Random telegraph processes with nonlocal memory

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We study two-state (dichotomous, telegraph) random ergodic continuous-time processes with dynamics depending on their past. We take into account the history of the process in an explicit form by introducing integral nonlocal memory term into conditional probability function. We start from an expression for the conditional transition probability function describing additive multistep binary random chain and show that the telegraph processes can be considered as continuous-time interpolations of discrete-time dichotomous random sequences. An equation involving the memory function and the two-point correlation function of the telegraph process is analytically obtained. This integral equation defines the correlation properties of the processes with given memory functions. It also serves as a tool for solving the inverse problem, namely for generation of a telegraph process with a prescribed pair correlation function. We obtain analytically the correlation functions of the telegraph processes with two exactly solvable examples of memory functions and support these results by numerical simulations of the corresponding telegraph processes.

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I. INTRODUCTION

The problems dealing with systems exhibiting long-range spatial and/or temporal correlations remain to be on the top of intensive research in physics, as well as in the theory of dynamical systems and in the theory of probability [1–6]. Nature offers a large number of examples of random processes. Moreover, they occur even more often than those with a deterministic behavior. A systematic research of these processes is necessary to describe a vast range of complex phenomena.

A need to generate a correlated random process of continuous or discrete variable appears in many areas of physics and engineering. The progress in this field of research may have a strong impact on design of a new class of electronic nanodevices, optic fibers, acoustic and electromagnetic wave guides with selective transport properties (see, e.g., Refs. [7–11]). The key ingredient of the theory of correlated disorder is the two-point (pair or binary) correlator of a random process. As was shown for a weak disorder, this correlator fully determines the transmission or reflection of classical or quantum waves through disordered structures. The algorithm proposed in publications [7–11] generates a statistical ensemble of random functions (trajectories of the process) all possessing the same pair correlator. Generally, the values of the random functions may take any number from $-\infty$ to ∞. In this work, we study a wide class of processes when random variable takes only two values, say a and b. Such processes are often found in nature; they are referred to as telegraph processes, also known us dichotomous random processes.

The study of telegraph processes has a long history and it is of great interest to researchers. For example, Kac writes in his classical publication on the theory of probability [12] "We will consider a very simple stochastic model, a random walk. Unfortunately, this model is little known. It has very interesting features and leads not to a diffusion equation but to a hyperbolic one. The model first appeared in the literature in a paper by Sidney Goldstein, known to you mostly because of his work in fluid dynamics. The model had first been proposed by G. I. Taylor—I think in an abortive, or at least not very successful, attempt to treat turbulent diffusion. But the model itself proved to be very interesting." At present, the telegraph process has been studied to a much greater extent than at the time Kac's work was published. Currently application of the theory of random telegraph processes can be found in a variety of complex phenomena. To mention a few, ion channel gating dynamics in biological transport processes and gene expression levels in cells, motion of bacteria, neuronal spike trains, disorder-induced spatial patterns, first-passage and thermally activated escape processes, some aspects of spin dynamics, hypersensitive transport, stochastic resonance, quantum multifractality, blinking quantum dots, rocking ratchets, and intermittent fluorescence. The diverse dichotomous systems may display nonstationarity, nonergodicity, and/or Lévy statistics. Links and references to these and many other important studies related to numerous applications of dichotomous processes can be found in Refs. [13–19].

The telegraph process is of interest not only from pure mathematical point of view, but also as a mechanism of specific noise affecting some dynamical systems. If the noise is neither Gaussian nor dichotomous, then it is generally impossible to analyze its effect on an dynamical system.

One of the methods to study statistical correlations in a dynamical system is mapping of its evolution to a sequence of symbols having the same statistical properties as the system itself. Several algorithms for generation of random sequences with long-range correlations are known in literature [7–11, 20–22]. Here we propose a powerful method based on statistics of multistep Markov chains. The additive Markov chain models [23–25] have shown their effectiveness in describing diverse objects, including literary texts and DNA sequences; therefore, it is of undoubted interest to obtain a generalization of these models to the class of systems characterized by continuous parameters.

A Markov process is a common and natural tool for describing random phenomena (see, e.g., Refs. [26–29]). Two well-known Gaussian Markov processes—Brownian motion and Ornstein-Uhlenbeck process [30]—have been used extensively in various applications from financial mathematics to natural sciences [31–33]. Both these processes can be described by the Langevin equation [31] for a random variable $V(t)$ (e.g., for the velocity of the particle),

$$dV(t) = -\nu V(t)dt + \sigma\, dW(t). \tag{1}$$

Here $dW(t)$ is standard centered white noise. The term $-\nu V(t)dt$ describes linear friction between particle and bath. It is important to note that such Eq. (1) is valid only if the external random applied force is a Gaussian white noise. In this case, the friction force is a linear function of the random variable $V(t)$. In a more general case, the friction force is a linear functional depending on the entire past dynamics of the system and can be written in the form,

$$dV(t)_{\mathrm{mem}} = \left(\int_0^\infty \mu(t')V(t-t')dt'\right)dt, \tag{2}$$

(see Refs. [34–37]). Thus, Eq. (1) with the additional term Eq. (2) containing the memory kernel $\mu(t')$ becomes an integrodifferential equation and describes a non-Markov process. By definition, all non-Markovian processes are history dependent.

While the nature and statistical characteristics of the random forces applied to the system may be unknown, it follows from the Langevin equation that if the applied force is not a δ-correlated process, then an additional terms should appear in Eq. (1), e.g., in the form of Eq. (2).

In this paper, we explicitly take into account the history of telegraph process by introducing integral nonlocal memory term into transition conditional probability function. A telegraph process with memory can be applied to a wider class of phenomena than a memoryless process.

The relation between the correlation and memory functions is given by a linear integrodifferential equation. It can be hardly analytically solved in general case. Here we demonstrate two interesting particular cases when the solution can be obtained analytically. Note that the equation considered here. for the telegraph process with added memory term does not describe a renewal process [38].

The structure of the paper is as follows. In Sec. II, we present some general definitions and provide a brief description of the models and some relevant previous results. We start from the expression for the transition conditional probability function describing additive multistep random chain and show that the proposed processes can be considered as a generalization to continuous variable of a discrete-time random Markovian sequence. In addition, equation connecting the memory function and the two-point correlation function of the process is derived. In Sec. III, we solve analytically the equations for the correlation function for two particular examples of the memory function. The direct and inverse problems for the weak memory function are studied in Sec. IV. The long-time asymptotics of the correlation function are derived in Sec. V. The last Sec. VI contains conclusions and the outline for further research.

II. TELEGRAPH PROCESS WITH MEMORY AS A GENERALIZATION OF THE DISCRETE MULTISTEP MARKOV CHAIN

A random process $N(t)$ that represents the total number of occurrences of some event within the time interval $[0, t]$ is called a renewal process, if the time intervals between events are independent and identically distributed random variables. The Poisson and telegraph processes with exponentially distributed intervals between events are examples of a renewal process.

