



# Dispersive estimates for 1D matrix Schrödinger operators with threshold resonance

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## Abstract

We establish dispersive estimates and local decay estimates for the time evolution of non-self-adjoint matrix Schrödinger operators with threshold resonances in one space dimension. In particular, we show that the decay rates in the weighted setting are the same as in the regular case after subtracting a finite rank operator corresponding to the threshold resonances. Such matrix Schrödinger operators naturally arise from linearizing a focusing nonlinear Schrödinger equation around a solitary wave. It is known that the linearized operator for the 1D focusing cubic NLS equation exhibits a threshold resonance. We also include an observation of a favorable structure in the quadratic nonlinearity of the evolution equation for perturbations of solitary waves of the 1D focusing cubic NLS equation.

**Mathematics Subject Classification** 35Q55

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## 1 Introduction

In this article, we establish dispersive estimates and local decay estimates for the (non-self-adjoint) matrix Schrödinger operators

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V} = \begin{bmatrix} -\partial_x^2 + \mu & 0 \\ 0 & \partial_x^2 - \mu \end{bmatrix} + \begin{bmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{bmatrix} \quad \text{on } L^2(\mathbb{R}) \times L^2(\mathbb{R}), \quad (1.1)$$

where  $\mu$  is a positive constant and  $V_1, V_2$  are real-valued sufficiently decaying potentials. The operator  $\mathcal{H}$  is closed on the domain  $D(\mathcal{H}) = H^2(\mathbb{R}) \times H^2(\mathbb{R})$ .

These matrix operators arise when linearizing a focusing nonlinear Schrödinger equation around a solitary wave. By our assumptions on  $V_1$  and  $V_2$ , Weyl's criterion implies that the essential spectrum of  $\mathcal{H}$  is the same as that of  $\mathcal{H}_0$ , given by  $(-\infty, -\mu] \cup [\mu, \infty)$ . As a core assumption in this paper, we suppose that the edges  $\pm\mu$  of the essential spectrum are irregular in the sense of Definition 4.4. This implies that there exist non-trivial bounded solutions to the equation  $\mathcal{H}\tilde{\Psi}_{\pm} = \pm\mu\tilde{\Psi}_{\pm}$ , see Lemma 4.5. The dispersive estimates for  $\mathcal{H}$  when the thresholds  $\pm\mu$  are regular have been obtained in Sections 7–8 of the paper by Krieger–Schlag [40], building on the scattering theory developed by Buslaev–Perel'man [9]. See also the recent work of Collot–Germain [11]. Our proof is instead based on the unifying approach to resolvent expansions first initiated by Jensen–Nenciu [36], and then further refined in Erdogan–Schlag [23] for matrix Schrödinger operators. We also adopt techniques from Erdogan–Green [20], where the authors prove similar dispersive estimates for one-dimensional Dirac operators.

### 1.1 Motivation

Our interest in developing dispersive estimates for (1.1) stems from the asymptotic stability problem for solitary wave solutions to nonlinear Schrödinger (NLS) equations. The NLS equation

$$i\partial_t\psi + \partial_x^2\psi + F(|\psi|^2)\psi = 0, \quad \text{where } \psi: \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{C}, \quad (1.2)$$

appears in many important physical contexts such as the propagation of a laser beam, the envelope description of water waves in an ideal fluid, or the propagation of light waves in nonlinear optical fibers. See, e.g., Sulem–Sulem [56] for physics background.

Under certain general conditions on the nonlinearity  $F(\cdot)$  (see, e.g., [7]), the equation (1.2) admits a parameterized family of localized, finite energy, traveling solitary waves of the form  $\psi(t, x) = e^{it\alpha^2}\phi(x; \alpha)$ , where  $\phi(\cdot; \alpha)$  is a ground state, i.e., a positive, decaying, real-valued solution to the (nonlinear) elliptic equation

$$-\partial_x^2\phi + \alpha^2\phi = F(\phi^2)\phi. \quad (1.3)$$

The existence and uniqueness of these ground state solutions are well-understood, see, e.g., [7, 41].

The solitary wave solutions (or simply, *solitons*) are of importance due to the special role they play for the long-time dynamics of the Cauchy problem (1.2). Consequently, over the last few decades there has been a significant interest in the study of stability (or instability) of such solitary waves under small perturbations. The primary notion of stability is that of orbital stability, and it is by now well-understood for the NLS equation. The pioneering works in this direction were due to Cazenave–Lions [14], Shatah–Strauss [55], and Weinstein [62]; see also [33] for the general theory. On the other hand, a stronger notion of stability

is that of asymptotic stability. There are two general approaches for the asymptotic stability problem. The first approach is to use integrability techniques, when the underlying partial differential equation is completely integrable and inverse scattering is available. A second approach is perturbative, which means that one studies the dynamics of the nonlinear flow in the neighborhood of the solitary wave, on a restricted set of the initial data. Generally, one starts by decomposing the perturbed solution into a sum of a solitary wave and a dispersive remainder term. For the perturbative approach, dispersive estimates for the linear flow are key.

Let us briefly describe the perturbative approach for the NLS equation. To keep our exposition short, we will not take into account any modulation aspects related to the Galilean invariance of the equation. For small  $\alpha > 0$ , consider the perturbation ansatz  $\psi(t, x) = e^{it\alpha^2}(\phi(x) + u(t, x))$  with the ground state  $\phi(\cdot) = \phi(\cdot; \alpha)$  and the dispersive remainder term  $u(t, x)$ . The linearization of (1.2) around the solitary wave  $e^{it\alpha^2}\phi(x)$  then leads to the following nonlinear partial differential equation

$$i\partial_t u = (-\partial_x^2 + \alpha^2 - V)u + W\bar{u} + N,$$

where  $N = N(\phi, u, \bar{u})$  is nonlinear in the variables  $(u, \bar{u})$ , and  $V = F(\phi^2) + F'(\phi^2)\phi^2$  and  $W = F'(\phi^2)\phi^2$  are real-valued potentials related to the ground state  $\phi$ . Equivalently, the above equation can be recast as a system for the vector  $U := (u, \bar{u})^\top$ , which is given by

$$i\partial_t U - \mathcal{H}U = \mathcal{N}, \quad (1.4)$$

where  $\mathcal{N}$  is a nonlinear term, and  $\mathcal{H}$  is a matrix Schrödinger operator of the form (1.1) with the parameters  $\mu = \alpha^2$ ,  $V_1 = V$ , and  $V_2 = W$ .

For the study of asymptotic stability of solitary waves for NLS, it is thus crucial to fully understand the spectral properties of the matrix operator  $\mathcal{H}$  as well as to derive dispersive estimates for the linear evolution operator  $e^{it\mathcal{H}}$ . One of the key steps in a perturbative analysis is to prove that the dispersive remainder (1.4) decays to zero in a suitable topology. Let us consider for example, the 1D focusing NLS with a pure power nonlinearity, i.e.

$$i\partial_t \psi + \partial_x^2 \psi + |\psi|^{2\sigma} \psi = 0, \quad \text{where } \sigma > 0. \quad (1.5)$$

The ground state  $\phi(x; 1)$  has an explicit formula for all  $\sigma > 0$  given by

$$\phi(x; 1) = (\sigma + 1)^{\frac{1}{2\sigma}} \operatorname{sech}^{\frac{1}{\sigma}}(\sigma x), \quad (1.6)$$

and the linearized operator around  $e^{it}\phi(x; 1)$  takes the form

$$\mathcal{H}_\sigma = \begin{bmatrix} -\partial_x^2 - (\sigma + 1)^2 \operatorname{sech}^2(\sigma x) + 1 & -\sigma(\sigma + 1) \operatorname{sech}^2(\sigma x) \\ \sigma(\sigma + 1) \operatorname{sech}^2(\sigma x) & \partial_x^2 + (\sigma + 1)^2 \operatorname{sech}^2(\sigma x) - 1 \end{bmatrix}.$$

For monomial nonlinearities, we may obtain  $\phi(x; \alpha)$  from rescaling by  $\phi(x; \alpha) = \alpha^{\frac{1}{\sigma}} \phi(\alpha x, 1)$ . The matrix operators when linearizing around  $e^{it\alpha^2}\phi(x; \alpha)$  are also equivalent to the matrix operator  $\mathcal{H}_\sigma$  by rescaling. The spectra for these matrix operators were investigated in [12]; see also Section 9 of [40]. For  $\sigma \geq 2$ , Krieger–Schlag [40] were able to construct finite co-dimensional center-stable manifolds around the solitary waves and prove asymptotic stability using dispersive and Strichartz estimates developed for the evolution operator  $e^{it\mathcal{H}}$ . However, for the completely integrable case ( $\sigma = 1$ ), it was shown in [12] that the matrix operator  $\mathcal{H}_1$  exhibits the threshold resonance  $\Psi(x) = (\tanh^2(x), -\operatorname{sech}^2(x))^\top$  at  $\lambda = 1$ . The dispersive estimates developed in [40] do not apply in this case. Furthermore, we note that a key assumption in the papers [9, 11, 32, 40] is that the linearized matrix operator

$\mathcal{H}$  does not possess threshold resonances at the edges of the essential spectrum. In these “generic” (regular) cases, it can be shown that the evolution operator enjoy improved decay estimates in weighted spaces; see, e.g., Proposition 8.1 in [40]. Thus, a meaningful motivation for this paper is to prove dispersive estimates in the presence of threshold resonances under some general spectral assumptions on the matrix operator  $\mathcal{H}$ , which are applicable to the 1D cubic NLS case ( $\sigma = 1$ ). We will discuss this particular case briefly in Sect. 1.4.

## 1.2 Main result

We are now in the position to state the main result of this paper. We begin by specifying some spectral assumptions on  $\mathcal{H}$ .

**Assumption 1.1** (A1)  $-\sigma_3 \mathcal{V}$  is a positive matrix, where  $\sigma_3$  is one of the Pauli matrices (c.f. (1.9)),

(A2)  $L_- := -\partial_x^2 + \mu - V_1 + V_2$  is non-negative,

(A3) there exists  $\beta > 0$  such that  $|V_1(x)| + |V_2(x)| \lesssim e^{-(\sqrt{2\mu}+\beta)|x|}$  for all  $x \in \mathbb{R}$ ,

(A4) there are no embedded eigenvalues in  $(-\infty, -\mu) \cup (\mu, \infty)$ .

Under these assumptions, we recall the general spectral theory for  $\mathcal{H}$  from [23].<sup>1</sup>

**Lemma 1.2** [23, Lemma 3] *Suppose Assumption 1.1 holds. The essential spectrum of  $\mathcal{H}$  equals  $(-\infty, -\mu] \cup [\mu, \infty)$ . Moreover,*

$$\text{spec}(\mathcal{H}) = -\text{spec}(\mathcal{H}) = \overline{\text{spec}(\mathcal{H})} = \text{spec}(\mathcal{H}^*), \quad (1.7)$$

and  $\text{spec}(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$ . The discrete spectrum of  $\mathcal{H}$  consists of eigenvalues  $\{z_j\}_{j=1}^N$ ,  $0 \leq N < \infty$ , of finite multiplicity. For each  $z_j \neq 0$ , the algebraic and geometric multiplicities coincide and  $\text{Ran}(\mathcal{H} - z_j)$  is closed. The zero eigenvalue has finite algebraic multiplicity, i.e., the generalized eigenspace  $\cup_{k=1}^{\infty} \ker(\mathcal{H}^k)$  has finite dimension. In fact, there is a finite  $m \geq 1$  so that  $\ker(\mathcal{H}^k) = \ker(\mathcal{H}^{k+1})$  for all  $k \geq m$ .

The symmetry (1.7) is due to the following commutation properties of  $\mathcal{H}$ ,

$$\mathcal{H}^* = \sigma_3 \mathcal{H} \sigma_3, \quad -\mathcal{H} = \sigma_1 \mathcal{H} \sigma_1, \quad (1.8)$$

with the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (1.9)$$

As a core assumption in this paper, we impose that the thresholds  $\pm\mu$  of the essential spectrum are irregular.

### Assumption 1.3

(A5) The thresholds  $\pm\mu$  are irregular in the sense of Definition 4.4. This implies that there exist non-trivial bounded solutions  $\vec{\Psi}_{\pm} = (\Psi_1^{\pm}, \Psi_2^{\pm})^{\top}$  to the equation  $\mathcal{H}\vec{\Psi}_{\pm} = \pm\mu\vec{\Psi}_{\pm}$ .

