A family of permutationally invariant quantum codes

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***Abstract*—We construct a new family of permutationally invariant codes that correct t Pauli errors for any $t \geqslant 1$. We also show that codes in the new family correct quantum deletion errors as well as spontaneous decay errors. Our construction contains some of the previously known permutationally invariant quantum codes as particular cases. In many cases, the codes in the new family are shorter than the best previously known explicit permutationally invariant codes for Pauli errors and deletions. This is an extended abstract of the preprint [1].**

1. Introduction

Quantum error correction is one of the essential components of quantum computing that aims to protect quantum information from errors caused by quantum noise, such as decoherence. Mapping a quantum state to be protected into a higher-dimensional Hilbert space of the physical system is a significant part of quantum error correction. A subspace of the Hilbert space of a physical system is called a quantum code for a given type of errors if it satisfies certain specific conditions for error correction [10]. In many applications, it is desirable to construct quantum codes that lie within the ground space of the system. Motivated by this goal, in this paper, we study permutation-invariant quantum codes whose codewords form ground states of the ferromagnetic Heisenberg model.

1.1 Why permutation-invariant codes?

Recall that Heisenberg's model characterizes interactions between spins in the system. Two spin-$1/2$ particles are coupled by an interaction given by $\boldsymbol{S}_i\boldsymbol{S}_j$, where $\boldsymbol{S}_i$ and $\boldsymbol{S}_j$ are the spin operators for the particles i and j, respectively. In the absence of the external magnetic field, the Heisenberg ferromagnetic model is described by the Hamiltonian $\hat{H}$ that can be written in the form $\hat{H} = -2\sum_{i<j} J_{ij}\boldsymbol{S}_i\boldsymbol{S}_j$, where $J_{ij} > 0$ is the exchange (coupling) constant between particles i and j in the system; see [2, Ch. 1,4] for a detailed discussion of this model. The swap operator $\boldsymbol{P}_{ij}$ exchanges spins i and j, essentially swapping the spin-$\frac{1}{2}$ particles i and j, e.g., $\boldsymbol{P}_{12}|\uparrow\downarrow\uparrow\uparrow\rangle = |\downarrow\uparrow\uparrow\uparrow\rangle$. The Hamiltonian of the Heisenberg model can be written in terms of the swap operators in the following way:

$$\hat{H} = -\sum_{i<j} J_{ij}\left(\boldsymbol{P}_{ij} - \frac{1}{2}\boldsymbol{I}\right).$$

A state $|\psi\rangle$ is called *permutation-invariant* if it is preserved by all swap operators $\boldsymbol{P}_{ij}$, i.e., $|\psi\rangle$ is a common eigenstate of the swap operators with eigenvalue 1. Denoting $J = \sum_{i<j} J_{ij}$, we observe that for any permutation-invariant state $|\psi\rangle$,

$$\left(\hat{H} - \frac{J}{2}\boldsymbol{I}\right)|\psi\rangle = -\sum_{i<j} J_{ij}\boldsymbol{P}_{ij}|\psi\rangle = -J|\psi\rangle.$$

Since $J_{ij} > 0$, the spectral norm of $\hat{H} - \frac{J}{2}\boldsymbol{I}$ is bounded above by J, so the smallest eigenvalue of the Hamiltonian is $-J/2$, and its corresponding eigenstate is $|\psi\rangle$. Therefore, any permutation-invariant state is a ground state in the ferromagnetic Heisenberg model [16].

1.2 Earlier work

Permutation-invariant codes were introduced in the works of Ruskai and Pollatsek [21], [23]. The codes they constructed encode a single logical qubit and are capable of correcting all one-qubit errors and certain types of two-qubit errors. Generalizing this construction, Ouyang [16] found a family of permutation-invariant codes that correct t arbitrary errors and t spontaneous decay errors. The family is parameterized by integers g, n, and u (hence the name "*gnu* codes"), and the shortest t-error-correcting codes in it are of length $(2t+1)^2$. Ouyang subsequently showed that permutation-invariant codes are capable of supporting reliable quantum storage, quantum sensing, and decoherence-free communication [18]–[20].

The connection between permutation-invariant codes and quantum deletions, along with the definition of the quantum deletion channel, was developed in the works of Nakayama and Hagiwara [8], [14], [15]. They also observed that permutation-invariant codes are capable of correcting deletion errors, and constructed single-deletion-correcting codes. Subsequently, works [17], [24] showed that Ouyang's *gnu* codes can correct t deletions. In particular, the shortest known code to correct t deletions comes from this family, and it has length $(t+1)^2$.

