Non-Markovian Models of Opinion Dynamics on Temporal Networks[*]

Weiqi Chu[†] and Mason A. Porter[‡]

Abstract. Traditional models of opinion dynamics, in which the nodes of a network change their opinions based on their interactions with neighboring nodes, consider how opinions evolve either on time-independent networks or on temporal networks with edges that follow Poisson statistics. Most such models are Markovian. However, in many real-life networks, interactions between individuals (and hence the edges of a network) follow non-Poisson processes and thus yield dynamics with memory-dependent effects. In this paper, we model opinion dynamics in which the entities of a temporal network interact and change their opinions via random social interactions. When the edges have non-Poisson interevent statistics, the corresponding opinion models have non-Markovian dynamics. We derive a family of opinion models that are induced by arbitrary waiting-time distributions (WTDs), and we illustrate a variety of induced opinion models from common WTDs (including Dirac delta distributions, exponential distributions, and heavy-tailed distributions). We analyze the convergence to consensus of these models and prove that homogeneous memory-dependent models of opinion dynamics in our framework always converge to the same steady state regardless of the WTD. We also conduct a numerical investigation of the effects of waiting-time distributions on both transient dynamics and steady states. We observe that models that are induced by heavy-tailed WTDs converge more slowly to a steady state than models that are induced by WTDs with light tails (or with compact support) and that entities with longer waiting times exert more influence on the mean opinion at steady state.

1. Introduction. The structure of networks has a major influence on the dynamics of complex systems of interacting entities in social, economic, information, biological, and physical systems [42]. In a network, entities interact via edges, which encode ties with time-dependent strengths. To model a networked system, it is important both to account for the time-dependence of edges and to examine the effects of (both time-independent and time-dependent) network structures on dynamical processes [45], such as opinion formation [44], the spread of infectious diseases [28], and e-mail communication [13].

When modeling real-life networks, it is convenient to assume that edges, which encode events between humans or other entities, appear randomly in a way that is captured well by a Poisson process. This assumption results in time-dependent networks (i.e., so-called

[†]Department of Mathematics, University of California, Los Angeles, CA 90095 USA (chu@math.ucla.edu).
[‡]Department of Mathematics, University of California, Los Angeles, CA 90095 USA and Santa Fe Institute, Santa Fe, NM 87501 USA (mason@math.ucla.edu).

"temporal networks") with memoryless stochastic effects [37]. It overlooks the non-Markovian and nonstationary nature of many systems [3], such as e-mail traffic and online communication. To incorporate memory effects and to model dynamics with bursty and heavy-tailed interevent times [27, 47], it is important to consider temporal networks with edges that appear (and are thus available for events) according to stochastic processes other than Poisson processes.

Many dynamical processes on networks are non-Markovian [14, 19], which introduces nontrivial memory-dependence into their dynamics. For example, in a social contagion, entities typically require multiple sources of influence (i.e., social reinforcement) to adopt some idea or behavior [32]. Such cumulative effects occur in the adoption of social norms and technologies [2, 7]. There have been a variety of efforts to incorporate memory effects into dynamical processes on networks. Such studies include generalizations of voter models [4, 8, 53], compartmental models of disease spread [14, 29, 50, 56], social-contagion models [58], and random walks [31].
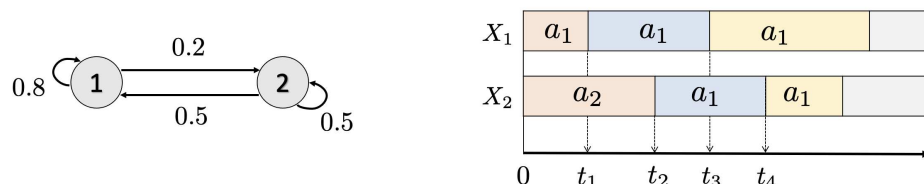
Most research on incorporating memory effects into dynamical processes on networks has focused on binary-state models, in which node states can take one of two values (e.g., susceptible or infected), but it is also important to examine memory effects in models in which nodes can take continuous values. Most research on these continuous-valued models incorporates memory-dependence directly into network structure through time-dependent edge weights and overlooks the effects of the previous node states. For example, Sugishita et al. considered an opinion model on "tie-decay networks" [51], in which interactions between entities are time-dependent and result in ties whose strengths increase instantaneously when an interaction occurs and decay exponentially between interactions. However, simply placing a dynamical process on a tie-decay network accounts only for the states during the most recent interaction; it ignores how states change with time to attain their current values. By contrast, we seek to account for the complete history of the states of a dynamical process.

Dynamical processes on networks are also impacted significantly by network architecture [45]. For example, Meng, Van Gorder, and Porter [41] studied how time-independent network structures affect the steady state and the convergence properties of a bounded-confidence model (BCM) of opinion dynamics. Sood and Redner [49] investigated the relationship between a network's degree distribution and the convergence time of a voter model on that network. Delvenne, Lambiotte, and Rocha [12] compared the effects of the structural and temporal features of networks on dynamical processes. In particular, they examined when a diffusion process is affected more strongly by network architecture or by a network's temporal features. Such investigations emphasize the importance of considering network structure when studying both transient and long-time qualitative behaviors of dynamical processes.

In the present study of non-Markovian opinion dynamics on networks, we consider arbitrary weighted networks and allow the waiting time between events to follow an arbitrary probability distribution, which is known as a "waiting-time distribution" (WTD). When the edges of a network satisfy Poisson statistics, the times between events follow independent exponential distributions [40] and thereby lead to memoryless models in which the dynamics depend only on a network's present state. Instead of using such a restrictive setting, we consider a generic WTD and use known results about interevent times [30] to systematically construct memory-dependent models of opinion dynamics. This setting allows us to study models that capture time-dependent interactions between entities and thus naturally incorporate memory effects.

Our paper proceeds as follows. In section 2, we propose a family of memory-dependent models of opinion dynamics on temporal networks in which the social interactions between two entities are determined by arbitrary WTDs. We illustrate the corresponding opinion models for several examples of both discrete and continuous WTDs. We prove that "homogeneous models" (in which all nodes have the same WTD) converge to the same steady state regardless of the WTD, and we give conditions for consensus (in which all entities hold the same opinion) in both homogeneous models and "heterogeneous models" (in which nodes have different WTDs). In section 3, we examine our memory-dependent opinion models on three types of graphs and investigate how WTDs affect both the transient dynamics and the steady states. We conclude in section 4. Our code is available at https://bitbucket.org/chuwq/non-markovian-models-of-opinion-dynamics-on-temporal-networks/src/main/.

2. Opinion models that are induced by waiting-time distributions. Let G be a weighted and directed graph with N nodes. We represent this graph using an adjacency matrix A. Entry A_{ij} of this matrix gives the weight of the edge from node i to node j; it encodes the interaction strength from entity i to entity j.[1] We assume that the entries of A are nonnegative real numbers and that each row sum of A is larger than 0. The row-normalized adjacency matrix $\widetilde{A}$ has entries $\widetilde{A}_{ij} = A_{ij}/\sum_{j=1}^{N} A_{ij}$. Entity i has a time-dependent continuous-valued opinion $X_i(t)$ and an internal clock τ_i, which indicates its waiting time. Each entity maintains its opinion until there is an event (which occurs at a random time that is determined by τ_i). The waiting time τ_i between two consecutive events of entity i follows a WTD $T_i(\tau)$. When an event occurs, entity i adopts the opinion of an adjacent node[2] j with probability $\widetilde{A}_{ij}$ and it resets its internal clock τ_i to 0. When $A_{ii} > 0$, entity i is adjacent to itself, so it can choose itself (with probability $\widetilde{A}_{ii}$) when selecting a node from which to adopt an opinion. If this occurs, entity i keeps its current opinion after the event and it resets its internal clock τ_i to 0. In Figure 1, we give an example of a two-node graph and illustrate how its entities update their opinions.

Figure 1. *(Left) A 2-node weighted graph with self-edges and (right) the trajectories of associated opinion trajectories (X_1 and X_2). Initially, the two entities of the graph have opinions a_1 and a_2. At time t_1, entity 1 experiences an event; it adopts opinion a_1 (which is the opinion of entity 1) with probability $\widetilde{A}_{11} = 0.8$, and it adopts opinion a_2 (which is the opinion of entity 2) with probability $\widetilde{A}_{12} = 0.2$. In the depicted scenario, entity 1 adopts opinion a_1 and holds that opinion until its next event, which occurs at time t_3. Entity 2 holds opinion a_2 until its first event occurs at time t_2. Entity 2 adopts the opinion of entity 1 at time t_2; this results in a consensus state in which both entities hold the same opinion a_1.*

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[1]To consider an unweighted graph, we let $A_{ij} = 1$ when there is an edge between nodes i and j and let $A_{ij} = 0$ when there is not an edge between them. In an undirected graph, $A_{ij} = A_{ji}$ for all i and j.

[2]When $A_{ij} > 0$, node j is adjacent to node i. When $A_{ij} = 0$, node j is not adjacent to node i.

The internal clock τ_i and WTD $T_i(\tau)$ determine when entity i can update its opinion, and the normalized weight $\widetilde{A}_{ij}$ gives the probability that entity i adopts the opinion of entity j. We assume that the events of different entities occur independently. If entity i adopts opinion $X_j(t)$ at time t, entity j still updates its opinion according to its internal clock τ_j without noticing that entity i has adopted its opinion. Therefore, we consider unidirectional interactions between entities. Such interactions arise in many social and biological systems [20]. For example, followers of a social-media account can update their opinions by reading posts without commenting or otherwise actively communicating with that account. As in [24], our graphs G are temporal networks because of the WTDs of the nodes.

In the following subsections, we study models of opinion dynamics that are induced by the WTDs of nodes on temporal networks. In section 2.1, we derive a master equation for the time-dependent opinions of the nodes, which can have different WTDs. In section 2.2, we examine models of opinion dynamics that use several well-known WTDs (Dirac delta distributions, exponential distributions, and heavy-tailed distributions) for the interevent times. In section 2.3, we examine convergence and consensus in our non-Markovian opinion models for both homogeneous and heterogeneous systems.

