cap, would need to "transfer the tension" equally from one rope to the other. Using *unscripted* embodiment and language of a metaphor that emerged from a student, Rafael demonstrated the transfer of tension by running his finger left and right and up and down the marker and its cap. Rafael shared that this equal distribution of tension was the "same mechanism" that complex differentiable functions have when they preserve stretching factors and angles. This episode demonstrates how students become part of the assemblages and Rafael's perception of complex differentiable functions as objects that do something because of the machinery that is part of their assemblage.

After this *unscripted* act and conversation Rafael reinforced the idea that complex differentiable functions have constant stretching factors in all directions near a reference point and he asked the students if there are other properties that such functions enjoy. Specifically, he asked, "After it has done that stretching, we have a scaled-up copy of the original, is there anything else that we could do to that image that would keep the size of it the same?" As he posed this question, he animated an imaginary domain in front of him that represented the stretching done by the function. Maintaining the shape of his imagined stretched domain, he answered his question and said, "It could **rotate** (gestured rotation)," (see Fig. 8).

5.3 Without given content

The creative act characteristic of *without given content* manifests when acts modify the way language or signs are integrated into a situation such as classroom shared meanings (Alibali et al., 2019). The student's invention of the fishing net is an example of without given content, and another is how Rafael's language of "rotate and dilate" permeated in his classroom. While illustrating the "mapping of a square at various reference points on the unit circle" under the function $f(z) = z^2$, using *Mathematica*, Rafael asked about the critical point and a student responded that it was zero. Rafael reminded the students that they must pick points on the periphery of the unit circle that are away from the origin and use those points to "march" around the unit circle to see the function behavior. As he said this, he brought his thumb and pointer finger together to pick an imaginary point from the air and then traversed his pointer finger around an imaginary circle in the air. To illustrate his air gestures, Rafael drew a unit circle on the board with a "bad spot" at the origin (marked with a star) and smaller circles on the circumference of the unit circle. Rafael showed the mapping of a small disk centered at (1,0) in the z-plane and demonstrated the "marching around like a clock hand" to depict how points get mapped (see Fig. 9). Pointing to (1,0), he asked "what happens if we multiply a displacement vector Δz by two?"

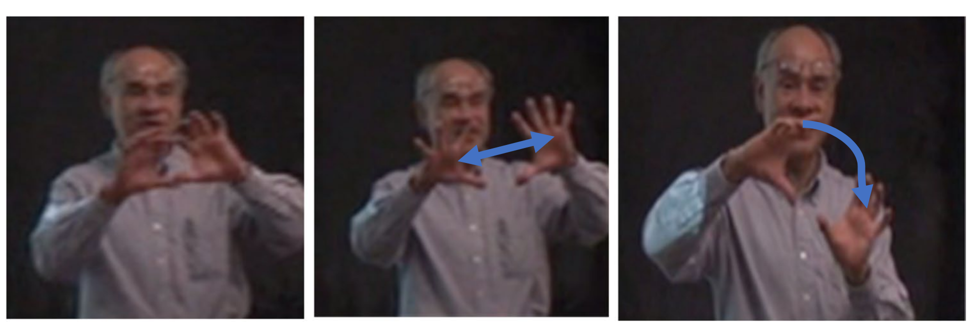


Fig. 8 Imagined Domain, Stretched, and Rotated

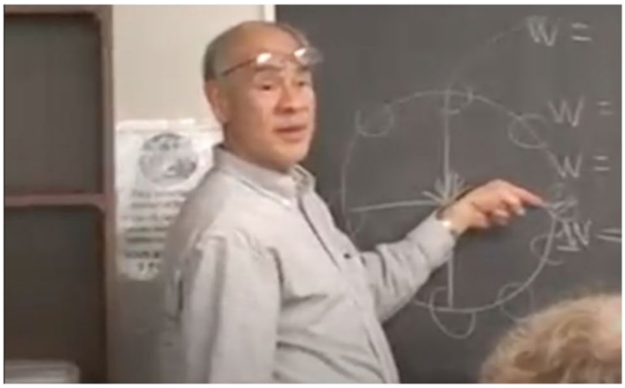


Fig. 9 Marching Around Like a Clock Hand

Several students correctly responded that the displacement vector dilates but does not rotate. Rafael then spread his hands out to depict the stretch. It appeared that Rafael's animation of a dynamic visualization, imagined description, and a sketched metaphor binded him and geometric and mobile language about the CR equations with his students.

