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ORIGINAL ARTICLE

The Derivation of the Compressible Euler Equation from Quantum Many-Body Dynamics

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Abstract

We study the three-dimensional many-particle quantum dynamics in mean-field setting. We forge together the hierarchy method and the modulated energy method. We prove rigorously that the compressible Euler equation is the limit as the particle number tends to infinity and the Planck's constant tends to zero. We improve the previous sufficient small time hierarchy argument to any finite time via a new iteration scheme and Strichartz bounds first raised by Klainerman and Machedon in this context. We establish strong and quantitative microscopic to macroscopic convergence of mass and momentum densities up to the 1st blow up time of the limiting Euler equation. We justify that the macroscopic pressure emerges from the space-time averages of microscopic interactions via the Strichartz-type bounds. We have hence found a physical meaning for Strichartz-type bounds.

Keywords Compressible Euler equation · BBGKY hierarchy · Quantum many-body dynamics · Klainerman–Machedon bounds · Modulated energy

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1 Introduction

The analysis of the nonlinear fluid equations like the Euler equations and the Navier–Stokes equations, is an important (if not vital) part of many areas of pure and applied mathematics, science, and engineering. On one hand, their validity has certainly been checked countless times against the experiments. On the other hand, the rigorous derivation of these macroscopic continuum equations from basic microscopic Newtonian/Maxwell/quantum particle models has largely remained open. It is certainly of fundamental interest in mathematics to establish such derivations and prove that macroscopic quantities like pressure emerge from the averaging of microscopic quantities. In this paper, we prove the derivation of the compressible Euler equation from the quantum N-body dynamic in the mean-field setting. We choose to start from the quantum theory as it is, at the moment, the most accurate microscopic model and such a derivation would also establish (again) that there is no obvious gap between the basic models in quantum and classical scales.

In the setting of classical mechanics, a strategy of the derivation of fluid equations from particle systems is to first pass to a mesoscopic Boltzmann equation, then derive the desired fluid equation from the Boltzmann equation. (See, for example, the standard monographs [8, 33, 54] and references within.) However, such a route may not suit our purpose here. On one hand, the validity of the classical Boltzmann equations is only justified up to a sufficiently small time and is not clear if it covers the 1st blow up time of the Euler equation. On the other hand, the derivation of the quantum Boltzmann equation is at a rudimentary stage. (See, for example, [10, 15, 28] and the references within.) Not to mention the possibility that one might need to pass to another classical Boltzmann equation if one takes such a route. Moreover, we would like to understand the fine interplay between $\hbar$ and N, the two fundamental constants, which differ by 10^{57} in SI units. In fact, starting from 2019, the mass unit is defined via the Planck's constant. Thus, we choose to derive the compressible Euler equation directly from quantum many-body dynamics.

We consider Bosons in this paper as it is more directly related to the Newton–Maxwell particles due to the assumption that particles are indistinguishable. In fact, N_2 and O_2 molecules are bosons (99.03% of air) and 99.05% H_2O molecules are bosons. (Fermions are also interesting, see for example, the survey [52].) We consider the 3D linear N-body bosonic Schrödinger equation:

$$i\hbar\partial_t\psi_{N,\hbar} = H_{N,\hbar}\psi_{N,\hbar} \tag{1.1}$$

with Hamiltonian $H_{N,\hbar}$ given by

$$H_{N,\hbar} = \sum_{j=1}^{N} -\frac{1}{2}\hbar^2\Delta_{x_j} + \frac{1}{N}\sum_{1\le j<k\le N} V_N(x_j - x_k) \tag{1.2}$$

where

$$V_N(x) = N^{3\beta}V(N^{\beta}x), \tag{1.3}$$

and the factor $1/N$ is to make sure the interactions grow like N instead N^2, a mean-field like scaling. The marginal densities $\gamma_{N,\hbar}^{(k)}$ associated with $\psi_{N,\hbar}$ in kernel form are given by

$$\gamma_{N,\hbar}^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k') = \int \psi_{N,\hbar}(t, \mathbf{x}_k, \mathbf{x}_{N-k})\overline{\psi_{N,\hbar}}(t, \mathbf{x}_k', \mathbf{x}_{N-k})d\mathbf{x}_{N-k} \tag{1.4}$$

where $\mathbf{x}_k = (x_1, \ldots, x_k) \in \mathbb{R}^{3k}$ and $\mathbf{x}_{N-k} = (x_{k+1}, \ldots, x_N) \in \mathbb{R}^{3(N-k)}$. Notably, one can derive cubic nonlinear Schrödinger equation (NLS) as the $N \to \infty$ limit of (1.1) with $\hbar$ fixed, then the well-known Madelung transform [50] relates Schrödinger type equation and the macroscopic Euler equations in a formal limit process as $\hbar$ tends to zero. That is, the macroscopic equations could formally emerge from (1.1) as an iterated limit: $\lim_{\hbar\to 0} \lim_{N\to\infty}$. Such an iterated limit is far from satisfactory in either mathematics or physics. Not only an iterated limit could lose information in any one limit, it kills the fine interplay between $\hbar$ and N and hence cannot show the $(N, \hbar)$ threshold at which classical behavior starts to dominate. In particular, the iterated limit cannot yield practical information like how large an N is enough for a fixed but small $\hbar$. Therefore, for a more complete and deeper understanding, we deal with the $(N, \hbar)$ double limit which is also a more challenging problem.

Our limiting macroscopic equation is the 3D compressible Euler equation,[1] which is,

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t u + (u \cdot \nabla)u + b_0 \nabla \rho = 0, \\ (\rho, u)|_{t=0} = (\rho^{\text{in}}, u^{\text{in}}), \end{cases} \tag{1.5}$$

if written in velocity form, or

$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0, \\ \partial_t J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) + \frac{1}{2}\nabla\left(b_0 \rho^2\right) = 0, \\ (\rho, J)|_{t=0} = (\rho^{\text{in}}, J^{\text{in}}), \end{cases} \tag{1.6}$$

if written in momentum form. Here, as usual, $\rho(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is the mass density, $u(t, x) = (u^1(t, x), u^2(t, x), u^3(t, x)) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes the velocity of the fluid, $J(t, x) = (\rho u)(t, x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes the momentum of the fluid with the coupling constant[2] $b_0 = \int V$ which is the macroscopic effect of the microscopic interaction V and hints that pressure $b_0 \rho^2$ should originate from the microscopic interaction between particles.[3]

[1] Equation (1.6) corresponds to a compressible inviscid liquid with the heat capacity ratio equal to 2. It is usually called a shallow water case. It can also describe liquid water under saturation pressure at around 600 K. (Liquid water's C_P/C_V changes against temperature like all real world fluids.)

[2] The Eqs. (1.5) and (1.6) are not hyperbolic if the microscopic potential V is focusing or $b_0 < 0$.

[3] One can see this from the iterated limit: the pressure terms comes from the nonlinear term in the NLS which comes from the interaction term in the N-body dynamics. This can also be seen in a formal hierarchy computation.

1.1 Statement of the Main Theorem

Theorem 1.1 *Let $d=3$, $\beta<\frac{2}{5}$, the marginal densities $\Gamma_{N,\hbar}=\{\gamma_{N,\hbar}^{(k)}\}$ associated with $\psi_{N,\hbar}$ be the solution to the N-body dynamics with a Schwarz even pair interaction $V\geq 0$. The N-body initial data satisfy the following condition:*

(a) $\psi_{N,\hbar}(0)$ *is normalized, that is,* $\|\psi_{N,\hbar}(0)\|_{L^2}=1$.
(b) *The N-body energy bounds hold:*

$$\langle \psi_{N,\hbar}(0),(H_{N,\hbar}/N+1)^k\psi_{N,\hbar}(0)\rangle \leq (E_{0,\hbar})^k \tag{1.7}$$

for $k\leq (\ln N)^{100}$.
(c) $\Gamma_{N,\hbar}(0)$ *is asymptotically factorized in the sense that*

$$\left\| \prod_{j=1}^{k}\langle \hbar\nabla_{x_j}\rangle\langle \hbar\nabla_{x_j'}\rangle \left[\gamma_{N,\hbar}^{(k)}(0)-|\phi_{N,\hbar}^{\rm in}\rangle\langle\phi_{N,\hbar}^{\rm in}|^{\otimes k}\right]\right\|_{L^2_{x,x'}} \leq (E_{0,\hbar})^k N^{\frac{5}{2}\beta-1} \tag{1.8}$$

for $k\leq (\ln N)^{100}$, *where* $\phi_{N,\hbar}^{\rm in}$ *is normalized that* $\|\phi_{N,\hbar}^{\rm in}\|_{L^2}=1$ *and has finite energy,*[4] *that is*

$$\frac{1}{2}\|\phi_{N,\hbar}^{\rm in}\|_{L^2}^2+\frac{1}{2}\|\hbar\nabla\phi_{N,\hbar}^{\rm in}\|_{L^2}^2+\frac{1}{2}\langle V_N*|\phi_{N,\hbar}^{\rm in}|^2,|\phi_{N,\hbar}^{\rm in}|^2\rangle\leq E_0. \tag{1.9}$$

(d) *The initial datum* $(\rho^{\rm in},u^{\rm in})$ *to* (1.5) *satisfies*

$$\rho^{\rm in}\geq 0,\quad \int\rho^{\rm in}(x)dx=1, \tag{1.10}$$

and is such that the Euler system (1.5) *has a solution* (ρ,u) *satisfying*

$$\begin{cases}(\rho,u)\in C([0,T_0];H^s)\cap C^1([0,T_0];H^{s-1}),\\ \rho\geq 0,\quad \int_{\mathbb{R}^d}\rho(t,x)dx=1,\end{cases} \tag{1.11}$$

where $s>\frac{d}{2}+3$. *The modulated/renormalized energy at initial time tends to zero:*

$$\int_{\mathbb{R}^d}|(i\hbar\nabla-u^{\rm in})\phi_{N,\hbar}^{\rm in}|^2dx+b_0\int_{\mathbb{R}^d}\left(|\phi_{N,\hbar}^{\rm in}|^2-\rho^{\rm in}\right)^2dx\leq C\hbar^2. \tag{1.12}$$

Then under the restriction that[5]

$$N\geq e^{(2)}\left(\left[C_V^2E_{0,\hbar}^2T_0/\hbar^7\right]^2\right), \tag{1.13}$$

[4] It is expected that $E_0\leq E_{0,\hbar}$ due to the correction structure.

[5] The composite function $e^{(n)}(x):=e^{(e^{(n-1)}(x))}$ and C_V is a constant which only depends on some Sobolev norms of V as needed in the proof.

for $N \geq N_0(\beta)$ and (ρ, u) satisfying (1.5), we have the quantitative estimates on the convergence of the mass density

$$\|\gamma^{(1)}_{N,\hbar}(t,x;x) - \rho(t,x)\|_{L^\infty_t[0,T_0]L^2_x(\mathbb{R}^d)} \leq C(T_0)\left(\frac{1}{\ln N} + \hbar\right), \tag{1.14}$$

on the convergence of the momentum density[6] *for $r \in (1, 4/3)$*

$$\left\|\operatorname{Im}\left(\hbar\nabla_{x_1}\gamma^{(1)}_{N,\hbar}\right)(t,x;x) - (\rho u)(t,x)\right\|_{L^\infty_t[0,T_0]L^r_x(\mathbb{R}^d)} \leq C(T_0)\left(\frac{1}{(\ln N)^{5(1-\frac{1}{r})}} + \hbar^{\frac{4-3r}{r}}\right), \tag{1.15}$$

and on the emergence of pressure

$$\left\|\int V_N(x-x_2)\gamma^{(2)}_{N,\hbar}(t,x,x_2;x,x_2)dx_2 - b_0\rho(t,x)^2\right\|_{L^1_t[0,T_0]L^1_x(B_R)} \leq C(T_0)\left(\frac{R^{d/2}}{\ln N} + \hbar\right), \tag{1.16}$$

where coupling constant is $b_0 = \int V$.

Theorem 1.1 is the first of its type and involves the up-to-date techniques in the hierarchy method as well as well-developed modulated energy approach and we can in fact see it from its assumptions. The N-body energy condition in (b) is inspired by purely factorized or statistically independent datum, and has been used since the first wave of work [1, 27, 29–32] on deriving NLS using hierarchy methods. It is usually cashed in as the H^1 bound on the marginals[7]

$$\left\|\prod_{j=1}^{k}\langle\hbar\nabla_{x_j}\rangle\langle\hbar\nabla_{x'_j}\rangle\gamma^{(k)}_{N,\hbar}(t)\right\|_{L^2_{x,x'}} \leq \left(2E_{0,\hbar}\right)^k \tag{1.17}$$

for $k \leq (\ln N)^{100}$, $N \geq N_0(\beta)$ which is independent of k and $\hbar$, and all $t \in (-\infty, +\infty)$. Here, we allow the $k \geq 2$ energy bound $E_{0,\hbar}$ to depend on $\hbar$ (the $k = 1$ case can be the same E_0 as in (1.9)) as long as it is finite for every nonzero $\hbar$, so that a larger variety of initial data is included at the cost of the restriction (1.13) with an unspecific factor $E_{0,h}$. This is a natural requirement as the $k > 2$ energy includes higher derivatives which do not play well with $\hbar$. Though the initially asymptotic statistically independent assumption (1.8) in (c) is like usual in this line of work, the optimal decay rate is believed (and proved in some cases, see for example, [3, 6]) to be $1/N$ for every given $\hbar$. We assume $N^{\frac{5}{2}\beta-1}$ here so that the paper is self-contained as we will prove this rate at the first step of bootstrapping argument. Indeed, for $\hbar = 1$, the convergence rate has been achieved in [22]. On the other hand, compared to the

[6] This convergence can be improved to $r \in (1, 3/2)$ with a new feedback technique in the modulated energy argument in our forthcoming paper [24]

[7] We include a proof as Proposition B.1 for completeness.

N-body energy bounds (1.7), the energy bound E_0 for $\phi_{N,h}^{\text{in}}$ is independent of $\hbar$ to be compatible with the modulated energy bound.

As for the assumptions regarding the initial datum of (1.5), the local well-posedness of compressible Euler equations has been studied by many authors, for example, see the monograph [51]. But we remark that, there are many variants/choices/constructions of the modulated energy (1.12) which look seemingly different but are intuitively and closely related up to an error term as the initial quantities like $|\phi_{N,\hbar}^{\text{in}}|^2$ and ρ^{in} are supposed to be close. In fact, the full modulated energy which we will use and is going to be controlled by (1.12) takes the form

$$\begin{aligned}\mathcal{M}\left[\phi_{N,\hbar},\rho,u\right](t) = \frac{1}{2}\int_{\mathbb{R}^d}|(i\hbar\nabla-u)\phi_{N,\hbar}(t)|^2dx + \frac{1}{2}\langle V_N*|\phi_{N,\hbar}|^2,|\phi_{N,\hbar}|^2\rangle \\ +\frac{b_0}{2}\int_{\mathbb{R}^d}\rho^2dx - b_0\int_{\mathbb{R}^d}\rho|\phi_{N,\hbar}|^2dx. \quad (1.18)\end{aligned}$$

We assume the convergence rate (1.12) to be $\hbar^2$ which should also be optimal, since the smallness factor in the modulated kinetic part is at most $\hbar^2$. Besides, the $\hbar^2$ rate can be achieved with WKB type initial datum.

Theorem 1.1 rigorously establishes the derivation of the macroscopic equation (1.5) in classical mechanics from the quantum many-body systems as a regional double limit and provides convergent rate estimates in the strong norm sense. It also justifies the emergence of the macroscopic pressure from the space-time averages of microscopic interactions, which are in fact, Strichartz-type bounds. Notice that, the microscopic quantity converging to the pressure ρ^2 is basically $\gamma_{N,\hbar}^{(2)}(x,x,x,x)$. It is not necessarily finite or defined a.e. if we are below $H^{9/8}$ in 3D by the Sobloev embeddings, and we only have H^1 here. The Strichartz bound, first raised by Klanerman–Machedon (KM) [46] in this context, makes this quantity well-defined and have unexpectedly verified the theory that pressure is the space-time averaging of the microscopic interactions under the physical H^1 assumption.[8] We have hence found the 1st physical meaning for Strichartz-type bounds since its original discovery in [61]. Such a discovery is part of the main novelty of this paper. On the other hand, the limit in Theorem 1.1 is taken within the region (1.13) which proves the dominance of classical behaviors when $N >> \hbar$. Such a requirement is physical as they indeed differ by 10^{57} in reality but we believe (1.13) is not optimal and searching for the sharp threshold (may not exist, some mesoscopic behaviors might happen) between classical and quantum behaviors is certainly of interest. However, it would not be surprising to have totally independent N and $\hbar$ in weak/weak* limits as a weak convergent sequence can be uniformly bounded away from its weak limit. To work with the 3D N-body equation smoothly in the physical H^1 energy space, we improvise and extend the up-to-date hierarchy method in KM format.

The hierarchy method in general was first suggested by Kac and proved to be successful in Lanford's work [47] regarding the Boltzmann equation. The hierarchy

[8] Such an averaging effect certainly cannot be observed if one assumes higher than $H^{9/8}$ regularity at the N-body level, but we remark that it cannot be observed either if one passes through the NLS in the H^1 setting as $|\phi|^4$ is already defined a.e. without any need to appeal to Strichartz.

method we use in the paper is actually more originated from the 1st wave of work [1, 30–32] by Adami–Golse–Teta and Erdős–Schlein–Yau on deriving NLS from quantum many-body dynamics around 2005 as suggested by Spohn [60]. At that time, the main difficulty lies in the uniqueness of the infinite Gross–Pitaevskii (GP) hierarchy. With a sophisticated Feynman graph analysis in the fundamental papers [30–32] which derived the 3D cubic defocusing NLS, Erdős, Schlein, and Yau proved the H^1-type unconditional uniqueness of the $\mathbb{R}^3$ cubic GP hierarchy. The first series of ground breaking papers have motivated a large amount of work.

Subsequently in 2007, by imposing an additional a-prior condition on space-time norm, Klainerman and Machedon [46], inspired by [30, 45], gave another uniqueness criterion of the GP hierarchy in a different space of density matrices defined by Strichartz-type norms. They provided a different combinatorial argument, the now so-called Klainerman–Machedon board game, to combine the inhomogeneous terms effectively reducing their numbers and then derived a space-time estimate to control these terms. At that time, it was open on how to prove that the limits coming from the N-body dynamics satisfy the now so-called KM space-time bound required for uniqueness. Nonetheless, [46] has made the delicate analysis of the GP hierarchy approachable from the perspective of PDE. Klainerman and Machedon also did not know the KM bound required for uniqueness, which is a usual product of Strichartz-type well-posedness theory, actually has a physical meaning.[9]

Later, Kirkpatrick et al. [44] obtained the KM space-time bound via a simple trace theorem in both $\mathbb{R}^2$ and $\mathbb{T}^2$ and derived the 2D cubic defocusing NLS from the 2D quantum many-body dynamic. Such a scheme also motivated many works [11, 13, 18, 20, 36, 39, 58, 59, 62] for the uniqueness of GP hierarchies and enables the hierarchy method on the derivation 1D or 2D NLS directly from 3D [16, 20, 56], which is quite different but has some similar flavor with our Theorem 1.1 here. However, how to verify the KM bound in the 3D cubic case remained fully open at that time.

Then in 2011, T. Chen and Pavlović proved that the 3D cubic KM space-time bound held for the defocusing $\beta < 1/4$ case in [12]. The result was quickly improved to $\beta < 2/7$ by X. Chen in [14] and then extended to the almost optimal case, $\beta < 1$, by X. Chen and Holmer in [17, 19], by lifting the $X_{1,b}$ space techniques from NLS theory into the field. Away from being the first work to prove the 3D KM bound, the work [12] hinted two unforeseen directions of the hierarchy method: one direction is to prove new NLS results via the more complicated hierarchies, while the other is that it is possible to derive NLS without a compactness or uniqueness argument as in the 1st wave of papers.

In 2013, by introducing the quantum de Finetti theorem from [48] to the field, T. Chen, Hainzl, Pavlović and Seiringer [9] provided a simplified proof of the $L_t^\infty H_x^1$-type 3D cubic uniqueness theorem as stated in [30]. This method motivated many work [26, 41, 42, 57] and has climbed to a climax recently as the previously open $\mathbb{T}^d$ energy-critical and supercritical NLS unconditional uniqueness problems progressed in [40] were completely and unifiedly resolved via the analysis of the supposedly more complicated GP hierarchy in [21, 23, 25] which used, the l^2 decoupling theorem [5] and has helped in the derivation of the energy-critical NLS [21, 23]. With these new

[9] Private communication with M. Machedon.

exciting developments, it seems that KM bound method is obsolete though the KM board game stays useful. Such an impression or conclusion is apparently wrong.

Recently, on the basis of [12, 14, 17, 19], X. Chen and Holmer in [22] reformatted the hierarchy method with KM space-time estimates and proved a bi-scattering theorem for the NLS to obtain almost optimal local in time convergence rate estimates under H^1 regularity. They integrate the idea from the Fock space approach (see, for example, [2, 4, 6, 37, 38] and references within[10]), that, using H-NLS as an intermediate dynamic, into the hierarchy method. Most notably, the work [22], though it did not use the KM bound, sheds light on our principal part in which we prove strong, quantitative, uniform in $\hbar$, estimates regarding the BBGKY hierarchy and the H-NLS hierarchy.

On the other hand, the behavior of the wave function of cubic defocusing NLS as the Planck's constant goes to zero is studied by many authors using various approaches. In [35], Grenier derived compressible Euler equations for small time from cubic NLS by WKB. Jin, Levermore and McLaughlin in [43] established the semiclassical limit of the 1D defocusing cubic NLS for all time by using the complete integrability. In [49], Lin and Zhang investigated Gross–Pitaevskii equation (a cubic Schrödinger equation nonzero at infinity) in 2D exterior domains by adopting the modulated energy method. For a more detailed survey related to semiclassical limits of NLS, see [7, 63] and references within.

As seen from above, it is highly nontrivial to derive Euler equations from NLS, let alone from quantum N-body dynamics. As the first breakthrough, Golse and Paul [34], with the help of Serfaty's inequality [55, Corollary 3.4], used the modulated energy method in the quantum N-body setting to justify the validity of the joint mean-field and classical limit of the quantum N-body dynamics leading to the pressureless Euler–Poisson with repulsive Coulomb potential. Subsequently, Rosenzweig complemented [34] in [53] by combining mean-field, semiclassical and quasi-neutral limits to reach a derivation of an incompressible Euler equation on $\mathbb{T}^d$ with binary Coulomb interactions.

Though both singular, the δ-interaction, which results in a compressible Euler equation, is substantially different from the Coulomb potential and calls for new ideas. The strong convergence and quantitative estimates are much more demanding as well. Our proof combines improvision and extension of up-to-date techniques in the hierarchy method and the well-developed modulated energy method. Compared to the methods in [34, 53], our method obtains strong convergence rates and establishes the emergence of the macroscopic pressure.

