# LIFTING METHODS IN MASS PARTITION PROBLEMS

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Abstract. Many results about mass partitions are proved by lifting $\mathbb{R}^d$ to a higher-dimensional space and dividing the higher-dimensional space into pieces. We extend such methods to use lifting arguments to polyhedral surfaces. Among other results, we prove the existence of equipartitions of $d+1$ measures in $\mathbb{R}^d$ by parallel hyperplanes and of $d+2$ measures in $\mathbb{R}^d$ by concentric spheres.

For measures whose supports are sufficiently well separated, we prove results where one can cut a fixed (possibly different) fraction of each measure either by parallel hyperplanes, concentric spheres, convex polyhedral surfaces of few facets, or convex polytopes with few vertices.

## 1. Introduction

In a standard mass partition problem, we are given measures or finite families of points in a Euclidean space, and we seek to partition the ambient space into regions that meet certain conditions. Some conditions determine how we split the measures and the sets of points. For instance, in an equipartition, we ask that each part has the same size in each measure or contains the same number of points of each set. Some conditions restrict the types of partitions allowed, such as partition by a single hyperplane. Determining whether such partitions always exist leads to a rich family of problems. Solutions to these problems often require topological methods and can have computational applications [Mat03, Živ17, RPS22]. The quintessential mass partition result is the ham sandwich theorem, conjectured by Steinhaus and proved by Banach [Ste38].

**Theorem 1.1** (Ham sandwich theorem)**.** *Let $d$ be a positive integer and $\mu_1, \ldots, \mu_d$ be finite Borel measures of $\mathbb{R}^d$. Then, there exists a hyperplane $H$ of $\mathbb{R}^d$ so that its two closed half-spaces $H^+$ and $H^-$ satisfy*

$$\mu_i(H^+) \geq \frac{1}{2}\mu_i(\mathbb{R}^d),$$
$$\mu_i(H^-) \geq \frac{1}{2}\mu_i(\mathbb{R}^d) \qquad \text{for } i = 1, \ldots, d.$$

If we further ask that $\mu_i(H') = 0$ for each hyperplane $H'$ and every $i = 1, \ldots, d$, the inequalities above are equalities. Stone and Tukey proved the ham sandwich theorem for general Borel measures [ST42]. They also proved the polynomial ham sandwich theorem which states that *any $\binom{d+k}{k} - 1$ Borel measures in $\mathbb{R}^d$ can be halved with a polynomial in $d$ variables of degree at most $k$.* Even though this is a far-reaching generalization of the ham sandwich theorem, its proof relies on a simple trick. We lift $\mathbb{R}^d$ to $\mathbb{R}^{\binom{d+k}{k}-1}$ by the Veronese map and apply the ham sandwich theorem in the higher-dimensional space.

In this paper, we prove several mass partition results by lifting $\mathbb{R}^d$ to higher-dimensional spaces, particularly $\mathbb{R}^{d+1}$, in new ways. In Section 2, we revisit a known result about equipartitions of measures with spheres and prove a new result

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about equipartitions of three measures in $\mathbb{R}^2$ using a sinusoidal curve of fixed period. Then, instead of lifting to higher-dimensional spaces via smooth maps, such as the Veronese maps, we lift to polyhedral surfaces in $\mathbb{R}^{d+1}$. This forces us to use the ham sandwich theorem for general measures, which is interesting on its own.

One of the advantages of using a lifting argument using polyhedral surfaces is that the boundary of the parts in the result partition is contained in the union of hyperplanes. Many mass partition results use regions with this property, such as partitions with wedges or partitions of $\mathbb{R}^d$ into convex pieces. We exhibit this in the following result, which is a ham sandwich theorem for parallel hyperplanes.

**Theorem 1.2.** *Let $d$ be a positive integer and $\mu_1, \ldots, \mu_{d+1}$ be $d+1$ finite Borel measures in $\mathbb{R}^d$, each absolutely continuous with respect to the Lebesgue measure. Then, there exist two parallel hyperplanes $H_1, H_2$ so that the region between them contains exactly half of each measure.*

If there is a hyperplane $H_1$ that halves all measures, we can consider $H_2$ to be at infinity. If $H_1$ and $H_2$ are not required to be parallel, we present with Theorem 3.2 a simple proof that *for any $d+1$ finite Borel measures in $\mathbb{R}^d$, there exist two closed half-spaces whose intersection contains exactly half of each measure.* The intersection of two half-spaces is called a wedge. The fact that $d+1$ measures in $\mathbb{R}^d$ can be halved by a wedge was first proved by Bárány and Matoušek in dimension two [BM01] and later generalized to $\mathbb{R}^d$ by Schnider [Sch19]. For $\mathbb{R}^2$, Bereg presented algorithmic approaches for the discrete version which show that more conditions can be imposed on the wedge [Ber05]. Before proving Theorem 1.2, we show how our methods simplify the proof of Schnider's extension of Bárány and Matoušek's result.

The proof requires a new Borsuk–Ulam type theorem about direct products of spheres and Stiefel manifolds, Theorem 3.4, which we describe in Section 3. As a corollary, we combine Theorem 1.2 with known lifting techniques. We nickname the following result the "bagel ham sandwich theorem", due to how it looks in $\mathbb{R}^2$.

**Corollary 1.3** (Bagel ham sandwich theorem)**.** *Let $d$ be a positive integer and $\mu_1, \ldots, \mu_{d+2}$ be $d+2$ finite Borel measures in $\mathbb{R}^d$, each absolutely continuous with respect to the Lebesgue measure. Then, we can find two concentric spheres or two parallel hyperplanes in $\mathbb{R}^d$ so that the closed region between them has exactly half of each measure.*

The case of parallel hyperplanes can be considered as a degenerate case of the concentric spheres, as spheres centered at infinity. Theorem 1.2 is optimal, as the region between the two hyperplanes is convex. One can simply take $d+1$ measures concentrated each around a vertex of a simplex and a final measure concentrated around the barycenter of the simplex to show that the result is impossible with $d+2$ measures. The problem of cutting the same fraction for a family of measures with a single convex set has been studied before [AK13, BB07], which we revisit in Section 6. Theorem 1.2 is also related to the problem of halving measures in $\mathbb{R}^d$ using hyperplane arrangements. Langerman conjectured that *any $dn$ measures in $\mathbb{R}^d$ can be simultaneously halved by a chessboard coloring induced by $n$ hyperplanes* [BPS19, HK20]. For $n=2$, this has been confirmed for $2d - O(\log d)$ measures [BBKK18]. If the hyperplanes are required to be parallel, this reduces the dimension of the space of possible partitions from $2d$ to $d+1$, matching the number of measures in Theorem 1.2.

Theorem 1.2 also provides a new direction to extend "necklace splitting results" to high dimensions. The necklace splitting theorem, due originally to Hobby and Rice [HR65, GW85, AW86] is a classic one-dimensional mass partition result. There exist several variations in high dimensions [LŽ08, KRPS16, BS18]. We establish the connection with this family of problems in section 6.

General mass partition results like the ham sandwich theorem can halve many measures simultaneously. If we want to cut a fixed (but possibly different) fraction of each measure, conditions need to be imposed. For example, if two measures coincide, it is impossible to find a half-space that contains exactly half of one and one third of the other.

The first result with arbitrary sizes for each measure was proved by Hugo Steinhaus in dimensions two and three [Ste45]. He required the support of the measures to be well separated, meaning that the supports of any set of measures could be separated from the supports of the rest by a hyperplane. This condition was sufficient to guarantee the existence of a half-space cutting a fixed fraction of several measures. This result was rediscovered and extended to high dimensions independently by Bárány, Hubard, and Jerónimo and by Breuer [BHJ08, Bre10].

**Theorem 1.4** (Bárány, Hubard, Jerónimo 2008; Breuer 2010)**.** *Let $d$ be a positive integer and $\mu_1, \dots, \mu_d$ be $d$ finite Borel measures in $\mathbb{R}^d$, each absolutely continuous with respect to the Lebesgue measure so that their supports $K_1, \dots, K_d$ are well separated. Let $\alpha_1, \dots, \alpha_d$ be real numbers in $(0,1)$. Then, there exists a half-space $H$ so that*

$$\mu_i(H) = \alpha_i \cdot \mu_i(\mathbb{R}^d) \qquad \textit{for } i = 1, \dots, d.$$

The proof of Bárány, Hubard, and Jerónimo uses Brouwer's fixed point theorem. Breuer's proof uses the Poincaré–Miranda theorem. The Poincare-Miranda theorem and Brouwer's fixed point theorem can easily be derived from one another. Steinhaus' proof is quite different and uses the Jordan curve theorem. In Section 4 we present a new proof of Theorem 1.4 that uses a degree argument. The method for this proof uses tools frequently used in extensions of the Knaster–Kuratowski–Mazurkiewicz theorem and its applications to "cake cutting". [KKM29, Su99, Gal84], as opposed to the equivariant topology tools commonly associated with high-dimensional mass partitions, combined with a parametrization argument of hyperplanes due to Hadwiger [Had57]. We extend Theorem 1.2 in a similar way for well separated measures.