In the conventional probability theory, the telegraph process is a memoryless continuous-time stochastic process where a random variable can take on two distinct values only, say a and b. It describes, for example, a one-dimensional random motion of a particle moving with a constant velocity $v = a$ along some direction for some random time interval drawn from an exponential distribution, and after that, the particle moves to the opposite direction with the velocity b, where $b = -a = -v$. Thus, we declare that, independently of prehistory of a particle motion, the probabilities to generate a random value of x_{t+dt} are

$$P(x_{t+dt} = b|x_t = a) = \lambda dt, \tag{3a}$$

$$P(x_{t+dt} = a|x_t = b) = \mu dt, \tag{3b}$$

where the random process is defined by two constants, λ and μ, representing the inverse average times of life $1/\bar{t}_a$ and $1/\bar{t}_b$ of the particle in the states a and b, correspondingly. The counterparts of these equations are the following relations:

$$P(x_{t+dt} = a|x_t = a) = 1 - \lambda dt, \tag{3c}$$

$$P(x_{t+dt} = b|x_t = b) = 1 - \mu dt. \tag{3d}$$

If the lifetime of the system (without memory) in the states a and b is governed by Eqs. (3), then it is possible to construct the process by two methods: step-by-step generation with infinitesimally small time step dt, or global generation of random time intervals t_a and t_b of the system to stay in the states a and b. These two ways are equivalent for the processes without memory. However, the first method allows us to adequately include memory into the process. Therefore, we use namely this method in our numerical simulations. A fragment

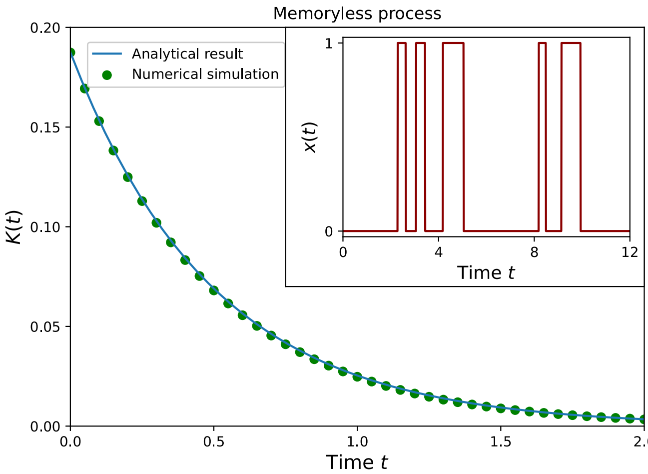

FIG. 1. Example of a telegraph process without memory. The correlation function, $K(t) = \mu\lambda/(\mu+\lambda)^2 \exp[-(\mu+\lambda)t]$, is plotted by solid line. The numerically generated results are shown by dots. The parameters of the generated process are: $\lambda = 1.5$, $\mu = 0.5$, $a = 1$, $b = 0$, $\bar{x} = 0.25$. The time extension of the process is 10^5 and the time step of generation is 10^{-2}. The inset shows the generated random variable $x(t)$.

of numerically constructed telegraph process without memory is shown in the insert to Fig. 1.

The conditional probabilities in Eqs. (3) know only the current value of the random variable taken at $t' = t$. To take into account the memory effects from the previous times $t' < t$, we have to introduce integral memory in the form similar to Eq. (2) [see also Eqs. (9)] to the right-hand sides of Eqs. (3). However, we prefer here a more transparent and clear way by considering the analogy of the telegraph process with a discrete additive memory-dependent Markov chain. The following step is a transition from the discrete random sequence to the continuous-time random process.

A convenient representation of a discrete random chain is to write down its transition conditional probability function [23–25]. This function completely determines the dynamics of the random chain as well as its correlation properties. The transition conditional probability function $P(x_{r+1}|x_{r+1-N},\dots,x_{r-1},x_r)$ of the binary N-step Markov chain is written as follows [25],

$$P\big(x_{r+1} = 1|x^r_{r+1-N}\big) = \bar{x} + \sum_{r'=1}^{N} F(r')(x_{r+1-r'} - \bar{x}), \qquad (4)$$

where $\bar{x}$ is the average value of the random variable x and a concise notation $x^r_{r+1-N} = x_{r+1-N},\dots,x_r$ for a sequence of N previous random values is used.

There is no common admitted name for the random sequences defined by Eq. (4). It can be referred to as categorical [39], higher-order [40,41], multistep or N-step [23,24] Markov's chains. One of the most important and interesting applications of the symbolic sequences is the probabilistic language model, which specializes in predicting the next item in a sequence by means of N previous known symbols. In this sense the Markov chains are known as the N-gram models. We refer to such sequences as the additive Markov chains and $F(r)$ as the memory function. It describes the strength of influence of the previous symbols $x_{r+1-r'}$ $(1 \leqslant r' \leqslant N)$ upon the generated one, x_{r+1}.

Let us rewrite Eq. (4) for the conditional probability function of the Markov chain containing two elements $\{a, b\} = \{1, 0\}$ assuming quasicontinuous process with infinitesimally short time step Δt, and putting $F(1) = (1 - 2\gamma\Delta t)$ and $F(r \geqslant 2) = \alpha_r(\Delta t)^2$

$$P(x_{r+1} = 1|x_r; x_{r'<r}) = \bar{x} + (1 - 2\gamma\Delta t)(x_r - \bar{x}) + \sum_{r'=1}^{\infty} \alpha_{r'}(\Delta t)^2(x_{r-r'} - \bar{x}), \qquad (5)$$

where

$$\bar{x} = \mu/(\mu+\lambda), \quad \gamma = (\mu+\lambda)/2. \qquad (6)$$

The term proportional to Δt describes the influence of the nearest term x_r on the generated symbol x_{r+1} and the terms proportional to $(\Delta t)^2$ are converted to the integral memory contributions in the limit $\Delta t \to 0$.

The random sequence defined by Eq. (5) is stationary since the conditional probability function $P(x_{r+1} = 1|x_r; x_{r'<r})$ does not depend explicitly on the discrete coordinate r. The sequence is ergodic if the conditional probability function satisfies the following strict inequalities:

$$0 < P(x_{r+1} = 1|x_r; x_{r'<r}) < 1, \qquad (7)$$

that impose certain restrictions on the sequence parameters λ, μ and the function α_r.

Transformation to the continuous time in Eq. (5) occurs in the limit $r \to \infty$, $\Delta t \to 0$ and

$$r\Delta t \to t, \quad r'\Delta t \to \tau, \quad \Delta t \to dt, \quad \alpha_{r'} \to \alpha(\tau). \qquad (8)$$

This transformation leads to the telegraph process with memory where the conditional probabilities are given by:

$$P(x_{t+dt} = 1|x_t = 1; x_{t'<t}) = 1 - \left[\lambda - \int_0^\infty \alpha(\tau)(x_{t-\tau} - \bar{x})d\tau\right]dt, \qquad (9a)$$

$$P(x_{t+dt} = 0|x_t = 1; x_{t'<t}) = \left[\lambda - \int_0^\infty \alpha(\tau)(x_{t-\tau} - \bar{x})d\tau\right]dt, \qquad (9b)$$

$$P(x_{t+dt} = 0|x_t = 0; x_{t'<t}) = 1 - \left[\mu + \int_0^\infty \alpha(\tau)(x_{t-\tau} - \bar{x})d\tau\right]dt, \qquad (9c)$$

$$P(x_{t+dt} = 1|x_t = 0; x_{t'<t}) = \left[\mu + \int_0^\infty \alpha(\tau)(x_{t-\tau} - \bar{x})d\tau\right]dt. \qquad (9d)$$

These equations are the generalization of the basic definitions (3). In what follows, we call $\alpha(\tau)$ the memory function of the telegraph process. The integral terms in Eqs. (9) describe effects of memory on the process. Note that memory does not change the average value $\bar{x}$, since the integral terms average to zero.