1.2 Outline of the Proof

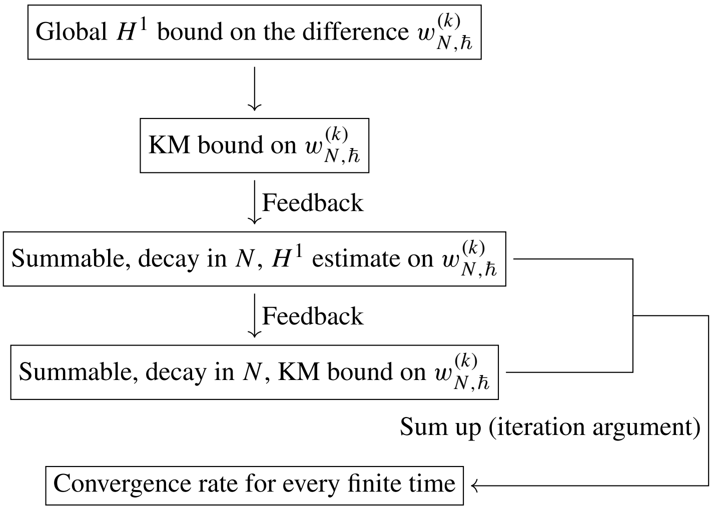
Equation (1.1) is very different from our goal (1.5) or (1.6), at least by the look of them. Key quantities of $\gamma_{N,\hbar}^{(k)}$ in (1.14)–(1.16) are all traces and thus as usual, are regularity thirsty and does not react well as $\hbar \to 0$, while solutions to (1.5) will blow up in finite time. Thus, we insert H-NLS (2.1)[11] as an intermediate dynamic.

[10] The Fock space approach is also a vast and deep subject right now. There are certainly more references available. But this paper is not directly related to that.

[11] We expect more NLS like behaviors from (2.1) due to the context and hence we call it H-NLS.

We hence divide the proof of Theorem 1.1 into two parts in Sects. 2 and 3 respectively. The first part is the quantitative estimate between the BBGKY hierarchy and the H-NLS using an improvised and extended version of cutting edge hierarchy methods, while the second part is comparing the H-NLS equation with the compressible Euler equation (1.5) by means of modulated energy approach. Here, we are using the BBGKY hierarchy directly satisfied by $\gamma_{N,\hbar}^{(k)}$. We are not using any Wigner transforms in this paper. Theorem 1.1 then follows from summing the concluding estimates in Sects. 2 and 3.

There are two main difficulties in Sect. 2. One is to make sure all the differences estimates are uniform in $\hbar$. The other one is to make sure the estimates hold for every finite time despite that the method [22] only works local in time. How to circumvent these two difficulties is also the main technical novelty of this paper. The key is to implement the Klainerman–Machedon space-time bound, which was thought of only as a part of uniqueness, to strengthen our local in time quantitative estimate via a new iteration scheme. We can then improve the previous sufficient small time hierarchy argument [22] to any finite time. The whole process is still very technical, we illustrate the principle logic of the proof of Sect. 2 by the following diagram.



The logic above looks quite like proving global well-posedness for an H^1 subcritical NLS. However, this is the 1st time such a diagram is carried out for the hierarchy analysis. The technical reason is exactly as mention before (and in almost all paper in this field), though the N-body equations and hierarchies are linear, we are dealing with traces instead of powers.

In Sect. 2.1, we first provide some preliminary or crude estimates for the difference between BBGKY hierarchy and H-NLS hierarchy. We then prove in Sect. 2.2 that $w_{N,\hbar}^{(k)}$ satisfies the Klainerman–Machedon bound by gathering information from the $(\ln N)^{10}$ coupling level. Subsequently in Sect. 2.3, we feed the KM bound/a Strichartz bound back, to strengthen the H^1 estimate for $k < (\ln N)^2$ to obtain summable and decay in N estimates. We can further feed the H^1 estimate of $w_{N,\hbar}^{(k)}$ back into the KM bound proof and deduce that the KM bound actually decays in N. Notice the difference between the given kth marginal and the selectable coupling level. For a given kth marginal, how to select a suitable coupling level to yield desired information is a fine technical point. Sections 2.2 to 2.3 addresses this issue. Finally, in Sect. 2.4, with the conclusion in Sect. 2.3, we can sacrifice some decays in N to bootstrap the quantitative estimates to every finite time by a clever but elementary manipulation.

As the N-body estimates have been set ready in Sect. 2, in Sect. 3, we adopt modulated energy method to compare directly the H-NLS equation with compressible Euler equations before the blowup time. The idea of proving convergence is via a Gronwall argument on modulated energies assuming and using the regularity of the limiting solution. Therefore, in Sect. 3.1, we compute the evolution of modulated energy. Subsequently in Sect. 3.2, we control the error term originating from the evolution of modulated energy to obtain a Gronwall type estimate. Due to the work in Sect. 2, we are able to have a close match inside the modulated energy, and hence the error term is very tractable.

The main novelty of the paper is Theorem 1.1 which establishes a strong microscopic to macroscopic derivation up to the 1st blow up time of the limiting Euler equation from the fundamental quantum N-body dynamics. The proof also combines the hierarchy method and the modulated energy method for the 1st time. We indeed anticipated more fusion of these two methods in the future. During the course of proof, we have implemented the Klainerman–Machedon Strichartz-type bound and hence verified the emergence of pressure as the space-time averagings of microscopic interaction. This argument thus discovers a physical meaning for Strichartz-type bounds for PDE and harmonic analysis.

2 BBGKY Hierarchy v.s. H-NLS: Long-Time Uniform in $\hbar$ Estimates

The main goal in this section is to establish long-time uniform in $\hbar$ estimate for the difference $\gamma_{N,\hbar}^{(k)} - |\phi_{N,\hbar}\rangle\langle\phi_{N,\hbar}|^{\otimes k}$ where $\phi_{N,\hbar}$ is the solution to H-NLS equation as below

$$\begin{cases} i\hbar\partial_t\phi_{N,\hbar} = -\frac{1}{2}\hbar^2\Delta\phi_{N,\hbar} + \big(V_N * |\phi_{N,\hbar}|^2\big)\phi_{N,\hbar}, \\ \phi_{N,\hbar}(0) = \phi_{N,\hbar}^{\mathrm{in}}. \end{cases} \tag{2.1}$$

Our strategy is to use the hierarchy approach. It is well-known that $\Gamma_{N,\hbar}(t) = \{\gamma_{N,\hbar}^{(k)}\}$ satisfies the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy

$$\begin{aligned} i\hbar\partial_t\gamma_{N,\hbar}^{(k)} &= \sum_{j=1}^{k}\left[-\frac{\hbar^2}{2}\Delta_{x_j}, \gamma_{N,\hbar}^{(k)}\right] + \frac{1}{N}\sum_{1\le i<j\le k}\big[V_N(x_i - x_j), \gamma_{N,\hbar}^{(k)}\big] \\ &\quad + \frac{N-k}{N}\sum_{j=1}^{k}\mathrm{Tr}_{k+1}\big[V_N(x_j - x_{k+1}), \gamma_{N,\hbar}^{(k+1)}\big]. \end{aligned} \tag{2.2}$$

In addition to (2.2), we will use the so-called H-NLS hierarchy which takes the form

$$i\hbar\partial_t \gamma_{H,\hbar}^{(k)} = \sum_{j=1}^{k}\left[-\frac{\hbar^2}{2}\Delta_{x_j}, \gamma_{H,\hbar}^{(k)}\right] + \sum_{j=1}^{k}\mathrm{Tr}_{k+1}\big[V_N(x_j - x_{k+1}), \gamma_{H,\hbar}^{(k+1)}\big], \tag{2.3}$$

generated by

$$\big\{\gamma_{H,\hbar}^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) = |\phi_{N,\hbar}\rangle\langle\phi_{N,\hbar}|^{\otimes k}\big\},$$

the tensor products[12] of solutions to H-NLS equation (2.1).

Denote the difference between the BBGKY hierarchy and the H-NLS hierarchy by

$$w_{N,\hbar}^{(k)} = \gamma_{N,\hbar}^{(k)} - \gamma_{H,\hbar}^{(k)}. \tag{2.4}$$

For convenience, we first set up some notations. Define

$$S_{\hbar}^{(1,k)} = \prod_{j=1}^{k}\langle\hbar\nabla_{x_j}\rangle\langle\hbar\nabla_{x'_j}\rangle, \tag{2.5}$$

the collision operator

$$\begin{aligned} B_{N,j,k+1}f^{(k+1)} &= B_{N,j,k+1}^{+}f^{(k+1)} - B_{N,j,k+1}^{-}f^{(k+1)} \\ &= \int V_N(x_j - x_{k+1}) f^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1})dx_{k+1} \\ &\quad - \int V_N(x'_j - x_{k+1}) f^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1})dx_{k+1}, \end{aligned} \tag{2.6}$$

and

$$B_{N,\hbar,j,k+1} = \frac{1}{\hbar}B_{N,j,k+1}, \quad B_{N,\hbar,j,k+1}^{\pm} = \frac{1}{\hbar}B_{N,j,k+1}^{\pm}. \tag{2.7}$$

Define the quantum mass density and momentum density in the quantum N-body setting

$$\gamma_{N,\hbar}^{(1)}(t, x; x), \quad J_{N,h}^{(1)}(t, x; x) = \mathrm{Im}\big(\hbar\nabla_{x_1}\gamma_{N,\hbar}^{(1)}\big)(t, x; x) \tag{2.8}$$

and

$$\rho_{N,\hbar}(t, x) = |\phi_{N,\hbar}(t, x)|^2, \quad J_{N,\hbar}(t, x) = \hbar\,\mathrm{Im}\left(\overline{\phi_{N,\hbar}}(t, x)\nabla_x\phi_{N,\hbar}(t, x)\right) \tag{2.9}$$

with respect to H-NLS equation.

Our main theorem of this section is the following.

[12] As it is indeed a tensor product, the energy bound (1.17) also holds for $\gamma_{H,\hbar}^{(k)}$ with $E_{0,\hbar}$ replaced by E_0.

Theorem 2.1 *Let $\phi_{N,\hbar}(t)$ be the solution to H-NLS equation with the initial data $\phi_{N,h}^{\mathrm{in}}$. Under the same conditions* (a), (b) *and* (c) *of Theorem* 1.1 *and the restriction that*

$$N \geq e^{(2)}\Big(\big[C_V^2 E_{0,\hbar}^2 T_0/\hbar^7\big]^2\Big), \tag{2.10}$$

then for $N \geq N_0(\beta)$ we have the quantitative estimates

$$\sup_{t\in[0,T_0]} \big\| S_\hbar^{(1,1)} w_{N,\hbar}^{(1)}(t)\big\|_{L^2_{x,x'}} \leq \left(\frac{1}{\ln N}\right)^{100}, \tag{2.11}$$

$$\int_{[0,T_0]} \big\| S_\hbar^{(1,1)} B_{N,\hbar,1,2}^{\pm} w_{N,\hbar}^{(2)}(t)\big\|_{L^2_{x,x'}} dt \leq \left(\frac{1}{\ln N}\right)^{100}, \tag{2.12}$$

which implies that

$$\|\gamma_{N,\hbar}^{(1)}(t,x;x) - \rho_{N,\hbar}(t,x)\|_{L_t^\infty[0,T_0]L_x^2(\mathbb{R}^d)} \leq \frac{C}{\ln N}, \tag{2.13}$$

$$\|J_{N,\hbar}^{(1)}(t,x;x) - J_{N,\hbar}(t,x)\|_{L_t^\infty[0,T_0]L_x^r(\mathbb{R}^d)} \leq \frac{C}{(\ln N)^{5\min\{1-\frac{1}{r},\frac{3}{r}-2\}}}, \tag{2.14}$$

$$\Big\|\big(B_{N,1,2}^{\pm}\gamma_{N,\hbar}^{(2)}\big)(t,x;x) - (\rho_{N,\hbar}V_N * \rho_{N,\hbar})(t,x)\Big\|_{L_t^1[0,T_0]L_x^1(B_R)} \leq C\frac{R^{d/2}+T_0}{\ln N}, \tag{2.15}$$

where $r \in (1, \frac{3}{2})$. Here $\pm$ does not matter as $(B_{N,1,2}^{+}\gamma_{N,\hbar}^{(2)})(t,x;x) = (B_{N,1,2}^{-}\gamma_{N,\hbar}^{(2)})(t,x;x)$.

Proof of Theorem 2.1 We prove (2.11) and (2.12) in Proposition 2.8. Here, we prove (2.13)–(2.15) using (2.11) and (2.12). For the mass density estimate (2.13), we split

$$w_{N,\hbar}^{(1)} = \big(P_{\leq M}^{1'} + P_{>M}^{1'}\big) w_{N,\hbar}^{(1)}, \tag{2.16}$$

where $P_{\leq M}$ denotes the Littlewood–Paley projection with M to be determined.

For the low frequency part, by Bernstein inequality and estimate (2.11), we have

$$\begin{aligned}\Big\|\big(P_{\leq M}^{1'} w_{N,\hbar}^{(1)}\big)(t,x;x)\Big\|_{L_x^2} &\leq \Big\|\big(P_{\leq M}^{1'} w_{N,\hbar}^{(1)}\big)(t,x;x')\Big\|_{L_x^2 L_{x'}^\infty} \\ &\lesssim M^{\frac{d}{2}} \big\|w_{N,\hbar}^{(1)}\big\|_{L^2_{x_1,x_1'}} \lesssim \frac{M^{\frac{d}{2}}}{(\ln N)^{100}}.\end{aligned}$$

For the high frequency part, by triangle inequality we have

$$\Big\|\big(P_{>M}^{1'} w_{N,\hbar}^{(1)}\big)(t,x;x)\Big\|_{L_x^2} \leq \Big\|\big(P_{>M}^{1'} \gamma_{N,\hbar}^{(1)}\big)(t,x;x)\Big\|_{L_x^2} + \Big\|\big(P_{<M}^{1'} \gamma_{H,\hbar}^{(1)}\big)(t,x;x)\Big\|_{L_x^2}.$$

It suffices to deal with $\gamma_{N,\hbar}^{(1)}$ as we can estimate $\gamma_{H,\hbar}^{(1)}$ in the same way. We use interpolation between L^1 and L^3

$$\left\|\left(P_{>M}^{1'}\gamma_{N,\hbar}^{(1)}\right)(t,x;x)\right\|_{L_x^2} \leq \left\|\left(P_{>M}^{1'}\gamma_{N,\hbar}^{(1)}\right)(t,x;x)\right\|_{L_x^1}^{\frac{1}{4}}\left\|\left(P_{>M}^{1'}\gamma_{N,\hbar}^{(1)}\right)(t,x;x)\right\|_{L_x^3}^{\frac{3}{4}}. \tag{2.17}$$

For the L_x^1 norm, we have, by definition of $\gamma_{N,\hbar}^{(1)}$ that

$$\left\|\left(P_{>M}^{1'}\gamma_{N,\hbar}^{(1)}\right)(t,x;x)\right\|_{L_x^1} = \int_{\mathbb{R}^d}\left|\int \psi_{N,\hbar}(t,x,\mathbf{x}_{2,N})\overline{P_{>M}^{1}\psi_{N,\hbar}}(t,x,\mathbf{x}_{2,N})d\mathbf{x}_{2,N}\right|dx,$$

where we have used $\mathbf{x}_{2,N}=(x_2,\ldots,x_N)$ for short. By Cauchy–Schwarz and Bernstein, the above

$$\begin{aligned}
&\leq \|\psi_{N,\hbar}\|_{L^2}\|P_{>M}^1\psi_{N,\hbar}\|_{L^2}\\
&\leq \|\psi_{N,\hbar}\|_{L^2}\frac{1}{\hbar M}\|\langle\hbar\nabla_{x_1}\rangle\psi_{N,\hbar}\|_{L^2}.
\end{aligned}$$

By the N-body energy bound (1.17), we reach

$$\left\|\left(P_{>M}^{1'}\gamma_{N,\hbar}^{(1)}\right)(t,x;x)\right\|_{L_x^1} \lesssim \frac{E_{0,\hbar}^{1/2}}{\hbar M}. \tag{2.18}$$

Similarly, for the L_x^3 norm, we have

$$\begin{aligned}
&\left\|\left(P_{>M}^{1'}\gamma_{N,\hbar}^{(1)}\right)(t,x;x)\right\|_{L_x^3}\\
&=\left[\int_{\mathbb{R}^d}\left|\int \psi_{N,\hbar}(t,x,\mathbf{x}_{2,N})\overline{P_{>M}^{1}\psi_{N,\hbar}}(t,x,\mathbf{x}_{2,N})d\mathbf{x}_{2,N}\right|^3dx\right]^{\frac{1}{3}}.
\end{aligned} \tag{2.19}$$

By Hölder, Minkowski, Sobolev, and the N-body energy bound (1.17), we get that the above

$$\begin{aligned}
&\leq \|\psi_{N,\hbar}\|_{L^2_{\mathbf{x}_{2,N}}L^6_{x_1}}\|P_{>M}^1\psi_{N,\hbar}\|_{L^2_{\mathbf{x}_{2,N}}L^6_{x_1}}\\
&\lesssim \|\langle\nabla_{x_1}\rangle\psi_{N,\hbar}\|_{L^2}\|\langle\nabla_{x_1}\rangle P_{>M}^1\psi_{N,\hbar}\|_{L^2}\lesssim\frac{E_{0,\hbar}}{\hbar^2}.
\end{aligned}$$

Combining (2.18) and (2.19), we obtain

$$\left\|\left(P_{>M}^{1'}\gamma_{N,\hbar}^{(1)}\right)(t,x;x)\right\|_{L_x^2} \lesssim \frac{1}{M^{1/4}}\left(\frac{E_{0,\hbar}}{\hbar^2}\right)^{\frac{7}{8}}. \tag{2.20}$$

By taking $M=(\ln N)^{10}$ and adopting the restriction (2.10), we obtain (2.13).

For the momentum estimate (2.14), we set

$$g_{N,\hbar}(t,x_1;x_1') = \hbar\nabla_{x_1}\gamma^{(1)}_{N,\hbar}(t,x_1;x_1') - \hbar\nabla_{x_1}\gamma^{(1)}_{H,\hbar}(t,x_1;x_1'). \tag{2.21}$$

We split

$$g_{N,\hbar} = \big(P^{1'}_{\le M} + P^{1'}_{>M}\big)g_{N,\hbar} \tag{2.22}$$

with M to be determined. By Interpolation,

$$\begin{aligned}\big\|(P^{1'}_{\le M}g_{N,\hbar})(t,x;x)\big\|_{L^r_x} &\le \big\|(P^{1'}_{\le M}g_{N,\hbar})(t,x;x)\big\|^{\frac{2}{r}-1}_{L^1_x}\big\|(P^{1'}_{\le M}g_{N,\hbar})(t,x;x)\big\|^{2-\frac{2}{r}}_{L^2_x}\\ &=: \mathrm{I}\cdot\mathrm{II}, \end{aligned}\tag{2.23}$$

$$\begin{aligned}\big\|(P^{1'}_{>M}g_{N,\hbar})(t,x;x)\big\|_{L^r_x} &\le \big\|(P^{1'}_{>M}g_{N,\hbar})(t,x;x)\big\|^{\frac{3}{r}-2}_{L^1_x}\big\|(P^{1'}_{>M}g_{N,\hbar})(t,x;x)\big\|^{3-\frac{3}{r}}_{L^{3/2}_x}\\ &=: \mathrm{III}\cdot\mathrm{IV}. \end{aligned}\tag{2.24}$$

Next, we separately estimate the above terms on the right hand side of (2.23) and (2.24).

For I, by triangle inequality we have

$$\begin{aligned}\big\|(P^{1'}_{\le M}g_{N,\hbar})(t,x;x)\big\|_{L^1_x} \le& \left\|\big(P^{1'}_{\le M}\hbar\nabla_{x_1}\gamma^{(1)}_{N,\hbar}\big)(t,x;x)\right\|_{L^1_x}\\ &+\left\|\big(P^{1'}_{\le M}\hbar\nabla_{x_1}\gamma^{(1)}_{H,\hbar}\big)(t,x;x)\right\|_{L^1_x}.\end{aligned}$$

By Cauchy–Schwarz and the N-body energy bound (1.17), we have

$$\left\|\big(P^{1'}_{\le M}\hbar\nabla_{x_1}\gamma^{(1)}_{N,\hbar}\big)(t,x;x)\right\|_{L^1_x} \le \|\hbar\nabla\psi_{N,\hbar}\|_{L^2}\|P^1_{\le M}\psi_{N,\hbar}\|_{L^2} \le E^{1/2}_{0,\hbar}. \tag{2.25}$$

Similarly, by Cauchy–Schwarz and the energy bound for $\phi_{N,\hbar}$, we have

$$\begin{aligned}\left\|\big(P^{1'}_{\le M}\hbar\nabla_{x_1}\gamma^{(1)}_{H,\hbar}\big)(t,x;x)\right\|_{L^1_x} &= \big\|(\hbar\nabla_{x_1}\phi_{N,\hbar})(t,x)\overline{P_{\le M}\phi_{N,\hbar}(t,x)}\big\|_{L^1_x}\\ &\le \|\hbar\nabla_{x_1}\phi_{N,\hbar}\|_{L^2_x}\|P_{\le M}\phi_{N,\hbar}\|_{L^2} \le E_0^{1/2}.\end{aligned}\tag{2.26}$$

With $E_0 \le E_{0,\hbar}$, we combine (2.25) and (2.26) to obtain

$$\mathrm{I} = \big\|(P^{1'}_{\le M}g_{N,\hbar})(t,x;x)\big\|^{\frac{2}{r}-1}_{L^1_x} \lesssim E^{\frac{1}{r}-\frac{1}{2}}_{0,\hbar}. \tag{2.27}$$

For II, we use Bernstein inequality and estimate (2.11) to get

$$\big\|(P^{1'}_{\le M}g_{N,\hbar})(t,x;x)\big\|_{L^2_x} \le \big\|(P^{1'}_{\le M}g_{N,\hbar})(t,x;x')\big\|_{L^2_xL^\infty_{x'}}$$

$$\lesssim M^{\frac{d}{2}} \| g_{N,\hbar}(t,x;x') \|_{L_x^2 L_{x'}^2} \lesssim \frac{M^{\frac{d}{2}}}{(\ln N)^{100}}, \tag{2.28}$$

and hence

$$\mathrm{II} \lesssim \left(\frac{M^{\frac{d}{2}}}{(\ln N)^{100}} \right)^{2-\frac{2}{r}}. \tag{2.29}$$

For III, by triangle inequality we have

$$\begin{aligned} \left\| (P^{1'}_{>M} g_{N,\hbar})(t,x;x) \right\|_{L_x^1} \leq & \left\| \left(P^{1'}_{>M} \hbar \nabla_{x_1} \gamma^{(1)}_{N,\hbar} \right)(t,x;x) \right\|_{L_x^1} \\ & + \left\| \left(P^{1'}_{>M} \hbar \nabla_{x_1} \gamma^{(1)}_{H,\hbar} \right)(t,x;x) \right\|_{L_x^1}. \end{aligned} \tag{2.30}$$