2.1. Master equations for opinion dynamics with arbitrary WTDs. Let $f_i(x,t)$ be the probability density function (PDF) of $X_i(t)$ on an opinion space Ω, and let $q_i^{(k)}(x,t)$ be a PDF on $\Omega \times \mathbb{R}_{\geq 0}$. The PDF $q_i^{(k)}(x,t)$ governs the probability that entity i adopts opinion x at time t in its kth event. For each entity i, we have

$$\int_\Omega f_i(x,t)\,dx = 1 \text{ for any } t \geq 0\,,$$

$$(2.1) \qquad \int_0^\infty \int_\Omega q_i^{(k)}(x,t)\,dx\,dt = 1 \text{ for any } k \in \mathbb{N} = \{0,1,\dots\}\,.$$

The 0th event of each entity occurs at time 0. At time 0, entity i updates its opinion according to the PDF $f_i(x,0)$ and sets its internal clock τ_i to 0. Suppose that entity i holds opinion x at time $t > 0$. This opinion arises from an earlier event at some time $t' \in [0,t)$, and entity i does not have any event between times t' and t. In mathematical terms,

$$(2.2) \qquad f_i(x,t) = \sum_{k=0}^\infty \int_0^t \phi_i(t-t')q_i^{(k)}(x,t')\,dt'\,,$$

where

$$(2.3) \qquad \phi_i(t) = 1 - \int_0^t T_i(t')\,dt'$$

is the probability that an event of entity i occurs after waiting for time t. The function $\phi_i(t)$ is the survival function with respect to T_i. To iterate between two consecutive events of the same entity i, we write

$$(2.4) \qquad q_i^{(k+1)}(x,t) = \sum_j \left[\int_0^t \int_\Omega q_i^{(k)}(y,t')T_i(t-t')\,dy\,dt' \right] \widetilde{A}_{ij} f_j^-(x,t)\,, \quad k \in \mathbb{N}\,,$$

$$q_i^{(0)}(x,t) = f_i(x,0)\delta(t)\,,$$

where $\delta(t)$ is the Dirac delta function.

The density $f_j^-(x, t)$ is the limit of $f_j(x, \tau)$ as $\tau \to t^-$. If entity i has an event at time t and entity i adopts the state of entity j, we change i's opinion X_i to $X_j(t^-)$, which is the opinion of entity j right before a possible event of entity j at time t. When $T_i(t)$ does not possess a point mass (i.e., $T_i(t)$ does not have a Dirac delta measure), f^- and f are the same because there is 0 probability that two events occur simultaneously. When $T_i(t)$ possesses a point mass (i.e., $T_i(t)$ has a positive probability at one or more isolated points), we need to distinguish between $X_i(t^-)$ and $X_i(t)$ to avoid ambiguity in situations when multiple events occur simultaneously.

Let $\bar{q}_i^{(k)}(t) = \int_\Omega q_i^{(k)}(x, t)\, dx$ be the PDF on $\mathbb{R}_{\geq 0}$ that the kth event of entity i occurs at time t. Using (2.4), we obtain

$$(2.5) \qquad \begin{aligned} \bar{q}_i^{(k+1)}(t) &= \int_0^t \bar{q}_i^{(k)}(t') T_i(t - t')\, dt', \quad k \in \mathbb{N}, \\ \bar{q}_i^{(0)}(t) &= \delta(t). \end{aligned}$$

Combining (2.2), (2.4), and (2.5) yields equations for the probability densities of the opinions:

$$(2.6) \qquad f_i(x, t) = \sum_j \phi_i(t) \star \left[\widetilde{A}_{ij} \theta_i(t) f_j^-(x, t) \right] + \phi_i(t) f_i(x, 0), \quad i \in \{1, \dots, N\},$$

where $\star$ denotes time convolution and

$$(2.7) \qquad \theta_i = \sum_{k=0}^\infty \bar{q}_i^{(k)} \star T_i.$$

Let the hat $\hat{\cdot}$ denote a Laplace transform. Using (2.5) and (2.7), we calculate that the Laplace transforms $\hat{\theta}_i$ and $\hat{T}_i$ satisfy

$$(2.8) \qquad \hat{\theta}_i = \left(1 - \hat{T}_i \right)^{-1} \hat{T}_i.$$

Let $x_i(t) = \int_\Omega x f_i(x, t)\, dx$ be the expectation of $X_i(t)$. A direct computation from (2.6) shows that $x_i(t)$ satisfies the integral equation

$$(2.9) \qquad x_i(t) = \sum_j \widetilde{A}_{ij} \left[\phi_i \star \left(\theta_i x_j^- \right) \right](t) + \phi_i(t) x_i(0),$$

where ϕ_i and θ_i are defined in (2.3) and (2.8), respectively. Equation (2.9) gives a family of *memory-dependent opinion models* that are induced by arbitrary WTDs $T_i(t)$. When $f_i(x, t)$ is continuous with respect to t, we take the Laplace transform of (2.9) and obtain

$$(2.10) \qquad \hat{x}_i = \sum_j \widetilde{A}_{ij} \hat{\phi}_i \widehat{\theta_i x_j^-} + \hat{\phi}_i x_i(0).$$

We use the equalities

$$(2.11) \qquad \hat{\phi}_i(s) = \frac{1}{s} \left(1 - \hat{T}_i(s) \right) \quad \text{and} \quad x_i(0) = s \hat{x}_i(s) - \hat{\dot{x}}_i(s)$$

and rewrite (2.10) as

$$(2.12) \qquad \hat{\dot{x}}_i = \sum_j \widetilde{A}_{ij} \widehat{\theta_i x_j^-} - \frac{s\hat{T}_i}{1 - \hat{T}_i} \hat{x}_i \,.$$

By taking the inverse Laplace transform of (2.12), we obtain a set of coupled integro-differential equations:

$$(2.13) \qquad \dot{x}_i(t) = \sum_j \widetilde{A}_{ij} \theta_i(t) x_j(t) - [\chi_i \star x_i](t) \,,$$

where the Laplace transform of χ_i satisfies $\hat{\chi}_i(s) = s\hat{T}_i(s)(1 - \hat{T}_i(s))^{-1}$. Equation (2.13) gives a family of memory-dependent opinion models that are induced by continuous-time WTDs.

We say that the opinion models (2.9) are *homogeneous* if all nodes (i.e., entities) of a network have the same WTD $T(t)$, which in turn leads to homogeneous survival functions $\phi(t)$ and $\theta(t)$; otherwise, we say that the opinion models (2.9) are *heterogeneous.* For the homogeneous case, we can simplify the memory-dependent opinion models (2.9) and write

$$(2.14) \qquad x_i(t) = \sum_j \widetilde{A}_{ij} \left[\phi \star \left(\psi x_j^- \right) \right](t) + \phi(t) \sum_j \left(\delta_{ij} - \widetilde{A}_{ij} \right) x_j(0) \,,$$

where δ_{ij} is the Kronecker delta function and $\psi(t)$ is related to the WTD $T(t)$ by $\hat{\psi}(s) = (1 - \hat{T}(s))^{-1}$. From a direct computation, we see for all $t \geq 0$ that

$$(2.15) \qquad \psi(t) = \theta(t) + \delta(t) \,, \quad (\phi \star \psi)(t) = 1 \,, \quad \psi(t) \geq 0 \,.$$

2.2. Models of opinion dynamics that are induced by common WTDs. We now examine opinion models (2.9) that are induced by several common WTDs, including both discrete-time and continuous-time distributions. Some WTDs, such as the exponential distribution, yield a Markovian dynamical process with a time discretization that matches the DeGroot model of opinion dynamics [11]. Other WTDs, such as the gamma distribution, yield non-Markovian dynamics; in these systems, the opinions depend on the entire history of all nodes' opinion values. We also examine models that arise from heavy-tailed WTDs and study approximations of them using a sum of Dirac delta measures (when we do not have explicit formulas for the inverse Laplace transforms).

2.2.1. Dirac delta WTD. Consider a situation in which events occur after entities wait for a fixed amount of time. That is, the WTD of each node is the Dirac delta distribution $T_i(t) = \delta(t - \triangle_i)$, which yields

$$(2.16) \qquad \hat{T}_i(s) = e^{-\triangle_i s} \,, \quad \theta_i(t) = \sum_{k=1}^{\infty} \delta(t - k\triangle_i) \,, \quad \phi_i(t) = \mathbb{1}_{[0,\triangle_i)}(t) \,,$$

where $\mathbb{1}_{[0,\triangle_i)}(t)$ is the indicator function on the interval $[0, \triangle_i)$.

In (2.9), $x_j^-(t)$ is the opinion right before entity j changes its opinion at time t. Because entity j updates its opinion after waiting for exactly time $\triangle_j$, we have $x_j^-(t) = x_j(t - \triangle_j)$. This yields the opinion model

$$
\begin{aligned}
x_i(t) &= x_i(0)\,, && t \in [0, \triangle_i)\,, \\
x_i(t) &= \sum_j \widetilde{A}_{ij} x_j(t - \triangle_j)\,, && t \in [\triangle_i, \infty)\,.
\end{aligned}
\tag{2.17}
$$

Suppose that all entities wait for the same amount of time $\triangle_i = 1$ before updating their opinions (i.e., the model is homogeneous). We can then write (2.17) in matrix form as the discrete-time dynamical system

$$
x(n+1) = P x(n)\,, \quad n \in \mathbb{N}\,,
\tag{2.18}
$$

where $x(n) \in \mathbb{R}^N$ is the vector of opinions and the transition matrix is $P = \widetilde{A}$, which is the row-normalized adjacency matrix. The model (2.18) has the same form as the DeGroot model of opinion dynamics [11].

2.2.2. Exponential WTD. We now suppose that each entity has an exponential WTD, which is closely related to a Poisson point process. In a Poisson point process, the time between two consecutive events follows an exponential distribution [40]. Poisson processes have been studied extensively both because they are mathematically convenient and because they are memoryless [21, 36]. We denote the exponential WTD of entity i by $T_i(t) = \lambda_i e^{-\lambda_i t}$, where the rate parameter $\lambda_i > 0$. From a direct computation, we obtain

$$
\theta_i(t) = \lambda_i\,, \quad \chi_i(t) = \lambda_i \delta(t)\,,
\tag{2.19}
$$

which we insert into (2.13) to obtain the opinion model

$$
\dot{x}_i(t) = \lambda_i \sum_j \widetilde{A}_{ij} x_j(t) - \lambda_i x_i(t)\,.
\tag{2.20}
$$

This yields a Markovian dynamical process $x(t)$ that satisfies

$$
x(t) = e^{\Lambda(\widetilde{A}-I)t} x(0)\,,
\tag{2.21}
$$

where Λ is a diagonal matrix with entries $\Lambda_{ii} = \lambda_i$, the matrix $\widetilde{A}$ is the row-normalized adjacency matrix, and I is the identity matrix. If we discretize the continuous-time opinion $x(t)$ at times $n = 0, \triangle t, 2\triangle t, \ldots$, we obtain a discrete-time description of $x(n)$ that satisfies the iterative relation

$$
x(n+1) = e^{\Lambda(\widetilde{A}-I)\triangle t} x(n)\,.
\tag{2.22}
$$

This discrete model is equivalent to the DeGroot model with a transition matrix $P = e^{\Lambda(\widetilde{A}-I)\triangle t}$ (instead of $\widetilde{A}$). We can thus view the model (2.20) as a continuous-time extension of the DeGroot model.