(A6) The vanishing (bilateral)-Laplace transform condition holds

$$\mathcal{L}[V_2\Psi_1^+ + V_1\Psi_2^+](\pm\sqrt{2\mu}) = \int_{-\infty}^{\infty} e^{\mp\sqrt{2\mu}y} (V_2\Psi_1^+ + V_1\Psi_2^+)(y) dy = 0. \quad (1.10)$$

<sup>1</sup> The results in Section 2 of [23] are stated for dimension 3, but they in fact hold for all dimensions. Moreover, only a polynomial decay on  $V_1$  and  $V_2$  is assumed in [23]. See also [35, Theorem 1.3].

For details about the characterization of the threshold functions  $\vec{\Psi}$ , we refer the reader to Definition 4.4 and Lemma 4.5 in Sect. 4. Due to the commutation identity (1.8), we have the relation  $\vec{\Psi}_+ = \sigma_1 \vec{\Psi}_-$ . We emphasize that assumption (A6) is used to infer that (non-trivial) bounded solutions  $\vec{\Psi}_\pm = (\Psi_1^\pm, \Psi_2^\pm)$  to the equation  $\mathcal{H}\vec{\Psi}_\pm = \pm\mu\vec{\Psi}_\pm$  satisfy  $\Psi_1^+ = \Psi_2^- \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$ .

Let  $P_d: L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})$  be the Riesz projection corresponding to the discrete spectrum of  $\mathcal{H}$ , and define  $P_s := I - P_d$ . We now state the main theorem of this article.

**Theorem 1.4** *Suppose assumptions (A1)–(A6) hold, and let  $\vec{\Psi} = (\Psi_1, \Psi_2)$  be the  $L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R}) \times L^2(\mathbb{R})$  distributional solution to*

$$\mathcal{H}\vec{\Psi} = \mu\vec{\Psi}, \quad (1.11)$$

*with the normalization<sup>2</sup>*

$$\lim_{x \rightarrow \infty} (|\Psi_1(x)|^2 + |\Psi_1(-x)|^2) = 2. \quad (1.12)$$

*Then, for any  $\vec{f} = (f_1, f_2) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ , we have*

*1. the unweighted dispersive estimate*

$$\left\| e^{it\mathcal{H}} P_s \vec{f} \right\|_{L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \left\| \vec{f} \right\|_{L^1(\mathbb{R}) \times L^1(\mathbb{R})}, \quad \forall |t| \geq 1, \quad (1.13)$$

*2. and the weighted dispersive estimate*

$$\left\| \langle x \rangle^{-2} (e^{it\mathcal{H}} P_s - F_t) \vec{f} \right\|_{L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{3}{2}} \left\| \langle x \rangle^2 \vec{f} \right\|_{L^1(\mathbb{R}) \times L^1(\mathbb{R})}, \quad \forall |t| \geq 1, \quad (1.14)$$

*where*

$$F_t \vec{f} := \frac{e^{it\mu}}{\sqrt{-4\pi it}} \langle \sigma_3 \vec{\Psi}, \vec{f} \rangle \vec{\Psi} - \frac{e^{-it\mu}}{\sqrt{4\pi it}} \langle \sigma_3 \sigma_1 \vec{\Psi}, \vec{f} \rangle \sigma_1 \vec{\Psi}. \quad (1.15)$$

We proceed with some remarks on the main theorem:

1. The estimate (1.14) is an analogue of the weighted dispersive estimates obtained by Goldberg [26] for the scalar Schrödinger operator  $H = -\partial_x^2 + V$  on the real line for non-generic potentials  $V$ ; see [26, Theorem 2]. The local decay estimate (1.14) shows that the bulk of the free wave  $e^{it\mathcal{H}} P_s$  enjoys improved local decay at the integrable rate  $\mathcal{O}(|t|^{-\frac{3}{2}})$ , and that the slow  $\mathcal{O}(|t|^{-\frac{1}{2}})$  local decay can be pinned down to the contribution of the finite rank operator  $F_t$ . Such sharp information can be useful for nonlinear asymptotic stability problems, see also Sect. 1.4 below.
2. We make some comments on the spectral hypotheses. The assumptions (A1)–(A4) are known to be satisfied by the linearized operator around the solitary wave for the 1D focusing power-type NLS (1.5). In the case of the 1D focusing cubic NLS ( $\sigma = 1$ ), the linearized operator  $\mathcal{H}_1$  satisfies the assumptions (A1)–(A6); see Sect. 1.4.1 below. More generally, in Lemma 4.5, we show for matrix operators  $\mathcal{H}$  of the form (1.1) satisfying assumptions (A1)–(A6) that the edges  $\pm\mu$  of the essential spectrum of  $\mathcal{H}$  cannot be eigenvalues, and that the non-trivial bounded solutions  $\vec{\Psi}_\pm = (\Psi_1^\pm, \Psi_2^\pm)^\top$  to

<sup>2</sup> By this, we mean to rescale the vector  $\vec{\Psi} \mapsto \lambda \vec{\Psi}$  by scalar multiplication where  $\lambda > 0$  is defined so that (1.12) holds. See (1.20) in Sect. 1.4 for an example.

$\mathcal{H}\vec{\Psi}_{\pm} = \pm\mu\vec{\Psi}_{\pm}$  belong to  $L^{\infty}\backslash L^2$  since  $\Psi_1^+(x) \equiv \Psi_2^-(x)$  has a non-zero limit as  $x \rightarrow \pm\infty$ . In this sense, we characterize the solutions  $\vec{\Psi}_{\pm}$  as threshold resonances. However, it is not yet clear to the author whether assumption (A6) is strictly needed to show that non-trivial bounded solutions  $\vec{\Psi}_{\pm}$  to  $\mathcal{H}\vec{\Psi}_{\pm} = \pm\mu\vec{\Psi}_{\pm}$  cannot be eigenfunctions. Moreover, an inspection of the proof of Lemma 4.5 reveals that the strong exponential decay assumption (A3) and the vanishing condition assumption (A6) are only used in a Volterra integral equation argument. In all other proofs, we only use some polynomial decay of the potentials  $V_1$  and  $V_2$ .

3. It might be possible to prove Theorem 1.4 using the scattering theory developed by [9]. However, one major difficulty for this approach is due to the fact that the matrix Wronskian associated with the vector Jost solutions is not invertible at the origin for cases where the matrix operators  $\mathcal{H}$  exhibit threshold resonances. Hence, the vector-valued distorted Fourier basis functions are not immediately well-defined at zero frequency. See Corollary 5.21 and Sect. 6 in [40] for further details.

### 1.3 Previous works

In this subsection, we collect references related to dispersive estimates for Schrödinger operators and to the study of the stability of solitary waves.

For dispersive estimates for the matrix Schrödinger operator  $\mathcal{H}$ , we refer to Sections 5–9 of [40] in dimension 1, and to [19, 23, 28, 46, 58] in higher dimensions. A comprehensive study on the spectral theory for the matrix operator arising from pure-power type NLS is given in [12]. See also [13, 49, 60] for related analytical and numerical studies. For dispersive estimates for the scalar Schrödinger operators, pioneering works include [37, 50, 61], and we refer to [4, 21, 22, 24–28, 31, 48, 52, 53] for a sample of recent works. Finally, we mention the papers [5, 36] on resolvent expansions for the scalar Schrödinger operator.

On the general well-posedness theory for the NLS Cauchy problem (1.2), we refer to the pioneering works [34, 38, 59]. Results on the orbital stability (or instability) of solitary waves for the NLS equation were first obtained by [2, 14, 55, 62, 63], and a general theory was established in [33]. Subsequent developments for general nonlinearities were due to [16, 29, 30, 45, 51]. Regarding the asymptotic stability of solitary waves, the first results were due to Buslaev–Perel’man [8, 9]. Subsequent works in this direction were due to [3, 11, 18, 32, 40, 47, 54]. For surveys on the stability of solitary waves, we refer to the reviews [15, 39] and the monographs [10, 56].

### 1.4 On the solitary wave for the 1D focusing cubic NLS

In this subsection, we present two observations related to the asymptotic stability problem for the solitary wave of the 1D focusing cubic NLS. First, we verify that the assumption (A6) holds for the linearized operator around the solitary wave of the 1D focusing cubic NLS. Second, we use the local decay estimate (1.14) to shed some light on the leading order structure of the quadratic nonlinearity in the perturbation equation for the solitary wave of the 1D focusing cubic NLS.

We note that a proof for the asymptotic stability problem has been given by Cuccagna–Pelinsonsky [17] via inverse scattering techniques. See also Borghese–Jenkins–McLaughlin [6]. On the other hand, a perturbative proof that does not explicitly rely on the integrable structure has not yet appeared in the literature to the best of the author’s knowledge. We now briefly discuss the evolution equation for perturbations of the solitary wave for the 1D

focusing cubic NLS. To keep our exposition short, we do not discuss the modulation aspects for the solitary wave. For simplicity, consider the perturbation ansatz

$$\psi(t, x) = e^{it} (Q(x) + u(t, x))$$

for the equation (1.5) ( $\sigma = 1$ ). The ground state has the explicit formula

$$Q(x) := \phi(x; 1) = \sqrt{2} \operatorname{sech}(x).$$

The evolution equation for the perturbation in vector form  $\vec{u} = (u_1, u_2) := (u, \bar{u})$  is given by

$$i \partial_t \vec{u} - \mathcal{H}_1 \vec{u} = \mathcal{Q}(\vec{u}) + \mathcal{C}(\vec{u}), \quad (1.16)$$

where

$$\mathcal{H}_1 = \mathcal{H}_0 + \mathcal{V}_1 = \begin{bmatrix} -\partial_x^2 + 1 & 0 \\ 0 & \partial_x^2 - 1 \end{bmatrix} + \begin{bmatrix} -4 \operatorname{sech}^2(x) & -2 \operatorname{sech}^2(x) \\ 2 \operatorname{sech}^2(x) & 4 \operatorname{sech}^2(x) \end{bmatrix}, \quad (1.17)$$

and

$$\mathcal{Q}(\vec{u}) := \begin{bmatrix} -Qu_1^2 - 2Qu_1u_2 \\ Qu_2^2 + 2Qu_1u_2 \end{bmatrix}, \quad \text{and} \quad \mathcal{C}(\vec{u}) := \begin{bmatrix} -u_1^2u_2 \\ u_1u_2^2 \end{bmatrix}. \quad (1.18)$$

Recall from [12] that the matrix operator  $\mathcal{H}_1$  has the essential spectrum  $(-\infty, -1] \cup [1, \infty)$ , and a four-dimensional generalized nullspace

$$\mathcal{N}_{\mathbb{R}}(\mathcal{H}_1) = \operatorname{span} \left\{ \begin{bmatrix} Q \\ -Q \end{bmatrix}, \begin{bmatrix} (1+x\partial_x)Q \\ (1+x\partial_x)Q \end{bmatrix}, \begin{bmatrix} \partial_x Q \\ \partial_x Q \end{bmatrix}, \begin{bmatrix} xQ \\ -xQ \end{bmatrix} \right\}, \quad (1.19)$$

as well as a threshold resonance at  $+1$  given by

$$\vec{\Psi} \equiv \vec{\Psi}_+ := \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}Q^2 \\ -\frac{1}{2}Q^2 \end{bmatrix} = \begin{bmatrix} \tanh^2(x) \\ -\operatorname{sech}^2(x) \end{bmatrix}. \quad (1.20)$$

By symmetry, there is also a threshold resonance function at  $-1$  given by

$$\vec{\Psi}_- = \sigma_1 \vec{\Psi}_+ = \begin{bmatrix} -\operatorname{sech}^2(x) \\ \tanh^2(x) \end{bmatrix}. \quad (1.21)$$

The eigenfunctions listed in (1.19) are related to the underlying symmetries for the NLS equation. Note that we have normalized the resonance function  $\vec{\Psi}$  to satisfy the condition (1.12) stated in Theorem 1.4.

#### 1.4.1 On assumption (A6) for the 1D focusing cubic NLS

Our first observation is that the assumption (A6) is satisfied by the matrix operator  $\mathcal{H}_1$ .