**Theorem 1.5.** *Let $d$ be a positive integer and $\mu_1, \dots, \mu_{d+1}$ be $d+1$ finite Borel measures in $\mathbb{R}^d$, each absolutely continuous with respect to the Lebesgue measure. Suppose that the supports $K_1, \dots, K_{d+1}$ of $\mu_1, \dots, \mu_{d+1}$ are well separated. Let $\alpha_1, \dots, \alpha_{d+1}$ be real numbers in $(0,1)$. Then, there exist two parallel hyperplanes $H_1, H_2$ in $\mathbb{R}^d$ so that the region $A$ between them satisfies*

$$\mu_i(A) = \alpha_i \cdot \mu_i(\mathbb{R}^d) \qquad \textit{for all } i = 1, \dots, d+1.$$

We also combine the results with partitions with few hyperplanes and those of partitions using a single convex set. We exhibit conditions for measures in $\mathbb{R}^d$ that guarantee the existence of a (possibly unbounded) convex polyhedron of few facets which contains a fixed fraction of each measure or the existence of a convex polytope with few vertices that contains a fixed fraction of each measure. This is done in Section 5. These results work with an arbitrary number of measures in $\mathbb{R}^d$.

Finally, we revisit a mass partition result by Akopyan and Karasev that uses a lifting argument in its proof. Akopyan and Karasev proved that for any positive integer $n$ and any $d+1$ Borel measures in $\mathbb{R}^d$, there exists a convex set $K$ whose measure is exactly $1/n$ of each measure. We extend the methods from Section 3 to bound the complexity of $K$ by writing it as the intersection of few half-spaces.

**Theorem 1.6.** *Let $n, d$ be positive integers and $\mu_1, \dots, \mu_{d+1}$ be $d+1$ finite Borel measures in $\mathbb{R}^d$, each absolutely continuous with respect to the Lebesgue measure.*

*There exists a convex set $K$, such that $K$ is the intersection of $\sum_{j=1}^{r} k_j(p_j-1)p_j$ half-spaces and*

$$\mu_i(K) = \frac{1}{n}\mu_i(\mathbb{R}^d) \qquad \text{for all } i = 1, \dots, d+1,$$

*where $n = p_1^{k_1} \dots p_r^{k_r}$ is the prime factorization of $n$.*

We conclude in Section 6 with remarks and open problems.

## 2. Equipartition with spheres and sine curves

While the traditional ham sandwich theorem simultaneously halves $d$ measures in $\mathbb{R}^d$ by a hyperplane, we can simultaneously halve $d+1$ or more measures in $\mathbb{R}^d$ if we increase the complexity of the cut. The following theorem is a consequence of Stone and Tukey's polynomial ham sandwich theorem [ST42] and was one of Stone and Tukey's first examples of their main results. It was also proved in dimension two by Hugo Steinhaus in 1945 [Ste45] using a particular parametrizations of the space of circles in $\mathbb{R}^2$. We present a new proof with a stereographic projection.

**Theorem 2.1.** *Let $d$ be a positive integer and let $\mu_1, \dots, \mu_{d+1}$ be $d+1$ finite Borel measures in $\mathbb{R}^d$, each absolutely continuous with respect to the Lebesgue measure. Then, there exists either a sphere or a hyperplane that simultaneously splits each measure in half.*

*Proof.* We first embed $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$ by appending a coordinate 1 to each point, so $x \mapsto (x, 1) \in \mathbb{R}^{d+1}$. Then, we apply $r : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{R}^{d+1} \setminus \{0\}$ the inversion centered at 0 with radius 1. This is a transformation that sends spheres containing the origin to hyperplanes and hyperplanes to spheres containing the origin. Hyperplanes containing the origin are fixed set-wise by the inversion. We consider them as degenerate spheres as well. Restricted to the embedding of $\mathbb{R}^d$, the inversion is a stereographic projection to the sphere $S$ of radius $1/2$ centered at $(0, \dots, 0, 1/2)$ by rays through the origin. We also know that $r \circ r$ is the identity.

When we lift the measures $\mu_1, \dots, \mu_{d+1}$ to $\mathbb{R}^{d+1}$ and apply $r$, we get measures $\sigma_1, \dots, \sigma_{d+1}$ on $S$. By the ham sandwich theorem in $\mathbb{R}^{d+1}$, there exists a hyperplane $H$ halving each of $\sigma_1, \dots, \sigma_{d+1}$. Since $r(H)$ is a sphere in $\mathbb{R}^{d+1}$, it intersects the embedding of $\mathbb{R}^d$ in a $(d-1)$-dimensional sphere halving each of $\mu_1, \dots, \mu_{d+1}$, as we wanted. The only exceptional case is if $H$ contains the origin, in which case $r(H) = H$, which gives us an equipartition by a hyperplane in $\mathbb{R}^d$. □

Similarly, the idea of lifting allows for visual and intuitive proofs of the existence of equipartitions of three sets in $\mathbb{R}^2$ by a sine wave. By a sine wave of period $\alpha$ we mean the graph of a function of the form $y = r + A\sin(2\pi x/\alpha + s)$, for real numbers $A, r, s$.

**Theorem 2.2.** *Let $\alpha > 0$ be a real number. Given three finite Borel measures $\mu_1, \mu_2, \mu_3$ in $\mathbb{R}^2$, each absolutely continuous with respect to the Lebesgue measure, there exists a sine wave of period $\alpha$ halving each measure.*

We allow "degenerate" sine waves of period $\alpha$. A degenerate sine wave of period $\alpha$ is formed by taking two vertical lines intersecting the $x$-axis in the interval $[0, \alpha)$ and making a translated copy in each interval of the form $n\alpha + [0, \alpha)$. This set induces a chessboard coloring of the plane into two regions. We can think of this as the limit of a sequence of sine waves of period $\alpha$ of increasing amplitude.

*Proof.* We prove the result for $\alpha = 2\pi$, as the two cases are equivalent after a scaling argument. We wrap $\mathbb{R}^2$ around the cylinder $C$ in $\mathbb{R}^3$ with equation $x^2 + z^2 = 1$ with the function

$$\begin{aligned} f : \mathbb{R}^2 &\to C \\ (x, y) &\mapsto (\cos(x), y, \sin(x)). \end{aligned}$$

Let $\sigma_1, \sigma_2, \sigma_3$ be the measures that $\mu_1, \mu_2, \mu_3$ induce on $C$ by this lifting, respectively. We apply the ham sandwich theorem to these three measures in $\mathbb{R}^3$. Therefore, we can find a plane $H = \{(x, y, z) : ax + by + cz = d\}$ that halves each of $\sigma_1, \sigma_2, \sigma_3$. When we pull $H \cap C$ back to $\mathbb{R}^2$, we get the set of points $(x, y)$ that satisfy $a\cos(x) + by + c\sin(x) = d$. Since a linear combination of the sine and cosine functions is a sinusoid with the same period but possibly different amplitude and phase shift, we have $a\sin(x) + c\cos(x) = A\sin(x + s)$ for some $A$ and $s$.

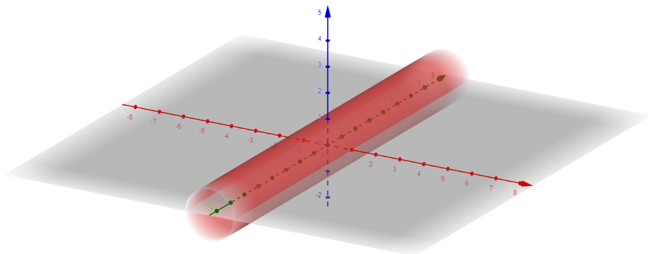


FIGURE 1. An example of a cylinder $C$ with period $\alpha = 2\pi$. The lift $\mathbb{R}^2 \to C$ is not an injective function, but this does not cause a problem.

□

The degenerate cases appear when $b$, the coefficient of $y$, is zero. One can prove a high-dimensional version of Theorem 2.2 by wrapping $\mathbb{R}^d$ around $S^{d-1} \times \mathbb{R}$ to find "sinusoidal surfaces of fixed period" that halve $d+1$ measures in $\mathbb{R}^d$.

## 3. Equipartitions with wedges and parallel hyperplanes

In this section, we prove results regarding equipartitions of $d+1$ mass distributions in $\mathbb{R}^d$ by a wedge. A *wedge* in $\mathbb{R}^d$ is the intersection of two closed half-spaces. Note that a single closed half-space is also considered a wedge.

We say that a measure $\mu$ with support $K$ in $\mathbb{R}^d$ is *absolutely continuous* if it is absolutely continuous with respect to the Lebesgue measure and $K$ is connected and not empty. Note that by the definition of support, for every open set $U \subset K$ we have $\mu(U) > 0$. This guarantees that there is a unique halving hyperplane in each direction for $\mu$ and that the halving hyperplane varies continuously as we change the direction of the cut. We first establish a lemma about halving hyperplanes. We only use the lemma below with $n = d + 1$, but it works in general.