2.2.3. Gamma WTD. Another WTD with exponential decay is the gamma distribution, which has been used for modeling a variety of phenomena, including human response times [26], earthquake interevent times [54], and delayed effects in pharmacodynamic responses [52].

Suppose that entity i has the gamma WTD $T_i(t) = \lambda_i^2 t e^{-\lambda_i t}$. From a direct computation, we obtain

$$(2.23) \qquad \theta_i(t) = \frac{\lambda_i}{2}\left(1 - e^{-2\lambda_i t}\right), \quad \chi_i(t) = \lambda_i^2 e^{-2\lambda_i t}.$$

We insert (2.23) into (2.13) to obtain the non-Markovian opinion model

$$(2.24) \qquad \dot{x}_i(t) = \sum_j \frac{1}{2}\widetilde{A}_{ij}\lambda_i\left(1 - e^{-2\lambda_i t}\right)x_j(t) - \int_0^t \lambda_i^2 e^{-2\lambda_i(t-t')}x_i(t')\,dt'\,,$$

which we rewrite in matrix form as

$$(2.25) \qquad \dot{x}(t) = \mathcal{K}(t)\left[\widetilde{A}x(t) - \tilde{x}\{t\}\right],$$

where $\tilde{x}_i\{t\}$ is a historically averaged opinion that is weighted by the exponential kernel $\kappa_i(t) = \lambda_i^2 e^{-2\lambda_i t}$, which weights recent opinions more heavily than older opinions, and $\mathcal{K}(t)$ is the diagonal matrix with entries $\mathcal{K}_{ij}(t) = \delta_{ij}\mathcal{K}_i(t)$. Specifically, $\mathcal{K}_i(t)$ and $\tilde{x}_i\{t\}$ are

$$(2.26) \qquad \mathcal{K}_i(t) = \int_0^t \kappa_i(t')\,dt'\,, \quad \tilde{x}_i\{t\} = \frac{\int_0^t \kappa_i(t-t')x_i(t')\,dt'}{\mathcal{K}_i(t)}\,.$$

We use the curly bracket $\{t\}$ to denote that $\tilde{x}_i\{t\}$ depends on entity i's entire opinion trajectory $\{x_i(t')\}_{t'\leq t}$. In comparison to the memoryless model (2.20) that is induced by exponential WTDs, the memory-dependent model (2.25) includes an exponential time-relaxation kernel $\mathcal{K}_i(t) = \frac{\lambda_i}{2}\left(1 - e^{-2\lambda_i t}\right)$, which approaches the constant $\lambda_i/2$ as $t \to \infty$. The model (2.25) also includes a damping term that drives the opinion of each entity to its historical mean; this promotes the self-consistency of each entity's opinion.

If we define the integral term in (2.24) as an auxiliary variable y_i, we obtain the Markovian system

$$(2.27) \qquad \begin{aligned} \dot{x}_i(t) &= \sum_j \frac{1}{2}\widetilde{A}_{ij}\lambda_i\left(1 - e^{-2\lambda_i t}\right)x_j(t) - y_i(t)\,, \\ \dot{y}_i(t) &= \lambda_i^2 x_i(t) - 2\lambda_i y_i(t)\,, \end{aligned}$$

where the variables in this extended system are x_i and y_i for all $i \in \{1, \ldots, N\}$.

In section 3, we study the non-Markovian opinion model (2.25) that is induced by the Gamma WTD. We are not aware of any existing opinion models that have the same form as (2.25).

2.2.4. Heavy-tailed WTDs.
Many real systems, such as e-mail communication [25] and the spread of infectious diseases [57], have bursty properties, which cannot be captured well by Poisson temporal statistics. In such situations, the time intervals between isolated events deviate from an exponential distribution. Instead, they follow a heavy-tailed distribution.

Suppose that each node has a Pareto WTD. The Pareto distribution has been used to model online participation inequality, distributions of wealth, website visits, and a variety of other phenomena [59]. We write the Pareto distribution in the form $T_i(t) = \lambda_i(t+1)^{-\lambda_i - 1}$

with $\lambda_i > 0$, where we have shifted the distribution so that its domain is $[0, \infty)$. A direct computation yields

$$(2.28) \qquad \hat{T}_i(s) = \lambda_i \int_0^\infty (t+1)^{-\lambda_i - 1} e^{-ts} \, dt \,, \quad \phi_i(t) = (t+1)^{-\lambda_i} \,.$$

The opinion model in (2.9) involves the inverse Laplace transform of $\hat{\theta}_i = (1 - \hat{T}_i)^{-1} \hat{T}_i$. Unfortunately, there is not a convenient formula for the inverse Laplace transform of $\hat{T}_i$.

Now suppose that the WTD of each node is a log-normal distribution, which is also heavy-tailed and has the PDF

$$(2.29) \qquad T_i(t) = \frac{1}{\sqrt{2\pi}\sigma_i t} \exp\left[-\frac{(\ln t - \mu_i)^2}{2\sigma_i^2} \right] ,$$

where μ_i and σ_i^2, respectively, are the mean and variance of the Gaussian distribution. A closed-form expression does not exist for the Laplace transform of a log-normal distribution [1]. Accordingly, we are unable to obtain closed-form expressions for related terms, such as $\hat{T}_i$ and $\hat{\theta}_i$ in (2.7), and their inverse Laplace transforms.

To the best of our knowledge, most common heavy-tailed distributions do not possess an explicit form for the opinion models (2.9). Instead of aiming to determine analytical expressions for models that are induced by heavy-tailed WTDs, we seek feasible numerical approaches to simulate opinion models (2.9) that are induced by them. We approximate the continuous-time WTD in (2.13) by a sum of Dirac delta distributions. This yields opinion models (2.9) that are induced by sums of Dirac delta distributions.

2.2.5. WTDs that are sums of Dirac delta distributions. When a WTD consists of a sum of Dirac delta distributions, the events take place at a set of discrete times. The WTD T_i of node i is

$$(2.30) \qquad T_i(t) = \sum_{k=1}^\infty m_k^i \, \delta(t - k\triangle_i) \,,$$

where m_k^i, with $\sum_{k=1}^\infty m_k^i = 1$, is the probability that an event occurs after entity i waits for time $k\triangle_i$. A direct computation yields

$$\phi_i(t) = 1 - \sum_{k=1}^{\lfloor t/\triangle_i \rfloor} m_k^i \,, \quad \theta_i(t) = \sum_{k=0}^\infty M_k^i \delta(t - k\triangle_i) \,,$$

$$M_k^i = \sum_{\alpha \in U_k} m_{\alpha_1}^i m_{\alpha_2}^i \cdots m_{\alpha_z}^i \,, \quad U_k = \{\alpha \in \mathbb{N}_+^z : \|\alpha\|_1 = k, \, z \in \mathbb{N}_+\} \,,$$

where $\|a\|_1 = |a_1| + \cdots + |a_z|$ is the discrete ℓ_1 norm, $\lfloor \cdot \rfloor$ is the floor function, and $\mathbb{N}_+ = \{1, 2, \ldots\}$. When $k = 0$, we define U_0 to be the empty set, which implies that $M_0^i = 0$. The first four terms of the sequence M_k^i are

$$M_0^i = 0, \quad M_1^i = m_1^i, \quad M_2^i = m_1^i m_1^i + m_2^i,$$
$$M_3^i = m_1^i m_1^i m_1^i + m_1^i m_2^i + m_2^i m_1^i + m_3^i.$$

When the node WTDs are sums of Dirac delta distributions, the associated memory-dependent opinion model is

$$(2.31) \qquad x_i(t) = \sum_j \widetilde{A}_{ij} \sum_{k=1}^{\lfloor t/\triangle_i \rfloor} \phi_i(t - k\triangle_i) M_k^i x_j^-(k\triangle_i) + \phi_i(t) x_i(0).$$

This model accounts for situations in which each entity updates its opinions at discrete times. One can use models of the form (2.31) as approximations of models that are induced by heavy-tailed WTDs. When the inverse Laplace transforms of the node WTDs are difficult to compute, we can discretize the WTDs (see (2.30)) with a small time step $\triangle_i$ and treat (2.31) as the resulting model of opinion dynamics.

When the time steps are uniform (i.e., $\triangle_i = \triangle t$ for all i), we write (2.31) in the matrix form

$$(2.32) \qquad x[n+1] = \sum_{k=0}^n \Lambda_\phi[n-k] \Lambda_M[k+1] \widetilde{A} x[k] + \Lambda_\phi[n+1] x[0],$$

where the bracket $[k]$ denotes evaluation at time $k\triangle t$ (e.g., $x[n] = x(n\triangle t)$) and Λ_ϕ and Λ_M are diagonal matrices with entries

$$(2.33) \qquad \left(\Lambda_\phi[k]\right)_{ii} = \phi_i(k), \quad \left(\Lambda_M[k]\right)_{ii} = M_k^i, \quad k \in \mathbb{N}.$$

In the classical Friedkin–Johnsen (FJ) model [17], the opinion updates follow the rule $x[n+1] = \eta P x[n] + (1-\eta) x[0]$ with a time-independent weight $(1-\eta)$ on the nodes' initial opinions. We can view the model (2.32) as an extension of the FJ model by writing $x[n+1]$ as a sum of all previous opinions $x[k]$ (with $k \in \{0, \ldots, n\}$) weighted by time-dependent matrices $P(n+1, k)$. That is,

$$(2.34) \qquad x[n+1] = \sum_{k=0}^n P(n+1, k) x[k].$$

Each time step generates a new opinion vector $x[n]$, yielding the collection $\{x[0], x[1], \ldots, x[n]\}$ of historical opinions that collectively determine $x[n+1]$. The model (2.34) renormalizes the weights of the historical opinions $x[k]$ at each time step and introduces time-dependence through the weight matrices $P(n+1, k)$. The model (2.32) is also related to a voter model with an exogenous updating rule [15]; that voter model is a special case of (2.32). In section 3, we implement the model (2.32) with weights M_k^i from continuous-time WTDs.

