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A bound for the exterior product of S-units

Shabnam Akhtari and Jeffrey D. Vaaler

We generalize an inequality for the determinant of a real matrix proved by A. Schinzel, to more general exterior products of vectors in Euclidean space. We apply this inequality to the logarithmic embedding of S-units contained in a number field k. This leads to a bound for the exterior product of S-units expressed as a product of heights. Using a volume formula of P. McMullen we show that our inequality is sharp up to a constant that depends only on the rank of the S-unit group but not on the field k. Our inequality is related to a conjecture of F. Rodriguez Villegas.

1. Introduction

Let k be an algebraic number field, $k^\times$ its multiplicative group of nonzero elements, and $h : k^\times \to [0, \infty)$ the absolute, logarithmic, Weil height (or simply the *height*). In [Akhtari and Vaaler 2016] we proved inequalities that compare the size of an S-regulator with the product of heights of a maximal collection of independent S-units. If $k \subseteq l$ are both number fields the results in [Akhtari and Vaaler 2022] extend inequalities of this sort to the multiplicative group of relative units. Here we prove analogous inequalities for the exterior product of a collection of independent S-units that is not a maximal collection.

At each place v of k we write k_v for the completion of k at v. We use two absolute values $\|\cdot\|_v$ and $|\cdot|_v$ from the place v. The absolute value $\|\cdot\|_v$ extends the usual archimedean or nonarchimedean absolute value on the subfield $\mathbb{Q}$. Then $|\cdot|_v$ must be a power of $\|\cdot\|_v$, and we set

$$|\cdot|_v = \|\cdot\|_v^{d_v/d}, \tag{1-1}$$

where $d_v = [k_v : \mathbb{Q}_v]$ is the local degree of the extension and $d = [k : \mathbb{Q}]$ is the global degree. With these normalizations the height of an algebraic number $\alpha \neq 0$ that belongs to k is given by

$$h(\alpha) = \sum_v \log^+ |\alpha|_v = \tfrac{1}{2} \sum_v \bigl|\log |\alpha|_v\bigr|. \tag{1-2}$$

Each sum in (1-2) is over the set of all places v of k, and the equality between the two sums follows from the product formula.

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Let S be a finite set of places of k such that S contains all the archimedean places. Then

$$O_S = \{\gamma \in k : \|\gamma\|_v \le 1 \text{ for all places } v \notin S\}$$

is the ring of S-integers in k, and

$$O_S^\times = \{\gamma \in k^\times : \|\gamma\|_v = 1 \text{ for all places } v \notin S\}$$

is the multiplicative group of S-units in O_S. The abelian group $O_S^\times$ has rank r, where $|S| = r+1$, and we assume that r is positive. We write $\boldsymbol{x} = (x_v)$ for a (column) vector in $\mathbb{R}^{r+1}$ where the coordinates of $\boldsymbol{x}$ are indexed by places v in S. We write

$$\|\boldsymbol{x}\|_1 = \sum_{v\in S} |x_v|$$

for the l^1-norm of $\boldsymbol{x}$. The *logarithmic embedding* of $O_S^\times$ into $\mathbb{R}^{r+1}$ is the homomorphism defined at each point α in $O_S^\times$ by

$$\alpha \mapsto \boldsymbol{\alpha} = (d_v \log \|\alpha\|_v), \tag{1-3}$$

where the rows of the vector $\boldsymbol{\alpha}$ on the right of (1-3) are indexed by places v in S. It follows from (1-1) and (1-2) that if α is a point in $O_S^\times$ and $\boldsymbol{\alpha}$ is the image of α in $\mathbb{R}^{r+1}$ using the logarithmic embedding (1-3), then

$$2[k:\mathbb{Q}]h(\alpha) = \sum_{v\in S} \bigl|d_v \log \|\alpha\|_v\bigr| = \|\boldsymbol{\alpha}\|_1. \tag{1-4}$$

The kernel of the logarithmic embedding (1-3) is the torsion subgroup

$$\bigl\{\alpha \in O_S^\times : (d_v \log \|\alpha\|_v) = \boldsymbol{0}\bigr\} = \mathrm{Tor}(O_S^\times) \tag{1-5}$$

of all roots of unity in $k^\times$. It is known that (1-5) is a finite, cyclic group, and from the S-unit theorem of Dirichlet, Chevalley, and Hasse (see [Narkiewicz 2004, Theorem 3.12]) we learn that the quotient

$$\mathfrak{U}_S(k) = O_S^\times / \mathrm{Tor}(O_S^\times)$$

is a free abelian group of rank r. Therefore the logarithmic embedding (1-3) induces an isomorphism from $\mathfrak{U}_S(k)$ onto the discrete subgroup

$$\Gamma_S(k) = \bigl\{(d_v \log \|\alpha\|_v) : \alpha \in O_S^\times\bigr\} \subseteq \mathbb{R}^{r+1},$$

which is a free group of rank r. It follows from the product formula

$$\sum_{v\in S} d_v \log \|\alpha\|_v = 0$$

that $\Gamma_S(k)$ is contained in the r-dimensional diagonal subspace

$$\mathcal{D}_r = \Bigl\{\boldsymbol{x} = (x_v) : \sum_{v\in S} x_v = 0\Bigr\} \subseteq \mathbb{R}^{r+1}.$$

The height h is constant on cosets of the quotient group $\mathfrak{U}_S(k)$ and therefore h is well defined as a map

$$h : \mathfrak{U}_S(k) \to [0, \infty).$$

Let $\eta_1, \eta_2, \dots, \eta_r$ be multiplicatively independent elements in $\mathfrak{U}_S(k)$ that form a basis for the free group $\mathfrak{U}_S(k)$. Let

$$\boldsymbol{\eta}_j = (d_v \log \|\eta_j\|_v) \quad \text{for } j = 1, 2, \dots, r$$

be the logarithmic embedding of these points in $\Gamma_S(k) \subseteq \mathcal{D}_r$. Working with the induced l^1-norm in the exterior algebra $\mathrm{Ext}(\mathbb{R}^{r+1})$ we find that

$$(r+1)\,\mathrm{Reg}_S(k) = \|\boldsymbol{\eta}_1 \wedge \boldsymbol{\eta}_2 \wedge \cdots \wedge \boldsymbol{\eta}_r\|_1, \tag{1-6}$$

where $\mathrm{Reg}_S(k)$ is the S-regulator. More generally, let $\alpha_1, \alpha_2, \dots, \alpha_r$ be multiplicatively independent elements in $\mathfrak{U}_S(k)$, and let $\mathfrak{A} \subseteq \mathfrak{U}_S(k)$ be the multiplicative subgroup of rank r which they generate. Let

$$\boldsymbol{\alpha}_j = (d_v \log \|\alpha_j\|_v) \quad \text{for } j = 1, 2, \dots, r$$

be the image of $\alpha_1, \alpha_2, \dots, \alpha_r$ in $\Gamma_S(k)$. It follows that there exists a unique $r \times r$ nonsingular matrix $B = (b_{ij})$ with entries in $\mathbb{Z}$ such that

$$\boldsymbol{\alpha}_j = \sum_{i=1}^{r} \boldsymbol{\eta}_i b_{ij} \quad \text{for } j = 1, 2, \dots, r. \tag{1-7}$$

Then the index of the subgroup $\mathfrak{A}$ in $\mathfrak{U}_S(k)$ is

$$[\mathfrak{U}_S(k) : \mathfrak{A}] = |\det B|. \tag{1-8}$$

Combining (1-6), (1-7), and (1-8), we find that

$$(r+1)\,\mathrm{Reg}_S(k)\,[\mathfrak{U}_S(k) : \mathfrak{A}] = \|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_r\|_1. \tag{1-9}$$

In [Akhtari and Vaaler 2016, Theorem 1.1] we proved an upper bound for the S-regulator that is equivalent to the identity (1-9) and the inequality

$$\|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_r\|_1 \le 2^{-r}(r+1) \prod_{j=1}^{r} \|\boldsymbol{\alpha}_j\|_1. \tag{1-10}$$

The following result provides a generalization of (1-10) to an exterior product of q independent vectors in the free group $\Gamma_S(k)$, where $1 \le q \le r$.

Theorem 1.1. *Let $\alpha_1, \alpha_2, \dots, \alpha_q$ be multiplicatively independent points in $\mathfrak{U}_S(k)$, and let*

$$\boldsymbol{\alpha}_j = (d_v \log \|\alpha_j\|_v) \quad \textit{for } j = 1, 2, \dots, q$$

be the logarithmic embedding of $\alpha_1, \alpha_2, \ldots, \alpha_q$ *in* $\Gamma_S(k)$. *Then we have*

$$\|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1 \le 2^{-q} C(q,r) \prod_{j=1}^{q} \|\boldsymbol{\alpha}_j\|_1, \tag{1-11}$$

where

$$C(q,r) = \min\left\{2^q, \left(\frac{r+1}{r+1-q}\right)^{r+1-q}\right\}. \tag{1-12}$$

We find that

$$C(q,r) = 2^q \quad \text{if } 2q \le r+1,$$

and

$$C(q,r) = \left(\frac{r+1}{r+1-q}\right)^{r+1-q} \quad \text{if } r+1 \le 2q.$$

In particular we have $C(r,r) = (r+1)$ so that (1-11) includes the inequality (1-10). By applying (1-4) it follows that (1-11) can be written using the Weil height as

$$\|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1 \le C(q,r) \prod_{j=1}^{q} ([k:\mathbb{Q}] h(\alpha_j)).$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_q$ be as in the statement of Theorem 1.1, and let $\mathfrak{A}$ be the subgroup of $\Gamma_S(k)$ generated by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_q$. Clearly $\mathfrak{A}$ is a free group of rank q. It is easy to show that the l^1-norm of the exterior product

$$\|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1 \tag{1-13}$$

depends on the subgroup $\mathfrak{A}$, but does not depend on the choice of generators. Because of (1-9) the l^1-norm of the exterior product (1-13) extends the S-regulator from the group $\Gamma_S(k)$ to subgroups of $\Gamma_S(k)$ having lower rank.