Before discussing our results, we share how Rafael used *usual* techniques to derive the CR equations on the last day. Verbally and in writing, he explained that $f\prime\left(z_0\right) = \frac{f(z+\Delta z)-f(z)}{\Delta z}$ and that the "limit has to exist regardless … of how Δz tends to zero." Rafael gestured Δz approaching zero along "a horizontal line, a vertical line, a spiral curve, a sequence of dots that hops around …." He then sketched a point on the complex plane and an arrow approaching it from the horizontal direction and another from the vertical direction. He explained that in the horizontal limit the quotient $\frac{\Delta f}{\Delta z}$ is evaluated only along Δx because "it's a derivative only in the x variable and if the limit exists, it has to converge to u_x plus iv_x." Rafael explained that the second case was more complicated because "we're not changing x, but Δz then is going to be i times Δy." He proceeded to compute the limit as $i\Delta y$ approached 0. Rafael simplified the expression and computed an expression for the derivative in terms of u_y and v_y and compared it with the previous computations, which had the derivative in terms u_x and v_x. Rafael then wrote down the CR equations and $f\prime(x, y) = A(x, y) + iB(x, y)$ in various ways. He wrote the CR equations as a matrix and compared the derivative to the matrix in terms of A and B. He stressed that the derivative could be written in four different ways depending on the partials of u and v and for a third day in a row he informed the students that this procedure does not work if we pick a random function u because of the "extra conditions." Instead of elaborating on this formal statement, he used the matrix form of the CR equations to find v if $u(x, y) = 2xy$. At this juncture, Rafael gestured looking through imaginary binoculars while he uttered the importance of getting the partial pieces to be self-consistent or come into focus as he stably displayed his hands in front of his face. The self-consistency animations could represent binding of the various symbolic representations indicating that the CR equations are satisfied.

6 Discussion

Recall our research question: *How does a mathematician use embodiment to animate and bind the CR equations and illustrate the virtual dimensions of the material world as they relate to the CR equations*? Our results indicate that this mathematician's introduction of matrices to teach the CR equations allowed him to discuss transformations which made the CR equations mobile. This mobility gave the mathematician a space to virtualize the CR equations by binding and animating the symbolic and formal aspects of the equations (especially conformality), metaphors, himself, and his class. Hence, our research illustrates how Rafael was the "body *in* and *of* mathematics" (de Freitas & Sinclair, 2014, p. 200) which engaged the CR equations in a process of becoming and how this process bound the equations with Rafael and his students.

Our results support de Freitas & Sinclair's (2014) thesis that conceptual mechanisms (Lakoff & Núñez, 2000) underscore the divide between the mathematically abstract and the physically concrete. On the other hand, virtuality binds the physical and the abstract and illustrates how a "body is a set of material relations that … structure[s] the other materials relations" (de Freitas & Sinclair, 2014, p. 34). It seems that virtuality binded Rafael's various conceptualizations of the CR equations (see Table 1) to himself and to his class by *introducing or catalyzing the new*. Specifically, Rafael's metaphors and animation braided his symbolism with his formal language through an assemblage of *unusual*, *unexpected, and unscripted*, and *without given content* creative acts and birthed the first creative act of *introducing or catalyzing the new*.

The *unusual* way of introducing the CR equations using matrices allowed Rafael to introduce the term transformation and immediately make the CR equations mobile. This introduction set the stage for Rafael to engage in a mathematical activity where he "*actualized the virtual"* (de Freitas & Sinclair, 2014) which occurs when we create something that links different relations but does not resemble the virtual i.e., *introducing or catalyzing the new*. This allowed Rafael to bind between and within his different conceptualizations and his students via animated gestures.