2.3. Theoretical analysis. We now discuss the properties—including opinion conservation, convergence, and conditions for consensus—of the proposed memory-dependent opinion models (2.9) for both homogeneous and heterogeneous scenarios.

2.3.1. Conservation of a weighted average of the opinions in homogeneous models. When all nodes have the same WTD, the opinion models (2.9) reduce to the homogeneous models (2.14). We state an opinion-conservation guarantee for the homogeneous models (2.14).

Theorem 2.1. *Let w be a left eigenvector of $\widetilde{A}$ with eigenvalue 1 (i.e., $w^T A = w^T$). Suppose that $x_i(t)$ are opinion trajectories of a homogeneous opinion model (2.14). It then follows that the averaged opinion*

$$\bar{x}(t) = \sum_i w_i x_i(t) \tag{2.35}$$

is conserved for any WTD.

Proof. Because $\widetilde{A}$ is row-normalized, it has the eigenvalue 1 and an associated real left eigenvector w. Using (2.14) and (2.35), we obtain the scalar integral equation

$$\bar{x}(t) = \int_0^t \phi(t - t')\psi(t')\bar{x}^-(t')\, dt', \tag{2.36}$$

where $\bar{x}^-(t) = \lim_{t' \to t_-} \bar{x}(t')$ and ϕ and ψ are defined in (2.3) and (2.15), respectively. Let $t_{\max}$ and $t_{\min}$ be the times that $\bar{x}(t)$ takes its maximum value and minimum value, respectively, in the time interval $[0, T]$. That is,

$$t_{\max} = \arg\max_{t \in [0,T]} \bar{x}(t), \quad t_{\min} = \arg\min_{t \in [0,T]} \bar{x}(t). \tag{2.37}$$

Recall that ϕ and ψ are positive and that $\phi \star \psi = 1$. Direct computations yield

$$\bar{x}(t_{\max}) \leq \bar{x}(t_{\max}) \int_0^{t_{\max}} \phi(t_{\max} - t')\psi(t')\, dt' = \bar{x}(t_{\max}), \tag{2.38a}$$

$$\bar{x}(t_{\min}) \geq \bar{x}(t_{\min}) \int_0^{t_{\min}} \phi(t_{\min} - t')\psi(t')\, dt' = \bar{x}(t_{\min}). \tag{2.38b}$$

The inequality in (2.38a) is an equality only if $\bar{x}(t) = \bar{x}(t_{\max})$ for all $t \in [0, t_{\max}]$, and the inequality in (2.38b) is an equality only if $\bar{x}(t) = \bar{x}(t_{\min})$ for all $t \in [0, t_{\min}]$. This implies that $\bar{x}(0) = \bar{x}(t_{\max}) = \bar{x}(t_{\min})$ and hence that $\bar{x}(t)$ is constant on any finite time interval $[0, T]$. The choice of T is arbitrary, so $\bar{x}(t)$ is constant. ∎

Researchers have previously noted the conservation of weighted averages of opinions in models of opinion dynamics [38, 48]. When the row-normalized adjacency matrix $\widetilde{A}$ is also column-normalized, Theorem 2.1 implies that the mean opinion $\bar{x}(t) = \sum_i x_i(t)/N$ is conserved (i.e., $\bar{x}(t)$ is constant). For an arbitrary heterogeneous model (which can include WTDs of versatile forms), it is not guaranteed that the mean opinion is conserved. In Figure 6 (see section 3.2), we give an example of a heterogeneous model in which the mean opinion is not conserved. In section 3, we discuss this example in detail.

2.3.2. Analysis of consensus for homogeneous models. One common question in models of opinion dynamics is whether or not a model converges to a consensus, in which all entities hold the same opinion. Researchers have successfully determined consensus criteria and the time to reach consensus in several types of models, including the classical DeGroot model [11], the FJ model [16], and BCMs on graphs [35] and hypergraphs [9, 23]. We analyze the steady-state opinions of the homogeneous memory-dependent models (2.14) and provide sufficient conditions that guarantee convergence to consensus.

Theorem 2.2. *If the row-normalized adjacency matrix $\widetilde{A}$ is diagonalizable and -1 is not an eigenvalue of $\widetilde{A}$, then the homogeneous models (2.14) that are induced by any WTD converge to the same steady state x^*. The steady state x^* satisfies*

$$(2.39) \qquad \lim_{t \to \infty} x(t) = x^* = \sum_{\{d:\,\nu_d=1\}} c_d^0 v_d \,,$$

where $\{v_d\}$ are the eigenvectors (which we assume are linearly independent) of $\widetilde{A}$ and $\{c_d^0\}$ are the associated coefficients of $x(0)$ with respect to the basis $\{v_d\}$.

Proof. Let $\{\nu_d\}$ be the eigenvalues of $\widetilde{A}$. Consider the decomposition $x(t) = \sum_d c_d(t) v_d$. From (2.14), we know that the basis coefficients $c_d(t)$ satisfy

$$(2.40) \qquad c_d(t) = \nu_d \Big[\phi \star \big(\psi c_d^- \big) \Big](t) + (1 - \nu_d)\phi(t)c_d^0 \,,$$

where ϕ and ψ are defined in (2.3) and (2.15), respectively. Because $\widetilde{A}$ is a right stochastic matrix (i.e., its row sums are 1), the eigenvalues satisfy $|\nu_d| \le 1$. By assumption, $\nu_d \ne -1$. We discuss the two cases $\nu_d = 1$ and $|\nu_d| < 1$ separately.

For eigenvalues $\nu_d = 1$, we rewrite (2.40) as

$$(2.41) \qquad c_d = \phi \star \big(\psi c_d^- \big) \,,$$

where the coefficients c_d satisfy the equation for $\bar{x}$ in (2.36). By Theorem 2.1, we know that $c_d(t)$ remains constant and hence always equals its initial value c_d^0.

For eigenvalues that satisfy $|\nu_d| < 1$, we consider a mapping F_d from an L^∞-function space to itself, with the norm $\|y\|_{L^\infty} = \sup_{t \in [0,\infty)} |y(t)|$. For all $y \in L^\infty$, we define the mapping F_d with the equation

$$(2.42) \qquad F_d[y](t) = \nu_d \big[\phi \star \big(\psi y^- \big) \big](t) + (1 - \nu_d)\phi(t)c_d^0 \,.$$

Because $\phi \star \psi = 1$, we have

$$(2.43) \qquad \begin{aligned} \|F_d(y_1 - y_2)\|_{L^\infty} &= \left\| \nu_d \phi \star \left[\psi \left(y_1^- - y_2^- \right) \right] \right\|_{L^\infty} \\ &\le |\nu_d| \|y_1 - y_2\|_{L^\infty} \|\phi \star \psi\|_{L^\infty} = |\nu_d| \|y_1 - y_2\|_{L^\infty} \,, \end{aligned}$$

which implies that F_d is a contraction mapping. According to the Banach fixed-point theorem, there exists a unique fixed point $c_* \in L^\infty$ that satisfies $F_d[c_*] = c_*$. Therefore, c_* is the unique solution of (2.40). For any fixed t, we choose $T > t$ and have

$$(2.44) \qquad |c_*(T)| \le \|\nu_d c_*\|_{L^\infty} \int_0^t \phi(T - t')\psi(t') \, dt' + |\nu_d| \sup_{t' \ge t} |c_*(t')| + |(1 - \nu_d)c_d^0|\phi(T) \,.$$

We let $T \to \infty$ in (2.44) to obtain

$$(2.45) \qquad \overline{\lim}_{T \to \infty} |c_*(T)| \le |\nu_d| \sup_{t' \ge t} |c_*(t')| \,,$$

where we have used the facts that $\phi(t)$ is nonincreasing and $\lim_{t\to\infty}\phi(t)=0$. The time t is arbitrary, so we let $t\to\infty$ in (2.45) and obtain

$$(2.46) \qquad \overline{\lim}_{T\to\infty}|c_*(T)| \le |\nu_d|\,\overline{\lim}_{t\to\infty}|c_*(t)|\,.$$

Because $|\nu_d|<1$, the inequality (2.46) holds if and only if $\lim_{t\to\infty}|c_*(t)|=0$. Therefore, for all $|\nu_d|<1$, there is a unique solution $c_d(t)$ of (2.40) with $c_d(0)=c_d^0$ and $\lim_{t\to\infty}c_d(t)=0$.

Combining the cases for $\nu_d=1$ and $|\nu_d|<1$, we have

$$(2.47) \qquad \lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left[\sum_{\{d:|\nu_d|<1\}} c_d(t)v_d + \sum_{\{d:\nu_d=1\}} c_d(t)v_d\right] = \sum_{\{d:\nu_d=1\}} c_d^0 v_d\,. \qquad\blacksquare$$

Remark 2.3. In Theorem 2.2, it is necessary to include the condition that $\widetilde{A}$ does not have the eigenvalue -1. Consider the 2×2 matrix

$$(2.48) \qquad \widetilde{A}=A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose eigenvalues are 1 and -1. The model (2.18) that is induced by the Dirac delta WTD never converges to a steady state if the two opinions are different initially. The two entities swap their opinions whenever an event occurs.

Using Theorem 2.2, we find the following sufficient conditions to guarantee that the homogeneous opinion models (2.14) converge to consensus (i.e., $\lim_{t\to\infty} x_i(t) = x_{\text{same}}$ for all i).

Corollary 2.4. *If the row-normalized adjacency matrix $\widetilde{A}$ is irreducible and does not have the eigenvalue -1, then all homogeneous models (2.14) converge to consensus regardless of the initial conditions and WTDs.*

Corollary 2.5. *Suppose that $\widetilde{A}$ is diagonalizable and does not have the eigenvalue -1. If $\sum_{\{d:\nu_d=1\}} c_d^0 v_d = c\mathbb{1}$, where c is a scalar and $\mathbb{1}$ is a vector in which each entry is 1, then the homogeneous models (2.14) converge to consensus with the opinion value c for any WTD.*

Corollary 2.6. *If the homogeneous DeGroot model (2.18) (which is induced by the Dirac delta WTD) converges to consensus, then all homogeneous models (2.14) converge to consensus.*

Corollaries 2.4–2.6 are direct consequences of Theorem 2.2. In Corollary 2.6, the convergence to consensus of any particular homogeneous model (2.14) other than the DeGroot model does not imply that the DeGroot model also converges to consensus. For example, consider the adjacency matrix in (2.48). The model (2.21) that is induced by the exponential WTD converges to consensus for any initial state, but the DeGroot model never converges if the initial opinions are different. In numerical computations, we observe for the adjacency matrix in (2.48) that the opinion models that are induced by the uniform WTD, the gamma WTD, and heavy-tailed WTDs also converge to consensus. For continuous WTDs, the events of two entities occur simultaneously with probability 0. With probability 1, the event of one entity occurs first, which causes an opinion adoption by the other entity and eventually leads to consensus.