The important statistical characteristics of a random process is the correlation function. In order to get the relation between the memory and correlation functions we start from obtaining similar equation for the random discrete Markov chains with memory using the well-known definition of the correlation function,

$$K_r = \overline{(x_{i+r} - \bar{x})(x_i - \bar{x})} = \overline{x_{i+r}x_i} - \bar{x}^2. \quad (10)$$

Multiplying Eq. (5) by x_0 and averaging over the ensemble of random sequences, we get the equation for the correlation function of a random sequence,

$$K_{r+1} = (1 - 2\gamma\Delta t)K_r + \sum_{r'=1}^{\infty} \alpha_{r'}(\Delta t)^2 K_{r-r'}, \quad r > 0. \quad (11)$$

This relation can be also obtained by averaging over the coordinate r along the chain (see Ref. [42]). These two methods of averaging give the same results due to ergodicity of the random sequences under study. Note that similar equation for the correlation function is valid also for the autoregressive random sequences (see Yule-Walker equations in Refs. [43–45]).

Rewriting Eq. (11) in the following form:

$$\frac{K_{r+1} - K_r}{\Delta t} = -2\gamma K_r + \sum_{r'=1}^{\infty} \alpha_{r'} \Delta t K_{r-r'}, \quad r > 0, \quad (12)$$

and taking the limit defined by (8) we obtain the final integrodifferential equation for the correlation function $K_r \to K(t)$ of random telegraph process,

$$\frac{dK(t)}{dt} + 2\gamma K(t) = \int_0^{\infty} \alpha(\tau)K(t-\tau)d\tau, \quad t > 0. \quad (13)$$

The solution of Eq. (13) is subject to the initial condition,

$$K(0) = \frac{\mu\lambda}{(\mu+\lambda)^2}, \quad (14)$$

and the requirement of parity,

$$K(-t) = K(t), \quad t > 0. \quad (15)$$

Note that because of the limits $[0, \infty]$ the integral term in Eq. (13) is neither Laplace nor Fourier convolution. It represents one of the non-Markovian kinetic equations. Recently different types of non-Markovian kinetic equations were derived and solved for stochastic processes with memory. The dichotomous non-Markovian sequences are widely used for modeling of photoluminescence in semiconductor heterostructures. The state of an exciton before and after radiative recombination can be characterized by 0 and 1. Relatively long exciton diffusion, presence of exciton traps, and dipole-dipole interactions violate the Markovianity of the relaxation process, giving rise to experimentally observable power-law tails in the decay of photoexcited states. A set of non-Markovian kinetic equations for evolution of excitons in 2D CdSe/CdS samples [46] looks similar to Eq. (13), but due to the standard limits of the integral term, $[0, t]$, it can be solved by the Laplace transformation.

The exponential solution,

$$K(t) = \frac{\mu\lambda}{(\mu+\lambda)^2} \exp[-(\mu+\lambda)t], \quad (16)$$

of Eq. (13) satisfying the conditions Eqs. (14)–(15) at $\lambda = 1.5$, $\mu = 0.5$ and $\alpha(\tau) = 0$ (Markovian limit) is presented by a solid line in the main panel in Fig. 1. The filled circles on this curve show the results obtained from the numerical simulation of the memoryless telegraph process shown in the inset. Note that, in principle, the term proportional to γ in Eq. (13) can be included into the integral term by adding the appropriate δ function to the memory function $\alpha(\tau)$.

The relation (13) can be considered as integral equation for the memory function $\alpha(t)$ provided that the correlation function $K(t)$ is known. In this case the integral equation (13) presents the inverse problem. Once $\alpha(t)$ is calculated, the telegraph sequence with the prescribed correlation function can be generated using the transition conditional probability functions Eqs. (9) as it was done in Ref. [25]. As a rule, the inverse problem is mathematically more difficult and not always solvable analytically. In Sec. IV we give the solution of the inverse problem for the limiting case of weak memory.

III. SPECIAL CASES FOR MEMORY FUNCTION

We propose two extreme forms—short and extended—for the memory function $\alpha(t)$. Both of them allow analytical solution of Eq. (13). The short memory form takes into account a single point located at a distance T from the current moment in time from the entire past process. This memory can be used to describe a fairly wide class of processes in which memory is taken into account in an area located at a distance T from the current moment in time, and the memory localization interval is significantly shorter than T. In the opposite case of extended memory the memory function is represented by a step function of finite length. It can also be used as some approximation for long-lasting nonlocal memory. Other mathematically more complicated memory functions can be considered as intermediate between these two extremes. While the explicit mathematical form of the correlator may be very different from our analytical expressions, statistical properties of the telegraph process with more complicated memory functions can be analyzed (at least qualitatively) using these two extreme cases. This is true for the memory functions, which exhibit fast decay at $t \to \infty$ when it is possible to introduce a characteristic time scale t_c defining the length of memory. We exclude memory functions with long power-law tails, which usually require application of Lévy statistics leading to nonstationary and nonergodic telegraph processes. Some practical examples of such processes are considered in Refs. [13,17–19,46]. In Sec. IV we give one more example of a process, with arbitrary, but weak memory function that allows exact analytical solution.

A. δ-delayed memory

We start from the case of memory function,

$$\alpha(\tau) = \zeta\delta(\tau - T), \quad (17)$$

which takes into account the memory of the process at only one point of the past at $t = T$. Then the conditional probabilities Eqs. (9) are rewritten as

$$P(x_{t+dt} = 1|x_t = 1; x_{t'<t}) = 1 - [\lambda - \zeta(x_{t-T} - \bar{x})]dt, \quad (18a)$$

$$P(x_{t+dt} = 0|x_t = 1; x_{t'<t}) = [\lambda - \zeta(x_{t-T} - \bar{x})]dt, \quad (18b)$$

$$P(x_{t+dt} = 0|x_t = 0; x_{t'<t}) = 1 - [\mu + \zeta(x_{t-T} - \bar{x})]dt, \quad (18c)$$

$$P(x_{t+dt} = 1|x_t = 0; x_{t'<t}) = [\mu + \zeta(x_{t-T} - \bar{x})]dt. \quad (18d)$$

It should be noted that the possible values of parameter ζ are constrained by the following conditions:

$$-\min\left(\frac{\lambda}{\mu}, \frac{\mu}{\lambda}\right)(\lambda+\mu) < \zeta < \lambda+\mu, \tag{19}$$

which guarantee the natural interval for the values of probability, $0 < P(\dots) < 1$.