**Lemma 3.1.** *Let $n$ be a positive integer, $\mu_1, \ldots, \mu_n$ be finite absolutely continuous Borel measures in $\mathbb{R}^d$, and $v$ be a unit vector in $\mathbb{R}^d$. There either exists a hyperplane $H$ orthogonal to $v$ that halves each of the $n$ measures or there exists a hyperplane $H$ such that its two closed half-spaces satisfy*

$$\begin{aligned} \mu_i(H^+) &< \frac{1}{2}\mu_i(\mathbb{R}^d) \qquad \textit{for some } i \in [n] \textit{ and} \\ \mu_{i'}(H^-) &< \frac{1}{2}\mu_{i'}(\mathbb{R}^d) \qquad \textit{for some } i' \in [n]. \end{aligned}$$

*Moreover, the hyperplane $H$ can be chosen continuously as a function of $v$ so that the same choice is made for $v$ and $-v$.*

*Proof.* For each $i$, let $H_i$ be the halving hyperplane for $\mu_i$ orthogonal to $v$. If all these hyperplanes coincide, we are done. Otherwise, we can order this set of hyperplanes by the direction $v$, and any hyperplane $H$ strictly between the first $H_i$ and the last $H_{i'}$ satisfies the conditions we want.

To show the continuity with respect to $v$, first note that each $H_i$ can be chosen uniquely and continuously. Each $\mu_i$ has a closed interval of possibilities for $H_i$ and we can choose the midpoint of such an interval. If we denote by $\ell$ the line through the origin with direction $v$, this makes $H_i \cap \ell$ a continuous function. We can then choose $H$ so that $H \cap \ell$ is the midpoint of the set $\{H_i \cap \ell : i \in [n]\}$, so the continuity on $v$ follows. The choice also makes it so that the same hyperplane is selected for $-v$. □

Through the rest of the manuscript, we denote the canonical basis of $\mathbb{R}^d$ by $e_1, \ldots, e_d$.

**Theorem 3.2.** *Let $d$ be a positive integer and $\mu_1, \ldots, \mu_{d+1}$ be $d+1$ finite absolutely continuous Borel measures in $\mathbb{R}^d$. Then, there exists a wedge that contains exactly half of each measure.*

Recall that the region between two parallel hyperplanes satisfies our definition of a wedge. This can be considered as a wedge where the intersection is a $(d-2)$-dimensional space at infinity.

*Proof.* Let $v$ be a unit vector in $\mathbb{R}^d$. If there is a hyperplane $H$ orthogonal to $v$ halving each measure we are done. Otherwise, by Lemma 3.1, we can find a hyperplane $H$ such that each side contains less than half of some measure.

Consider the lifting of $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$ where we append an additional coordinate to every point $x \in \mathbb{R}^d$. Formally, we lift via the map $x \mapsto (x, \operatorname{dist}(x, H))$.

We denote by $S(H)$ the image of $\mathbb{R}^d$ in this embedding. Note that the function $x \mapsto \operatorname{dist}(x, H)$ is affine on each side of $H$, so $S(H)$ is contained in the union of two hyperplanes that contain $\{(x, 0) : x \in H\}$. See Fig. 2 for an illustration of the case $d = 2$.

We lift each measure $\mu_i$ in $\mathbb{R}^d$ to a measure $\sigma_i$ in $\mathbb{R}^{d+1}$. The measures $\sigma_1, \ldots, \sigma_{d+1}$ are no longer absolutely continuous. We now apply the ham sandwich theorem for general measures in $\mathbb{R}^{d+1}$. Therefore, we can find a hyperplane $H'$ in $\mathbb{R}^{d+1}$ so that its two closed half-spaces $(H')^+, (H')^-$ satisfy $\sigma_i((H')^+) \geq \frac{1}{2}\sigma_i(\mathbb{R}^{d+1})$ and $\sigma_i((H')^-) \geq \frac{1}{2}\sigma_i(\mathbb{R}^{d+1})$ for all $i = 1, \ldots, d+1$.

By construction, each side of $H$ has strictly less than half of one of the measures $\mu_i$. If the hyperplane $H'$ coincides with one of the two hyperplanes whose union contains $S(H)$, the half-space bounded by $H'$ that contains infinite rays in the direction $-e_{d+1}$ would have less than half of the corresponding measure $\sigma_i$. Therefore $H'$ is not one of the two hyperplanes forming $S(H)$.

As the two components of $S(H)$ were the only hyperplanes with non-zero measure for each $\sigma_i$, we conclude that $H'$ halves each of the measures in $\mathbb{R}^{d+1}$. As a final observation, $H' \cap S(H)$ projects back to $\mathbb{R}^d$ as the boundary of a wedge that halves all measures. □

In the proof of Theorem 3.2 we could choose the direction $v$ arbitrarily. We now use this degree of freedom to strengthen the result. Even though Theorem 1.2 implies Theorem 3.2, we state it separately as the proof requires more technical tools. In particular, a simple application of the ham sandwich theorem is insufficient. We require some additional topological tools in lieu of the Borsuk–Ulam theorem.

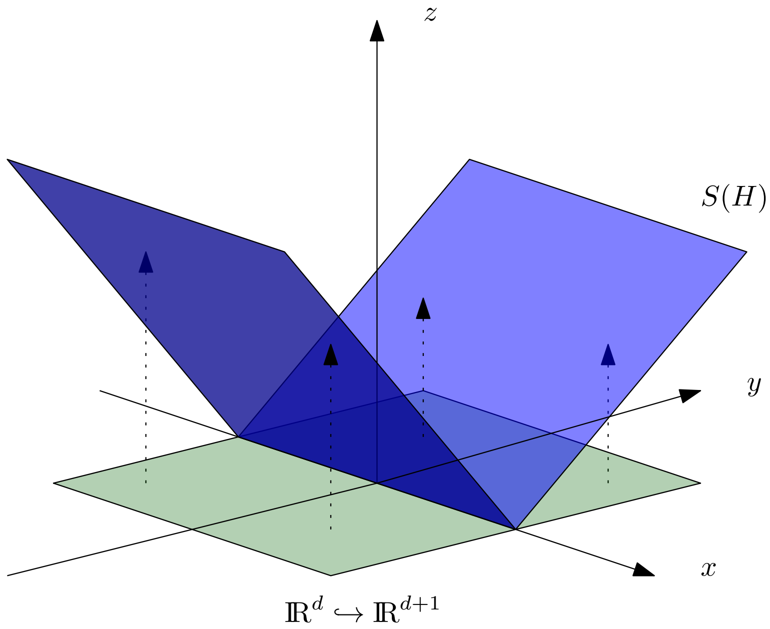



FIGURE 2. An example of lifting when given three measures in $\mathbb{R}^2$. Measures on the $xy$-plane are sent to measures on this surface. The hyperplane $H$ in this case is the $x$-axis.

Let $V_k(\mathbb{R}^d)$ be the Stiefel manifold of orthonormal $k$-frames in $\mathbb{R}^d$. Formally,

$$V_k(\mathbb{R}^d) = \{(v_1, \ldots, v_k) : v_1, \ldots, v_k \in \mathbb{R}^d \text{ are orthonormal}\}.$$

The space $V_k(\mathbb{R}^d)$ has a free action of the group $(\mathbb{Z}_2)^k$, where we consider $\mathbb{Z}_2 = \{+1, -1\}$ with multiplication. Given $(v_1, \ldots, v_k) \in V_k(\mathbb{R}^d)$ and $(\lambda_1, \ldots, \lambda_k) \in (\mathbb{Z}_2)^k$, we define

$$(\lambda_1, \ldots, \lambda_k) \cdot (v_1, \ldots, v_k) = (\lambda_1 v_1, \ldots, \lambda_k v_k) \in V_k(\mathbb{R}^d).$$

A similar action of $(\mathbb{Z}_2)^k$ can be defined in $\mathbb{R}^{d-1} \times \ldots \times \mathbb{R}^{d-k}$, as the direct product of the actions of $\mathbb{Z}_2$ on each $\mathbb{R}^{d-i}$. A recent result of Chan, Chen, Frick, and Hull describes properties of $(\mathbb{Z}_2)^k$-equivariant maps between these two spaces.

**Theorem 3.3** (Chan, Chen, Frick, Hull 2020 [CCFH20])**.** *Let $k, d$ be positive integers. Every continuous $(\mathbb{Z}_2)^k$-equivariant map $f : V_k(\mathbb{R}^d) \to \mathbb{R}^{d-1} \times \ldots \times \mathbb{R}^{d-k}$ has a zero.*

Manta and Soberón recently found an elementary proof of Theorem 3.3 [MS21]. We use the result above in Section 4. For this section, we need a slight modification. We use the product of the actions of $\mathbb{Z}_2$ on the $d$-dimensional sphere $S^d$ and of $(\mathbb{Z}_2)^k$ on $V_k(\mathbb{R}^d)$ to define a free action of $(\mathbb{Z}_2)^{k+1}$ on $S^d \times V_k(\mathbb{R}^d)$.

**Theorem 3.4.** *Let $k, d$ be positive integers. Every continuous $(\mathbb{Z}_2)^{k+1}$-equivariant map $f : S^d \times V_k(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^{d-1} \times \ldots \times \mathbb{R}^{d-k}$ has a zero.*

There are several ways to prove the result above. The dimension of the image and the domain are the same, and the action of $(\mathbb{Z}_2)^{k+1}$ is free on $S^d \times V_k(\mathbb{R}^d)$. Therefore, Theorem 3.4 is a consequence of the general Borsuk–Ulam type results of Musin [Mus12]. We present this as our second proof. Alternatively, one can use the methods of Chan et al. [CCFH20] to prove Theorem 3.4. It suffices to note that $S^d \times V_k(\mathbb{R}^d)$ is a space in which their topological invariants can be applied and the particular function $g$ in our second proof is all that's needed to replace [CCFH20, second proof of Lemma 3.2]. We only use Theorem 3.4 for $k = d-1$.