Alternatively, if $\alpha \ne 1$ belongs to $O_S^\times$ and $\boldsymbol{\alpha} \ne \mathbf{0}$ is the image of α with respect to the logarithmic embedding (1-3), then $\boldsymbol{\alpha}$ and $-\boldsymbol{\alpha}$ are the unique pair of generators of a subgroup of rank 1 in $\Gamma_S(k)$. In view of (1-4) we may regard $\|\boldsymbol{\alpha}\|_1$ as the height of this subgroup. Then (1-13) extends the height to more general subgroups $\mathfrak{A} \subseteq \Gamma_S(k)$ having rank q. This definition of a height on subgroups is similar to the definition stated in [Vaaler 2014, equation (6.14)].

In [Akhtari and Vaaler 2016, Theorem 1.2] we showed that if $\mathfrak{A} \subseteq \Gamma_S(k)$ is a subgroup with full rank r, then there exist r linearly independent points in $\mathfrak{A}$ such that the product of their heights is bounded by a number depending only on r multiplied by

$$\mathrm{Reg}_S(k)\, [\mathfrak{U}_S(k) : \mathfrak{A}]. \tag{1-14}$$

The following result generalizes [Akhtari and Vaaler 2016, Theorem 1.2] to arbitrary subgroups $\mathfrak{A} \subseteq \Gamma_S(k)$ having positive rank q where $1 \le q \le r$. In this result the S-regulator (1-14) is replaced by the l^1-norm (1-13) of the exterior product of a set of generators for the subgroup $\mathfrak{A}$.

Theorem 1.2. *Let $\mathfrak{A} \subseteq \Gamma_S(k)$ be a subgroup of positive rank q, and let the points*

$$\boldsymbol{\alpha}_j = (d_v \log \|\alpha_j\|_v), \quad \textit{where } j = 1, 2, \ldots, q,$$

generate the subgroup $\mathfrak{A}$. Then there exists a subgroup $\mathfrak{B} \subseteq \mathfrak{A}$ of rank q and a set of generators

$$\boldsymbol{\beta}_j = (d_v \log \|\beta_j\|_v), \quad \textit{where } j = 1, 2, \ldots, q,$$

for $\mathfrak{B}$ such that

$$\|\boldsymbol{\beta}_1 \wedge \boldsymbol{\beta}_2 \wedge \cdots \wedge \boldsymbol{\beta}_q\|_1 = [\mathfrak{A} : \mathfrak{B}]\, \|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1 \tag{1-15}$$

and

$$\prod_{j=1}^{q} \|\boldsymbol{\beta}_j\|_1 \le q!\, \|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1. \tag{1-16}$$

We have $[\mathfrak{A} : \mathfrak{B}] \le q!$.

By applying (1-4) we find that the product on the left of (1-16) can be written using the Weil height as

$$\prod_{i=1}^{q} \|\boldsymbol{\beta}_i\|_1 = 2^q \prod_{j=1}^{q} ([k : \mathbb{Q}]\, h(\beta_j)).$$

Because the subgroups $\mathfrak{B} \subseteq \mathfrak{A}$ both have rank q, the identity (1-15) follows as in our derivation of (1-8) from (1-7).

It would be of interest to know if there exist absolute constants $b_0 > 0$ and $b_1 > 1$ such that the factor $q!$ on the right of (1-16) could be replaced by $b_0 b_1^q$. This could have implications for a conjecture of F. Rodriguez Villegas which we discuss in Section 2.

2. A conjecture of F. Rodriguez Villegas

In a well-known paper D. H. Lehmer [1933] proposed an important problem about the roots of irreducible polynomials in $\mathbb{Z}[x]$. An equivalent form of Lehmer's problem stated using the absolute, logarithmic, Weil height (1-2) is this: does there exist an absolute constant $c > 0$ such that

$$c \le [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

whenever $\alpha \neq 0$ is an algebraic number and not a root of unity? If $\alpha \neq 0$ and α is not a unit, the lower bound

$$\log 2 \le [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

follows easily. Therefore when considering Lehmer's problem we may restrict our attention to algebraic units α which are not roots of unity. Further information about Lehmer's problem can be found in [Bombieri and Gubler 2006, Section 1.6.15; Smyth 2008; Waldschmidt 2000, Section 3.6].

Let S_∞ be the set of archimedean places of k and assume that $|S_\infty| \ge 2$. We continue to write $|S_\infty| = r + 1$ so that the logarithmic embedding (1-3) is an isomorphism from the free group

$$\mathfrak{U}_{S_\infty}(k) = O_{S_\infty} / \operatorname{Tor}(O_{S_\infty}^{\times})$$

onto the discrete subgroup $\Gamma_{S_\infty}(k)$ of rank r contained in the diagonal subspace $\mathcal{D}_r \subseteq \mathbb{R}^{r+1}$. Then Lehmer's problem asks if there exists an absolute constant $c > 0$ such that the inequality

$$c \le 2[k : \mathbb{Q}]h(\alpha) = \|\boldsymbol{\alpha}\|_1 \tag{2-1}$$

holds at all points $\boldsymbol{\alpha} \neq \mathbf{0}$ in $\Gamma_{S_\infty}(k)$. A generalization of this conjecture to independent subsets $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_q$ in $\Gamma_{S_\infty}(k)$ with $2 \le q \le r$ was proposed by Bertrand [1997]. More precisely, Bertrand asked if for each integer $2 \le q$ there exists a constant $c_q > 0$ such that

$$c_q \le \|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_2, \tag{2-2}$$

where the l^2-norm of the wedge product on the right of (2-2) is the covolume of the subgroup of $\Gamma_{S_\infty}(k)$ generated by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_q$. Examples found by Siegel [1969] show that the inequality (2-2) cannot hold for $q = 1$. However, a positive answer for $q \ge 3$ was established by Amoroso and David [1999].

An alternative generalization of Lehmer's problem to subgroups of rank q has been proposed in a conjecture of F. Rodriguez Villegas stated in [Chinburg et al. 2022, Appendix], and also discussed in [Amoroso and David 2021]. We state a special case of this conjecture for pure wedges.

Conjecture 2.1 (F. Rodriguez Villegas). *There exist two absolute constants $c_0 > 0$ and $c_1 > 1$ with the following property. If q is an integer such that*

$$1 \le q \le r = \operatorname{rank} \Gamma_{S_\infty}(k),$$

and if $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_q$ are linearly independent points in $\Gamma_{S_\infty}(k)$, then

$$c_0 c_1^q \le \|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1. \tag{2-3}$$

If $q = 1$ then the truth of (2-3) would solve the problem originally proposed by Lehmer, and if $q = r$ then (2-3) follows from a known lower bound for the regulator proved by R. Zimmert [1981]. Thus the conjecture of Rodriguez Villegas interpolates between the unsolved problem of Lehmer and Zimmert's result. It follows from earlier work of Pohst [1978] and Schinzel [1973] that Conjecture 2.1 holds for the collection of totally real algebraic number fields k.

Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_q$ be linearly independent points in $\Gamma_{S_\infty}(k)$ and let $\mathfrak{A} \subseteq \Gamma_{S_\infty}(k)$ be the subgroup of rank q that they generate. We have already observed in connection with (1-13) that the l^1-norm

$$\|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1$$

depends on the subgroup $\mathfrak{A}$, but does *not* depend on the choice of generators. Thus Conjecture 2.1 can be regarded as a generalization of Lehmer's problem (reformulated as a conjecture) from subgroups of rank 1 to more general subgroups of rank q where $1 \le q \le r$.

Here is a related conjecture.

Conjecture 2.2. *There exist two absolute constants $d_0 > 0$ and $d_1 > 1$ with the following property. If q is an integer such that*

$$1 \le q \le r = \operatorname{rank} \Gamma_{S_\infty}(k),$$

and if $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_q$ *are linearly independent points in* $\Gamma_{S_\infty}(k)$, *then*

$$d_0 d_1^q \le \|\boldsymbol{\alpha}_1\|_1 \|\boldsymbol{\alpha}_2\|_1 \cdots \|\boldsymbol{\alpha}_q\|_1.$$

It follows from (1-12) that the constant on the right of (1-11) satisfies

$$2^{-q} C(q, r) \le 1.$$

Therefore if the conjectured inequality (2-3) is correct, then from Theorem 1.1 we also get

$$c_0 c_1^q \le \|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1 \le \prod_{j=1}^{q} \|\boldsymbol{\alpha}_j\|_1.$$

Thus Conjecture 2.1 implies Conjecture 2.2 with $d_0 = c_0$ and $d_1 = c_1$.

Now assume that Conjecture 2.2 is correct. Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_q$ be linearly independent points in the logarithmic embedding $\Gamma_{S_\infty}(k)$, and let $\mathfrak{A}$ be the subgroup of rank q that they generate. By Theorem 1.2 there exist linearly independent points $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_q$ in $\mathfrak{A}$ such that

$$d_0 d_1^q \le \|\boldsymbol{\beta}_1\|_1 \|\boldsymbol{\beta}_2\|_1 \cdots \|\boldsymbol{\beta}_q\|_1 \le q!\, \|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1, \tag{2-4}$$

where the inequality on the left of (2-4) follows from Conjecture 2.2, and the inequality on the right of (2-4) follows from (1-16). However, as $q!$ grows faster than an exponential function of q, at present we are unable to conclude that Conjecture 2.2 implies Conjecture 2.1. This could change if the factor $q!$ in the inequality (1-16) could be replaced by a factor of the form $b_0 b_1^q$, where $b_0 > 0$ and $b_1 > 1$ are absolute constants.

