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Propagation phenomena for a nonlocal reaction-diffusion model with bounded phenotypic traits

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Abstract

In this paper we study the stability of cylinder front waves and the propagation of solutions of a nonlocal Fisher-type model describing the propagation of a population with nonlocal competition among bounded and continuous phenotypic traits. By applying spectral analysis and separation of variables we prove the spectral and local exponential stability of the cylinder waves with the noncritical speeds in some exponentially weighted spaces. By combining the detailed analysis with the spectral expansion and the special construction of sub-supersolutions, we further prove the uniform boundedness of the solutions and the global asymptotic stability of the cylinder waves for more general nonnegative bounded initial data, and prove that the spreading speeds and the asymptotic behavior of the solutions are determined by the decay rates of the initial data. Our results also extend some classical results on the stability of planar waves for Fisher-KPP equation to the nonlocal Fisher model in multi-dimensional cylinder case.
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1. Introduction and statement of main results

To investigate the intra-specific competition among multiple phenotypes within a single population, the following nonlocal reaction-diffusion model was proposed in [17],

$$\begin{cases} \partial_t u(t,x,y) - d_x \triangle_x u(t,x,y) - d_y \triangle_y u(t,x,y) & \\ \quad = \left[1 - \alpha g(y-\theta) - \int_\Omega K(x,y,y')u(t,x,y')\mathrm{d}y'\right] u(t,x,y), & (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \partial\Omega, \\ u(0,x,y) = u_0(x,y), & (x,y) \in \mathbb{R} \times \Omega. \end{cases} \tag{1.1}$$

Here $u(t,x,y)$ represents the density of a population that is structured by a continuous spatial variable $x \in \mathbb{R}$ and the continuous bounded phenotypical traits $y \in \Omega \subset \mathbb{R}^n$, with Ω being the set of all possible traits. The traits represent within species variations, e.g. rate of food intake, average litter size, flowering time, or age at maturity. The terms $d_x \triangle_x u$ and $d_y \triangle_y u$ measure the spatial diffusion and the mutations, respectively. The nonlocal term $\int_\Omega K(x,y,y')u(t,x,y')\mathrm{d}y'$ indicates that the intra-specific competition occurs among all the individuals at each location x. The birth rate of the population is given by the fitness function $1 - \alpha g(y-\theta)$ where g is positive except $g(0) = 0$, this assumption takes into account the impact of natural selection on the population survival. Here α is a parameter that quantifies the intensity of selection towards the optimal value θ. More detailed information about the biological background of the nonlocal model (1.1) can be found in [15,27,30].

Over the past decade, the propagation phenomena arising from the model (1.1) have attracted tremendous attention among mathematicians. For model (1.1) in unbounded domains, planar waves and cylinder waves can describe the simplest and the typical wave phenomena. A traveling front solution (or a cylinder front solution) of equation (1.1) is a solution $u(t,x,y)$ in the form of $\phi(x-ct,y)$, which connects zero to a non-trivial state with a constant speed $c \in \mathbb{R}$ and $\phi(z,y)$ is monotone in z for each $y \in \bar{\Omega}$. For the nonlocal model (1.1) with the simplified kernel $K(x,y,y') = K(y')$ and $\theta = 0$ in the whole space $(x,y) \in \mathbb{R} \times \mathbb{R}^n$ or with bounded traits $y \in \Omega$, by applying spectral expansion (or separation of variables), H. Berestycki et al. [4] obtained the existence and the uniqueness of the cylinder front solution $\phi_c(x-ct,y)$ to (1.1) for $c \geq c^*$, and showed that $\phi_c(x-ct,y)$ must be in the form of $V_c(x-ct)\phi_0(y)$. Under some additional assumptions on $K(y)$ in the whole $\mathbb{R}^n$, in [4] it is also proved that the minimal speed c^* is the spreading speed of the solution with a compactly supported initial datum.

For the nonlocal model (1.1) in the whole space $(x,y) \in \mathbb{R} \times \mathbb{R}^n$ with more general kernel $K(x,y,y')$ and $\theta(x) = bx$, M. Alfaro et al. [1] proved the existence of cylinder waves by employing Harnack's inequality and topological fixed-point argument. Subsequently, accelerating invasion has been analyzed in [28] if the initial datum displays a heavy tail in the direction $y - bx = 0$. M. Alfaro and G. Peltier [3] proved the existence of steady-state solutions and pulsating fronts for the case when θ is periodic in x. For the model (1.1) in moving environment with $\theta = b \cdot (x - c_m t)$, M. Alfaro et al. [2] investigated the existence of waves and the spreading speeds of solutions. For the nonlocal model (1.1) with a bounded Ω and with a constant kernel $K \equiv 1$, by applying Hamilton-Jacobi approach, E. Bouin and S. Mirrahimi [9] investigated the asymptotic spreading speed and the asymptotic behavior of the solution $u(t,x,y)$ or those of $\int_\Omega u(t,x,y)dy$.

When the spatial diffusion rate of a population varies (see [31][35]) and is measured by the trait variable y such as the leg length of cane toads, O. Bénichou et al. [6] proposed the following biological diffusion model

$$\partial_t u(t,x,y) - y\triangle_x u(t,x,y) - d\triangle_y u(t,x,y) = r\left[1 - \int_\Omega u(t,x,y')\mathrm{d}y'\right]u(t,x,y), \tag{1.2}$$

where Ω is a bounded or unbounded interval in $\mathbb{R}^+$, d and r are positive constants.

There are some deep and interesting theoretical works on the wave propagation and the spreading speed of solution to the model (1.2) when the set of traits Ω is bounded. The spreading speed of the solution to model (1.2) with bounded Ω was investigated in [7] and [36] by applying Hamilton-Jacobi framework. By applying the Leray-Schauder degree argument similar to [1], E. Bouin and V. Calvez [8] proved the existence of traveling wave with a minimal speed to model (1.2) with bounded Ω. Subsequently, E. Bouin et al. [11] proved that the spreading speed of the solution with a compactly supported initial datum is the minimal wave speed with a Bramson's logarithmic delay.

For the model (1.2) with the unbounded $\Omega = \mathbb{R}^+$, N. Berestycki et al. [5] applied the probabilistic techniques and E. Bouin et al. [10] applied the PDE method to prove that the spreading speed of the solution with a compactly supported initial datum is unbounded, and the associated population front travels super-linearly in time (in order of $t^{3/2}$), see also [14] for more detailed estimates on the accelerated propagation.

Another related nonlocal Fisher model is in the form of $u_t = u_{xx} + (1 - \int_\mathbb{R} \phi(x-y)u(t,y)\mathrm{d}y)u$, where x is a spatial variable, and the nonlocal competition term characterizes the long range intra-specific competition. Some recent work on the existence of traveling waves and the spreading speed of solutions for this type of nonlocal Fisher equations can be referred to [12,20,23,29] and the references therein.

It is worth mentioning that different from the investigation on the classical reaction-diffusion models, the comparison principle can not be applied directly to the aforementioned models with nonlinearly coupled nonlocal reaction terms, thus the sub-supersolution method or some techniques such as sliding method or monotone iteration schemes can not be applied directly to such nonlocal models, which leads to some additional difficulties in establishing sharp estimates on the bound of solution in time and in determining the asymptotic behavior of solution in time with more general initial datum, and as far as we know even for the simplest nonlocal model (1.1) with bounded Ω there are no theoretical results on the stability of waves or the asymptotic behavior of solutions with more general initial data except the case when the initial data have compact supports.

This paper focuses on the nonlocal reaction-diffusion model (1.1) in the cylinder domain $\mathbb{R}\times\Omega$, where Ω is bounded and $K(x,y,y') = K(y')$. By rescaling of x and y, we may assume without loss of generality that $d_x = d_y = 1$. Thus, we may recast (1.1) as follows

$$\begin{cases} \partial_t u(t,x,y) - \triangle_{x,y} u(t,x,y) & \\ = \left[1 - g(y) - \int_\Omega K(y')u(t,x,y')\mathrm{d}y'\right]u(t,x,y), & (t,x,y)\in\mathbb{R}^+\times\mathbb{R}\times\Omega, \\ \frac{\partial u}{\partial \nu} = 0, & (t,x,y)\in\mathbb{R}^+\times\mathbb{R}\times\partial\Omega, \\ u(0,x,y) = u_0(x,y), & (x,y)\in\mathbb{R}\times\Omega. \end{cases} \tag{1.3}$$

Next, we introduce the assumptions on K and g. The function $g(y)$ is bounded and measurable (and can be sign-changing). Furthermore, let $\{\lambda_j\}_{j=0}^{+\infty}$ denote all the eigenvalues of the operator $-\triangle_y + g(y)$ under homogeneous Neumann boundary condition on $\partial\Omega$, with $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \ldots$. It is well known that the first eigenvalue λ_0 is simple and corresponds to a positive eigenfunction $\psi_0(y)$, and denote $\{\psi_j(y)\}_{j=0}^{+\infty}$ be a sequence of the eigenfunctions which forms an orthonormal basis of $L^2(\Omega)$, i.e. $\int_\Omega \psi_j^2(y)\mathrm{d}y = 1$, and $\int_\Omega \psi_i(y)\psi_j(y)\mathrm{d}y = 0$ for $i, j \ge 0$ and $i \ne j$.

In this paper the assumptions on K and g can be summarized as follows

$$\textbf{(H1)}\quad g \in \mathrm{L}_\infty(\Omega), \quad \lambda_0 < 1, \quad K \in \mathrm{L}_2(\Omega); \quad K(y) \ge 0 \text{ and } K(y) \not\equiv 0,\ y \in \Omega.$$

It is easy to check that for any $c \ge 2\sqrt{1-\lambda_0}$ the expression $V_c(x-ct)\psi_0(y)$ is a traveling front solution of (1.3), where $V_c(x-ct)$ is the planar front solution satisfying the following Fisher-KPP equation

$$\begin{cases} V_c''(\xi) + V_c'(\xi) + \left((1-\lambda_0) - V_c(\xi)\displaystyle\int_\Omega \psi_0(y)K(y)\mathrm{d}y\right)V_c(\xi) = 0, \ \xi \in \mathbb{R}, \\ V_c(-\infty) = \mu_0, \ V_c(+\infty) = 0, \end{cases} \tag{1.4}$$

with $\mu_0 = (1-\lambda_0)\left(\int_\Omega K(y)\psi_0(y)\mathrm{d}y\right)^{-1} > 0$.

By applying the argument based on separation of variables and detailed asymptotic estimates, it is also proved in [4] that, under the assumption of **(H1)**, the problem (1.3) has a positive and bounded cylinder front solution $\phi_c(z, y)$ (where $z = x - ct$) with $\phi_c(z, y)$ decreasing in z if and only if $c \ge 2\sqrt{1-\lambda_0}$, and the cylinder front $\phi_c(z, y)$ is unique (neglecting the shift in z) and thus $\phi_c(x-ct, y) = V_c(x-ct)\psi_0(y)$.

In this paper, we study the local and global asymptotic stability of the cylinder waves $V(x-ct)\psi_0(y)$ to model (1.3) in various settings.

For the remainder of this paper, we further assume $\mu_0 = 1$ without loss of generality, i.e. $\int_\Omega \psi_0(y)K(y)\mathrm{d}y = 1-\lambda_0$. This is possible by replacing $u(t,x,y)$ by $\frac{1}{\mu_0}u(t,x,y)$ (and accordingly for the cylinder wave) for the original model (1.3). Then the re-scaled $V_c(\xi)$ satisfies

$$\begin{cases} V_c''(\xi) + V_c'(\xi) + (1-\lambda_0)(1-V_c(\xi))V_c(\xi) = 0, \ \xi \in \mathbb{R}, \\ V_c(-\infty) = 1, \ V_c(+\infty) = 0. \end{cases} \tag{1.5}$$

By applying detailed spectral analysis and the classical stability theories of traveling waves based on analytic semigroup theories, in the following section we shall prove that all the cylinder waves with noncritical speeds are spectrally stable and nonlinearly exponentially stable in some appropriate spaces. Our results on the nonlinear exponential stability of cylinder waves are stated as follows.