2.3.3. Analysis of consensus for heterogeneous models with Poisson statistics. When the WTDs are exponential, the interevent times arise from Poisson point processes. In the following theorem, we state a convergence criterion for this situation.

Theorem 2.7. Suppose that all nodes have exponential WTDs, which we parameterize by the rate parameter $\lambda_i > 0$ for node i. Let Λ be the diagonal matrix with entries $\Lambda_{ii} = \lambda_i$, and let $\widetilde{A}$ be a row-normalized adjacency matrix. If the matrix $Z = \Lambda(\widetilde{A} - I)$ is diagonalizable with eigenvalue–eigenvector pairs $\{\nu_d, v_d\}_{d=1,\dots,N}$ and the eigenvectors $\{v_d\}$ are linearly independent, then the model (2.20) converges to a steady state x^*. Additionally, x^* satisfies

$$(2.49) \qquad \lim_{t \to \infty} x(t) = x^* = \sum_{\{d: \nu_d = 0\}} c_d^0 v_d,$$

where c_d^0 is the coefficient of v_d in the decomposition of $x(0)$ in terms of the basis $\{v_d\}_{d=1,\dots,N}$.

Proof. The solution of (2.20) is $x(t) = e^{Zt}x(0)$. If we express $x(t)$ using the basis $\{v_d\}$, then the coefficients $c_d(t)$ are $c_d(t) = e^{\nu_d t}c_d^0$. Let $\mathrm{eig}_{\max}(\mathcal{M})$ denote the maximum eigenvalue of a matrix $\mathcal{M}$. For all eigenvalues ν_d, we have

$$(2.50) \qquad \nu_d \le \mathrm{eig}_{\max}(Z) = \mathrm{eig}_{\max}\left(\Lambda(\widetilde{A} - I)\right) \le \mathrm{eig}_{\max}(\Lambda)\,\mathrm{eig}_{\max}(\widetilde{A} - I).$$

By the Gershgorin circle theorem, the maximum eigenvalue of $\widetilde{A} - I$ is less than or equal to 0, which implies that $\nu_d \le 0$. Consequently, the coefficients $c_d(t) = e^{\nu_d t}c_d^0$ satisfy $\lim_{t\to\infty} c_d(t) = 0$ for $\nu_d < 0$ and $c_d(t) = c_d^0$ for $\nu_d = 0$. This concludes the proof. ∎

Theorem 2.7 gives a convergence criterion for a heterogeneous model (2.20) that is induced by exponential WTDs. The matrix Z and the initial condition together determine if a model converges to a consensus state. However, the rate parameter λ_i of the exponential WTD affects the speed of convergence; a larger λ_i results in faster convergence. Using the same notation as in Theorem 2.7, the following corollary guarantees convergence to consensus.

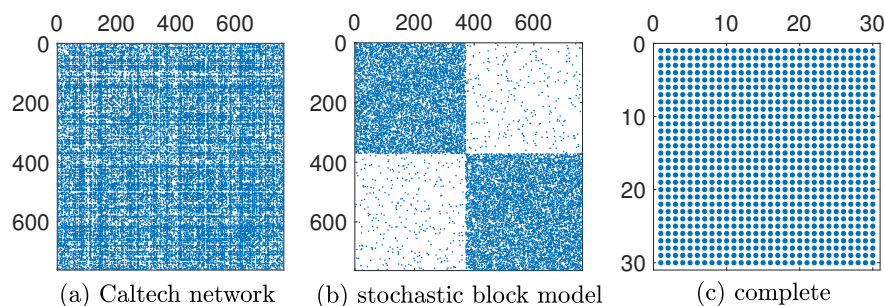*Corollary 2.8. If Z is diagonalizable and irreducible, then the model (2.20) converges to consensus.*

Corollary 2.8 is a direct consequence of Theorem 2.7. Because $\widetilde{A}$ is row-normalized, $Z = \Lambda(\widetilde{A} - I)$ has the eigenvalue 0 with the associated eigenvector $\mathbb{1}$. The irreducibility condition in Corollary 2.8 guarantees that $\mathbb{1}$ is the only eigenvector of the eigenvalue 0 and guarantees convergence to consensus.

There are other sufficient conditions that guarantee convergence to consensus. For example, requiring the initial opinion to have a decomposition that satisfies the constraint in Corollary 2.5 also guarantees convergence to consensus for the heterogeneous model (2.20).

3. Numerical computations. In this section, we numerically investigate the time evolution of the opinion models that we proposed in section 2. We study how WTDs affect opinion dynamics by examining steady-state opinion clusters and the time to converge to a steady state for both homogeneous and heterogeneous models on a variety of graphs.

We compare opinion models (2.9) that are induced by different WTDs on three types of graphs. The first graph is the largest connected component of the Caltech network from the

(a) Caltech network (b) stochastic block model (c) complete

Figure 2. *Sparsity patterns of the adjacency matrices of three graphs. Each blue dot signifies a nonzero entry and each white dot signifies a 0 entry. (a) The Caltech network has $N = 762$ nodes and 16,651 edges. (b) In our stochastic-block-model network, there are $N = 762$ nodes, $s = 2$ communities, an edge probability of $p \approx 0.0554$ for edges within communities, and an edge probability of $q = 0.002$ for edges between communities. With these probabilities, the expected total number of edges matches the number of edges in the Caltech network. (c) A complete weighted graph with $N = 30$ nodes and weights that we draw independently from the uniform distribution on $[0, 1]$.*

FACEBOOK100 data set [46, 55]. The nodes are individuals and the edges encode Facebook "friendships" between those individuals on one day in fall 2005. The second type of network is a graph that we generate using the stochastic block model (SBM) $G(N, s, p, q)$, where N denotes the number of nodes. We assign nodes uniformly at random to one of the s communities. We place edges between nodes in the same community with a homogeneous and independent probability p, and we place edges between nodes in different communities with a homogeneous and independent probability q. We run our simulations on only one SBM graph, but we expect to obtain similar results on other graphs that are generated by the same SBM. The third type of network is a complete weighted graph with random edge weights, which we draw independently from the uniform distribution on $[0, 1]$. We generate 100 graphs from this network ensemble and use them in the study of heterogeneous opinion models (2.9). In Figure 2, we show the sparsity patterns of the adjacency matrices of the three types of graphs.

We consider six different WTDs: Dirac delta, exponential, gamma, uniform, Pareto, and log-normal distributions. The WTDs are given by the formulas

$$(3.1a) \qquad T_{\text{delta}}(t) = \delta(t - \mu),$$

$$(3.1b) \qquad T_{\text{uniform}}(t) = \mathbb{1}_{[0, 2\mu]}(t),$$

$$(3.1c) \qquad T_{\text{gamma}}(t) = \frac{4t}{\mu^2} \exp(-2t/\mu),$$

$$(3.1d) \qquad T_{\text{exp}}(t) = \frac{1}{\mu} \exp(-t/\mu),$$

$$(3.1e) \qquad T_{\text{LN}}(t) = \frac{1}{\sqrt{2\pi} t \sigma} \exp\left(-\frac{(\ln(t) - \mu)^2}{2\sigma^2}\right),$$
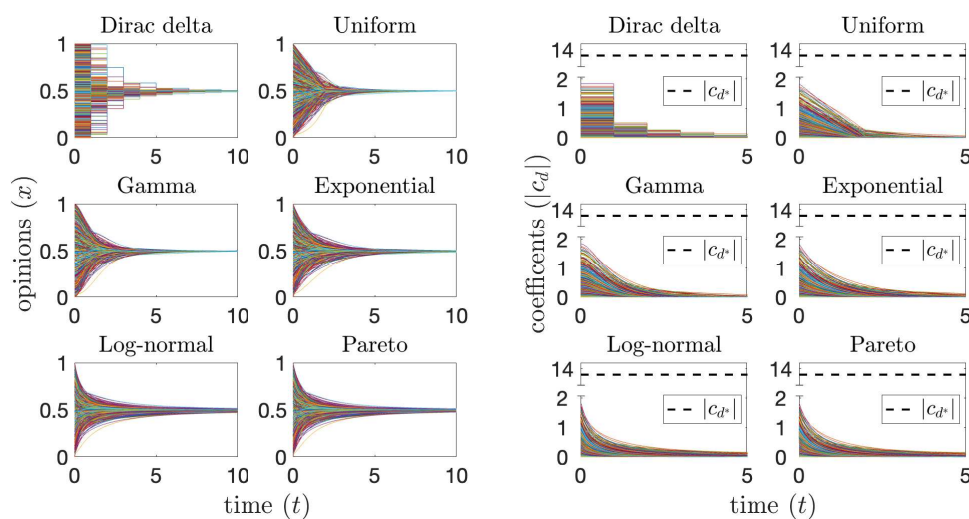
$$(3.1f) \qquad T_{\text{pareto}}(t) = \frac{\alpha}{(1 + t)^{\alpha+1}}.$$

The mean of the WTDs in (3.1a)–(3.1d) is μ. The mean of the log-normal WTD in (3.1e) is $\exp(\mu + \frac{\sigma^2}{2})$, where the parameters μ and σ are the mean and standard deviation of a normal distribution. The mean of the Pareto WTD in (3.1f) is $1/(\alpha-1)$, where the parameter $\alpha > 1$ is called the Pareto index. The Dirac delta and uniform distributions are compactly supported, the gamma and exponential distributions have light tails (specifically, they decay exponentially), and the log-normal and Pareto distributions have heavy tails. We discretize the above WTDs (except for the Dirac delta distribution, which is already discrete) using uniform grids (with a spacing of 0.01 between grid points) and approximate them by a sum of Dirac delta distributions (see (2.30)).