3. Generalization of Schinzel's inequality, I

For a real number x we write

$$x^+ = \max\{0, x\} \quad \text{and} \quad x^- = \max\{0, -x\},$$

so that $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Let $\boldsymbol{x} = (x_n)$ be a (column) vector in $\mathbb{R}^N$. As in [Akhtari and Vaaler 2016, equation (4.3)], the *Schinzel norm* is the function

$$\delta : \mathbb{R}^N \to [0, \infty)$$

defined by

$$\delta(\boldsymbol{x}) = \max\left\{ \sum_{m=1}^{N} x_m^+,\ \sum_{n=1}^{N} x_n^- \right\} = \tfrac{1}{2}\left|\sum_{n=1}^{N} x_n\right| + \tfrac{1}{2}\sum_{n=1}^{N} |x_n|.$$

It is clear that δ is in fact a norm on $\mathbb{R}^N$, and we write

$$K_N = \{\boldsymbol{x} \in \mathbb{R}^N : \delta(\boldsymbol{x}) \le 1\}$$

for the corresponding closed unit ball. Then K_N is a compact, convex, symmetric subset of $\mathbb{R}^N$ with a nonempty interior. The N-dimensional volume of K_N was computed in [Akhtari and Vaaler 2016, Lemma 4.1]. The connection between the Schinzel norm and the Weil height follows from (1-4) and (5-2) (see also [Akhtari and Vaaler 2016, Lemma 5.1]).

In Lemma 3.2 we will determine the finite collection of extreme points of K_N. Then a combinatorial argument in Section 4 applied to the extreme points of K_N will lead to a proof of the following inequalities.

Theorem 3.1. *Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_L$ be linearly independent vectors in $\mathbb{R}^N$. If $L = N$ then*

$$|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_N| \le \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\cdots\delta(\boldsymbol{x}_N), \tag{3-1}$$

if $L < N \le 2L$ then

$$\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1 \le \left(\frac{N}{N-L}\right)^{N-L} \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\cdots\delta(\boldsymbol{x}_L), \tag{3-2}$$

and if $2L \le N$ then

$$\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1 \le 2^L\, \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\cdots\delta(\boldsymbol{x}_L). \tag{3-3}$$

Alternatively, for $L < N$ we have

$$\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1 \le \min\left\{2^L, \left(\frac{N}{N-L}\right)^{N-L}\right\} \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\cdots\delta(\boldsymbol{x}_L). \tag{3-4}$$

If $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N$, are (column) vectors in $\mathbb{R}^N$, then Schinzel [1978] proved the inequality

$$|\det(\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \cdots \;\; \boldsymbol{x}_N)| \le \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\cdots\delta(\boldsymbol{x}_N), \tag{3-5}$$

which is equivalent to (3-1). It can be shown that there exist nontrivial cases of equality in the inequality (3-2) whenever the integer $N - L$ is a divisor of N. And it can be shown that there always exist nontrivial cases of equality in the inequality (3-3). It is instructive to define the function

$$g_L : [L, \infty] \to [1, e^L]$$

by

$$g_L(x) = \begin{cases} 1 & \text{if } x = L, \\ \left(\dfrac{x}{x-L}\right)^{x-L} & \text{if } L < x < \infty, \\ e^L & \text{if } x = \infty. \end{cases}$$

It follows that $x \mapsto g_L(x)$ is continuous, and has a continuous, positive derivative on (L, ∞). Then $x \mapsto g_L(x)$ is strictly increasing on $[L, \infty]$. We have $g_L(2L) = 2^L$, and this clarifies the behavior of the function

$$x \mapsto \min\{2^L, g_L(x)\}$$

which occurs on the right of (3-4).

We recall that a point $\boldsymbol{k}$ in K_N is an *extreme point* of K_N if $\boldsymbol{k}$ cannot be written as a proper convex combination of two distinct points in K_N. Obviously all extreme points of K_N occur on the boundary of K_N. Let

$$\varphi : \mathbb{R}^N \to \mathbb{R}$$

be a continuous linear functional, and write

$$\delta^*(\varphi) = \sup\{\varphi(\boldsymbol{x}) : \delta(\boldsymbol{x}) \le 1\}$$

for the dual norm of φ. As K_N is compact there exists a point $\boldsymbol{\eta}$ in K_N such that

$$\delta^*(\varphi) = \varphi(\boldsymbol{\eta}).$$

If there exists a linear functional φ such that

$$\{\boldsymbol{\eta} \in K_N : \delta^*(\varphi) = \varphi(\boldsymbol{\eta})\} = \{\boldsymbol{k}\},$$

then $\boldsymbol{k}$ is an *exposed point* of K_N. It is known (see [Eggleston 1958, section 1.8, exercise 3]) that an exposed point of K_N is also an extreme point of K_N.

We define two finite, disjoint subsets of $\mathbb{R}^N$ by

$$E_N = \{\pm\boldsymbol{e}_m : 1 \le m \le N\} \quad \text{and} \quad F_N = \{\boldsymbol{e}_m - \boldsymbol{e}_n : m \ne n\}, \tag{3-6}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_N$ are the standard basis vectors in $\mathbb{R}^N$. Clearly we have

$$|E_N| = 2N \quad \text{and} \quad |F_N| = N^2 - N.$$

It follows easily that each point of $E_N \cup F_N$ is on the boundary of K_N.

Lemma 3.2. *The subset $E_N \cup F_N$ is the collection of all extreme points of K_N.*

Proof. For $1 \le m \le N$ let $\varphi_m : \mathbb{R}^N \to \mathbb{R}$ be the linear functional defined by

$$\varphi_m(\boldsymbol{x}) = \tfrac{1}{2}\sum_{n=1}^{N} x_n + \tfrac{1}{2}x_m.$$

Then we have

$$\varphi_m(\boldsymbol{x}) \le \tfrac{1}{2}\left|\sum_{n=1}^{N} x_n\right| + \tfrac{1}{2}|x_m|, \tag{3-7}$$

and there is equality in the inequality (3-7) if and only if

$$0 \le \sum_{n=1}^{N} x_n \quad \text{and} \quad 0 \le x_m.$$

We also have

$$\tfrac{1}{2}\left|\sum_{n=1}^{N} x_n\right| + \tfrac{1}{2}|x_m| \le \delta(\boldsymbol{x}), \tag{3-8}$$

and there is equality in the inequality (3-8) if and only if

$$x_n = 0 \quad \text{for each } n \ne m.$$

Combining (3-7) and (3-8) we find that

$$\varphi_m(\boldsymbol{x}) \le \delta(\boldsymbol{x}) \tag{3-9}$$

for all $\boldsymbol{x}$ in $\mathbb{R}^N$, and there is equality in the inequality (3-9) if and only if $\boldsymbol{x} = t\boldsymbol{e}_m$ with $0 \le t$. Therefore

$$\delta^*(\varphi_m) = \sup\{\varphi_m(\boldsymbol{x}) : \delta(\boldsymbol{x}) \le 1\} = \varphi_m(\boldsymbol{e}_m) = 1$$

and

$$\{\boldsymbol{\eta} \in K_N : \delta^*(\varphi_m) = \varphi_m(\boldsymbol{\eta})\} = \{\boldsymbol{e}_m\}.$$

This shows that $\boldsymbol{e}_m$ is an exposed point of K_N, and therefore $\boldsymbol{e}_m$ is an extreme point of K_N. As K_N is symmetric, we find that $-\boldsymbol{e}_m$ is also an extreme point.

Next we suppose that $m \ne n$, and we define the linear functional $\psi_{mn} : \mathbb{R}^N \to \mathbb{R}$ by

$$\psi_{mn}(\boldsymbol{x}) = \tfrac{1}{2}(x_m - x_n).$$

Then we have

$$\psi_{mn}(\boldsymbol{x}) \le \tfrac{1}{2}\left|\sum_{\ell=1}^{N} x_\ell\right| + \tfrac{1}{2}|x_m| + \tfrac{1}{2}|x_n|, \tag{3-10}$$

and there is equality in the inequality (3-10) if and only if

$$\sum_{\ell=1}^{N} x_\ell = 0, \quad 0 \le x_m \text{ and } x_n \le 0.$$

We get

$$\tfrac{1}{2}\left|\sum_{\ell=1}^{N} x_\ell\right| + \tfrac{1}{2}|x_m| + \tfrac{1}{2}|x_n| \le \delta(\boldsymbol{x}), \tag{3-11}$$

with equality in the inequality (3-11) if and only if

$$x_\ell = 0 \quad \text{for all } \ell \ne m \text{ and } \ell \ne n.$$

By combining (3-10) and (3-11) we find that

$$\psi_{mn}(\boldsymbol{x}) \le \delta(\boldsymbol{x}), \tag{3-12}$$

and there is equality in the inequality (3-12) if and only if $\boldsymbol{x} = t(\boldsymbol{e}_m - \boldsymbol{e}_n)$ with $0 \le t$. As in the previous case we conclude that

$$\delta^*(\psi_{mn}) = \sup\{\psi_{mn}(\boldsymbol{x}) : \delta(\boldsymbol{x}) \le 1\} = \psi_{mn}(\boldsymbol{e}_m - \boldsymbol{e}_n) = 1$$

and

$$\{\boldsymbol{\eta} \in K : \delta^*(\psi_{mn}) = \psi_{mn}(\boldsymbol{\eta})\} = \{\boldsymbol{e}_m - \boldsymbol{e}_n\}.$$

This shows that $\boldsymbol{e}_m - \boldsymbol{e}_n$ is an exposed point of K_N, and therefore $\boldsymbol{e}_m - \boldsymbol{e}_n$ is an extreme point of K_N.