We use Cauchy–Schwarz, Bernstein, and the N-body energy bound (1.17) to obtain

$$\begin{aligned} & \left\| \left(P^{1'}_{>M} \hbar \nabla_{x_1} \gamma^{(1)}_{N,\hbar} \right)(t,x;x) \right\|_{L_x^1} \\ & = \int \left| \int \hbar \nabla_{x_1} \psi_{N,\hbar}(t,x,x_{2,N}) \overline{P^1_{>M} \psi_{N,\hbar}}(t,x,x_{2,N}) dx_{2,N} \right| dx \\ & \leq \| \hbar \nabla_{x_1} \psi_{N,\hbar} \|_{L^2} \| P^1_{>M} \psi_{N,\hbar} \|_{L^2} \\ & \leq M^{-1} \| \hbar \nabla_{x_1} \psi_{N,\hbar} \|_{L^2} \| \langle \nabla_{x_1} \rangle P^1_{>M} \psi_{N,\hbar} \|_{L^2} \\ & \lesssim \frac{E_{0,\hbar}}{\hbar M}. \end{aligned} \tag{2.31}$$

In the same method, we use the energy bound for $\phi_{N,\hbar}$ to get

$$\left\| \left(P^{1'}_{>M} \hbar \nabla_{x_1} \gamma^{(1)}_{H,\hbar} \right)(t,x;x) \right\|_{L_x^1} \lesssim \frac{E_0}{\hbar M}. \tag{2.32}$$

Combining (2.31) with (2.32), we have

$$\mathrm{III} \lesssim \left(\frac{E_{0,\hbar}}{\hbar M} \right)^{\frac{3}{r}-2}. \tag{2.33}$$

For IV, we use Hölder, Minkowski, Sobolev, and the N-body energy bound (1.17) to obtain

$$\begin{aligned} & \left\| \left(P^{1'}_{>M} \hbar \nabla_{x_1} \gamma^{(1)}_{N,\hbar} \right)(t,x;x) \right\|_{L_x^{3/2}} \\ & = \left[\int \left| \int \hbar \nabla_{x_1} \psi_{N,\hbar}(t,x,x_{2,N}) \overline{P^1_{>M} \psi_{N,\hbar}}(t,x,x_{2,N}) dx_{2,N} \right|^{\frac{3}{2}} dx \right]^{\frac{2}{3}} \\ & \leq \| \hbar \nabla_{x_1} \psi_{N,\hbar} \|_{L^2_{x_1} L^2_{\mathbf{x}_{2,N}}} \| P_{>M} \psi_{N,\hbar} \|_{L^6_{x_1} L^2_{\mathbf{x}_{2,N}}} \end{aligned}$$

$$
\begin{aligned}
&\leq \|\hbar\nabla_{x_1}\psi_{N,\hbar}\|_{L^2_{x_1}L^2_{\mathbf{x}_{2,N}}}\|P_{>M}\psi_{N,\hbar}\|_{L^2_{\mathbf{x}_{2,N}}L^6_{x_1}}\\
&\leq \|\hbar\nabla_{x_1}\psi_{N,\hbar}\|_{L^2_{x_1}L^2_{\mathbf{x}_{2,N}}}\|\langle\nabla_{x_1}\rangle\psi_{N,\hbar}\|_{L^2_{\mathbf{x}_{2,N}}L^2_{x_1}}\\
&\lesssim \frac{E_{0,\hbar}}{\hbar}.
\end{aligned}
\tag{2.34}
$$

In the same method, we use the energy bound for $\phi_{N,\hbar}$ to get

$$
\left\|\left(P^{1'}_{>M}\hbar\nabla_{x_1}\gamma^{(1)}_{H,\hbar}\right)(t,x;x)\right\|_{L^{3/2}_x}\lesssim\frac{E_0}{\hbar}.
\tag{2.35}
$$

Combining (2.34) with (2.35), we have

$$
\mathrm{IV}\lesssim\left(\frac{E_{0,\hbar}}{\hbar}\right)^{3-\frac{3}{r}}.
\tag{2.36}
$$

Putting together with estimates (2.27), (2.29), (2.33) and (2.36), we arrive at

$$
\begin{aligned}
&\left\|J^{(1)}_{N,\hbar}(t,x;x)-J_{N,\hbar}(t,x)\right\|_{L^r_x}\\
&\quad\lesssim E_{0,\hbar}^{\frac{1}{r}-\frac{1}{2}}\left(\frac{M^{\frac{d}{2}}}{(\ln N)^{100}}\right)^{2-\frac{2}{r}}+\left(\frac{E_{0,\hbar}}{\hbar M}\right)^{\frac{3}{r}-2}\left(\frac{E_{0,\hbar}}{\hbar}\right)^{3-\frac{3}{r}}.
\end{aligned}
\tag{2.37}
$$

Setting $M=(\ln N)^{20}$, the above

$$
\leq\frac{E_{0,\hbar}}{\hbar}\left(\frac{1}{(\ln N)^{10}}\right)^{\min\{1-\frac{1}{r},\frac{3}{r}-2\}}.
$$

For fixed $r\in(1,3/2)$, we make use of the restriction (2.10) to obtain

$$
\left\|J^{(1)}_{N,\hbar}(t,x;x)-J_{N,\hbar}(t,x)\right\|_{L^r_x}\lesssim\frac{1}{(\ln N)^{5\min\{1-\frac{1}{r},\frac{3}{r}-2\}}},
\tag{2.38}
$$

which completes the proof of (2.14).

For the pressure estimate (2.15), we set

$$
p^{\pm}_{N,\hbar}(t,x_1,x_1')=\left[B^{\pm}_{N,1,2}\left(\gamma^{(2)}_{N,\hbar}-\gamma^{(2)}_{H,\hbar}\right)\right](t,x_1,x_1').
\tag{2.39}
$$

Again we split

$$
p^{\pm}_{N,\hbar}=\left(P^{1'}_{\leq M}+P^{1'}_{>M}\right)p^{\pm}_{N,\hbar}
\tag{2.40}
$$

with M to be determined. We use Hölder and Bernstein inequalities to obtain

$$
\left\|\left(P^{1'}_{\leq M}p^{\pm}_{N,\hbar}\right)(t,x;x)\right\|_{L^1([0,T_0];L^1_x(B_R))}
$$

$$
\begin{aligned}
&\leq R^{\frac{d}{2}} \left\| \left(P^{1}_{\leq M} p^{\pm}_{N,\hbar}\right)(t,x;x) \right\|_{L^1([0,T_0];L^2_x(B_R))} \\
&\leq R^{\frac{d}{2}} \left\| \left(P^{1'}_{\leq M} p^{\pm}_{N,\hbar}\right)(t,x;x') \right\|_{L^1([0,T_0];L^2_x L^\infty_{x'}(\mathbb{R}^d))} \\
&\leq M^{\frac{d}{2}} R^{\frac{d}{2}} \left\| \left(P^{1'}_{\leq M} p^{\pm}_{N,\hbar}\right)(t,x;x') \right\|_{L^1([0,T_0];L^2_x L^2_{x'}(\mathbb{R}^d))}.
\end{aligned} \tag{2.41}
$$

By estimate (2.12), we arrive at

$$
\left\| \left(P^{1'}_{\leq M} p^{\pm}_{N,\hbar}\right)(t,x;x) \right\|_{L^1([0,T_0];L^1_x(B_R))} \leq \frac{\hbar M^{\frac{d}{2}} R^{\frac{d}{2}}}{(\ln N)^{100}}. \tag{2.42}
$$

On the other hand, we note that

$$
\begin{aligned}
&P^{1'}_{>M} B^{+}_{N,1,2} \gamma^{(2)}_{N,\hbar}(t,x_1;x_1') \\
&\quad = \int V_N(x_1-x_2)\psi_{N,\hbar}(t,x_1,\mathbf{x}_{2,N}) \overline{P^{1}_{>M}\psi_{N,\hbar}}(t,x_1',\mathbf{x}_{2,N}) d\mathbf{x}_{2,N}.
\end{aligned}
$$

Hence, by Cauchy–Schwarz we have

$$
\begin{aligned}
&\int |P^{1'}_{>M} B^{+}_{N,1,2} \gamma^{(2)}_{N,\hbar}(t,x_1;x_1)| dx_1 \\
&\quad \leq \langle \psi_{N,\hbar}, V_N(x_1-x_2)\psi_{N,\hbar}\rangle^{1/2} \langle P^{1}_{>M}\psi_{N,\hbar}, V_N(x_1-x_2) P^{1}_{>M}\psi_{N,\hbar}\rangle^{1/2}.
\end{aligned} \tag{2.43}
$$

By estimate (A.17) in Lemma A.7, Bernstein inequality, and the N-body energy bound (1.17), the above

$$
\begin{aligned}
&\lesssim \langle \psi_{N,\hbar}, (1-\Delta_{x_1})(1-\Delta_{x_2})\psi_{N,\hbar}\rangle^{1/2} \left\langle P^{1}_{>M}\psi_{N,\hbar}, \left[(1-\Delta_{x_1})(1-\Delta_{x_2})\right]^{\frac{d}{4}+} P^{1}_{>M}\psi_{N,\hbar}\right\rangle^{1/2} \\
&\lesssim \frac{1}{M^{(1-\frac{d}{4})-}} \|\langle\nabla_{x_1}\rangle\langle\nabla_{x_2}\rangle\psi_{N,\hbar}\|_{L^2} \|\langle\nabla_{x_1}\rangle\langle\nabla_{x_2}\rangle P^{1}_{>M}\psi_{N,\hbar}\|_{L^2} \\
&\lesssim \frac{E^2_{0,\hbar}}{\hbar^4 M^{(1-\frac{d}{4})-}}.
\end{aligned}
$$

In the same method, we use the energy bound for $\phi_{N,\hbar}$ to get

$$
\| P^{1'}_{>M} B^{\pm}_{N,1,2} \gamma^{(2)}_{H,\hbar}(t,x;x) \|_{L^1_x} \lesssim \frac{E_0^2}{\hbar^4 M^{(1-\frac{d}{4})-}}. \tag{2.44}
$$

Estimates (2.42), (2.43), and (2.44) together give

$$
\begin{aligned}
&\left\| \left(B^{\pm}_{N,1,2}\gamma^{(2)}_{N,\hbar}\right)(t,x,x) - \left(\rho_{N,\hbar} V_N * \rho_{N,\hbar}\right)(t,x) \right\|_{L^1([0,T_0];L^1(B_R))} \\
&\quad = \left\| \left[B^{\pm}_{N,1,2}\left(\gamma^{(2)}_{N,\hbar} - \gamma^{(2)}_{H,\hbar}\right)\right](t,x;x) \right\|_{L^1([0,T_0];L^1(B_R))} \lesssim \frac{\hbar M^{\frac{d}{2}} R^{\frac{d}{2}}}{(\ln N)^{100}} + \frac{T_0 E^2_{0,\hbar}}{\hbar^4 M^{(1-\frac{d}{4})-}}.
\end{aligned} \tag{2.45}
$$

By taking $M = (\ln N)^{50}$, the above

$$\leq \frac{\hbar R^{\frac{d}{2}}}{(\ln N)^{10}} + \frac{T_0 E_{0,\hbar}^4}{\hbar^4 (\ln N)^{10}}.$$

For fixed T_0, we utilize the restriction (2.10) to get

$$\left\| \left(B_{N,1,2}^{\pm}\gamma_{N,\hbar}^{(2)}\right)(t,x;x) - \left(\rho_{N,\hbar} V_N * \rho_{N,\hbar}\right)(t,x) \right\|_{L^1([0,T_0];L^1(B_R))} \lesssim \frac{R^{d/2}+T_0}{\ln N},$$

which completes the proof of (2.15). □

The proof of Theorem 2.1 is hence concluded assuming (2.11) and (2.12) included in Proposition 2.8. The rest of Sect. 2 is to prove Proposition 2.8.

2.1 A Tool Box of Space-time Estimates

We reproduce and rewrite [22, Section 2] with $\hbar$ for our purpose here and provide some preliminary estimates for $w_{N,\hbar}^{(k)}$. We start by rewriting the $3D$ cubic BBGKY hierarchy (2.2) in integral form

$$\begin{aligned} \gamma_{N,\hbar}^{(k)} &= U_\hbar^{(k)}\gamma_{N,\hbar}^{(k)}(0) + \int_0^{t_k} U_\hbar^{(k)}(t_k - t_{k+1}) V_{N,\hbar}^{(k)}\gamma_{N,\hbar}^{(k)}(t_{k+1})dt_{k+1} \\ &\quad + \frac{N-k}{N}\int_0^{t_k} U_\hbar^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)}\gamma_{N,\hbar}^{(k+1)}(t_{k+1})dt_{k+1} \end{aligned} \tag{2.46}$$

where we have adopted the shorthands[13]

$$U_\hbar^{(k)}(t) = \prod_{j=1}^{k} e^{it\hbar\Delta_{x_j}/2} e^{-it\hbar\Delta_{x_j'}/2}, \tag{2.47}$$

$$V_{N,\hbar}^{(k)}\gamma_{N,\hbar}^{(k)} = \frac{1}{N}\sum_{1\leq i<j\leq k}\left[V_{N,\hbar}(x_i - x_j), \gamma_{N,\hbar}^{(k)}\right], \tag{2.48}$$

$$V_{N,\hbar}(x) = \frac{1}{\hbar} N^{d\beta} V(N^\beta x), \tag{2.49}$$

$$B_{N,\hbar}^{(k+1)}\gamma_{N,\hbar}^{(k+1)} = \sum_{j=1}^{k} B_{N,\hbar,j,k+1}\gamma_{N,\hbar}^{(k+1)} = \sum_{j=1}^{k} \mathrm{Tr}_{k+1}\left[V_{N,\hbar}(x_j - x_{k+1}), \gamma_{N,\hbar}^{(k+1)}\right], \tag{2.50}$$

and we have omitted the $(-i)$ in front of the second and third terms in the right hand side of (2.46) as it serves as 1 in our estimates. In addition to (2.46), we write (2.3) in

[13] Please notice that we have divided by $\hbar$ to use (2.47).

integral form

$$\gamma_{H,\hbar}^{(k)}(t_k) = U_\hbar^{(k)}(t_k)\gamma_{H,\hbar}^{(k)}(0) + \int_0^{t_k} U_\hbar^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)} \gamma_{H,\hbar}^{(k+1)}(t_{k+1}) dt_{k+1}. \tag{2.51}$$

The difference $w_{N,\hbar}^{(k)} = \gamma_{N,\hbar}^{(k)} - \gamma_{H,\hbar}^{(k)}$ solves the hierarchy

$$\begin{aligned} w_{N,\hbar}^{(k)}(t_k) &= U_\hbar^{(k)}(t_k) w_{N,\hbar}^{(k)}(0) + \int_0^{t_k} U_\hbar^{(k)}(t_k - t_{k+1}) V_{N,\hbar}^{(k)} \gamma_{N,\hbar}^{(k)}(t_{k+1}) dt_{k+1} \\ &\quad - \frac{k}{N} \int_0^{t_k} U_\hbar^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)} \gamma_{N,\hbar}^{(k+1)}(t_{k+1}) dt_{k+1} \\ &\quad + \int_0^{t_k} U_\hbar^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)} w_{N,\hbar}^{(k+1)}(t_{k+1}) dt_{k+1}. \end{aligned} \tag{2.52}$$

Iterating hierarchy (2.52) l_c times[14] at the last term of (2.52), we have

$$w_{N,\hbar}^{(k)}(t_k) = \mathrm{FP}^{(k,l_c)}(t_k) + \mathrm{DP}^{(k,l_c)}(t_k) + \mathrm{EP}^{(k,l_c)}(t_k) + \mathrm{IP}^{(k,l_c)}(t_k), \tag{2.53}$$

where we have grouped the terms in $w_{N,\hbar}^{(k)}(t_k)$ into four parts: the free/driving/error/interaction parts. We remark that (2.53) holds for all $l_c \geq 1$ and we will select l_c depending on what aspect of $w_{N,\hbar}^{(k)}$ we need in Sects. 2.2–2.4. To write out the four parts of $w_{N,\hbar}^{(k)}$, we define the notation that, for $j \geq 1$,

$$\begin{aligned} &J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)})(f^{(k+j)}(t_{k+j})) \\ &\quad = \big(U_\hbar^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)}\big) \cdots \big(U_\hbar^{(k+j-1)}(t_{k+j-1} - t_{k+j}) B_{N,\hbar}^{(k+j)}\big) f^{(k+j)}(t_{k+j}), \end{aligned} \tag{2.54}$$

and $J_{N,\hbar}^{(k,0)}(t_k, \underline{t}_{(k,0)})(f^{(k)}(t_k)) = f^{(k)}(t_k)$, where $\underline{t}_{(k,j)} = (t_{k+1}, \ldots, t_{k+j})$ for $j \geq 1$. In this notation, the free part of $w_{N,\hbar}^{(k)}$ at l_c coupling level is

$$\begin{aligned} \mathrm{FP}^{(k,l_c)}(t_k) &= U_\hbar^{(k)}(t_k) w_{N,\hbar}^{(k)}(0) \\ &\quad + \sum_{j=1}^{l_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} U_\hbar^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)} \cdots \\ &\quad \times U_\hbar^{(k+j-1)}(t_{k+j-1} - t_{k+j}) B_{N,\hbar}^{(k+j)} \big(U_\hbar^{(k+j)}(t_{k+j}) w_{N,\hbar}^{(k+j)}(0)\big) d\underline{t}_{(k,j)} \\ &= \sum_{j=0}^{l_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)}) \big(f_{\mathrm{FP}}^{(k,j)}(t_{k+j})\big) d\underline{t}_{(k,j)}, \end{aligned} \tag{2.55}$$

[14] l_c means "coupling level".

where in the $j=0$ case, it is meant that there are no time integrals and $J_{N,\hbar}^{(k,0)}$ is the identity operator, and

$$f_{\mathrm{FP}}^{(k,j)}(t_{k+j}) = U_{\hbar}^{(k+j)}(t_{k+j}) w_{N,\hbar}^{(k+j)}(0). \tag{2.56}$$

The driving part is given by

$$\begin{aligned}
\mathrm{DP}^{(k,l_c)}(t_k) &= \int_0^{t_k} U_{\hbar}^{(k)}(t_k - t_{k+1}) V_{N,\hbar}^{(k)} \gamma_{N,\hbar}^{(k)}(t_{k+1}) dt_{k+1} \\
&\quad + \sum_{j=1}^{l_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} U_{\hbar}^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)} \cdots U_{\hbar}^{(k+j-1)}(t_{k+j-1} - t_{k+j}) B_{N,\hbar}^{(k+j)} \\
&\quad \times \left(\int_0^{t_{k+j}} U_{\hbar}^{(k+j)}(t_{k+j} - t_{k+j+1}) V_{N,\hbar}^{(k+j)} \gamma_{N,\hbar}^{(k+j)}(t_{k+j+1}) dt_{k+j+1} \right) d\underline{t}_{(k,j)} \\
&= \sum_{j=0}^{l_c} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_{N,\hbar}^{(k,j)}(\underline{t}_{(k,j)}) \big(f_{\mathrm{DP}}^{(k,j)}(t_{k+j}) \big) d\underline{t}_{(k,j)},
\end{aligned} \tag{2.57}$$

where in the $j=0$ case, it is meant that there are no time integrals and $J_{N,\hbar}^{(k,0)}$ is the identity operator, and

$$f_{\mathrm{DP}}^{(k,j)}(t_{k+j}) = \int_0^{t_{k+j}} U_{\hbar}^{(k+j)}(t_{k+j} - t_{k+j+1}) V_{N,\hbar}^{(k+j)} \gamma_{N,\hbar}^{(k+j)}(t_{k+j+1}) dt_{k+j+1}. \tag{2.58}$$

The error part is given by

$$\begin{aligned}
&\mathrm{EP}^{(k,l_c)}(t_k) \\
&= -\frac{k}{N} \int_0^{t_k} U_{\hbar}^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)} \gamma_{N,\hbar}^{(k+1)}(t_{k+1}) dt_{k+1} \\
&\quad - \sum_{j=1}^{l_c} \frac{k+j}{N} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} U_{\hbar}^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)} \cdots U_{\hbar}^{(k+j-1)}(t_{k+j-1} - t_{k+j}) B_{N,\hbar}^{(k+j)} \\
&\quad \times \left(\int_0^{t_{k+j}} U_{\hbar}^{(k+j)}(t_{k+j} - t_{k+j+1}) B_{N,\hbar}^{(k+j+1)} \gamma_{N,\hbar}^{(k+j+1)}(t_{k+j+1}) dt_{k+j+1} \right) d\underline{t}_{(k,j)} \\
&= \sum_{j=1}^{l_c+1} \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_{N,\hbar}^{(k,j)}(\underline{t}_{(k,j)}) \big(f_{\mathrm{EP}}^{(k,j)}(t_{k+j}) \big) d\underline{t}_{(k,j)},
\end{aligned} \tag{2.59}$$

where

$$f_{\mathrm{EP}}^{(k,j)}(t_{k+j}) = -\frac{k+j-1}{N} \gamma_{N,\hbar}^{(k+j)}. \tag{2.60}$$

The interaction part is given by

$$\mathrm{IP}^{(k,l_c)}(t_k) = \int_0^{t_k} \cdots \int_0^{t_{k+l_c}} U_{\hbar}^{(k)}(t_k - t_{k+1}) B_{N,\hbar}^{(k+1)} \cdots$$

$$\times U^{(k+l_c)}(t_{k+l_c} - t_{k+l_c+1}) B_{N,\hbar}^{(k+l_c+1)}\big(w_{N,\hbar}^{(k+l_c+1)}(t_{k+l_c+1})\big) dt_{k+1} \cdots dt_{k+l_c+1}$$
$$= \int_0^{t_k} \cdots \int_0^{t_{k+l_c}} J_{N,\hbar}^{(k,l_c+1)}(t_k, \underline{t}_{(k,l_c+1)})\big(w_{N,\hbar}^{(k+l_c+1)}(t_{k+l_c+1})\big) d\underline{t}_{(k,l_c+1)}, \tag{2.61}$$

where

$$f_{\mathrm{IP}}^{(k,l_c+1)} = w_{N,\hbar}^{(k+l_c+1)}(t_{k+l_c+1}). \tag{2.62}$$

There are around $\frac{(k+l_c)!}{k!}$ many summands in each part. They can be grouped together by using the KM board game argument [46], which is below.

Lemma 2.2 [46, Lemma 2.1][15] *For $j \geq 1$, one can express*

$$\int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)})(f^{(k+j)}) d\underline{t}_{(k,j)}$$
$$= \sum_m \int_D J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)}, \mu_m)(f^{(k+j)}) d\underline{t}_{(k,j)}. \tag{2.63}$$

Here $D \subset [0, t_k]^j$, μ_m are a set of maps from $\{k+1, \ldots, k+j\}$ to $\{1, \ldots, k+j-1\}$ and $\mu_m(l) < l$ for all l, and

$$J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)}, \mu_m)(f^{(k+j)})$$
$$= \big(U_\hbar^{(k)}(t_k - t_{k+1}) B_{N,\hbar,\mu_m(k+1),k+1}\big) \cdots$$
$$\times \big(U_\hbar^{(k+j-1)}(t_{k+j-1} - t_{k+j}) B_{N,\hbar,\mu_m(k+j),k+j}\big) f^{(k+j)}(t_{k+j}). \tag{2.64}$$

The summing number can be controlled by 2^{k+2j-2}.