Such a memory yields the following delay differential equation for the correlation function:

$$\frac{dK(t)}{dt} + 2\gamma K(t) = \zeta K(t-T), \quad t > 0. \tag{20}$$

We solve this equation in three steps:

(i) Find the solution $K(t) = K_0(t)$ for $0 < t < T$ with

$$K_0(t) = K(0)\frac{\cosh(\phi_0 - \eta t)}{\cosh\phi_0},$$
$$\phi_0 = \frac{\eta T}{2} + \operatorname{arctanh}\frac{2\gamma-\zeta}{\eta}, \quad \eta = \sqrt{4\gamma^2-\zeta^2} > 0. \tag{21}$$

(ii) Present the solutions for the time intervals, $nT < t < (n+1)T$, in the following form,

$$K_n(t) = K(0)\frac{\cosh[\phi_n - \eta(t-nT)]}{\cosh\phi_0} + P_n(t)\exp[-2\gamma(t-nT)], \tag{22}$$

where $P_n(t)$ are the $(n-1)$th degree polynomials, and

$$\phi_n = \phi_0 + n\operatorname{arctanh}\frac{\eta}{2\gamma}.$$

(iii) Obtain the recurrence relation for the polynomials $P_n(t)$,

$$\frac{dP_n(t)}{dt} = \zeta P_{n-1}(t-T), \tag{23}$$

and the continuity condition for the correlation function $K(t)$ at $t = nT$,

$$P_n(nT) = P_{n-1}(nT)\exp(-2\gamma T) + \frac{K(0)}{\cosh\phi_0}[\cosh(\eta T - \phi_{n-1}) - \cosh\phi_n] \tag{24}$$

and analyze them. The mathematical details of calculations leading to the explicit form for $K(t)$ are presented in Appendix A. The final results for the correlation functions $K_1(t)$ and $K_2(t)$ are written in the following form:

$$K_1(t) = K(0)\frac{\cosh[\phi_1 - \eta(t-T)]}{\cosh\phi_0} + A_1\exp[-2\gamma(t-T)], \quad T < t < 2T, \tag{25}$$

$$K_2(t) = K(0)\frac{\cosh[\phi_2 - \eta(t-2T)]}{\cosh\phi_0} + [\zeta A_1(t-2T) + A_1\exp(-2\gamma T) + A_2] \times \exp[-2\gamma(t-2T)], \quad 2T < t < 3T, \tag{26}$$

with

$$A_1 = \frac{K(0)}{\cosh\phi_0}[\cosh(\eta T - \phi_0) - \cosh\phi_1],$$
$$A_2 = \frac{K(0)}{\cosh\phi_0}[\cosh(\eta T - \phi_1) - \cosh\phi_2]. \tag{27}$$

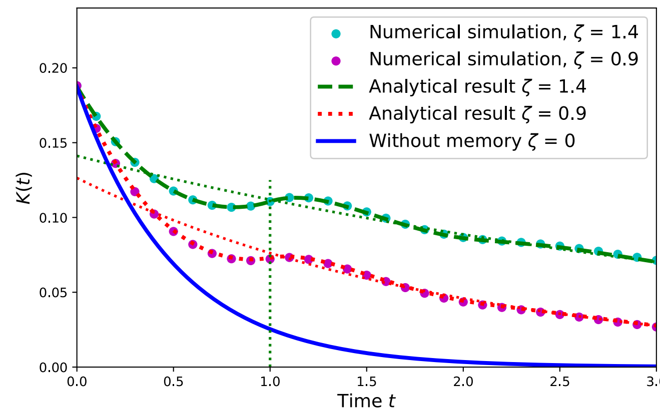

FIG. 2. The correlation functions of the telegraph processes with δ-delayed memory (dashed and dotted lines with symbols) and without memory (solid line). The parameters of the generated processes are the same as in Fig. 1. The dashed and dotted lines present the correlation functions given by Eqs. (21) and (25), the symbols are the results of the corresponding numerical simulations. Thin dotted lines represent the long-time asymptotics of the correlation function obtained in Sec. V. The values of memory parameter ζ are shown in the legend. The vertical line at $t = T = 1$ indicates the singular point position for a memory-dependent processes.

The correlation functions $K_0(t)$, $K_1(t)$, and $K_2(t)$ of the process with different values of the memory constant ζ are presented in Fig. 2. Let us pay attention to the specific property of the correlation function $K(t)$ of the process with δ-delayed memory. The function $K(t)$ being itself continuous has a discontinuity of its $(n+1)$th derivative at $t = nT$, $n = 0, 1, 2, \dots$ Indeed, one can see in Fig. 2 that $K'(t)$ is discontinuous at $t = 0$ [recall that $K(-t) = K(t)$] and $K''(t)$ is discontinuous at $t = T = 1$.

B. Stepwise memory

In this section, we study the telegraph process with a stepwise memory function,

$$\alpha(\tau) = \xi[\theta(\tau) - \theta(\tau - T)], \tag{28}$$

where $\theta(\tau)$ is the Heaviside step function. In this case, the transition conditional probability functions can be written in the form:

$$P(x_{t+dt}=1|x_t=1;x_{t'<t}) = 1 - \left[\lambda - \xi\int_0^T(x_{t-\tau}-\bar{x})d\tau\right]dt, \tag{29a}$$

$$P(x_{t+dt}=0|x_t=1;x_{t'<t}) = \left[\lambda - \xi\int_0^T(x_{t-\tau}-\bar{x})d\tau\right]dt, \tag{29b}$$

$$P(x_{t+dt}=0|x_t=0;x_{t'<t}) = 1 - \left[\mu + \xi\int_0^T(x_{t-\tau}-\bar{x})d\tau\right]dt, \tag{29c}$$

$$P(x_{t+dt}=1|x_t=0;x_{t'<t}) = \left[\mu + \xi\int_0^T(x_{t-\tau}-\bar{x})d\tau\right]dt. \tag{29d}$$

The possible values of the parameter ξ are constrained by the conditions

$$-\min\left(\frac{\lambda}{\mu}, \frac{\mu}{\lambda}\right)(\lambda+\mu) < \xi T < \lambda+\mu. \quad (30)$$

From Eq. (13), we obtain the following integrodifferential equation for the correlation function of telegraph process with the stepwise memory function:

$$\frac{dK(t)}{dt} + 2\gamma K(t) = \xi \int_0^T K(t-\tau)d\tau, \quad t > 0. \quad (31)$$

The detailed solution of this equation is given in Appendix B. Here we present the result for $K(t)$ in two first time intervals:

$$K(t) = K(0)\frac{2\xi \cosh\phi_0 + \eta(2\gamma - \xi T)\sinh(\phi_0 - \eta t)}{2\xi \cosh\phi_0 + \eta(2\gamma - \xi T)\sinh\phi_0} \quad (32)$$

for $0 < t < T$ and

$$K(t) = K(0)\frac{2\xi \cosh\phi_0 - \eta(2\gamma - \xi T)\sinh[\phi_1 - \eta(t-T)]}{2\xi \cosh\phi_0 + \eta(2\gamma - \xi T)\sinh\phi_0} + 4K(0)\frac{\cosh\phi_0 \exp[-\gamma(t-T)](2\gamma - \xi T)(2\gamma^2+\xi)}{\xi[2\xi\cosh\phi_0 + \eta(2\gamma - \xi T)\sinh\phi_0]} \times \left\{2\gamma\cosh[\kappa(t-T)] - \frac{2\gamma^2+\xi}{\kappa}\sinh[\kappa(t-T)]\right\} \quad (33)$$

for $T < t < 2T$. Here

$$\eta = \sqrt{4\gamma^2 + 2\xi} > 0, \quad \kappa = \sqrt{\gamma^2+\xi},$$
$$\phi_0 = \frac{\eta T}{2} + \operatorname{arctanh}\frac{2\gamma}{\eta}, \quad \phi_1 = \phi_0 + \operatorname{arcsinh}\frac{2\gamma\eta}{\xi}. \quad (34)$$