For example, Rafael often integrated pointing gestures when he engaged with algebraic inscriptions via formal language or logic and deduction. In general, when Rafael uttered a formal statement, he also wrote it down and pointed to pieces that needed highlighting. This occurred when he

Table 1 Conceptualizations of CR equations

	Examples
Symbolism	$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}, \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix},$ $R^2 vs C$
Metaphors & Animation	Elastic fiber; silly putty; fishing net; rubber patch; dough; clock hand; pick it up, stretch it out, and slap it down; pound down on it, oozes out in all directions; carve circular patches, curve them around, make a pirouette, and lay it back on top of itself; march around unit circle; right angles map to right angles; not random; transfer of tension; rotation and dilation,
Formal Language	Linear transformations; matrix multiplication; symmetric matrix; unification of real derivative of a single real variable, complex derivative of a single complex variable, and multivariable real derivative; conformal maps; hierarchy of differentiability (Venn diagram)

introduced the words CR equations and pointed to the partials as he mentioned the symmetries of the matrix and the connection to complex differentiable. It also occurred when Rafael traced the vertical and horizontal lines in his input plane and then conveyed the notion of conformality. Likewise, when Rafael uttered and wrote the definition of complex differentiable, he pointed to the part of the algebraic symbolism that was continuous. Moreover, via gestures Rafael distinguished between real and complex differentiability, which he expressed was not generally distinguished. Thus, in an *unusual* way he bound and animated the formal and symbolic aspects of the CR equations (e.g., symmetries of the matrix, the words CR equations, conformal maps, definitions of real and complex differentiable, and multivariable calculus).

Rafael's *unusual* metaphors also allowed him to bind the conformal aspect of the CR equations with the metaphors' virtual motions, which he animated through gesture. For example, in explaining the function, $w = f(z)$, Rafael introduced the "elastic fabric" metaphor to describe the z-plane and described how the function moved, warped, tossed, stretched, and slapped points or patches to obtain the output in the w-plane, as he gestured all the actions. He also introduced the silly putty metaphor and viscerally described how it oozes out in all directions as he pounded on the desk and spread out his fingers to represent the oozing. With dough in hand, pulled, curved it around to make a pirouette, and laid it back on top of itself so that it overlapped. In an *unexpected and unscripted* way Rafael also bound and animated his student's metaphor of a fishing net using markers as he gestured the action of "transfer" of tension. Thus, Rafael bound his animated metaphors which included sound (e.g., the pounding), to conformal mappings, the CR equations, and a student.

Besides animating his metaphors, Rafael also animated an idea that he presented with geometric inscriptions. For example, he moved his hand from one plane to the other to denote the dynamic aspect of a complex function, pinched his thumb and index finger to denote infinitesimally close, positioned his hands to illustrate perpendicular lines, illustrated carving out circular patches on his *Mathematica* image, used his pointer finger to march around a circle to depict how points get mapped locally, made a pinching gesture to denote the pointwise property of differentiability, and so forth. Rafael's gestures animated the static and made it dynamic; Rafael had a talent for binding different mediums using representational gestures to convey the same abstract concept. This was prevalent when he showcased the dynamism in *Mathematica*, followed by gesturing his imagined points, and concluding with the clock metaphor to paint a picture of how a function behaves around a given point. This last episode also showcased the creative act *without given content* because the students adopted the language of "rotate and dilate." Thus, through animation Rafael bound geometric inscriptions, mathematical ideas embedded in the notion of conformal maps, and his students.

A hallmark of Rafael's teaching was that he stayed close to the document camera or the projected content when he discussed symbolic or formal concepts. This allowed him to point to new and specialized content that aligns with other's research (e.g., Alibali & Nathan, 2007, 2012). On the other hand, Rafael moved closer to the students when he uttered informal explanations with an imagined object which prompted his animations—this seemed to be a strategy for binding the CR equations, his class, and himself. Thus, a possible theoretical contribution is that much of Rafael's lesson could not be displayed with written materials (e.g., texts, board) because fundamentally mathematics is a human activity. This also highlights how "the body is an assemblage of human and non-human components, always in a process of becoming that belies any centralized control" (de Freitas & Sinclair, 2014, p. 25). We acknowledge that there is a possibility that some students did not attend to Rafael's gesture and hence resulted in missed opportunities to learn, as shown in other work (Fukawa-Connelly, et al., 2017; Melhuish et al., 2022).