Then we are able to estimate $J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)})(f^{(k+j)})$ via collapsing estimates in Lemma A.2.

Lemma 2.3 *Let $d = 3$ and $\alpha = d + 1/2$. For $j \geq 1$,*

$$\left\| \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} S_\hbar^{(1,k)} J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)})(f^{(k+j)}) d\underline{t}_{(k,j)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}}$$
$$\leq 2^k 4^j (C_V \hbar^{-\alpha} T^{1/2})^{j-1} \int_{[0,T]} \left\| S_\hbar^{(1,k+j-1)} B_{N,\hbar,1,k+j}(f^{(k+j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j}, \tag{2.65}$$

$$\left\| S_\hbar^{(1,k)} B_{N,\hbar,1,k+1} \int_0^{t_{k+1}} \cdots \int_0^{t_{k+j}} J_{N,\hbar}^{(k+1,j)}(t_{k+1}, \underline{t}_{(k+1,j)})(f^{(k+j+1)}) d\underline{t}_{(k+1,j)} \right\|_{L^1_{t_{k+1}}[0,T] L^2_{x,x'}}$$

[15] More advanced version of this combinatoric is now available, see [23, 25].

$$\leq 2^{k+1}4^j (C_V \hbar^{-\alpha} T^{1/2})^j \int_{[0,T]} \left\| S_\hbar^{(1,k+j)} B_{N,\hbar,1,k+j+1}(f^{(k+j+1)}(t_{k+j+1})) \right\|_{L^2_{x,x'}} dt_{k+j+1}. \tag{2.66}$$

Proof This is well-known for $\hbar = 1$. We include a proof for completeness. For (2.65), we start by using Lemma 2.2,

$$\begin{aligned} &\left\| \int_0^{t_k} \cdots \int_0^{t_{k+j-1}} S_\hbar^{(1,k)} J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)})(f^{(k+j)}) d\underline{t}_{(k,j)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\ &\leq 2^k 4^j \left\| \int_D S_\hbar^{(1,k)} J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)}, \mu_m)(f^{(k+j)}) d\underline{t}_{(k,j)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\ &\leq 2^k 4^j \int_{[0,T]^j} \left\| S_\hbar^{(1,k)} J_{N,\hbar}^{(k,j)}(t_k, \underline{t}_{(k,j)}, \mu_m)(f^{(k+j)}) \right\|_{L^2_{x,x'}} d\underline{t}_{(k,j)}. \end{aligned} \tag{2.67}$$

By Cauchy–Schwarz at dt_{k+1}, the above

$$\leq 2^k 4^j T^{1/2} \int_{[0,T]^{j-1}} \left\| S_\hbar^{(1,k)} B_{N,\hbar,\mu_m(k+1),k+1} U_\hbar^{(k+1)}(t_{k+1} - t_{k+2}) \cdots \right\|_{L^2_{t_{k+1}}[0,T] L^2_{x,x'}} d\underline{t}_{(k+1,j-1)}. \tag{2.68}$$

By Lemma A.2, the above

$$\leq 2^k 4^j C_V \hbar^{-\alpha} T^{1/2} \int_{[0,T]^{j-1}} \left\| S_\hbar^{(1,k+1)} B_{N,\hbar,\mu_m(k+2),k+2} U_\hbar^{(k+2)}(t_{k+2} - t_{k+3}) \cdots \right\|_{L^2_{x,x'}} d\underline{t}_{(k+1,j-1)}. \tag{2.69}$$

Repeating such a process gives that the above

$$\leq 2^k 4^j (C_V \hbar^{-\alpha} T^{1/2})^{j-1} \int_{[0,T]} \left\| S_\hbar^{(1,k+j-1)} B_{N,\hbar,\mu_m(k+j),k+j}(f^{(k+j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j}. \tag{2.70}$$

By symmetry, the above

$$= 2^k 4^j (C_V \hbar^{-\alpha} T^{1/2})^{j-1} \int_{[0,T]} \left\| S_\hbar^{(1,k+j-1)} B_{N,\hbar,1,k+j}(f^{(k+j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j}. \tag{2.71}$$

For (2.66), we apply Lemma 2.2 again to obtain

$$\begin{aligned} &\left\| \int_0^{t_{k+1}} \cdots \int_0^{t_{k+j}} S_\hbar^{(1,k)} B_{N,\hbar,1,k+1} J_{N,\hbar}^{(k+1,j)}(t_{k+1}, \underline{t}_{(k+1,j)})(f^{(k+j+1)}) d\underline{t}_{(k+1,j)} \right\|_{L^1_{t_{k+1}}[0,T] L^2_{x,x'}} \\ &\leq 2^{k+1} 4^j \left\| \int_D S_\hbar^{(1,k)} B_{N,\hbar,1,k+1} J_{N,\hbar}^{(k+1,j)}(t_{k+1}, \underline{t}_{(k+1,j)}, \mu_m)(f^{(k+j+1)}) d\underline{t}_{(k+1,j)} \right\|_{L^1_{t_{k+1}}[0,T] L^2_{x,x'}} \end{aligned}$$

$$\leq 2^{k+1}4^j \int_{[0,T]^j} \left\| S_\hbar^{(1,k)} B_{N,\hbar,1,k+1} J_{N,\hbar}^{(k+1,j)}(t_{k+1}, \underline{t}_{(k+1,j)}, \mu_m)(f^{(k+j+1)}) \right\|_{L^1_{t_{k+1}}[0,T]L^2_{x,x'}} d\underline{t}_{(k+1,j)}. \tag{2.72}$$

By Cauchy–Schwarz at dt_{k+1}, the above

$$\leq 2^{k+1}4^j T^{1/2} \int_{[0,T]^j} \left\| S_\hbar^{(1,k)} B_{N,\hbar,1,k+1} U_\hbar^{(k+1)}(t_{k+1} - t_{k+2}) \cdots \right\|_{L^2_{t_{k+1}}[0,T]L^2_{x,x'}} d\underline{t}_{(k+1,j)}.$$

Iterating the same process as (2.68), we obtain that the above

$$\leq 2^{k+1}4^j (C_V \hbar^{-\alpha} T^{1/2})^j \int_{[0,T]} \left\| S_\hbar^{(1,k+j)} B_{N,\hbar,1,k+j+1}(f^{(k+j+1)}(t_{k+j+1})) \right\|_{L^2_{x,x'}} dt_{k+j+1}. \tag{2.73}$$

□

Away from Lemma 2.3, we obtain below crude estimates of the driving part, error part and the interaction part.

Lemma 2.4 *Let $k \leq (\ln N)^{10}$ and $j \leq (\ln N)^{10}$. For the driving part, we have*

$$\left\| S_\hbar^{(1,k)} f_{\mathrm{DP}}^{(k,0)}(t_k) \right\|_{L^\infty_{t_k}[0,T]L^2_{x,x'}} \leq N^{\frac{5}{2}\beta-1}(C_V \hbar^{-\alpha} T^{1/2}) k^2 (2E_{0,\hbar})^k \tag{2.74}$$

and

$$\int_{[0,T]} \left\| S_\hbar^{(1,k+j-1)} B_{N,\hbar,1,k+j}(f_{\mathrm{DP}}^{(k,j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j} \leq N^{\frac{5}{2}\beta-1}(C_V \hbar^{-\alpha} T^{1/2})^2 (k+j)^2 (2E_{0,\hbar})^{k+j}. \tag{2.75}$$

For the error part, we have

$$\int_{[0,T]} \left\| S_\hbar^{(1,k+j-1)} B_{N,\hbar,1,k+j}(f_{\mathrm{EP}}^{(k,j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j} \leq N^{\frac{5}{2}\beta-1}(C_V \hbar^{-\alpha} T^{1/2})(k+j)(2E_{0,\hbar})^{k+j}. \tag{2.76}$$

For the interaction part, we have

$$\int_{[0,T]} \left\| S_\hbar^{(1,k+j-1)} B_{N,\hbar,1,k+j}(f_{\mathrm{IP}}^{(k,j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j} \leq N^{\frac{5}{2}\beta}(C_V \hbar^{-\alpha} T^{1/2})(4E_{0,\hbar})^{k+j}. \tag{2.77}$$

Proof For (2.74), plugging in $f_{\mathrm{DP}}^{(k,0)}$, we need to estimate

$$\left\| S_\hbar^{(1,k)} \int_0^{t_k} U_\hbar^{(k)}(t_k - t_{k+1}) V_{N,\hbar}^{(k)} \gamma_{N,\hbar}^{(k)}(t_{k+1}) dt_{k+1} \right\|_{L^\infty_{t_k}[0,T]L^2_{x,x'}}. \tag{2.78}$$

By (A.7) in Lemma A.4, the above

$$\leq N^{\frac{5}{2}\beta-1}\hbar(C_V\hbar^{-\alpha}T^{1/2})k^2\|S_{\hbar}^{(1,k)}\gamma_{N,\hbar}^{(k)}(t_{k+1})\|_{L^{\infty}_{t_{k+1}}L^2_{x,x'}}.$$

Using the N-body energy bound (1.17) and discarding the unimportant factor[16] $\hbar$, we arrive at

$$\begin{aligned}&\left\|S_{\hbar}^{(1,k)}\int_0^{t_k}U_{\hbar}^{(k)}(t_k-t_{k+1})V_{N,\hbar}^{(k)}\gamma_{N,\hbar}^{(k)}(t_{k+1})dt_{k+1}\right\|_{L^{\infty}_{t_k}[0,T]L^2_{x,x'}}\\&\leq N^{\frac{5}{2}\beta-1}(C_V\hbar^{-\alpha}T^{1/2})k^2(2E_{0,\hbar})^k,\end{aligned}\tag{2.79}$$

which completes the proof of (2.74).

For (2.75), we insert $f_{\mathrm{DP}}^{(k,j)}$ defined in (2.58) to obtain

$$\begin{aligned}&\int_{[0,T]}\left\|S_{\hbar}^{(1,k+j-1)}B_{N,\hbar,1,k+j}(f_{\mathrm{DP}}^{(k,j)}(t_{k+j}))\right\|_{L^2_{x,x'}}dt_{k+j}\\&=\left\|S_{\hbar}^{(1,k+j-1)}B_{N,\hbar,1,k+j}\int_0^{t_{k+j}}U_{\hbar}^{(k+j)}(t_{k+j}-t_{k+j+1})\right.\\&\quad\left.\times V_{N,\hbar}^{(k+j)}\gamma_{N,\hbar}^{(k+j)}(t_{k+j+1})dt_{k+j+1}\right\|_{L^1_{t_{k+j}}[0,T]L^2_{x,x'}}.\end{aligned}\tag{2.80}$$

Utilizing (A.8) in Lemma A.4, the above

$$\leq N^{\frac{5}{2}\beta-1}\hbar(C_V\hbar^{-\alpha}T^{1/2})^2(k+j)^2\left\|S_{\hbar}^{(1,k+j)}\gamma_{N,\hbar}^{(k+j)}(t_{k+j+1})\right\|_{L^{\infty}_{t_{k+j+1}}L^2_{x,x'}}.$$

Making use of the N-body energy bound (1.17) and discarding the unimportant small factor $\hbar$, (2.75) is then proved.

For the error part (2.76), inserting $f_{\mathrm{EP}}^{(k,j)}$ we have

$$\begin{aligned}&\int_{[0,T]}\left\|S_{\hbar}^{(1,k+j-1)}B_{N,\hbar,1,k+j}(f_{\mathrm{EP}}^{(k,j)}(t_{k+j}))\right\|_{L^2_{x,x'}}dt_{k+j}\\&=\frac{k+j-1}{N}\int_{[0,T]}\left\|S_{\hbar}^{(1,k+j-1)}B_{N,\hbar,1,k+j}(\gamma_{N,\hbar}^{(k+j)}(t_{k+j}))\right\|_{L^2_{x,x'}}dt_{k+j}.\end{aligned}\tag{2.81}$$

By (A.11) in Lemma A.5 and the N-body energy bound (1.17), the above

$$\leq N^{\frac{5}{2}\beta-1}\hbar^2T^{1/2}(C_V\hbar^{-\alpha}T^{1/2})(k+j)(2E_{0,\hbar})^{k+j}.$$

Discarding the unimportant small factor $\hbar^2T^{1/2}$, we complete the proof of (2.76).

[16] Keeping this $\hbar$ does not give much better estimate in the end. In fact, as we will see, $\hbar^{-\alpha}$ accumulates but this $\hbar$ stays as just one factor.

For the interaction part (2.77), inserting $f_{\mathrm{IP}}^{(k,j)}$ we have

$$\begin{aligned}
&\int_{[0,T]} \left\| S_{\hbar}^{(1,k+j-1)} B_{N,\hbar,1,k+j} (f_{\mathrm{IP}}^{(k,j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j} \\
&\quad \leq \int_{[0,T]} \left\| S_{\hbar}^{(1,k+j-1)} B_{N,\hbar,1,k+j} (\gamma_{N,\hbar}^{(k+j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j} \\
&\qquad + \int_{[0,T]} \left\| S_{\hbar}^{(1,k+j-1)} B_{N,\hbar,1,k+j} (\gamma_{H,\hbar}^{(k+j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j}.
\end{aligned} \tag{2.82}$$

By (A.11) in Lemma A.5 and the N-body energy bound (1.17), the above

$$\leq N^{\frac{5}{2}\beta} \hbar^2 T^{1/2} (C_V \hbar^{-\alpha} T^{1/2}) (4E_{0,\hbar})^{k+j}.$$

By discarding the unimportant small factor $\hbar^2 T^{1/2}$, we complete the proof of (2.77). □

2.2 A Klainerman–Machedon Bound First

Via the preliminary estimates in Sect. 2.1, we are able to provide a "preliminary" Klainerman–Machedon bound for $w_{N,\hbar}^{(k)}$. Here, "preliminary" certainly means, "not final" as we will improve it once we have used it to prove (2.11).

Lemma 2.5 *Let $t_0 \in [0, \infty)$, $T \leq \frac{\hbar^{2\alpha}}{(64 E_{0,\hbar} C_V e)^2}$, and $\alpha = d + \frac{1}{2}$. For $k \leq (\ln N)^{10}$, we have*

$$\int_{[t_0, t_0+T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} w_{N,\hbar}^{(k+1)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1} \leq (16 E_{0,\hbar})^k. \tag{2.83}$$

It holds for sufficiently small T but independent of the initial time.

Proof We give a proof following the method in [14, 17, 19] which was inspired by [12]. We might as well take $t_0 = 0$ for convenience, as the general case also holds from time translation. Decomposing $w_{N,\hbar}^{(k)}$ as in (2.53), it suffices to prove that

$$\int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} \mathrm{FP}^{(k+1,l_c)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1} \leq (8 E_{0,\hbar})^k, \tag{2.84}$$

$$\int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} \mathrm{DP}^{(k+1,l_c)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1} \leq (8 E_{0,\hbar})^k, \tag{2.85}$$

$$\int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} \mathrm{EP}^{(k+1,l_c)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1} \leq (8 E_{0,\hbar})^k, \tag{2.86}$$

$$\int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} \mathrm{IP}^{(k+1,l_c)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1} \leq (8 E_{0,\hbar})^k. \tag{2.87}$$

For the FP part (2.84), we start by using estimate (2.66) in Lemma 2.3 to obtain

$$\begin{aligned}
&\int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} \mathrm{FP}^{(k+1,l_c)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1} \\
&\le \int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} f_{\mathrm{FP}}^{(k+1,0)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1} \\
&\quad + \sum_{j=1}^{l_c} 2^{k+1} 4^j (C_V \hbar^{-\alpha} T^{1/2})^j \int_{[0,T]} \left\| S_{\hbar}^{(1,k+j)} B_{N,\hbar,1,k+j+1} \big(f_{\mathrm{FP}}^{(k+1,j)}(t_{k+j+1}) \big) \right\|_{L^2_{x,x'}} dt_{k+j+1}.
\end{aligned}$$

Plugging in $f_{\mathrm{FP}}^{(k+1,j)}$, applying Cauchy–Schwarz at dt_{k+j+1} and then Lemma A.2, the above

$$\le 2^{k+1} \sum_{j=0}^{l_c} (4C_V \hbar^{-\alpha} T^{1/2})^{j+1} \left\| S_{\hbar}^{(1,k+j+1)} w_{N,\hbar}^{(k+j+1)}(0) \right\|_{L^2_{x,x'}}.$$

We have required that $l_c \le \ln N$ thus we can use the N-body energy bound (1.17) to obtain that the above

$$\le (8E_{0,\hbar})^k \sum_{j=0}^{l_c} (16 E_{0,\hbar} C_V \hbar^{-\alpha} T^{1/2})^{j+1} \le (8E_{0,\hbar})^k$$

if we plug in $T \le \frac{\hbar^{2\alpha}}{(64 E_{0,\hbar} C_V e)^2}$.

For the DP part (2.85), the above process gives

$$\begin{aligned}
&\int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} \mathrm{DP}^{(k+1,l_c)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1} \\
&\le \int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} f_{\mathrm{DP}}^{(k+1,0)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1} + 2^{k+1} \sum_{j=1}^{l_c} 4^j (C_V \hbar^{-\alpha} T^{1/2})^j \\
&\quad \times \int_{[0,T]} \left\| S_{\hbar}^{(1,k+j)} B_{N,\hbar,1,k+j+1} \big(f_{\mathrm{DP}}^{(k+1,j)}(t_{k+j+1}) \big) \right\|_{L^2_{x,x'}} dt_{k+j+1}.
\end{aligned}$$

As $k \le (\ln N)^{10}$ and $j \le l_c \le \ln N$, we can use estimate (2.75) in Lemma 2.4 to get that the above

$$\begin{aligned}
&\le N^{\frac{5}{2}\beta - 1} 2^{k+1} \sum_{j=0}^{l_c} (4C_V \hbar^{-\alpha} T^{1/2})^{j+2} (k+j+1)^2 (2E_{0,\hbar})^{k+j+1} \\
&\le N^{\frac{5}{2}\beta - 1} (8E_{0,\hbar})^k \sum_{j=0}^{l_c} (16 E_{0,\hbar} C_V \hbar^{-\alpha} T^{1/2})^{j+2} \\
&\le (8E_{0,\hbar})^k
\end{aligned}$$

if we plug in $T \leq \frac{\hbar^{2\alpha}}{(64E_{0,\hbar}C_V e)^2}$.

Similarly, for the error part (2.86), we have

$$\int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} \mathrm{EP}^{(k+1,l_c)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1}$$
$$\leq 2^{k+1} \sum_{j=1}^{l_c+1} 4^j (C_V \hbar^{-\alpha} T^{1/2})^j \int_{[0,T]} \left\| S_{\hbar}^{(1,k+j)} B_{N,\hbar,1,k+j+1} \big(f_{\mathrm{EP}}^{(k+1,j)}(t_{k+j+1})\big) \right\|_{L^2_{x,x'}} dt_{k+j+1}.$$

Plugging in $f_{\mathrm{EP}}^{(k,j)}$ and using estimate (2.76) in Lemma 2.4, the above

$$\leq N^{\frac{5}{2}\beta-1} 2^{k+1} \sum_{j=1}^{l_c+1} (4C_V \hbar^{-\alpha} T^{1/2})^{j+1} (k+j+1)(2E_{0,\hbar})^{k+j+1}$$
$$\leq N^{\frac{5}{2}\beta-1} (8E_{0,\hbar})^k \sum_{j=1}^{l_c+1} (16E_{0,\hbar} C_V \hbar^{-\alpha} T^{1/2})^{j+1} \leq (8E_{0,\hbar})^k$$

if we plug in $T \leq \frac{\hbar^{2\alpha}}{(64E_{0,\hbar}C_V e)^2}$.

Finally, for the interaction part (2.87), we have

$$\int_{[0,T]} \left\| S_{\hbar}^{(1,k)} B_{N,\hbar,1,k+1} \mathrm{IP}^{(k+1,l_c)}(t_{k+1}) \right\|_{L^2_{x,x'}} dt_{k+1}$$
$$\leq 2^{k+1} 4^{l_c+1} (C_V \hbar^{-\alpha} T^{1/2})^{l_c+1} \int_{[0,T]} \left\| S_{\hbar}^{(1,k+l_c+1)} B_{N,\hbar,1,k+l_c+2} w_{N,\hbar}^{(k+l_c+2)}(t_{k+l_c+2}) \right\|_{L^2_{x,x'}} dt_{k+l_c+2}.$$

By estimate (2.77) in Lemma 2.4, the above

$$\leq N^{\frac{5}{2}\beta} 2^{k+1} (4C_V \hbar^{-\alpha} T^{1/2})^{l_c+2} (4E_{0,\hbar})^{k+l_c+2}.$$

Plugging in $T \leq \frac{\hbar^{2\alpha}}{(64E_{0,\hbar}C_V e)^2}$ and taking $l_c + 1 = \ln N$, the above

$$\leq 2N^{\frac{5}{2}\beta-1} (8E_{0,\hbar})^k,$$

and we have completed the proof of Lemma 2.5. □

2.3 Feeding the Strichartz Bound into the H^1 Estimate

In the section, we first provide estimates for the four parts in the expansion of $w_{N,\hbar}^{(k)}$ via the preliminary crude estimates established in Sect. 2.1. Then with the help of the KM bound we prove in Sect. 2.2, we can establish a strong stepping estimate for $w_{N,\hbar}^{(k)}$ which is Proposition 2.7.

Lemma 2.6 *Let $\alpha = d + 1/2$. For $k \leq (\ln N)^2$ and $l_c \leq \ln N$, we have the following estimates.*

For the free part,

$$\sup_{t_k \in [t_0, t_0+T]} \left\| S_\hbar^{(1,k)} \mathrm{FP}^{(k,l_c)}(t_k) \right\|_{L^2_{x,x'}} \leq 2^k \sum_{j=0}^{l_c} (4C_V \hbar^{-\alpha} T^{1/2})^j \left\| S_\hbar^{(1,k+j)} w_{N,\hbar}^{(k+j)}(t_0) \right\|_{L^2_{x,x'}}. \tag{2.88}$$

For the driving part,

$$\sup_{t_k \in [t_0, t_0+T]} \left\| S_\hbar^{(1,k)} \mathrm{DP}^{(k,l_c)}(t_k) \right\|_{L^2_{x,x'}} \leq (8E_{0,\hbar})^k N^{\frac{5}{2}\beta - 1} \sum_{j=0}^{l_c} (16 E_{0,\hbar} C_V \hbar^{-\alpha} T^{1/2})^{j+1}. \tag{2.89}$$

For the error part,

$$\sup_{t_k \in [t_0, t_0+T]} \left\| S_\hbar^{(1,k)} \mathrm{EP}^{(k,l_c)}(t_k) \right\|_{L^2_{x,x'}} \leq (8E_{0,\hbar})^k N^{\frac{5}{2}\beta - 1} \sum_{j=0}^{l_c} (16 E_{0,\hbar} C_V \hbar^{-\alpha} T^{1/2})^{j+1}. \tag{2.90}$$

For the interaction part,

$$\begin{aligned} &\sup_{t_k \in [t_0, t_0+T]} \left\| S_\hbar^{(1,k)} \mathrm{IP}^{(k,l_c)}(t_k) \right\|_{L^2_{x,x'}} \\ &\leq 2^k 4^{l_c+1} (C_V \hbar^{-\alpha} T^{1/2})^{l_c} \int_{[t_0, t_0+T]} \left\| S_\hbar^{(1,k+l_c)} B_{N,\hbar,1,k+l_c+1} w_{N,\hbar}^{(k+l_c+1)}(t_{k+l_c+1}) \right\|_{L^2_{x,x'}} dt_{k+l_c+1}. \end{aligned} \tag{2.91}$$

Proof For convenience, we might as well take $t_0 = 0$ as the proof works the same for general case by time translation.