Theorem 1. *Under the assumption of* **(H1)**, *for each* $c > c^* = 2\sqrt{1-\lambda_0}$ *and* $a > 0$ *satisfying*

$$0 < \frac{c - \sqrt{c^2 - 4(1-\lambda_0)}}{2} < a < \frac{c + \sqrt{c^2 - 4(1-\lambda_0)}}{2},$$

the cylinder traveling front $V_c(x-ct)\psi_0(y)$ of (1.3) is locally exponentially stable in the following exponentially weighted space

$$X_a = \{u(x,y) \in X:\ w_a(x)u(x,y) \in X,\ \ \|u\|_{X_a} = \|w_a u\|_X < \infty\},\ \ X = \mathrm{C}_{\mathrm{unif}}(\mathbb{R}\times\overline{\Omega}),$$

where $w_a(x) = 1 + \mathrm{e}^{ax}$. In other words, if the initial perturbation $\|u_0(x,y) - V_c(x)\psi_0(y)\|_{X_a}$ is sufficiently small, then there exist positive constants M and σ_c such that the problem (1.3) admits a unique global solution $u(t, z+ct, y)$ satisfying

$$\|w_a(z)(u(t,z+ct,y) - V_c(z)\psi_0(y))\|_X \leqslant M\mathrm{e}^{-\sigma_c t},\ \ \forall t > 0.$$

In this paper we also investigate the uniform boundedness and the long time behavior of the solution to the nonlocal problem (1.3) with more general nonnegative initial datum, where the nonnegative initial datum needs not to be a small perturbation of a cylinder wave. In Section 3, under the assumption of **(H1)**, by applying spectral expansion $u(t,x,y) = \sum_{j=0}^{\infty} v_j(t,x)\psi_j(y)$, we investigate the related Cauchy problem of the coupled system of $v_j(t,x)$, and by detailed spectral analysis and applying comparison principle to some auxiliary linear evolutional models, we can prove that the boundedness and the long time behavior of the solution $u(t,x,y)$ to the nonlinear problem (1.3) are determined by that of $v_0(t,x) = \int_\Omega u(t,x,y)\psi_0(y)\mathrm{d}y$, then by investigating the Cauchy problem of $v_0(t,x)$ with $x \in \mathbb{R}$, we can prove that $\|v_0(t,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})}$ and $\|u(t,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)}$ are uniformly bounded in time for any nonnegative initial data, which can be stated as follows.

Theorem 2. *There exist positive constants δ_0, M_0 and M, such that for any given nonzero and nonnegative bounded initial datum $u_0(x,y) \in \mathrm{L}_\infty(\mathbb{R}\times\Omega)$, there exists a unique global positive solution $u(t,x,y)$ of (1.3), which is also uniformly bounded in time and satisfies*

$$\|u(t,x,y) - v_0(t,x)\psi_0(y)\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)} \leqslant M\mathrm{e}^{-\delta_0 t}(\|u_0\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)} + 1),\ \ t > 0, x \in \mathbb{R}, \tag{1.6}$$

and

$$0 < v_0(t,x) \leqslant M_0(\|u_0\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)} + 1),\ \ t > 0, x \in \mathbb{R}, \tag{1.7}$$

where $v_0(t,x)$ satisfies the following initial value problem

$$\begin{cases} \frac{\partial}{\partial t}v_0 - \frac{\partial^2}{\partial x^2}v_0 = (1-\lambda_0)[(1 - v_0 - b_0(t,x)]v_0, & t > 0,\ x \in \mathbb{R}, \\ v_0(0,x) = \langle u_0(x,\cdot), \psi_0(\cdot)\rangle, & x \in \mathbb{R}. \end{cases} \tag{1.8}$$

Furthermore, the term $b_0(t,x) = \frac{1}{(1-\lambda_0)}\int_\Omega K(y)(u(t,x,y) - v_0(t,x)\psi_0(y))\,dy$ decays exponentially in time, uniformly in x:

$$\sup_{x\in\mathbb{R}} |b_0(t,x)| \leqslant M\mathrm{e}^{-\delta_0 t}(\|u_0\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)} + 1),\ \ t > 0. \tag{1.9}$$

In Section 4, we further investigate the global asymptotic stability of cylinder waves and the asymptotic behavior of the solutions to (1.3) in the x direction as $t \to \infty$ for more general nonnegative initial data which can decay exponentially or vanishes at either ends. By virtue of Theorem 2, we focus on investigating the asymptotic behavior of $v_0(t,x) = \langle u(t,x,\cdot), \psi_0(\cdot)\rangle$ as $t \to +\infty$, where $v_0(t,x)$ satisfies a projected PDE incorporating a nonlocal dependence on $u(t,x,\cdot)$. After spectral expansion, this can be treated as a Fisher-KPP equation in one dimensional space $v_t - v_{xx} = (1-\lambda_0)v(1 - b_0(t,x) - v)$, with a nonlocal heterogeneous perturbation term $b_0(t,x) = \frac{1}{(1-\lambda_0)} \int_\Omega K(y)(u(t,x,y) - v_0(t,x)\psi_0(y))\,\mathrm{d}y$ satisfying $|b_0(t,x)| \leqslant M\mathrm{e}^{-\delta_0 t}$ for any $t > 0$ and $x \in \mathbb{R}$.

For the Cauchy problem of the classical Fisher-KPP model $u_t = \triangle_{x,y}u + u(1-u)$ in higher dimensional cylinder space or in one dimensional space, there is an extensive literature (see [13,18,21,25,33,34,37] for some classical results) which demonstrate how long time behavior of solutions can be classified in terms of the initial data. This includes the remarkable fact that the spreading speed of the solution in the positive x direction can be fully determined by the decaying rate of the initial datum $u_0(x,y)$ at $x = +\infty$. Recently for Fisher-type equations with some special types of heterogeneous resource terms depending only on $x - ct$ or t or x (e.g. periodic in t or x) or for some asymptotic Fisher equations as $x \to +\infty$, there are many interesting works on some new wave phenomena induced by heterogeneity and the long time behavior of solution especially when the initial datum has a compact support; see [22] for the case when the reaction term is in the form of $g(x)f(u)$ with $g(x)$ periodic in x, see [18,39] for the asymptotic Fisher equation in one or higher dimensional space with the reaction term $g(x,u) \to f(u)$ as $|x| \to \infty$, and see [16,24,26] for the case when the reaction term is in the form of $u(a(x-ct) - u)$. However for the Fisher-KPP equation with more general heterogenous resource term $b(t,x)$ or with nonlocal competition term, as far as we know, there are few works on the stability of waves or long time behavior of the solutions with general initial data, and it is not clear whether the spreading speed of the solution can still be determined by the decay rate of the initial datum.

Now we state our main results on the asymptotic behavior of solution as follows.

Theorem 3. *(Global asymptotic stability of cylinder waves with more general initial data) Assume **(H1)** holds and let* $\int_\Omega K(y)\psi_0(y)\,dy = 1 - \lambda_0$. *For any nonnegative initial datum* $u_0(x,y) \in \mathrm{L}_\infty(\mathbb{R} \times \Omega)$ *satisfying*

$$\lim_{x\to-\infty} \inf \int_\Omega u_0(x,y)\psi_0(y)\mathrm{d}y > 0 \text{ and } \lim_{x\to\infty} \mathrm{e}^{\sigma x} \int_\Omega u_0(x,y)\psi_0(y)\mathrm{d}y = r > 0, \tag{1.10}$$

with $0 < \sigma < \sqrt{1-\lambda_0}$, *(1.3) has a unique global solution* $u(t,x,y)$, *which satisfies*

$$\lim_{t\to\infty} \|u(t,z+ct,y) - V_c\left(z - \tfrac{1}{\sigma}\ln r\right)\psi_0(y)\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)} = 0, \tag{1.11}$$

where $c = \sigma + \frac{\sqrt{1-\lambda_0}}{\sigma} \in (2\sqrt{1-\lambda_0}, \infty)$, *and* $V_c(z)$ *is the unique planar wave solution of* (1.5) *satisfying* $\lim_{z\to+\infty} \mathrm{e}^{\sigma z} V_c(z) = 1$.

Theorem 4. *(Global exponential stability of cylinder waves in exponentially weighted space) Under the assumption of Theorem 3, if the initial datum* $u_0(x,y)$ *satisfies, for some* $a > \sigma$ *with* $0 < a - \sigma \ll 1$,

$$\int_\Omega u_0(x,y)\psi_0(y)dy \sim r\mathrm{e}^{-\sigma x} + O(\mathrm{e}^{-ax}),\ x \to +\infty,\ r > 0,\ \ 0 < \sigma < \sqrt{1-\lambda_0},$$

then there exist positive constants M and δ_a such that the problem (1.3) admits a unique global solution $u(t, z+ct, y)$ satisfying

$$\|(1+e^{az})(u(t,z+ct,y) - V_c(z - \tfrac{1}{\sigma}\ln r)\psi_0(y))\|_{L_\infty(\mathbb{R}\times\Omega)} \leqslant M\mathrm{e}^{-\delta_a t},\ t > 0.$$

Theorem 5. *Under the assumption of **(H1)**, if the initial datum $u_0(x, y)$ is nonnegative and has a compact support in the cylinder, then there exist two functions $\xi_-(t)$ and $\xi_+(t)$ such that the solution $u(t, x, y)$ of (1.3) satisfies*

$$\begin{aligned} &\lim_{t\to+\infty} \|u(t,x,y) - V_{c^*}(x - \xi_+(t))\psi_0(y)\|_{\mathrm{L}_\infty(\mathbb{R}^+\times\Omega)} = 0, \\ &\lim_{t\to+\infty} \|u(t,x,y) - V_{c^*}(-x - \xi_-(t))\psi_0(y)\|_{\mathrm{L}_\infty(\mathbb{R}^-\times\Omega)} = 0, \end{aligned} \tag{1.12}$$

with $c^* = 2\sqrt{1-\lambda_0}$. In addition, there exists a positive constant C such that

$$\left|\xi_\pm(t) - 2\sqrt{1-\lambda_0}t + \frac{3}{2}\log t\right| \leqslant C,\ \text{ for } t >> 1.$$

This paper is organized as follows. In section 2, by applying the spectral analysis we prove the spectral and local exponential stability of the cylinder waves with noncritical speeds to the problem (1.3) in some weighted spaces. In section 3, by combining the spectral expansion method and the detailed asymptotic analysis with the sub-supersolution method, we prove the uniform boundedness of the solution to the problem (1.3) in time for any nonnegative initial datum. In Section 4, we investigate the long time behavior of the solution with more general bounded initial datum, which decays with some exponential rates at one end or with a compact support, and prove Theorems 3-5.