We have now shown that each point in $E_N \cup F_N$ is an extreme point of K_N. To complete the proof we will show that if $\boldsymbol{x}$ is a point on the boundary of K_N, then $\boldsymbol{x}$ can be written as a convex combination of points in $E_N \cup F_N$. Thus we assume that

$$\delta(\boldsymbol{x}) = \max\left\{\sum_{m=1}^{N} x_m^+,\ \sum_{n=1}^{N} x_n^-\right\} = 1, \tag{3-13}$$

and we write

$$\sigma^+ = \sum_{m=1}^{N} x_m^+ \quad \text{and} \quad \sigma^- = \sum_{n=1}^{N} x_n^-.$$

Then we have

$$\begin{aligned}
\sum_{\substack{m=1\\ m\neq n}}^{N}\sum_{n=1}^{N} x_m^+ x_n^- (\boldsymbol{e}_m - \boldsymbol{e}_n) &= \left(\sum_{n=1}^{N} x_n^-\right)\sum_{m=1}^{N} x_m^+ \boldsymbol{e}_m - \left(\sum_{m=1}^{N} x_m^+\right)\sum_{n=1}^{N} x_n^- \boldsymbol{e}_n \\
&= \sigma^- \sum_{m=1}^{N} x_m^+ \boldsymbol{e}_m - \sigma^+ \sum_{n=1}^{N} x_n^- \boldsymbol{e}_n \\
&= \sum_{m=1}^{N} x_m^+ \boldsymbol{e}_m - \sum_{n=1}^{N} x_n^- \boldsymbol{e}_n - (1-\sigma^-)\sum_{m=1}^{N} x_m^+ \boldsymbol{e}_m + (1-\sigma^+)\sum_{n=1}^{N} x_n^- \boldsymbol{e}_n \\
&= \boldsymbol{x} - (1-\sigma^-)\sum_{m=1}^{N} x_m^+ \boldsymbol{e}_m - (1-\sigma^+)\sum_{n=1}^{N} x_n^- (-\boldsymbol{e}_n),
\end{aligned}$$

and therefore

$$\boldsymbol{x} = (1-\sigma^-)\sum_{m=1}^{N} x_m^+ \boldsymbol{e}_m + (1-\sigma^+)\sum_{n=1}^{N} x_n^- (-\boldsymbol{e}_n) + \sum_{\substack{m=1\\ m\neq n}}^{N}\sum_{n=1}^{N} x_m^+ x_n^- (\boldsymbol{e}_m - \boldsymbol{e}_n). \tag{3-14}$$

The identity (3-14) shows that $\boldsymbol{x}$ is a linear combination of points in $E_N \cup F_N$ with nonnegative coefficients. Using (3-13), the sum of the coefficients in (3-14) is

$$\begin{aligned}
(1-\sigma^-)\sum_{m=1}^{N} x_m^+ + (1-\sigma^+)\sum_{n=1}^{N} x_n^- + \sum_{\substack{m=1\\ m\neq n}}^{N}\sum_{n=1}^{N} x_m^+ x_n^- &= (1-\sigma^-)\sigma^+ + (1-\sigma^+)\sigma^- + \sigma^+\sigma^- \\
&= 1 - (1-\sigma^+)(1-\sigma^-) \\
&= 1.
\end{aligned}$$

It follows that $\boldsymbol{x}$ is a convex combination of points in $E_N \cup F_N$. We have shown that if $\boldsymbol{x}$ is on the boundary of K_N, then $\boldsymbol{x}$ is a convex combination of points in $E_N \cup F_N$. Therefore the only extreme points of K_N are the points in $E_N \cup F_N$. □

Let

$$I = \{i_1 < i_2 < \cdots < i_L\} \subseteq \{1, 2, \ldots, N\}$$

be a subset of positive cardinality L. If $\boldsymbol{x} = (x_n)$ is a point in $\mathbb{R}^N$ we write $\boldsymbol{x}_I$ for the point in $\mathbb{R}^L$ given by $\boldsymbol{x}_I = (x_{i_\ell})$. Alternatively, $\boldsymbol{x}_I$ is the $L \times 1$ submatrix of $\boldsymbol{x}$ having rows indexed by the integers in the subset I. The following result is now an immediate consequence of Lemma 3.2.

Corollary 3.3. *Let $\boldsymbol{\xi}$ be an element in the set of extreme points $E_N \cup F_N$, and let*

$$I \subseteq \{1, 2, \dots, N\}$$

be a subset of positive cardinality L. Then either $\boldsymbol{\xi}_I = \boldsymbol{0}$ in $\mathbb{Z}^L$, or $\boldsymbol{\xi}_I$ belongs to the set of extreme points $E_L \cup F_L$.

Let

$$\Phi_{L,N} : \mathbb{R}^N \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N \to \mathbb{R}^M, \quad \text{where } M = \binom{N}{L},$$

be the continuous, alternating, multilinear function taking values in $\mathbb{R}^M$ and defined by

$$\Phi_{L,N}(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_L) = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L.$$

By compactness the continuous, nonnegative function

$$(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_L) \mapsto \|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1$$

assumes its maximum value on the L-fold product

$$K_N \times K_N \times \cdots \times K_N.$$

We write

$$\mu_{L,N} = \max\{\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1 : \boldsymbol{x}_\ell \in K_N \text{ for } \ell = 1, 2, \dots, L\} \tag{3-15}$$

for this maximum value. We show that $\mu_{L,N}$ can be determined by restricting each variable $\boldsymbol{x}_\ell$ to the set $E_N \cup F_N$ of extreme points in K_N.

Lemma 3.4. *There exist points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_L$ in the set of extreme points $E_N \cup F_N$ such that*

$$\mu_{L,N} = \|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1. \tag{3-16}$$

If $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_L$ are vectors in $\mathbb{R}^N$ then

$$\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1 \le \mu_{L,N}\, \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2) \cdots \delta(\boldsymbol{x}_L). \tag{3-17}$$

Proof. Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots, \boldsymbol{\eta}_L$ be points in K_N such that

$$\mu_{L,N} = \|\boldsymbol{\eta}_1 \wedge \boldsymbol{\eta}_2 \wedge \cdots \wedge \boldsymbol{\eta}_L\|_1. \tag{3-18}$$

Because $\Phi_{L,N}$ is linear in each variable, it is easy to show that $\delta(\boldsymbol{\eta}_\ell) = 1$ for each $\ell = 1, 2, \dots, L$. Also, among all the collections of L points from the boundary of K_N that satisfy (3-18), we may assume that the collection $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots, \boldsymbol{\eta}_L$ contains the maximum number of extreme points. If this maximum number is L then we are done. Therefore we may assume that the maximum number of extreme points is less than L.

If, for example, $\boldsymbol{\eta}_1$ is not an extreme point, then there exist extreme points $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_J$ in K_N, and positive numbers $\theta_1, \theta_2, \ldots, \theta_J$, such that

$$\boldsymbol{\eta}_1 = \sum_{j=1}^{J} \theta_j \boldsymbol{u}_j \quad \text{and} \quad \sum_{j=1}^{J} \theta_j = 1.$$

It follows that

$$\mu_{L,N} = \biggl\| \sum_{j=1}^{J} \theta_j (\boldsymbol{u}_j \wedge \boldsymbol{\eta}_2 \wedge \cdots \wedge \boldsymbol{\eta}_L) \biggr\|_1 \le \sum_{j=1}^{J} \theta_j \|\boldsymbol{u}_j \wedge \boldsymbol{\eta}_2 \wedge \cdots \wedge \boldsymbol{\eta}_L\|_1 \le \mu_{L,N} \sum_{j=1}^{J} \theta_j = \mu_{L,N} \tag{3-19}$$

Hence there is equality throughout the inequality (3-19), and we conclude that

$$\mu_{L,N} = \|\boldsymbol{u}_j \wedge \boldsymbol{\eta}_2 \wedge \cdots \wedge \boldsymbol{\eta}_L\|_1$$

for each $j = 1, 2, \ldots, J$. But each collection of points $\boldsymbol{u}_j, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_L$ plainly contains one more extreme point than the collection $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_L$. The contradiction shows that there exists a collection of points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ from the boundary of K_N such that (3-16) holds and each $\boldsymbol{\xi}_\ell$ is an extreme point of K_N.

Next we verify the inequality (3-17). If one of the vectors in the collection $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_L$ is the zero vector, then both sides of (3-17) are zero. Thus we may assume that $\boldsymbol{x}_\ell \neq \boldsymbol{0}$ for each $\ell = 1, 2, \ldots, L$. Let

$$\boldsymbol{y}_\ell = \delta(\boldsymbol{x}_\ell)^{-1} \boldsymbol{x}_\ell, \tag{3-20}$$

so that $\delta(\boldsymbol{y}_\ell) = 1$ for each $\ell = 1, 2, \ldots, L$. Then we certainly have

$$\|\boldsymbol{y}_1 \wedge \boldsymbol{y}_2 \wedge \cdots \wedge \boldsymbol{y}_L\|_1 \le \mu_{L,N} \tag{3-21}$$

by the definition of $\mu_{L,N}$. Then (3-17) follows using (3-20), (3-21), and the multilinearity of the exterior product. □

The extreme points $E_N \cup F_N$ for the δ-unit ball K_N have the following useful property.

Lemma 3.5. *Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ be extreme points in the set $E_N \cup F_N$, and let*

$$\Xi = (\boldsymbol{\xi}_1 \;\; \boldsymbol{\xi}_2 \;\; \cdots \;\; \boldsymbol{\xi}_L)$$

be the $N \times L$ matrix having $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ as columns. If

$$I \subseteq \{1, 2, \ldots, N\}$$

is a subset of cardinality $|I| = L$, and Ξ_I is the $L \times L$ submatrix having rows indexed by I, then the integer $\det \Xi_I$ *belongs to the set* $\{-1, 0, 1\}$.

Proof. Clearly the columns of the $L \times L$ submatrix Ξ_I are the $L \times 1$ column vectors $(\boldsymbol{\xi}_1)_I, (\boldsymbol{\xi}_2)_I, \ldots, (\boldsymbol{\xi}_L)_I$. If a column of Ξ_I is $\boldsymbol{0}$, then $\det \Xi_I = 0$ is obvious. If each column of Ξ_I is not $\boldsymbol{0}$, then it follows from

Corollary 3.3 that each column of Ξ_I belongs to the set of extreme points $E_L \cup F_L$. Applying Schinzel's determinant inequality (3-5) to the matrix Ξ_I, we get

$$|\det \Xi_I| \le \delta((\boldsymbol{\xi}_1)_I)\delta((\boldsymbol{\xi}_2)_I)\cdots\delta((\boldsymbol{\xi}_L)_I) = 1.$$

As $\det \Xi_I$ is an integer, the lemma is proved. □

If $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ are extreme points in $E_N \cup F_N$, then it follows from Lemma 3.5 that

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 = \sum_{\substack{I \subseteq \{1,2,\ldots,N\} \\ |I|=L}} |\det \Xi_I| \le \binom{N}{L}. \tag{3-22}$$

Using (3-16) we get the simple upper bound

$$\mu_{L,N} \le \binom{N}{L} \quad \text{for } 1 \le L \le N. \tag{3-23}$$

It follows from (3-5) that there is equality in (3-23) when $L = N$. There is also equality in (3-23) when $L + 1 = N$; this follows from the example

$$\Xi = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix}.$$

By squaring each of the subdeterminants in the sum (3-22) we can determine the value of $\mu_{L,N}$ for $2L \le N$.