For the free part, applying estimate (2.65) in Lemma 2.3, we arrive at

$$\begin{aligned} &\left\| S_\hbar^{(1,k)} \mathrm{FP}^{(k,l_c)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\ &\leq \left\| S_\hbar^{(1,k)} f_{\mathrm{FP}}^{(k,0)}(t_k) \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} + \sum_{j=1}^{l_c} 2^k 4^j (C_V \hbar^{-\alpha} T^{1/2})^{j-1} \\ &\quad \times \int_{[0,T]} \left\| S_\hbar^{(1,k+j-1)} B_{N,\hbar,1,k+j} (f_{\mathrm{FP}}^{(k,j)}(t_{k+j})) \right\|_{L^2_{x,x'}} dt_{k+j}. \end{aligned}$$

Plugging in $f_{\mathrm{FP}}^{(k,j)}$ and applying Cauchy–Schwarz at dt_{k+j}, the above

$$\begin{aligned} &\leq \left\| S_\hbar^{(1,k)} U_\hbar^{(k)}(t_k) w_{N,\hbar}^{(k)}(0) \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} + \sum_{j=1}^{l_c} 2^k 4^j (C_V \hbar^{-\alpha} T^{1/2})^{j-1} T^{1/2} \\ &\quad \times \left\| S_\hbar^{(1,k+j-1)} B_{N,\hbar,1,k+j} U_\hbar^{(k+j)}(t_{k+j}) w_{N,\hbar}^{(k+j)}(0) \right\|_{L^2_{t_{k+j}}[0,T] L^2_{x,x'}}. \end{aligned}$$

Applying the KM collapsing estimate (Lemma A.2) for $j \geq 1$, the above

$$\leq \sum_{j=0}^{l_c} 2^k (4C_V \hbar^{-\alpha} T^{1/2})^j \left\| S_{\hbar}^{(1,k+j)} w_{N,\hbar}^{(k+j)}(0) \right\|_{L^2_{x,x'}}. \tag{2.92}$$

We have (2.88) as claimed.

For the driving part, the same process yields

$$\begin{aligned}
&\left\| S_{\hbar}^{(1,k)} \mathrm{DP}^{(k,l_c)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\
&\leq \left\| S_{\hbar}^{(1,k)} f_{\mathrm{DP}}^{(k,0)}(t_k) \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} + 2^k \sum_{j=1}^{l_c} 4^j (C_V \hbar^{-\alpha} T^{1/2})^{j-1} \\
&\quad \times \int_{[0,T]} \left\| S_{\hbar}^{(1,k+j-1)} B_{N,\hbar,1,k+j} \big(f_{\mathrm{DP}}^{(k,j)}(t_{k+j}) \big) \right\|_{L^2_{x,x'}} dt_{k+j}.
\end{aligned}$$

Plugging in $f_{\mathrm{DP}}^{(k,j)}$ and using estimates (2.74) and (2.75) gives that the above

$$\begin{aligned}
&\leq N^{\frac{5}{2}\beta-1} 2^k \sum_{j=0}^{l_c} (k+j)^2 (4C_V \hbar^{-\alpha} T^{1/2})^{j+1} (2E_{0,\hbar})^{k+j} \\
&\leq N^{\frac{5}{2}\beta-1} (8E_{0,\hbar})^k \sum_{j=0}^{l_c} (16 E_{0,\hbar} C_V \hbar^{-\alpha} T^{1/2})^{j+1},
\end{aligned}$$

which completes the proof for the driving part.

For the error part, it reads

$$\begin{aligned}
&\left\| S_{\hbar}^{(1,k)} \mathrm{EP}^{(k,l_c)} \right\|_{L^\infty_{t_k}[0,T] L^2_{x,x'}} \\
&\leq 2^k \sum_{j=1}^{l_c+1} 4^j (C_V \hbar^{-\alpha} T^{1/2})^{j-1} \int_{[0,T]} \left\| S_{\hbar}^{(1,k+j-1)} B_{N,\hbar,1,k+j} \big(f_{\mathrm{EP}}^{(k,j)}(t_{k+j}) \big) \right\|_{L^2_{x,x'}} dt_{k+j}.
\end{aligned}$$

Plugging in $f_{\mathrm{EP}}^{(k,j)}$ and using estimate (2.76) provides that the above

$$\begin{aligned}
&\leq N^{\frac{5}{2}\beta-1} 2^k \sum_{j=1}^{l_c+1} (k+j) (4C_V \hbar^{-\alpha} T^{1/2})^j (2E_{0,\hbar})^{k+j} \\
&\leq N^{\frac{5}{2}\beta-1} (8E_{0,\hbar})^k \sum_{j=1}^{l_c+1} (16 E_{0,\hbar} C_V \hbar^{-\alpha} T^{1/2})^j,
\end{aligned}$$

which completes the proof for the error part.

For the interaction part, we have similarly

$$\begin{aligned}
&\left\| S_{\hbar}^{(1,k)} \mathrm{IP}^{(k,l_c)} \right\|_{L^\infty_{t_k}[0,T]L^2_{x,x'}} \\
&= \left\| \int_0^{t_k} \cdots \int_0^{t_{k+l_c}} S_{\hbar}^{(1,k)} J_{N,\hbar}^{(k,l_c+1)}(t_k, \underline{t}_{(k,l_c+1)}) \big(w_{N,\hbar}^{(k+l_c+1)}(t_{k+l_c+1}) \big) d\underline{t}_{(k,l_c+1)} \right\|_{L^\infty_{t_k}[0,T]L^2_{x,x'}} \\
&\leq 2^k 4^{l_c+1} (C_V \hbar^{-\alpha} T^{1/2})^{l_c} \int_{[0,T]} \left\| S_{\hbar}^{(1,k+l_c)} B_{N,\hbar,1,k+l_c+1} w_{N,\hbar}^{(k+l_c+1)}(t_{k+l_c+1}) \right\|_{L^2_{x,x'}} dt_{k+l_c+1},
\end{aligned}$$

which is (2.91). □

Notice that, we are not using the crude estimates in Lemma 2.4 for (2.91). We will use the KM bound we refined in Lemma 2.5 to strengthen our estimate in Proposition 2.7. Before we start, we recall that (2.53) is true for all $l_c \geq 1$, hence properties regarding $w_{N,\hbar}^{(k)}$ using l_c equal to some number A can be fed into the proof of another property of $w_{N,\hbar}^{(k)}$ using l_c equal to some number B.

Proposition 2.7 *Let $T \leq \frac{\hbar^{2\alpha}}{(64E_{0,\hbar}C_V e)^2}$ and $\alpha = d + 1/2$. For $k \leq (\ln N)^2$, $l_c \leq \ln N$, we have*

$$\begin{aligned}
&\sup_{t\in[t_0,t_0+T]} \left\| S_{\hbar}^{(1,k)} w_{N,\hbar}^{(k)}(t) \right\|_{L^2_{x,x'}} \\
&\leq 2^k \sum_{j=0}^{l_c} (4C_V \hbar^{-\alpha} T^{1/2})^j \left\| S_{\hbar}^{(1,k+j)} w_{N,\hbar}^{(k+j)}(t_0) \right\|_{L^2_{x,x'}} + (C_{0,\hbar})^k N^{\frac{5}{2}\beta-1} + (C_{0,\hbar})^k \left(\frac{1}{e}\right)^{l_c+1},
\end{aligned} \tag{2.93}$$

and

$$\begin{aligned}
&\sup_{t\in[t_0,t_0+T]} \left\| S_{\hbar}^{(1,k)} w_{N,\hbar}^{(k)}(t) \right\|_{L^2_{x,x'}} \\
&\leq 2^k \sum_{j=0}^{l_c} (4C_V \hbar^{-\alpha} T^{1/2})^j \left\| S_{\hbar}^{(1,k+j)} w_{N,\hbar}^{(k+j)}(t_0) \right\|_{L^2_{x,x'}} + (C_{0,\hbar})^k N^{\frac{5}{2}\beta-1} + (C_{0,\hbar})^k \left(\frac{1}{e}\right)^{l_c+1},
\end{aligned} \tag{2.94}$$

where $C_{0,\hbar} = 64E_{0,\hbar}$. Notice that (2.94) is stronger than (2.83).

Proof The conclusion of Lemma 2.6 reads

$$\begin{aligned}
&\sup_{t\in[t_0,t_0+T]} \left\| S_{\hbar}^{(1,k)} w_{N,\hbar}^{(k)}(t) \right\|_{L^2_{x,x'}} \\
&\leq 2^k \sum_{j=0}^{l_c} (4C_V \hbar^{-\alpha} T^{1/2})^j \left\| S_{\hbar}^{(1,k+j)} w_{N,\hbar}^{(k+j)}(t_0) \right\|_{L^2_{x,x'}} \\
&\quad + 2(8E_{0,\hbar})^k N^{\frac{5}{2}\beta-1} \sum_{j=0}^{l_c} (16E_{0,\hbar} C_V \hbar^{-\alpha} T^{1/2})^{j+1} + 2^k 4^{l_c+1} (C_V \hbar^{-\alpha} T^{1/2})^{l_c}
\end{aligned}$$

$$\times \int_{[t_0,t_0+T]} \left\| S_{\hbar}^{(1,k+l_c)} B_{N,\hbar,1,k+l_c+1} w_{N,\hbar}^{(k+l_c+1)}(t_{k+l_c+1}) \right\|_{L^2_{x,x'}} dt_{k+l_c+1}. \tag{2.95}$$

Since $k + l_c \leq (\ln N)^{10}$ and $T \leq \frac{\hbar^{2\alpha}}{(64E_{0,\hbar}C_V e)^2}$, we can employ KM bound in Lemma 2.5 to get that the above

$$\begin{aligned}
&\leq 2^k \sum_{j=0}^{l_c} (4C_V \hbar^{-\alpha} T^{1/2})^j \left\| S_{\hbar}^{(1,k+j)} w_{N,\hbar}^{(k+j)}(t_0) \right\|_{L^2_{x,x'}} \\
&\quad + 2(8E_{0,\hbar})^k N^{\frac{5}{2}\beta-1} \sum_{j=0}^{l_c} (16E_{0,\hbar} C_V \hbar^{-\alpha} T^{1/2})^{j+1} \\
&\quad + 2^k 4^{l_c+1} (C_V \hbar^{-\alpha} T^{1/2})^{l_c} (16E_{0,\hbar})^{k+l_c}.
\end{aligned}$$

Plugging in $T \leq \frac{\hbar^{2\alpha}}{(64E_{0,\hbar}C_V e)^2}$ and $C_{0,\hbar} = 64E_{0,\hbar}$, we obtain (2.93).

For (2.94), repeating the proof of KM bound in Lemma 2.5, we have

$$\begin{aligned}
&\int_{[t_0,t_0+T]} \left\| S_{\hbar}^{(1,1)} B_{N,\hbar,1,2}^{\pm} w_{N,\hbar}^{(2)}(t) \right\|_{L^2_{x,x'}} dt \\
&\leq 4 \sum_{j=0}^{l_c} (4C_V \hbar^{-\alpha} T^{1/2})^{j+1} \left\| S_{\hbar}^{(1,2+j)} w_{N,\hbar}^{(2+j)}(t_0) \right\|_{L^2_{x,x'}} + 2N^{\frac{5}{2}\beta-1} (8E_{0,\hbar})^2 \\
&\quad + 4^{l_c+2} (C_V \hbar^{-\alpha} T^{1/2})^{l_c+1} \int_{[t_0,t_0+T]} \left\| S_{\hbar}^{(1,2+l_c)} B_{N,\hbar,1,3+l_c} w_{N,\hbar}^{(3+l_c)}(t_{3+l_c}) \right\|_{L^2_{x,x'}} dt_{3+l_c}.
\end{aligned} \tag{2.96}$$

Since $2 + l_c \leq (\ln N)^{10}$ and $T \leq \frac{\hbar^{2\alpha}}{(64E_{0,\hbar}C_V e)^2}$, we can employ KM bound in Lemma 2.5 to get that the above

$$\begin{aligned}
&\leq 4 \sum_{j=0}^{l_c} (4C_V \hbar^{-\alpha} T^{1/2})^{j+1} \left\| S_{\hbar}^{(1,2+j)} w_{N,\hbar}^{(2+j)}(t_0) \right\|_{L^2_{x,x'}} + 2N^{\frac{5}{2}\beta-1} (8E_{0,\hbar})^2 \\
&\quad + 4^{l_c+2} (C_V \hbar^{-\alpha} T^{1/2})^{l_c+1} (16E_{0,\hbar})^{3+l_c}.
\end{aligned}$$

Plugging in $T \leq \frac{\hbar^{2\alpha}}{(64E_{0,\hbar}C_V e)^2}$ and $C_{0,\hbar} = 64E_{0,\hbar}$, we obtain (2.94). □

2.4 Convergence Rate for Every Finite Time

In the section, we will iteratively use Proposition 2.7 to obtain the convergence rate for every finite time at the price of weakening the convergence rate.

Proposition 2.8 *Let $T_0 < +\infty$ and $\alpha = d + 1/2$. For*

$$k \leq (\ln N)^2 - \left(1 - \frac{5}{2}\beta\right) \sum_{j=0}^{n(T_0,\hbar)} \frac{\ln N}{2^j j!},$$

we have

$$\sup_{t\in[0,T_0]} \left\| S_\hbar^{(1,k)} w_{N,\hbar}^{(k)}(t) \right\|_{L^2_{x,x'}} \leq (e^{n(T_0,\hbar)} C_{0,\hbar})^k N^{\frac{\frac{5}{2}\beta - 1}{2^{n(T_0,\hbar)} n(T_0,\hbar)!}} \tag{2.97}$$

and

$$\int_{[0,T_0]} \left\| S_\hbar^{(1,1)} B^{\pm}_{N,\hbar,1,2} w_{N,\hbar}^{(2)}(t) \right\|_{L^2_{x,x'}} dt \leq 8n(T_0,\hbar) C_{0,\hbar}^2 N^{\frac{\frac{5}{2}\beta - 1}{2^{n(T_0,\hbar)} n(T_0,\hbar)!}}, \tag{2.98}$$

where $n(T_0,\hbar) = (8eC_V C_{0,h})^2 T_0/\hbar^{2\alpha}$ and $C_{0,\hbar} = 64E_{0,\hbar}$ as defined in Proposition 2.7. Moreover, under the restriction (2.10) that

$$N \geq e^{(2)}\big([C_V^2 E_{0,\hbar}^2 T_0/\hbar^{2\alpha}]^2\big), \tag{2.99}$$

for $N \geq N_0(\beta)$ we have (2.11) and (2.12) which we restate here

$$\sup_{t\in[0,T_0]} \left\| S_\hbar^{(1,1)} w_{N,\hbar}^{(1)}(t) \right\|_{L^2_{x,x'}} \leq \left(\frac{1}{\ln N}\right)^{100},$$

$$\int_{[0,T_0]} \left\| S_\hbar^{(1,1)} B^{\pm}_{N,\hbar,1,2} w_{N,\hbar}^{(2)}(t) \right\|_{L^2_{x,x'}} dt \leq \left(\frac{1}{\ln N}\right)^{100}.$$

Proof Step 0. Set $\lambda = \frac{1}{8eC_V C_{0,\hbar}}$. Then for

$$k \leq (\ln N)^2 - \left(1 - \frac{5}{2}\beta\right) \ln N, \quad l_c \leq \left(1 - \frac{5}{2}\beta\right) \ln N,$$

by estimate (2.93) in Proposition 2.7, we have

$$\begin{aligned} &\sup_{t\in[0,\lambda^2\hbar^{2\alpha}]} \left\| S_\hbar^{(1,k)} w_{N,\hbar}^{(k)}(t) \right\|_{L^2_{x,x'}} \\ &\leq 2^k \sum_{j=0}^{l_c} (4C_V\lambda)^j \left\| S_\hbar^{(1,k+j)} w_{N,\hbar}^{(k+j)}(0) \right\|_{L^2_{x,x'}} + (C_{0,\hbar})^k N^{\frac{5}{2}\beta - 1} + (C_{0,\hbar})^k \left(\frac{1}{e}\right)^{l_c+1}. \end{aligned} \tag{2.100}$$

By initial condition (1.8) in condition (c), we plug in $\lambda = \frac{1}{8eC_V C_{0,\hbar}}$ and take $l_c = (1 - \frac{5}{2}\beta)\ln N$ to get

$$\sup_{t\in[0,\lambda^2\hbar^{2\alpha}]}\left\|S_\hbar^{(1,k)}w_{N,\hbar}^{(k)}(t)\right\|_{L^2_{x,x'}}\leq 4(C_{0,\hbar})^k N^{\frac{5}{2}\beta-1} \tag{2.101}$$

for every $k\leq(\ln N)^2-(1-\frac{5}{2}\beta)\ln N$.

Step 1. Let $t_1=\lambda^2\hbar^{2\alpha}$. For

$$k\leq(\ln N)^2-\left(1-\frac{5}{2}\beta\right)\left(\ln N+\frac{\ln N}{2}\right),\quad l_c\leq\left(1-\frac{5}{2}\beta\right)\ln N,$$

we make use of estimate (2.93) in Proposition 2.7 again to obtain

$$\begin{aligned}&\sup_{t\in[t_1,t_1+\lambda^2\hbar^{2\alpha}]}\left\|S_\hbar^{(1,k)}w_{N,\hbar}^{(k)}(t)\right\|_{L^2_{x,x'}}\\&\leq 2^k\sum_{j=0}^{l_c}(4C_V\lambda)^j\left\|S_\hbar^{(1,k+j)}w_{N,\hbar}^{(k+j)}(t_1)\right\|_{L^2_{x,x'}}+(C_{0,\hbar})^kN^{\frac{5}{2}\beta-1}+(C_{0,\hbar})^k\left(\frac{1}{e}\right)^{l_c+1}.\end{aligned}$$

Since $k+l_c\leq(\ln N)^2-(1-\frac{5}{2}\beta)\ln N$, one can adopt estimate (2.101) in Step 0 to reach that the above

$$\leq N^{\frac{5}{2}\beta-1}4(C_{0,\hbar})^k\sum_{j=0}^{l_c}(4C_VC_{0,\hbar}\lambda)^j+(C_{0,\hbar})^kN^{\frac{5}{2}\beta-1}+(C_{0,\hbar})^k\left(\frac{1}{e}\right)^{l_c+1}.$$

Recalling $\lambda=\frac{1}{8eC_VC_{0,\hbar}}$, the above

$$\leq N^{\frac{5}{2}\beta-1}8(C_{0,\hbar})^k+(C_{0,\hbar})^kN^{\frac{5}{2}\beta-1}+(C_{0,\hbar})^k\left(\frac{1}{e}\right)^{l_c+1}.$$

By taking $l_c=(1-\frac{5}{2}\beta)\ln N/2$, we arrive at

$$\sup_{t\in[t_1,t_1+\lambda^2\hbar^{2\alpha}]}\left\|S_\hbar^{(1,k)}w_{N,\hbar}^{(k)}(t)\right\|_{L^2_{x,x'}}\leq(eC_{0,\hbar})^kN^{\frac{\frac{5}{2}\beta-1}{2}} \tag{2.102}$$

for every $k\leq(\ln N)^2-(1-\frac{5}{2}\beta)\left(\ln N+\frac{\ln N}{2}\right)$.

Step m. Let $t_m=m\lambda^2\hbar^{2\alpha}$. Now we assume (2.102) is true for the case $n=m$, that is,

$$\sup_{t\in[t_1,t_1+\lambda^2\hbar^{2\alpha}]}\left\|S_\hbar^{(1,k)}w_{N,\hbar}^{(k)}(t)\right\|_{L^2_{x,x'}}\leq(eC_{0,\hbar})^kN^{\frac{\frac{5}{2}\beta-1}{2}} \tag{2.103}$$

for every $k\leq(\ln N)^2-(1-\frac{5}{2}\beta)\sum_{j=0}^m\frac{\ln N}{2^j j!}$. Then we will prove it for $n=m+1$.

For

$$k \le (\ln N)^2 - \left(1 - \frac{5}{2}\beta\right)\sum_{j=0}^{m+1} \frac{\ln N}{2^j j!}, \quad l_c \le \frac{(1-\frac{5}{2}\beta)\ln N}{2^{m+1}(m+1)!},$$

one can employ estimate (2.93) in Proposition 2.7 to reach

$$\begin{aligned}
&\sup_{t\in[t_{m+1},t_{m+1}+\lambda^2\hbar^{2\alpha}]} \left\| S_\hbar^{(1,k)} w_{N,\hbar}^{(k)}(t)\right\|_{L^2_{x,x'}} \\
&\le 2^k \sum_{j=0}^{l_c} (4C_V\lambda)^j \left\| S_\hbar^{(1,k+j)} w_{N,\hbar}^{(k+j)}(t_{m+1})\right\|_{L^2_{x,x'}} + (C_{0,\hbar})^k N^{\frac{5}{2}\beta-1} + (C_{0,\hbar})^k \left(\frac{1}{e}\right)^{l_c+1}.
\end{aligned}$$

Since $k + l_c \le (\ln N)^2 - (1-\frac{5}{2}\beta)\sum_{j=0}^m \frac{\ln N}{2^j j!}$, one can use estimate (2.103) in the case $n = m$ to get that the above

$$\le N^{\frac{\frac{5}{2}\beta-1}{2^m m!}} (2e^m C_{0,\hbar})^k \sum_{j=0}^{l_c} (4C_V\lambda)^j (e^m C_{0,\hbar})^j + (C_{0,\hbar})^k N^{\frac{5}{2}\beta-1} + (C_{0,\hbar})^k \left(\frac{1}{e}\right)^{l_c+1}.$$

Recalling $\lambda = \frac{1}{8eC_V C_{0,\hbar}}$, the above

$$\le (2e^m C_{0,\hbar})^k N^{\frac{\frac{5}{2}\beta-1}{2^m m!}} \left(e^m\right)^{l_c+1} + (C_{0,\hbar})^k N^{\frac{5}{2}\beta-1} + (C_{0,\hbar})^k \left(\frac{1}{e}\right)^{l_c+1}.$$

Taking $l_c + 1 = \frac{(1-\frac{5}{2}\beta)\ln N}{2^{m+1}(m+1)!}$, we arrive at

$$\begin{aligned}
&\sup_{t\in[t_{m+1},t_{m+1}+\lambda^2\hbar^{2\alpha}]} \left\| S_\hbar^{(1,k)} w_{N,\hbar}^{(k)}(t)\right\|_{L^2_{x,x'}} \\
&\le (2e^m C_{0,\hbar})^k N^{\frac{\frac{5}{2}\beta-1}{2^{m+1}m!}} + (C_{0,\hbar})^k N^{\frac{5}{2}\beta-1} + (C_{0,\hbar})^k N^{\frac{\frac{5}{2}\beta-1}{2^{m+1}(m+1)!}} \\
&\le (e^{m+1} C_{0,\hbar})^k N^{\frac{\frac{5}{2}\beta-1}{2^{m+1}(m+1)!}}.
\end{aligned}$$

This proves (2.103) and completes the proof of (2.97) as we can take $m = n(T_0,\hbar) = (8eC_V C_{0,\hbar})^2 T_0/\hbar^{2\alpha}$.