2. Local exponential stability of cylinder waves in some weighted spaces

In this section, we investigate the spectral and local exponential stability of cylinder wave solution $\phi_c(x-ct, y)$ with $c > c^*$ for the problem (1.3) in some appropriate spaces, where the cylinder wave solution $\phi_c(\xi, y)$ satisfies the following boundary value problem

$$\begin{cases} \triangle_{\xi,y}\phi_c(\xi,y) + c\dfrac{\partial}{\partial\xi}\phi_c(\xi,y) - g(y)\phi_c(\xi,y) \\ + \left[1 - \displaystyle\int_\Omega K(y')\phi_c(\xi,y')\mathrm{d}y'\right]\phi_c(\xi,y) = 0, & (\xi,y)\in\mathbb{R}\times\Omega, \\ \dfrac{\partial}{\partial\nu}\phi_c(\xi,y) = 0, & (\xi,y)\in\mathbb{R}\times\partial\Omega, \\ \displaystyle\lim_{\xi\to+\infty}\phi_c(\xi,\cdot) = 0, \liminf_{\xi\to-\infty}\phi_c(\xi,\cdot) > 0. \end{cases} \tag{2.1}$$

Let λ_0 be the principal eigenvalue of $-\Delta_y + g(y)$ in Ω under homogeneous Neumann boundary condition on $\partial\Omega$. It is proved in [4] that if $\lambda_0 < 1$ then for any $c \geq 2\sqrt{1-\lambda_0}$, (2.1) has a unique positive bounded solution with separate variable expression $\phi_c(\xi, y) = V_c(\xi)\psi_0(y)$, where under the assumption $\int_\Omega \psi_0(y)K(y)\mathrm{d}y = 1 - \lambda_0$ (after the re-scaling of $\phi_c(\xi, y)$), $V_c(\xi)$ satisfies

$$\begin{cases} V_c''(\xi) + cV_c'(\xi) + (1-\lambda_0)(1-V_c(\xi))V_c(\xi) = 0, \ \ \xi \in \mathbb{R}, \\ V_c(-\infty) = 1, \ V_c(+\infty) = 0. \end{cases} \tag{2.2}$$

It is well known that for $c \geqslant 2\sqrt{1-\lambda_0}$, the planar wave solution $V_c(\xi)$ of (2.2) decays exponentially at both ends uniformly in $y \in \Omega$ and satisfies

$$\begin{cases} \text{if } c \geqslant c^*, \ V_c(\xi) - 1 \sim \mathrm{e}^{\mu^+\xi}, & \text{as } \xi \to -\infty, \\ \text{if } c > c^*, \ V_c(\xi) \sim \mathrm{e}^{-\sigma^-\xi}, & \text{as } \xi \to +\infty, \\ \text{if } c = c^*, \ V_c(\xi) \sim \xi\mathrm{e}^{-\sigma^+\xi}, & \text{as } \xi \to +\infty, \end{cases} \tag{2.3}$$

where

$$\mu^+ = \frac{-c + \sqrt{c^2 + 4(1-\lambda_0)}}{2} > 0, \ \ \sigma^\pm = \frac{c \pm \sqrt{c^2 - 4(1-\lambda_0)}}{2} > 0.$$

In moving coordinate (ξ, y, t) ($\xi = x - ct$) the initial boundary value problem (1.3) can be rewritten as follows

$$\begin{cases} \partial_t u = \triangle_{\xi,y} u + c\partial_\xi u - g(y)u \\ \qquad + \left[1 - \int\limits_\Omega K(y')u(t,\xi,y')\mathrm{d}y'\right]u, & t > 0, (\xi, y) \in \mathbb{R} \times \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & t > 0, \ \ (\xi, y) \in \mathbb{R} \times \partial\Omega, \\ u(0, \xi, y) = u_0(\xi, y), & (\xi, y) \in \mathbb{R} \times \Omega. \end{cases} \tag{2.4}$$

To prove the local asymptotic stability of the cylinder waves in some appropriate space, we first investigate the following linearized evolutional equation of (2.4) around the cylinder wave $\phi_c(\xi, y)$

$$\begin{aligned} \partial_t v = \mathcal{L}_c\, v \triangleq \triangle_{\xi,y} v + c\partial_\xi v - g(y)v + \left(1 - \int\limits_\Omega K(y')\phi_c(\xi, y')\mathrm{d}y'\right) v \\ - \phi_c \int\limits_\Omega K(y')v(t,\xi,y')\mathrm{d}y'. \end{aligned} \tag{2.5}$$

It is easy to check that the operator $\mathcal{L}_c$ generates an analytic semigroup in the Banach space $L_2(\Sigma)$, $\Sigma = \mathbb{R} \times \Omega$ with the domain $D(\mathcal{L}_c) = \mathrm{H}^2_\nu(\Sigma)$, and respectively in the Banach space $\mathrm{C}_{\mathrm{unif}}(\overline{\Sigma})$ with domain $D(\mathcal{L}_c) = X^2$ given by

$$X^2 = \left\{ u \in \mathrm{C_{unif}}(\overline{\Sigma}) \bigcap (\bigcap_{q \geq 1} W^{2,q}_{\mathrm{loc}}(\Sigma)),\ \triangle_{x,y} u \in \mathrm{C_{unif}}(\overline{\Sigma}),\ \text{and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Sigma \right\}.$$

By applying the analytic semigroup theories and stability theories of traveling waves, to prove the local exponential stability/instability of cylinder waves in space $X = \mathrm{C_{unif}}(\overline{\Sigma})$ or $\mathrm{H}^k(\Sigma)$, it suffices to investigate the spectral distribution of the linear operator $\mathcal{L}_c$ in X or $\mathrm{H}^k(\Sigma)$.

For convenience of our investigation on the nonlinear local stability of the waves, in the following of this paper we choose the working space of $\mathcal{L}_c$ as $X = C_{\mathrm{unif}}(\overline{\Sigma})$, with domain $D(\mathcal{L}_c) = X^2$.

Let $\sigma(\mathcal{L}_c)$ be the spectral set of $\mathcal{L}_c$ in X, $\sigma_n(\mathcal{L}_c)$ the set consisting of the isolated eigenvalues of $\mathcal{L}_c$ with finite algebraic multiplicity and $\sigma_{ess}(\mathcal{L}_c) = \sigma(\mathcal{L}_c) \backslash \sigma_n(\mathcal{L}_c)$ the essential spectral set of $\mathcal{L}_c$.

2.1. Location of $\sigma_{ess}(\mathcal{L}_c)$

By applying the essential spectral theories to the elliptic operator $\mathcal{L}_c$ in $\mathrm{C_{unif}}(\overline{\Sigma})$ (see [32]) or in $\mathrm{H}^k(\Sigma)$ (see [38]), it is known that the boundaries of the essential spectra of $\mathcal{L}_c$ are determined by the location of the spectra of the limiting operators $\mathcal{L}_c^{\pm}$ of $\mathcal{L}_c$ as $\xi \to \pm\infty$, with $\mathcal{L}_c^{\pm}$ defined by

$$\begin{aligned} &\mathcal{L}_c^{+} u \triangleq \triangle_{\xi,y} u + c\partial_\xi u - g(y)u + u, \quad u \in X^2, \\ &\mathcal{L}_c^{-} u \triangleq \triangle_{\xi,y} u + c\partial_\xi u - g(y)u - \psi_0(y) \textstyle\int_\Omega K(y')u(\xi,y')\mathrm{d}y' + \lambda_0 u, \quad u \in X^2, \end{aligned} \tag{2.6}$$

where $(\lambda_0, \psi_0(y))$ is defined in Section 1 and $\int_\Omega K(y')\psi_0(y')\mathrm{d}y' = 1 - \lambda_0$.

Without loss of generality, we investigate the essential spectral set of $\mathcal{L}_c$ in $\mathrm{C_{unif}}(\overline{\Sigma}) \cap L_2(\Sigma)$, after applying Fourier transform to $\mathcal{L}_c^-$ and $\mathcal{L}_c^+$ with respect to ξ, in the following we first investigate the location of eigenvalues of the corresponding operators $\widehat{\mathcal{L}_c^-}$ and $\widehat{\mathcal{L}_c^+}$ with a parameter τ, i.e. the following eigenvalue problems

$$\begin{aligned} &\lambda^-(\tau)v(y) = \widehat{\mathcal{L}_c^-} v(y) \\ &\triangleq \triangle_y v(y) - g(y)v(y) + (-\tau^2 + ic\tau + \lambda_0)v(y) - (\textstyle\int_\Omega K(y')v(y')\mathrm{d}y')\psi_0(y), \end{aligned} \tag{2.7}$$

and

$$\lambda^+(\tau)v(y) = \widehat{\mathcal{L}_c^+} v(y) \triangleq \triangle_y v(y) - g(y)v(y) + (-\tau^2 + ic\tau + 1)v(y), \tag{2.8}$$

with an eigenfunction $v(y) \in \mathrm{H}^2_\nu(\Omega)$.

For any given parameter $\tau \in \mathbb{R}$, let $\lambda^-(\tau)$ be an eigenvalue of (2.7) with an eigenfunction $v(y)$, note that we can represent the nonzero function $v(y)$ as

$$v(y) = \sum_{k=0}^{\infty} c_k \psi_k(y), \quad \text{with constant } c_{k_0} \neq 0, \text{ for some } k_0 \geq 0. \tag{2.9}$$

Substituting (2.9) into (2.7), we have

$$\begin{aligned}\lambda^-(\tau)\sum_{k=0}^{\infty}c_k\psi_k(y)=&\sum_{k=0}^{\infty}c_k(\Delta_y-g(y))\psi_k(y)+(-\tau^2+ic\tau+\lambda_0)\sum_{k=0}^{\infty}c_k\psi_k(y)\\&-\sum_{k=0}^{\infty}c_k\psi_0(y)\int_{\Omega}K(y')\psi_k(y')\mathrm{d}y'.\end{aligned}\tag{2.10}$$

For the case when there exists some $k_0 \geq 1$ such that $c_{k_0} \neq 0$ in (2.9), multiplying (2.10) by $\psi_{k_0}(y)$ and integrating on Ω, it yields

$$\lambda^-(\tau) = -\lambda_{k_0} - \tau^2 + ic\tau + \lambda_0, \text{ for some } k_0 \geq 1. \tag{2.11}$$

For the remaining case when the eigenfunction $v(y) = \psi_0(y)$, multiplying (2.10) by $\psi_0(y)$ and integrating on Ω, we have

$$\lambda^-(\tau) = -\tau^2 + ic\tau - 1 + \lambda_0. \tag{2.12}$$

(2.11) and (2.12) imply that there exists $\delta_0 \geq \min\{1-\lambda_0, \lambda_1 - \lambda_0\} > 0$ such that for any given $\tau \in \mathbb{R}$ all the eigenvalues of (2.7) with the parameter τ denoted by $\lambda^-(\tau)$ satisfy $\mathrm{Re}\,\lambda^-(\tau) \leq -\delta_0 < 0$. Thus

$$\sigma(\mathcal{L}_c^-) \subset \{\mathrm{Re}\,\lambda \leq -\delta_0 < 0\}. \tag{2.13}$$

It is easy to see that $\lambda^+(\tau)$ is an eigenvalue of (2.8) with the parameter τ, if and only if

$$-\tau^2 + ic\tau + 1 - \lambda^+(\tau) = \lambda_k, \text{ for some } k \geq 0,$$

thus

$$\sup\{\mathrm{Re}\,\lambda^+(\tau), \tau \in \mathbb{R}\} = 1 - \lambda_0 > 0, \tag{2.14}$$

where $\lambda^+(0) = 1 - \lambda_0$ with the eigenfunction $\psi_0(y)$.

The fact $\sigma(\mathcal{L}_c^+)\bigcap\{\mathrm{Re}\,\lambda > 0\} \neq \emptyset$ further means

$$\sigma_{\mathrm{ess}}(\mathcal{L}_c)\bigcap\{\mathrm{Re}\,\lambda > 0\} \neq \emptyset,$$

which is also true when the working space of $\mathcal{L}_c$ is $\mathrm{C}_{\mathrm{unif}}(\overline{\Sigma})$ or $L_2(\Sigma)$, thus for any $c \geq c^* = 2\sqrt{1-\lambda_0}$ the cylinder waves $\phi_c(x - ct, y)$ are spectrally unstable and nonlinearly unstable in $\mathrm{C}_{\mathrm{unif}}(\overline{\Sigma})$ or in $\mathrm{H}^k(\Sigma)$.

In the following we try to prove that the cylinder waves $V_c(x - ct)\psi_0(y)$ with noncritical speed $c > 2\sqrt{1-\lambda_0}$ are spectrally stable and nonlinearly exponentially stable in some exponentially weighted spaces of X with an exponential weight near $\xi = +\infty$. Let $w_a(\xi) = 1 + \mathrm{e}^{a\xi}$, define the exponentially weighted space X_a by

$$X_a = \{u(\xi, y) : w_a(\xi)u(\xi, y) \in X, \ \|u\|_{X_a} = \|w_a u\|_X < \infty\}, \tag{2.15}$$

and we can define the associated exponentially weighted space of X^2 similarly, which is denoted by X_a^2.