3.1. Models of opinion dynamics with homogeneous WTDs. In this subsection, we investigate the time evolution of opinion models (2.14) with homogeneous WTDs on the Caltech network and the SBM network. These two networks have the same number of nodes and a similar number of edges, but they have different community structures. In the WTDs in (3.1), we use the following parameter values: $\mu = 1$ in (3.1a)–(3.1d), $\mu = -1$ and $\sigma = \sqrt{2}$ in (3.1e), and $\alpha = 2$ in (3.1f). With these choices, these WTDs all have the same mean, which is equal to 1.

We first examine the homogeneous opinion models (2.14) on the Caltech network. Because the row-normalized adjacency matrix $\widetilde{A}$ is irreducible and does not have the eigenvalue -1, by Corollary 2.4, we expect the opinions to converge to consensus. In Figure 3, we show the opinion trajectory $x_i(t)$ of each entity i for each WTD. We also show the time-dependent

Figure 3. *Opinion trajectories $x_i(t)$ and their associated basis coefficients $c_d(t)$ on the Caltech network for the homogeneous opinion models (2.14) with different WTDs. All of the models have the same initial opinion, which we draw randomly from the uniform distribution on $[0,1]$. In the right panels, we plot the magnitudes $|c_d(t)|$ of the basis coefficients as a function of time. We use solid colored curves for the coefficients that are associated with eigenvalues that are smaller than 1, and we use dashed black lines (which we label with $|c_{d^*}|$) to plot the coefficients that are associated with the leading eigenvalue $\nu_{d^*} = 1$. We observe that $|c_{d^*}(t)| \approx 13.79$ for each of the WTDs.*
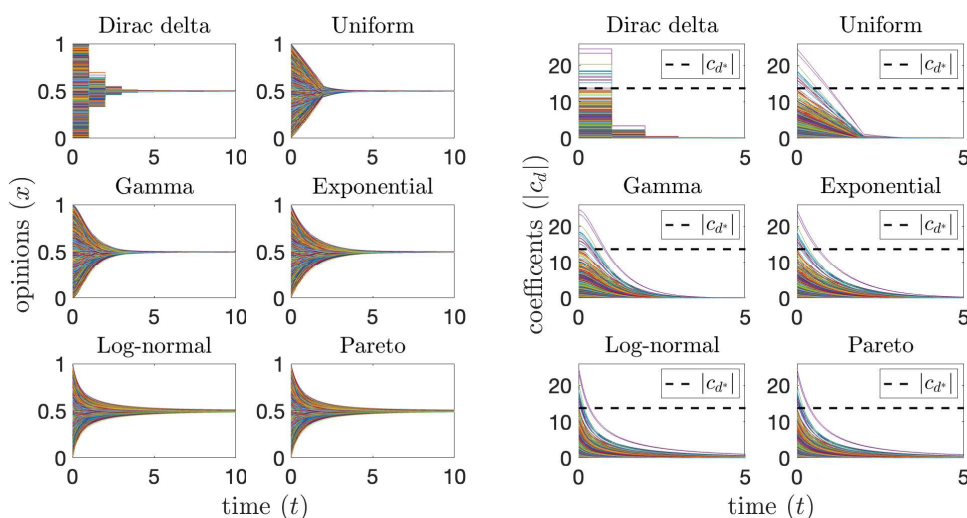
basis coefficients $c_d(t)$ that we obtain by expressing the time-dependent opinion vector $x(t)$ in terms of the eigenvectors $\{v_d\}$ of $\widetilde{A}$.

As expected, the opinions of all entities converge to a single opinion cluster for each type of WTD. We also observe that the coefficient $c_{d^*}(t)$ that is associated with the eigenvalue $\nu_{d^*} = 1$ is constant with respect to time and that the magnitudes $|c_d(t)|$ of the other coefficients decay to 0 for all WTDs. For different WTDs, the coefficient magnitudes $|c_d(t)|$ have different dynamics as they decay to 0. For example, the coefficient magnitudes $|c_d(t)|$ for the uniform WTD $T_{\text{uniform}}(t) = \mathbb{1}_{[0,2]}(t)$ decay linearly at first. Recall expression (2.40) in the proof of Theorem 2.2. The basis coefficients $c_d(t)$ satisfy the bound
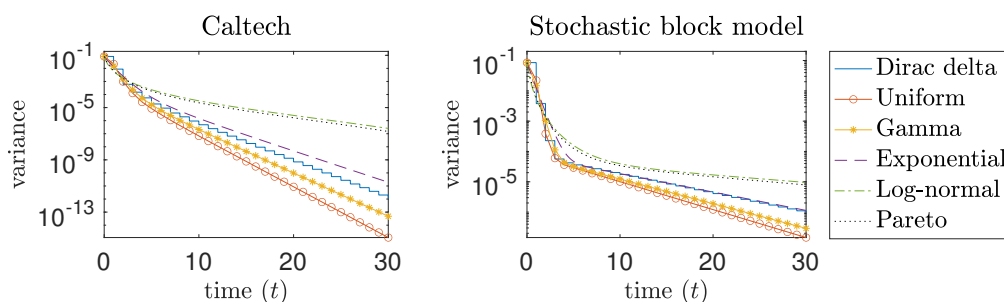
$$(3.2) \qquad |c_d(t)| \leq |\nu_d| \|c_d\|_{L^\infty} + \phi(t)|(1 - \nu_d)c_d^0|.$$

When T is a uniform PDF, we compute from (2.3) that $\phi(t) = 1 - t/2$ for $t \in [0,2]$ and $\phi(t) = 0$ otherwise. This formula for ϕ explains the associated linear decay of the coefficients $c_d(t)$ in Figure 3.

We generate a single two-community SBM graph (see Figure 2(b)) and examine the homogeneous opinion models (2.14) on that SBM graph. We compute the eigenvalues of the row-normalized adjacency matrix $\widetilde{A}$. Despite the two-community structure of this network, $\widetilde{A}$ has only one eigenvalue that is equal to 1. As guaranteed by Theorem 2.2, the opinions converge to consensus for all initial opinions and all WTDs. In Figure 4, we show the opinion trajectories $x_i(t)$ and the behavior of their associated coefficients $c_d(t)$. For each WTD, the opinions converge to consensus and the magnitudes $|c_d(t)|$ of the coefficients decay to 0 when $\nu_d \neq 1$.

Figure 4. *Opinion trajectories $x_i(t)$ and their associated basis coefficients $c_d(t)$ on the two-community SBM network for the homogeneous opinion models (2.14) with different WTDs. All of the models have the same initial opinion, which we draw randomly from the uniform distribution on $[0,1]$. In the right panels, we plot the magnitudes $|c_d(t)|$ of the basis coefficients as a function of time. We use solid colored curves for the coefficients that are associated with eigenvalues that are smaller than 1, and we use dashed black lines (which we label with $|c_{d^*}|$) to plot the coefficients that are associated with the leading eigenvalue $\nu_{d^*} = 1$. We observe that $|c_{d^*}(t)| \approx 13.42$ for each of the WTDs.*
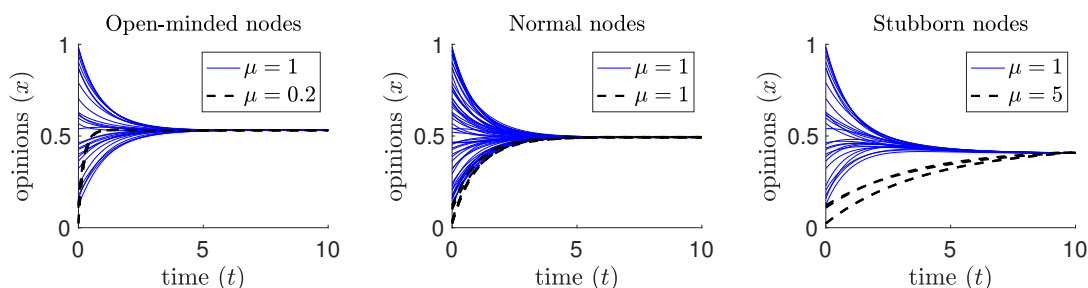
Figure 5. *Time-dependent variances of the node opinions for opinion models with homogeneous WTDs on (left) the Caltech network and (right) the two-community SBM network.*

In the above simulations, we observe that the homogeneous opinion models (2.14) that are induced by any WTD converge to consensus for both the Caltech and SBM networks, but different models have different convergence rates. To compare the convergence rates of the homogeneous opinion models (2.14) that are induced by our different WTDs, we plot time-dependent variances of the opinions in Figure 5. For both networks, the variance decays exponentially with time and the opinion models that are induced by the heavy-tailed WTDs (i.e., the log-normal and Pareto distributions) have the slowest convergence to steady state; the models that are induced by the uniform and gamma distributions have the fastest convergence.

In both networks, the row-normalized adjacency matrices $\widetilde{A}$ have the same largest eigenvalue of 1. Because the coefficient c_{d^*} that is associated with the eigenvalue 1 is constant as a function of time, the second-largest eigenvalue of $\widetilde{A}$ determines the convergence rate. In the Caltech network, the second-largest eigenvalue of $\widetilde{A}$ is 0.7229. In the SBM network, the second-largest eigenvalue is 0.9299, which is closer to 1 and hence leads to a slower convergence than in the Caltech network (see Figure 5). The second-largest eigenvalue of $\widetilde{A}$ is related to the Fiedler value of the adjacency matrix A. The Fiedler value has a strong influence on the time that it takes for random walks and diffusion processes on a network to converge to a steady state [39]. In our simulations on the SBM network, we observe a scale separation of the variance (see Figure 5). This may depend both on network community structure and on the WTD.

3.2. Models of opinion dynamics with heterogeneous WTDs. In this subsection, we discuss the effect of WTD heterogeneity on the memory-dependent opinion models (2.9). We consider (1) an example in which all nodes have the same WTD type but different WTD mean values and (2) an example in which different nodes have different types of WTDs.