**Lemma 1.5** *Let  $V_1(x) = 4 \operatorname{sech}^2(x)$ ,  $V_2(x) = 2 \operatorname{sech}^2(x)$ , and  $(\Psi_1(x), \Psi_2(x)) = (\tanh^2(x), -\operatorname{sech}^2(x))$ . Then, we have*

$$\int_{\mathbb{R}} e^{\pm\sqrt{2}y} (V_2(y)\Psi_1(y) + V_1(y)\Psi_2(y)) dy = 0. \quad (1.22)$$

**Proof** We denote the (two-sided) Laplace transform by

$$\mathfrak{L}[f](s) = \int_{-\infty}^{\infty} e^{-sy} f(y) dy, \quad s \in \mathbb{C}, \quad (1.23)$$

which is formally related to the Fourier transform by

$$\mathfrak{L}[f](s) = \sqrt{2\pi} \mathcal{F}[f](is).$$

By direct computation,

$$(V_1 \Psi_2 + V_2 \Psi_1)(x) = 2 \operatorname{sech}^2(x) - 6 \operatorname{sech}^4(x),$$

and

$$\operatorname{sech}^4(x) = \frac{2}{3} \operatorname{sech}^2(x) - \frac{1}{6} \partial_x^2 (\operatorname{sech}^2(x)). \quad (1.24)$$

Recall from [43, Corollary 5.7] that as equalities in  $\mathcal{S}(\mathbb{R})$ ,

$$\mathcal{F}[\operatorname{sech}^2](\xi) = \sqrt{\frac{\pi}{2}} \frac{\xi}{\sinh(\frac{\pi}{2}\xi)}. \quad (1.25)$$

Hence, using the basic property  $\mathcal{F}[-\partial_x^2 f](\xi) = \xi^2 \mathcal{F}[f](\xi)$  and (1.24), we obtain

$$\mathcal{F}[\operatorname{sech}^4](\xi) = \frac{1}{6} \sqrt{\frac{\pi}{2}} \frac{\xi(4 + \xi^2)}{\sinh(\frac{\pi}{2}\xi)}. \quad (1.26)$$

As complex functions, we recall that  $\sinh(iz) = i \sin(z)$  and that  $z \mapsto \frac{z}{\sin(z)}$  is analytic<sup>3</sup> in the strip  $\{s + i\sigma : s \in (-\pi, \pi), \sigma \in \mathbb{R}\}$ . Thus, by analytic continuation,

$$\mathfrak{L}[V_1 \Psi_2 + V_2 \Psi_1](s) = \sqrt{2\pi} (2 \mathcal{F}[\operatorname{sech}^2](is) - 6 \mathcal{F}[\operatorname{sech}^4](is)) = \frac{\pi s(-2 + s^2)}{\sin(\frac{\pi s}{2})},$$

for any  $s \in \mathbb{C}$  with  $\Re(s) \in (-2, 2)$ , which in particular proves the vanishing condition (1.22).  $\square$

The other assumptions (A1)–(A5) for  $\mathcal{H}_1$  are also satisfied by either checking directly or invoking the results from Section 9 in [40].

### 1.4.2 Null structure for perturbations of the solitary wave of the 1D focusing cubic NLS

Due to the slow local decay of the Schrödinger waves in the presence of a threshold resonance, the spatially localized quadratic nonlinearity in (1.16) may pose significant difficulties for proving decay of small solutions to (1.16). The weighted dispersive estimate (1.14) shows that the slow local decay is only due to the finite rank projection  $F_L$ . To shed some light on the expected leading order behavior of the quadratic nonlinearity  $\mathcal{Q}(\vec{u})$  in (1.16), it is instructive to insert a free Schrödinger wave

$$\vec{u}_{\text{free}}(t) := e^{-it\mathcal{H}} P_s \vec{f},$$

for some fixed  $\vec{f} \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ . By Theorem 1.4, we have

$$\vec{u}_{\text{free}}(t) = c_- \frac{e^{-it}}{\sqrt{t}} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} + c_+ \frac{e^{it}}{\sqrt{t}} \begin{bmatrix} \Psi_2 \\ \Psi_1 \end{bmatrix} + \vec{r}(t), \quad (1.27)$$

<sup>3</sup> to be pedantic, there is a removable singularity at  $z = 0$  which we can remove by setting the function  $\frac{z}{\sin(z)}$  equal to 1 at  $z = 0$ .

with

$$c_- = \frac{1}{\sqrt{-4\pi i}} \langle \sigma_3 \vec{\Psi}, \vec{f} \rangle, \quad c_+ = -\frac{1}{\sqrt{4\pi i}} \langle \sigma_3 \sigma_1 \vec{\Psi}, \vec{f} \rangle, \quad (1.28)$$

and where the remainder  $\vec{r}(t)$  satisfies

$$\|\langle x \rangle^{-2} \vec{r}(t)\|_{L_x^\infty(\mathbb{R}) \times L_x^\infty(\mathbb{R})} \lesssim |t|^{-\frac{3}{2}} \|\langle x \rangle^2 \vec{f}\|_{L_x^1(\mathbb{R}) \times L_x^1(\mathbb{R})}. \quad (1.29)$$

Thus, owing to the spatial localization of the quadratic nonlinearity, we have

$$\mathcal{Q}(\vec{u}_{\text{free}}(t)) = \frac{c_+^2 e^{2it}}{t} \mathcal{Q}_1(\vec{\Psi}) + \frac{c_+ c_-}{t} \mathcal{Q}_2(\vec{\Psi}) + \frac{c_-^2 e^{-2it}}{t} \mathcal{Q}_3(\vec{\Psi}) + \mathcal{O}_{L^\infty}(|t|^{-2}), \quad (1.30)$$

where

$$\mathcal{Q}_1(\vec{\Psi}) = \begin{bmatrix} -Q\Psi_2^2 - 2Q\Psi_1\Psi_2 \\ Q\Psi_1^2 + 2Q\Psi_1\Psi_2 \end{bmatrix}, \quad (1.31)$$

$$\mathcal{Q}_2(\vec{\Psi}) = \begin{bmatrix} -2Q\Psi_1\Psi_2 - 2Q(\Psi_1^2 + \Psi_2^2) \\ 2Q\Psi_1\Psi_2 + 2Q(\Psi_1^2 + \Psi_2^2) \end{bmatrix}, \quad (1.32)$$

$$\mathcal{Q}_3(\vec{\Psi}) = -\sigma_1 \mathcal{Q}_1(\vec{\Psi}) = \begin{bmatrix} -Q\Psi_1^2 - 2Q\Psi_1\Psi_2 \\ Q\Psi_2^2 + 2Q\Psi_1\Psi_2 \end{bmatrix}. \quad (1.33)$$

Due to the critical  $\mathcal{O}(|t|^{-1})$  decay of the leading order terms on the right-hand side of (1.30), it is instructive to analyze the long-time behavior of small solutions to the inhomogeneous matrix Schrödinger equation with such a source term

$$\begin{cases} i\partial_t \vec{u}_{\text{src}} - \mathcal{H}_1 \vec{u}_{\text{src}} = P_s \left( \frac{c_+^2 e^{2it}}{t} \mathcal{Q}_1(\vec{\Psi}) + \frac{c_+ c_-}{t} \mathcal{Q}_2(\vec{\Psi}) + \frac{c_-^2 e^{-2it}}{t} \mathcal{Q}_3(\vec{\Psi}) \right), & t \geq 1, \\ \vec{u}_{\text{src}}(1) = \vec{0}. \end{cases} \quad (1.34)$$

To this end, it will be useful to exploit a special conjugation identity for the matrix Schrödinger operator  $\mathcal{H}_1$ . It was recently pointed out by Martel, see [47, Section 2.3], that the matrix operator  $\mathcal{H}_1$  can be conjugated to the flat matrix Schrödinger operator  $\mathcal{H}_0$ . By first conjugating  $\mathcal{H}_1$  with the unitary matrix  $\mathcal{J} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$ , we obtain the equivalent matrix Schrödinger operator

$$\begin{aligned} \mathcal{L}_1 &= -i\mathcal{J}^{-1}\mathcal{H}_1\mathcal{J} := \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix} \\ &= \mathcal{L}_0 + \mathcal{W} := \begin{bmatrix} 0 & -\partial_x^2 + 1 \\ \partial_x^2 - 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2\operatorname{sech}^2(x) \\ 6\operatorname{sech}^2(x) & 0 \end{bmatrix}. \end{aligned}$$

Introducing the operator

$$\mathcal{D} := \begin{bmatrix} 0 & (-\partial_x^2 + 1)S^2 \\ -S^2L_+ & 0 \end{bmatrix}, \quad \text{where } S := Q \cdot \partial_x \cdot Q^{-1} = \partial_x + \tanh(x), \quad (1.35)$$

one has the conjugation identity (see also [12, Section 3.4])

$$\mathcal{D}\mathcal{L}_1 = \mathcal{L}_0\mathcal{D}. \quad (1.36)$$

We then transfer the above identity to the matrix operator  $\mathcal{H}$  by setting  $\tilde{\mathcal{D}} := \mathcal{J}\mathcal{D}\mathcal{J}^{-1}$  to obtain the conjugation identity

$$\tilde{\mathcal{D}}\mathcal{H}_1 = \mathcal{H}_0\tilde{\mathcal{D}}. \quad (1.37)$$

Moreover, it can be checked directly that  $\tilde{\mathcal{D}}\tilde{\eta} = 0$  for any generalized eigenfunction  $\tilde{\eta} \in \mathcal{N}_g(\mathcal{H}_1)$ , and this implies that  $\tilde{\mathcal{D}}P_d \equiv 0$ , which is equivalent to saying that  $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}P_s$ . Hence, by applying the transformation  $\tilde{\mathcal{D}}$  to the equation (1.34), we obtain the transformed equation

$$i\partial_t \tilde{v}_{\text{src}} - \mathcal{H}_0 \tilde{v}_{\text{src}} = \tilde{\mathcal{D}} \left( \frac{c_+^2 e^{2it}}{t} Q_1(\tilde{\Psi}) + \frac{c_+ c_-}{t} Q_2(\tilde{\Psi}) + \frac{c_-^2 e^{-2it}}{t} Q_3(\tilde{\Psi}) \right), \quad (1.38)$$

where  $\tilde{v}_{\text{src}} := \tilde{\mathcal{D}}u_{\text{src}}$  is the transformed variable. Note that the above equation features the flat operator  $\mathcal{H}_0$  on the left. The Duhamel formula for  $\tilde{v}_{\text{src}}(t)$  at times  $t \geq 1$  reads

$$\tilde{v}_{\text{src}}(t) = -i \int_1^t e^{-i(t-s)\mathcal{H}_0} \tilde{\mathcal{D}} \left( \frac{c_+^2 e^{2is}}{s} Q_1(\tilde{\Psi}) + \frac{c_+ c_-}{s} Q_2(\tilde{\Psi}) + \frac{c_-^2 e^{-2is}}{s} Q_3(\tilde{\Psi}) \right) ds. \quad (1.39)$$

The flat, self-adjoint, matrix operator  $\mathcal{H}_0$  has the benefit that the semigroup  $e^{-it\mathcal{H}_0}$  can be represented in terms of the standard Fourier transform by the formula

$$(e^{-it\mathcal{H}_0} \vec{g})(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it(\xi^2+1)} \widehat{g}_1(\xi) e^{ix\xi} d\xi \underline{e}_1 + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{it(\xi^2+1)} \widehat{g}_2(\xi) e^{ix\xi} d\xi \underline{e}_2, \quad (1.40)$$

where  $\vec{g} = (g_1, g_2)^\top$  and  $\underline{e}_1, \underline{e}_2$  are the standard unit vectors in  $\mathbb{R}^2$ . The profile of  $\tilde{v}_{\text{src}}(t)$  is given by

$$\vec{f}_{\text{src}}(t) := e^{it\mathcal{H}_0} \tilde{v}_{\text{src}}(t). \quad (1.41)$$

Setting

$$\tilde{\mathcal{D}}Q_j(\tilde{\Psi}) =: (G_{j,1}, G_{j,2})^\top \quad \text{for } 1 \leq j \leq 3,$$

we have for times  $t \geq 1$  that

$$\begin{aligned} \mathcal{F}[\vec{f}_{\text{src}}(t)](\xi) &= c_+^2 \int_1^t \frac{e^{is(\xi^2+3)}}{s} \widehat{G_{1,1}}(\xi) ds \underline{e}_1 \\ &\quad + c_+ c_- \int_1^t \frac{e^{is(\xi^2+1)}}{s} \widehat{G_{2,1}}(\xi) ds \underline{e}_1 + c_-^2 \int_1^t \frac{e^{is(\xi^2-1)}}{s} \widehat{G_{3,1}}(\xi) ds \underline{e}_1 \\ &\quad + c_+^2 \int_1^t \frac{e^{-is(\xi^2-1)}}{s} \widehat{G_{1,2}}(\xi) ds \underline{e}_2 \\ &\quad + c_+ c_- \int_1^t \frac{e^{-is(\xi^2+1)}}{s} \widehat{G_{2,2}}(\xi) ds \underline{e}_2 + c_-^2 \int_1^t \frac{e^{-is(\xi^2+3)}}{s} \widehat{G_{3,2}}(\xi) ds \underline{e}_2. \end{aligned} \quad (1.42)$$

The uniform-in-time boundedness in  $L_\xi^\infty$  of the Fourier transform of the profile  $\mathcal{F}[\vec{f}_{\text{src}}(t)](\xi)$  is related to recovering the free decay rate for  $\tilde{v}_{\text{src}}(t)$ . However, in view of the critical decay of the integrand, this requires favorable time oscillations. Observe that the above terms with time phases  $e^{\pm is(\xi^2+1)}$ ,  $e^{\pm is(\xi^2+3)}$  are non-stationary for any  $s \in \mathbb{R}$  which implies that they have a better decay rate using integration by parts in the variable  $s$ . On the other hand, the terms with the phases  $e^{\pm is(\xi^2-1)}$  are stationary at the points  $\xi = \pm 1$ . Thus, it is important to

know if the Fourier coefficients  $\widehat{G_{3,1}}(\pm 1)$  and  $\widehat{G_{1,2}}(\pm 1)$  vanish. Indeed, this is true due to the following lemma.