For (2.98), we can use estimate (2.94) in Proposition 2.7 to get to

$$\begin{aligned}
&\int_{[t_m,t_m+\lambda^2\hbar^{2\alpha}]} \left\| S_\hbar^{(1,1)} B^{\pm}_{N,\hbar,1,2} w_{N,\hbar}^{(2)}(t)\right\|_{L^2_{x,x'}} dt \\
&\le 4\sum_{j=0}^{l_c} (4C_V\lambda)^{j+1} \left\| S_\hbar^{(1,2+j)} w_{N,\hbar}^{(2+j)}(t_m)\right\|_{L^2_{x,x'}} + C_{0,\hbar}^2 N^{\frac{5}{2}\beta-1} + C_{0,\hbar}^2 \left(\frac{1}{e}\right)^{l_c+1}.
\end{aligned} \tag{2.104}$$

Plugging in estimate (2.103), the above

$$\leq 4\sum_{j=0}^{l_c}(4C_V\lambda)^{j+1}\left(e^m C_{0,\hbar}\right)^{j+2}N^{\frac{\frac{5}{2}\beta-1}{2^m m!}}+C_{0,\hbar}^2 N^{\frac{5}{2}\beta-1}+C_{0,\hbar}^2\left(\frac{1}{e}\right)^{l_c+1}.$$

Recalling $\lambda=\frac{1}{8eC_V C_{0,\hbar}}$, the above

$$\leq 4C_{0,\hbar}(e^m)^{l_c+2}N^{\frac{\frac{5}{2}\beta-1}{2^m m!}}+C_{0,\hbar}^2 N^{\frac{5}{2}\beta-1}+C_{0,\hbar}^2\left(\frac{1}{e}\right)^{l_c+1}.$$

Setting $l_c+2=\frac{(1-\frac{5}{2}\beta)\ln N}{2^{m+1}(m+1)!}$, we arrive at that the above

$$\begin{aligned}&\leq 4C_{0,\hbar}N^{\frac{\frac{5}{2}\beta-1}{2^{m+1}m!}}+C_{0,\hbar}^2 N^{\frac{5}{2}\beta-1}+eC_{0,\hbar}^2 N^{\frac{\frac{5}{2}\beta-1}{2^{m+1}(m+1)!}}\\&\leq 8C_{0,\hbar}^2 N^{\frac{\frac{5}{2}\beta-1}{2^{m+1}(m+1)!}}.\end{aligned}$$

Then by summing the integration time domain, we obtain

$$\begin{aligned}&\int_{[0,T_0]}\left\|S_\hbar^{(1,1)}B_{N,\hbar,1,2}^{\pm}w_{N,\hbar}^{(2)}(t)\right\|_{L^2_{x,x'}}dt\\&\leq\sum_{m=0}^{n(T_0,\hbar)}\int_{[t_m,t_{m+1}]}\left\|S_\hbar^{(1,1)}B_{N,\hbar,1,2}^{\pm}w_{N,\hbar}^{(2)}(t)\right\|_{L^2_{x,x'}}dt\\&\leq\sum_{m=0}^{n(T_0,\hbar)}8C_{0,\hbar}^2 N^{\frac{\frac{5}{2}\beta-1}{2^{m+1}(m+1)!}}\\&\leq 8n(T_0,\hbar)C_{0,\hbar}^2 N^{\frac{\frac{5}{2}\beta-1}{2^{n(T_0,\hbar)}n(T_0,\hbar)!}}.\end{aligned}\tag{2.105}$$

This completes the proof of (2.98).

For estimates (2.11) and (2.12), under the restriction (2.10) that

$$N\geq e^{(2)}\left([C_V^2E_{0,\hbar}^2T_0/\hbar^7]^2\right),\tag{2.106}$$

which implies that $n(T_0,\hbar)\leq\sqrt{C\ln\ln N}$ with an absolute constant C, we have

$$2^{n(T_0,\hbar)}n(T_0,\hbar)!\leq n(T_0,\hbar)^{n(T_0,\hbar)}\leq(\sqrt{C\ln\ln N})^{\sqrt{C\ln\ln N}}\leq\sqrt{\ln N}.$$

Also, we have

$$8n(T_0,\hbar)C_{0,\hbar}^2\leq e^{n(T_0,\hbar)}C_{0,\hbar}\leq n(T_0,\hbar)^{n(T_0,\hbar)}\leq\sqrt{\ln N}.$$

Hence, we obtain

$$\sup_{t\in[0,T_0]} \left\| S_\hbar^{(1,1)} w_{N,\hbar}^{(1)}(t) \right\|_{L^2_{x,x'}} \le e^{n(T_0,\hbar)} C_{0,\hbar} N^{\frac{\frac{5}{2}\beta-1}{2^{n(T_0,\hbar)} n(T_0,\hbar)!}} \le \frac{\sqrt{\ln N}}{N^{\frac{1-\frac{5}{2}\beta}{\sqrt{\ln N}}}} \le \left(\frac{1}{\ln N}\right)^{100},$$

$$\int_{[0,T_0]} \left\| S_\hbar^{(1,1)} B^{\pm}_{N,\hbar,1,2} w_{N,\hbar}^{(2)}(t) \right\|_{L^2_{x,x'}} dt \le 8n(T_0,\hbar) C_{0,\hbar}^2 N^{\frac{\frac{5}{2}\beta-1}{2^{n(T_0,\hbar)} n(T_0,\hbar)!}} \le \left(\frac{1}{\ln N}\right)^{100}$$

for $N \ge N_0(\beta)$. This completes the proof of estimates (2.11) and (2.12). □

3 H-NLS v.s. the Compressible Euler Equation: A Modulated Energy Approach

We will compare the H-NLS equation (2.1) and the compressible Euler equation (1.5) before its blowup time by the method of modulated energy. Recall the H-NLS equation (2.1)

$$\begin{cases} i\hbar\partial_t \phi_{N,\hbar} = -\frac{1}{2}\hbar^2 \Delta\phi_{N,\hbar} + (V_N * |\phi_{N,\hbar}|^2)\phi_{N,\hbar}, \\ \phi_{N,\hbar}(0) = \phi^{\text{in}}_{N,\hbar}, \end{cases}$$

with the mass density and momentum density defined by (2.9)

$$\rho_{N,\hbar}(t,x) = |\phi_{N,\hbar}(t,x)|^2, \quad J_{N,\hbar}(t,x) = \hbar \operatorname{Im}\left(\overline{\phi_{N,\hbar}}(t,x)\nabla\phi_{N,\hbar}(t,x)\right),$$

and the compressible Euler equation (1.5)

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho u) = 0, \\ \partial_t u + (u\cdot\nabla)u + b_0\nabla\rho = 0, \\ (\rho,u)|_{t=0} = (\rho^{\text{in}}, u^{\text{in}}). \end{cases}$$

Here is the main theorem of the section.

Theorem 3.1 *Let $\phi_{N,\hbar}(t)$ be the solution to H-NLS equation with the initial data $\phi^{\text{in}}_{N,h}$. Under the same conditions of Theorem 1.1), then we have*[17]

$$\|\rho_{N,\hbar} - \rho\|_{L^\infty([0,T_0];L^2(\mathbb{R}^d))} \le C(T_0)\left(\frac{1}{\hbar^4 N^\beta} + \hbar^2\right)^{\frac{1}{2}}, \tag{3.1}$$

$$\|J_{N,\hbar} - \rho u\|_{L^\infty([0,T_0];L^r(\mathbb{R}^d))} \le C(T_0)\left(\frac{1}{\hbar^4 N^\beta} + \hbar^2\right)^{\frac{1}{2}\left(\frac{4}{r}-3\right)}, \tag{3.2}$$

[17] Under the restriction (1.13), the smallness factor $\frac{1}{\hbar^4 N^\beta}$ can be absorbed into $\hbar^2$.

where $r \in [1, 4/3)$,

$$\|\rho_{N,\hbar} V_N * \rho_{N,\hbar}(t,x) - b_0\rho(t,x)^2\|_{L^1([0,T_0];L^1(\mathbb{R}^d))} \leq C(T_0)\left(\frac{1}{\hbar^4 N^\beta} + \hbar^2\right)^{\frac{1}{2}}. \tag{3.3}$$

Proof of Theorem 3.1 By (3.37) and (3.38) in Proposition 3.5, we have

$$\|\rho_{N,\hbar} - \rho\|_{L^\infty([0,T_0];L^2(\mathbb{R}^d))} \leq C(T_0)\left(\frac{1}{\hbar^4 N^\beta} + \hbar^2\right)^{\frac{1}{2}},$$
$$\|(i\hbar\nabla - u)\phi_{N,\hbar}\|_{L^\infty([0,T_0];L^2(\mathbb{R}^d))} \leq C(T_0)\left(\frac{1}{\hbar^4 N^\beta} + \hbar^2\right)^{\frac{1}{2}},$$

which directly completes the proof of (3.1).

For (3.2), by the triangle and Hölder's inequalities as well as estimates (3.37) and (3.38) we have

$$\begin{aligned}
\|J_{N,\hbar} - \rho u\|_{L^1(\mathbb{R}^d)} &\leq \|J_{N,\hbar} - \rho_{N,\hbar} u\|_{L^1(\mathbb{R}^d)} + \|\rho_{N,\hbar} u - \rho u\|_{L^1(\mathbb{R}^d)} \\
&= \left\|\mathrm{Im}\big(\overline{\phi_{N,\hbar}}(\hbar\nabla - iu)\phi_{N,\hbar}\big)\right\|_{L^1(\mathbb{R}^d)} + \|\rho_{N,\hbar} u - \rho u\|_{L^1(\mathbb{R}^d)} \\
&\leq \|\phi_{N,\hbar}\|_{L^2(\mathbb{R}^d)} \|(i\hbar\nabla - u)\phi_{N,\hbar}\|_{L^2(\mathbb{R}^d)} \\
&\quad + \|u\|_{L^2(\mathbb{R}^d)} \|\rho_{N,\hbar} - \rho\|_{L^2(\mathbb{R}^d)} \\
&\leq C(T_0)\left(\frac{1}{\hbar^4 N^\beta} + \hbar^2\right)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, by the energy bound for $\phi_{N,\hbar}$ and the uniform bound for $\|\rho_{N,\hbar}\|_{L^2}$ we have

$$\|J_{N,\hbar}\|_{L^{4/3}} \leq \|\hbar\nabla\phi_{N,\hbar}\|_{L^2} \|\phi_{N,\hbar}\|_{L^4} \lesssim E_0, \tag{3.4}$$

where we used energy bound and uniform bound for $\|\rho_{N,\hbar}\|_{L^2}$ in the last inequality. Hence, by interpolation inequality we obtain

$$\begin{aligned}
&\|J_{N,\hbar} - \rho u\|_{L^\infty([0,T_0];L^r(\mathbb{R}^d))} \\
&\quad \leq \|J_{N,\hbar} - \rho u\|^{1-\alpha}_{L^\infty([0,T_0];L^1(\mathbb{R}^d))} \|J_{N,\hbar} - \rho u\|^{\alpha}_{L^\infty([0,T_0];L^{4/3}(\mathbb{R}^d))} \\
&\quad \leq C\left(\frac{1}{\hbar^4 N^\beta} + \hbar^2\right)^{\frac{1-\alpha}{2}} E_0^\alpha,
\end{aligned} \tag{3.5}$$

where $\alpha = 4 - 4/r$. This completes the proof of (3.2).

For (3.3), by triangle inequality we have

$$\|\rho_{N,\hbar}V_N*\rho_{N,\hbar}-b_0\rho^2\|_{L^1(\mathbb{R}^d)}\leq\|\rho_{N,\hbar}V_N*\rho_{N,\hbar}-b_0(\rho_{N,\hbar})^2\|_{L^1(\mathbb{R}^d)}\\ +b_0\|(\rho_{N,\hbar})^2-\rho^2\|_{L^1(\mathbb{R}^d)}. \tag{3.6}$$

By the approximation of identity estimate (3.13) which reads

$$\|\rho_{N,\hbar}V_N*\rho_{N,\hbar}-b_0(\rho_{N,\hbar})^2\|_{L^1(\mathbb{R}^d)}\lesssim\frac{1}{\hbar^4N^\beta} \tag{3.7}$$

and estimate (3.1), we have

$$\begin{aligned}\|\rho_{N,\hbar}V_N*\rho_{N,\hbar}-b_0\rho^2\|_{L^1(\mathbb{R}^d)}&\lesssim\frac{1}{\hbar^4N^\beta}+\|\rho_{N,\hbar}-\rho\|_{L^2(\mathbb{R}^d)}\big(\|\rho_{N,\hbar}\|_{L^2(\mathbb{R}^d)}+\|\rho\|_{L^2(\mathbb{R}^d)}\big)\\&\leq C(T_0)\left(\frac{1}{\hbar^4N^\beta}+\hbar^2\right)^{\frac{1}{2}}.\end{aligned}$$

By taking L^∞ norm at dt, we complete the proof of (3.3). Thus we have proved Theorem 3.1 assuming Proposition 3.5 and (3.13). The rest of this section is to prove them. □

3.1 The Evolution of the Modulated Energy

We consider the following modulated energy

$$\mathcal{M}[\phi_{N,\hbar},\rho,u](t)=\frac{1}{2}\int_{\mathbb{R}^d}|(i\hbar\nabla-u)\phi_{N,\hbar}(t)|^2dx\\ +\frac{1}{2}\langle V_N*\rho_{N,\hbar},\rho_{N,\hbar}\rangle+\frac{b_0}{2}\int_{\mathbb{R}^d}\rho^2dx-b_0\int_{\mathbb{R}^d}\rho\rho_{N,\hbar}dx. \tag{3.8}$$

We need to derive a time evolution equation for $\mathcal{M}[\phi_{N,\hbar},\rho,u](t)$. The related quantities for $\phi_{N,\hbar}$ are given as the following.

Lemma 3.2 *We have the following estimates regarding $\phi_{N,\hbar}$:*

$$\partial_t\rho_{N,\hbar}+\operatorname{div}J_{N,\hbar}=0, \tag{3.9}$$

$$\partial_tJ^j_{N,\hbar}+\sum_{j,k}\partial_k\left[\hbar^2\mathrm{Re}\big(\partial_j\overline{\phi_{N,\hbar}}\partial_k\phi_{N,\hbar}\big)-\frac{\hbar^2}{4}\partial_{jk}\rho_{N,\hbar}\right]+\big(\partial_j(V_N*\rho_{N,\hbar})\big)\rho_{N,\hbar}=0, \tag{3.10}$$

$$E_{N,\hbar}(t)\equiv E_{N,\hbar}(0), \tag{3.11}$$

where the energy $E_{N,\hbar}(t)$ is defined by

$$E_{N,\hbar}(t)=\frac{1}{2}\|\hbar\nabla\phi_{N,\hbar}(t)\|^2_{L^2}+\frac{1}{2}\langle V_N*\rho_{N,\hbar},\rho_{N,\hbar}\rangle(t). \tag{3.12}$$

We also have the approximation of identity estimate:

$$\|\rho_{N,\hbar}V_N * \rho_{N,\hbar} - b_0(\rho_{N,\hbar})^2\|_{L^1(\mathbb{R}^d)} \lesssim \frac{1}{\hbar^4 N^\beta}. \tag{3.13}$$

Proof We omit the proof of (3.9)–(3.11) as this is a direct computation and is well-known in H^1 well-posedness theory. For (3.13), we set $W_N = V_N - b_0\delta$ and rewrite

$$\|\rho_{N,\hbar}V_N * \rho_{N,\hbar} - b_0(\rho_{N,\hbar})^2\|_{L^1(\mathbb{R}^d)} = \|\rho_{N,\hbar}W_N * \rho_{N,\hbar}\|_{L^1(\mathbb{R}^d)}.$$

By Hölder, the above

$$\leq \|W_N * \rho_{N,\hbar}\|_{L^{3/2}}\|\rho_{N,\hbar}\|_{L^3}.$$

By Lemma 3.6, the above

$$\lesssim N^{-\beta}\|\langle\nabla\rangle\rho_{N,\hbar}\|_{L^{3/2}}\|\rho_{N,\hbar}\|_{L^3}.$$

By fractional Leibniz rule in Lemma A.6 and Sobolev inequality, the above

$$\lesssim N^{-\beta}\|\phi_{N,\hbar}\|_{H^1}^4.$$

By the energy bound for $\phi_{N,\hbar}$, the above

$$\lesssim \frac{1}{\hbar^4 N^\beta},$$

which completes the proof of (3.13). □

Next let us derive the time derivative of $\mathcal{M}[\phi_{N,\hbar}, \rho, u](t)$.

Proposition 3.3 *There holds*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{M}\left[\phi_{N,\hbar}, \rho, u\right](t) \\
&= -\int_{\mathbb{R}^d} \partial_k u^j \mathrm{Re}\Big((\hbar\partial_k - iu^k)\phi_{N,\hbar}\overline{(\hbar\partial_j - iu^j)\phi_{N,\hbar}}\Big) \\
&\quad - \frac{b_0}{2}\int_{\mathbb{R}^d} \operatorname{div} u(\rho_{N,\hbar}-\rho)^2 dx - \frac{\hbar^2}{4}\int_{\mathbb{R}^d}\rho_{N,\hbar}(\Delta \operatorname{div} u)dx + \mathrm{Er},
\end{aligned} \tag{3.14}$$

where the summation convention for repeated indices is used and the error term is given by

$$\mathrm{Er} = \int_{\mathbb{R}^d} u^j(\partial_j(V_N * \rho_{N,\hbar}))\rho_{N,\hbar}dx + \frac{b_0}{2}\int_{\mathbb{R}^d}\operatorname{div} u(\rho_{N,\hbar})^2 dx. \tag{3.15}$$

Proof By energy conservation law (3.11) in Lemma 3.2, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{M}[\phi_{N,\hbar},\rho,u](t) &= \frac{1}{2}\frac{d}{dt}\|\hbar\nabla\phi_{N,\hbar}(t)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|u|^2\rho_{N,\hbar}dx - \frac{d}{dt}\int_{\mathbb{R}^d}J_{N,\hbar}udx \\ &\quad + \frac{1}{2}\frac{d}{dt}\langle V_N*\rho_{N,\hbar},\rho_{N,\hbar}\rangle + \frac{b_0}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\rho^2dx - b_0\frac{d}{dt}\int_{\mathbb{R}^d}\rho_{N,\hbar}\rho dx \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|u|^2\rho_{N,\hbar}dx - \frac{\mathrm{d}}{dt}\int_{\mathbb{R}^d}J_{N,\hbar}udx \\ &\quad + \frac{b_0}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\rho^2dx - b_0\frac{d}{dt}\int_{\mathbb{R}^d}\rho_{N,\hbar}\rho dx.\end{aligned}$$

Next, we calculate the above four terms separately. For the first term, by (1.5) and (3.9) we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|u|^2\rho_{N,\hbar}dx &= \int_{\mathbb{R}^d}\left(u\partial_t u\rho_{N,\hbar} + \frac{1}{2}|u|^2\partial_t\rho_{N,\hbar}\right)dx \\ &= \int_{\mathbb{R}^d}\left(\partial_t u^j\rho_{N,\hbar}u^j - \frac{1}{2}|u|^2\mathrm{div}\,J_{N,\hbar}\right)dx \\ &= \int_{\mathbb{R}^d}\big(-\rho_{N,\hbar}u^ju^k\partial_k u^j - b_0\rho_{N,\hbar}u^j\partial_j\rho + J^j_{N,\hbar}u^k\partial_j u^k\big)dx\end{aligned} \tag{3.16}$$

where we have used integration by parts in the last equality.

For the second term, via (3.10) and (1.5) we have

$$\begin{aligned}&-\frac{d}{dt}\int_{\mathbb{R}^d}J_{N,\hbar}u\,dx \\ &= \int_{\mathbb{R}^d}(-\partial_t J_{N,\hbar}u - J_{N,\hbar}\partial_t u)dx \\ &= \int_{\mathbb{R}^d}\left(\partial_k\left(\hbar^2\mathrm{Re}\big(\partial_j\overline{\phi_{N,h}}\partial_k\phi_{N,h}\big) - \frac{\hbar^2}{4}\partial^2_{jk}\rho_{N,\hbar}\right) + (\partial_j(V_N*\rho_{N,\hbar}))\rho_{N,\hbar}\right)u^j dx \\ &\quad + \int_{\mathbb{R}^d}J^j_{N,\hbar}u^k\partial_k u^j\,dx + b_0\int_{\mathbb{R}^d}J^j_{N,\hbar}\partial_j\rho\,dx.\end{aligned} \tag{3.17}$$

Integrating by parts and using (3.15), the above

$$\begin{aligned}&= \int_{\mathbb{R}^d}-\hbar^2\partial_k u^j\big[\mathrm{Re}(\partial_j\overline{\phi_{N,h}}\partial_k\phi_{N,h})\big]dx - \int_{\mathbb{R}^d}\frac{\hbar^2}{4}\rho_{N,h}\partial^2_{jk}\partial_k u^j\,dx \\ &\quad - \frac{b_0}{2}\int\mathrm{div}\,u(\rho_{N,\hbar})^2dx + \mathrm{Er} + \int_{\mathbb{R}^d}J^j_{N,\hbar}u^k\partial_k u^j\,dx + b_0\int_{\mathbb{R}^d}J^j_{N,\hbar}\partial_j\rho\,dx.\end{aligned}$$

For the third term, using (1.5) and integration by parts, we obtain

$$\frac{b_0}{2}\frac{d}{dt}\int_{\mathbb{R}^d}\rho^2dx = b_0\int_{\mathbb{R}^d}\rho\partial_t\rho dx = -b_0\int_{\mathbb{R}^d}\rho\mathrm{div}(\rho u)dx$$

$$= b_0 \int_{\mathbb{R}^d} (\partial_j \rho) \rho u^j \, dx = -\frac{b_0}{2} \int_{\mathbb{R}^d} \rho^2 \mathrm{div}\, u \, dx. \tag{3.18}$$

For the forth term, plugging in (1.5) and (3.9), we integrate by parts to get

$$\begin{aligned} -b_0 \frac{d}{dt} \int_{\mathbb{R}^d} \rho_{N,\hbar} \rho \, dx &= b_0 \int_{\mathbb{R}^d} (-\rho \partial_t \rho_{N,\hbar} - \rho_{N,\hbar} \partial_t \rho) dx \\ &= b_0 \int_{\mathbb{R}^d} [\rho \mathrm{div} J_{N,\hbar} + \rho_{N,\hbar} \mathrm{div}(\rho u)] dx \\ &= b_0 \int_{\mathbb{R}^d} (-\partial_j \rho J^j_{N,\hbar} + \rho_{N,\hbar} \rho \mathrm{div}\, u + \rho_{N,\hbar} u^j \partial_j \rho) dx. \end{aligned} \tag{3.19}$$

Summing up (3.16)–(3.19), we conclude

$$\begin{aligned} &\frac{d}{dt} \mathcal{M}[\phi_{N,\hbar}, \rho, u](t) \\ &= \int_{\mathbb{R}^d} \big[- \rho_{N,\hbar} u^j u^k \partial_k u^j - b_0 \rho_{N,\hbar} u^j \partial_j \rho + J^j_{N,\hbar} u^k \partial_j u^k \big] dx \\ &\quad + \int_{\mathbb{R}^d} -\hbar^2 \partial_k u^j \big[\mathrm{Re}(\partial_j \overline{\phi_{N,h}} \partial_k \phi_{N,h}) \big] dx - \int_{\mathbb{R}^d} \frac{\hbar^2}{4} \rho_{N,h} \partial^2_{jk} \partial_k u^j \, dx \\ &\quad - \frac{b_0}{2} \int \mathrm{div}\, u (\rho_{N,\hbar})^2 dx + \mathrm{Er} + \int_{\mathbb{R}^d} J^j_{N,\hbar} u^k \partial_k u^j \, dx + \int_{\mathbb{R}^d} b_0 J^j_{N,\hbar} \partial_j \rho \, dx \\ &\quad - \frac{b_0}{2} \int_{\mathbb{R}^d} \rho^2 \mathrm{div}\, u \, dx + \int_{\mathbb{R}^d} b_0 \rho_{N,\hbar} \rho \mathrm{div}\, u + b_0 \rho_{N,\hbar} u^j \partial_j \rho - b_0 \partial_j \rho J^j_{N,\hbar} dx \\ &= - \int_{\mathbb{R}^d} \partial_k u^j \Big\{ \rho_{N,\hbar} u^j u^k + \hbar^2 \big[\mathrm{Re}(\partial_j \overline{\phi_{N,h}} \partial_k \phi_{N,h}) \big] - J^j_{N,\hbar} u^k - J^k_{N,\hbar} u^j \Big\} dx \\ &\quad - \frac{b_0}{2} \int_{\mathbb{R}^d} \mathrm{div}\, u (\rho_{N,\hbar} - \rho)^2 dx - \frac{\hbar^2}{4} \int_{\mathbb{R}^d} \rho_{N,\hbar} (\Delta \mathrm{div}\, u) dx + \mathrm{Er}, \end{aligned}$$

which is equivalent to (3.14). This completes the proof. □

3.2 Modulated Energy Estimate

We first estimate the error term (3.15) and then establish Gronwall's inequality for the modulated energy $\mathcal{M}[\phi_{N,\hbar}, \rho, u](t)$.