In our first example, we consider the heterogeneous exponential model (2.20), in which the nodes have exponential WTDs (3.1d) with different values of the mean μ. We examine a scenario in which 90% of the nodes have a WTD with a mean of $\mu = 1$ and the remaining 10% of the nodes have a mean μ that we vary. This "90-10 decomposition," which also has been used in a BCM with heterogeneous node-activity levels [33], is motivated by the so-called "90-9-1 rule" of participation inequality. The 90-9-1 heuristic was proposed as a rule of thumb [43] that 90% of users consume content online but do not contribute to it, 9% of users occasionally contribute content, and 1% of users account for most contributions of content. In our example, we consider a 90-10 decomposition of a population for simplicity, but one can also consider other decompositions (such as a 90-9-1 decomposition).
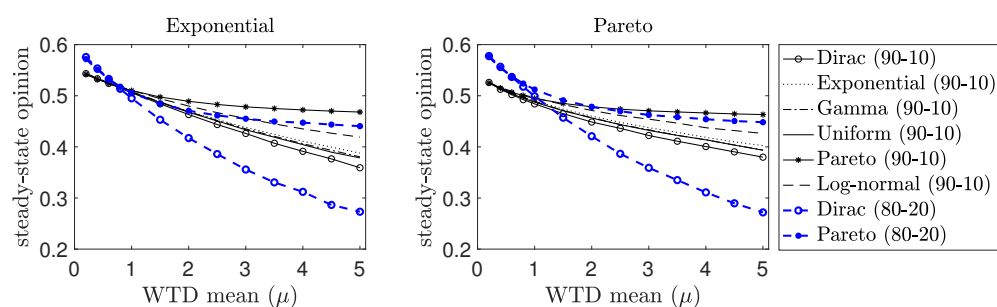
Figure 6. *Opinion trajectories of the opinion model* (2.20), *in which the nodes have exponential WTDs* (3.1d), *on a complete weighted graph (see Figure* 2(c)*). The graph has* 30 *nodes and all nodes have an exponential WTD. The WTD of* 27 *nodes has a mean of* $\mu = 1$*; the* 3 *nodes with the smallest initial opinion values have means of (left)* $\mu = 0.2$*, (center)* $\mu = 1$*, and (right)* $\mu = 5$*. The steady-state mean opinions are (left)* 0.5407*, (center)* 0.5015*, and (right)* 0.4096*.*

When the mean $\mu < 1$, the minority nodes represent "open-minded" entities that tend to change their opinions more frequently than normal nodes. When $\mu > 1$, these nodes represent "stubborn" entities that tend to preserve their opinions by interacting less frequently than normal nodes. We interpret nodes with long waiting times as stubborn entities, but similar ideas arise in a variety of contexts. Prior studies have considered examples such as immune nodes in compartmental models [34], media nodes in opinion models [6], and (as in the present work) stubborn entities in opinion models [18].

We consider the parameter values $\mu = 0.2$, $\mu = 1$, and $\mu = 5$ for the minority nodes as example scenarios with open-minded nodes, normal nodes, and stubborn nodes, respectively. We generate one complete graph with random weights (see Figure 2(c)) and examine the above three scenarios on this graph. All three scenarios have the same initial opinions, which we draw independently from the uniform distribution on $[0, 1]$ for each node. We show the opinion trajectories in Figure 6. Open-minded nodes tend to converge quickly to their steady-state opinions, whereas stubborn nodes change their opinions slowly and attract the mean opinion toward their opinions.

In our second example, we examine the effect of WTD heterogeneity on steady-state opinion clusters when nodes have different types of WTDs. We consider both a 90-10 decomposition and an 80-20 decomposition of the node WTD types. The majority of the nodes have one WTD (which we call the "majority WTD") and the remaining nodes have another WTD (which we call the "minority WTD"). The nodes with the largest initial opinion values—either 90% of the nodes or 80% of the nodes—have the majority WTD. We suppose that the majority WTD is either an exponential distribution (3.1d) with $\mu = 1$ or a Pareto distribution (3.1f) with $\alpha = 2$, so the majority WTD always has a mean of 1. We suppose that the minority WTD is one of the six distributions in (3.1). We vary the mean of the minority distribution and examine the steady-state opinion clusters of (2.9) on a complete graph with random edge weights (see Figure 2(c)).

We plot the steady-state opinion clusters in Figure 7. For each steady-state opinion in Figure 7, we examine 100 realizations and compute the mean steady-state opinions of the nodes across the realizations. In each realization, we generate a new complete weighted graph

Figure 7. *Steady-state opinions when a majority of the nodes of a network have (left) an exponential WTD with a mean of* 1 *and (right) a Pareto WTD with a mean of* 1*. The horizontal axis is the mean value of the minority WTD, and the vertical axis gives the mean steady-state opinions across* 100 *realizations of complete graphs with random edge weights. In each realization, we generate a new graph and new initial opinions, which we draw independently from the uniform distribution on* [0, 1]*. We use black curves for the mean steady-state opinions in scenarios with a* 90-10 *population decomposition (in which* 90% *of the nodes have the majority WTD and* 10% *of the nodes have the minority WTD) and blue curves for the mean steady-state opinions in scenarios with an* 80-20 *population decomposition.*

and new initial opinions, which we draw independently from the uniform distribution on $[0, 1]$. We terminate each simulation of the models (2.9) once the opinion variance is less than 10^{-7}, and we assume that the system has reached a steady state.

In all of the combinations of WTDs, when the minor nodes are stubborn (i.e., their mean waiting time is longer than that of the normal nodes), the steady-state opinions are smaller than 0.5, which is the expected value of the initial mean opinion. The steady-state opinion decreases as we increase the mean waiting time of the stubborn nodes. Of the examined WTDs, stubborn nodes exert the least influence when they have heavy-tailed WTDs and exert the most influence when they have a Dirac delta WTD. The homogeneous model in which all nodes have the same WTD (either an exponential distribution or a Pareto distribution), which has a mean of 1, has an expected steady-state mean opinion of 0.5. When we increase the percentage of special nodes (whether they are stubborn or open-minded) from 10% to 20%, the heterogeneous models deviate more from their corresponding homogeneous models than when only 10% of the nodes are special.

3.3. A short remark about polarized and fragmented steady states. In our numerical simulations of both homogeneous opinion models (see section 3.1) and heterogeneous opinion models (see section 3.2), we studied examples that converge to a consensus state. However, one can construct examples that converge to a polarized state (which has two distinct opinion clusters) or to a fragmented state (which has three or more distinct opinion clusters) by using graphs whose row-normalized adjacency matrices have more eigenvalues that are equal to 1.

4. Conclusions and discussion. We proposed a family of memory-dependent models of opinion dynamics that depend on the waiting-time distributions (WTDs) of the nodes of a network. Our models have continuous-valued opinions and account for memory effects in opinion dynamics. By contrast, to the best of our knowledge, all existing opinion models with continuous-valued opinions yield Markovian descriptions of the time evolution of opinions. In our models, the effects of memory emerge naturally from the non-Poisson interevent statistics of the edges of a network. We illustrate our memory-dependent opinion models using several

examples of common WTDs (specifically, Dirac delta distributions, exponential distributions, gamma distributions, and heavy-tailed distributions). When the nodes have a Dirac delta WTD or an exponential WTD, our models have Markovian dynamics and are equivalent to the DeGroot model. When the nodes have a gamma WTD, we obtain a non-Markovian model in which each entity of a network tries to maintain the self-consistency of its opinion as it interacts with other entities. We also approximated heavy-tailed continuous-time WTDs with sums of Dirac delta functions and derived an associated set of discrete-time opinion models.

We examined the convergence to steady states in our models both theoretically and numerically for both homogeneous and heterogeneous scenarios. In homogeneous scenarios, in which all nodes of a network have the same WTD, the time-independent adjacency matrix of the network determines the steady-state opinion. However, the WTD affects the transient dynamics and the rate of convergence to a steady state. We also observed that models with heavy-tailed WTDs converge more slowly than models with exponentially decaying WTDs. In heterogeneous scenarios, in which nodes have different WTDs (either the same type of WTD with different parameter values or WTDs of different types), "stubborn" nodes (i.e., nodes with longer mean waiting times than normal nodes) dominate the dynamics by attracting the mean opinion in a network toward their opinions.

In the present paper, we proposed non-Markovian models of opinion dynamics with continuous-valued opinions, and we studied some of the properties of these models. There are many interesting ways to extend our investigation. We considered WTDs that do not depend on the states (i.e., opinions) of the nodes, and we also assumed that the weights A_{ij} are time-independent. It seems fruitful to examine state-dependent weights (i.e., $A_{ij}(t) = w(x_i(t), x_j(t))$) and other time-dependent weights. For example, one can generalize bounded-confidence models of opinion dynamics [10, 22] to incorporate memory effects. In particular, after an interaction, one can suppose that entity i updates its opinion to that of entity j only if their opinions differ from each other by no more than some threshold. In our study, we also assumed that all entities of a network have independent interevent-time statistics, so we did not account for interactions that affect multiple entities at once. One can relax this independence assumption and examine the interdependence that results from coupled stochastic processes. It is also worth considering interdependence between entity opinions and network structure in the form of adaptive (i.e., coevolving) networks [5]. In our memory-dependent model that was induced by the gamma WTD, a damping term arose naturally; it promotes self-consistency of the opinion of each entity. It is also worthwhile to explore other memory-dependent models that account for the self-consistency of individual opinions. In our study, we examined some scenarios in which the entities of a network have heterogeneous WTDs. However, we only allowed nodes to have one of two different WTDs. It is important to investigate more diverse types of heterogeneity, such as systems with several WTD types or systems with WTDs with randomly determined parameter values.

REFERENCES

[1] S. ASMUSSEN, J. L. JENSEN, AND L. ROJAS-NANDAYAPA, *On the Laplace transform of the lognormal distribution*, Methodol. Comput. Appl., 18 (2016), pp. 441–458.

[2] A. BANDURA AND F. J. MCDONALD, *Influence of social reinforcement and the behavior of models in shaping children's moral judgment*, J. Abnorm. Soc. Psychol., 67 (1963), pp. 274–281.

[3] A.-L. BARABÁSI, *The origin of bursts and heavy tails in human dynamics*, Nature, 435 (2005), pp. 207–211.

[4] J. W. BARON, A. F. PERALTA, T. GALLA, AND R. TORAL, *Analytical and numerical treatment of continuous ageing in the voter model*, Entropy, 24 (2022), 1331.

[5] R. BERNER, T. GROSS, C. KUEHN, J. KURTHS, AND S. YANCHUK, *Adaptive Dynamical Networks*, preprint, https://arXiv.org/abs/2304.05652, 2023.

[6] H. Z. BROOKS AND M. A. PORTER, *A model for the influence of media on the ideology of content in online social networks*, Phys. Rev. Res., 2 (2020), 023041.

[7] D. CENTOLA, *The spread of behavior in an online social network experiment*, Science, 329 (2010), pp. 1194–1197.