Correcting deletions is an established research area in classical coding theory, see [7] and [3] for recent overviews. In quantum coding theory, a deletion can be modeled as a partial trace operation where the traced-out qubits are unknown. It turns out that the performance of permutation-invariant codes for correcting errors or deletions is sometimes amenable to analysis. Focusing on this code family, we study the error correction (Knill–Laflamme) conditions for general permutation-invariant codes. Using deletion correction as motivation, we propose a new family of permutation-invariant codes defined by the parameters g, m, δ, and ϵ. The shortest codes in this

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family have length $(2t+1)^2-2t$ and can correct all t patterns of qubit errors and $2t$ deletion errors. The shortest t-error-correcting permutation-invariant codes known previously are due to Ouyang and require $2t$ more physical qubits than the codes that we propose. Specializing our construction to $t=1$, we observe that the length of our code is the same as the Pollatsek–Ruskai's $((7,2,3))$ permutation-invariant code [21], although the two codes are different.

In Sections 2 and 3 we collect the necessary definitions and some basic facts about quantum deletions. In particular, in Sec. 3 we recall the definition of deletion operators [25] and prove some of their properties. Sec. 4 contains a detailed form of the error-correcting conditions for permutation-invariant codes. Our main result (the new code family) is presented in Sec. 5. Details and proofs omitted from this extended abstract can be found in [1].

2. Preliminaries

Throughout this paper, we use the following notation. Let $|\Psi\rangle = |\psi_1\psi_2,\ldots,\psi_n\rangle \subset \mathbb{C}^{2\otimes n}$ be a pure state, where $\mathbb{C}^{2\otimes n}$ is a shorthand for $(\mathbb{C}^2)^{\otimes n}$, and we assume that $\langle\psi_i\,|\,\psi_i\rangle = 1$ for all $i=1,2,\ldots,n$. A general quantum state is identified with its density matrix, i.e., a positive semidefinite Hermitian matrix of trace 1. The density matrix of a pure state is simply $\rho = |\psi\rangle\langle\psi|$. For a collection of pure states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ such that $\Pr(|\psi_i\rangle) = p_i$ for all i and $\sum_i p_i = 1$, the density matrix is defined as $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Denote by $S(\mathbb{C}^{2\otimes n})$ the set of all density matrices of order 2^n.

Definition 2.1: Consider an $n\times n$ matrix

$$A = \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} a_{\boldsymbol{x},\boldsymbol{y}} |\boldsymbol{x}\rangle\langle\boldsymbol{y}|,$$

where $a_{\boldsymbol{x},\boldsymbol{y}} \in \mathbb{C}$. For an integer $i \in \{1,2,\ldots,n\}$, the *partial trace* of A is the mapping

$$\begin{aligned}\mathrm{Tr}_i : S(\mathbb{C}^{2\otimes n}) &\to S(\mathbb{C}^{2\otimes(n-1)})\\ A &\mapsto \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} a_{\boldsymbol{x},\boldsymbol{y}} \,\mathrm{Tr}(|x_i\rangle\langle y_i|)|\boldsymbol{x}'\rangle\langle\boldsymbol{y}'|,\end{aligned}$$

where $|\boldsymbol{x}'\rangle = |x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n\rangle$ and $|\boldsymbol{y}'\rangle = |y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n\rangle$.
Permutation-invariant quantum states are conveniently described in terms of Dicke states [5], [9], [13].

Definition 2.2: A *Dicke state* $|D_w^n\rangle$ is a linear combination of all qubit states of length n and "Hamming weight" w, i.e.

$$|D_w^n\rangle = \frac{1}{\sqrt{\binom{n}{w}}} \sum_{\substack{\boldsymbol{x}\in\{0,1\}^n\\ |\boldsymbol{x}|=w}} |\boldsymbol{x}\rangle.$$

Sometimes we also use unnormalized Dicke states given by $|H_w^n\rangle = \sqrt{\binom{n}{w}}|D_w^n\rangle$. Note that $\langle D_i^n\,|\,D_j^n\rangle = \delta_{ij}$.
For spin-$\frac{1}{2}$ particles, a Dicke state $|D_n^w\rangle$ can be viewed as a superposition of the tensor product of states of an n-particle system in which w particles are in the spin-up, and $n-w$ are in the spin-down configuration; for instance,

$$|D_1^3\rangle = \frac{|001\rangle+|010\rangle+|100\rangle}{\sqrt{3}} = \frac{|\downarrow\downarrow\uparrow\rangle+|\downarrow\uparrow\downarrow\rangle+|\uparrow\downarrow\downarrow\rangle}{\sqrt{3}}.$$

A quantum code $\mathscr{C}$ maps a 2^k-dimensional Hilbert space into a subspace of the 2^n-dimensional Hilbert space $\mathbb{C}^{2\otimes n}$, i.e., it encodes k logical qubits into n physical qubits. Throughout this paper, we will be dealing with two-dimensional codes and denote their basis codewords by $|\boldsymbol{c}_0\rangle$ and $|\boldsymbol{c}_1\rangle$.

The following definition originates with [21].