Correlation function $K(t)$ of the process with different values of memory constant ξ is shown in Fig. 3. It is seen from Figs. 2 and 3 that the correlation functions obtained from the numerically generated telegraph processes with different memory functions $\alpha(t)$ are in excellent agreement with the corresponding results for $K(t)$ calculated analytically. However, we note that as the memory function increases, fluctuations increase both during the generation of the process itself and during the calculation of the correlation function. Qualitatively this can be explained by the fact that in the absence of correlations, the sum of N random independent variables follows the central limit theorem with the relative amplitude of fluctuations being proportional to $1/\sqrt{N}$. In a correlated sequence, random variables belonging to different subsequences of lengths of the order of the correlation radius r_c can be considered as statistically independent. In this case, the effective sequence length is reduced to $\sim N/r_c$. For fixed total length N, the amplitude of relative fluctuations of the sum of random variables grows with increasing r_c. Despite this, it is always possible to choose the length of the random sequence so that the amplitude of relative fluctuations becomes as small as desired. However, analytical estimation of the magnitude of these fluctuations is a rather complicated problem, since it depends on the nature of the interactions of random variables within the sequence.

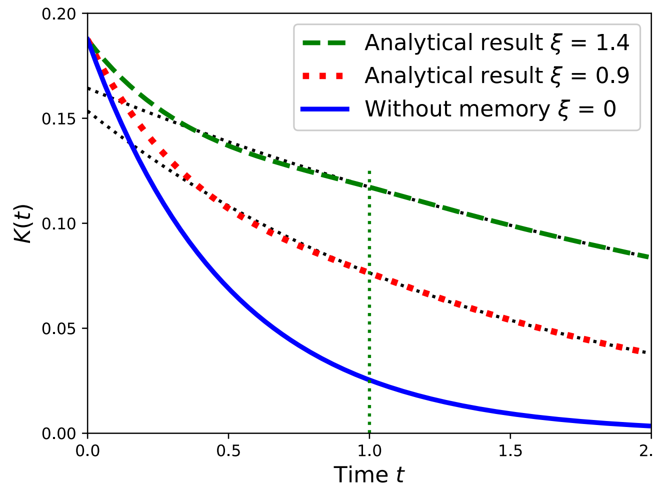

FIG. 3. The correlation functions of the telegraph processes with stepwise memory (dashed and dotted lines with symbols) and without memory (solid curve). The parameters of generated processes are the same as in Fig. 1. The dashed and dotted lines present the correlation functions given by Eqs. (32) and (33), the symbols are the results of numerical simulations. Thin dotted lines represent the long-time asymptotics of the correlation function obtained in Sec. V. The values of memory parameter ξ are shown in the legend. The vertical line indicates the singular point position, $t = T = 1$, for memory-dependent processes.

IV. SOLUTION OF THE INVERSE PROBLEM FOR WEAK MEMORY FUNCTION

The inverse problem for Eq. (13), i.e., calculation of the memory function $\alpha(t)$ for a prescribed correlator $K(t)$, is reduced to a Fredholm integral equation of the first kind. This equation belongs to so-called ill-posed problems, which may have no solution in general case. Below we solve it analytically in the case of arbitrary, but weak memory.

Let us assume that the correlator in Eq. (13) is only slightly different from the correlator of a memoryless process [see Eq. (16)] and can be written in the following form:

$$K(t) = \exp(-2\gamma t)[1 + k(t)], \quad |k(t)| \ll 1. \quad (35)$$

Then it is naturally to suppose that the memory function $\alpha(t)$ is also small, $\alpha(t) \sim |k(t)| \ll 1$. Substituting Eq. (35) into Eq. (13) and keeping the linear over $\alpha(t)$ and $k(t)$ terms, we get the following integral equation:

$$\frac{dk(t)}{dt} = \int_0^\infty \alpha(\tau)\exp[-2\gamma(|t-\tau| - t)]d\tau, \ t > 0. \quad (36)$$

To avoid the sign of modulus in the integrand, the integral from 0 to ∞ is split into two parts: from 0 to t and from t to ∞. Each of these integrals with t-dependent limit becomes an algebraic function after two sequential differentiations. After this the solution of the inverse problem is easily obtained,

$$\alpha(t) = \exp(-2\gamma t)\left[k''(t) - \frac{k'''(t)}{4\gamma} + O\left(\frac{k^{IV}(t)}{\gamma^2}\right)\right]. \quad (37)$$

The omitted terms in this solutions are negligibly small if

$$t_c \gg 1/\gamma, \quad (38)$$

where t_c is the characteristic scale of the disturbance $k(t)$ and $1/\gamma$ is the correlation length of the telegraph process in the absence of memory [see Eq. (35)].

Note that the direct problem for Eq. (36) is also easily solved. Indeed, integrating Eq. (36) from 0 to t and substituting the result for $k(t)$ to Eq. (35), the correlator $K(t)$ can be expressed through arbitrary (but weak) memory function,

$$K(t) = \exp(-2\gamma t) + \int_0^\infty Q(t,\tau)\alpha(\tau)d\tau, \quad t > 0. \tag{39}$$

Here the kernel of the integral term is

$$Q(t,\tau) = (t-\tau)\exp[-2\gamma(t-\tau)]\theta(t-\tau) + \frac{1}{2\gamma}\{\exp(-2\gamma|t-\tau|) - \exp[-2\gamma(t+\tau)]\}. \tag{40}$$

V. LONG-TIME ASYMPTOTICS FOR THE CORRELATION FUNCTION

In the long-time limit the memory effects disappear and the correlator of any process with nonlocal memory approaches the exponential form. Then, at $t\to\infty$ the solution of Eq. (13) can be sought in the form of an exponential function with unknown parameters A and κ,

$$K(t)|_{t\to\infty} \to A\exp(-\kappa t). \tag{41}$$

Substitution of this anzatz into Eq. (13) gives

$$-\kappa + 2\gamma = \int_0^t \alpha(\tau)e^{\kappa\tau}d\tau + \int_t^\infty \alpha(\tau)e^{\kappa(2t-\tau)}d\tau. \tag{42}$$

In the limit $t\to\infty$ the last integral approaches zero due to equal limits, $[\infty,\infty]$. The memory function in this limit can be approximated by Eq. (37) and the exponentially growing factor $e^{2\kappa t}$ is compensated by the corresponding decaying exponent $e^{-2\gamma\tau}$ at the weak memory function ($\gamma > \kappa$, $\tau > t$). Due to such compensation the integrand in the first integral decays exponentially at $\tau\to\infty$ and the upper limit can be extended to infinity. Finally, after neglecting the second term, the following equation for the parameter κ is obtained:

$$\kappa + \int_0^\infty \alpha(\tau)\exp(\kappa\tau)d\tau = 2\gamma. \tag{43}$$

This relation means that the larger the positive memory function is, the smaller the parameter κ is. Consequently, the longer the correlation time $t_c \sim 1/\kappa$ is.