Define the operator $\mathcal{L}_{c,a} : X_a^2 \to X_a$ as the restriction of $\mathcal{L}_c$ on X_a^2, and defined $\tilde{\mathcal{L}}_{c,a} : X^2 \to X$ as $\tilde{\mathcal{L}}_{c,a} v(\xi, y) = w_a(\xi)\mathcal{L}_c(w_a^{-1}(\xi)v(\xi, y))$ for $v(\xi, y) \in X^2$, obviously

$$\sigma_{\text{ess}}(\tilde{\mathcal{L}}_{c,a}) = \sigma_{\text{ess}}(\mathcal{L}_{c,a}), \sigma_n(\tilde{\mathcal{L}}_{c,a}) = \sigma_n(\mathcal{L}_{c,a}),$$

and $\|(\lambda I - \tilde{\mathcal{L}}_{c,a})^{-1}\|_{X\to X} = \|(\lambda I - \mathcal{L}_{c,a})^{-1}\|_{X_a \to X_a}$.

For $a > 0$ it is easy to check that the limiting operator of $\tilde{\mathcal{L}}_{c,a}$ as $\xi \to -\infty$ is still $\mathcal{L}_c^-$, while the limiting operator of $\tilde{\mathcal{L}}_{c,a}$ as $\xi \to +\infty$ denoted by $\tilde{\mathcal{L}}_{c,a}^+$ has the following expression

$$\tilde{\mathcal{L}}_{c,a}^+ v = \triangle_{\xi,y} v - 2av_\xi + a^2 v + cv_\xi - cav - g(y)v + v, \quad v \in X^2.$$

To obtain the location of $\sigma_{\text{ess}}(\mathcal{L}_{c,a})$, it remains to investigate the location of $\sigma(\tilde{\mathcal{L}}_{c,a}^+)$, by applying Fourier transform to $\tilde{\mathcal{L}}_{c,a}^+$ with respect to ξ, we investigate the following eigenvalue problem with a parameter $\tau \in \mathbb{R}$

$$\lambda^+(\tau)v(y) = \widehat{\tilde{\mathcal{L}}_{c,a}^+} v \triangleq \Delta_y v(y) - g(y)v(y) + (-\tau^2 + ic\tau - 2ia\tau + a^2 - ca + 1)v(y), \quad (2.16)$$

under the zero Neumann boundary condition on $\partial\Omega$.

Obviously for any given $\tau \in \mathbb{R}$, $\lambda^+(\tau)$ is an eigenvalue of (2.16) if and only if

$$\lambda^+(\tau) = -\tau^2 + ic\tau - 2ia\tau + a^2 - ca + 1 - \lambda_k, \quad \text{for some } k \ge 0. \quad (2.17)$$

For any given $c > 2\sqrt{1-\lambda_0}$, if we choose $a > 0$ satisfying

$$\frac{c - \sqrt{c^2 - 4(1-\lambda_0)}}{2} < a < \frac{c + \sqrt{c^2 - 4(1-\lambda_0)}}{2}, \quad (2.18)$$

then by (2.17) it follows that there exists a positive constant δ_+ depending only on a and c such that for any given $\tau \in \mathbb{R}$ it holds that

$$\text{Re}\,\lambda^+(\tau) < -\delta_+ < 0,$$

which with the location of $\sigma(\mathcal{L}_c^-)$ in (2.13) guarantees that

$$\sup\{\text{Re}\,\{\sigma_{ess}(\mathcal{L}_{c,a})\}\} \leqslant -\delta < 0, \quad \delta = \min\{\delta_+, \lambda_1 - \lambda_0, 1 - \lambda_0\} > 0.$$

Thus we have the following spectral result.

Lemma 2.1. *For any given $c > c^*$ and $a > 0$ satisfying (2.18), let $\mathcal{L}_{c,a}$ be the restriction of $\mathcal{L}_c$ on the weighted space X_a, with a weight function defined by $w_a(x) = 1 + \mathrm{e}^{ax}$, then there exists a small enough $\delta > 0$ such that*

$$\sup\{\text{Re}\,\{\sigma_{ess}(\mathcal{L}_{c,a})\}\} \leqslant -\delta < 0. \quad (2.19)$$

2.2. Location of isolated eigenvalues of $\mathcal{L}_{c,a}$

By Lemma 2.1, to prove the spectral stability and the nonlinear exponential stability of the cylinder wave $V_c(x-ct)\psi_0(y)$ with $c>c^*$ in the weighted space X_a, it remains to prove the non-existence of unstable eigenvalues of $\mathcal{L}_{c,a}$. For this purpose, in this subsection we investigate the location of eigenvalues of $\mathcal{L}_{c,a}$ in the range $\Omega_\delta=\{\lambda\in\mathbf{C}:\mathrm{Re}\,\lambda\geqslant-\delta/2\}$ with small enough $\delta>0$ satisfying (2.19).

Consider the eigenvalue problem

$$\begin{aligned}\lambda u(\xi,y)=\mathcal{L}_{c,a}u(\xi,y)&\triangleq\triangle_{\xi,y}u+c\partial_\xi u-g(y)u+(1-V_c(\xi)(1-\lambda_0))\,u\\&-\psi_0(y)V_c(\xi)\int_\Omega K(y')u(\xi,y')\mathrm{d}y'\end{aligned}\tag{2.20}$$

with an eigenvalue λ satisfying $\mathrm{Re}\,\lambda\geqslant-\delta/2$ and an eigenfunction $u(\xi,y)\in X_a^2$.

We express the eigenfunction $u(\xi,y)\in X_a^2$ of (2.20) by spectral expansion

$$u(\xi,y)=\sum_{i=0}^{\infty}v_i(\xi)\psi_i(y)\tag{2.21}$$

with $\psi_i(y)$ defined as in Section 1.

Substituting (2.21) into (2.20), it is easy to check that if λ is an eigenvalue of (2.20) with $\mathrm{Re}\,\lambda\geqslant-\delta/2$, then there exists some $k\geq 0$ such that $v_k(\xi)\not\equiv 0$ in (2.21) and $(\lambda,v_k(\xi))$ must be an eigenpair of the following eigenvalue problem

$$v_k''(\xi)+cv_k'(\xi)+[1-\lambda_k-(1-\lambda_0)V_c(\xi)]v_k(\xi)=\lambda v_k(\xi),\ \ \text{if } v_k\neq 0,\ \ \text{for some } k\geq 1;\tag{2.22}$$

or

$$v_0''(\xi)+cv_0'(\xi)+[1-\lambda_0-2(1-\lambda_0)V_c(\xi)]v_0(\xi)=\lambda v_0(\xi),\tag{2.23}$$

if $v_0(\xi)\psi_0(y)$ is an eigenfunction of (2.20).

Theorem 2.1. *For any given $c>2\sqrt{1-\lambda_0}$ and a satisfying (2.18), let $\delta>0$ be small enough chosen as in Lemma 2.1.*
(i) If λ is an eigenvalue of $\mathcal{L}_{c,a}$ with $\mathrm{Re}\,\lambda\geq-\delta/2$, then λ must be real and the eigenfunction must be in the form of $v_0(\xi)\psi_0(y)$.
(ii) There exists small enough $\delta_{c,a}>0$ such that there is no eigenvalue of $\mathcal{L}_{c,a}$ with $\mathrm{Re}\,\lambda\geqslant-\delta_{c,a}$.

Proof. Let λ be an isolated eigenvalue of $\mathcal{L}_{c,a}$ with an eigenfunction $u(\xi,y)\in X_a^2$ expressed by (2.21), and assume $\mathrm{Re}\,\lambda\geq-\delta/2$ for small enough $\delta>0$ chosen as in Lemma 2.1.

We first assume that there exists some $k\geq 1$ such that $v_k(\xi)\neq 0$ in (2.21), i.e. $(\lambda,v_k(\xi))$ is an eigenpair of (2.22) with $v_k(\xi)(1+\mathrm{e}^{a\xi})\in C_{\mathrm{unif}}(\mathbb{R})$. Using the fact that for any $\mathrm{Re}\,\lambda\geq-\delta/2$

$$\mathrm{Re}\left(\frac{c-\sqrt{c^2+4(\lambda+\lambda_k-1)}}{2}\right)<a<\mathrm{Re}\left(\frac{c+\sqrt{c^2+4(\lambda+\lambda_k-1)}}{2}\right),$$

then by applying the classical asymptotic analysis to (2.22) it holds that

$$v_k(\xi) \sim C_k \exp\left\{\frac{-c-\sqrt{c^2+4(\lambda+\lambda_k-1)}}{2}\right\}, \text{ as } \xi \to +\infty, \text{ if } \operatorname{Re}\lambda \geq -\delta/2, \tag{2.24}$$

for some $C_k > 0$.

Let $\widetilde{v}_k(\xi) = \mathrm{e}^{\frac{c}{2}\xi} v_k(\xi)$, by (2.22) and (2.24), it is easy to check that $\widetilde{v}_k(\xi) \in \mathrm{H}^2(\mathbb{R})$ and satisfies the differential equation

$$\widetilde{v}_k''(\xi) + [-\frac{c^2}{4} + 1 - \lambda_k - (1-\lambda_0)V_c(\xi)]\widetilde{v}_k(\xi) = \lambda \widetilde{v}_k(\xi), \text{ for some } k \geq 1, \tag{2.25}$$

which means that λ must be a real eigenvalue of the differential operator $L_k = \frac{\partial^2}{\partial \xi^2} + b_k(\xi)$ with an eigenfunction $\widetilde{v}_k(\xi) = \mathrm{e}^{\frac{c}{2}\xi} v_k(\xi) \in \mathrm{H}^2(\mathbb{R})$ and note that

$$b_k(\xi) \triangleq -\frac{c^2}{4} + 1 - \lambda_k - (1-\lambda_0)V_c(\xi) < \lambda_0 - \lambda_1 \leq -\delta < 0, \ \forall k \geq 1, \ c > 2\sqrt{1-\lambda_0}. \tag{2.26}$$

(2.26) further implies that

$$\sigma\left(\frac{\partial^2}{\partial \xi^2} + b_k(\xi)\right) \subset (-\infty, -\delta], \ \forall k \geq 1,$$

which contradicts with the assumptions $\operatorname{Re}\lambda \geq -\delta/2$ and $v_k(\xi) \neq 0$ for some $k \geq 1$, this proves that if λ is an eigenvalue of $\mathcal{L}_{c,a}$ with $\operatorname{Re}\lambda \geqslant -\delta/2$, then the eigenfunction in X_a^2 must be in the form of $v_0(\xi)\psi_0(y)$ and $(\lambda, v_0(\xi))$ is an eigenpair of (2.23).

By applying nearly the same argument as above, it can be proved that the eigenvalue λ must be real and λ is an eigenvalue of L_0 with an eigenfunction $v_0 \in \mathrm{H}^2(\mathbb{R})$ and L_0 defined by

$$L_0 = \frac{\partial^2}{\partial \xi^2} - \frac{c^2}{4} + 1 - \lambda_0 - 2(1-\lambda_0)V_c(\xi).$$

Using the fact that

$$-\frac{c^2}{4} + 1 - \lambda_0 - 2(1-\lambda_0)V_c(\xi) \leq -\frac{c^2}{4} + 1 - \lambda_0 = -\delta_c < 0, \text{ for } c > 2\sqrt{1-\lambda_0},$$

which means that $\sigma(L_0) \subset (-\infty, -\delta_c]$, this completes the proof of Theorem 2.1 and Theorem 1 □

Remark 2.1. Note that the estimates (2.13) and (2.17) are still valid for the critical speed case $c = c^* = 2\sqrt{1-\lambda_0}$, thus if we choose $a = \sqrt{1-\lambda_0}$, then $\sigma_{\mathrm{ess}}(\mathcal{L}_{c^*,a}) \subset \{\operatorname{Re}\lambda < 0\} \cup \{0\}$, and it can be further proved that $\sigma(\mathcal{L}_{c^*,a}) \setminus \{0\} \subset \{\operatorname{Re}\lambda < 0\}$ and zero is not an eigenvalue of $\mathcal{L}_{c^*,a}$, but $0 \in \sigma_{\mathrm{ess}}(\mathcal{L}_{c^*,a})$, the above stated spectral results of $\mathcal{L}_{c^*,a}$ are nearly the same as that for the linearized operator around the planar wave front with the critical speed for Fisher equation $u_t = u_{xx} + (1-\lambda_0)u(1-u)$. By applying Green function method with detailed point-wise semigroup estimate, it was proved in [19] that for the Fisher equation $u_t = u_{xx} + (1-\lambda_0)u(1-u)$ if the small initial perturbation of $V_{c^*}(x)$ in X_a ($a = \sqrt{1-\lambda_0}$) decays faster than $x^{-2}\mathrm{e}^{-ax}$ at $x = +\infty$, then the solution tends to the planar wave $V_{c^*}(x - c^*t)$ in X_a and the perturbation of the wave decays algebraically in time. However in the multi-dimensional cylinder case, even for the classical nonlinear parabolic equation, it is still an open problem whether the above mentioned weak spectral stability of the cylinder wave with the critical speed can still guarantee some types of asymptotically stability of the wave.