Definition 2.3: A permutation-invariant code is a linear subspace with basis vectors of the form

$$|\boldsymbol{c}_0\rangle = \sum_{j=0}^{n} \alpha_j |D_j^n\rangle \quad\text{and}\quad |\boldsymbol{c}_1\rangle = \sum_{j=0}^{n} \beta_j |D_j^n\rangle, \tag{1}$$

where $\alpha_j,\beta_j \in \mathbb{C}, j=0,1,\ldots,n$ and $\sum_j \bar{\alpha}_j\beta_j = 0$.

2.1 Kraus Operators and the Knill–Laflamme conditions

A *quantum channel* $\mathcal{A}$ is a linear operator acting on density matrices such that it admits the Kraus decomposition

$$\mathcal{A}(\rho) = \sum_{\boldsymbol{A}\in\mathfrak{K}_{\mathcal{A}}} \boldsymbol{A}\rho\boldsymbol{A}^\dagger,$$

where $\sum_{\mathfrak{K}_{\mathcal{A}}} \boldsymbol{A}\boldsymbol{A}^\dagger = \boldsymbol{I}$ and $\mathfrak{K}_{\mathcal{A}}$ is the Kraus set of the channel. Elements of this set are called *Kraus operators*.

The necessary and sufficient conditions for the quantum error correction were formulated by Knill and Laflamme [10].

*Theorem 2.1 (*Knill–Laflamme conditions*):* Let $\mathscr{C}$ be a quantum code with an orthonormal basis $|\boldsymbol{c}_0\rangle, |\boldsymbol{c}_1\rangle, \ldots, |\boldsymbol{c}_{k-1}\rangle$, and let $\mathcal{A}$ be a quantum channel with Kraus operators $\boldsymbol{A}_i$. There exists a quantum recovery operator $\mathcal{R}$ such that $\mathcal{R}(\mathcal{A}(\rho)) = \rho$ for every density matrix supported on $\mathscr{C}$ if and only if for every a,b,

$$\langle\boldsymbol{c}_i|\boldsymbol{A}_a^\dagger\boldsymbol{A}_b|\boldsymbol{c}_j\rangle = 0 \quad \text{for all } i\neq j\ , \tag{2}$$

$$\langle\boldsymbol{c}_i|\boldsymbol{A}_a^\dagger\boldsymbol{A}_b|\boldsymbol{c}_i\rangle = g_{ab} \quad \text{for all } i=0,1,\ldots,k-1, \tag{3}$$

for some constants $g_{ab}\in\mathbb{C}$.

3. Quantum Deletion Channel

Definition 3.1: [25] (t-Deletion channel) Let $t \in \{1,2,\ldots,n\}$ and let $\rho \in S(\mathbb{C}^{2\otimes n})$ be a quantum state. For a set $E = \{e_1,e_2\ \ldots,e_t\} \subset \{1,2,\ldots,n\}$ with $e_1 < e_2 < \ldots < e_t$, define a map $D_E : S(\mathbb{C}^{2\otimes n}) \to S(\mathbb{C}^{2\otimes(n-t)})$ as

$$D_E^n(\rho) := \mathrm{Tr}_{e_1}\circ\cdots\circ\mathrm{Tr}_{e_t}(\rho).$$

The action of D_E deletes the qubits in locations contained in E, and is called a t-deletion error. A *t-deletion channel* Del_t^n is a convex combination of all t-deletion errors, where t is a fixed integer with $|E| = t$, i.e.,

$$\mathrm{Del}_t^n(\rho) = \sum_{E:|E|=t} p(E) D_E^n(\rho),$$

where $p(E)$ is a probability distribution.

Let $n \geqslant t \geqslant 1$ be integers. Define the set $E = \{e_1,e_2,\ldots,e_t\} \subset \{1,2,\ldots n\}$ with $e_1 < e_2 < \ldots < e_t$. Let $|\boldsymbol{c}\rangle = |c_1c_2\ldots c_t\rangle$ be a pure quantum state with $\boldsymbol{c}\in\{0,1\}^t$. Define the operator $\boldsymbol{A}_{E,\langle c|}^n = A_1\otimes A_2\otimes\ldots\otimes A_n$ [25], where

$$A_j = \begin{cases}\langle c_i| & j = e_i \in E,\\ \boldsymbol{I} & j\notin E.\end{cases}$$

Lemma 3.1 ([25], Lemma 3.1): Let $t\geqslant 1, n\geqslant t$ be integers. Define the set $E = \{e_1,e_2,\ldots,e_t\} \subset \{1,2,\ldots n\}$ with $e_1 <$

$e_2 < \ldots < e_t$. Let $|\boldsymbol{c}\rangle = |c_1 c_2 \ldots c_t\rangle$ be a pure quantum state with $(c_1 c_2 \ldots c_t) \in \{0,1\}^t$. Then,

$$\boldsymbol{A}^n_{E,\langle \boldsymbol{c}|} = \boldsymbol{A}^{n-t+1}_{e_1,\langle c_1|} \boldsymbol{A}^{n-t+2}_{e_2,\langle c_2|} \ldots \boldsymbol{A}^{n-1}_{e_{t-1},\langle c_{t-1}|} \boldsymbol{A}^{n}_{e_t,\langle c_t|}.$$

Lemma 3.1 can be easily proved by direct calculations. We are now in a position to describe the Kraus decomposition of the deletion channel. First, we cite another auxiliary result.