**Lemma 1.6** *It holds that*

$$\widehat{G_{3,1}}(\pm 1) = \widehat{G_{1,2}}(\pm 1) = 0. \quad (1.43)$$

**Proof** First, to ease notation, we write

$$\tilde{\mathcal{D}} = \frac{i}{2} \begin{bmatrix} (-D_1 - D_2) (D_1 - D_2) \\ (-D_1 + D_2) (D_1 + D_2) \end{bmatrix}, \quad (1.44)$$

where

$$\begin{aligned} D_1 &:= (-\partial_x^2 + 1)S^2 = (-\partial_x^2 + 1)(\partial_x + \tanh(x))(\partial_x + \tanh(x)), \\ D_2 &:= S^2L_+ = (\partial_x + \tanh(x))(\partial_x + \tanh(x))(-\partial_x^2 - 6\operatorname{sech}^2(x) + 1). \end{aligned} \quad (1.45)$$

Since  $\sigma_1 \tilde{\mathcal{D}} = -\tilde{\mathcal{D}}\sigma_1$  and  $\mathcal{Q}_3(\tilde{\Psi}) = -\sigma_1 \mathcal{Q}_1(\tilde{\Psi})$  (c.f. (1.33)), it follows that  $G_{3,1} \equiv G_{1,2}$  as functions. Note that

$$G_{3,1} = \frac{i}{2} (D_1(Q\Psi_1^2) + D_1(Q\Psi_2^2) + 2D_1(2Q\Psi_1\Psi_2) + D_2(Q\Psi_1^2) - D_2(Q\Psi_2^2)) \quad (1.46)$$

where

$$\begin{aligned} (Q\Psi_1^2)(x) &= \sqrt{2} \operatorname{sech}(x) \tanh^4(x), \\ (Q\Psi_1\Psi_2)(x) &= -\sqrt{2} \operatorname{sech}^3(x) \tanh^2(x), \\ (Q\Psi_2^2)(x) &= \sqrt{2} \operatorname{sech}^5(x). \end{aligned}$$

By using the trigonometric identity  $\operatorname{sech}^2(x) + \tanh^2(x) = 1$ , we may simplify the expression for  $G_{3,1}$  into

$$G_{3,1}(x) = \frac{i\sqrt{2}}{2} (D_1(\operatorname{sech}(x) - 6\operatorname{sech}^3(x) + 6\operatorname{sech}^5(x)) + D_2(\operatorname{sech}(x) - 2\operatorname{sech}^3(x))).$$

By patient direct computation, we find

$$\begin{aligned} F_1(x) &:= D_1(\operatorname{sech}(x) - 6\operatorname{sech}^3(x) + 6\operatorname{sech}^5(x)) \\ &= 192\operatorname{sech}^3(x) - 3456\operatorname{sech}^5(x) + 9720\operatorname{sech}^7(x) - 6720\operatorname{sech}^9(x) \end{aligned} \quad (1.47)$$

and

$$F_2(x) := D_2(\operatorname{sech}(x) - 2\operatorname{sech}^3(x)) = 48\operatorname{sech}^3(x) - 264\operatorname{sech}^5(x) + 240\operatorname{sech}^7(x). \quad (1.48)$$

Moreover, using the identities

$$\begin{aligned} (\partial_x^2 \operatorname{sech})(x) &= \operatorname{sech}(x) - 2\operatorname{sech}^3(x), \\ (\partial_x^4 \operatorname{sech})(x) &= \operatorname{sech}(x) - 20\operatorname{sech}^3(x) + 24\operatorname{sech}^5(x), \\ (\partial_x^6 \operatorname{sech})(x) &= \operatorname{sech}(x) - 182\operatorname{sech}^3(x) + 840\operatorname{sech}^5(x) - 720\operatorname{sech}^7(x), \\ (\partial_x^8 \operatorname{sech})(x) &= \operatorname{sech}(x) - 1640\operatorname{sech}^3(x) \\ &\quad + 23184\operatorname{sech}^5(x) - 60480\operatorname{sech}^7(x) + 40320\operatorname{sech}^9(x), \end{aligned} \quad (1.49)$$

we obtain

$$F_1(x) = -\frac{1}{6}(-\partial_x^2 + 3\partial_x^4 - 3\partial_x^6 + \partial_x^8) \operatorname{sech}(x) = -\frac{1}{6}(-\partial_x^2 + 1)^3(-\partial_x^2) \operatorname{sech}(x), \quad (1.50)$$

and

$$F_2(x) = \frac{1}{3}(-\partial_x^2 + 2\partial_x^4 - \partial_x^6) \operatorname{sech}(x) = \frac{1}{3}(-\partial_x^2 + 1)^2(-\partial_x^2) \operatorname{sech}(x). \quad (1.51)$$

Thus, using the property  $\mathcal{F}[-\partial_x^2 f] = \xi^2 \mathcal{F}[f](\xi)$  and the fact that

$$\widehat{\operatorname{sech}}(\xi) = \sqrt{\frac{\pi}{2}} \operatorname{sech}\left(\frac{\pi\xi}{2}\right),$$

we compute that

$$\widehat{G_{3,1}}(\xi) = \frac{i\sqrt{2}}{2}(\widehat{F_1}(\xi) + \widehat{F_2}(\xi)) = -\frac{i\sqrt{\pi}}{12}(\xi^2 - 1)\xi^2(\xi^2 + 1)^2 \operatorname{sech}\left(\frac{\pi\xi}{2}\right), \quad (1.52)$$

which implies (1.43) as claimed.  $\square$

**Remark 1.7** We determined the identities (1.47)–(1.51) with the aid of the Wolfram Mathematica software.

The above lemma shows that the localized quadratic resonant terms are well-behaved for the nonlinear perturbation equation (1.16). The presence of this null structure is potentially a key ingredient for a perturbative proof of the asymptotic stability of the solitary wave solutions to the 1D focusing cubic NLS. We end this subsection with the following closing remark.

**Remark 1.8** The motivation for analyzing the quadratic nonlinearity in the perturbation equation (1.16) and for uncovering the null structure for the localized quadratic resonant terms in Lemma 1.6 is due to the recent work by Lührmann–Schlag [43], where the authors investigate the asymptotic stability of kink solutions to the 1D sine-Gordon equation under odd perturbations. In [43], the authors employ a similar conjugation identity like the one we used in (1.37) to transform the scalar Schrödinger operator  $H_1 := -\partial_x^2 - 2\operatorname{sech}^2(x)$  to the flat operator  $H_0 := -\partial_x^2$  for the perturbation equation. In fact, it is easy to check that one has the conjugation identity  $SH_1 = H_0S$ , where  $S = \partial_x + \tanh(x)$ . Moreover, an analogue of Lemma 1.6 on the non-resonant property for the localized quadratic resonant terms in the perturbation equation for the sine-Gordon kink was first obtained in [42, Remark 1.2]. This remarkable null structure for the sine-Gordon model played a key role in the asymptotic stability proof in [43]. In [44], the same authors obtained long-time decay estimates for even perturbation of the soliton of the 1D focusing cubic Klein–Gordon equation. The absence of the null structure in the nonlinearity of the perturbation equation in the focusing cubic Klein–Gordon model is a major obstruction to full co-dimension one asymptotic stability result under even perturbations.

Our short discussion on the effects of the threshold resonance on the quadratic term for (1.16) suggests that the localized quadratic resonant terms are well-behaved for the perturbation equation in the 1D cubic NLS model. However, note that a full perturbative proof of the asymptotic stability problem for this model has to encompass the modulation theory associated to the moving solitary wave, and take into account the long-range (modified) scattering effects due to the non-localized cubic nonlinearities in the perturbation equation. We point out that Collot–Germain [11] recently obtained general such asymptotic stability

results for solitary waves for 1D nonlinear Schrödinger equations under the assumption that the linearized matrix Schrödinger operator does not exhibit threshold resonances. See also [47] for related local asymptotic stability results.

## 1.5 Organization of the article

The remaining sections of this paper are devoted to the proof of Theorem 1.4. In Sect. 2, we state a few stationary phase lemmas, which will be heavily utilized in Sections 5 and 6, and we will also provide an analogue of Theorem 1.4 for the free matrix operator  $\mathcal{H}_0$ . In Sect. 3, we employ the symmetric resolvent expansion following the framework in [23], and in Sect. 4, we carefully extract the leading operators for these resolvent expansions. A characterization of the threshold resonance is stated in Lemma 4.5 under the spectral assumptions (A1)–(A6). Then, in Sect. 5, we prove dispersive estimates for the evolution operator  $e^{it\mathcal{H}}$  in the low energy regime. The approach taken in Sect. 5 largely follows the techniques employed in [20] for one-dimensional Dirac operators. In Sect. 6, we prove dispersive estimates for the remaining energy regimes and finish the proof of Theorem 1.4.

## 1.6 Notation

For any  $\vec{f} = (f_1, f_2)^\top$ ,  $\vec{g} = (g_1, g_2)^\top \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ , we use the inner product

$$\langle \vec{f}, \vec{g} \rangle := \int_{\mathbb{R}} \vec{f}^* \vec{g} \, dx = \int_{\mathbb{R}} (\bar{f}_1 g_1 + \bar{f}_2 g_2) \, dx, \quad \text{where } \vec{f}^* := (\bar{f}_1, \bar{f}_2). \quad (1.53)$$

The Schwartz space is denoted by  $\mathcal{S}(\mathbb{R})$  and we use the weighted  $L^2$ -spaces

$$X_\sigma := \langle x \rangle^{-\sigma} L^2(\mathbb{R}) \times \langle x \rangle^{-\sigma} L^2(\mathbb{R}), \quad \|\vec{f}\|_{X_\sigma} := \|\langle x \rangle^\sigma \vec{f}\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R})}, \quad \text{where } \sigma \in \mathbb{R}. \quad (1.54)$$

Note that for any  $\alpha > \beta > 0$ , one has the continuous inclusions

$$X_\alpha \subset X_\beta \subset X_0 = L^2(\mathbb{R}) \times L^2(\mathbb{R}) \subset X_{-\beta} \subset X_{-\alpha}, \quad (1.55)$$

and the duality  $X_\alpha^* = X_{-\alpha}$ . Our convention for the Fourier transform is

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx, \quad \mathcal{F}^{-1}[f](x) = \check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(\xi) \, d\xi.$$

We denote by  $C > 0$  an absolute constant whose value is allowed to change from line to line. In order to indicate that the constant depends on a parameter, say  $\theta$ , we will use the notation  $C_\theta$  or  $C(\theta)$ . For non-negative  $X, Y$  we write  $X \lesssim Y$  if  $X \leq CY$ . We use the Japanese bracket notation  $\langle x \rangle = (1 + x^2)^{\frac{1}{2}}$  for  $x \in \mathbb{R}$ . The standard tensors on  $\mathbb{R}^2$  are denoted by

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{e}_{11} = \underline{e}_1 \underline{e}_1^\top = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{e}_{22} = \underline{e}_2 \underline{e}_2^\top = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (1.56)$$

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**Data availability.** Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