Lemma 3.4 *Let Er be defined as in* (3.15)*. We have*

$$|\mathrm{Er}| \lesssim \frac{1}{\hbar^4 N^\beta}. \tag{3.20}$$

Proof For (3.20), we decompose

$$
\begin{aligned}
\mathrm{Er} &= \sum_{j=1}^{3} \int_{\mathbb{R}^d} u^j (\partial_j (V_N * \rho_{N,\hbar}))\rho_{N,\hbar} dx + \frac{b_0}{2} \int_{\mathbb{R}^d} \operatorname{div} u (\rho_{N,\hbar})^2 dx \\
&= I_1 + I_2,
\end{aligned} \tag{3.21}
$$

where

$$
\begin{aligned}
I_1 &= \sum_{j=1}^{3} \int_{\mathbb{R}^d} u^j (\partial_j (V_N * \rho_{N,\hbar}))\rho_{N,\hbar} dx \\
&\quad - \sum_{j=1}^{3} \frac{1}{2} \int \partial_j u^j (y)[x^j - y^j]\partial_j [V_N (x-y)]\rho_{N,\hbar}(y)\rho_{N,\hbar}(x) dxdy
\end{aligned} \tag{3.22}
$$

and

$$
\begin{aligned}
I_2 &= \frac{b_0}{2} \int_{\mathbb{R}^d} \operatorname{div} u (\rho_{N,\hbar})^2 dx \\
&\quad + \sum_{j=1}^{3} \frac{1}{2} \int \partial_j u^j (y)[x^j - y^j]\partial_j [V_N (x-y)]\rho_{N,\hbar}(y)\rho_{N,\hbar}(x) dxdy
\end{aligned} \tag{3.23}
$$

with $x = (x^1, x^2, x^3)$ and $y = (y^1, y^2, y^3)$.

First, we deal with I_1. Note that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} u^j (\partial_j (V_N * \rho_{N,\hbar}))\rho_{N,\hbar} dx \\
&\quad = \int u^j (x)(\partial_j V_N)(x-y)\rho_{N,\hbar}(x)\rho_{N,\hbar}(y) dxdy \\
&\quad = \int u^j (y)(\partial_j V_N)(y-x)\rho_{N,\hbar}(y)\rho_{N,\hbar}(x) dxdy.
\end{aligned} \tag{3.24}
$$

By the anti-symmetry of $\partial_j V_N$, the above

$$
= - \int u^j (y)(\partial_j V_N)(x-y)\rho_{N,\hbar}(x)\rho_{N,\hbar}(y) dxdy.
$$

Hence we obtain

$$
\begin{aligned}
I_1 &= \frac{1}{2} \sum_{j=1}^{3} \int (u^j (x) - u^j (y))\partial_j V_N (x-y)\rho_{N,\hbar}(y)\rho_{N,\hbar}(x) dxdy \\
&\quad - \frac{1}{2} \sum_{j=1}^{3} \int \partial_j u^j (y)[x^j - y^j]\partial_j V_N (x-y)\rho_{N,\hbar}(y)\rho_{N,\hbar}(x) dxdy.
\end{aligned}
$$

It suffices to estimate the $j = 1$ case. By Taylor's expansion, we get

$$(u^1(x) - u^1(y)) = \sum_{i=1}^{3} \partial_i u^1(y)[x^i - y^i] + \frac{1}{2}((x-y)\cdot\nabla)^2 u^1(y + \theta(x-y)), \tag{3.25}$$

so we can rewrite

$$I_1 = A_1 + A_2 + A_3,$$

where

$$A_1 = \frac{1}{2}\int \frac{1}{2}((x-y)\cdot\nabla)^2 u^1(y+\theta(x-y))\partial_1 V_N(x-y)\rho_{N,\hbar}(y)\rho_{N,\hbar}(x)dxdy, \tag{3.26}$$

$$A_2 = \frac{1}{2}\int \partial_2 u^1(y)[x^2 - y^2]\partial_1 V_N(x-y)\rho_{N,\hbar}(y)\rho_{N,\hbar}(x)dxdy, \tag{3.27}$$

$$A_3 = \frac{1}{2}\int \partial_3 u^1(y)[x^3 - y^3]\partial_1 V_N(x-y)\rho_{N,\hbar}(y)\rho_{N,\hbar}(x)dxdy. \tag{3.28}$$

For A_1,

$$|A_1| \lesssim \frac{\|D^2 u\|_{L^\infty}}{N^{2\beta}} \int (N^\beta|x-y|)^2 |\partial_1 V_N(x-y)|\rho_{N,\hbar}(x)\rho_{N,\hbar}(y)dxdy.$$

By Hölder, the above

$$\lesssim \frac{\|D^2 u\|_{L^\infty}}{N^\beta} \left\| |(|x|^2\partial_1 V)_N| * \rho_{N,\hbar}\right\|_{L^2} \left\|\rho_{N,\hbar}\right\|_{L^2}.$$

By Young's inequality, interpolation inequality, and the energy bound for $\phi_{N,\hbar}$, the above

$$\lesssim \frac{\|D^2 u\|_{L^\infty} \left\| |x|^2 \partial_1 V\right\|_{L^1}}{\hbar^4 N^\beta}.$$

For A_2,

$$A_2 = \frac{1}{2}\int \partial_2 u^1(y)[x^2 - y^2]\partial_1 V_N(x-y)\rho_{N,\hbar}(y)\rho_{N,\hbar}(x)dxdy. \tag{3.29}$$

By integration by parts, the above

$$= -\frac{1}{2}\int \partial_2 u^1(y)[x^2 - y^2] V_N(x-y)\rho_{N,\hbar}(y)\partial_1\rho_{N,\hbar}(x)dxdy$$

$$= -\frac{1}{2N^\beta} \int \partial_2 u^1(y) \big[N^\beta (x^2 - y^2) \big] V_N(x-y) \rho_{N,\hbar}(y) \partial_1 \rho_{N,\hbar}(x) dx dy.$$

So we get

$$|A_2| \lesssim \frac{\|Du\|_{L^\infty}}{N^\beta} \|\widetilde{V}_N * \rho_{N,\hbar}\|_{L^3} \|\partial_1 \rho_{N,\hbar}\|_{L^{3/2}}, \tag{3.30}$$

where we use the notation that $\widetilde{V}(x) = x^2 V(x)$. By Young's inequality and Hölder inequality, the above

$$\lesssim \frac{\|Du\|_{L^\infty}}{N^\beta} \|\widetilde{V}_N\|_{L^1} \|\rho_{N,\hbar}\|_{L^3} \|\phi_{N,\hbar}\|_{L^6} \|\nabla \phi_{N,\hbar}\|_{L^2}.$$

By Sobolev, the above

$$\lesssim \frac{\|Du\|_{L^\infty}}{N^\beta} \|\widetilde{V}\|_{L^1} \|\phi_{N,\hbar}\|_{H^1}^4.$$

By the energy bound for $\phi_{N,\hbar}$, the above

$$\lesssim \frac{\|Du\|_{L^\infty}}{\hbar^4 N^\beta} \|\widetilde{V}\|_{L^1}.$$

For A_3, we deal with it in the same way and obtain

$$A_3 \lesssim \frac{\|Du\|_{L^\infty}}{\hbar^4 N^\beta} \|\widetilde{V}\|_{L^1}. \tag{3.31}$$

For I_2, it suffices to treat the case $j = 1$. Let

$$\widetilde{\widetilde{V}}(x) = -x^1 \partial_1 V(x), \tag{3.32}$$

then we have

$$\begin{aligned} |I_2| &= \frac{1}{2} \left| \left\langle \partial_1 u^1 \widetilde{\widetilde{V}}_N * \rho_{N,\hbar}, \rho_{N,\hbar} \right\rangle - b_0 \langle \partial_1 u^1 \rho_{N,\hbar}, \rho_{N,\hbar} \rangle \right| \\ &= \frac{1}{2} \left| \left\langle \partial_1 u^1 (\widetilde{\widetilde{V}}_N - b_0 \delta) * \rho_{N,\hbar}, \rho_{N,\hbar} \right\rangle \right|. \end{aligned} \tag{3.33}$$

Since $\int \widetilde{\widetilde{V}} \mathrm{d}x = \int V \mathrm{d}x = b_0$, we can repeat the proof of the approximation of identity estimate (3.13) to get that the above

$$\begin{aligned} &\lesssim \|Du\|_{L^\infty} \|W_N * \rho_{N,\hbar}\|_{L^{3/2}} \|\rho_{N,\hbar}\|_{L^3} \\ &\lesssim \frac{1}{\hbar^4 N^\beta}. \end{aligned}$$

Putting together the estimates of I_1 and I_2 completes the proof. □

We can now provide a closed estimate for the modulated energy.

Proposition 3.5 *Let $\mathcal{M}[\phi_{N,\hbar}, \rho, u](t)$ be defined as in (3.8). We have the lower bound estimate*

$$\mathcal{M}[\phi_{N,\hbar}, \rho, u](t) + \frac{C}{\hbar^4 N^\beta} \geq 0 \tag{3.34}$$

and the following Gronwall's inequality

$$\frac{d}{dt}\mathcal{M}\left[\phi_{N,\hbar}, \rho, u\right](t) \lesssim \mathcal{M}\left[\phi_{N,\hbar}, \rho, u\right](t) + \frac{1}{\hbar^4 N^\beta} + \hbar^2. \tag{3.35}$$

Moreover, we have

$$\mathcal{M}[\phi_{N,\hbar}, \rho, u](t) + \frac{C}{\hbar^4 N^\beta} \leq \exp(CT_0)\left(\mathcal{M}[\phi_{N,\hbar}, \rho, u](0) + \frac{C}{\hbar^4 N^\beta} + C\hbar^2 t\right) \tag{3.36}$$

and

$$\|\rho_{N,\hbar} - \rho\|_{L^\infty([0,T_0];L^2(\mathbb{R}^d))} \leq C(T_0)\left(\frac{1}{\hbar^4 N^\beta} + \hbar^2\right)^{1/2}, \tag{3.37}$$

$$\|(i\hbar\nabla - u)\phi_{N,\hbar}\|_{L^\infty([0,T_0];L^2(\mathbb{R}^d))} \leq C(T_0)\left(\frac{1}{\hbar^4 N^\beta} + \hbar^2\right)^{1/2}. \tag{3.38}$$

Proof For (3.34), we rewrite

$$\begin{aligned}
&\mathcal{M}[\phi_{N,\hbar}, \rho, u](t) \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^d} |(i\hbar\nabla - u)\phi_{N,\hbar}(t)|^2 dx + \frac{b_0}{2}\int \left(\rho_{N,\hbar} - \rho\right)^2 dx + \frac{1}{2}\langle W_N * \rho_{N,\hbar}, \rho_{N,\hbar}\rangle,
\end{aligned} \tag{3.39}$$

where $W_N = V_N - b_0\delta$. By estimate (3.13), we arrive at

$$\mathcal{M}[\phi_{N,\hbar}, \rho, u](t) \gtrsim -\frac{1}{\hbar^4 N^\beta}, \tag{3.40}$$

which completes the proof of (3.34).

For (3.35), we make use of Proposition 3.3 to obtain[18]

$$\begin{aligned}
&\frac{d}{dt}\mathcal{M}[\phi_{N,\hbar}, \rho, u](t) \\
&\quad = -\int_{\mathbb{R}^d} \partial_k u^j \mathrm{Re}\Big((\hbar\partial_k - iu^k)\phi_{N,\hbar}\overline{(\hbar\partial_j - iu^j)\phi_{N,\hbar}}\Big)
\end{aligned}$$

[18] The regularity requirement that $s > \frac{d}{2} + 3$ comes from $\|\Delta \mathrm{div}\, u\|_{L^\infty}$, the second term on the right side of (3.41). One can reduce one derivative in requirement (d) of Theorem 1.1 by integration by parts at the price of weakening the convergence rate.

$$
\begin{aligned}
&-\frac{b_0}{2}\int_{\mathbb{R}^d}\operatorname{div}u(\rho_{N,\hbar}-\rho)^2dx-\frac{\hbar^2}{4}\int_{\mathbb{R}^d}\rho_{N,\hbar}(\Delta\operatorname{div}u)dx+\mathrm{Er}\\
&\lesssim\|Du\|_{L^\infty}\left(\int_{\mathbb{R}^d}|(i\hbar\nabla-u)\phi_{N,\hbar}(t)|^2dx+b_0\int(\rho_{N,\hbar}-\rho)^2dx\right)\\
&+\hbar^2\|\rho_{N,\hbar}\|_{L^1}\|\Delta\operatorname{div}u\|_{L^\infty}+|\mathrm{Er}|. \qquad (3.41)
\end{aligned}
$$

By the error term estimate (3.20), we reach

$$
\frac{d}{dt}\mathcal{M}[\phi_{N,\hbar},\rho,u](t)\lesssim\mathcal{M}[\phi_{N,\hbar},\rho,u](t)+\hbar^2+\frac{1}{\hbar^4N^\beta}, \qquad (3.42)
$$

which completes the proof of (3.35).

Combining (3.34) and (3.35), we have

$$
\begin{aligned}
&\mathcal{M}[\phi_{N,\hbar},\rho,u](t)+\frac{C}{\hbar^4N^\beta}\\
&=\mathcal{M}[\phi_{N,\hbar},\rho,u](0)+\frac{C}{\hbar^4N^\beta}+\int_0^t\frac{d}{d\tau}\left(\mathcal{M}[\phi_{N,\hbar},\rho,u](\tau)+\frac{C}{\hbar^4N^\beta}\right)d\tau\\
&\le\mathcal{M}[\phi_{N,\hbar},\rho,u](0)+\frac{C}{\hbar^4N^\beta}+C\int_0^t\mathcal{M}[\phi_{N,\hbar},\rho,u](\tau)+\frac{C}{\hbar^4N^\beta}+\hbar^2d\tau\\
&=\left(\mathcal{M}[\phi_{N,\hbar},\rho,u](0)+\frac{C}{\hbar^4N^\beta}+C\hbar^2t\right)+C\int_0^t\mathcal{M}[\phi_{N,\hbar},\rho,u](\tau)+\frac{C}{\hbar^4N^\beta}d\tau. \qquad (3.43)
\end{aligned}
$$

Then by Gronwall's inequality, we obtain estimate (3.36).

Finally, we deal with (3.37) and (3.38). By error estimate (3.13), we note that

$$
\begin{aligned}
&\int_{\mathbb{R}^d}|(i\hbar\nabla-u)\phi_{N,\hbar}(t)|^2dx+b_0\int(\rho_{N,\hbar}-\rho)^2dx\lesssim\mathcal{M}[\phi_{N,\hbar},\rho,u](t)+\frac{1}{\hbar^4N^\beta},\\
&\mathcal{M}[\phi_{N,\hbar},\rho,u](0)\lesssim\int_{\mathbb{R}^d}\left|(i\hbar\nabla-u^{\mathrm{in}})\phi_{N,\hbar}^{\mathrm{in}}\right|^2dx+b_0\int_{\mathbb{R}^d}\left(\rho_{N,\hbar}^{\mathrm{in}}-\rho^{\mathrm{in}}\right)^2dx+\frac{1}{\hbar^4N^\beta}.
\end{aligned}
$$

Hence, we can appeal to estimate (3.36) and the initial condition (1.12) to get

$$
\begin{aligned}
&\int_{\mathbb{R}^d}|(i\hbar\nabla-u)\phi_{N,\hbar}(t)|^2dx+b_0\int(\rho_{N,\hbar}-\rho)^2dx\\
&\le C\left(\mathcal{M}\left[\phi_{N,\hbar},\rho,u\right](t)+\frac{1}{\hbar^4N^\beta}\right)\\
&\le C(T_0)\left(\hbar^2+\frac{1}{\hbar^4N^\beta}\right). \qquad (3.44)
\end{aligned}
$$

This completes the proof of estimates (3.37) and (3.38). □

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Appendix A: Miscellaneous Lemmas

A.1 Collapsing Estimate and Strichartz Estimates

Lemma A.1 ([12, 14, 46], KM Collapsing Estimate)[19] *There is a C independent of V, j, k and N such that,*

$$\left\| S^{(1,k)} B^{\pm}_{N,j,k+1} U^{(k+1)}(t) f^{(k+1)} \right\|_{L^2_t L^2_{x,x'}} \le C \|V\|_{L^1} \left\| S^{(1,k+1)} f^{(k+1)} \right\|_{L^2_{x,x'}}, \tag{A.1}$$

where $f^{(k+1)}(\boldsymbol{x}_{k+1}; \boldsymbol{x}'_{k+1})$ is independent of t.

Lemma A.2 *Let $d \le 3$ and $\alpha = d + 1/2$. Then we have*

$$\left\| S^{(1,k)}_{\hbar} B^{\pm}_{N,\hbar,j,k+1} U^{(k+1)}_{\hbar}(t) f^{(k+1)} \right\|_{L^2_t L^2_{x,x'}} \le \frac{C\|V\|_{L^1}}{\hbar^{\alpha}} \left\| S^{(1,k+1)}_{\hbar} f^{(k+1)} \right\|_{L^2_{x,x'}}. \tag{A.2}$$

Proof Let us define

$$(\delta^a_x f)(x) = f(ax), \quad (\delta^a_t f)(t) = f(at). \tag{A.3}$$

By scaling,

$$\begin{aligned} &\left\| S^{(1,k)}_{\hbar} \mathrm{Tr}_{k+1}\big(V_{N,\hbar}(x_j - x_{k+1}) U^{(k+1)}_{\hbar}(t) f^{(k+1)}\big) \right\|_{L^2_t L^2_{x,x'}} \\ &\quad = \hbar^{kd+\frac{1}{2}} \left\| \delta^{\hbar}_t \delta^{\hbar}_x \big[S^{(1,k)}_{\hbar} \mathrm{Tr}_{k+1}\big(V_{N,\hbar}(x_j - x_{k+1}) U^{(k+1)}_{\hbar}(t) f^{(k+1)}\big)\big] \right\|_{L^2_t L^2_{x,x'}}. \end{aligned} \tag{A.4}$$

Noting that $V_{N,\hbar}$ carries $\hbar^{-1}$, the above

$$= \hbar^{kd-\frac{1}{2}} \left\| S^{(1,k)} \mathrm{Tr}_{k+1}\big(\hbar^d V_N(\hbar(x_j - x_{k+1}))\big) U^{(k+1)}(t) \big(\delta^{\hbar}_x [f^{(k+1)}]\big) \right\|_{L^2_t L^2_{x,x'}}.$$

By estimate (A.1), the above

$$\begin{aligned} &\le \hbar^{kd-\frac{1}{2}} C \left\| \hbar^d V_N(\hbar x) \right\|_{L^1} \left\| S^{(1,k+1)} \delta^{\hbar}_x [f^{(k+1)}] \right\|_{L^2_{x,x'}} \\ &= \frac{C\|V\|_{L^1}}{\hbar^{d+\frac{1}{2}}} \left\| S^{(1,k+1)}_{\hbar} f^{(k+1)} \right\|_{L^2_{x,x'}}, \end{aligned}$$

which completes the proof. □

[19] See also [11, 13, 36, 39, 44, 62] for many different versions of estimates of this type.

Lemma A.3 [19, Lemmas 4.1, 4.3, and 4.6][20] *Let* θ *and* $\widetilde{\theta}$ *are cutoff functions supported* $[-1,1]$ *and* $\widetilde{\theta}_T(t)=\widetilde{\theta}(t/T)$. *For the case* $\hbar=1$, *we have*

$$\begin{aligned}&\left\|S^{(1,k)}\theta(t_k)\int_0^{t_k}U^{(k)}(t_k-t_{k+1})V_N(x_1-x_2)\widetilde{\theta}_T(t_{k+1})\gamma_N^{(k)}(t_{k+1})dt_{k+1}\right\|_{L^\infty_{t_k}L^2_{x,x'}}\\&\quad\le N^{\frac{5}{2}\beta}C_VC_\theta\left\|S^{(1,k)}\widetilde{\theta}_T(t_{k+1})\gamma_N^{(k)}(t_{k+1})\right\|_{L^2_{t_{k+1}}L^2_{x,x'}}\end{aligned}\tag{A.5}$$

and

$$\begin{aligned}&\left\|S^{(1,k+j-1)}B_{N,1,k+j}\theta(t_{k+j})\int_0^{t_{k+j}}U^{(k+j)}(t_{k+j}-t_{k+j+1})V_{N,12}\widetilde{\theta}_T\gamma_N^{(k+j)}(t_{k+j+1})dt_{k+j+1}\right\|_{L^2_{t_{k+j}}L^2_{x,x'}}\\&\quad\le N^{\frac{5}{2}\beta}C_VC_\theta\left\|S^{(1,k+j)}\widetilde{\theta}_T(t_{k+j+1})\gamma_N^{(k+j)}(t_{k+j+1})\right\|_{L^2_{t_{k+j+1}}L^2_{x,x'}},\end{aligned}\tag{A.6}$$

where $V_{N,12}=N^{d\beta}V(N^\beta(x_1-x_2))$ *and*

$$C_\theta=|\mathrm{Supp}(\theta)|\left(\|\theta\|_{L^2_t}+\|\theta'\|_{L_t^{\frac{4}{3}}}\right)+\|\theta\|_{L_t^{\frac{4}{3}}}+\|\langle\nabla_t\rangle^{\frac{3}{4}}\theta\|_{L^2_t}+\|\theta\|_{L^\infty_t}$$

with $|\mathrm{Supp}(\theta)|$ *denoting the Lebesgue measure of the support of* θ.