Lemma 3.2 ([25], Lemma 4.2): Let $\rho \in S\left(\mathbb{C}^{2\otimes n}\right)$ be a quantum state. The output state after deleting the qubits on the positions labeled by the set $E \subset \{1,2,\ldots,n\}$ can be expressed as

$$D^n_E(\rho) = \sum_{\boldsymbol{c}\in\{0,1\}^t} \boldsymbol{A}^n_{E,\langle \boldsymbol{c}|} \rho \boldsymbol{A}^{n^\dagger}_{E,\langle \boldsymbol{c}|}.$$

Lemma 3.2 together with Definition 3.1 imply that the Kraus decomposition of the quantum t-deletion channel is given by

$$\mathrm{Del}^N_t(\rho) = \sum_{E,\boldsymbol{c}} p(E) \boldsymbol{A}^n_{E,\langle \boldsymbol{c}|} \rho \boldsymbol{A}^n_{E,\langle \boldsymbol{c}|}{}^\dagger,$$

where $p(E)$ is a probability distribution.

It will be convenient to distinguish between two types of deletions.

Definition 3.2: (Deletion operators) A 0-*type deletion* applied to the i-th qubit is the operator $\boldsymbol{F}_i := \boldsymbol{A}^n_{i,\langle 0|}$. Likewise, a 1-*type deletion* is the operator $\boldsymbol{G}_i := \boldsymbol{A}^n_{i,\langle 1|}$. In other words, given $\boldsymbol{x} \in \{0,1\}^n$, the action of these operators on the state $|\boldsymbol{x}\rangle$ is

$$\boldsymbol{F}_i|\boldsymbol{x}\rangle = \langle 0\,|\,x_i\rangle|\boldsymbol{x}'\rangle \quad \text{and} \quad \boldsymbol{G}_i|\boldsymbol{x}\rangle = \langle 1\,|\,x_i\rangle|\boldsymbol{x}'\rangle,$$

where $|\boldsymbol{x}'\rangle = |x_1 \ldots x_{i-1} x_{i+1} \ldots x_n\rangle$.

In the next lemma, we show how these deletion operators transform Dicke states.

Lemma 3.3: Let $|D^n_w\rangle$ be a Dicke state and let $a \in \{0,1,\ldots,n\}$. Then for any $i = 1,\ldots,n$

$$(\boldsymbol{F}_i)^a |D^n_w\rangle = \sqrt{\frac{\binom{n-a}{w}}{\binom{n}{w}}}\, |D^{n-a}_w\rangle$$

$$(\boldsymbol{G}_i)^a |D^n_w\rangle = \sqrt{\frac{\binom{n-a}{w-a}}{\binom{n}{w}}}\, |D^{n-a}_{w-a}\rangle,$$

where by definition $\binom{n}{k} = 0$ if $n < k$ or $k < 0$.

Proof: Acting by $\boldsymbol{F}_i$ on the state $|H^n_w\rangle = \sum_{\boldsymbol{x}:|\boldsymbol{x}|=w} |\boldsymbol{x}\rangle$ annihilates the terms $|\boldsymbol{x}\rangle$ with $x_i = 1$ and deletes one zero from the states $|\boldsymbol{x}\rangle$ with $x_i = 0$. Thus, the only retained states are those with $x_i = 0$, and $\boldsymbol{F}_i|H^n_w\rangle = |H^{n-1}_w\rangle$. Likewise, $\boldsymbol{G}_i|H^n_w\rangle = |H^{n-1}_{w-1}\rangle$, $i = 1,\ldots,n$. The full claim now follows by induction. ∎

By the nature of permutation-invariant states, the statements we make below in the paper do not depend on the location of the deleted qubit, and we write 0-type and 1-type deletions simply as $\boldsymbol{F}, \boldsymbol{G}$, omitting the subscript i from the notation.

Lemmas 3.1, 3.2, and 3.3 imply that the Kraus set of the t-deletion channel for a permutation-invariant code has the form $\{\boldsymbol{G}^i \boldsymbol{F}^{t-i} : i \in \{0,1,\ldots,t\}\}$. Throughout the paper, we will denote the t-deletion error set for a permutation-invariant code as $\varepsilon_t = \{\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_t\}$, where $\mathbb{E}_i = \boldsymbol{G}^i \boldsymbol{F}^{t-i}$. The following lemma describes the action of the error operator $\mathbb{E}_a \in \varepsilon_t$ on a permutation-invariant state.