3. Uniform boundedness of solutions with more general initial data

In this section under the assumption of **(H1)**, we investigate the following initial boundary value problem

$$\begin{cases} u_t(t,x,y) - \triangle_{x,y}u(t,x,y) + g(y)u(t,x,y) \\ \quad = [1-m(t,x)]\,u(t,x,y), & (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega, \\ m(t,x) = \displaystyle\int_\Omega K(y')u(t,x,y')\mathrm{d}y', & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \dfrac{\partial u}{\partial \nu} = 0, & (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \partial\Omega, \\ u(0,x,y) = u_0(x,y), & (x,y) \in \mathbb{R} \times \Omega, \end{cases} \tag{3.1}$$

with a nonnegative bounded initial datum $u_0(x, y)$.

Lemma 3.1. *For any given nonzero and nonnegative initial datum $u_0 \in L_\infty(\mathbb{R} \times \Omega)$, the problem (3.1) admits a unique global positive classical solution $u(t,x,y) \in C^\infty(\mathbb{R}^+ \times \mathbb{R} \times \Omega)$, which satisfies*

$$0 < u(t,x,y) \leqslant \mathrm{e}^t ||u_0||_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)}, \;\; t > 0, \;\; (x,y) \in (\mathbb{R} \times \Omega). \tag{3.2}$$

Proof. By applying comparison principle to (3.1) in the linear form, obviously $u(t,x,y) > 0$ for any $t > 0$ and $(x,y) \in \mathbb{R} \times \Omega$. It is easy to see that $\mathrm{e}^t ||u_0||_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)}$ is a supersolution of the following linear initial boundary value problem

$$\begin{cases} w_t - \triangle_{x,y}w + g(y)w = w, \; (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0, & (t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \partial\Omega, \\ w(t,x,y) = u_0(x,y), \; (x,y) \in \mathbb{R} \times \Omega, \end{cases} \tag{3.3}$$

and $u(t,x,y)$ is a subsolution of (3.3), then by comparison principle we have

$$u(t,x,y) \leq \mathrm{e}^t ||u_0||_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)}, \;\; t > 0, \; (x,y) \in \mathbb{R} \times \Omega. \quad \square$$

By Lemma 3.1, we denote $m(t,x) = \int_\Omega K(y')u(t,x,y')\mathrm{d}y'$ and $v_j(t,x) = \langle u(t,x,y), \psi_j(y)\rangle = \int_\Omega u(x,y,t)\psi_j(y)\mathrm{d}y$, i.e. $u(t,x,y) = \sum_{j=0}^{\infty} v_j(t,x)\psi_j(y)$, then $v_j(t,x)$ $(j \geq 0)$ is the unique global solution of the following nonlinear initial value problem

$$\begin{cases} \frac{\partial}{\partial t}v_j - \frac{\partial^2}{\partial x^2}v_j + (\lambda_j - 1 + m(t,x))v_j = 0, \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v_j(0,x) = \int_\Omega u_0(x,y)\psi_j(y)\mathrm{d}y, \ x \in \mathbb{R}. \end{cases} \tag{3.4}$$

Lemma 3.2. *Let $(\lambda_j, \psi_j(y))$, $(j = 0, 1, \dots)$ be the eigenpair stated as in Section 1, there exists $J \geq 1$ such that $\lambda_j \geq 2$ for all $j \geq J+1$, and denote $u^\perp(t,x,y) = \sum\limits_{j=J+1}^{\infty} v_j(x,t)\psi_j(y)$. Then for any given nonnegative bounded initial datum $u_0(x,y)$, it holds that*

$$\sup_{x\in\mathbb{R}} ||u^\perp(t,x,\cdot)||_{\mathrm{L}^2(\Omega)} \leqslant \mathrm{e}^{-t}||u_0||_{\mathrm{L}_{\infty,2}(\mathbb{R}\times\Omega)}, \ \ t \geq 0. \tag{3.5}$$

Proof. Using the fact that $\lambda_j \geq 2$ for any $j \geq J+1$ and $m(t,x) \geq 0$ for $x \in \mathbb{R}$, it is easy to check that $\mathrm{e}^{-t}\|v_j(0,x)\|_{L_\infty(\mathbb{R})}$ and $-\mathrm{e}^{-t}\|v_j(0,x)\|_{L_\infty(\mathbb{R})}$ are super and subsolutions of (3.4) respectively for any $j \geq J+1$, thus

$$\sup_{x\in\mathbb{R}} |v_j(t,x)| \leq \mathrm{e}^{-t}\|v_j(0,x)\|_{L_\infty(\mathbb{R})}, \ \ t \geq 0, j \geq J+1,$$

and

$$\begin{aligned} \sup_{x\in\mathbb{R}} ||u^\perp(t,x,\cdot)||_{\mathrm{L}_2(\Omega)} &= \sup_{x\in\mathbb{R}} \sum_{j\geq J+1}^{+\infty} |v_j(t,x)| \leq \mathrm{e}^{-t} \sum_{j\geq J+1}^{+\infty} \|v_j(0,x)\|_{L_\infty(\mathbb{R})} \\ &\leq \mathrm{e}^{-t}||u_0||_{\mathrm{L}_{\infty,2}(\mathbb{R}\times\Omega)}, \ \ t \geq 0, j \geq J+1. \quad \square \end{aligned}$$

Next, we consider the finite sum $u(t,x,y) - u^\perp(t,x,y) = \sum\limits_{j=0}^{J} v_j(t,x)\psi_j(y)$, we only need to deal with the functions $v_j(t,x)$ for $j = 0, 1, \cdots, J$. Denote by $c_j = ||\frac{\psi_j}{\psi_0}||_{\mathrm{L}_\infty(\Omega)}$, then

$$|v_j(0,x)| \leqslant \int_\Omega u_0(x,y)|\psi_j(y)|\mathrm{d}y \leqslant c_j \int_\Omega u_0(x,y)\psi_0(y)\mathrm{d}y = c_j v_0(0,x). \tag{3.6}$$

By (3.6) and the fact $\lambda_j - \lambda_0 \geq \delta_0 > 0$ for $j \geq 1$, applying sub-supersolution method to the linear problem (3.4), it can be proved that

$$|v_j(t,x)| \leqslant c_j \mathrm{e}^{-(\lambda_j-\lambda_0)t} v_0(t,x) \leq c_j \mathrm{e}^{-\delta_0 t} v_0(t,x), \ \ x \in \mathbb{R}, \ t > 0, \ j \geq 1. \tag{3.7}$$

Define

$$\begin{aligned} m_0(t,x) &= v_0(t,x) \int_\Omega K(y)\psi_0(y)\mathrm{d}y = (1-\lambda_0)v_0(t,x), \\ m_j(t,x) &= v_j(t,x) \int_\Omega K(y)\psi_j(y)\mathrm{d}y, \ \ j = 1, 2, \cdots, J, \end{aligned} \tag{3.8}$$

then

$$|m_j(t,x)| \leqslant \int_\Omega K(y)|\psi_j(y)|\mathrm{d}y \cdot |v_j(t,x)| \leqslant (1-\lambda_0)c_j^2 \mathrm{e}^{-(\lambda_j-\lambda_0)t} v_0(t,x). \tag{3.9}$$

By Lemma 3.2 and (3.7)-(3.9), now we are ready to complete the proof of Theorem 2.

Proof of Theorem 2: Define

$$m^\perp(t,x) = \int_\Omega K(y)u^\perp(t,x,y)\mathrm{d}y,$$

then by Lemma 3.2 and (3.7), for any $t \geq 0$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} |m^\perp(t,x)| &\leqslant ||K||_{\mathrm{L}_2(\Omega)} ||u^\perp(t,x,\cdot)||_{\mathrm{L}_2(\Omega)} \\ &\leqslant \mathrm{e}^{-t} ||K||_{\mathrm{L}_2(\Omega)} ||u_0||_{\mathrm{L}_{\infty,2}(\mathbb{R}\times\Omega)}. \end{aligned} \tag{3.10}$$

Denote $b_0(t,x) = \frac{1}{(1-\lambda_0)}\big(m(t,x) - m_0(t,x)\big)$, then the equation of $v_0(t,x)$ in the system of (3.4) can be written as

$$\frac{\partial}{\partial t}v_0 - \frac{\partial^2}{\partial x^2}v_0 = (1-\lambda_0)(1 - b_0(t,x) - v_0)v_0, \ t > 0, x \in \mathbb{R}. \tag{3.11}$$

(3.9) and (3.10) imply that for any $t \geq 0$ and $x \in \mathbb{R}$,

$$\begin{aligned} |b_0(t,x)| &= \frac{1}{(1-\lambda_0)} \left| \sum_{j=1}^{\infty} v_j(t,x) \int_\Omega K(y)\psi_i(y)\mathrm{d}y \right| \\ &= \frac{1}{(1-\lambda_0)} \left| \sum_{j=1}^{J} m_j(t,x) + m^\perp(t,x) \right| \\ &\leqslant \sum_{j=1}^{J} c_j^2 \mathrm{e}^{-(\lambda_j-\lambda_0)t} |v_0(t,x)| + \frac{\|K\|_{\mathrm{L}_2(\Omega)}}{(1-\lambda_0)} \mathrm{e}^{-t} \|u_0(x,\cdot)\|_{\mathrm{L}_2(\Omega)}. \end{aligned} \tag{3.12}$$

In particular, there exists a positive constant C_0 independent of u_0 such that

$$|b_0(t,x)| \leq C_0 \mathrm{e}^{-\delta_0 t} \left(|v_0(t,x)| + \|u_0(x,\cdot)\|_{L_2(\Omega)}\right), \quad t \geq 0, x \in \mathbb{R}, \tag{3.13}$$

with $\delta_0 = \min\{1, \lambda_1 - \lambda_0\} > 0$.