Let us evaluate the parameter κ for the considered above processes with δ and stepwise memories. For the δ-delayed memory function, Eq. (17), we get

$$\kappa + \zeta\exp(\kappa T) = 2\gamma. \tag{44}$$

The numerical evaluations at $T = 1$ give $\kappa \approx 0.506$ for $\zeta = 0.9$ and $\kappa \approx 0.233$ for $\zeta = 1.4$. The thin dotted lines in Fig. 2 represent the asymptotic behavior of the correlation function. It is clear that the exact result approaches the asymptotic immediately at $t > T = 1$.

For the memory function, Eq. (28), we have

$$\kappa + \xi\frac{\exp(\kappa T) - 1}{\kappa} = 2\gamma. \tag{45}$$

The numerical evaluations at $T = 1$ give $\kappa \approx 0.698$ for $\xi = 0.9$ and $\kappa \approx 0.336$ for $\xi = 1.4$. We emphasize that each of the equations (44) and (45) has a single root.

Note that the estimate for the correlation length $t_c \sim 1/\kappa$ is valid if $1/\kappa \gg 1/2\gamma$, where $1/2\gamma$ is the correlation length of the telegraph process without memory [see the exponent in Eq. (16)]. In the intermediate case, it is better to use the interpolation formula

$$t_c \sim \frac{1}{2\gamma} + \frac{1}{\kappa_m}, \tag{46}$$

which is valid for both cases of small and large κ_m as compared to 2γ. In Eq. (46) κ_m is the smallest root of Eq. (43).

VI. CONCLUSION

In conclusion, we propose a mathematical approach based on additive Markov chain, Refs. [24,25,47], to study telegraph random ergodic processes with dynamics depending on the past. We took into account the history of the process in the explicit form introducing an integral nonlocal memory term into conditional probability function. We showed that the proposed processes can be considered as continuous-time interpolations of discrete-time higher-order random sequences. An equation connecting the memory function and the two-point correlation function of the telegraph process is obtained. This equation can be considered as a direct problem, if solved for the correlation function provided that the memory function is given. At the same time, it is an inverse problem for the unknown memory function. Solution of the inverse problem is of great practical interest since it gives the algorithm of generation of the telegraph process with a prescribed pair correlation function. We found analytically solutions of integral equations for the correlation functions of telegraph processes with δ-delayed and stepwise memory functions. As an illustration, some examples of numerical simulation of the processes with nonlocal memory are presented.

Natural continuation of this study is expansion of the proposed method to the processes with time-dependent quantities λ and μ. This will allow, in particular, consideration of the effects of memory on telegraph processes with Lévy distributions of system lifetimes in states a and b (see, e.g., Refs. [48,49]). An interesting and separate problem is finding applications of the telegraph process with memory to specific random processes. In particular, the telegraph process can describe the information transcription in DNA molecules [50] where the memory effects play extremely important role.

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APPENDIX A: SOLUTION OF EQ. (20) FOR THE PROCESSES WITH δ-CORRELATED MEMORY

Solution for $K_0(t)$. Recalling parity condition (15) we replace $K(t-T)$ by $K(T-t)$ for the interval $0<t<T$ and rewrite Eq. (20) as

$$K_0'(t)+2\gamma K_0(t)=\zeta K_0(T-t). \tag{A1}$$

Applying operator $d/dt-2\gamma$ to both sides of this equation we get

$$K_0''(t)-4\gamma^2K_0(t)=\zeta^2K_0(t). \tag{A2a}$$

Its solution is

$$\begin{aligned}K_0(t)&=C_+\exp(\eta t)+C_-\exp(-\eta t),\\ \eta&=\sqrt{4\gamma^2-\zeta^2}>0.\end{aligned} \tag{A2b}$$

Substituting this solution into Eq. (A1), equating the coefficients at the exponents, after some algebra the Eq. (21) is obtained.

General solution by iterating procedure. Let us denote $K(t)=K_n(t)$ for the interval $nT<t<(n+1)T$. Then Eq. (20) naturally transforms in a sort of recurrence relation for the functions $K_n(t)$,

$$\begin{aligned}&K_n'(t)+2\gamma K_n(t)=\zeta K_{n-1}(t-T),\\ &nT<t<(n+1)T,\end{aligned} \tag{A3}$$

with boundary conditions $K_n(nT)=K_{n-1}(nT)$.

Then we solve the problem iteratively:

(i) $K_0(t)$ is defined in Eq. (21).

(ii) Find $K_1(t)$ from Eq. (A3) in the form of superposition of exponential functions $\exp(\pm\eta t)$ [the particular solution originated from the hyperbolic cosine in $K_0(t)$ in the right-hand side] and $\exp(-2\gamma t)$ (general solution of the homogeneous equation).

(iii) Find $K_2(t)$ from Eq. (A3) in the form of superposition of exponential functions $\exp(\pm\eta t)$, $\exp(-2\gamma t)$, and $t\exp(-2\gamma t)$ [the particular solution originated from $\exp(-2\gamma t)$ in $K_1(t)$ in the right-hand side].

Continuing this procedure it is easy to see that the function $K_n(t)$ can be written in the following form:

$$\begin{aligned}K_n(t)&=C_n\cosh[\eta(t-nT)-\phi_n]\\ &\quad+P_n(t)\exp[-2\gamma(t-nT)],\end{aligned} \tag{A4}$$

where C_n and ϕ_n are constants and $P_n(t)$ is the polynomial function.

Substituting the last expression into Eq. (A3) and equating prefactors for $\exp(\pm\eta t)$ and $\exp(-2\gamma t)$, we obtain the coefficients C_n and the recurrence relation for ϕ_n,

$$C_n=\frac{K(0)}{\cosh\phi_0},\qquad \phi_n=\phi_{n-1}+\operatorname{arctanh}\frac{\eta}{2\gamma}, \tag{A5}$$

as well as the recurrence relation (23) with the continuity condition (24) for the polynomials $P_n(t)$.

Iterative scheme for the polynomials $P_n(t)$. Proceeding iteratively, one can see that $P_n(t)$ is the polynomial of the $(n-1)$th degree. Then we can look for its explicit form as

$$P_n(t)=\sum_{m=0}^{n-1}c_{nm}(t-nT)^m. \tag{A6}$$

According to the recurrence relation (23), we have

$$\sum_{m=0}^{n-1}c_{nm}m(t-nT)^{m-1}=\zeta\sum_{m=0}^{n-2}c_{(n-1)m}[t-T-(n-1)T]^m. \tag{A7}$$

Then coefficients c_{nm} with $m>0$ can be expressed via coefficients $c_{(n-1)(m-1)}$, and, iteratively, via $c_{(n-m)0}$:

$$\begin{aligned}c_{nm}&=\frac{\zeta}{m}c_{(n-1)(m-1)}=\frac{\zeta^2}{m(m-1)}c_{(n-2)(m-2)}\\ &=\cdots=\frac{\zeta^m}{m!}c_{(n-m)0}.\end{aligned} \tag{A8}$$