We present first a short proof using the existing computations of the Fadell–Husseini index of these spaces on $\mathbb{Z}_2$ cohomology [FH88]. Given spaces $X$ and $Y$ with actions of $(\mathbb{Z}_2)^{k+1}$, their indices $\mathrm{Ind}^{(\mathbb{Z}_2)^{k+1}}(X)$, $\mathrm{Ind}^{(\mathbb{Z}_2)^{k+1}}(Y)$ are ideals in the polynomial ring $\mathbb{Z}_2[t_0, t_1, \ldots, t_k]$. Moreover, if there exists a continuous $(\mathbb{Z}_2)^{k+1}$-equivariant map $f : X \to Y$, we must have $\mathrm{Ind}^{(\mathbb{Z}_2)^{k+1}}(Y) \subset \mathrm{Ind}^{(\mathbb{Z}_2)^{k+1}}(X)$. More details on this index and its computation for spaces and group actions common in discrete geometry can be found in recent work of Blagojević, Lück, and Ziegler [BLZ15].

*First proof of Theorem 3.4.* The result is equivalent to showing that there exists no continuous $(\mathbb{Z}_2)^{k+1}$-equivariant map $f : S^d \times V_k(\mathbb{R}^d) \to (\mathbb{R}^d \times \mathbb{R}^{d-1} \times \ldots \times \mathbb{R}^{d-k}) \setminus \{0\}$. The space $(\mathbb{R}^d \times \ldots \times \mathbb{R}^{d-k}) \setminus \{0\}$ is homotopy equivalent to the join of spheres $S^{d-1} * \ldots * S^{d-k-1}$. We know $\mathrm{Ind}^{(\mathbb{Z}_2)^{k+1}}(S^{d-1} * \ldots * S^{d-k-1}) \subset \mathbb{Z}_2[t_0, t_1, \ldots, t_k]$ is the ideal generated by the single monomial $t_0^d t_1^{d-1} \ldots t_k^{d-k}$.

On the other hand, $\mathrm{Ind}^{(\mathbb{Z}_2)^{k+1}}(S^d \times V_k(\mathbb{R}^d)) \subset \mathbb{Z}_2[t_0, \ldots, t_k]$ is the ideal generated by the polynomials $t_0^{d+1}, f_1, \ldots, f_k$ where $f_1, \ldots, f_k \subset \mathbb{Z}_2[t_1, \ldots, t_k]$ generate $\mathrm{Ind}^{(\mathbb{Z}_2)^k}(V_k(\mathbb{R}^d))$. These polynomials were described completely by Fadell and Husseini [FH88, Thm. 3.16]. Notably

$$f_i = t_i^{n-i+1} + w_{i,n-i} t_i^{n-i} + \ldots + w_{i,0},$$

where $w_{i,j} \in \mathbb{Z}_2[t_1, \ldots, t_{i-1}]$ and the degree of $w_{i,j} t_i^j$ is $n - i + 1$. In particular,

$$t_0^d t_1^{d-1} \ldots t_k^{d-k} \notin \mathrm{Ind}^{(\mathbb{Z}_2)^{k+1}}(S^d \times V_k(\mathbb{R}^d)),$$

which shows that no continuous $(\mathbb{Z}_2)^{k+1}$-equivariant map $f : S^d \times V_k(\mathbb{R}^d) \to (\mathbb{R}^d \times \mathbb{R}^{d-1} \times \ldots \times \mathbb{R}^{d-k}) \setminus \{0\}$ exists. $\square$

*Second proof of Theorem 3.4.* We construct a second map $g : S^d \times V_k(\mathbb{R}^d) \to \mathbb{R}^d \times \ldots \times \mathbb{R}^{d-k}$. For $(u, v) \in S^d \times V_k$, we denote $g(u, v) = (x_0, x_1, \ldots, x_k)$ where $x_i \in \mathbb{R}^{d-i}$ for $0 \leq i \leq k$. We take $x_0$ formed by the last $d$ entries of $u \in S^d \subset \mathbb{R}^{d+1}$, and for $1 \leq i \leq k$ we take $x_i$ to be formed by the last $d-i$ entries of the $i$-the vector in $v \in V_k(\mathbb{R}^d)$. This map is $(\mathbb{Z}_2)^{k+1}$-equivariant, continuous, and there is a single orbit of zeroes.

Now consider a new map

$$\begin{aligned} F : S^d \times V_k(\mathbb{R}^d) \times [0,1] &\to \mathbb{R}^d \times \ldots \times \mathbb{R}^{d-k} \\ (u, v, t) &\mapsto t g(u, v) + (1-t) f(u, v) \end{aligned}$$

This is a continuous $(\mathbb{Z}_2)^{k+1}$-equivariant map. For any $\varepsilon$, Thom's transversality theorem [Tho54; GP10, pp 68-69] implies that there exist a map $F_\varepsilon : S^d \times V_k(\mathbb{R}^d) \times [0,1]$ such that the two following conditions hold: for each $(u, v) \in S^d \times V_k(\mathbb{R}^d)$, we have $F_\varepsilon(u, v, 1) = F(u, v, 1) = g(u, v)$, 0 is a regular value of $F_\varepsilon$, and $\|F_\varepsilon - F\|_\infty < \varepsilon$. Note that, since the domain is compact, the infinity norm is well defined for any continuous map from $S^d \times V_k(\mathbb{R}^d)$ to $\mathbb{R}^d \times \ldots \times \mathbb{R}^{d-k}$.

The set $F_\varepsilon^{-1}(0) \in S^d \times V_k(\mathbb{R}^d) \times [0,1]$ is therefore a one-dimensional manifold with a free action of $(\mathbb{Z}_2)^{k+1}$. Its connected components are diffeomorphic to circles or intervals. The interval components must have their endpoints in $S^d \times V_k(\mathbb{R}^d) \times \{0, 1\}$. Any continuous function from an interval to itself must have a fixed point, so the group $(\mathbb{Z}_2)^{k+1}$ acts freely on the set of intervals in $F_\varepsilon^{-1}(0)$.

Therefore, the parity of $(\mathbb{Z}^2)^{k+1}$ orbits on $F_\varepsilon^{-1}(0)$ with $t = 0$ and the parity of $(\mathbb{Z}^2)^{k+1}$ orbits on $F_\varepsilon^{-1}(0)$ with $t = 1$ must be the same. Since for $t = 1$ there is a single orbit of zeros, there must be at least one orbit of zeroes for $t = 0$. The function $F_\varepsilon$ restricted to $t = 1$ is very similar to $f$. As we let $\varepsilon$ tend to zero, the compactness of $S^d \times V_k(\mathbb{R}^d)$ implies that $f^{-1}(0)$ is not empty. □

The second tool we require is a minor modification of the lift from $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$. In the previous proof, given a hyperplane $H \subset \mathbb{R}^d$ we lifted $\mathbb{R}^d$ directly to $S(H) \subset \mathbb{R}^{d+1}$. This has the inconvenience that the lift of an absolutely continuous measure is no longer absolutely continuous in $\mathbb{R}^{d+1}$. We do not require such a strong condition, but we do require the lifted measures to assign mass zero to any hyperplane. To avoid this problem, we lift each measure $\mu_i$ to $S(H)^\varepsilon$, the region between $S(H) - \varepsilon \cdot e_{d+1}$ and $S(H) + \varepsilon \cdot e_{d+1}$, which we formalize below.

We say that a measure $\mu$ in $\mathbb{R}^d$ is smooth if it is the integral of a continuous positive function $f : \mathbb{R}^d \to \mathbb{R}$ (i.e., $f(A) = \int_A f$ for any measurable set $A$). We "lift" $f$ to a function

$$\tilde{f} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$$
$$(x,t) \mapsto \begin{cases} \left(\frac{1}{2\varepsilon}\right) f(x) & \text{if } |\operatorname{dist}(x,H) - t| \le \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

We say that the measure $\sigma^\varepsilon$ defined as the integral of $\tilde{f}$ in $\mathbb{R}^d$ is the lift of $\mu$ to $S(H)^\varepsilon$. Notice that as $\varepsilon \to 0$, the measure $\sigma^\varepsilon$ converges weakly to the lift of $\mu$ to $S(H)$. For $\varepsilon > 0$, the measure $\sigma^\varepsilon$ is not absolutely continuous, but it has value 0 on each hyperplane. Note that the same lifting process can be done for absolutely continuous measures having densities, which may generalize the proof below for a wider family of problems.

*Proof of Theorem 1.2.* We first assume that no hyperplane simultaneously halves all measures, or we are done. Since the set of smooth measures is dense in the set of absolutely continuous measures, we may assume without loss of generality that the measures $\mu_1, \dots, \mu_{d+1}$ are smooth. Let $\varepsilon > 0$. For $v \in S^d$ and $(v_1, \dots, v_{d-1}) \in V_{d-1}(\mathbb{R}^d)$, consider the element $(v, v_1, \dots, v_{d-1}) \in S^d \times V_{d-1}(\mathbb{R}^d)$.

Let $H$ be the translate of the hyperplane $T = \operatorname{span}\{v_1, \dots, v_{d-1}\}$ chosen from Lemma 3.1. Let $\sigma_1^\varepsilon, \dots, \sigma_{d+1}^\varepsilon$ be the lifts of $\mu_1, \dots, \mu_{d+1}$ to $S(H)^\varepsilon$, respectively.