Next, we claim that

$$\|v_0(t,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})} \leq C, \quad \text{uniformly in } t \geq 0. \tag{3.14}$$

By Lemma 3.1 the assertion holds for finite t. For t large enough, we set $T >> 1$ such that

$$C_0 \mathrm{e}^{-\delta_0 T} \leqslant \frac{1}{2}, \tag{3.15}$$

then (3.11) and (3.12) yield that

$$\begin{aligned}\frac{\partial}{\partial t}v_0-\frac{\partial^2}{\partial x^2}v_0&=(1-\lambda_0)(1-b_0(t,x)-v_0)v_0\\&\leq(1-\lambda_0)\left[1-\frac{1}{2}v_0(t,x)+\frac{1}{2}\|u_0(x,\cdot)\|_{\mathrm{L}_2(\Omega)}\right]v_0(t,x),\ t\geq T,x\in\mathbb{R}.\end{aligned}\tag{3.16}$$

It follows from the maximum principle that

$$\|v_0(t,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})}\leqslant\max\{\|v_0(0,\cdot)\|_{\mathrm{L}_\infty(\mathbb{R})},2+\|u_0(\cdot,\cdot)\|_{\mathrm{L}_{\infty,2}(\mathbb{R}\times\Omega)}\},\ \ t\geqslant T.\tag{3.17}$$

This proves (3.14). Lemma 3.2 and (3.7) also imply that

$$\begin{aligned}&\|u(t,x,\cdot)-v_0(t,x)\psi_0(\cdot)\|_{\mathrm{L}_2(\Omega)}\\&\leqslant\sum_{j=1}^{J}c_j\mathrm{e}^{-\delta_0 t}|v_0(t,x)|+\mathrm{e}^{-t}\|u_0(x,\cdot)\|_{\mathrm{L}_2(\Omega)}\leq C\mathrm{e}^{-\delta_0 t},\quad t\geqslant 0,x\in\mathbb{R},\end{aligned}\tag{3.18}$$

which with (3.14) further implies

$$\sup_{t>0}\|u(t,\cdot,\cdot)\|_{\mathrm{L}_{\infty,2}(\mathbb{R}\times\Omega)}\leq M_0.\tag{3.19}$$

By virtue of (3.19), the nonlinear equation (3.1) can be written in the form of a linear heterogeneous parabolic equation $u_t=\triangle_{x,y}u-u+f(t,x,y)$ with $f(t,x,y)=u(2-g(y)-m(t,x))$ satisfying $\|f(t,x,y)\|_{\mathrm{L}_\infty(\mathbb{R}^+,\mathrm{L}_{\infty,2}(\mathbb{R}\times\Omega))}\leq M_1$, and note that $\sigma(L_0)\subset\{\mathrm{Re}\,\lambda\leq-1\}$ with $L_0=\triangle_{x,y}-I$, and

$$\|\mathrm{e}^{L_0 t}\|_{L_p(\mathbb{R}\times\Omega)\to W_p^1(\mathbb{R}\times\Omega)}\leq C_p t^{-1/2}e^{-1/2t},\ \ t>0,\ 1<p<+\infty,\tag{3.20}$$

then by the decay estimate (3.20) and by applying a recursive argument to (3.1) it is easy to show that there exist positive constants θ and C_θ such that for any $u_0\in\mathrm{L}_\infty(\mathbb{R}\times\Omega)$ the unique classical solution of u of (3.1) also satisfies

$$\|u(t,x,y)\|_{C^\theta(\mathbb{R}\times\overline{\Omega})}\leq C_\theta(\|u_0\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)}+M_0),\ \ t\geq 1.\tag{3.21}$$

Estimate (3.21) can be similarly proved by applying interior $W^{1,2}_{p,p}$ estimates and bootstrap argument. By interpolation, (3.18) can be improved to

$$\sup_{(x,y)\in\mathbb{R}\times\Omega}|u(t,x,y)-v_0(t,x)\psi_0(y)|\leqslant C'\mathrm{e}^{-\delta_0' t},\ \ t\geq 1,\tag{3.22}$$

for some positive constants C' and δ_0'. This proves Theorem 2.

4. Asymptotic behavior of solution with more general initial datum

By virtue of the uniform boundedness of the solution and the estimates (1.6)-(1.7) obtained in Theorem 2, to investigate the spreading speed and asymptotic behavior of the solution $u(t,x,y)$ in higher dimensional cylinder to the problem (3.1) with a more general initial datum, it suffices to investigate the long time behavior of $v_0(t,x) = \langle u(t,x,\cdot), \psi_0(\cdot)\rangle$ as $t \to +\infty$, where $v_0(t,x)$ satisfies the nonlinear equation (3.11) in one dimensional space, i.e. $v_t - v_{xx} = (1-\lambda_0)v(1-b_0(t,x)-v)$, with $b_0(t,x) = \frac{1}{1-\lambda_0}\int_\Omega K(y)(u(t,x,y)-v_0(t,x)\psi_0(y))\mathrm{d}y$. Due to the exponential decay in time of the coupled term $b_0(t,x)$ obtained in (3.13), the equation (3.11) of $v_0(t,x)$ can be treated as a Fisher-KPP equation with a heterogenous term $b_0(t,x)$.

In this section we shall focus on the investigation of the long time behavior of the solution of Fisher-KPP equation (3.11) with more general heterogenous resource term $b_0(t,x)$, using the decaying estimate (3.13) of $b_0(t,x)$ in time, we shall prove that for the more general initial datum the spreading speed of the solution to the problem (3.1) or the problem (3.11) is still determined by the decay rate of the initial datum and the solution may still tend to the wave with some noncritical speed or the critical speed in some appropriate sense.

4.1. Global asymptotic stability of the waves with the noncritical speeds

In this subsection we investigate the Cauchy problem of (3.11), i.e.

$$\begin{cases} v_t - v_{xx} = (1-\lambda_0)v(1-b_0(t,x)-v), & x\in\mathbb{R},\ t>0,\\ v(0,x) = v_0^*(x), & x\in\mathbb{R}. \end{cases} \tag{4.1}$$

For any given $c > 2\sqrt{1-\lambda_0}$, let $V_c(x-ct)$ be the traveling front solution connecting 1 and 0 of (the limiting problem of) (4.1) with $b_0(t,x)\equiv 0$ (as $t\to+\infty$), and without loss of generality, we choose $V_c(z)$ be the unique wave solution satisfying the following boundary value problem

$$\begin{cases} cV_c' + V_c'' + (1-\lambda_0)V_c(1-V_c) = 0, \ \ z\in\mathbb{R},\\ V_c(-\infty) = 1, \quad V_c(+\infty) = 0,\\ \lim\limits_{z\to+\infty} \mathrm{e}^{\sigma z}V_c(z) = 1, \ \text{ with } \sigma = \frac{c-\sqrt{c^2-4(1-\lambda_0)}}{2}. \end{cases} \tag{4.2}$$

Observe that $\sigma \mapsto c = \sigma + \frac{1-\lambda_0}{\sigma}$ is a bijection from $(0, \sqrt{1-\lambda_0})$ to $(2\sqrt{1-\lambda_0}, \infty)$.

For any given $c > 2\sqrt{1-\lambda_0}$ and $r>0$, let $\psi_c(x-ct; r)$ be the unique planar wave solution connecting r and 0 to the following boundary value problem:

$$\begin{cases} c\psi_c' + \psi_c'' + (1-\lambda_0)\psi_c(1-\frac{\psi_c}{r}) = 0, \ \ z\in\mathbb{R},\\ \psi_c(-\infty, r) = r, \quad \psi_c(+\infty, r) = 0,\\ \lim\limits_{z\to+\infty} \mathrm{e}^{\sigma z}\psi_c(z) = 1, \ \text{ with } \sigma = \frac{c-\sqrt{c^2-4(1-\lambda_0)}}{2}. \end{cases} \tag{4.3}$$

Obviously $\psi_c(z;1) = V_c(z)$ and $\psi_c(z;r) = rV_c(z-z_r)$ with $r\mathrm{e}^{\sigma z_r} = 1$.

In this subsection we always assume that the initial datum $v_0^*(x)$ is nonnegative, bounded and stays away from zero at $x=-\infty$, i.e.

$$0 < \underline{q}_0 < \liminf_{x\to-\infty} v_0^*(x) \le \limsup_{x\to-\infty} v_0^*(x) < \overline{q}_0; \tag{4.4}$$

and assume that the nonnegative bounded initial datum $v_0^*(x)$ decays to zero exponentially as $x \to +\infty$ with the same decay rate of a wave with a noncritical speed for (4.2) or (4.3), which means that for some $c > 2\sqrt{1-\lambda_0}$

$$\lim_{x\to+\infty} \frac{v_0^*(x)}{V_c(x+x_0)} = 1, \quad \text{for some } x_0 \in \mathbb{R}, \tag{4.5}$$

or equivalently and without loss of generality, we assume that the initial datum $v_0^*(x)$ satisfies the following decay estimate

$$\lim_{x\to+\infty} v_0^*(x)\mathrm{e}^{\sigma x} = 1, \quad \text{for some } \sigma \in (0, \sqrt{1-\lambda_0}). \tag{4.6}$$

For the heterogeneous term $b_0(t,x)$, we assume that

$$|b_0(t,x)| \le C_0\mathrm{e}^{-\delta t}(v(t,x) + \mathrm{e}^{-\sigma(x-ct)} \wedge 1), \qquad \text{for some } C_0, \delta, \sigma > 0. \tag{4.7}$$

Note that the decay estimate (3.13) implies (4.7).

Lemma 4.1. *Let $v(t,x)$ be a solution to (4.1) with an initial datum satisfying (4.6) for some $\sigma \in (0, \sqrt{1-\lambda_0})$. Assume in addition that $b_0(t,x)$ satisfies (4.7) for some positive constants C_0 and δ. Then the following statements hold true.*

(a) *For each $t > 0$, we have $\lim_{x\to+\infty} \mathrm{e}^{\sigma x} v(t,x) = \mathrm{e}^{\sigma ct}$, where $c = \sigma + \frac{1-\lambda_0}{\sigma}$.*
(b) *There exist positive constants t_1 and r_1 such that*

$$v(t,x) \ge r_1(\mathrm{e}^{-\sigma(x-ct)} \wedge 1), \qquad t \ge t_1,\ x \in \mathbb{R}.$$

(c) *For each $\varepsilon > 0$, there exists $t_2 > 0$ such that*

$$|b_0(t,x)| \le \frac{\varepsilon}{1+\varepsilon} v(t,x), \qquad t \ge t_2,\ x \in \mathbb{R}.$$

Proof. To prove (a), we first observe that $v_t - v_{xx} \le \kappa v$, where $\kappa = (1-\lambda_0)(1 + \|b_0\|_{L_\infty(\mathbb{R}^+\times\mathbb{R})})$. Hence, the comparison principle yields, for each $t > 0$,

$$0 \le v(s,x) \le \sup_{y\in\mathbb{R}} (\mathrm{e}^{\sigma y} v_0^*(y))\mathrm{e}^{-\sigma x + (\sigma^2+\kappa)s} \le C_t \mathrm{e}^{-\sigma x}, \qquad (s,x) \in [0,t] \times \mathbb{R}. \tag{4.8}$$

Next, observe that by Duhamel's principle:

$$v(t,x) = \mathrm{e}^{(1-\lambda_0)t}(p_t * v_0^*)(x) + E(t,x), \tag{4.9}$$

where $p_t(x) = p(t,x) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{\frac{-x^2}{4t}}$ and

$$E(t,x) = \int_0^t \mathrm{e}^{(1-\lambda_0)(t-s)} \int_{\mathbb{R}} p(t-s, x-x')(-b_0(s,x') - v(s,x'))v(s,x')\, dx' ds.$$

Thanks to (4.7) and (4.8), it follows that for each fixed $t > 0$,

$$|E(t,x)| \le C_t \int_0^t \mathrm{e}^{(1-\lambda_0)(t-s)} \int_{\mathbb{R}} p(t-s, x-x')\mathrm{e}^{-2\sigma x'}\, dx' ds \le C'_t \mathrm{e}^{-2\sigma x},$$

so that $\lim_{x\to+\infty} \mathrm{e}^{\sigma x}|E(t,x)| = 0$ for any $t > 0$. Therefore, using (4.6) and (4.9) again, we have

$$\lim_{x\to+\infty} \mathrm{e}^{\sigma x} v(t,x) = \lim_{x\to+\infty} \mathrm{e}^{\sigma x}\mathrm{e}^{(1-\lambda_0)t}(p_t * v_0^*)(x) = \mathrm{e}^{\sigma c t}, \quad t > 0. \tag{4.10}$$

To see the last equality, we note that $v_0^*(x) = \mathrm{e}^{-\sigma x}(1 + h(x))$ with $h(+\infty) = 0$, so that

$$\begin{aligned}
\mathrm{e}^{\sigma x}\mathrm{e}^{(1-\lambda_0)t}(p_t * v_0^*)(x) &= \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{\sigma x+(1-\lambda_0)t} \int_{\mathbb{R}} \mathrm{e}^{-\frac{(x-x')^2}{4t}} \mathrm{e}^{-\sigma x'}(1 + h(x'))\, dx' \\
&= \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{(1-\lambda_0)t} \int_{\mathbb{R}} \mathrm{e}^{-\frac{(x-x'-2\sigma t)^2}{4t}+\sigma^2 t}(1 + h(x'))\, dx' \\
&= \mathrm{e}^{(1-\lambda_0)t+\sigma^2 t} \int_{\mathbb{R}} p_t(x - x' - 2\sigma t)(1 + h(x'))\, dx' \\
&= \mathrm{e}^{\sigma c t} \int_{\mathbb{R}} p_t(\tilde{x})(1 + h(x - \tilde{x} - 2\sigma t))\, d\tilde{x}, \quad t > 0.
\end{aligned}$$

Then one can take $x \to +\infty$ in the above by the dominant convergence theorem to obtain the last equality in (4.10). This completes the proof of (a).