Using the last relation and condition (24), we obtain the recurrence relation for c_{n0},

$$c_{n0}=\sum_{m=0}^{n-2}\frac{(\zeta T)^m}{m!}c_{(n-m-1)0}\exp(-2\gamma T)+A_n,$$

where

$$A_n=\frac{K(0)}{\cosh\phi_0}[\cosh(\eta T-\phi_{n-1})-\cosh\phi_n].$$

The scheme to calculate the coefficients c_{nm} is presented in the following diagram:

$$\begin{tikzcd}
 & & & c_{10} \arrow[dl, "\zeta"'] \arrow[d] & A_1 \arrow[l] \\
 & & c_{21} \arrow[dl, "\zeta/2"'] & c_{20} \arrow[dl, "\zeta"'] \arrow[d] & A_2 \arrow[l] \\
 & c_{32} \arrow[dl, "\zeta/3"'] & c_{31} \arrow[dl, "\zeta/2"'] & c_{30} \arrow[dl, "\zeta"'] \arrow[d] & A_3 \arrow[l] \\
\iddots & \iddots & \iddots & \vdots &
\end{tikzcd}$$

For example, we calculate several first polynomials,

$$\begin{aligned}
P_0(t)&\equiv 0,\quad P_1(t)=\underbrace{A_1}_{c_{10}},\\
P_2(t)&=\underbrace{\zeta A_1}_{c_{21}}(t-2T)+\underbrace{A_1\exp(-2\gamma T)+A_2}_{c_{20}},\\
P_3(t)&=\underbrace{\frac{\zeta^2}{2}A_1}_{c_{32}}(t-3T)^2+\underbrace{\zeta[A_1\exp(-2\gamma T)+A_2]}_{c_{31}}(t-3T)+\underbrace{\{[A_1\exp(-2\gamma T)+A_2]+\zeta TA_1\}\exp(-2\gamma T)+A_3}_{c_{30}}.
\end{aligned} \tag{A9}$$

APPENDIX B: SOLUTION OF EQ. (31) FOR THE PROCESSES WITH STEP-WISE MEMORY

Solution for $K_0(t)$. Using the parity condition (15), $K(t-\tau) = K(\tau - t)$, we split the region of integration in Eq. (31) into two parts, $0 < \tau < t$ and $t < \tau < T$, and change the variables,

$$\int_0^T K(t-\tau)d\tau = \int_0^t K(t-\tau)d\tau + \int_t^T K(\tau - t)d\tau = \int_0^t K(\tau)d\tau + \int_0^{T-t} K(\tau)d\tau. \tag{B1}$$

Then differentiating Eq. (31) over t we arrive at the delay differential equation,

$$K''(t) + 2\gamma K'(t) - \xi K(t) = \xi K(T-t), \quad t > 0. \tag{B2}$$

Applying operator $d^2/dt^2 - 2\gamma d/dt - \xi$ to Eq. (B2), we get the differential equation with constant coefficients,

$$K_0''''(t) - (4\gamma^2 + 2\xi)K_0''(t) = 0. \tag{B3}$$

Its solution is

$$K_0(t) = C_+ \exp(\eta t) + C_- \exp(-\eta t) + C_0 + C_1 t, \qquad \eta = \sqrt{4\gamma^2 + 2\xi} > 0. \tag{B4}$$

Substituting this solution into Eq. (31), equating the coefficients at t^0, t^1 and $\exp(\pm\eta t)$, Eq. (32) is obtained after some simplification.

General solution by iterating procedure. Let us denote $K(t) = K_n(t)$ for the interval $nT < t < (n+1)T$. Then Eq. (31) transforms in a sort of recurrence relation for the functions $K_n(t)$,

$$K_n'(t) + 2\gamma K_n(t) = \xi\left[\int_0^{t-nT} K_n(t-\tau)d\tau + \int_{t-nT}^T K_{n-1}(t-\tau)d\tau\right], \quad 0 < n. \tag{B5}$$

Changing variables of integrations and differentiating over t we obtain the differential equation with a time shift [compare to Eq. (B2)],

$$K_n''(t) + 2\gamma K_n'(t) - \xi K_n(t) = -\xi K_{n-1}(t-T). \tag{B6}$$

This differential equation is subject to two boundary conditions,

$$K_n(nT) = K_{n-1}(nT), \quad K_n'(nT) = -2\gamma K_{n-1}(nT) + \xi \int_0^T K_{n-1}(nT - \tau)d\tau. \tag{B7}$$

Then the problem can be solved iteratively:

(i) $K_0(t)$ is defined by Eq. (32).

(ii) Find $K_1(t)$ from Eq. (B6) in the form of superposition of constant and exponential functions $\exp(\pm\eta t)$ [the particular solution originates from the constant term and the hyperbolic sine in $K_0(t)$ in the right-hand side], and $\exp[-(\gamma \pm \kappa)t]$ (the general solution of the homogeneous equation), with

$$\kappa = \sqrt{\gamma^2 + \xi}. \tag{B8}$$

(iii) Find $K_2(t)$ from Eq. (B6) in the form of superposition of a constant term and the exponential functions $\exp(\pm\eta t)$, $\exp[-(\gamma \pm \kappa)t]$, and $t\exp[-(\gamma \pm \kappa)t]$ [the particular solution originates from $\exp[-(\gamma \pm \kappa)t]$ in $K_1(t)$ in the right-hand side].

It can be seen from the procedure that the function $K_n(t)$ can be presented in the following form,

$$K_n(t) = B_n + C_n \sinh[\phi_n - \eta(t - nT)] + P_n^+(t)\exp[-(\gamma+\kappa)(t-nT)] + P_n^-(t)\exp[-(\gamma-\kappa)(t-nT)], \tag{B9}$$

where B_n, C_n, and ϕ_n are some indefinite constants, while $P_n^+(t)$ and $P_n^-(t)$ are some polynomial functions of the $(n-1)$th order.

Substituting the last expression into Eq. (B6), equating prefactors at $\exp(\pm\eta t)$, $\exp[-(\gamma \pm \kappa)t]$, and at the constant terms, we obtain the coefficients B_n, C_n, as well as the recurrence relations for ϕ_n. Then the function $K_n(t)$ can be expressed in the following form:

$$K_n(t) = \bar{K}_n(t) + P_n^+(t)\exp[-(\gamma+\kappa)(t-nT)] + P_n^-(t)\exp[-(\gamma-\kappa)(t-nT)], \tag{B10}$$

$$\bar{K}_n(t) = K(0)\frac{2\xi\cosh\phi_0 + \eta(2\gamma - \xi T)(-1)^n \sinh[\phi_n - \eta(t-nT)]}{2\xi\cosh\phi_0 + \eta(2\gamma - \xi T)\sinh\phi_0}, \tag{B11}$$

$$\phi_n = \phi_0 + n\,\mathrm{arctanh}\frac{2\gamma\eta}{\eta^2 - \xi}. \tag{B12}$$

Iterative scheme for the polynomials $P_n^{\pm}(t)$. Substituting Eq. (B10) for $K_n(t)$ into Eq. (B5), we get the recurrence relations for $P_n^+(t)$ and $P_n^-(t)$,

$$P_n^{\pm\prime\prime}(t) \mp 2\kappa P_n^{\pm\prime}(t) = -\xi P_{n-1}^{\pm}(t-T). \tag{B13}$$