Let $\lambda$ be the value so that the half-spaces

$$A = \{x \in \mathbb{R}^{d+1} : \langle x, v \rangle \ge \lambda\} \quad \text{and}$$
$$B = \{x \in \mathbb{R}^{d+1} : \langle x, v \rangle \le \lambda\}$$

have the same $\sigma_{d+1}^\varepsilon$-measure. Now we are ready to define a map

$$f : S^d \times V_{d-1}(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^{d-1} \times \ldots \times \mathbb{R}^1$$
$$(v, v_1, \dots, v_{d-1}) \mapsto (x_d, \dots, x_1)$$

where $x_i \in \mathbb{R}^i$ for $i = 1, \dots, d$. First, we consider

$$x_d = \begin{bmatrix} \sigma_1^\varepsilon(A) - \sigma_1^\varepsilon(B) \\ \vdots \\ \sigma_d^\varepsilon(A) - \sigma_d^\varepsilon(B) \end{bmatrix}.$$

For $i = 1, \dots, d-1$, the first coordinate of $x_i$ is $\langle v, e_{d+1} \rangle \langle v, (v_{d-i}, 0) \rangle$ and the rest are zero.

This function is continuous by the construction of the lift of the measures. It is also $(\mathbb{Z}_2)^{k+1}$-equivariant: if we flip the sign of $v$, only $x_d$ changes sign and if we flip the sign of $v_i$ only $x_{d-i}$ changes sign for $i = 1, \dots, d-1$.

By Theorem 3.4 the function $f$ has a zero. The condition $x_d = 0$ tells us that $A$ and $B$ each have exactly half of each $\sigma_i^\varepsilon$ for $i = 1, \dots, d+1$. The conditions of the rest of the $x_i$ being zero vectors means that either $v$ is orthogonal to $e_{d+1}$ or $v$ is orthogonal to each $(v_i, 0)$ for $i = 1, \dots, d-1$.

If the first condition happens, then when we project $\mathbb{R}^{d+1}$ to the hyperplane $e_{d+1} = 0$, each $\sigma_i^\varepsilon$ projects to $\mu_i$ and $A$, $B$ project onto two half-spaces of $\mathbb{R}^d$. This would mean we have a hyperplane halving each of the original measures, contradicting our initial assumption. Therefore $v$ is orthogonal to $(v_i, 0)$ for all $i$.

We take a sequence of positive real numbers $\varepsilon_k \to 0$. For each of them, we find a zero of the function induced above. As $S^d \times V_{d-1}(\mathbb{R}^d)$ is compact, the zeros must have a converging subsequence. In the limit, we obtain two complementary half-spaces $A, B$ so that each contains at least half of $\sigma_i$ for $i = 1, \dots, d+1$ on $S(H)$ (where the direction of $H$ is determined by the limit of $(v_1, \dots, v_{d-1})$). Let $H'$ be the hyperplane at the boundary of $A$ and $B$.

In the limit, the vector $v$ normal to $H'$ is orthogonal to the subspace $T_0 = \operatorname{span}\{(v_1, 0), \dots, (v_{d-1}, 0)\}$. Note that $T_0$ is a copy of $H$ with a coordinate zero appended to each point. By the construction of $H$, the hyperplane $H'$ cannot be one of the two hyperplane components of $S(H)$. The orthogonality mentioned before implies that $H'$ contains a translate of $T_0$ through each of its points, and so does each of the two hyperplane components of $S(H)$. Therefore, $H' \cap S(H)$ must be two $(d-1)$-dimensional affine spaces parallel to $T_0$. When we project back to $\mathbb{R}^d$, these parallel intersections form $H_1$ and $H_2$ and the region between them has exactly half of each $\mu_i$. □

The configuration space used in this proof, $S^d \times V_{d-1}(\mathbb{R}^d)$ is larger than needed for our purposes. Indeed, many elements of $V_{d-1}(\mathbb{R}^d)$ lead us to the same construction for the affine space $H$. The benefit of this space is that it might be possible to extend such results to spaces of partitions that are more sensitive to the orthonormal set given by $V_{d-1}(\mathbb{R}^d)$ than simply its span. A clear example are Yao–Yao partitions of $\mathbb{R}^d$ into $2^d$ convex sets, which depend on a choice of an orthonormal basis [YY85]. It would be interesting to know which results of that kind are amenable with lifting arguments as presented.

*Proof of Corollary 1.3.* We lift $\mathbb{R}^d$ to the paraboloid

$$\mathcal{P} = \left\{ \left( x_1, \dots, x_d, \sum_{i=1}^{d} x_i^2 \right) \in \mathbb{R}^{d+1} \right\}.$$

We now have $d+2$ measures in $\mathbb{R}^{d+1}$, and every hyperplane has measure zero in each of them. We can apply Theorem 1.2 and find two parallel hyperplanes $H_1, H_2$ so that the region between them contains exactly half of each measure. Notice that $H_1 \cap \mathcal{P}$ and $H_2 \cap \mathcal{P}$ project onto concentric spheres in $\mathbb{R}^d$. They can also project to two parallel hyperplanes, which we consider as a valid degenerate case of our result. □

Of course, similar results can be obtained by applying general Veronese maps as Stone and Tukey did to prove the polynomial ham sandwich theorem [ST42].

**Corollary 3.5.** *Let $d, k$ be positive integers. For any set of $\binom{d+k}{k}$ finite absolutely continuous Borel measures in $\mathbb{R}^d$, there exists a polynomial $P$ on $d$ variables and constants $\lambda_1, \lambda_2$ so that $\{x \in \mathbb{R}^d : \lambda_1 \le P(x) \le \lambda_2\}$ has exactly half of each measure.*

The polynomial ham sandwich usually splits $\binom{d+k}{k} - 1$ measures using a polynomial of degree at most $k$. If we want to restrict which monomials are used in the splitting polynomial, we just have to reduce the number of measures accordingly.

## 4. Fixed size partitions for well separated measures

As noted by Bárány et al. [BHJ08], it is known that for well separated convex subsets $K_1, \ldots, K_d$ in $\mathbb{R}^d$, there are $2^d$ hyperplanes tangent to all of them. If none of the hyperplanes are vertical (i.e., perpendicular to $e_d$), the tangent hyperplanes are in one-to-one correspondence with the sets of $K_i$ below the hyperplane. This is was also proved by Klee, Lewis, and Hohenbalken [KLVH97]. We use this fact in our proof of Theorem 1.4.

For the sake of completeness, we prove a technical lemma. This argument is often used to prove envy-free results in dimension 1, so-called "cake partitions". See, e.g., [Wel85, BT96, Su99, Bar05, Pro16] and the references therein for more applications of this method. It is interesting to see this argument used in mass partition results where the usual methods come from equivariant topology.

**Lemma 4.1.** *Let $P$ be a convex polytope and $f : \partial P \to \partial P$ be a continuous function on its boundary. If $f(\sigma) \subset \sigma$ for each proper face $\sigma$ of $P$, then $f$ is of degree one.*

*Proof.* It suffices to exhibit a homotopy between $f$ and the identity map on the boundary of $P$. For $\lambda \in [0,1]$ we define

$$\begin{aligned} f_\lambda : \partial P &\to P \\ x &\mapsto \lambda x + (1-\lambda) f(x). \end{aligned}$$

We have $f_0 = f$ and $f_1$ equal to the identity. Moreover, for $x \in \partial P$, there exists a proper face $\sigma$ containing it. The convexity of $\sigma$ implies that $f_\lambda(x) \in \sigma \subset \partial P$. Therefore, this provides the homotopy we wanted. $\square$

*Proof of Theorem 1.4.* We assume without loss of generality that $\mu_1, \ldots, \mu_d$ are probability measures. Consider the hypercube $Q = [0,1]^d$. Each vertex of $Q$ can be assigned to a subset of $I \subset [d]$ uniquely. We denote

$$v_I = (p_1, \ldots, p_d) \qquad \text{where} \qquad p_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \notin I. \end{cases}$$

For a point $q = (q_1, \ldots, q_d) \in Q$ and $I \subset [d]$ we consider the coefficients

$$\lambda_I(q) = \prod_{i \in I} q_i \prod_{i \notin I} (1 - q_i).$$

Expanding the product $\prod_{i \in [d]} (q_i + (1 - q_i))$ shows that the sum of the $\lambda_I(q)$ is 1. Therefore, the coefficients $\lambda_I(q)$ are the coefficients of a convex combination, as they are also non-negative. Suppose we have a function $f : \{0,1\}^d \to \mathbb{R}^d$. We can extend it to a function $\tilde{f} : Q \to \mathbb{R}^d$ by mapping

$$q \mapsto \sum_{I \subset [d]} \lambda_I(q) f(v_I).$$

Notice that if $\sigma$ is a face of $Q$, then $\tilde{f}(\sigma) \subset \operatorname{conv}(\{f(v_I) : v_I \in \sigma\})$. In particular, $\tilde{f}(v_I) = f(v_I)$.

Now, suppose we are given $d$ well separated convex sets $K_1, \ldots, K_d$ in $\mathbb{R}^d$ and measures $\mu_1, \ldots, \mu_d$ so that the support of $\mu_i$ is $K_i$. We may assume without loss of generality that there is no vertical hyperplane tangent to each of $K_1, \ldots, K_d$.