Lemma 3.6. *If $1 \le L < N$ then*

$$\mu_{L,N} \le 2^L. \tag{3-24}$$

If $2L \le N$ then there is equality in the inequality (3-24).

Proof. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ be extreme points in $E_N \cup F_N$, and let

$$\Xi = (\boldsymbol{\xi}_1 \;\; \boldsymbol{\xi}_2 \;\; \cdots \;\; \boldsymbol{\xi}_L)$$

be the $N \times L$ matrix having $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ as columns. It follows from Lemma 3.5 that

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 = \sum_{\substack{I \subseteq \{1,2,\ldots,N\} \\ |I|=L}} |\det \Xi_I| = \sum_{\substack{I \subseteq \{1,2,\ldots,N\} \\ |I|=L}} (\det \Xi_I)^2.$$

Then from the Cauchy–Binet identity we get

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 = \sum_{\substack{I \subseteq \{1,2,\dots,N\} \\ |I|=L}} (\det \Xi_I)^2 = \det(\Xi^T \Xi). \tag{3-25}$$

The $L \times L$ matrix in the determinant on the right of (3-25) is

$$\Xi^T \Xi = (\boldsymbol{\xi}_k^T \boldsymbol{\xi}_\ell),$$

where $k = 1, 2, \dots, L$ indexes rows and $\ell = 1, 2, \dots, L$ indexes columns. As $\Xi^T \Xi$ is an $L \times L$ real, symmetric matrix, we can apply Hadamard's inequality to estimate its determinant. We find that

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 = \det(\Xi^T \Xi) \le \prod_{\ell=1}^{L} \|\boldsymbol{\xi}_\ell\|_2^2 \le 2^L. \tag{3-26}$$

This proves the inequality (3-24).

If the columns of the matrix Ξ are orthogonal, then there is equality in Hadamard's inequality. Therefore, if $2L \le N$ we select $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_L$ in F_N so that

$$\Xi = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For this choice of Ξ the columns of Ξ are orthogonal. Hence for this choice of Ξ there is equality in (3-26), and equality in (3-24). □

If $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_L$ belong to $\mathbb{R}^N$ and $2L \le N$, then it follows from (3-17) and the case of equality in (3-24) that

$$\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1 \le 2^L \, \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2) \cdots \delta(\boldsymbol{x}_L). \tag{3-27}$$

This proves the inequality (3-3) in the statement of Theorem 3.1.

The following lemma, together with combinatorial arguments in Section 4, will be used in the proof of the inequality (3-2).

Lemma 3.7. *Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_L$ be linearly independent extreme points in the set $E_N \cup F_N$. Assume that exactly K of the points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_L$ belong to the subset E_N, where $1 \le K < L$. Then there exist linearly independent extreme points $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots, \boldsymbol{\eta}_{L-K}$ in the set $E_{N-K} \cup F_{N-K}$ such that*

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 = \|\boldsymbol{\eta}_1 \wedge \boldsymbol{\eta}_2 \wedge \cdots \wedge \boldsymbol{\eta}_{L-K}\|_1.$$

Proof. By using a suitable permutation of the points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_L$, we may assume that

$$\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_K\} \subseteq E_N \quad \text{and} \quad \{\boldsymbol{\xi}_{K+1}, \boldsymbol{\xi}_{K+2}, \dots, \boldsymbol{\xi}_L\} \subseteq F_N.$$

We may further assume that for $k = 1, 2, \dots, K$ we have

$$\boldsymbol{\xi}_k = \pm \boldsymbol{e}_{m_k}, \quad \text{where } 1 \le m_1 < m_2 < \cdots < m_K \le N.$$

It will be convenient to write

$$M = \{m_1, m_2, \dots, m_K\}.$$

Now let

$$\Xi = (\boldsymbol{\xi}_1 \;\; \boldsymbol{\xi}_2 \;\; \cdots \;\; \boldsymbol{\xi}_L)$$

be the $N \times L$ matrix having $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_L$ as columns. We partition Ξ into submatrices

$$\Xi = (U \;\; V),$$

where

$$U = (\boldsymbol{\xi}_1 \;\; \boldsymbol{\xi}_2 \;\; \cdots \;\; \boldsymbol{\xi}_K) \quad \text{and} \quad V = (\boldsymbol{\xi}_{K+1} \;\; \boldsymbol{\xi}_{K+2} \;\; \cdots \;\; \boldsymbol{\xi}_L)$$

are $N \times K$ and $N \times (L-K)$, respectively. We suppose that $I \subseteq \{1, 2, \dots, N\}$ is a subset of cardinality $|I| = L$ such that

$$\det \Xi_I = \det(U_I \;\; V_I) \ne 0. \tag{3-28}$$

On the right of (3-28) the submatrix U_I is $L \times K$ and the submatrix V_I is $L \times (L-K)$. If the integer m_k, which occurs in M, does not belong to I, then the k-th column of Ξ_I is identically zero and (3-28) cannot hold. Therefore (3-28) implies that

$$M \subseteq I.$$

Next we apply the Laplace expansion of the determinant to Ξ_I partitioned as in (3-28). In view of our previous remarks we find that

$$\det \Xi_I = \sum_{\substack{J \subseteq I \\ |J| = K}} (-1)^{\varepsilon(J)} (\det U_J)(\det V_{\widetilde{J}}), \tag{3-29}$$

where

$$\widetilde{J} = I \setminus J$$

is the complement of J in I, and $\varepsilon(J)$ is an integer that depends on J. As before, if the integer m_k which occurs in M does not belong to the subset J, then the k-th column of U_J is identically zero and therefore

$\det U_J = 0$. As $|J| = |M| = K$, we conclude that there is exactly one nonzero term in the sum on the right of (3-29), and the nonzero term occurs when $J = M$. From these observations we conclude that the Laplace expansion (3-29) is simply

$$\det \Xi_I = (-1)^{\varepsilon(M)}(\det U_M)(\det V_{I\setminus M}). \tag{3-30}$$

It is obvious that $\det U_M = \pm 1$, and therefore (3-30) leads to the identity

$$|\det \Xi_I| = |\det V_{I\setminus M}|.$$

Let

$$V' = (\boldsymbol{\xi}'_{K+1} \;\; \boldsymbol{\xi}'_{K+2} \;\; \cdots \;\; \boldsymbol{\xi}'_L)$$

be the $(N-K)\times(L-K)$ submatrix of V obtained by removing the rows of V that are indexed by the integers m_k in the subset M. It follows from Lemma 3.4 that the columns of V' belong to the set of extreme points $E_{N-K} \cup F_{N-K}$. Moreover, we have

$$|\det \Xi_I| = |\det V_{I\setminus M}| = |\det V'_J|, \tag{3-31}$$

where

$$J = I \setminus M \subseteq \{1, 2, \dots, N\} \setminus M \quad \text{and} \quad |J| = L - K.$$

We note that

$$I \mapsto J = I \setminus \{m_1, m_2, \dots, m_K\}$$

is a bijection from the set of subsets I that contain M onto the set of subsets of $\{1, 2, \dots, N\} \setminus M$ that have cardinality $L - K$. Using (3-31) we find that

$$\sum_{\substack{I\subseteq\{1,2,\dots,N\}\\ M\subseteq I}} |\det \Xi_I| = \sum_{\substack{J\subseteq\{1,2,\dots,N\}\setminus M\\ |J|=L-K}} |\det V'_J|. \tag{3-32}$$

Because the rows of V' are indexed by the elements of the set $\{1, 2, \dots, N\}\setminus M$, it follows from (3-32) that

$$\|\boldsymbol{\xi}_1\wedge\boldsymbol{\xi}_2\wedge\cdots\wedge\boldsymbol{\xi}_L\|_1 = \sum_{\substack{I\subseteq\{1,2,\dots,N\}\\ M\subseteq I}} |\det \Xi_I| = \sum_{\substack{J\subseteq\{1,2,\dots,N\}\setminus M\\ |J|=L-K}} |\det V'_J| = \|\boldsymbol{\xi}'_{K+1}\wedge\boldsymbol{\xi}'_{K+2}\wedge\cdots\wedge\boldsymbol{\xi}'_L\|_1. \tag{3-33}$$

As the columns of V' belong to $E_{N-K}\cup F_{N-K}$ and satisfy (3-33), they are linearly independent. Therefore we set

$$\boldsymbol{\eta}_\ell = \boldsymbol{\xi}'_{K+\ell} \quad \text{for } \ell = 1, 2, \dots, L-K,$$

and the lemma is proved. □

4. Generalization of Schinzel's inequality, II

We develop a combinatorial method which leads to an asymptotically sharp upper bound for the quantity $\mu_{L,N}$ defined in (3-15). The bound we prove here applies when $L < N \le 2L$, and will be used to verify the inequality (3-2) in the statement of Theorem 3.1.