Lemma 3.4: Let $\mathbb{E}_a \in \varepsilon_t$. For any permutation-invariant state $|D^n_w\rangle$,

$$\mathbb{E}_a |D^n_w\rangle = \sqrt{\frac{\binom{n-t}{w-a}}{\binom{n}{w}}}\, |D^{n-t}_{w-a}\rangle.$$

Lemma 3.4 can be easily shown by direct calculations using Lemma 3.3.

We make an important observation concerning correction of Pauli errors. Upon applying a deletion error to a permutation-invariant state, we obtain a permutation-invariant state (on fewer qubits). Clearly, this does not hold for Pauli errors. Indeed, generally $X_i|D^n_w\rangle \neq X_j|D^n_w\rangle$ if $i \neq j$, so the Kraus set for Pauli errors is much larger than for deletions, complicating the analysis. A workaround proposed in [21] suggests averaging Pauli errors, but general constructions look difficult. At the same time, invariance with respect to permutations plays the defining role for deletions, and it is also the main property supporting the code construction we propose. Further, given a deletion-correcting permutation-invariant code, we can argue about its distance and make claims about its properties with respect to correcting Pauli errors. Indeed, the following proposition is true.

Proposition 3.5: A permutation-invariant code that corrects $2t$ deletions also corrects all combinations of t Pauli errors.

Proof: For a permutation-invariant state, deleting any $2t$ qubits is equivalent to deleting the first $2t$ qubits in an n-qubit state, so deletions are equivalent to erasures. Of course, a code that corrects $2t$ erasures has quantum distance of at least $2t+1$ (see, e.g., [22]). Thus, correcting deletions is tied to the code distance, and distance $d = 2t+1$ is a sufficient condition for correcting t qubit errors. ∎

4. Error Correction Conditions for Permutation-Invariant Codes

Sufficient conditions for any code to correct deletions were previously derived in [24]. In this section, we focus on permutation-invariant codes and derive the necessary and sufficient conditions for such a code to correct deletions by showing the equivalence between them and the Knill–Laflamme conditions for the $2t$-deletion channel. By Proposition 3.5, this also implies that they correct t qubit errors.

Theorem 4.1: [1] Let $\mathscr{C}$ be a permutation-invariant quantum error correction code as given in Definition 2.3, and suppose that the coefficients α_j and β_j, $j = 1,\ldots,n$ in the codewords (1) are real. Then the code $\mathscr{C}$ corrects all t-qubit errors if and only if its coefficient vectors $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_n)$ satisfy the following conditions:

(C1) $\displaystyle\sum_{j=0}^{n} \alpha_j \beta_j = 0;$

(C2) $\displaystyle\sum_{j=0}^{n} \alpha_j^2 = \sum_{j=0}^{n} \beta_j^2 = 1;$

(C3) For all $0 \leqslant a, b \leqslant 2t$,

$$\sum_{j=0}^{n-2t} \frac{\binom{n-2t}{j}}{\sqrt{\binom{n}{j+a}\binom{n}{j+b}}} \alpha_{j+a}\beta_{j+b} = 0;$$

(C4) For all $0 \leqslant a, b \leqslant 2t$,

$$\sum_{j=0}^{n-2t} \frac{\binom{n-2t}{j}}{\sqrt{\binom{n}{j+a}\binom{n}{j+b}}} (\alpha_{j+a}\alpha_{j+b} - \beta_{j+a}\beta_{j+b}) = 0.$$

Proof outline: Since the Dicke states are orthonormal by construction, conditions (C1), (C2) are required for the codewords $|\boldsymbol{c}_0\rangle, |\boldsymbol{c}_1\rangle$ to form orthonormal states. It can be shown that conditions (C3) and (C4) are equivalent to the Knill–Laflamme conditions for the $2t$-deletion channel. By Proposition 3.5 this suffices to prove the theorem. ■

Example 1: Ouyang's codes [16] with parameters (g, m, u) can be defined via the logical computational basis

$$|\boldsymbol{c}_0\rangle = \sum_{\substack{l \text{ even} \\ 0 \leqslant l \leqslant m}} \sqrt{\frac{\binom{m}{l}}{2^{m-1}}} |D^n_{gl}\rangle, \quad |\boldsymbol{c}_1\rangle = \sum_{\substack{l \text{ odd} \\ 0 \leqslant l \leqslant m}} \sqrt{\frac{\binom{m}{l}}{2^{m-1}}} |D^n_{gl}\rangle,$$

where $n = gmu$ is the code length. Consider a $(2t+1, 2t+1, 1)$ code from this family. Its coefficient vectors trivially satisfy conditions (C1)-(C3) because of the choice of the gap parameter $g = 2t+1$, and condition (C4) turns into

$$\sum_{l=0}^{m} (-1)^l \binom{m}{l} \frac{\binom{n-2t}{gl-a}}{\binom{n}{gl}},$$

which is zero for all $0 \leqslant a \leqslant 2t$ (see Lemmas 1 and 2 in [16]). Coupled with Theorem 4.1, this shows that this code corrects t qubit errors, recovering one of the results in [16].