It should be emphasized that the general solutions of Eq. (B13), except polynomial summands, contain exponential terms with $\exp(\pm 2\kappa t)$. Such terms should be omitted. Therefore, two differential recurrence relations (B13) (with superscripts $\pm$) should be supplied with only two rather cumbersome boundary conditions that follow from Eqs. (B7),

$$\begin{aligned}
&P_n^+(nT)+P_n^-(nT)\\
&\quad = P_{n-1}^+(nT)\mathrm{e}^{-(\gamma+\kappa)T}+P_{n-1}^-(nT)\mathrm{e}^{-(\gamma-\kappa)T}+A_n,\\
&(\gamma-\kappa)P_n^+(nT)+(\gamma+\kappa)P_n^-(nT)\\
&\quad = A_n' - P_n^{+\prime}(nT) - P_n^{-\prime}(nT)\\
&\qquad +\xi\int_0^T \{P_{n-1}^+(nT-\tau)\mathrm{e}^{-(\gamma+\kappa)(T-\tau)}\\
&\qquad + P_{n-1}^-(nT-\tau)\mathrm{e}^{-(\gamma-\kappa)(T-\tau)}\}d\tau, \qquad \text{(B14)}
\end{aligned}$$

$$\begin{aligned}
A_n &= [\bar{K}_{n-1}(nT)-\bar{K}_n(nT)],\\
A_n' &= -\bar{K}_n(nT) - 2\gamma \bar{K}_n'(nT) + \xi\int_0^T \bar{K}_n(nT-\tau)d\tau. \qquad \text{(B15)}
\end{aligned}$$

We look for its explicit polynomial form as

$$P_n^{\pm}(t) = \sum_{m=0}^{n-1} c_{nm}^{\pm}(t-nT)^m.$$

According to Eq. (B13),

$$\begin{aligned}
&c_{n(m+2)}^{\pm}(m+2)(m+1) \mp 2\kappa c_{n(m+1)}^{\pm}(m+1) = -\xi c_{(n-1)m}^{\pm},\\
&\quad 0\leqslant m\leqslant n-3, \quad \mp 2\kappa c_{n(n-1)}^{\pm}(n-1) = -\xi c_{(n-1)(n-2)}^{\pm}.
\end{aligned}$$

The second relation here can be reduced to

$$c_{n(n-1)}^{\pm} = \left(\pm\frac{\xi}{2\kappa}\right)^{n-1}\frac{c_{10}^{\pm}}{(n-1)!}.$$

Therefore, each coefficient $c_{n(m>0)}^{\pm}$ can be expressed via $c_{(n'<n)0}^{\pm}$. Here the coefficients $c_{n'0}^{\pm}$ can be found iteratively from the boundary condition,

$$\begin{aligned}
&c_{n0}^+ + c_{n0}^-\\
&\quad = A_n + \sum_{m=0}^{n-2} T^m e^{-\gamma T}\big[c_{(n-1)m}^+\mathrm{e}^{-\kappa T} + c_{(n-1)m}^-\mathrm{e}^{\kappa T}\big],\\
&(\gamma-\kappa)c_{n0}^+ + (\gamma+\kappa)c_{n0}^-\\
&\quad = A_n' - (c_{n1}^+ + c_{n1}^-) + \xi\sum_{m=0}^{n-2}\int_0^T \tau^m\big[c_{(n-1)m}^+\mathrm{e}^{-(\gamma+\kappa)\tau}\\
&\qquad + c_{(n-1)m}^-\mathrm{e}^{-(\gamma-\kappa)\tau}\big]d\tau.
\end{aligned}$$

The diagram below illustrates the scheme of the described calculations.

$$\begin{tikzcd}
 & & c_{10}^+ \arrow[r, bend left] \arrow[dl] \arrow[d] & c_{10}^- \arrow[l, bend left] \arrow[d] \arrow[dr] & & \\
 & c_{21}^+ \arrow[r] \arrow[dl] & c_{20}^+ \arrow[r, bend left] \arrow[dl] \arrow[d] & c_{20}^- \arrow[l, bend left] \arrow[d] \arrow[dr] & c_{21}^- \arrow[l] \arrow[dr] & \\
c_{32}^+ \arrow[r] & c_{31}^+ \arrow[r] \arrow[dl] & c_{30}^+ \arrow[r, bend left] \arrow[dl] \arrow[d] & c_{30}^- \arrow[l, bend left] \arrow[d] \arrow[dr] & c_{31}^- \arrow[l] \arrow[dr] & c_{32}^- \arrow[l] \\
\iddots & \iddots & \vdots & \vdots & \ddots & \ddots
\end{tikzcd}$$

Here we present the first instances of the recurrence relations that allow to calculate successively the coefficients $c_{10}^{\pm}$, $c_{21}^{\pm}$, $c_{20}^{\pm}$, etc.,

$$A_1 = c_{10}^+ + c_{10}^-, \quad A_1' = (\gamma-\kappa)c_{10}^+ + (\gamma+\kappa)c_{10}^-,$$

$$c_{21}^{\pm} = \pm\frac{\xi}{2\kappa}c_{10}^{\pm}, \quad c_{20}^+ + c_{20}^- = A_2 + \big[c_{10}^+\mathrm{e}^{-(\gamma+\kappa)T} + c_{10}^-\mathrm{e}^{-(\gamma-\kappa)T}\big],$$

$$(\gamma-\kappa)c_{20}^+ + (\gamma+\kappa)c_{20}^- = A_2' - (c_{21}^+ + c_{21}^-) + \xi\left[c_{10}^+\frac{1-\mathrm{e}^{-(\gamma+\kappa)T}}{\gamma+\kappa} + c_{10}^-\frac{1-\mathrm{e}^{-(\gamma-\kappa)T}}{\gamma-\kappa}\right],$$

$$c_{32}^{\pm} = \pm\frac{\xi}{2\kappa}c_{21}^{\pm} = \left(\frac{\xi}{2\kappa}\right)^2 c_{10}^{\pm}, \quad c_{31}^{\pm} = \frac{\xi c_{20}^{\pm} + 2c_{32}^{\pm}}{\pm 2\kappa} = \frac{(2\kappa)^2\xi c_{20}^{\pm} + 2\xi^2 c_{10}^{\pm}}{\pm(2\kappa)^3},$$

$$c_{30}^+ + c_{30}^- = A_3 + \big[(c_{20}^+ + c_{21}^+T)\mathrm{e}^{-(\gamma+\kappa)T} + (c_{20}^- + c_{21}^-T)\mathrm{e}^{-(\gamma-\kappa)T}\big],$$

$$\begin{aligned}
(\gamma-\kappa)c_{30}^+ + (\gamma+\kappa)c_{30}^- &= A_3' - (c_{31}^+ + c_{31}^-) + \xi\Bigg\{c_{20}^+\frac{1-\mathrm{e}^{-(\gamma+\kappa)T}}{\gamma+\kappa} + c_{20}^-\frac{1-\mathrm{e}^{[-(\gamma-\kappa)T]}}{\gamma-\kappa}\\
&\quad + c_{21}^+\frac{1-\mathrm{e}^{-(\gamma+\kappa)T}[1+T(\gamma+\kappa)]}{(\gamma+\kappa)^2} + c_{21}^-\frac{1-\mathrm{e}^{-(\gamma-\kappa)T}[1+T(\gamma-\kappa)]}{(\gamma-\kappa)^2}\Bigg\}.
\end{aligned}$$

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