## 2 Free matrix Schrödinger estimates

In this section, we derive dispersive estimates for the free evolution semigroup  $e^{it\mathcal{H}_0}$ . We recall that the free matrix Schrödinger operator

$$\mathcal{H}_0 = \begin{bmatrix} -\partial_x^2 + \mu & 0 \\ 0 & \partial_x^2 - \mu \end{bmatrix},$$

has a purely continuous spectrum

$$\text{spec}(\mathcal{H}_0) = \sigma_{\text{ac}}(\mathcal{H}_0) = (-\infty, -\mu] \cup [\mu, \infty),$$

and the resolvent operator of  $\mathcal{H}_0$  is given by

$$(\mathcal{H}_0 - \lambda)^{-1} = \begin{bmatrix} R_0(\lambda - \mu) & 0 \\ 0 & -R_0(-\lambda - \mu) \end{bmatrix}, \quad \lambda \in \mathbb{C} \setminus (-\infty, -\mu] \cup [\mu, \infty), \quad (2.1)$$

where  $R_0$  is the resolvent operator for the one-dimensional Laplacian, with an integral kernel given by

$$R_0(\zeta^2)(x, y) := (-\partial^2 - \zeta^2)^{-1}(x, y) = \frac{-e^{i\zeta|x-y|}}{2i\zeta}, \quad \zeta \in \mathbb{C}_+, \quad (2.2)$$

where  $\mathbb{C}_+$  is the upper half-plane. We obtain from the scalar resolvent theory due to Agmon [1] that the limiting resolvent operators

$$(\mathcal{H}_0 - (\lambda \pm i0))^{-1} = \lim_{\varepsilon \downarrow 0} (\mathcal{H}_0 - (\lambda \pm i\varepsilon))^{-1}, \quad \lambda \in (-\infty, -\mu) \cup (\mu, \infty),$$

are well defined as operators from  $X_\sigma \rightarrow X_{-\sigma}$  for any  $\sigma > \frac{1}{2}$ . Here, the matrix operator  $\mathcal{H}_0$  is self-adjoint and Stone's formula applies:

$$e^{it\mathcal{H}_0} = \frac{1}{2\pi i} \int_{|\lambda| \geq \mu} e^{it\lambda} \left[ (\mathcal{H}_0 - (\lambda + i0))^{-1} - (\mathcal{H}_0 - (\lambda - i0))^{-1} \right] d\lambda. \quad (2.3)$$

Let us focus on the spectrum on the positive semi-axis  $[\mu, \infty)$ , as the negative part can be treated using the symmetric properties of  $\mathcal{H}$  (c.f. Remark 3.3). By invoking the change of variables  $\lambda \mapsto \lambda = \mu + z^2$  with  $0 < z < \infty$ , the kernel of  $e^{it\mathcal{H}_0} P_s^+$  is then given by

$$\begin{aligned} e^{it\mathcal{H}_0} P_s^+(x, y) &= \frac{e^{it\mu}}{\pi i} \int_0^\infty e^{itz^2} z \left[ (\mathcal{H}_0 - (\mu + z^2 + i0))^{-1} - (\mathcal{H}_0 - (\mu + z^2 - i0))^{-1} \right] (x, y) dz. \end{aligned}$$

Here, the notation  $P_s^+$  means that we restrict the free evolution  $e^{it\mathcal{H}_0}$  to the positive semi-axis in the integral representation (2.3). By (2.1) and (2.2), we have

$$(\mathcal{H}_0 - (\mu + z^2 \pm i0))^{-1}(x, y) = \begin{bmatrix} \frac{\pm i e^{\pm iz|x-y|}}{2z} & 0 \\ 0 & -\frac{e^{-\sqrt{z^2+2\mu}|x-y|}}{2\sqrt{z^2+2\mu}} \end{bmatrix}, \quad 0 < z < \infty, \quad (2.4)$$

and thus,

$$e^{it\mathcal{H}_0} P_s^+(x, y) = \frac{e^{it\mu}}{2\pi} \int_{\mathbb{R}} e^{itz^2} e^{iz|x-y|} \underline{e}_{11} dz. \quad (2.5)$$

Note that the above integral is to be understood in the principal value sense, due to the pole in (2.4). To this end, we recall the following standard stationary phase results. The first lemma is a direct consequence of the classic van der Corput lemma.

**Lemma 2.1** *Let  $r \in \mathbb{R}$ , and let  $\psi(z)$  be a compactly supported smooth function. Then for any  $|t| > 0$ ,*

$$\left| \int_{\mathbb{R}} e^{itz^2 + izr} \psi(z) \, dz \right| \leq C|t|^{-\frac{1}{2}} \|\partial_z \psi\|_{L_z^1(\mathbb{R})}. \quad (2.6)$$

Moreover, if  $\psi(z)$  is supported away from zero, then for all  $|t| > 0$ ,

$$\left| \int_{\mathbb{R}} e^{itz^2 + izr} \psi(z) \, dz \right| \leq C|t|^{-\frac{3}{2}} \left\| [\partial_z^2 + ir\partial_z] \left( \frac{\psi}{z} \right) \right\|_{L_z^1(\mathbb{R})}. \quad (2.7)$$

**Proof** The bound (2.6) follows from the van der Corput lemma (see e.g. [57, VIII Proposition 2]) by observing that the phase  $\phi(z) = z^2 + \frac{zr}{t}$  satisfies  $|\partial_z^2 \phi(z)| = 2 > 0$ . The last bound follows by first integrating by parts

$$\begin{aligned} \int_{\mathbb{R}} e^{itz^2} e^{izr} \psi(z) \, dz &= -\frac{1}{2it} \int_{\mathbb{R}} e^{itz^2} \partial_z \left[ e^{izr} \frac{\psi(z)}{z} \right] \, dz \\ &= -\frac{1}{2it} \int_{\mathbb{R}} e^{itz^2 + izr} [ir + \partial_z] \left[ \frac{\psi(z)}{z} \right] \, dz, \end{aligned}$$

and then invoking the van der Corput lemma.  $\square$

We will also need the following sharper stationary phase lemma, which may be found in many text on oscillatory integrals with a Fresnel phase.

**Lemma 2.2** *Let  $\chi(z)$  be a smooth, non-negative, even cut-off function such that  $\chi(z) = 1$  for  $z \in [-1, 1]$  and  $\chi(z) = 0$  for  $|z| \geq 2$ . For  $r, t \in \mathbb{R}$ , define*

$$G_t(r) := \int_{\mathbb{R}} e^{itz^2 + izr} \chi(z^2) \, dz. \quad (2.8)$$

Then there exists  $C = C(\|\chi(z^2)\|_{W^{4,1}(\mathbb{R})}) > 0$  such that for any  $r \in \mathbb{R}$  and for any  $|t| > 0$ ,

$$\left| G_t(r) - \frac{\sqrt{\pi}}{\sqrt{-it}} e^{-i\frac{r^2}{4t}} \right| \leq C|t|^{-\frac{3}{2}} \langle r \rangle. \quad (2.9)$$

Moreover, if  $r_1, r_2 \geq 0$ , then

$$\left| G_t(r_1 + r_2) - \frac{\sqrt{\pi}}{\sqrt{-it}} e^{-i\frac{r_1^2}{4t}} e^{-i\frac{r_2^2}{4t}} \right| \leq C|t|^{-\frac{3}{2}} \langle r_1 \rangle \langle r_2 \rangle. \quad (2.10)$$

**Proof** First, the phase  $\phi(z) := z^2 + \frac{zr}{t}$  has a critical point at  $z_* = -\frac{r}{2t} \in \mathbb{R}$  with  $\phi''(z) = 2 > 0$ . We use Taylor expansion of  $\phi(z)$  and shift the integral by the change of variables  $z \mapsto z + z_*$  to obtain

$$\begin{aligned} G_t(r) &= \int_{\mathbb{R}} e^{it\phi(z)} \chi(z^2) \, dz = \int_{\mathbb{R}} e^{it\phi(z_*) + \phi''(z_*)(z-z_*)^2} \chi(z^2) \, dz \\ &= e^{-i\frac{r^2}{4t}} \int_{\mathbb{R}} e^{itz^2} \chi((z + z_*)^2) \, dz. \end{aligned} \quad (2.11)$$

Using the Fourier transform of the free Schrödinger group and the Plancherel's identity, we have

$$\begin{aligned}
 & \int_{\mathbb{R}} e^{itz^2} \chi((z+z_*)^2) dz \\
 &= \frac{1}{\sqrt{-2it}} \int_{\mathbb{R}} e^{-i\frac{\xi^2}{4t}} \mathcal{F}_{z \rightarrow \xi} [\chi((z+z_*)^2)](\xi) d\xi \\
 &= \frac{1}{\sqrt{-2it}} \int_{\mathbb{R}} \mathcal{F}_{z \rightarrow \xi} [\chi((z+z_*)^2)](\xi) d\xi \\
 &\quad + \frac{1}{\sqrt{-2it}} \int_{\mathbb{R}} \left( e^{-i\frac{\xi^2}{4t}} - 1 \right) \mathcal{F}_{z \rightarrow \xi} [\chi((z+z_*)^2)](\xi) d\xi \\
 &= \frac{\sqrt{2\pi}}{\sqrt{-2it}} \chi(z_*^2) + \frac{1}{\sqrt{-2it}} \int_{\mathbb{R}} \left( e^{-i\frac{\xi^2}{4t}} - 1 \right) e^{iz_*\xi} \mathcal{F}[\chi((z+z_*)^2)](\xi) d\xi.
 \end{aligned}$$

Using the bound  $|e^{i\frac{\xi^2}{4t}} - 1| \leq C|t|^{-1}\xi^2$  and the Hölder's inequality, we bound the remainder term by

$$\begin{aligned}
 \left| \frac{1}{\sqrt{-2it}} \int_{\mathbb{R}} \left( e^{-i\frac{\xi^2}{4t}} - 1 \right) e^{iz_*\xi} \mathcal{F}[\chi((z+z_*)^2)](\xi) d\xi \right| &\leq C|t|^{-\frac{3}{2}} \int_{\mathbb{R}} |\xi^2 \mathcal{F}[\chi(z^2)](\xi)| d\xi \\
 &\leq C|t|^{-\frac{3}{2}} \|\chi(z^2)\|_{W^{4,1}(\mathbb{R})} \leq C|t|^{-\frac{3}{2}}.
 \end{aligned}$$

Next, we use the fact that  $|1 - \chi(z^2)| \leq C|z|$  for all  $z \in \mathbb{R}$  and for some  $C > 0$  large enough so that

$$|1 - \chi(z_*^2)| \leq C|z_*| \leq C|t|^{-1}\langle r \rangle. \quad (2.12)$$

Then (2.9) follows (2.11)–(2.12). Finally, we use the estimate (2.9) to obtain

$$\left| G_t(r_1 + r_2) - \frac{\sqrt{2\pi}}{\sqrt{-2it}} e^{-i\frac{(r_1-r_2)^2}{4t}} \right| \leq C|t|^{-\frac{3}{2}} \langle r_1 - r_2 \rangle \leq C|t|^{-1} \langle r_1 \rangle \langle r_2 \rangle.$$

Thus, by the triangle inequality and the bound

$$\left| e^{-i\frac{(r_1-r_2)^2}{4t}} - e^{-i\frac{r_1^2}{4t}} e^{-i\frac{r_2^2}{4t}} \right| = \left| e^{-i\frac{r_1^2}{4t}} e^{-i\frac{r_2^2}{4t}} \right| \left| e^{i\frac{r_1 r_2}{2t}} - 1 \right| \leq C|t|^{-1} \langle r_1 \rangle \langle r_2 \rangle,$$

we conclude (2.10).  $\square$

Next, we prove the analogue of Theorem 1.4 for the free evolution. We emphasize that the free matrix Schrödinger operator  $\mathcal{H}_0$  has threshold resonances  $\mathcal{H}_0 e_1 = \mu e_1$  and  $\mathcal{H}_0 e_2 = -\mu e_2$ .

**Proposition 2.3** For any  $\vec{u} = (u_1, u_2) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$  and for any  $|t| \geq 1$ , we have

$$\left\| e^{it\mathcal{H}_0} P_s^+ \vec{u} \right\|_{L_x^\infty \times L_x^\infty} \lesssim |t|^{-\frac{1}{2}} \|\vec{u}\|_{L_x^1 \times L_x^1}, \quad (2.13)$$

and

$$\left\| \langle x \rangle^{-1} \left( e^{it\mathcal{H}_0} P_s^+ - F_t^0 \right) \vec{u} \right\|_{L_x^\infty \times L_x^\infty} \lesssim |t|^{-\frac{3}{2}} \|\langle x \rangle \vec{u}\|_{L_x^1 \times L_x^1}, \quad (2.14)$$

where

$$F_t^0(x, y) := \frac{e^{it\mu}}{\sqrt{-4\pi it}} e^{-i\frac{x^2}{4t}} e_1 e^{-i\frac{y^2}{4t}} e_1^\top. \quad (2.15)$$

**Proof** We first begin by splitting the evolution operator into low and high energy parts<sup>4</sup>:

$$\begin{aligned} e^{it\mathcal{H}_0} P_s^+(x, y) &= e^{it\mathcal{H}_0} \chi(\mathcal{H}_0 - \mu I) P_s^+(x, y) + e^{it\mathcal{H}_0} (1 - \chi(\mathcal{H}_0 - \mu I)) P_s^+(x, y) \\ &= \frac{e^{it\mu}}{2\pi} \int_{\mathbb{R}} e^{itz^2 + iz|x-y|} \chi(z^2) dz e_{11} \\ &\quad + \frac{e^{it\mu}}{2\pi} \int_{\mathbb{R}} e^{itz^2 + iz|x-y|} (1 - \chi(z^2)) dz e_{11}, \end{aligned} \quad (2.16)$$

where  $\chi(z)$  is a standard smooth, even, non-negative cut-off function satisfying  $\chi(z) = 1$  for  $|z| \leq 1$  and  $\chi(z) = 0$  for  $|z| \geq 2$ .