5. A NEW FAMILY OF PERMUTATION-INVARIANT CODES

In this section, we present a new family of permutation-invariant codes, defined by the parameters g, m, δ, and ϵ. The code we construct encodes one logical qubit into $n = 2gm + \delta + 1$ physical qubits. The following combinatorial identity, proved in [1, Appendix], plays a role in the construction.

Lemma 5.1: Let n, g, m, a, r be integers such that $g > 0$ and $0 \leqslant a \leqslant r \leqslant 2m < n/g$. Then

$$\sum_{l=0}^{m} (-1)^l \frac{\binom{m}{l}}{\binom{n/g-l}{m+1}} \left[\frac{\binom{n-r}{gl-a}}{\binom{n}{gl}} - \frac{\binom{n-r}{gl-r+a}}{\binom{n}{gl}} \right] = 0. \tag{4}$$

Construction 5.1: Let g, m, δ be nonnegative integers, and let $\epsilon \in \{-1, +1\}$. Define a permutation-invariant code $\mathcal{Q}_{g,m,\delta,\epsilon}$ via its logical computational basis

$$|\boldsymbol{c}_0\rangle = \sum_{\substack{l \text{ even} \\ 0 \leqslant l \leqslant m}} \gamma b_l |D^n_{gl}\rangle + \sum_{\substack{l \text{ odd} \\ 0 \leqslant l \leqslant m}} \gamma b_l |D^n_{n-gl}\rangle,$$

$$|\boldsymbol{c}_1\rangle = \sum_{\substack{l \text{ odd} \\ 0 \leqslant l \leqslant m}} \gamma b_l |D^n_{gl}\rangle + \epsilon \sum_{\substack{l \text{ even} \\ 0 \leqslant l \leqslant m}} \gamma b_l |D^n_{n-gl}\rangle,$$

where $n = 2gm + \delta + 1$, $b_l = \sqrt{\binom{m}{l} / \binom{n/g-l}{m+1}}$, and $\gamma = \sqrt{\binom{n/(2g)}{m} \frac{n-2gm}{g(m+1)}}$ is the normalizing factor.

The next theorem establishes the error correction properties of the code $\mathcal{Q}_{g,m,\delta,\epsilon}$.

Theorem 5.2: [1] Let t be a nonnegative integer and let $m \geqslant t$ and $\delta \geqslant 2t$. If

$$(g \geqslant 2t, \epsilon = -1) \text{ or } (g \geqslant 2t+1, \epsilon = +1),$$

then the code $\mathcal{Q}_{m,l,\delta,\epsilon}$ encodes one qubit into $n = 2gm + \delta + 1$ qubits and corrects any t qubit errors.

Proof outline: We need to prove that the coefficients $\boldsymbol{\alpha}, \boldsymbol{\beta}$ of the basis states satisfy conditions (C1)-(C4). Writing these coefficients for the code $\mathcal{Q}_{m,l,\delta,-}$ explicitly, we obtain

$$\alpha_j = \sum_{\substack{l \text{ even} \\ 0 \leqslant l \leqslant m}} f(l)\delta_{j,gl} + \sum_{\substack{l \text{ odd} \\ 0 \leqslant l \leqslant m}} f(l)\delta_{j,n-gl},$$

$$\beta_j = \sum_{\substack{l \text{ odd} \\ 0 \leqslant l \leqslant m}} f(l)\delta_{j,gl} - \sum_{\substack{l \text{ even} \\ 0 \leqslant l \leqslant m}} f(l)\delta_{j,n-gl},$$

where $f(l) = \gamma b_l$, and $\delta_{\cdot,\cdot}$ is the Kronecker delta. Condition (C1) is trivially satisfied owing to the choice of the gap parameter g. Condition (C2) turns into $\gamma^2 \sum_{l=0}^{m} b_l^2$, which can be shown to be 1, so it also is satisfied. Condition (C3) holds since all the terms with positive signs also appear with the same magnitude and the negative sign, which makes the sum in condition (C3) zero. Finally, condition (C4) transforms into the combinatorial identity

$$\gamma^2 \sum_{l=0}^{m} (-1)^l \frac{\binom{m}{l}}{\binom{n/g-l}{m+1}} \left(\frac{\binom{n-2t}{gl-a}}{\binom{n}{gl}} - \frac{\binom{n-2t}{gl-2t+a}}{\binom{n}{gl}} \right),$$

which is zero by Lemma 5.1. Following the same sequence of steps, it is possible to show the error correction property of the code $\mathcal{Q}_{m,l,\delta,+}$. ■

Example 2: The permutation-invariant code $\mathcal{Q}_{2,1,2,-}$ with logical codewords

$$\left.\begin{aligned} |\boldsymbol{c}_0\rangle &= \sqrt{\frac{3}{10}} |D^7_0\rangle + \sqrt{\frac{7}{10}} |D^7_5\rangle \\ |\boldsymbol{c}_1\rangle &= \sqrt{\frac{7}{10}} |D^7_2\rangle - \sqrt{\frac{3}{10}} |D^7_7\rangle. \end{aligned}\right\} \tag{5}$$

has the same length as the 7-qubit permutation-invariant code of [21], and it can correct a single error.