Each non-vertical hyperplane $H$ can be written as

$$\{(x_1, \ldots, x_d) : x_d = \alpha_1 x_1 + \ldots + \alpha_{d-1} x_{d-1} + \alpha_d\}$$

for some constants $\alpha_1, \ldots, \alpha_d$. We assign the vector $r(H) = (\alpha_1, \ldots, \alpha_d)$ to the hyperplane $H$. We say that a point is above $H$ if $x_d \geq \alpha_1 x_1 + \ldots + \alpha_{d-1} x_{d-1} + \alpha_d$ and below $H$ if $x_d \leq \alpha_1 x_1 + \ldots + \alpha_{d-1} x_{d-1} + \alpha_d$. Notice that if a point $x$ is below a set of hyperplanes $H_1, \ldots, H_k$, then it is also below the hyperplane $r^{-1}(y)$ for any

$y \in \operatorname{conv}(r(H_1), \dots, r(H_k))$, as a direct manipulation of the inequalities induced on $x_d$ shows. Similarly, if a point $x$ is above a set of hyperplanes $H_1, \dots, H_k$, then it is also above the hyperplane $r^{-1}(y)$ for any $y \in \operatorname{conv}(r(H_1), \dots, r(H_k))$.

We know that for each subset $I \subset [d]$, there is a unique hyperplane $H_I$ that is tangent to each $K_i$ and so that $K_i$ is below $H$ if and only if $i \in I$. This defines a function $f : \{0,1\}^d \to \mathbb{R}^d$ by simply taking $f(v_I) = r(H_I)$. We extend $f$ to a function $\tilde{f} : Q \to \mathbb{R}^d$ as described above. For $q \in Q$, let $H(q)$ be the set of points below $r^{-1}(\tilde{f}(q))$.

We define a final function

$$\begin{aligned} g : Q &\to Q \\ q &\mapsto (\mu_1(H(q)), \dots, \mu_d(H(q))) \end{aligned}$$

The function $g$ is continuous. Notice, for example, that $g(v_I) = v_I$ for each $I \subset [d]$. Let us show that for every face $\sigma \subset Q$, we have $g(\sigma) \subset \sigma$.

It is sufficient to prove this when $\sigma$ is a facet, as every face is an intersection of facets. Take $i \in [d]$ and consider the facet $\sigma$ formed by all the vertices $V_I$ for which the $i$-th coordinate is 1. Let $q \in \sigma$. Since $K_i$ is below each vertex $v \in \sigma$, we know that $K_i$ is below $H(q)$. Therefore, the $i$-coordinate of $g(q)$ satisfies $1 \ge \mu_i(H(q)) \ge \mu_i(K_i) = 1$. Therefore, $g(q) \in \sigma$. For facets defined by a coordinate equal to zero, the same argument holds, now with $K_i$ above each hyperplanes involved.

This means that $g$ is of degree one on the boundary, so it is surjective. In particular, there is a point $q_0 \in Q$ such that $g(q_0) = (\alpha_1, \dots, \alpha_d)$. Therefore, the hyperplane $H(q_0)$ is the hyperplane we were looking for. □

Bárány, Hubard, and Karasev also showed that, under simple conditions, the half-space $H$ from Theorem 1.4 is unique. It suffices that

- each measure $\mu_i$ assigns a positive value to each open set in its support $K_i$,
- Each $K_i$ is connected and not empty,
- no vertical hyperplane is tangent to $K_1, \dots, K_d$, and
- the half-space $H$ contains infinite rays in direction $-e_d$.

*Proof of Theorem 1.5.* We follow a process similar to the proof of Theorem 1.2. First we need an additional observation about our construction of $S(H)$. In $\mathbb{R}^d$, there is no hyperplane $H$ intersecting each of $K_1, \dots, K_{d+1}$. Otherwise, we can take a point $p_i \in K_i \cap H$. This gives us $d+1$ points in a $(d-1)$-dimensional space. By Radon's lemma, we can find a partition of them into two subsets $A, B$ whose convex hulls intersect. This implies that $\{K_i : p_i \in A\}$ cannot be separated from $\{K_i : p_i \in B\}$, contradicting the hypothesis.

Therefore, when we are given a vector $v \in \mathbb{R}^d$ and we construct $S(H)$ for $H \perp v$, each side of $H$ must have measure zero for some $\mu_i$. This is much stronger than simply having less than half of some measure.

The main idea will be to lift each measure to a surface $S(H)$ for an appropriate $H$ and use Theorem 1.4. We show that by choosing $H$ carefully, we can deduce the existence of the two parallel hyperplanes we seek.

Consider the Stiefel manifold $V_{d-1}(\mathbb{R}^d)$. Given $(v_1, \dots, v_{d-1}) \in V_{d-1}(\mathbb{R}^d)$, we lift $\mathbb{R}^d$ to $S(H) \subset \mathbb{R}^{d+1}$ as in Lemma 3.1 where $H$ is parallel to $\operatorname{span}\{v_1, \dots, v_{d-1}\}$. This lifts each measure $\mu_i$ in $\mathbb{R}^d$ to a measure $\sigma_i$ in $S(H)$. Every hyperplane in $\mathbb{R}^d$ separating the support vectors of two sets of the measures $\mu_i$ can be extended vertically in $\mathbb{R}^{d+1}$ to separate the corresponding measures $\sigma_i$. A small tilting can ensure that the separating hyperplane is not vertical. Therefore, the measures $\sigma_i$ are well separated.

The measures $\sigma_i$ do not satisfy the requirements of Theorem 1.4, so an additional step is necessary. For an $\varepsilon > 0$, we lift the measures to $S(H)^\varepsilon$ as in the proof of Theorem 1.2, apply Theorem 1.4, and then take $\varepsilon \to 0$.

This ensures that we get a half-space $H^+$ that has infinite rays in the direction $-e_{d+1}$ and such that $\sigma_i(H^+) \geq \alpha_i \cdot \sigma_i(\mathbb{R}^d)$ and $\sigma_i(H^-) \geq (1-\alpha_i) \cdot \sigma_i(\mathbb{R}^d)$, where $H^-$ is the complementary closed half-space of $H^+$. Since $\sigma_i(H^+) > 0$ for all $i$, we know that the boundary of $H^+$ cannot be one of the two hyperplane components of $S(H)$. Therefore the measure $\sigma_i$ of the boundary of $H^+$ is zero for all $i$ and $\sigma_i(H^+) = \alpha_i \cdot \mu_i(H^+)$. The same arguments that Bárány, Hubard, and Jerónimo used to show the uniqueness in their theorem can be applied to show that $H^+$ is uniquely defined. The uniqueness also implies that $H^+$ changes continuously as we modify $(v_1, \ldots, v_{d-1})$.

Let $n \in S^d \subset \mathbb{R}^{d+1}$ be the normal vector to the boundary of $H^+$ that points in the direction of $H^+$. We can use this to construct a function

$$\begin{aligned} g : V_{d-1}(\mathbb{R}^d) &\to \mathbb{R}^{d-1} \times \ldots \times \mathbb{R}^1 \\ (v_1, \ldots, v_{d-1}) &\mapsto (x_1, \ldots, x_d). \end{aligned}$$

For each $i = 1, \ldots, d-1$, the first coordinate of $x_i \in \mathbb{R}^{d-i}$ is $\langle n, (v_i, 0)\rangle$ and the rest are zero. This function is well defined and continuous. If we flip the sign of $v_i$, the surface $S(H)$ does not change. The vector $n \in S^d$ is not affected by this change, so only the sign of $x_i$ changes. Therefore, the function $g$ is $(\mathbb{Z}_2)^{d-1}$-equivariant. By Theorem 3.3, the function $g$ must have a zero. This implies that the projection of $H^+ \cap S(H)$ onto $\mathbb{R}^d$ is the region between two hyperplanes parallel to $H$. □

The construction of the function $g$ only uses $d-1$ out of the $d(d-1)/2$ coordinates that Theorem 3.3 makes available. It would be interesting to know if much stronger conditions can be imposed on $H$.

We also have consequences similar to Corollary 1.3. We say that a family of sets $K_1, \ldots, K_{d+2}$ in $\mathbb{R}^d$ is *well separated by spheres* if for any subset way to split them into two families $I, J$, there is a sphere that separates $I$ and $J$, i.e., it contains the union of one of the sets and leaves out the union of the other set.

**Corollary 4.2.** *Let $d$ be a positive integer and $\mu_1, \ldots, \mu_{d+2}$ be finite Borel measures in $\mathbb{R}^d$ absolutely continuous with respect to the Lebesgue measure. Suppose that the supports $K_1, \ldots, K_{d+2}$ of $\mu_1, \ldots, \mu_{d+2}$ are well separated by spheres. Let $\alpha_1, \ldots, \alpha_{d+2}$ be real numbers in $(0, 1)$. Then, there exist two concentric spheres $S_1, S_2$ or two parallel hyperplanes in $\mathbb{R}^d$ so that the region $A$ between them satisfies*

$$\mu_i(A) = \alpha_i \cdot \mu_i(\mathbb{R}^d) \qquad \textit{for all } i = 1, \ldots, d+2.$$

*Proof.* We lift $\mathbb{R}^d$ to the paraboloid

$$\mathcal{P} = \left\{ \left( x_1, \ldots, x_d, \sum_{i=1}^d x_i^2 \right) \in \mathbb{R}^{d+1} \right\}.$$

A sphere in $\mathbb{R}^d$ separating two families $I, J$ of measure supports translates to a hyperplane in $\mathbb{R}^{d+1}$ separating the lift of those supports. We apply Theorem 1.5 to the family of measures induced on $\mathcal{P}$ and we are done.