In the high energy part in (2.16), following the ideas from [26, 31], we prove the estimate

$$\left| \int_{\mathbb{R}} e^{itz^2 + iz|x-y|} (1 - \chi(z^2)) dz \right| \lesssim \min\{|t|^{-\frac{1}{2}}, |t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle\}. \quad (2.17)$$

For a more rigorous treatment, we instead use a truncated cutoff  $\chi_L(z) = (1 - \chi(z^2))\chi(z/L)$ , where  $L \geq 1$ , and we prove the uniform estimate

$$\sup_{L \geq 1} \left| \int_{\mathbb{R}} e^{itz^2 + iz|x-y|} \chi_L(z) dz \right| \leq C \min\{|t|^{-\frac{1}{2}}, |t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle\}, \quad (2.18)$$

with a constant  $C > 0$  independent of  $L$ . This estimate will imply (2.17). Indeed for any  $|t| > 0$ , by the Plancherel's identity, we have

$$\begin{aligned} &\sup_{a \in \mathbb{R}} \left| \int_{\mathbb{R}} e^{itz^2 + iaz} \chi_L(z) dz \right| \\ &= \sup_{a \in \mathbb{R}} \left| \int_{\mathbb{R}} \mathcal{F}^{-1}[e^{itz^2 + iaz}](\xi) \mathcal{F}[\chi_L(z)](\xi) d\xi \right| \leq C |t|^{-\frac{1}{2}} \|\mathcal{F}[\chi_L]\|_{L_{\xi}^1(\mathbb{R})}. \end{aligned}$$

Here, we use that the Fourier transform of the tempered distribution  $e^{itz^2 + iaz}$  has  $|t|^{-\frac{1}{2}}$  decay. Using the definition of  $\chi_L$ , the scaling properties of the Fourier transform, and Young's convolution inequality, we obtain

$$\begin{aligned} \|\mathcal{F}[\chi_L]\|_{L_{\xi}^1(\mathbb{R})} &\leq \|\mathcal{F}[\chi(z/L)]\|_{L_{\xi}^1(\mathbb{R})} + \|\mathcal{F}[\chi(z/L)]\|_{L_{\xi}^1(\mathbb{R})} \|\mathcal{F}[\chi(z^2)]\|_{L_{\xi}^1(\mathbb{R})} \\ &\leq C \|L\mathcal{F}[\chi](L\xi)\|_{L_{\xi}^1(\mathbb{R})} = C \|\mathcal{F}[\chi](\xi)\|_{L_{\xi}^1(\mathbb{R})} \leq C \|\chi\|_{W^{2,1}(\mathbb{R})} \lesssim 1. \end{aligned} \quad (2.19)$$

For the high-energy weighted dispersive estimate, we use integration by parts to find that

$$\left| \int_{\mathbb{R}} e^{itz^2} e^{iz|x-y|} \chi_L(z) dz \right| \leq C |t|^{-1} \left| \int_{\mathbb{R}} e^{itz^2} \partial_z \left( e^{iz|x-y|} z^{-1} \chi_L(z) \right) dz \right|.$$

When the derivative falls onto  $e^{iz|x-y|}$ , the weights  $\langle x \rangle \langle y \rangle$  appear, whereas the term  $z^{-1} \chi_L(z)$  is smooth since  $\chi_L$  is compactly supported away from the interval  $[-1, 1]$ . By following the previous argument, we conclude the  $\mathcal{O}(|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle)$  bound for (2.18) in the high-energy regime.

Next we turn to the low-energy estimates. For the low-energy unweighted estimate, we employ Lemma 2.1 to obtain

$$\left| \int_{\mathbb{R}} e^{itz^2 + iz|x-y|} \chi(z^2) dz \right| \leq C |t|^{-\frac{1}{2}} \|\partial_z \chi(z^2)\|_{L^1(\mathbb{R})} \leq C |t|^{-\frac{1}{2}}. \quad (2.20)$$

<sup>4</sup> Symbols like  $\chi(\mathcal{H}_0 - \mu I)$  are only used in a formal way to represent the cut-off  $\chi(z^2)$  in the  $z$ -integrals, where they arise.

On the other hand, for the low-energy weighted estimate, we observe that by Lemma 2.2,

$$\left| \int_{\mathbb{R}} e^{itz^2 + iz|x-y|} \chi(z^2) dz - \frac{\sqrt{2\pi}}{\sqrt{-2it}} e^{-i\frac{x^2}{4t}} e^{-i\frac{y^2}{4t}} \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle.$$

Hence, using that  $\underline{e}_{11} = \underline{e}_1 \underline{e}_1^\top$ , we arrive at the kernel estimate

$$\left| e^{it\mathcal{H}_0} \chi(\mathcal{H}_0 - \mu) P_s^+(x, y) - F_t^0(x, y) \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle, \quad (2.21)$$

where  $F_t^0$  is given by (2.15). Thus, by combining the high energy bounds (2.17) and the low energy bounds (2.20)–(2.21) we conclude the dispersive estimates (2.13) and (2.14).  $\square$

### 3 Symmetric resolvent identity

By assumption (A1), we can factorize the matrix potential

$$\mathcal{V} = -\sigma_3 v v = v_1 v_2, \quad (3.1)$$

with

$$v_1 = -\sigma_3 v := \begin{bmatrix} -a & -b \\ b & a \end{bmatrix} \quad \text{and} \quad v_2 = v := \begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

where

$$a := \frac{1}{2} \left( \sqrt{V_1 + V_2} + \sqrt{V_1 - V_2} \right) \quad \text{and} \quad b := \frac{1}{2} \left( \sqrt{V_1 + V_2} - \sqrt{V_1 - V_2} \right).$$

It will be helpful in later sections to keep in mind that

$$V_1 = a^2 + b^2, \quad V_2 = 2ab. \quad (3.2)$$

We denote the resolvent of  $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$  by  $(\mathcal{H} - z)^{-1}$  for  $z \in \rho(\mathcal{H})$ . The resolvent identity states that

$$(\mathcal{H} - z)^{-1} = (I + (\mathcal{H}_0 - z)^{-1} \mathcal{V})^{-1} (\mathcal{H}_0 - z)^{-1}, \quad \forall z \in \rho(\mathcal{H}_0) \cap \rho(\mathcal{H}).$$

This identity was used in [23] to establish that there is a limiting absorption principle for the resolvent of  $\mathcal{H}$  on the semi-axes  $(-\infty, -\mu) \cup (\mu, \infty)$  in the weighted  $L^2$ -spaces  $X_\sigma \rightarrow X_{-\sigma}$ ,  $\sigma > \frac{1}{2}$ . Note that the lemma below applies in any spatial dimension.

**Lemma 3.1** ([23, Lemma 4-Corollary 6], see also the proof in [40, Lemma 6.8]) *Suppose assumptions (A1)–(A4) hold. Then, the following holds.*

1. For  $\sigma > \frac{1}{2}$ , and  $|\lambda| > \mu$ , the operator

$$(\mathcal{H}_0 - (\lambda \pm i0))^{-1} \mathcal{V} : X_{-\sigma} \rightarrow X_{-\sigma} \quad (3.3)$$

is compact and  $I + (\mathcal{H}_0 - (\lambda \pm i0))^{-1} \mathcal{V}$  is boundedly invertible on  $X_{-\sigma}$ .

2. For  $\sigma > \frac{1}{2}$  and  $\lambda_0 > \mu$  arbitrary, we have

$$\sup_{|\lambda| \geq \lambda_0, \varepsilon > 0} |\lambda|^{\frac{1}{2}} \left\| (\mathcal{H} - (\lambda \pm i\varepsilon))^{-1} \right\|_{X_\sigma \rightarrow X_{-\sigma}} < \infty. \quad (3.4)$$

3. For  $|\lambda| > \mu$ , define

$$(\mathcal{H} - (\lambda \pm i0))^{-1} := \left( I + (\mathcal{H}_0 - (\lambda \pm i0))^{-1} \mathcal{V} \right)^{-1} (\mathcal{H}_0 - (\lambda \pm i0))^{-1}. \quad (3.5)$$

Then, as  $\varepsilon \searrow 0$ ,

$$\left\| (\mathcal{H} - (\lambda \pm i\varepsilon))^{-1} - (\mathcal{H} - (\lambda \pm i0))^{-1} \right\|_{X_\sigma \rightarrow X_{-\sigma}} \longrightarrow 0 \quad (3.6)$$

for any  $\sigma > \frac{1}{2}$ .

We recall the following spectral representation of  $e^{it\mathcal{H}}$  from [23].

**Lemma 3.2** ([23, Lemma 12]) *Under assumptions (A1)–(A4), there is the representation*

$$e^{it\mathcal{H}} = \frac{1}{2\pi i} \int_{|\lambda| \geq \mu} e^{it\lambda} \left[ (\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1} \right] d\lambda + \sum_j e^{it\mathcal{H}} P_{z_j}, \quad (3.7)$$

where the sum runs over the entire discrete spectrum and  $P_{z_j}$  is the Riesz projection corresponding to the eigenvalue  $z_j$ . The formula (3.7) and the convergence of the integral are to be understood in the sense that if  $\phi, \psi \in [W^{2,2}(\mathbb{R}) \times W^{2,2}(\mathbb{R})] \cap [(x)^{-1-L^2}(\mathbb{R}) \times (x)^{-1-L^2}(\mathbb{R})]$ , then

$$\begin{aligned} \langle e^{it\mathcal{H}} \phi, \psi \rangle &= \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{R \geq |\lambda| \geq \mu} e^{it\lambda} \left\langle \left[ (\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1} \right] \phi, \psi \right\rangle d\lambda \\ &\quad + \sum_j \langle e^{it\mathcal{H}} P_{z_j} \phi, \psi \rangle, \end{aligned} \quad (3.8)$$

for all  $t \in \mathbb{R}$ .

We write  $P_s = P_s^+ + P_s^-$ , where the signs  $\pm$  refer to the positive and negative halves of the essential spectrum  $(-\infty, -\mu] \cup [\mu, \infty)$ . In the following sections, we will focus on the analysis on the positive semi-axis part of the essential spectrum. We can treat the negative semi-axis of the essential spectrum by taking advantage of the symmetry properties of  $\mathcal{H}$ , see Remark 3.3 below. In view of the spectral representation of  $e^{it\mathcal{H}}$  from Lemma 3.2, we use the change of variables  $\lambda \mapsto \lambda = \mu + z^2$  with  $0 < z < \infty$  to write

$$e^{it\mathcal{H}} P_s^+ = \frac{e^{it\mu}}{\pi i} \int_0^\infty e^{itz^2} z \left[ (\mathcal{H} - (\mu + z^2 + i0))^{-1} - (\mathcal{H} - (\mu + z^2 - i0))^{-1} \right] dz.$$

For the upcoming dispersive estimates, it is convenient to first open up the domain of integration for the above integral to the entire real line by means of analytic continuation for the perturbed resolvent. Following the framework of Sect. 5 in [23], we introduce the operator

$$\begin{aligned} \mathcal{R}(z) &:= (\mathcal{H} - (\mu + z^2 + i0))^{-1}, \quad \text{for } z > 0, \\ \mathcal{R}(z) &:= (\mathcal{H} - (\mu + z^2 - i0))^{-1} = \overline{(\mathcal{H} - (\mu + z^2 + i0))^{-1}}, \quad \text{for } z < 0, \end{aligned} \quad (3.9)$$

so that

$$e^{it\mathcal{H}} P_s^+ = \frac{e^{it\mu}}{\pi i} \int_{\mathbb{R}} e^{itz^2} z \mathcal{R}(z) dz. \quad (3.10)$$

Here, the integral should be understood in the principal value sense due to the pole associated with the resolvent  $\mathcal{R}(z)$  at the origin. We also set

$$\begin{aligned}\mathcal{R}_0(z) &:= (\mathcal{H}_0 - (\mu + z^2 + i0))^{-1}, \quad \text{for } z > 0, \\ \mathcal{R}_0(z) &:= (\mathcal{H}_0 - (\mu + z^2 + i0))^{-1}, \quad \text{for } z < 0.\end{aligned}\quad (3.11)$$

In particular, with this definition, we have by (2.4) for all  $z \in \mathbb{R} \setminus \{0\}$  that

$$\mathcal{R}_0(z)(x, y) = (\mathcal{H}_0 - (\mu + z^2 + i0))^{-1}(x, y) = \begin{bmatrix} \frac{ie^{iz|x-y|}}{2z} & 0 \\ 0 & -\frac{e^{-\sqrt{z^2+2\mu}|x-y|}}{2\sqrt{z^2+2\mu}} \end{bmatrix}. \quad (3.12)$$

As in [23], we employ the symmetric resolvent identity

$$\mathcal{R}(z) = \mathcal{R}_0(z) - \mathcal{R}_0(z)v_1(M(z))^{-1}v_2\mathcal{R}_0(z), \quad (3.13)$$

where

$$M(z) = I + v_2\mathcal{R}_0(z)v_1, \quad z \in \mathbb{R} \setminus \{0\}. \quad (3.14)$$

By inserting the above identity, one checks that

$$e^{it\mathcal{H}}P_s^+ = \frac{e^{it\mu}}{\pi i} \int_{\mathbb{R}} e^{itz^2} z \mathcal{R}_0(z) dz - \frac{e^{it\mu}}{\pi i} \int_{\mathbb{R}} e^{itz^2} z \mathcal{R}_0(z) v_1 (M(z))^{-1} v_2 \mathcal{R}_0(z) dz. \quad (3.15)$$

In the next section, we will investigate the invertibility of the matrix operator  $M(z)$  near the origin. We give the following remark for the evolution operator in the negative part of the essential spectrum.