Example 3: The permutation-invariant code $\mathcal{Q}_{4,2,4,-}$ with logical codewords

$$|\boldsymbol{c}_0\rangle = \sqrt{\frac{5}{68}} |D^{21}_0\rangle + \sqrt{\frac{7}{12}} |D^{21}_8\rangle + \sqrt{\frac{35}{102}} |D^{21}_{17}\rangle,$$

$$|\boldsymbol{c}_1\rangle = \sqrt{\frac{35}{102}} |D^{21}_4\rangle - \sqrt{\frac{7}{12}} |D^{21}_{13}\rangle - \sqrt{\frac{5}{68}} |D^{21}_{21}\rangle$$

has length 21 and is shorter than all currently known explicit families permutation-invariant codes that correct double errors.

Note that the permutation-invariant code $\mathcal{Q}_{2t,t,2t,-}$ with length $(2t+1)^2 - 2t$ corrects arbitrary t qubit errors, and it has the best code parameters among all the previously known permutation-invariant codes that can correct t qubit errors. The following proposition describes the relation between our code family and Ouyang's *gnu* codes [16].

Proposition 5.3: For all odd integers $m > 0$ and for all integers $g > 0$, the code $\mathcal{Q}_{g,\frac{m-1}{2},g-1,+}$ coincides with Ouyang's *gnu* code with parameters $(g, m, 1)$.

5.1 Deletion Correction Property

We already know that codes $\mathcal{Q}_{g,m,\delta,\epsilon}$ correct deletion. A precise formulation of this claim is given in the following proposition.

Proposition 5.4: If $m \geqslant \lceil \frac{s}{2} \rceil, \delta \geqslant s$ and

$$(g \geqslant s, \epsilon = -1) \text{ or } (g \geqslant s+1, \epsilon = +1),$$

then the code $\mathcal{Q}_{g,m,\delta,\epsilon}$ corrects all patters of s deletions.

This claim follows from the fact that conditions (C3) and (C4) are equivalent to the Knill–Laflamme conditions for correcting $2t$ deletions, specialized to permutation-invariant codes.

For an odd number of deletions, the shortest code $\mathcal{Q}_{g,m,\delta,\epsilon}$ has length $(s+1)^2$. This coincides with the length of Ouyang's *gnu* codes, although the code families are different. For any even number of deletions > 0, the code $\mathcal{Q}_{s,\lceil s/2 \rceil,s,-}$ has length $(s+1)^2 - s$, which is shorter than the existing constructions.

In [15], Nakayama and Hagiwara showed that the smallest length of single quantum deletion-correcting codes is 4. They also constructed a code that meets this bound with equality. We note that code $\mathcal{Q}_{1,1,1,-}$ gives another construction of an optimal code correcting one deletion. Its logical codewords are

$$|\boldsymbol{c}_0\rangle = \sqrt{\frac{1}{3}}|D_0^4\rangle + \sqrt{\frac{2}{3}}|D_3^4\rangle, \quad |\boldsymbol{c}_1\rangle = \sqrt{\frac{2}{3}}|D_1^4\rangle - \sqrt{\frac{1}{3}}|D_4^4\rangle.$$

5.2 Transversality

Permutation-invariant codes were linked to transversal gate sets in a recent paper [11], based on the results of [6]. Among other results, [6] constructed spin codes as representations of the groups $2O$ and $2I$ (the binary octahedral and icosahedral groups [4]) that can be mapped onto permutation-invariant codes. For instance, [6] constructed a code spanned by the basis states

$$\left.\begin{aligned} |c_0\rangle &= \sqrt{\frac{3}{10}}\left|\frac{7}{2},\frac{7}{2}\right\rangle + \sqrt{\frac{7}{1}0}\left|\frac{7}{2},-\frac{3}{2}\right\rangle \\ |c_1\rangle &= \sqrt{\frac{7}{10}}\left|\frac{7}{2},\frac{3}{2}\right\rangle - \sqrt{\frac{3}{10}}\left|\frac{7}{2},-\frac{7}{2}\right\rangle. \end{aligned}\right\} \tag{6}$$

Following up on this work, the authors of [11] defined a *Dicke state mapping* $\mathscr{D}$ that converts a state of a spin-j system into a permutation-invariant state on $n = 2j$ qubits. It can be defined as follows: $\mathscr{D} : |j, m\rangle \to |D_{j-m}^{2j}\rangle$. This mapping converts the logical gates of a spin code into the logical *transversal* gates of a permutation-invariant code. To link this line of work to our paper, observe that applying $\mathscr{D}$ to the spin code of (6), we obtain exactly our code $Q_{2,1,2,-}$ (5). Hence, the permutation-invariant $Q_{2,1,2,-}$ code admits the $2I$ group gates transversally.