We suppose throughout this section that

$$\{S(1), S(2), S(3), \dots, S(L)\} \tag{4-1}$$

is a collection of L distinct subsets of $\{1, 2, \dots, N\}$ such that

$$|S(\ell)| = 2 \quad \text{for each } \ell = 1, 2, \dots, L \tag{4-2}$$

and

$$\bigcup_{\ell=1}^{L} S(\ell) = \{1, 2, \dots, N\}. \tag{4-3}$$

It follows from (4-2) and (4-3) that

$$N \le 2L \le N(N-1),$$

but for our later applications we will make the more restrictive assumption that

$$L < N \le 2L. \tag{4-4}$$

Let $\mathcal{A}$ be the collection of *all* subsets $A \subseteq \{1, 2, \dots, N\}$. We define a map $\eta : \mathcal{A} \to \mathcal{A}$ by

$$\eta(A) = \bigcup_{\substack{\ell=1 \\ S(\ell)\cap A \neq \varnothing}}^{L} S(\ell). \tag{4-5}$$

Then it follows from (4-3) that

$$A \subseteq \eta(A) \quad \text{for each subset } A \in \mathcal{A}. \tag{4-6}$$

We are interested in subsets A in $\mathcal{A}$ that satisfy $\eta(A) = A$. Obviously $\varnothing$ and $\{1, 2, \dots, N\}$ have this property. More generally we define

$$\mathcal{P} = \{A \in \mathcal{A} : \eta(A) = A\}. \tag{4-7}$$

If A belongs to the collection $\mathcal{P}$ and $S(\ell) \cap A \neq \varnothing$, then $S(\ell) \subseteq A$. Thus a nonempty subset A in $\mathcal{P}$ must have $2 \le |A|$. We show that the collection $\mathcal{P}$ forms an algebra of subsets.

Lemma 4.1. *Let $\mathcal{P} \subseteq \mathcal{A}$ be the collection of subsets defined by* (4-7).

(i) *If A_1 belongs to $\mathcal{P}$ then its complement*

$$A_2 = \{1, 2, \dots, N\} \setminus A_1$$

also belongs to $\mathcal{P}$.

(ii) *If A_3 and A_4 belong to $\mathcal{P}$ then $A_3 \cup A_4$ belongs to $\mathcal{P}$.*

(iii) *If A_5 and A_6 belong to $\mathcal{P}$ then $A_5 \cap A_6$ belongs to $\mathcal{P}$.*

Proof. Assume that $S(\ell) \cap A_2 \neq \varnothing$. Then $S(\ell) \cap A_1 \neq \varnothing$ is impossible. Hence we have $S(\ell) \subseteq A_2$, and this implies that A_2 belongs to $\mathcal{P}$.

Let $S(\ell) \cap (A_3 \cup A_4) \neq \varnothing$. Then either $S(\ell) \cap A_3 \neq \varnothing$ or $S(\ell) \cap A_4 \neq \varnothing$. Hence either $S(\ell) \subseteq A_3$ or $S(\ell) \subseteq A_4$, and therefore $S(\ell) \subseteq A_3 \cup A_4$. It follows that $A_3 \cup A_4$ belongs to $\mathcal{P}$.

By what we have already proved the sets

$$A_7 = \{1, 2, \ldots, N\} \setminus A_5 \quad \text{and} \quad A_8 = \{1, 2, \ldots, N\} \setminus A_6$$

both belong to $\mathcal{P}$, and therefore the set

$$A_5 \cap A_6 = \{1, 2, \ldots, N\} \setminus (A_7 \cup A_8)$$

belongs to $\mathcal{P}$. □

Lemma 4.2. *Let A_1 be a nonempty subset in $\mathcal{A}$, and let B be a subset in $\mathcal{P}$. Assume that $A_1 \subseteq B$. Define an increasing sequence of subsets*

$$A_1, A_2, A_3, \ldots$$

from $\mathcal{A}$ inductively by

$$A_{n+1} = \eta(A_n) \quad \textit{for each } n = 1, 2, 3, \ldots.$$

Then

$$A_n \subseteq B \quad \textit{for each } n = 1, 2, 3, \ldots.$$

Proof. We argue by induction on n. If $n = 1$ then $A_1 \subseteq B$ by hypothesis. Now assume that $2 \leq n$ and $A_{n-1} \subseteq B$. Then we have

$$A_n = \eta(A_{n-1}) = \bigcup_{\substack{\ell=1 \\ S(\ell) \cap A_{n-1} \neq \varnothing}}^{L} S(\ell). \tag{4-8}$$

If $S(\ell) \cap A_{n-1} \neq \varnothing$ then $S(\ell)$ contains a point of B, and therefore $S(\ell) \subseteq B$. It follows from (4-8) that $A_n \subseteq B$. This proves the lemma. □

We say that a subset A in $\mathcal{A}$ is *minimal* if A is not empty and belongs to $\mathcal{P}$, but no proper subset of A belongs to $\mathcal{P}$. That is, a nonempty set A in $\mathcal{P}$ is *minimal* if for every nonempty subset $B \subseteq A$ such that $B \neq A$, we have $\eta(B) \neq B$. We will show that each element of $\{1, 2, \ldots, N\}$ is contained in a minimal subset in $\mathcal{P}$.

Lemma 4.3. *Let A_1 in $\mathcal{A}$ have cardinality* 1. *Define an increasing sequence of subsets*

$$A_1, A_2, A_3, \ldots$$

from $\mathcal{A}$ inductively by

$$A_{n+1} = \eta(A_n) \quad \textit{for } n = 1, 2, 3, \ldots. \tag{4-9}$$

Let K be the smallest positive integer such that

$$A_K = \eta(A_K) = A_{K+1}. \tag{4-10}$$

Then K exists, $2 \le K$, and the subset A_K is minimal.

Proof. From (4-6) we get

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n \subseteq \cdots .$$

As $|A_n| \le N$ for each $n = 1, 2, \ldots,$ it is obvious that K exists.

Let $A_1 = \{k_1\}$ where $1 \le k_1 \le N$. It follows from (4-3) that there exists a subset $S(\ell_1)$ that contains k_1. Write $S(\ell_1) = \{k_1, k_2\}$ where $k_1 \ne k_2$. From (4-5) we conclude that

$$S(\ell_1) = \{k_1, k_2\} \subseteq \eta(A_1) = A_2,$$

and therefore $A_1 = \{k_1\}$ is a proper subset of $\eta(A_1) = A_2$. Hence we have $2 \le K$.

If A_K is not minimal there exists a proper subset $B \subseteq A_K$ such that $\eta(B) = B$, and therefore B belongs to $\mathcal{P}$. Let

$$C = A_K \setminus B = A_K \cap (\{1, 2, \ldots, N\} \setminus B) \tag{4-11}$$

be the complement of B in A_K. It follows from Lemma 4.1, and the representation on the right of (4-11), that C is a proper subset of A_K and C belongs to $\mathcal{P}$. Thus we have the disjoint union of proper subsets

$$A_K = B \cup C, \quad \text{where } B \in \mathcal{P} \text{ and } C \in \mathcal{P}. \tag{4-12}$$

Plainly $A_1 = \{k_1\}$ is a subset of either B or C, and by renaming these sets if necessary we may assume that $A_1 = \{k_1\}$ is contained in B. Then it follows from Lemma 4.2 that

$$A_n \subseteq B \quad \text{for each } n = 1, 2, 3, \ldots .$$

But this is inconsistent with the representation of A_K as the disjoint union (4-12). We conclude that B and C do not exist, and therefore A_K is minimal. □

It follows from Lemma 4.3 that each element of $\{1, 2, \ldots, N\}$ is contained in a minimal subset. This minimal subset is unique, and leads to a partition of $\{1, 2, \ldots, N\}$ into a disjoint union of minimal subsets.

Lemma 4.4. *Let B and C be nonempty, minimal subsets in $\mathcal{P}$. Then either*

$$B = C \quad \textit{or} \quad B \cap C = \varnothing.$$

Proof. If $B \cap C = \varnothing$ we are done. Therefore we assume that k_1 is a point in $B \cap C$. Let $A_1 = \{k_1\}$, and let $A_1, A_2, A_3, \ldots$ be the sequence of subsets defined by (4-9). Let K be the smallest positive integer such that (4-10) holds. By Lemma 4.3 the subset A_K is minimal, and by Lemma 4.2 we have both $A_K \subseteq B$ and $A_K \subseteq C$. But A_K is minimal and therefore A_K cannot be a proper subset of the minimal subset B. Similarly, A_K cannot be a proper subset of the minimal subset C. We conclude that

$$B = A_K = C.$$

□

Lemma 4.5. *Let (4-1) be a collection of distinct subsets of* $\{1, 2, \ldots, N\}$ *such that*

$$|S(\ell)| = 2 \quad \textit{for each } \ell = 1, 2, \ldots, L$$

and

$$\bigcup_{\ell=1}^{L} S(\ell) = \{1, 2, \ldots, N\}.$$

Let $\mathcal{P} \subseteq \mathcal{A}$ *be the collection of subsets of* $\{1, 2, \ldots, N\}$ *defined by (4-7), and let* $A_1, A_2, \ldots, A_r$ *be the collection of all distinct, minimal subsets in* $\mathcal{P}$*. Then the subsets* $A_1, A_2, \ldots, A_r$ *are disjoint and*

$$A_1 \cup A_2 \cup \cdots \cup A_r = \{1, 2, \ldots, N\}.$$

Proof. The subsets $A_1, A_2, \ldots, A_r$ exist by Lemma 4.3. Then it follows from Lemma 4.4 that the subsets $A_1, A_2, \ldots, A_r$ are disjoint. Therefore we get

$$A_1 \cup A_2 \cup \cdots \cup A_r \subseteq \{1, 2, \ldots, N\}. \tag{4-13}$$

It follows from Lemma 4.3 that each point in $\{1, 2, \ldots, N\}$ is contained in a minimal subset, hence there is equality in (4-13). □

We continue to assume that L and N are positive integers that satisfy (4-4). Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ be vectors from the set of extreme points F_N, and write

$$\Xi = (\boldsymbol{\xi}_1 \ \ \boldsymbol{\xi}_2 \ \ \cdots \ \ \boldsymbol{\xi}_L)$$

for the $N \times L$ matrix having $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ as columns. We assume that no row of the matrix Ξ is identically zero, and we assume that rank $\Xi = L$. We write $\boldsymbol{\xi}_\ell = (\xi_{n\ell})$ and use the vectors $\boldsymbol{\xi}_\ell$ to define a collection of subsets

$$S(\ell) \subseteq \{1, 2, \ldots, N\} \quad \text{for each } \ell = 1, 2, \ldots, L. \tag{4-14}$$

More precisely, we define

$$S(\ell) = \{n : 1 \le n \le N \text{ and } \xi_{n\ell} \ne 0\} \quad \text{for each } \ell = 1, 2, \ldots, L. \tag{4-15}$$

As each column vector $\boldsymbol{\xi}_\ell$ belongs to the set of extreme points F_N, it follows that each subset $S(\ell)$ has cardinality 2 and

$$\sum_{n=1}^{N} \xi_{n\ell} = \sum_{n \in S(\ell)} \xi_{n\ell} = 0.$$

Because no row of the matrix Ξ is identically zero, we find that

$$\bigcup_{\ell=1}^{L} S(\ell) = \{1, 2, \ldots, N\}.$$

Therefore the subsets $S(\ell)$ defined by (4-15) satisfy the conditions (4-2) and (4-3) that were assumed in the previous lemmas. We continue to write $\mathcal{A}$ for the collection of all subsets of $\{1, 2, \dots, N\}$, and we write $\mathcal{P}$ for the collection of subsets defined by (4-7).