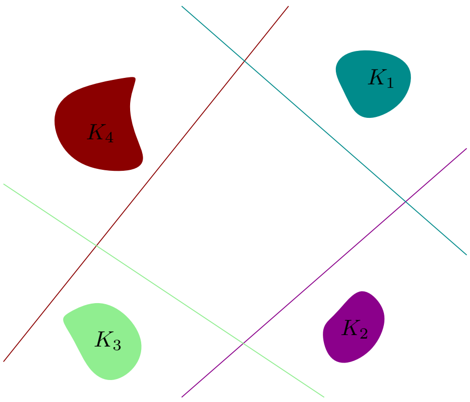
Even though the set of lifted measures do not satisfy the conditions of Theorem 1.5, a standard approximation argument fixes this problem. □

## 5. Equipartition with Polytopes and polyhedral surfaces of bounded complexity

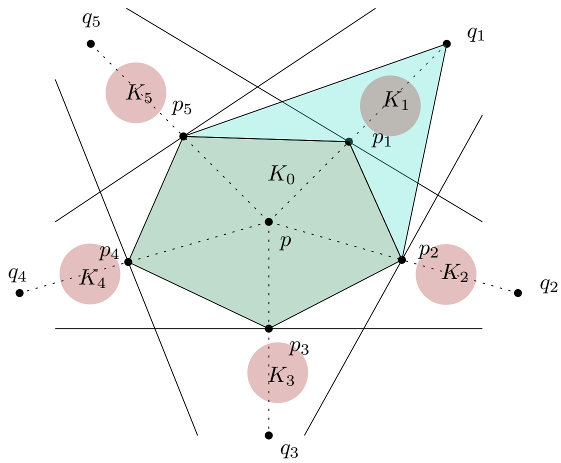
In previous sections, the number of measures to be partitioned was constrained by the dimension of the ambient space, while the boundaries of the partition were relatively simple. In this section, we consider mass partitions of a family of $n$ measures in $\mathbb{R}^d$, where $n$ can be much larger than $d$. We do so by increasing the complexity of the boundary of the partition. We focus on partitions by polyhedral surfaces.

**Definition 5.1.** Let $\mathcal{F} = \{\mu_1, \ldots, \mu_n\}$ be a family of finite absolutely continuous Borel measures in $\mathbb{R}^d$ with support $K_i$ for each $1 \le i \le n$. The supports are called *convexly separated* if for each $1 \le i \le n$, there exists a hyperplane $H_i$ such that $K_i \cap H_i^+ = \emptyset$ and $K_j \cap H_i^- = \emptyset$ for all $j \neq i$.

The maximum number of well separated measures is $d+1$, due to Radon's theorem. For convexly separated measures we only want to be able to separate any measure from the union of the other $n-1$, and not any two subsets. An example of convexly separated measures are $n$ measures such that each is concentrated near a vertex of a convex polytope.




$(a)$




$(b)$

FIGURE 3. (a) An example of four convexly separated measures in $\mathbb{R}^2$. (b) An example of five convexly separated and concentrated measures in $\mathbb{R}^2$. Notice that if we take $q_1$ instead of $p_1$ to form the convex hull, the resulting polygon contains all of $K_1$.

We define a polyhedron in $\mathbb{R}^d$ to be a finite intersection of closed half-spaces. A facet of a polyhedron is a $(d-1)$-dimensional face, and a vertex of a polyhedron is a zero-dimensional face.

**Theorem 5.2.** *Let $\mathcal{F} = \{\mu_1, \ldots, \mu_n\}$ be a family of finite absolutely continuous Borel measures in $\mathbb{R}^d$ with convexly separated supports $K_i$ for all $1 \le i \le n$, and let $\alpha_1, \ldots, \alpha_n$ be real numbers in $(0,1)$. Then, there exists a polyhedron $P$ with at most $n$ facets such that $\mu_i(P) = \alpha_i \cdot \mu_i(\mathbb{R}^d)$ for every $1 \le i \le n$.*

*Proof.* Because the supports are convexly separated, for each $1 \le i \le n$, we can fix a hyperplane $H_i$ with $K_i \cap H_i^+ = \emptyset$ and $K_j \cap H_i^- = \emptyset$ for all $j \neq i$. Notice that a polyhedron $P = \bigcap_{i=1}^n H_i^+$ has the property $\mu_i(P) = 0$ for every $1 \le i \le n$.

Now, consider $\mu_1$. Let $v$ be the normal vector to the hyperplane $H_1$ pointing in the direction of $H^-$. We can move $H_1$ in the direction of $v$ until we have the desired portion of the measure $\mu_1$, so we can fix $H_1' \parallel H_1$ with $\mu_1(H_1'^+) = \alpha \cdot \mu_1(\mathbb{R}^d)$. By letting $P' = \left(\bigcap_{i=2}^n H_i^+\right) \cap H_1'^+$, we have $\mu_1(P') = \alpha_1 \cdot \mu_1(\mathbb{R}^d)$ because

$\mu_1\big(\bigcap_{i=2}^n H_i^+\big) = \mu_1(\mathbb{R}^d)$. Moreover, because $K_j \cap H_1^- = \emptyset$ for each $j \neq 1$, moving $H_1$ to the direction of $H_1^-$ does not interfere with the rest of the measures $\mu_2, \dots, \mu_n$. We can repeat the same process for $\mu_2, \dots, \mu_n$ to find a convex polyhedron of at most $n$ facets with the desired property. □

While Theorem 5.2 allows for a mass partition with a polyhedron of $n$ facets, we can quantify the complexity of a compact polyhedron by the number of vertices as well. Theorem 5.3 proves a mass partition with a polyhedron of $n$ vertices, but this time for $n$ measures with a slightly stronger separation condition. We will use a similar idea to the proof of Theorem 1.4.

Let $\mu_1, \dots, \mu_n$ be a family of convexly separated measures in $\mathbb{R}^d$. Let $H_i$ be the hyperplane separating $K_i$ from the rest of the supports, as in Definition 5.1. Let $H_i^+$ be the closed side of $H_i$ that does not contain $K_i$. For $n \geq d+2$, any $d+1$ or fewer sets $H_i^+$ have non-empty intersection, so Helly's theorem we know that $P = \bigcap_{i=1}^n H_i^+ \neq \emptyset$. We say that the measures are *concentrated* if the following happens. There exists a point $p \in P$ and points $p_i, q_i$ for $i = 1, \dots, n$ so that the following holds.

- For each $i = 1, \dots, n$, $p_i \in H_i \cap P$. We denote $K_0 = \operatorname{conv}\{p_1, \dots, p_n\}$.
- We have $p \in K_0$.
- For each $i = 1, \dots, n$, $q_i$ is in the ray $pp_i$ and in $\bigcap_{i' \neq i} H_{i'}^+$.
- For each $i = 1, \dots, n$, we have $K_i \subset \operatorname{conv}(\{q_i\} \cup K_0)$.

An example is illustrated in Fig. 3(b).

**Theorem 5.3.** *Let $n, d$ be positive integers. Let $\mathcal{F} = \{\mu_1, \dots, \mu_n\}$ be a family of convexly separated and concentrated Borel measures in $\mathbb{R}^d$, each absolutely continuous. Let $\alpha_1, \dots, \alpha_n$ be real numbers in $(0,1)$. Then, there exists a polytope $K$ with $n$ vertices such that $\mu_i(K) = \alpha_i \cdot \mu_i(\mathbb{R}^d)$ for every $1 \leq i \leq n$.*

Note that the intuitive idea we used to prove Theorem 5.2 would indicate that we should slide each $p_i$ towards $q_i$ until we have the desired measure. The issue with this is that the values of other measures are no longer fixed.

*Proof.* Consider the hypercube $Q = [0,1]^n$. For $x = (x_1, \dots, x_n) \in Q$, and $i = 1, \dots, n$, let $y_i = (1-x_i)q_i + x_i p_i$. We define

$$K(x) = \operatorname{conv}\{y_1, \dots, y_n\}.$$

This convex set allows us to construct a function

$$\begin{aligned} f : Q &\to Q \\ x &\mapsto \left( \frac{\mu_1(K(x))}{\mu_1(\mathbb{R}^d)}, \dots, \frac{\mu_n(K(x))}{\mu_n(\mathbb{R}^d)} \right). \end{aligned}$$

The function is continuous. From the conditions of the measures, we can see that for every vertex $v$ of $Q$, we have $f(v) = v$. However, we have a stronger condition. For every face $\sigma \subset Q$, we have $f(\sigma) \subset \sigma$. This is because if a coordinate $x_i$ of $x$ equals zero, the $K(x) \subset H_i^+$, so $\mu_i(K(x)) = 0$. If $x_i = 1$, then $K(x) \supset \operatorname{conv}\{\{q_i\} \cup K_0\}$, so $\mu_i(K(x)) = \mu_i(\mathbb{R}^d)$. Therefore $f$ is of degree one on the boundary and must be surjective. There is a point $x \in Q$ such that $f(x) = (\alpha_1, \dots, \alpha_n)$, which implies that $K(x)$ is the polytope we were looking for. □

## 6. Remarks and open problems

To prove Theorem 1.6, we need to strengthen Lemma 3.1.