7 Didactic implications

These data provided us with lessons where in an entangled but bound and creative manner, Rafael engaged in body-assemblage with the CR equations. Through creative acts, animation, and binding Rafael told a story about the CR equations that does not and cannot appear in traditional texts or in his projected written sources. This further illustrates how mathematics is fundamentally a human activity. One might ask if students understood the CR equations better through Rafael's virtuality of the CR equations, but our data do not offer an answer to such a question. Instead, our data illustrate how students may come to *understand differently* through Rafael's virtuality of the CR equations because such teaching does not privilege the traditional DTP model and has the potential to support learning. This seems relevant especially because the work of de Freitas & Sinclair (2014) is with children and ours is with undergraduate students.

Rafael's virtuality of the CR equations did not privilege the formal and stagnant product view of mathematics which tends to be perceived as the ideal and which enforces a disconnect between abstract concepts (de Freitas & Sinclair, 2014; Nemirovsky, 2020). Instead, Rafael's binding and animation favor the tactile, the real-life dimensions, imagination, touch, and movement which give rise to new mathematical ideas. Bunn et al. (2022) argue that such teaching/learning provides a space to recognize that mathematics is everywhere and thus, has the potential to tap into the aesthetics and affect domain of the *inclusive materialism* framework. This phenomenon emerged in our data when the student invented the fishing net metaphor for conformality; de Freitas & Sinclair (2014) argue that such inventiveness is an example of learning. As such a possible didactic implication of our work is to adopt Rafael's conceptualizations of the CR equations in one's classroom *and* ask students to bind them. This could allow students to engage in their own creative acts which attend to the aesthetics and affect domain of our framework, and which could produce instances of the *without given content* creative act. Moreover, it could further stress the mobility of conformal maps which are generally not taught in an undergraduate class. We believe these findings are valuable as research generally portrays a deficit representation of mathematicians' teaching. Furthermore, Rafael's teaching methods have the potential to inform the revitalization of the teaching of complex analysis to be more geometric and dynamic in nature, especially because the course has not changed in decades (MAA, 2015) except for Needham's (1997) book. Along with providing dynamic interpretations for concepts, authors may want to clearly distinguish between real and complex differentiability. For example, the CR equation theorem is more exact if stated as: A **complex-valued** function $f(z) = u(x, y) + iv(x, y)$ is **complex differentiable** at a point $z = x + iy$ *if and only if* at (x, y) the first-order **real partial derivative** of the **real-valued** functions u and v exist and satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This version highlights the formal differences between real and complex differentiability.

8 Limitations and future work

One of the limitations of our work is that we did not member-check (Merriam, 1998) because we had interviewed Rafael several times prior to videotaping him while he taught. Such a practice could have provided insight into the *unexpected and unscripted*. Interviewing students could have also provided evidence of the creative act, *without given conten*t, which was limited in our data. Given that classroom discourse and interactions are an integral binding component of virtuality, it is important that future work attend to this component. Other research could entail binding Rafael's conceptualizations along with Troup et al.'s (2023) CR equations lab. Although the undergraduates from Troup et al. did not discover the *amplitwist* concept, they did discover the conformal aspects of the CR equations. The technology could serve as an added material to be bound with speech, gesture, tangible materials, inscriptions, and possibly unveil Rafael's three unification pieces, which Troup et al.'s participants failed to unify. Another line of inquiry is to offer professional development for educators to invent how mathematical concepts imbue mobility, learn about creative acts, recognize students' embodiment that conveys mathematical concepts, and engineer didactic practices that support binding and animating mathematical concepts.

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