Next we suppose that $A_1, A_2, \dots, A_r$ is the collection of distinct, nonempty, minimal subsets in $\mathcal{P}$. Then it follows from Lemma 4.5 that

$$A_1 \cup A_2 \cup \cdots \cup A_r = \{1, 2, \dots, N\} \tag{4-16}$$

is a disjoint union of nonempty sets. Because each subset A_j is minimal we have

$$A_j = \bigcup_{\substack{\ell=1 \\ S(\ell)\subseteq A_j}}^{L} S(\ell) = \bigcup_{\substack{\ell=1 \\ S(\ell)\cap A_j \neq \varnothing}}^{L} S(\ell). \tag{4-17}$$

We use each subset A_j to define a subset $D_j \subseteq \{1, 2, \dots, L\}$ by

$$D_j = \{\ell : 1 \le \ell \le L \text{ and } S(\ell) \subseteq A_j\} \quad \text{for } j = 1, 2, \dots, r. \tag{4-18}$$

Then it follows from (4-16), (4-17), and (4-18), that

$$D_1 \cup D_2 \cup \cdots \cup D_r = \{1, 2, \dots, L\} \tag{4-19}$$

is a disjoint union of nonempty sets. For each $j = 1, 2, \dots, r$ we write Y_j for the $N \times |D_j|$ submatrix of Ξ having columns indexed by the integers in D_j. That is, we define

$$Y_j = (\boldsymbol{\xi}_\ell), \quad \text{where } \ell \in D_j \text{ indexes columns.} \tag{4-20}$$

We assemble the matrices $Y_1, Y_2, \dots, Y_r$ as $N \times |D_j|$ blocks so as to define the $N \times L$ matrix

$$Z = (Y_1 \;\; Y_2 \;\; \cdots \;\; Y_r). \tag{4-21}$$

Because of the disjoint union (4-19), the columns of the matrix Z can also be obtained by permuting the columns of the matrix Ξ. That is, there exists an $L \times L$ permutation matrix P such that

$$\Xi = ZP.$$

As $\det P = \pm 1$ and the columns of Ξ are linearly independent, it follows that the matrix Y_j has rank $|D_j|$ for each $j = 1, 2, \dots, r$. We also find that

$$\det(\Xi^T \Xi) = \det(P^T Z^T Z P) = \det(Z^T Z)$$

is a positive integer.

Now suppose that $1 \le i \le r$, that $1 \le j \le r$, and $i \ne j$. It follows from (4-14), (4-18), and (4-19), that each nonzero row of the matrix Y_i is indexed by an integer in the set A_i, and each nonzero row of the matrix Y_j is indexed by an integer in the set A_j. As A_i and A_j are disjoint we conclude that $Y_i^T Y_j$ is a zero matrix. Because we have organized Z into blocks as in (4-21), we find that

$$\det(\Xi^T \Xi) = \det(Z^T Z) = \prod_{j=1}^{r} \det(Y_j^T Y_j). \tag{4-22}$$

Since the extreme points $\boldsymbol{\xi}_l$ that form the columns of Ξ belong to F_N, it follows that

$$\sum_{n=1}^{N} \xi_{n\ell} = 0 \quad \text{for each } \ell = 1, 2, \dots, L.$$

For each $j = 1, 2, \dots, r$ the nonzero rows of Y_j are indexed by the elements of A_j, and so we get

$$\sum_{n \in A_j} \xi_{n\ell} = 0 \quad \text{for each } \ell \in D_j. \tag{4-23}$$

As Y_j has rank $|D_j|$ we find that

$$|D_j| + 1 \le |A_j|. \tag{4-24}$$

Next we will show that there is equality in the inequality (4-24). Each subset A_j is minimal in $\mathcal{P}$ and therefore no proper subset of A_j belongs to $\mathcal{P}$. It follows from (4-23) that the $|A_j|$ distinct (row) vectors

$$\{(\xi_{n\ell}) : n \in A_j\} \tag{4-25}$$

are linearly dependent. Let $f : A_j \to \mathbb{Z}$ be a function that is supported on the subset

$$B = \{n \in A_j : f(n) \ne 0\},$$

where B is a proper subset of A_j. As B does not belong to $\mathcal{P}$ it follows that there exists ℓ_1 in D_j such that

$$|S(\ell_1) \cap B| = 1.$$

We conclude that

$$\sum_{n \in A_j} f(n) \xi_{n\ell_1} = \sum_{n \in B} f(n) \xi_{n\ell_1} \ne 0,$$

because this sum contains exactly one nonzero term. This shows that no proper subset of the collection of (row) vectors (4-25) is linearly dependent. In particular, each subset of the (row) vectors in (4-25) with cardinality $|A_j| - 1$ is linearly independent. As the rank of the matrix Y_j is $|D_j|$ we conclude by (4-24) that

$$|D_j| + 1 = |A_j| \quad \text{for each } j = 1, 2, \dots, r. \tag{4-26}$$

We also get the identity

$$L + r = \sum_{j=1}^{r} (|D_j| + 1) = \sum_{j=1}^{r} |A_j| = N, \tag{4-27}$$

which determines the value of r.

Lemma 4.6. *Let the columns of the $N \times L$ matrix*

$$\Xi = (\xi_1 \;\; \xi_2 \;\; \cdots \;\; \xi_L)$$

be vectors from the set of extreme points F_N defined in (3-6). If $L < N \le 2L$ then

$$\|\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_L\|_1 \le \left(\frac{N}{N-L}\right)^{N-L}. \tag{4-28}$$

Proof. Clearly we may assume that rank $\Xi = L$. We assume to begin with that no row of the matrix Ξ is identically zero. As in our proof of Lemma 3.6 we have

$$\|\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_L\|_1 = \sum_{\substack{I \subseteq \{1,2,\ldots,N\} \\ |I| = L}} (\det \Xi_I)^2 = \det(\Xi^T \Xi) \tag{4-29}$$

by the Cauchy–Binet identity. By combining (4-22) and (4-29) we find that

$$\|\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_L\|_1 = \prod_{j=1}^{r} \det(Y_j^T Y_j),$$

where each $N \times |D_j|$ matrix Y_j is defined as in (4-20). Let W_j be the $|A_j| \times |D_j|$ submatrix of Y_j obtained by removing all rows which are identically zero. Because there is equality in the inequality (4-24) the submatrix W_j is also $(|D_j|+1) \times |D_j|$. That is, W_j is an $(M+1) \times M$ matrix with columns in the set of extreme points F_M, where $M = |D_j|$. Then it follows from the inequality (3-23) and (4-26) that

$$\prod_{j=1}^{r} \det(Y_j^T Y_j) = \prod_{j=1}^{r} \det(W_j^T W_j) \le \prod_{j=1}^{r} (|D_j| + 1) = \prod_{j=1}^{r} |A_j|. \tag{4-30}$$

We estimate the product on the right of (4-30) by applying the arithmetic/geometric mean inequality and using the identity (4-27). In this way we arrive at the inequality

$$\prod_{j=1}^{r} \det(Y_j^T Y_j) \le \left(r^{-1} \sum_{j=1}^{r} |A_j|\right)^r = (r^{-1} N)^r = \left(\frac{N}{N-L}\right)^{N-L}.$$

This proves (4-28) under the assumption that no row of Ξ is identically zero.

Next we suppose that $L < N \le 2L$, that

$$\Xi = (\xi_1 \;\; \xi_2 \;\; \cdots \;\; \xi_L)$$

is an $N \times L$ matrix with columns $\xi_1, \xi_2, \ldots, \xi_L$ from F_N, that rank $\Xi = L$, and that Ξ has exactly $N - M > 0$ rows that are identically zero. Because rank $\Xi = L$, we find that $L \le M < N \le 2L$. We write

$$\Xi' = (\xi'_1 \;\; \xi'_2 \;\; \cdots \;\; \xi'_L)$$

for the $M \times L$ matrix obtained from Ξ by removing the rows of Ξ that are identically zero. It follows from Lemma 3.4 that each column $\xi'_1, \xi'_2, \ldots, \xi'_L$ belongs to F_M. Clearly each $L \times L$ submatrix of Ξ

with a row that is identically zero has a zero determinant. Thus we have

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 = \|\boldsymbol{\xi}'_1 \wedge \boldsymbol{\xi}'_2 \wedge \cdots \wedge \boldsymbol{\xi}'_L\|_1.$$

If $L = M$ then Ξ' is $L \times L$, and it follows from Lemma 3.5 that

$$\|\boldsymbol{\xi}'_1 \wedge \boldsymbol{\xi}'_2 \wedge \cdots \wedge \boldsymbol{\xi}'_L\|_1 = 1 \le \left(\frac{N}{N-L}\right)^{N-L}.$$

If $L < M < N \le 2L$ then by the case already considered we get

$$\|\boldsymbol{\xi}'_1 \wedge \boldsymbol{\xi}'_2 \wedge \cdots \wedge \boldsymbol{\xi}'_L\|_1 \le \left(\frac{M}{M-L}\right)^{M-L} < \left(\frac{N}{N-L}\right)^{N-L}.$$

This verifies the bound (4-28) in general. □

We now combine Lemma 3.7 and Lemma 4.6 to obtain the inequality (4-28) in full generality.