For (b), note that $v(t,x)$ satisfies

$$v_t - v_{xx} \ge (1-\lambda_0)v(1 - |b_0| - v) \ge (1-\lambda_0)v[1 - C_b\mathrm{e}^{-\delta t}(v + (\mathrm{e}^{-\sigma(x-ct)} \wedge 1) - v]. \tag{4.11}$$

By choosing $t_1 > 1$ large enough, we see that $v(t,x)$ is a supersolution of

$$w_t - w_{xx} = (1-\lambda_0)w[1 - \tfrac{\varepsilon}{2}\psi_c(x-ct;1) - (1+\tfrac{\varepsilon}{2})w], \quad t \ge t_1,\ x \in \mathbb{R}, \tag{4.12}$$

where we used

$$(\mathrm{e}^{-\sigma x} \wedge 1) \le B\psi_c(x;1) \quad \text{for some } B > 1. \tag{4.13}$$

Next, observe that $\underline{w}(t,x) = r\psi_c(x-ct;1)$ is a subsolution of (4.12) for any $r \in (0,1)$. Finally, we can choose $r = r(\varepsilon)$ small enough so that

$$v(t_1,x) \geq r\psi_c(x-ct_1;1), \quad x \in \mathbb{R},$$

which is due to (a) and $\lim_{x\to+\infty} \mathrm{e}^{\sigma x}\psi_c(x-ct_1;1) = \mathrm{e}^{\sigma ct_1}$. We can then conclude by the comparison principle that

$$v(t,x) \geq r\psi_c(x-ct;1) \geq \frac{r}{B}(\mathrm{e}^{-\sigma(x-ct)} \wedge 1), \quad t \geq t_1,\ x \in \mathbb{R}. \tag{4.14}$$

This proves (b). Assertion (c) follows from (4.7) and assertion (b). □

Theorem 4.1. *Let $v(t,x)$ be a solution to (4.1) with the initial datum $v_0^*(x)$ satisfying (4.4) and (4.6) for some $\sigma \in (0, \sqrt{1-\lambda_0})$. Suppose, in addition, that (4.7) holds, then*

$$\lim_{t\to\infty}\left[\sup_{z\in\mathbb{R}} |v(t,z+ct) - V_c(z)|\right] = 0. \tag{4.15}$$

Proof. Fix $\varepsilon > 0$, then by Lemma 4.1 (a) and (c), there exists $t_\varepsilon \geq 1$, such that

$$(1-\lambda_0)v\left(1-\frac{v}{1-\varepsilon}\right) \leq v_t - v_{xx} \leq (1-\lambda_0)v\left(1-\frac{v}{1+\varepsilon}\right), \quad t \geq t_\varepsilon,\ x \in \mathbb{R}, \tag{4.16}$$

and

$$\lim_{x\to+\infty} \mathrm{e}^{\sigma x} v(t,x) = \mathrm{e}^{\sigma ct}, \ \ t \geq 0. \tag{4.17}$$

By comparison principle, we have

$$\tilde{v}^-(t,x) \leq v(t,x) \leq \tilde{v}^+(t,x), \quad t \geq t_\varepsilon,\ x \in \mathbb{R},$$

where $\tilde{v}^\pm(t,x)$ are, respectively, the solutions of the Cauchy problem of the following classical Fisher equation

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = (1-\lambda_0)\tilde{v}\left(1-\frac{\tilde{v}}{1\pm\varepsilon}\right), & t \geq t_\varepsilon,\ x \in \mathbb{R}, \\ \tilde{v}(t_\varepsilon,x) = v(t_\varepsilon,x), & x \in \mathbb{R}. \end{cases} \tag{4.18}$$

Notice that by (4.17) and (4.18), the initial datum $\tilde{v}(t_\varepsilon,x)$ satisfies

$$\lim_{z\to+\infty} \mathrm{e}^{\sigma z}\tilde{v}(t_\varepsilon, z+ct_\varepsilon) = 1,$$

then by [37, Theorem 9.3] it follows that the solution $\tilde{v}^\pm(t,x)$ converges to the planar wave solution $\psi_c(x-ct;1\pm\varepsilon)$ of (4.18) uniformly in the moving coordinate $z = x-ct$ as $t \to +\infty$; precisely speaking, we have

$$\lim_{t\to+\infty} \sup_{z\in\mathbb{R}} |\tilde{v}^\pm(t,z+ct) - \psi_c(z;1\pm\varepsilon)| = 0,$$

where $\psi_c(z;r)$ is given in (4.3). Thus

$$\psi_c(z;1-\varepsilon) \le \liminf_{t\to+\infty} v(t,z+ct) \le \limsup_{t\to+\infty} v(t,z+ct) \le \psi_c(z;1-\varepsilon), z\in\mathbb{R}. \tag{4.19}$$

The proof is completed by letting $\varepsilon \searrow 0$ in (4.19). □

Obviously Theorem 3 follows from Theorem 2 and Theorem 4.1.

Remark 4.1. If the decay assumption (4.6) on the initial datum is weakened to

$$v_0^*(x) = \mathrm{e}^{-(\sigma+o(1))x}, \;\; x \to +\infty,$$

we conjecture that

$$\lim_{t\to\infty}\left[\sup_{x\in\mathbb{R}} |v(t,x) - \psi_c\,(x - ct + \xi(t))|\right] = 0, \tag{4.20}$$

where $\xi(t)$ is in general a bounded function.

Proof of Theorem 4: Under the assumption that the initial datum $u_0(x,y)$ satisfies the assumption (1.10) and

$$\int_\Omega u_0(x,y)\psi_0(y)\mathrm{d}y \sim r\mathrm{e}^{-\sigma x} + O(\mathrm{e}^{-ax}), \;\; \text{as } x\to+\infty, \tag{4.21}$$

for some $r>0$, $\sigma\in(0,\sqrt{1-\lambda_0})$ and $a>\sigma$, which means $u_0(x,y) - V_c(x - \frac{1}{\sigma}\ln r)\psi_0(y) \in X_a$, for $c = \sigma + \frac{1-\lambda_0}{\sigma} > 2\sqrt{1-\lambda_0}$ and $a>\sigma$. By virtue of the local exponential stability of the wave $V_c(x+x_0)\psi_0(y)$ in some weighted space X_a (see Theorem 1), it suffices to consider the case $r=1$ in (4.21) and prove that $\|u(t,z+ct,y) - V_c(z)\psi_0(y)\|_{X_a} \to 0$ as $t\to+\infty$ if $a-\sigma$ is small enough.

By Theorem 3, it is known that under the assumption (1.10),

$$\|u(t,z+ct,y) - V_c(z)\psi_0(y)\|_{\mathrm{L}_\infty(\mathbb{R}\times\Omega)} \to 0 \quad \text{as } t\to+\infty.$$

Denote $v(t,x) = \langle u(t,x,\cdot), \psi_0(\cdot)\rangle$, and let $\tilde v(t,z) = v(t,z+ct)$, then in the moving coordinate $z = x-ct$, $\tilde v(t,z)$ satisfies the following heterogeneous Fisher type equation:

$$\tilde v_t = \tilde v_{zz} + c\tilde v_z + (1-\lambda_0)\tilde v(1 - b_0(t,z+ct) - \tilde v),$$

then $\hat v(t,z) = \tilde v(t,z) - V_c(z)$, satisfies the nonlinear equation

$$\hat v_t = \hat v_{zz} + c\hat v_z + (1-\lambda_0)\hat v + F(t,z,\hat v),$$

with the initial datum $\hat v_0(z) = \int_\Omega u_0(z,y)\psi_0(y)\mathrm{d}y - V_c(z) \in \mathrm{L}_\infty(\mathbb{R})$, and $\hat v_0(z) = O(\mathrm{e}^{-az})$ for $z >> 1$ and $a>\sigma$, where

$$F(t,z,\hat{v}) = (1-\lambda_0)[-b_0(t,z+ct)(\hat{v}+V_c(z)) - \hat{v}^2(t,z) - 2V_c(z)\hat{v}].$$

Under the assumption (1.10), by (4.7) and Theorem 3 we know that

$$|F(t,z,\hat{v})| \leq C_t \mathrm{e}^{-2\sigma z} \wedge \eta(t), \ z \in \mathbb{R}, t > 0,$$

where $\eta(t) \to 0^+$ as $t \to +\infty$.

Note that

$$\hat{v}(t,z) = \mathrm{e}^{(1-\lambda_0)t} \int_{\mathbb{R}} p(t, z+ct-z')\hat{v}_0(z')dz' + \hat{F}(t,z,\hat{v}),$$

where $p(t,z) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{z^2}{4t}}$ and

$$\hat{F}(t,z,\hat{v}) = \int_0^t \mathrm{e}^{(1-\lambda_0)(t-s)} \int_{\mathbb{R}} p(t-s, z+ct-z') F(s,z',\hat{v}(s,z'))\, dz' ds.$$

Choosing $a \in (\sigma, \sigma_+)$, with $\sigma_+ = \frac{c+\sqrt{c^2-4(1-\lambda_0)}}{2} > \sqrt{1-\lambda_0} > \sigma$, by detailed computation it can be verified that

$$\mathrm{e}^{(1-\lambda_0)t} \int_{\mathbb{R}} p(t, z+ct-z')\mathrm{e}^{-az'} dz' = \mathrm{e}^{-\delta_a t} \mathrm{e}^{-az},$$

with $-\delta_a = a^2 - ca + 1 - \lambda_0 < 0$, if $a \in (\sigma, \sigma_+)$, and it can be proved that

$$|\hat{F}(t,z,\hat{v})| \leq C\mathrm{e}^{-2\sigma z}, \quad t \geq 0, z \geq 0,$$

thus for any given $\hat{v}_0 \in \mathrm{L}_\infty(\mathbb{R})$ satisfying $\hat{v}_0(z) = O(\mathrm{e}^{-az})$ for $z >> 1$ with $a \in (\sigma, \sigma_+)$ and $a < 2\sigma$, we have

$$\lim_{z \to +\infty} |\mathrm{e}^{az}\hat{v}(t,z)| \leq C_0 \mathrm{e}^{-\delta_a t} \|\mathrm{e}^{az}\hat{v}_0(z)\|_{\mathrm{L}_\infty(\mathbb{R})}, \ t > 0,$$

which with Theorem 3 further implies that

$$\|u(t, z+ct, y) - V_c(z)\psi_0(y)\|_{X_a} \to 0 \ \text{as}\ t \to +\infty,$$

then for large enough t we can apply Theorem 1 to yield the exponential decay in time of $\|u(t,z+ct,y) - V_c(z)\psi_0(y)\|_{X_a}$. This completes the proof of Theorem 4.