[8] H. CHEN, S. WANG, C. SHEN, H. ZHANG, AND G. BIANCONI, *Non-Markovian majority-vote model*, Phys. Rev. E, 102 (2020), 062311.

[9] W. CHU AND M. A. PORTER, *A density description of a bounded-confidence model of opinion dynamics on hypergraphs*, SIAM J. Appl. Math. (2023), in press; preprint available at https://arXiv.org/abs/2203.12189.

[10] G. DEFFUANT, D. NEAU, F. AMBLARD, AND G. WEISBUCH, *Mixing beliefs among interacting agents*, Adv. Complex. Syst., 3 (2000), pp. 87–98.

[11] M. H. DEGROOT, *Reaching a consensus*, J. Amer. Statist. Assoc., 69 (1974), pp. 118–121.

[12] J.-C. DELVENNE, R. LAMBIOTTE, AND L. E. C. ROCHA, *Diffusion on networked systems is a question of time or structure*, Nat. Commun., 6 (2015), 7366.

[13] J.-P. ECKMANN, E. MOSES, AND D. SERGI, *Entropy of dialogues creates coherent structures in e-mail traffic*, Proc. Natl. Acad. Sci. USA, 101 (2004), pp. 14333–14337.

[14] M. FENG, S.-M. CAI, M. TANG, AND Y.-C. LAI, *Equivalence and its invalidation between non-Markovian and Markovian spreading dynamics on complex networks*, Nat. Commun., 10 (2019), 3748.

[15] J. FERNÁNDEZ-GRACIA, V. M. EGUÍLUZ, AND M. SAN MIGUEL, *Update rules and interevent time distributions: Slow ordering versus no ordering in the voter model*, Phys. Rev. E, 84 (2011), 015103.

[16] N. E. FRIEDKIN AND E. C. JOHNSEN, *Social influence and opinions*, J. Math. Sociol., 15 (1990), pp. 193–206.

[17] N. E. FRIEDKIN AND E. C. JOHNSEN, *Social Influence Network Theory: A Sociological Examination of Small Group Dynamics*, Cambridge University Press, Cambridge, UK, 2011.

[18] J. GHADERI AND R. SRIKANT, *Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate*, Automatica, 50 (2014), pp. 3209–3215.

[19] J. P. GLEESON, D. CELLAI, J.-P. ONNELA, M. A. PORTER, AND F. REED-TSOCHAS, *A simple generative model of collective online behavior*, Proc. Natl. Acad. Sci. USA, 111 (2014), pp. 10411–10415.

[20] V. GUTTAL AND I. D. COUZIN, *Social interactions, information use, and the evolution of collective migration*, Proc. Natl. Acad. Sci. USA, 107 (2010), pp. 16172–16177.

[21] J. HALLAS, D. GAIST, AND L. BJERRUM, *The waiting time distribution as a graphical approach to epidemiologic measures of drug utilization*, Epidemiology, (1997), pp. 666–670.

[22] R. HEGSELMANN AND U. KRAUSE, *Opinion dynamics and bounded confidence models, analysis, and simulation*, J. Artif. Societies Social Simul., 5(3) (2002), 2.

[23] A. HICKOK, Y. KUREH, H. Z. BROOKS, M. FENG, AND M. A. PORTER, *A bounded-confidence model of opinion dynamics on hypergraphs*, SIAM J. Appl. Dyn. Syst., 21 (2022), pp. 1–32.

[24] T. HOFFMANN, M. A. PORTER, AND R. LAMBIOTTE, *Generalized master equations for non-Poisson dynamics on networks*, Phys. Rev. E, 86 (2012), 046102.

[25] J. L. IRIBARREN AND E. MORO, *Impact of human activity patterns on the dynamics of information diffusion*, Phys. Rev. Lett., 103 (2009), 038702.

[26] J. L. IRIBARREN AND E. MORO, *Branching dynamics of viral information spreading*, Phys. Rev. E, 84 (2011), 046116.

[27] M. KARSAI, M. KIVELÄ, R. K. PAN, K. KASKI, J. KERTÉSZ, A.-L. BARABÁSI, AND J. SARAMÄKI, *Small but slow world: How network topology and burstiness slow down spreading*, Phys. Rev. E, 83 (2011), 025102.

[28] I. Z. KISS, J. C. MILLER, AND P. L. SIMON, *Mathematics of Epidemics on Networks: From Exact to Approximate Models*, Springer, Cham, Switzerland, 2017.

[29] I. Z. KISS, G. RÖST, AND Z. VIZI, *Generalization of pairwise models to non-Markovian epidemics on networks*, Phys. Rev. Lett., 115 (2015), 078701.

[30] M. KIVELÄ AND M. A. PORTER, *Estimating interevent time distributions from finite observation periods in communication networks*, Phys. Rev. E, 92 (2015), 052813.

[31] R. LAMBIOTTE, V. SALNIKOV, AND M. ROSVALL, *Effect of memory on the dynamics of random walks on networks*, J. Complex Netw., 3 (2015), pp. 177–188.

[32] S. LEHMANN AND Y.-Y. AHN, EDS., *Complex Spreading Phenomena in Social Systems: Influence and Contagion in Real-World Social Networks*, Springer, Cham, Switzerland, 2018.

[33] G. J. LI AND M. A. PORTER, *Bounded-confidence model of opinion dynamics with heterogeneous node-activity levels*, Phys. Rev. Res., 5 (2023), 023179.

[34] Q. LIU, T. LI, AND M. SUN, *The analysis of an SEIR rumor propagation model on heterogeneous network*, Phys. A Stat. Mech. Appl., 469 (2017), pp. 372–380.

[35] J. LORENZ, *A stabilization theorem for dynamics of continuous opinions*, Phys. A, 355 (2005), pp. 217–223.

[36] T. MAHMUD, M. HASAN, A. CHAKRABORTY, AND A. K. ROY-CHOWDHURY, *A Poisson process model for activity forecasting*, in Proceedings of the 2016 IEEE International Conference on Image Processing (ICIP), IEEE, 2016, pp. 3339–3343.

[37] N. MASUDA AND R. LAMBIOTTE, *A Guide to Temporal Networks*, World Scientific, Singapore, 2016.

[38] N. MASUDA AND H. OHTSUKI, *Evolutionary dynamics and fixation probabilities in directed networks*, New J. Phys., 11 (2009), 033012.

[39] N. MASUDA, M. A. PORTER, AND R. LAMBIOTTE, *Random walks and diffusion on networks*, Phys. Rep., 716–717 (2017), pp. 1–58.

[40] J. MEDHI, *Waiting time distribution in a Poisson queue with a general bulk service rule*, Management Sci., 21 (1975), pp. 777–782.

[41] X. F. MENG, R. A. VAN GORDER, AND M. A. PORTER, *Opinion formation and distribution in a bounded-confidence model on various networks*, Phys. Rev. E, 97 (2018), 022312.

[42] M. E. J. NEWMAN, *Networks*, 2nd ed., Oxford University Press, Oxford, UK, 2018.

[43] J. NIELSEN, *The 90-9-1 Rule for Participation Inequality in Social Media and Online Communities*, Nielsen Norman Group, 2006, https://www.nngroup.com/articles/participation-inequality/.

[44] H. NOORAZAR, K. R. VIXIE, A. TALEBANPOUR, AND Y. HU, *From classical to modern opinion dynamics*, Internat. J. Modern. Phys. C, 31 (2020), 2050101.

[45] M. A. PORTER AND J. P. GLEESON, *Dynamical Systems on Networks: A Tutorial*, Front. Appl. Dyn. Syst. Rev. Tutor. 4, Springer, Cham, Switzerland, 2016.

[46] V. RED, E. D. KELSIC, P. J. MUCHA, AND M. A. PORTER, *Comparing community structure to characteristics in online collegiate social networks*, SIAM Rev., 53 (2011), pp. 526–543.

[47] I. SCHOLTES, N. WIDER, R. PFITZNER, A. GARAS, C. J. TESSONE, AND F. SCHWEITZER, *Causality-driven slow-down and speed-up of diffusion in non-Markovian temporal networks*, Nat. Commun., 5 (2014), 5024.

[48] V. SOOD, T. ANTAL, AND S. REDNER, *Voter models on heterogeneous networks*, Phys. Rev. E, 77 (2008), 041121.

[49] V. SOOD AND S. REDNER, *Voter model on heterogeneous graphs*, Phys. Rev. Lett., 94 (2005), 178701.

[50] M. STARNINI, J. P. GLEESON, AND M. BOGUÑÁ, *Equivalence between non-Markovian and Markovian dynamics in epidemic spreading processes*, Phys. Rev. Lett., 118 (2017), 128301.

[51] K. SUGISHITA, M. A. PORTER, M. BEGUERISSE-DÍAZ, AND N. MASUDA, *Opinion dynamics on tie-decay networks*, Phys. Rev. Res., 3 (2021), 023249.

[52] Y.-N. SUN AND W. J. JUSKO, *Transit compartments versus gamma distribution function to model signal transduction processes in pharmacodynamics*, J. Pharm. Sci., 87 (1998), pp. 732–737.

[53] T. TAKAGUCHI AND N. MASUDA, *Voter model with non-Poissonian interevent intervals*, Phys. Rev. E, 84 (2011), 036115.

[54] S. TOUATI, M. NAYLOR, AND I. G. MAIN, *Origin and nonuniversality of the earthquake interevent time distribution*, Phys. Rev. Lett., 102 (2009), 168501.

[55] A. L. TRAUD, P. J. MUCHA, AND M. A. PORTER, *Social structure of Facebook networks*, Phys. A, 391 (2012), pp. 4165–4180.

[56] P. VAN MIEGHEM AND R. VAN DE BOVENKAMP, *Non-Markovian infection spread dramatically alters the susceptible–infected–susceptible epidemic threshold in networks*, Phys. Rev. Lett., 110 (2013), 108701.

[57] A. VAZQUEZ, B. RACZ, A. LUKACS, AND A.-L. BARABÁSI, *Impact of non-Poissonian activity patterns on spreading processes*, Phys. Rev. Lett., 98 (2007), 158702.

[58] W. WANG, M. TANG, H.-F. ZHANG, AND Y.-C. LAI, *Dynamics of social contagions with memory of nonredundant information*, Phys. Rev. E, 92 (2015), 012820.

[59] S. WOJCIK AND A. HUGHES, *Sizing up Twitter users*, Pew Research Center, 2019, https://www.pewresearch.org/internet/2019/04/24/sizing-up-twitter-users/.