Theorem 4.7. *Let the columns of the $N \times L$ matrix*

$$\Xi = (\boldsymbol{\xi}_1 \;\; \boldsymbol{\xi}_2 \;\; \cdots \;\; \boldsymbol{\xi}_L)$$

be vectors in the set of extreme points $E_N \cup F_N$. If $L < N \le 2L$ then

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 \le \left(\frac{N}{N-L}\right)^{N-L}. \tag{4-31}$$

Proof. We argue by induction on the positive integer L. If $L = 1$ then $N = 2$ and the result is trivial to check. Next we assume that $2 \le L$, and we assume that (4-31) holds for all pairs (L', N') such that $L' < N' \le 2L'$ and $1 \le L' < L$.

If the extreme points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ all belong to the set of extreme points E_N, then

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 = 1$$

and the inequality (4-31) is trivial. If the extreme points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ all belong to the set of extreme points F_N, then the inequality (4-31) follows from Lemma 4.6. To complete the proof we assume that K of the extreme points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ belong to E_N and $L-K$ extreme points $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_L$ belong to F_N, where $1 \le K < L$. In this case the set of extreme points satisfies the hypotheses of Lemma 3.7. It follows from the conclusion of Lemma 3.7 that there exist linearly independent extreme points $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \ldots, \boldsymbol{\eta}_{L-K}$ in the set $E_{N-K} \cup F_{N-K}$ such that

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 = \|\boldsymbol{\eta}_1 \wedge \boldsymbol{\eta}_2 \wedge \cdots \wedge \boldsymbol{\eta}_{L-K}\|_1. \tag{4-32}$$

We write $L' = L - K$, $N' = N - K$, and we consider two cases. First we suppose that

$$N' \le 2L'.$$

In this case we apply the inductive hypothesis and conclude that

$$\|\boldsymbol{\eta}_1 \wedge \boldsymbol{\eta}_2 \wedge \cdots \wedge \boldsymbol{\eta}_{L-K}\|_1 \le \left(\frac{N'}{N'-L'}\right)^{N'-L'} = \left(\frac{N-K}{N-L}\right)^{N-L} < \left(\frac{N}{N-L}\right)^{N-L}. \tag{4-33}$$

Next we suppose that

$$2L' \le N'.$$

In this case we appeal to the inequality (3-27) which we have already proved. By that result we have

$$\begin{aligned}\|\boldsymbol{\eta}_1 \wedge \boldsymbol{\eta}_2 \wedge \cdots \wedge \boldsymbol{\eta}_{L-K}\|_1 \le 2^{L'} &= \min\left\{2^{L'}, \left(\frac{N'}{N'-L'}\right)^{N'-L'}\right\} \\ &\le \left(\frac{N'}{N'-L'}\right)^{N'-L'} = \left(\frac{N-K}{N-L}\right)^{N-L} < \left(\frac{N}{N-L}\right)^{N-L}.\end{aligned} \tag{4-34}$$

Combining (4-32), (4-33), and (4-34), establishes the inequality

$$\|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_L\|_1 \le \left(\frac{N}{N-L}\right)^{N-L}$$

whenever $L < N \le 2L$. This proves the lemma. □

If $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_L$ belong to $\mathbb{R}^N$ and $L < N \le 2L$, then it follows from (4-31) that

$$\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1 \le \left(\frac{N}{N-L}\right)^{N-L} \delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\cdots\delta(\boldsymbol{x}_L).$$

This proves the inequality (3-2), and so completes the proof of Theorem 3.1.

5. Proof of Theorem 1.1

We apply Theorem 3.1 with $N = r + 1$ and $L = q$, and we apply the theorem to the collection of linearly independent points $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_q$ in

$$\Gamma_S(k) \subseteq \mathcal{D}_r \subseteq \mathbb{R}^{r+1}.$$

From (3-4) we find that

$$\begin{aligned}\|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1 &\le \min\left\{2^q, \left(\frac{r+1}{r+1-q}\right)^{r+1-q}\right\} \delta(\boldsymbol{\alpha}_1)\delta(\boldsymbol{\alpha}_2)\cdots\delta(\boldsymbol{\alpha}_q) \\ &= C(r,q)\,\delta(\boldsymbol{\alpha}_1)\delta(\boldsymbol{\alpha}_2)\cdots\delta(\boldsymbol{\alpha}_q).\end{aligned} \tag{5-1}$$

By the product formula the points $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_q$ belong to the diagonal subspace $\mathcal{D}_r$. Therefore we get

$$\delta(\boldsymbol{\alpha}_j) = \tfrac{1}{2}\|\boldsymbol{\alpha}_j\|_1 \quad \text{for each } j = 1, 2, \ldots, q. \tag{5-2}$$

Combining (5-1) and (5-2) establishes the inequality (1-11).

6. Proof of Theorem 1.2

Let $1 \leq L < N$ and let

$$X = (\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \cdots \;\; \boldsymbol{x}_L)$$

be an $N \times L$ real matrix with columns $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_L$. We assume that the columns of X are $\mathbb{R}$-linearly independent so that rank $X = L$ and

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L \neq \boldsymbol{0}.$$

We use the matrix X to define a norm on $\mathbb{R}^L$ by

$$\boldsymbol{y} \mapsto \|X\boldsymbol{y}\|_1. \tag{6-1}$$

The unit ball associated to the norm (6-1) is obviously the set

$$B_X = \{\boldsymbol{y} \in \mathbb{R}^L : \|X\boldsymbol{y}\|_1 \leq 1\}.$$

It is not difficult to show that the dual unit ball is

$$B_X^* = \{X^T\boldsymbol{w} : \boldsymbol{w} \in \mathbb{R}^N \text{ and } \|\boldsymbol{w}\|_\infty \leq 1\}.$$

It can be shown (see [Bolker 1969; Schneider and Weil 1983] or, for a more general result, [Vaaler 2014, Lemma 2]) that the dual unit ball B_X^* is an example of a zonoid. Therefore by an inequality of S. Reisner [1985, Theorem 2], we have

$$\frac{4^L}{L!} \leq \mathrm{Vol}_L(B_X)\,\mathrm{Vol}_L(B_X^*). \tag{6-2}$$

An identity for the L-dimensional volume of B_X^* was established by P. McMullen [1984] and C. G. Shephard [1974, equation (57)]. These results assert that

$$\mathrm{Vol}_L(B_X^*) = 2^L \sum_{|I|=L} |\det X_I| = 2^L \|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1. \tag{6-3}$$

By combining Reisner's inequality (6-2) and the volume formula (6-3), we obtain the lower bound

$$\frac{2^L}{L!} \leq \mathrm{Vol}_L(B_X)\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1. \tag{6-4}$$

Now let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_L < \infty$$

be the successive minima for the convex symmetric set B_X and the integer lattice $\mathbb{Z}^L$. By Minkowski's theorem on successive minima (see [Cassels 1959, Section VIII.4.3]) we have

$$\mathrm{Vol}_L(B_X)\lambda_1\lambda_2 \cdots \lambda_L \leq 2^L. \tag{6-5}$$

We combine the lower bound (6-4) and the upper bound (6-5), and obtain the inequality

$$\lambda_1\lambda_2 \cdots \lambda_L \leq L!\,\|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1. \tag{6-6}$$

This leads to the following general result.

Theorem 6.1. *Let $\mathcal{X} \subseteq \mathbb{R}^N$ be the free group of rank L generated by the linearly independent vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_L$. Then there exist linearly independent points $\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_L$ in $\mathcal{X}$ such that*

$$\|\boldsymbol{y}_1\|_1 \|\boldsymbol{y}_2\|_1 \cdots \|\boldsymbol{y}_L\|_1 \le L!\, \|\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_L\|_1. \tag{6-7}$$

If $\mathcal{Y} \subseteq \mathcal{X}$ is the subgroup generated by the points $\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_L$, then $[\mathcal{X} : \mathcal{Y}] \le L!$.

Proof. By Minkowski's theorem on successive minima there exist linearly independent points $\boldsymbol{m}_1, \boldsymbol{m}_2, \dots, \boldsymbol{m}_L$ in the integer lattice $\mathbb{Z}^L$ such that

$$\|X\boldsymbol{m}_\ell\|_1 = \lambda_\ell \quad \text{for } \ell = 1, 2, \dots, L. \tag{6-8}$$

As rank $X = L$ the points

$$\{X\boldsymbol{m}_\ell : \ell = 1, 2, \dots, L\}$$

are linearly independent points in the free abelian group $\mathcal{X}$. We write $\boldsymbol{y}_\ell = X\boldsymbol{m}_\ell$ for each $\ell = 1, 2, \dots, L$. Then (6-7) follows from (6-6) and (6-8). The bound $[\mathcal{X} : \mathcal{Y}] \le L!$ also follows from Minkowski's theorem. □

Now let $L = q$, $N = r + 1$ and let $\mathfrak{A} \subseteq \mathbb{R}^{r+1}$ be the subgroup of rank q generated by the linearly independent vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_q$. By Theorem 6.1 there exist linearly independent vectors $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_q$ in $\mathfrak{A}$ such that

$$\|\boldsymbol{\beta}_1\|_1 \|\boldsymbol{\beta}_2\|_1 \cdots \|\boldsymbol{\beta}_q\|_1 \le q!\, \|\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \cdots \wedge \boldsymbol{\alpha}_q\|_1.$$

Moreover, the free group $\mathfrak{B} \subseteq \mathfrak{A}$ generated by the vectors $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_q$ has rank q and index

$$[\mathfrak{A} : \mathfrak{B}] \le q!.$$

This proves Theorem 1.2.

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