**Remark 3.3** Using the identities

$$\mathcal{H} = -\sigma_1 \mathcal{H} \sigma_1, \quad \mathcal{V} = -\sigma_1 \mathcal{V} \sigma_1, \quad (3.16)$$

we infer that

$$e^{it\mathcal{H}}P_s^- = \sigma_1 e^{-it\mathcal{H}}P_s^+ \sigma_1. \quad (3.17)$$

Furthermore, since these identities also hold for  $\mathcal{H}_0$ , the analogue of Proposition 2.3 for the weighted estimate of the free evolution  $e^{it\mathcal{H}_0}P_s^-$  is given by

$$\left\| \langle x \rangle^{-1} \left( e^{it\mathcal{H}_0} P_s^- - \tilde{F}_t^0 \right) \tilde{u} \right\|_{L_x^\infty} \leq C |t|^{-\frac{3}{2}} \|\langle x \rangle \tilde{u}\|_{L_x^1}, \quad |t| \geq 1, \quad (3.18)$$

where

$$\tilde{F}_t^0(x, y) := \frac{e^{-it\mu}}{\sqrt{4\pi it}} e^{i\frac{x^2}{4t}} \underline{e}_2 e^{i\frac{y^2}{4t}} \underline{e}_2^\top. \quad (3.19)$$

Note that  $\tilde{F}_t^0 = \sigma_1 F_{-t}^0 \sigma_1$ .

## 4 Laurent expansion of the resolvent near the threshold

In this section we study asymptotic expansions of the perturbed resolvent operators near the thresholds of the essential spectrum, closely following the framework of the seminal paper [36] for the scalar Schrödinger operators  $H = -\partial_x^2 + V$  on the real line. As specified in the introduction, we are interested in the irregular case, where the matrix Schrödinger operator  $\mathcal{H}$  exhibits a threshold resonance. See Definition 4.4 for a precise definition. This means that there exist globally bounded non-trivial solutions of  $\mathcal{H}\Psi = \pm\mu\Psi$ . In this context, we mention that the threshold regularity can also be characterized by the scattering theory introduced by [9]; see Lemma 5.20 of [40]. We begin with the terminology used in [52].

**Definition 4.1** (*Absolutely bounded operators*) We say an operator  $A: L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})$  with an integral kernel  $A(x, y) \in \mathbb{C}^{2 \times 2}$  is absolutely bounded if the operator with the kernel  $|A(x, y)| := (|A(x, y)_{i,j}|)_{i,j=1}^2 \in \mathbb{R}^{2 \times 2}$  is bounded from  $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})$ . In particular, Hilbert-Schmidt and finite rank operators are absolutely bounded.

To investigate the asymptotic expansions of the operator  $M(z)$  (c.f. (3.14)), we start with the following Taylor expansions of the free resolvent around the origin  $z = 0$ .

**Lemma 4.2** *Let  $z_0 := \min\{1, \sqrt{2\mu}\}$ . For any  $0 < |z| < z_0$ , we have the following expansion*

$$\mathcal{R}_0(z)(x, y) = \frac{i}{2z} e_{11} + \mathcal{G}_0(x, y) + z\mathcal{G}_1(x, y) + E(z)(x, y) \quad (4.1)$$

where

$$\mathcal{G}_0(x, y) := \begin{bmatrix} -\frac{|x-y|}{2} & 0 \\ 0 & -\frac{e^{-\sqrt{2\mu}|x-y|}}{2\sqrt{2\mu}} \end{bmatrix}, \quad (4.2)$$

$$\mathcal{G}_1(x, y) := \begin{bmatrix} \frac{|x-y|^2}{4i} & 0 \\ 0 & 0 \end{bmatrix}, \quad (4.3)$$

and  $E(z)$  is an error term which satisfies the estimate

$$|z|^k |\partial_z^k E(z)(x, y)| \leq C_{\mu,k} |z|^2 \langle x \rangle^{3+k} \langle y \rangle^{3+k}, \quad \forall k = 0, 1, 2, \quad (4.4)$$

for any  $|z| < z_0$ .

**Proof** Recall from (3.12) that

$$\mathcal{R}_0(z)(x, y) = \begin{bmatrix} \frac{ie^{iz|x-y|}}{2z} & 0 \\ 0 & -\frac{e^{-\sqrt{z^2+2\mu}|x-y|}}{2\sqrt{z^2+2\mu}} \end{bmatrix}.$$

For  $0 < |z| < 1$ , we have the Laurent expansion

$$\frac{ie^{iz|x-y|}}{2z} = \frac{i}{2z} + \frac{-|x-y|}{2} + \frac{|x-y|^2}{4i}z + r_1(z, |x-y|), \quad (4.5)$$

where the remainder term is

$$r_1(z, |x-y|) := \frac{i}{2z} \tilde{r}_1(z, |x-y|), \quad \tilde{r}_1(z, |x-y|) := \frac{(iz|x-y|)^3}{2!} \int_0^1 e^{is_z|x-y|} (1-s)^2 ds.$$

By direct computation, for any  $x, y \in \mathbb{R}$  and for any  $|z| < 1$ , we have the estimate

$$|z|^k |\partial_z^k r_1(z, |x - y|)| \lesssim |z|^2 \langle x \rangle^{3+k} \langle y \rangle^{3+k}, \quad k = 0, 1, 2. \quad (4.6)$$

In the lower component of the resolvent kernel, for  $|z| < 2\mu$ , we have the Taylor expansion

$$-\frac{e^{-\sqrt{z^2+2\mu}|x-y|}}{2\sqrt{z^2+2\mu}} = -\frac{e^{-\sqrt{2\mu}|x-y|}}{2\sqrt{2\mu}} + r_2(z, |x - y|), \quad (4.7)$$

where we denote the remainder term by

$$r_2(z, |x - y|) := \frac{z^2}{2!} \int_0^1 (1-s) (\partial_z^2 g_\mu)(sz, |x - y|) ds, \quad g_\mu(z, |x - y|) := -\frac{e^{-\sqrt{z^2+2\mu}|x-y|}}{2\sqrt{z^2+2\mu}}.$$

Using the fact that for any  $\eta \in \mathbb{R}$ ,  $\langle \eta \rangle := (1 + \eta^2)^{\frac{1}{2}}$ , one has the bounds

$$|\partial_\eta^k \langle \eta \rangle^{-1}| \leq C_k \langle \eta \rangle^{-1-k} \quad |\partial_\eta^k \langle \eta \rangle| \leq C_k \langle \eta \rangle^{1-k}, \quad k = 0, 1, 2, \dots,$$

it follows that all derivatives of  $\sqrt{z^2 + 2\mu}$  and  $2(z^2 + 2\mu)^{-\frac{1}{2}}$  are uniformly bounded in  $z$  up to a constant depending only on  $\mu$  and the number of derivatives. Therefore, by the Leibniz formula, we have the estimate

$$\sup_{z \in \mathbb{R}} \left| \partial_z^k g_\mu(z, |x - y|) \right| \leq C_{\mu,k} \langle x \rangle^k \langle y \rangle^k, \quad k = 0, 1, \dots, 4,$$

which in turn implies that

$$|z|^k |\partial_z^k r_2(z, |x - y|)| \lesssim |z|^2 \langle x \rangle^{2+k} \langle y \rangle^{2+k}, \quad k = 0, 1, 2. \quad (4.8)$$

Thus, by using (4.6) and (4.8), the error term given by

$$E(z)(x, y) := \begin{bmatrix} r_1(z, |x - y|) & 0 \\ 0 & r_2(z, |x - y|) \end{bmatrix} \quad (4.9)$$

satisfies (4.4) as claimed.  $\square$

We insert the above asymptotic expansion into the operator  $M(z) = I + v_2 \mathcal{R}_0(z) v_1$ . First, we have the *transfer operator*  $T$  on  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  with a kernel given by

$$T(x, y) = I + v_2(x) \mathcal{G}_0(x, y) v_1(y). \quad (4.10)$$

Note that  $T$  is self-adjoint because

$$(v_2 \mathcal{G}_0 v_1)^* = v_1^* \mathcal{G}_0 v_2 = (-v \sigma_3) \mathcal{G}_0 v = v \mathcal{G}_0 (-\sigma_3 v) = v_2 \mathcal{G}_0 v_1.$$

Since the potentials  $v_1$  and  $v_2$  have exponential decay by assumption (A3), it follows that  $v_2 \mathcal{G}_0 v_1$  is a Hilbert-Schmidt operator on  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ . Hence,  $T$  is a compact perturbation of the identity, and therefore the dimension of  $\ker(T)$  is finite by the Fredholm alternative. Recalling the formulas for  $v_1$  and  $v_2$  from (3.1), we have the identity

$$v_2 \underline{e}_{11} v_1 = - \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = - \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}. \quad (4.11)$$

Next, we define the orthogonal projection onto the span of the vector  $(a, b)^\top \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$  by

$$P \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} (x) := \frac{\int_{\mathbb{R}} (a(y) f_1(y) + b(y) f_2(y)) dy}{\|a^2 + b^2\|_{L^1(\mathbb{R})}} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix}$$

$$= \frac{1}{\|V_1\|_{L^1(\mathbb{R})}} \langle (a, b)^\top, \vec{f} \rangle \begin{bmatrix} a(x) \\ b(x) \end{bmatrix}. \quad (4.12)$$

Note that we use the identity (3.2) above. From (3.14), the contribution of the singular term  $\frac{i}{2z} e_{11}$  of  $\mathcal{R}_0(z)$  to  $M(z)$  will be associated to the following integral operator with the kernel

$$\frac{i}{2z} v_2(x) e_{11} v_1(y) = \frac{-i}{2z} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix} [a(y) \ b(y)] =: g(z) P(x, y), \quad (4.13)$$

where

$$g(z) := -\frac{i}{2z} \|V_1\|_{L^1(\mathbb{R})}. \quad (4.14)$$

Lastly, we denote the orthogonal projection to the complement of the span of  $(a, b)^\top$  by

$$Q := I - P. \quad (4.15)$$

In summary, we have the following proposition.

**Proposition 4.3** Suppose  $|a(x)|, |b(x)| \lesssim \langle x \rangle^{-5.5-}$ , and let  $z_0 := \min\{1, \sqrt{2\mu}\}$ . Then, for any  $0 < |z| < z_0$ , we have

$$M(z) = g(z)P + T + zM_1 + \mathcal{M}_2(z), \quad (4.16)$$

where  $M_1$  and  $\mathcal{M}_2(z)$  are Hilbert-Schmidt operators on  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  defined by

$$M_1(x, y) := v_2(x) \mathcal{G}_1(x, y) v_1(y) = \frac{|x - y|^2}{4i} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix} [a(y) \ b(y)], \quad (4.17)$$

$$\mathcal{M}_2(z)(x, y) := v_2(x) E(z)(x, y) v_1(y), \quad (4.18)$$

with  $G_1$  and  $E(z)$  defined in Lemma 4.2. Moreover, the error term  $\mathcal{M}_2(z)$  and its derivatives satisfy the absolute bound

$$|z|^k \left\| |\partial_z^k \mathcal{M}_2(z)| \right\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})} \lesssim |z|^2, \quad k = 0, 1, 2, \quad (4.19)$$

for all  $|z| < z_0$ .