Even more recently, paper [12] introduced a family of permutation- invariant codes of distance 3 that admits transversal gates from BD_{2b} (the binary dihedral group of degree $2b$). The group BD_{2b} is a non-abelian subgroup of $SU(2)$ of order $8b$ with generators $X, Z, \begin{pmatrix} e^{-i\pi/2b} & 0 \\ 0 & e^{i\pi/2b} \end{pmatrix}$. For instance, $BD_2 = \langle X, Z\rangle$, $BD_4 = \langle X, Z, S\rangle$, and $BD_8 = \langle X, Z, S, T\rangle$ It is well known that $[[2^{r+1} - 1, 1, 3]]$ Reed-Muller codes implement the BD_{2^r} group gates transversally.

Proposition 5.5: Let $b > 0$ be an integer that is not of the form 2^r or $3(2^r)$. The codes in the family $Q_{3,1,2b-4,+}$ implement the group BD_{2b} transversally when $3 \nmid b$ and implement the group $BD_{2b/3}$ transversally when $3|b$. The codes $\mathcal{Q}_{3,1,2^r-4,+}$ implement the group BD_{2^r} transversally for all integers $r \geqslant 3$.

This follows because the first code family in the proposition offers an alternative construction of the codes in Family 1 in [12], where the transversality properties are proved. The second code family is the same as Family 2 in [12].

For example, the code $\mathcal{Q}_{3,1,4,+}$ of length $n = 11$ with its basis codewords

$$|\boldsymbol{c}_0\rangle = \frac{\sqrt{5}}{4}|D_0^{11}\rangle + \frac{\sqrt{11}}{4}|D_8^{11}\rangle,$$
$$|\boldsymbol{c}_1\rangle = \frac{\sqrt{11}}{4}|D_3^{11}\rangle + \frac{\sqrt{5}}{4}|D_{11}^{11}\rangle$$

can correct one error and it implements the T gate transversally. For comparison, the $[[15, 1, 3]]$ Reed-Muller code, which also has this property, is longer than our construction. Furthermore, the code $\mathcal{Q}_{3,1,12,+}$ with its codewords

$$|\boldsymbol{c}_0\rangle = \sqrt{\frac{13}{32}}|D_0^{19}\rangle + \sqrt{\frac{19}{32}}|D_{16}^{19}\rangle,$$
$$|\boldsymbol{c}_1\rangle = \sqrt{\frac{19}{32}}|D_3^{19}\rangle + \sqrt{\frac{13}{32}}|D_{19}^{19}\rangle.$$

can correct one error, implements the $\sqrt{T}$ gate transversally, and has better code parameters than the $[[31, 1, 3]]$ RM code that implements a tranversal $\sqrt{T}$.

5.3 Spontaneous decay errors

Codes of construction 5.1 also correct errors arising from spontaneous photon emission. The noise process in the *amplitude damping channel* models decay of the photon from its excited state $|1\rangle$ to the ground state $|0\rangle$. Assuming that the probability of decay is p, the behavior of the noise process on a single qubit system is defined as

$$\mathcal{E}_p(\rho) = \boldsymbol{A}_0\rho\boldsymbol{A}_0^\dagger + \boldsymbol{A}_1\rho\boldsymbol{A}_1^\dagger, \tag{7}$$

where

$$\boldsymbol{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad \boldsymbol{A}_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}.$$

We clearly have $\boldsymbol{A}_0|0\rangle = |0\rangle, \boldsymbol{A}_0|1\rangle = \sqrt{1-p}|1\rangle$ and $\boldsymbol{A}_1|0\rangle = 0, \boldsymbol{A}_1|1\rangle = \sqrt{p}|0\rangle$. For an n-qubit system, a typical error in this channel has the form $\otimes_{i=1}^n \boldsymbol{K}_i$, where $\boldsymbol{K}_i \in \{\boldsymbol{A}_0, \boldsymbol{A}_1\}$. We say the error has *multiplicity* t if t (or fewer) of the terms in this product equal A_1. In [1] we prove the following result.

Theorem 5.6: Let t be a nonnegative integer. Let $g \geqslant t+1$, $m \geqslant \lceil \frac{3t}{2} \rceil$, $\delta \geqslant t$, and $\epsilon = \pm 1$. Then the code $\mathcal{Q}_{m,l,\delta,\epsilon}$ corrects t amplitude-damping errors.

The proof relies on general conditions for error correction established in [16] and the specific form of the codes constructed in this paper.

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