4.2. *Spreading speed of the solutions with the Bramson logarithmic delay when the initial data have compact supports*

In this subsection we investigate the spreading speed and the asymptotic behavior of the solutions of (1.3) with nonnegative compactly supported initial data, in [4] it has been proved that the spreading speed of the solution must be the minimal speed $2\sqrt{1-\lambda_0}$, in this paper we try to prove that the propagation of the solution to problem (1.3) with a bounded domain Ω still has the Bramson's type of delay estimate, which also extends some classical results for the scalar Fisher-KPP equation to the nonlocal model (1.3). By Theorem 2 and estimate (3.13), to prove Theorem 5 it suffices to investigate the asymptotic behavior of solution $v(t,x)$ to the heterogenous Fisher type equation (4.1) with compactly supported initial datum. After re-scaling of the coordinates: $x \mapsto \sqrt{1-\lambda_0}x$ and $t \mapsto (1-\lambda_0)t$, it is easy to see that in the new coordinates $v(t,x)$ satisfies the equation (4.1) with $\lambda_0 = 0$, thus in the following of this subsection we just investigate the Cauchy problem of (4.1) with $\lambda_0 = 0$, i.e.

$$\begin{cases} v_t - v_{xx} = v(1 - b_0(t,x) - v), & t > 0,\ x \in \mathbb{R}, \\ v(0,x) = v_0^*(x), & x \in \mathbb{R}, \end{cases} \tag{4.22}$$

where $b_0(t,x)$ satisfies (3.13), which with Theorem 2 implies that for any given nonnegative bounded initial datum $u_0(x,y)$ there exist positive constants C_0 and δ_0, such that

$$\|b_0(t,\cdot)\|_{L_\infty(\mathbb{R})} \le C_0 e^{-\delta_0 t},\ \ t > 0,$$

thus

$$\tilde{b}(t) = \sup_{x\in\mathbb{R}} |b_0(t,x)| \in L_1(\mathbb{R}^+). \tag{4.23}$$

Denote $b_1(t) = \tilde{b}(t)$, $b_2(t) = -\tilde{b}(t)$, and $v_i(t,x) = e^{\int_0^t b_i(s)ds} v(t,x)$ $(i = 1,2)$, it is easy to see that $v_i(t,x)$ satisfies

$$\frac{\partial}{\partial t} v_1 - \frac{\partial^2}{\partial x^2} v_1 \ge v_1(1 - v_1),\ \ t > 0, x \in \mathbb{R},$$

and

$$\frac{\partial}{\partial t} v_2 - \frac{\partial^2}{\partial x^2} v_2 \le v_2(1 - v_2),\ \ t > 0, x \in \mathbb{R}.$$

Let $\Phi(x - 2t)$ be the traveling wave solution with the minimal speed of Fisher equation $u_t = u_{xx} + u(1-u)$ satisfying

$$\begin{cases} \Phi''(z) + 2\Phi'(z) + \Phi(z)(1 - \Phi(z)) = 0, & z \in \mathbb{R}, \\ \Phi(-\infty) = 1, \quad \Phi(+\infty) = 0, \\ \Phi(0) = \frac{1}{2}. \end{cases} \tag{4.24}$$

Let $\tilde{v}(t,x)$ be the unique solution of

$$\begin{cases} \tilde{v}_t - \tilde{v}_{xx} = \tilde{v}(1-\tilde{v}), & t>0,\ x\in\mathbb{R}, \\ \tilde{v}(0,x) = v_0^*(x), & x\in\mathbb{R}. \end{cases}$$

Then by comparison principle it yields that

$$e^{-\int_0^t \tilde{b}(s)\,ds}\tilde{v}(t,x) \le v(t,x) \le e^{\int_0^t \tilde{b}(s)\,ds}\tilde{v}(t,x). \tag{4.25}$$

Theorem 5 is a consequence of Theorem 2 and the following theorem.

Theorem 4.2. *Let $b_0(t,x)$ satisfy (3.13) and let v be a solution of (4.22). There exists a constant $C \ge 0$ and two functions $\xi_\pm(t)$, $t\in\mathbb{R}^+$ such that $|\xi(t)| \le C$ for all $t>0$, and*

$$\lim_{t\to+\infty}\sup_{z\in\mathbb{R}^+}\left|v(z+2t-\tfrac{3}{2}\log t, t) - \Phi(z+\xi_+(t))\right| = 0, \tag{4.26}$$

and

$$\lim_{t\to+\infty}\sup_{z\in\mathbb{R}^-}\left|v(z-(2t-\tfrac{3}{2}\log t), t) - \Phi(-z+\xi_-(t))\right| = 0. \tag{4.27}$$

Proof. By Theorem 1.1 of [21] and (4.25), we have

$$\begin{aligned} &\liminf_{t\to+\infty}\left(\min_{0\le x\le 2t-\frac{3}{2}\log t - C} v(t,x)\right) \\ \ge e^{-\|\tilde{b}\|_{L^1(\mathbb{R}^+)}} &\liminf_{t\to+\infty}\left(\min_{0\le x\le 2t-\frac{3}{2}\log t - C} \tilde{v}(t,x)\right) > 0, \quad \text{as} \quad C\to+\infty, \end{aligned}$$

and

$$\begin{aligned} &\limsup_{t\to+\infty}\left(\max_{x\ge 2t-\frac{3}{2}\log t + C} v(t,x)\right) \\ \le e^{\|\tilde{b}\|_{L^1(\mathbb{R}^+)}} &\limsup_{t\to+\infty}\left(\max_{x\ge 2t-\frac{3}{2}\log t + C} \tilde{v}(t,x)\right) \to 0, \quad \text{for } C >> 1. \end{aligned}$$

Furthermore, by Propositions 2.3 and 3.1 of [21], there exist positive constants κ and ρ such that

$$\kappa z e^{-z} \le v(t, 2t-\tfrac{3}{2}\log t + z) \le \rho(z+1)e^{-z}, \quad t\ge 1, 0\le z\le \sqrt{t}. \tag{4.28}$$

Based on the above three estimates, one can repeat the proof of [21, Theorem 1.2] to prove (4.26). Indeed, by passing to a subsequence, the functions

$$v_n(t,z) = v(t+t_n, z+2t_n - \frac{3}{2}\log t_n)$$

converge locally uniformly to some solution $0 \leq v_\infty(t,x) \leq 1$ of $v_t - v_{zz} = v(1-v)$ as $n \to +\infty$, and such that

$$\lim_{C\to+\infty}\left[\sup_{z\geq 2t+C} v_\infty(t,z)\right]=0 \quad \text{and} \quad \lim_{C\to+\infty}\left[\sup_{z\leq 2t-C} v_\infty(t,z)\right]=1, \tag{4.29}$$

and also

$$\kappa z e^{-z} \leq v_\infty(t, 2t+z) \leq \rho(z+1)e^{-z}, \ \ t\in\mathbb{R} \text{ and } z\geq 0. \tag{4.30}$$

From a Liouville result [21, Lemma 4.1], $v_\infty(t,x) = \Phi(x-2t+\xi_0)$ for some ξ_0 that is uniformly bounded. The convergence to the translated critical front follows. This proves (4.26). Finally, (4.27) can be proved by the same argument. □

Remark 4.2. It is worth mentioning that Ducrot [18] investigated the asymptotic behavior of solutions with compacted supported initial data for Fisher-KPP equation and some types of asymptotic Fisher-KPP equations in one or higher dimensional space, where the zero-th order coefficient is asymptotically constant in x, which in the formulation of (4.22) corresponds to assuming $b_0(t,x) \to 0$ as $|x| \to \infty$. Under the assumption that $b_0(t,x) \in L_{\infty,1}(\mathbb{R}^+\times\mathbb{R})$, Ducrot [18] proves that the logarithmic delay of the solution for the asymptotic Fisher equation is the same as that for the homogeneous Fisher-KPP equation, he also proved that the logarithmic delay will be changed for the critical case when $b_0(t,x) = \frac{\gamma}{1+|x|}$ due to $b_0(t,x) \notin L_{\infty,1}(\mathbb{R}^+\times\mathbb{R})$.

In contrast, we treated the nonlinear heterogeneous Fisher equation (4.22) under the assumption $b_0(t,x) \in L_{1,\infty}(\mathbb{R}^+\times\mathbb{R})$. Thanks to the estimate (4.7) or (4.23) the nonlocal model (1.2) can be reduced to this case.

For our nonlocal model (1.2) and the corresponding nonlinear heterogeneous Fisher equation (4.22), the heterogeneous term $b_0(t,x)$ is a nonlocal coupled term, where the spatial decay of $b_0(t,x)$ highly depends on the spatial decaying of $v_0(t,x)$ and the decay rate of the initial datum, in this sense the argument in [18] can not be applied to our case directly before proving the fast spatial decay of $v_0(t,x)$. In our proof of Theorem 4.2 by virtue of the uniform boundedness of $v_0(t,x)$ and the exponential decay in t of $b_0(t,x)$, by applying sub-supersolution method and applying some classical results obtained in [21] for Fisher-KPP equation, we can prove that the decaying estimate (4.28) is still valid for the asymptotic Fisher equation in t and then the estimates in Theorem 4.2 can be similarly proved as in [21].

Remark 4.3. If the nonnegative initial datum $v_0^*(x)$ satisfies the assumption

$$0 < \underline{q}_0 < \liminf_{x\to-\infty} v_0^*(x) \leq \limsup_{x\to-\infty} v_0^*(x) < \overline{q}_0, \tag{4.31}$$

and

$$v_0^*(x) \equiv 0, \ \ \text{for} \ \ x >> 1,$$

then estimate (4.26) is still valid and it can be proved that the solution $v(t,x)$ of (4.22) tends to $V_{c^*}(x-c^*t)$ in the following weak way

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|v(t,x)-V_{c^*}\left(x-c^*t-\eta(t)\right)\right|=0, \tag{4.32}$$

with $\eta(t)=-\frac{3}{2}\log t+O(1)$ for all $t>0$, which with Theorem 2 also means that the solution $u(t,x,y)$ of the nonlocal equation (1.3) tends to the cylinder wave $V_{c^*}(x-c^*t)\psi_0(y)$ in the similar weak sense.

Remark 4.4. If the initial datum $v_0^*(x)$ satisfies (4.31) and decays with the same exponential rate as that of $V_{c^*}(x)$ at $x=+\infty$, i.e.

$$v_0^*(x)\sim xe^{-x}\sim V_{c^*}(x),\;\; x\to+\infty,\; c^*=2,$$

due to the exponential decay of $b_0(t,x)$ in time and the initial assumption, by constructing appropriate sub-supersolutions to heterogeneous Fisher type equation $v_t=v_{xx}+v(1-b_0(t,x)-v)$, it is naturally expected that the shift $\eta(t)$ in (4.32) can be uniformly bounded for all $t>0$, and we conjecture that the shift $\eta(t)$ has a limit as $t\to+\infty$ if $v_0^*(x)\equiv V_{c^*}(x),$ for $x>>1$.

If $v_0^*(x)$ decays faster than $V_{c^*}(x)$ at $x=+\infty$, such as

$$v_0^*(x)=o(V_{c^*}(x)),\;\; x\to+\infty, \tag{4.33}$$

by virtue of (4.25) and Theorems 5 and 4.2, by applying comparison argument it can be proved that the spreading speed of the solution is still the critical speed. We conjecture that the estimate (4.32) is still valid with $\eta(t)$ satisfying $\frac{\eta(t)}{t}\to 0$ as $t\to+\infty$.

It is well known that for the classical Fisher equation in one dimensional space, in [13,25] the authors give more detailed description on the asymptotic behavior of solution and the spreading of the level set of solution, which are classified by the decay rate of the initial datum $v_0^*(x)$ near $z=+\infty$. However due to the fact that the comparison principle can not be applied directly to the nonlocal model (1.3) or to the nonlinear heterogenous equation (4.22), some powerful techniques applied in [13,25], which are based on the comparison principle for nonlinear homogeneous parabolic equation, can not be applied directly to the nonlocal model (1.3) or to equation (4.22) with a heterogeneous term. For the typical case when the initial datum is compactly supported (or a heaviside function), it is unknown whether the bounded shifts $\xi_\pm(t)$ in Theorem 4.2 have limits, which may be not true for the nonlocal model (1.3) and the above mentioned conjectures are also open problems.

Data availability

No data was used for the research described in the article.

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