**Lemma 6.1.** *Let $m, n$ be positive integers, $\mu_1, \ldots, \mu_n$ be $n$ finite absolutely continuous Borel measures in $\mathbb{R}^d$, and $v$ be a unit vector in $\mathbb{R}^d$. Then there either exists $m-1$ hyperplanes orthogonal to $v$ that divide $\mathbb{R}^d$ into $m$ regions $R_1, \ldots, R_m$ of equal measure for each $\mu_i$ simultaneously or there exist $m-1$ hyperplanes orthogonal to $v$ such that they divide $\mathbb{R}^d$ into $m$ regions $R_1, \ldots, R_m$ such that for every $j = 1, \ldots, m$ there exists an $i$ such that*

$$\mu_i(R_j) < \frac{1}{m}\mu_i(\mathbb{R}^d).$$

*Proof.* Given parallel hyperplanes $H_1, \ldots, H_{m-1}$ in this order, we denote by $R_1, \ldots, R_m$ the regions they divide $\mathbb{R}^d$ into such that $R_j$ is bounded by $H_{j-1}$ and $H_j$. The unbounded regions $R_1$, $R_m$ are bounded only by $H_1$ and $H_m$ respectively.

We can find $m-1$ hyperplanes such that $\mu_1(R_j) = (1/m)\mu_1(\mathbb{R}^d)$ for every $j$. If these regions also form an equipartition for every other $\mu_i$, we are done. Otherwise, there is an $i$ and a $j$ such that $\mu_i(R_j) < (1/m)\mu_i(\mathbb{R}^d)$. We can widen $R_j$ by moving $R_{j-1}$ and $R_j$ slightly apart so that we still have $\mu_i(R_j) < (1/m)\mu_i(\mathbb{R}^d)$.

Then, $\mu_1(R_{j-1}) < (1/m)\mu_1(\mathbb{R}^d)$ and $\mu_1(R_{j+1}) < (1/m)\mu_1(\mathbb{R}^d)$. We can translate $H_{j-2}$ and $H_{j+1}$ away from $H_{j-1}$ and $H_j$ respectively so that these inequalities are preserved. This makes $\mu_1(R_{j-2})$ and $\mu_1(R_{j+2})$ to be strictly reduced. We continue this way until we are done. □

Now, given $\mu_1, \ldots, \mu_{d+1}$ finite absolutely continuous measures in $\mathbb{R}^d$, we construct a surface in $\mathbb{R}^{d+1}$. We take $v = e_d$ and find the $m-1$ hyperplanes $H_1, \ldots, H_m$ such that

$$H_j = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d = \lambda_j\}.$$

For some $\lambda_1 < \ldots < \lambda_{m-1}$. We define $\lambda_0 = -\infty$ and $\lambda_m = \infty$. Let $h : \mathbb{R} \to \mathbb{R}$ be a convex function that is linear between $\lambda_j$ and $\lambda_{j+1}$ for each $j = 0, \ldots, m-1$, but not between $\lambda_j$ and $\lambda_{j+2}$ for each $j = 0, \ldots, m-2$.

Let $V$ be the surface in $\mathbb{R}^{d+1}$ defined by the equation $x_{d+1} = h(x_d)$. The set of points on or above $V$ is the intersection of $m$ closed half-spaces. To prove Theorem 1.6 we repeat the proof of Akopyan and Karasev but we lift $\mathbb{R}^d$ to $V$ instead of a paraboloid.

*Proof of Theorem 1.6.* By a subdivision argument, it suffices to prove the result when $n = p$ a prime number. We apply Lemma 6.1 with $m = p$. If there are $p-1$ parallel hyperplanes that form an equipartition of the measures, we are done. Otherwise, we lift $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$ by lifting it to the surface $V$ defined above. Let $\sigma_1, \ldots, \sigma_{d+1}$ be the measures induced by $\mu_1, \ldots, \mu_{d+1}$ on $V$. It's known that we can split $\mathbb{R}^{d+1}$ into $p$ convex sets $C_1, \ldots, C_p$ that form an equipartition of $\mu_1, \ldots, \mu_{d+1}$ [Sob12, KHA14, BZ14]. Since each of the regions $R_1, \ldots, R_p$ we constructed in $\mathbb{R}^d$ have less than a $(1/p)$-fraction of some $\mu_i$ and $V$ is the boundary of a convex set, none of the boundaries between the sets $C_j$ can coincide with the hyperplanes defining $V$.

Moreover, for a prime number of parts the sets are $C_1, \ldots, C_p$ are induced by a generalized Voronoi diagram [KHA14, BZ14]. In other words, there are points (called sites) $s_1, \ldots, s_p$ in $\mathbb{R}^{d+1}$ and real number $\beta_1, \ldots, \beta_p$ such that the $p$ convex regions

$$C_j = \{x \in \mathbb{R}^{d+1} : ||x - s_j||^2 - \beta_j \le ||x - s_{j'}||^2 - \beta_{j'} \text{ for } j' = 1, \ldots, p\}$$

form an equipartition of $\mu_1, \ldots, \mu_{d+1}$. Since the set of points above $V$ is convex, if we take the region $C_j$ whose site $s_j$ has minimal $(d+1)$-th coordinate, when we project $C_j \cap V$ back to $\mathbb{R}^d$ we get a convex set. This is the set $K$ we are looking for. The boundary of the corresponding $C_j$ is the union of at most $p-1$ hyperplanes (the ones dividing it from each other $C_{j'}$). Each of those $p-1$ hyperplanes can intersect each of the $p$ hyperplanes defining $V$, forming at most $p(p-1)$ linear components of the boundary of $C_j \cap V$. This gives us the bound on the number of half-spaces whose intersection is $K$. □

When $n$ is a prime power, the number of half-spaces we used grows logarithmically with $n$. We wonder if this holds in general.

**Question 6.2.** Let $d$ be a fixed integer. Determine if for every positive integer $n$ and any $d+1$ finite absolutely continuous measures $\mu_1, \dots, \mu_{d+1}$ in $\mathbb{R}^d$ there exists a convex set $K \subset \mathbb{R}^d$ formed by the intersection of $O(\log n)$ half-spaces that contains exactly a $(1/n)$-fraction of each $\mu_i$.

We nicknamed Corollary 1.3 the bagel ham sandwich theorem due to its drawing in $\mathbb{R}^2$. However, since the set used is the region between two concentric spheres, it certainly does not look like a Bagel in $\mathbb{R}^3$. We define a *regular torus* in $\mathbb{R}^3$ to be the any set of the form $\{x \in \mathbb{R}^3 : \operatorname{dist}(x, S) \le \alpha\}$ where $S$ is a flat circle in $\mathbb{R}^3$ and $\alpha$ is a positive real number.

**Question 6.3** (Three-dimensional bagels)**.** Is it true that for any five absolutely continuous finite measures in $\mathbb{R}^3$ there exists a regular torus containing exactly half of each measure?

With four measures the result holds, since when $S$ degenerates to a point the regular torus is a sphere.

One of the questions that motivated the work on this manuscript was inspired by a conjecture by Mikio Kano. Kano conjectured that for any $n$ smooth measures in $\mathbb{R}^2$ there exists a path formed only by horizontal and vertical segments that takes at most $n-1$ turns, that simultaneously halves each measure. The conjecture is only known for $k = 1, 2$ or if the path is allowed to go through infinity [UKK09, KRPS16]. We wonder if the following way to mix Kano's conjecture with Theorem 2.1 holds.

**Question 6.4** (Existence of square sandwiches)**.** Is is true that for any three finite absolutely continuous measures in $\mathbb{R}^2$ there exists a square that contains exactly half of each measure?

Theorem 1.2 shows that we have a positive answer for rectangles (if the support of the measures are compact, we can cut the two lines given by Theorem 1.2 by perpendicular segments sufficiently far away, otherwise we have degenerate rectangles). However, it is still possible that for squares the answer to Question 6.4 is affirmative.

One additional connection of Theorem 1.2 with earlier results is with the Hobby–Rice theorem, also known as the necklace splitting problem due to interpretations of its discrete versions [HR65, GW85, AW86]. Hobby and Rice proved that *for any $k$ absolutely continuous measures in $\mathbb{R}^1$ there exists a partition of $\mathbb{R}^1$ into $k+1$ intervals so that they can be distributed among two sets, each receiving exactly half of each measure.* The intervals can be unbounded, so we are cutting $\mathbb{R}^1$ using $k$ points. Combining Theorem 1.2 and the Hobby–Rice theorem leads to the following natural conjecture.

**Conjecture 6.5.** Let $k, d$ be positive integers. We are given $d + k - 1$ absolutely continuous finite Borel measures in $\mathbb{R}^d$. There exist a set of $k$ parallel hyperplanes that divide $\mathbb{R}^d$ into $k+1$ regions which can be distributed among two sets so that each set has exactly half of each measure.

Theorem 1.2 confirms the case $k = 2$, the ham sandwich theorem is the case $k = 1$, and the Hobby–Rice theorem is the case $d = 1$. This would be a way to extend the Hobby–Rice theorem in higher dimensions in new ways (compare with [LŽ08, KRPS16, BS18]).

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