Lemma A.4 *For* $j\ge 0$ *and* $k\ge 1$, *we have the following estimates*

$$\begin{aligned}&\left\|S_\hbar^{(1,k)}\int_0^{t_k}U_\hbar^{(k)}(t_k-t_{k+1})V_{N,\hbar}^{(k)}\gamma_{N,\hbar}^{(k)}(t_{k+1})dt_{k+1}\right\|_{L^\infty_{t_k}[0,T]L^2_{x,x'}}\\&\quad\le N^{\frac{5}{2}\beta-1}\hbar(C_V\hbar^{-\alpha}T^{1/2})k^2\left\|S_\hbar^{(1,k)}\gamma_{N,\hbar}^{(k)}(t_{k+1})\right\|_{L^\infty_{t_{k+1}}L^2_{x,x'}}\end{aligned}\tag{A.7}$$

and

$$\begin{aligned}&\left\|S_\hbar^{(1,k+j-1)}B_{N,\hbar,1,k+j}\int_0^{t_{k+j}}U_\hbar^{(k+j)}(t_{k+j}-t_{k+j+1})V_{N,\hbar}^{(k+j)}\gamma_{N,\hbar}^{(k+j)}(t_{k+j+1})dt_{k+j+1}\right\|_{L^1_{t_{k+j}}[0,T]L^2_{x,x'}}\\&\quad\le N^{\frac{5}{2}\beta-1}\hbar(C_V\hbar^{-\alpha}T^{1/2})^2(k+j)^2\left\|S_\hbar^{(1,k+j)}\gamma_{N,\hbar}^{(k+j)}(t_{k+j+1})\right\|_{L^\infty_{t_{k+j+1}}L^2_{x,x'}}.\end{aligned}\tag{A.8}$$

Proof For (A.7), we have

$$\left\|S_\hbar^{(1,k)}\int_0^{t_k}U_\hbar^{(k)}(t_k-t_{k+1})V_{N,\hbar}^{(k)}\gamma_{N,\hbar}^{(k)}(t_{k+1})dt_{k+1}\right\|_{L^\infty_{t_k}[0,T]L^2_{x,x'}}$$

[20] These are $X_{s,b}$ estimates in disguise. As we are not using the $X_{s,b}$ spaces directly in this paper, we will not go into the details.

$$\leq \left\| S_{\hbar}^{(1,k)}\theta(t_k)\int_0^{t_k} U_{\hbar}^{(k)}(t_k-t_{k+1})V_{N,12}\widetilde{\theta}_T(t_{k+1})\gamma_{N,\hbar}^{(k)}(t_{k+1})dt_{k+1}\right\|_{L^\infty_{t_k}L^2_{x,x'}}, \tag{A.9}$$

where θ and $\widetilde{\theta}$ are cutoff functions supported $[-1, 1]$ and $\widetilde{\theta}_T(t) = \widetilde{\theta}(t/T)$. For simplicity, we set

$$V_{N\hbar,12} = (N^\beta\hbar)^d V\big(N^\beta\hbar(x_1-x_2)\big).$$

Then by scaling argument, we arrive at

$$\begin{aligned}
&\left\| S_{\hbar}^{(1,k)}\theta(t_k)\int_0^{t_k} U_{\hbar}^{(k)}(t_k-t_{k+1})V_{N,12}\widetilde{\theta}_T(t_{k+1})\gamma_{N,\hbar}^{(k)}(t_{k+1})dt_{k+1}\right\|_{L^\infty_{t_k}L^2_{x,x'}}\\
&= \hbar^{kd}\left\|\delta_x^\hbar\delta_t^\hbar\left[S_{\hbar}^{(1,k)}\theta(t_k)\int_0^{t_k} U_{\hbar}^{(k)}(t_k-t_{k+1})V_{N,12}\widetilde{\theta}_T(t_{k+1})\gamma_{N,\hbar}^{(k)}(t_{k+1})dt_{k+1}\right]\right\|_{L^\infty_{t_k}L^2_{x,x'}}\\
&= \hbar\hbar^{kd}\left\|S^{(1,k)}(\delta_t^\hbar\theta)(t_k)\int_0^{t_k} U^{(k)}(t_k-t_{k+1})\delta_x^\hbar\left[V_{N,12}\delta_t^\hbar\left[\widetilde{\theta}_T(t_{k+1})\gamma_{N,\hbar}^{(k)}(t_{k+1})\right]\right]dt_{k+1}\right\|_{L^\infty_{t_k}L^2_{x,x'}}\\
&= \frac{\hbar\hbar^{kd}}{\hbar^d}\left\|S^{(1,k)}(\delta_t^\hbar\theta)(t_k)\int_0^{t_k} U^{(k)}(t_k-t_{k+1})V_{N\hbar,12}\delta_x^\hbar\delta_t^\hbar\left[\widetilde{\theta}_T(t_{k+1})\gamma_{N,\hbar}^{(k)}(t_{k+1})\right]dt_{k+1}\right\|_{L^\infty_{t_k}L^2_{x,x'}}.
\end{aligned} \tag{A.10}$$

By using estimate (A.5), the above

$$\begin{aligned}
&\leq \frac{\hbar\hbar^{kd}(N^\beta\hbar)^{\frac{5}{2}}C_V C_{\delta_t^\hbar\theta}}{\hbar^d}\left\|S^{(1,k)}\delta_x^\hbar\delta_t^\hbar\left[\widetilde{\theta}_T(t_{k+1})\gamma_{N,\hbar}^{(k)}(t_{k+1})\right]\right\|_{L^2_{t_{k+1}}L^2_{x,x'}}\\
&= \frac{\hbar(N^\beta\hbar)^{\frac{5}{2}}C_V C_{\delta_t^\hbar\theta}}{\hbar^d\hbar^{1/2}}\left\|S^{(1,k)}\widetilde{\theta}_T(t_{k+1})\gamma_{N,\hbar}^{(k)}(t_{k+1})\right\|_{L^2_{t_{k+1}}L^2_{x,x'}}.
\end{aligned}$$

By taking L^∞ at dt_{k+1} and using the estimate that $\hbar^{\frac{3}{2}}C_{\delta_t^\hbar\theta} \leq C$, the above

$$\leq \frac{N^{\frac{5}{2}\beta}\hbar^2 C_V T^{1/2}}{\hbar^d\hbar^{1/2}}\left\|S^{(1,k)}\gamma_{N,\hbar}^{(k)}(t_{k+1})\right\|_{L^\infty_{t_{k+1}}L^2_{x,x'}}.$$

We note that the N^{-1}, k^2 and $\hbar^{-1}$ factors come from the expansion of $V_{N,\hbar}^{(k)}$ and then arrive at (A.7).

Next, we deal with (A.8). With the help of estimate (A.6), we can use scaling argument in the same way as above to arrive at (A.8), where the N^{-1}, $(k+j)^2$, and $\hbar^{-1}$ factors come from the expansion of $V_{N,\hbar}^{(k+j)}$ and another $\hbar^{-1}$ factor comes from $B_{N,\hbar,1,k+j}$. □

Lemma A.5 *For $j \geq 0$ and $k \geq 1$, we have*

$$\int_{[0,T]} \left\| S_{\hbar}^{(1,k+j)} B_{N,\hbar,1,k+j+1} \gamma_{N,\hbar}^{(k+j+1)}(t_{k+j+1}) \right\|_{L^2_{x,x'}} dt_{k+j+1} \\ \leq N^{\frac{5}{2}\beta} \hbar^2 T^{1/2} (C_V \hbar^{-\alpha} T^{1/2}) \left\| S_{\hbar}^{(1,k+j+1)} \gamma_{N,\hbar}^{(k+j+1)}(t_{k+j+1}) \right\|_{L^\infty_{t_{k+j+1}} L^2_{x,x'}}. \tag{A.11}$$

Proof By taking L^∞ at $\mathrm{d}t_{k+j+1}$, it suffices to prove that

$$\left\| S_{\hbar}^{(1,k+j)} B_{N,\hbar,1,k+j+1} \gamma_{N,\hbar}^{(k+j+1)}(t_{k+j+1}) \right\|_{L^2_{x,x'}} \leq N^{\frac{5}{2}\beta} \hbar^2 C_V \hbar^{-\alpha} \left\| S_{\hbar}^{(1,k+j+1)} \gamma_{N,\hbar}^{(k+j+1)} \right\|_{L^2_{x,x'}}. \tag{A.12}$$

For $\hbar = 1$, we have

$$\left\| S^{(1,k+j)} B_{N,1,k+j+1} \gamma_{N}^{(k+j+1)}(t_{k+j+1}) \right\|_{L^2_{x,x'}} \leq N^{\frac{5}{2}\beta} C_V \left\| S^{(1,k+j+1)} \gamma_{N}^{(k+j+1)} \right\|_{L^2_{x,x'}}. \tag{A.13}$$

By scaling, we arrive at (A.12). □

A.2 Convolution and Commutator Estimates

Lemma 3.6 [22, Lemma A.5] *Let $W_N(x) = N^{d\beta} V(N^\beta x) - b_0 \delta$, where $b_0 = \int V(x) dx$. For any $0 \leq s \leq 1$,*

$$\| W_N * f \|_{L^p} \lesssim N^{-\beta s} \| \langle \nabla \rangle^s f \|_{L^p} \tag{A.14}$$

for any $1 < p < \infty$. The implicit constant depends only on $\| \langle x \rangle V(x) \|_{L^1}$.

Lemma A.6 (Fractional Leibniz Rule)

$$\| \langle \nabla \rangle^s (fg) \|_{L^r} \lesssim \| \langle \nabla \rangle^s f \|_{L^{p_1}} \| g \|_{L^{p_2}} + \| f \|_{L^{q_1}} \| \langle \nabla \rangle^s g \|_{L^{q_2}}, \tag{A.15}$$

where

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}, \tag{A.16}$$

$r \in [1, \infty)$, $p_i, q_i \in (1, \infty]$, $s > 0$.

Lemma A.7 [21, 30] *Let $d = 3$, $\eta > d/4$ and $V_N(x) = N^{3\beta} V(N^\beta x)$. Then*

$$V_N(x_1 - x_2) \leq C(\eta) \| V \|_{L^1} (1 - \Delta_{x_1})^\eta (1 - \Delta_{x_2})^\eta, \tag{A.17}$$

$$V_N(x_1 - x_2) \leq C N^\beta \| V \|_{L^{3/2}} (1 - \Delta_{x_1}), \tag{A.18}$$

$$V_N(x_1 - x_2) \leq C N^{3\beta} \| V \|_{L^\infty}. \tag{A.19}$$

Proof For (A.17) with $\eta = 1$, (A.18) and (A.19), see [30, Lemma A.3]. For (A.17) with $\eta > 3/4$, see [21] by using the low-high frequency decomposition. $\square$

Appendix B: Energy Estimate

Recall the Hamiltonian (1.2)

$$H_{N,\hbar} = \sum_{j=1}^{N} -\frac{1}{2}\hbar^2 \Delta_{x_j} + \frac{1}{N} \sum_{1 \leq j < k \leq N} V_N(x_j - x_k)$$

and the derivative involving $\hbar$ in (2.5)

$$S_{\hbar,j}^2 = 1 - \frac{\hbar^2}{2} \Delta_{x_j}.$$

Proposition B.1 *Let $\beta < \frac{3}{5}$, $k \leq (\ln N)^{100}$ and $\hbar^{-1} \leq \ln N$[21]. There exists $N_0(\beta)$ independent of k and $\hbar$, such that*

$$\langle \psi, (H_{N,\hbar} + N)^k \psi \rangle \geq \frac{N^k}{2^k} \langle \psi, S_{\hbar,1}^2 S_{\hbar,2}^2 \cdots S_{\hbar,k}^2 \psi \rangle \tag{B.1}$$

for every $N \geq N_0(\beta)$.

Proof This proof has been done by many authors in many work. We include one here solely for completeness purposes. For $k = 0$ and $k = 1$, the claim is trivial because of the positivity of the potential. Now we assume the proposition is true for all $k \leq n$, and we prove it for $k = n + 2$.

$$\begin{aligned}\left\langle \psi, (H_{N,\hbar} + N)^{n+2} \psi \right\rangle &= \left\langle (H_{N,\hbar} + N)\psi, (H_{N,\hbar} + N)^n (H_{N,\hbar} + N)\psi \right\rangle \\ &\geq \frac{N^n}{2^n} \left\langle \psi, (H_{N,\hbar} + N) S_{\hbar,1}^2 \cdots S_{\hbar,n}^2 (H_{N,\hbar} + N)\psi \right\rangle. \end{aligned} \tag{B.2}$$

We set

$$H_{N,\hbar}^{(n)} = \sum_{j=1}^{n} S_{\hbar,j}^2 + \frac{1}{N} \sum_{j<m}^{N} V_{jm}$$

with $V_{jm} = N^{3\beta} V(N^\beta (x_j - x_m))$. Then we have

$$\begin{aligned}&\left\langle \psi, (H_{N,\hbar} + N) S_{\hbar,1}^2 \cdots S_{\hbar,n}^2 (H_{N,\hbar} + N)\psi \right\rangle \\ &\quad = \sum_{j_1, j_2 \geq n+1} \left\langle \psi, S_{\hbar,j_1}^2 S_{\hbar,1}^2 \cdots S_{\hbar,n}^2 S_{\hbar,j_2}^2 \psi \right\rangle\end{aligned}$$

[21] The restriction that $\hbar^{-1} \leq \ln N$ is not necessary and it can be removed at the price of reducing down the parameter β.

$$+ \sum_{j\geq n+1} \left(\left\langle \psi, S_{\hbar,j}^2 S_{\hbar,1}^2 \cdots S_{\hbar,n}^2 H_{N,\hbar}^{(n)} \psi \right\rangle + c.c. \right) + \left\langle \psi, H_{N,\hbar}^{(n)} S_{\hbar,1}^2 \cdots S_{\hbar,n}^2 H_{N,\hbar}^{(n)} \psi \right\rangle.$$

where $c.c.$ denotes the complex conjugate. Since $H_{N,\hbar}^{(n)} S_{\hbar,1}^2 \cdots S_{\hbar,n}^2 H_{N,\hbar}^{(n)} \geq 0$, we have, using the symmetry with respect to permutations,

$$\begin{aligned}
&\left\langle \psi, (H_{N,\hbar} + N) S_{\hbar,1}^2 \cdots S_{\hbar,n}^2 (H_{N,\hbar} + N) \psi \right\rangle \\
&\quad \geq (N-n)(N-n-1) \left\langle \psi, S_{\hbar,1}^2 S_{\hbar,2}^2 \cdots S_{\hbar,n+2}^2 \psi \right\rangle \\
&\qquad + (2n+1)(N-n) \left\langle \psi, S_{\hbar,1}^4 S_{\hbar,2}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle \\
&\qquad + \frac{n(n+1)(N-n)}{2N} \left(\left\langle \psi, V_{12} S_{\hbar,1}^2 S_{\hbar,2}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle + c.c. \right) \\
&\qquad + \frac{(n+1)(N-n)(N-n-1)}{N} \left(\left\langle \psi, V_{1,n+2} S_{\hbar,1}^2 S_{\hbar,2}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle + c.c. \right).
\end{aligned} \tag{B.3}$$

Here we also used the fact that

$$\left\langle \psi, V_{jm} S_{\hbar,1}^2, \cdots S_{\hbar,n+1}^2 \psi \right\rangle \geq 0$$

if $j, m > n+1$, because of the positivity of the potential. Next, we will bound the last two terms on the r.h.s of (B.3) from below, so we might as well set $S_{\hbar,j}^2 = 1 - \hbar^2 \Delta_{x_j}$ for simplicity. Then we have

$$\begin{aligned}
&\left\langle \psi, V_{12} S_{\hbar,1}^2 S_{\hbar,2}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle + c.c. \\
&\quad = \left\langle \psi, V_{12} (1 - \hbar^2 \Delta_{x_1})(1 - \hbar^2 \Delta_{x_2}) S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle + c.c. \\
&\quad \geq \left\langle \psi, \hbar \nabla V_{12} \hbar \nabla_{x_2} S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle + c.c. \\
&\qquad + \left\langle \hbar \nabla_{x_2} \psi, \hbar \nabla V_{12} \hbar \nabla_{x_1} \hbar \nabla_{x_2} S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle + c.c. \\
&\qquad + \left\langle \psi, \hbar \nabla V_{12} \hbar^2 \Delta_{x_1} \hbar \nabla_{x_2} S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle + c.c. \\
&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned}$$

where $\nabla V_{12} = N^{4\beta} (\nabla V)(N^\beta (x_1 - x_2))$. Applying Cauchy–Schwarz, we get

$$\begin{aligned}
\mathrm{I} &\geq -2 \Big\{ \alpha_1 \left\langle \psi, |\hbar \nabla V_{12}| S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle \\
&\qquad + \alpha_1^{-1} \left\langle |\hbar \nabla_{x_2}| \psi, |\hbar \nabla V_{12}| S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 |\hbar \nabla_{x_2}| \psi \right\rangle \Big\}, \\
\mathrm{II} &\geq -2 \Big\{ \alpha_2 \left\langle |\hbar \nabla_{x_2}| \psi, |\hbar \nabla V_{12}| S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 |\hbar \nabla_{x_2}| \psi \right\rangle \\
&\qquad + \alpha_2^{-1} \left\langle |\hbar \nabla_{x_1}| |\hbar \nabla_{x_2}| \psi, |\hbar \nabla V_{12}| S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 |\hbar \nabla_{x_1}| |\hbar \nabla_{x_2}| \psi \right\rangle \Big\}, \\
\mathrm{III} &\geq -2 \Big\{ \alpha_3 \left\langle \psi, |\hbar \nabla V_{12}| S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 \psi \right\rangle \\
&\qquad + \alpha_3^{-1} \left\langle |\hbar \nabla_{x_1}|^2 |\hbar \nabla_{x_2}| \psi, |\hbar \nabla V_{12}| S_{\hbar,3}^2 \cdots S_{\hbar,n+1}^2 |\hbar \nabla_{x_1}|^2 |\hbar \nabla_{x_2}| \psi \right\rangle \Big\}.
\end{aligned}$$

By Lemma A.7,

$$
\begin{aligned}
\mathrm{I} &\geq -C\Big\{\alpha_1 N^{\beta}\hbar^{-3}\langle\psi, S_{\hbar,1}^2\cdots S_{\hbar,n+1}^2\psi\rangle + \alpha_1^{-1}N^{4\beta}\hbar\langle\psi, S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle\Big\},\\
\mathrm{II} &\geq -C\Big\{\alpha_2 N^{2\beta}\hbar^{-1}\langle\psi, S_{\hbar,1}^2\cdots S_{\hbar,n+1}^2\psi\rangle + \alpha_2^{-1}N^{2\beta}\hbar^{-1}\langle\psi, S_{\hbar,1}^4 S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle\Big\},\\
\mathrm{III} &\geq -C\Big\{\alpha_3 N^{\beta}\hbar^{-3}\langle\psi, S_{\hbar,1}^2\cdots S_{\hbar,n+1}^2\psi\rangle + \alpha_3^{-1}N^{4\beta}\hbar\langle\psi, S_{\hbar,1}^4 S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle\Big\}.
\end{aligned}
$$

Optimizing the choice of α_1, α_2 and α_3, we find that

$$
\begin{aligned}
&\langle\psi, V_{12}S_{\hbar,1}^2 S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle + c.c.\\
&\quad\geq -CN^{-3/2}N^{\frac{5}{2}\beta}\hbar^{-1}\Big\{N^2\langle\psi, S_{\hbar,1}^2\cdots S_{\hbar,n+1}^2\psi\rangle + N\langle\psi, S_{\hbar,1}^4 S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle\Big\}.
\end{aligned}
$$

As for the last term on the r.h.s of (B.3), we have

$$
\begin{aligned}
&\langle\psi, V_{1,n+2}S_{\hbar,1}^2 S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle + c.c.\\
&\quad\geq \langle\psi, V_{1,n+2}(-\hbar^2\Delta_{x_1})S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle + c.c.\\
&\quad\geq \langle\psi, |\hbar\nabla V_{1,n+2}||\hbar\nabla_{x_1}|S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle + c.c.\\
&\quad\geq -\alpha\langle\psi, |\hbar\nabla V_{1,n+1}|S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle\\
&\qquad - \alpha^{-1}\langle|\hbar\nabla_{x_1}|\psi, |\hbar\nabla V_{1,n+2}|S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2|\hbar\nabla_{x_1}|\psi\rangle\\
&\quad\geq -C\big(\alpha N^{\beta}\hbar^{-3} + \alpha^{-1}N^{2\beta}\hbar^{-1}\big)\langle\psi, S_{\hbar,1}^2\cdots S_{\hbar,n+2}^2\psi\rangle\\
&\quad\geq -CN^{\frac{3}{2}\beta}\hbar^{-2},
\end{aligned}
$$

where we optimized the choice of α. Then we get

$$
\begin{aligned}
&\langle\psi, (H_{N,\hbar}+N)S_{\hbar,1}^2\cdots S_{\hbar,n}^2(H_{N,\hbar}+N)\psi\rangle\\
&\quad\geq (N-n)(N-n-1)\left(1-\frac{CN^{\frac{5}{2}\beta}\hbar^{-1}n^2}{N^{1/2}(N-n)}-\frac{CN^{\frac{3}{2}\beta}\hbar^{-2}n}{N}\right)\langle\psi, S_{\hbar,1}^2\cdots S_{\hbar,n+2}^2\psi\rangle\\
&\qquad + (2n+1)(N-n)\left(1-\frac{CN^{\frac{5}{2}\beta}\hbar^{-1}n}{N^{3/2}}\right)\langle\psi, S_{\hbar,1}^4 S_{\hbar,2}^2\cdots S_{\hbar,n+1}^2\psi\rangle.
\end{aligned}
$$

Since $\beta < \frac{3}{5}$, $n \leq (\ln N)^{100}$ and $\hbar^{-1} \leq \ln N$, we can find $N_0(\beta)$ which is independent of n and $\hbar$, so that

$$
\langle\psi, (H_{N,\hbar}+N)S_{\hbar,1}^2\cdots S_{\hbar,n}^2(H_{N,\hbar}+N)\psi\rangle \geq \frac{N^2}{4}\langle\psi, S_{\hbar,1}^2\cdots S_{\hbar,n+2}^2\psi\rangle
$$

for every $N \geq N_0(\beta)$. Together with (B.2), this completes the proof. □

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