**Proof** The identity on the right of (4.17) follows from (4.11). We recall that operators of the following type

$$U(x) \langle x \rangle^k \langle y \rangle^k W(y)$$

are Hilbert-Schmidt operators on  $L^2(\mathbb{R})$  whenever  $U$  and  $W$  are bounded potentials with polynomial decay  $|U(x)|, |W(x)| \lesssim \langle x \rangle^{-k-\frac{1}{2}-}$ , for  $k \in \mathbb{N}$ . Hence, under the assumptions on  $a(x)$  and  $b(x)$ , and using the fact that

$$|\mathcal{G}_1(x, y)| \lesssim |x - y|^2 \leq \langle x \rangle^2 \langle y \rangle^2,$$

it follows that  $M_1$  is Hilbert-Schmidt. The same argument can be applied to the error term  $\mathcal{M}_2(z)$  and its derivatives using the remainder estimates in (4.4) and we are done.  $\square$

The next definition characterizes the regularity of the endpoint  $\mu$  of the essential spectrum.

**Definition 4.4** 1. We say that the threshold  $\mu$  is a regular point of the spectrum of  $\mathcal{H}$  provided that the operator  $QTQ$  is invertible on the subspace  $Q(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ .

2. Suppose  $\mu$  is not a regular point. Let  $S_1$  be the Riesz projection onto the kernel of  $QTQ$ , and we define  $D_0 = (Q(T + S_1)Q)^{-1}$ . Note that  $QD_0Q$  is an absolutely bounded operator on  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ . The proof for this follows from Lemma 8 of [52] with minor changes. See also [24, Lemma 2.7].

Note that since we impose symmetry assumptions on the potential  $\mathcal{V}$ , the thresholds  $\mu$  and  $-\mu$  are either both regular or irregular. The invertibility of  $QTQ$  is related to the absence of distributional  $L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$  solutions to  $\mathcal{H}\Psi = \mu\Psi$ . The following lemma establishes the equivalent definitions. See [36, Lemma 5.4] for the analogue in the scalar case.

**Lemma 4.5** *Suppose assumptions (A1)–(A5) hold. Then the following holds.*

1. Let  $\Phi \in S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R})) \setminus \{0\}$ . If  $\Psi = (\Psi_1, \Psi_2)^\top$  is defined by

$$\Psi(x) := -\mathcal{G}_0[v_1\Phi](x) + c_0\underline{e}_1, \quad (4.20)$$

with

$$c_0 = \frac{\langle (a, b)^\top, T\Phi \rangle}{\|V_1\|_{L^1(\mathbb{R})}}, \quad (4.21)$$

then

$$\Phi = v_2\Psi, \quad (4.22)$$

and  $\Psi \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$  is a distributional solution to

$$\mathcal{H}\Psi = \mu\Psi. \quad (4.23)$$

Furthermore, if additionally assumption (A6) holds, i.e.,

$$c_{2,\pm} := \frac{1}{2\sqrt{2\mu}} \int_{\mathbb{R}} e^{\pm\sqrt{2\mu}y} (V_2(y)\Psi_1(y) + V_1(y)\Psi_2(y)) \, dy = 0, \quad (4.24)$$

then

$$\lim_{x \rightarrow \pm\infty} \Psi_1(x) = c_0 \mp c_1, \quad (4.25)$$

where

$$c_1 := \frac{1}{2} \langle x(a(x), b(x))^\top, \Phi(x) \rangle = \frac{1}{2} \int_{\mathbb{R}} x(a(x)\Phi_1(x) + b(x)\Phi_2(x)) \, dx. \quad (4.26)$$

In particular,

$$\Psi_1 \notin L^2(\mathbb{R}). \quad (4.27)$$

More precisely, the constants  $c_0$  and  $c_1$  cannot both be zero.

2. Conversely, suppose there exists  $\Psi \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$  satisfying (4.23) in the distributional sense. Then

$$\Phi = v_2\Psi \in S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R})). \quad (4.28)$$

3. Suppose assumptions (A1)–(A6) hold. Then,  $\dim S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R})) \leq 1$ . In the case  $\dim S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R})) = 1$ , i.e.,  $S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R})) = \text{span}\{\Phi\}$  for some  $\Phi = (\Phi_1, \Phi_2)^\top \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \setminus \{0\}$ , we have the following identities

$$S_1 T P T S_1 = |c_0|^2 \|\Phi\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R})}^{-2} \|V_1\|_{L^1(\mathbb{R})} S_1, \quad (4.29)$$

$$PT S_1 T P = |c_0|^2 \|\Phi\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R})}^{-2} \|V_1\|_{L^1(\mathbb{R})} P, \quad (4.30)$$

$$S_1 M_1 S_1 = -2i |c_1|^2 \|\Phi\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R})}^{-2} S_1, \quad (4.31)$$

where the constants  $c_0$  and  $c_1$  are given by (4.21) and (4.26) respectively for this  $\Phi$ .

**Proof** (Proof of (1)) Let  $\Phi = (\Phi_1, \Phi_2) \in S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$  with  $\Phi \neq 0$ . Since  $S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$  is a subspace of  $Q(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ , we have  $Q\Phi = \Phi$ . Using the fact that  $\Phi \in \ker(QTQ)$  and the definition of  $T$  (c.f. (4.10)), we obtain

$$0 = QTQ\Phi = (I - P)T\Phi = (I + v_2\mathcal{G}_0v_1)\Phi - PT\Phi. \quad (4.32)$$

Since  $(a, b)^\top = v_2e_1$  and  $P$  is the orthogonal projection onto the span of  $(a, b)^\top$ , we have

$$PT\Phi = \frac{\langle (a, b)^\top, T\Phi \rangle}{\|V_1\|_{L^1(\mathbb{R})}} (a, b)^\top = c_0 v_2 e_1, \quad (4.33)$$

with  $c_0$  defined in (4.21). It follows that

$$\Phi = -v_2\mathcal{G}_0v_1\Phi + c_0v_2e_1 = v_2(-\mathcal{G}_0v_1\Phi + c_0e_1) = v_2\Psi.$$

This proves (4.22). Next, we show (4.23). Denoting  $\Phi = (\Phi_1, \Phi_2)^\top$  and using the definition of  $\mathcal{G}_0$  (c.f. (4.2)), we have

$$(\mathcal{H}_0 - \mu I)\mathcal{G}_0(v_1\Phi) = v_1\Phi,$$

i.e.,

$$\begin{cases} (-\partial_x^2) \int_{\mathbb{R}} \frac{-|x-y|}{2} (-a(y)\Phi_1(y) - b(y)\Phi_2(y)) dy = -a(x)\Phi_1(x) - b(x)\Phi_2(x), \\ (\partial_x^2 - 2\mu) \int_{\mathbb{R}} \frac{-e^{-\sqrt{2\mu}|x-y|}}{2\sqrt{2\mu}} (b(y)\Phi_1(y) + a(y)\Phi_2(y)) dy = b(x)\Phi_1(x) + a(x)\Phi_2(x). \end{cases}$$

This equation is well-defined, since  $v_1\Phi \in \langle x \rangle^{-1-L^1(\mathbb{R})} \times \langle x \rangle^{-1-L^1(\mathbb{R})}$ . Using (4.20), (4.22), and  $(H_0 - \mu I)(c_0e_1) = 0$ , we have

$$(\mathcal{H}_0 - \mu I)\Psi = (H_0 - \mu I)[- \mathcal{G}_0(v_1\Phi) + c_0e_1] = -v_1\Phi = -v_1v_2\Psi = -\mathcal{V}\Psi,$$

which implies (4.23). We now show that  $\Psi = (\Psi_1, \Psi_2)^\top$  is in  $L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ . Noting that

$$\Psi_1(x) = c_0 + \frac{1}{2} \int_{\mathbb{R}} |x-y| (a(y)\Phi_1(y) + b(y)\Phi_2(y)) dy,$$

by employing the orthogonality condition  $\langle (a, b)^\top, \Phi \rangle = 0$ , we have

$$\Psi_1(x) = c_0 + \frac{1}{2} \int_{\mathbb{R}} (|x-y| - |x|) (a(y)\Phi_1(y) + b(y)\Phi_2(y)) dy.$$

Using  $||x-y| - |x|| \leq |y|$  and  $|a(y)| + |b(y)| \lesssim \langle y \rangle^{-2}$ , we have the uniform bound

$$\sup_{x \in \mathbb{R}} |\Psi_1(x)| \leq |c_0| + \frac{1}{2} \int_{\mathbb{R}} |y| |a(y)\Phi_1(y) + b(y)\Phi_2(y)| dy \lesssim \|\Phi\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} \lesssim 1.$$

Since  $(a, b)^\top$  and  $\Phi$  are in  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ , we have the uniform bound on  $\Psi_2$  by the Cauchy-Schwarz inequality

$$\begin{aligned} \sup_{x \in \mathbb{R}} |\Psi_2(x)| &\lesssim \int_{\mathbb{R}} |b(y)\Phi_1(y) + a(y)\Phi_2(y)| dy \\ &\leq \|b\|_{L^2(\mathbb{R})} \|\Phi_1\|_{L^2(\mathbb{R})} + \|a\|_{L^2(\mathbb{R})} \|\Phi_2\|_{L^2(\mathbb{R})} \lesssim 1. \end{aligned}$$

Thus, we have shown that  $\Psi = (\Psi_1, \Psi_2)^\top \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ . Finally, we now assume  $c_{2,\pm} = 0$  and show that  $\Psi_1$  cannot be in  $L^2(\mathbb{R}) \setminus \{0\}$  by a Volterra argument. Using  $\langle (a, b)^\top, \Phi \rangle = 0$ , for  $x \geq 0$  large, we write

$$\Psi_1(x) = c_0 - c_1 + \int_x^\infty (y - x)(a(y)\Phi_1(y) + b(y)\Phi_2(y)) dy. \quad (4.34)$$

Using  $c_{2,\pm} = 0$ , we insert  $-e^{-\sqrt{2\mu}x}c_{2,+} = 0$  to write

$$\Psi_2(x) = \frac{1}{2\sqrt{2\mu}} \int_x^\infty (e^{-\sqrt{2\mu}(y-x)} - e^{-\sqrt{2\mu}(x-y)})(V_2(y)\Psi_1(y) + V_1(y)\Psi_2(y)) dy. \quad (4.35)$$

Similarly, for  $x < 0$ , using  $e^{\sqrt{2\mu}x}c_{2,-} = 0$ , we have

$$\Psi_1(x) = c_0 + c_1 + \int_{-\infty}^x (x - y)(V_1(y)\Psi_1(y) + V_2(y)\Psi_2(y)) dy, \quad (4.36)$$

$$\Psi_2(x) = \frac{1}{2\sqrt{2\mu}} \int_{-\infty}^x (e^{-\sqrt{2\mu}(x-y)} - e^{-\sqrt{2\mu}(y-x)})(V_2(y)\Psi_1(y) + V_1(y)\Psi_2(y)) dy. \quad (4.37)$$

Suppose now that  $c_0 = c_1 = 0$ . Owing to the exponential decay of  $V_1, V_2$  by assumption (A3), we obtain from (4.34) and (4.34) a homogeneous Volterra equation for  $\Psi = (\Psi_1, \Psi_2)^\top$  satisfying

$$\Psi(x) = \int_{\mathbb{R}} K(x, y)\Psi(y) dy, \quad x \geq 0,$$

where  $|K(x, y)| \lesssim e^{-\gamma|y|} \mathbb{1}_{y>x}$  for some  $0 < \gamma < \beta$ , which is a quasi-nilpotent operator. By performing a standard contraction on  $L^\infty(M, \infty)$ , with  $M > 0$  sufficiently large, one arrives at a solution  $\Psi(x) \equiv 0$  for all  $x \geq M$ . By the uniqueness theorem for ODEs, this implies that  $\Psi \equiv 0$  on  $\mathbb{R}$ . Then, by the relation  $\Phi = v_2\Psi$  and the fact that  $v_2$  is a positive matrix, one finds that  $\Phi \equiv 0$ , which contradicts the hypothesis  $\Phi \neq 0$ . Thus, the conclusion is that  $c_0$  and  $c_1$  cannot be both zero. In particular, it follows from (4.34) and (4.36) that

$$\lim_{x \rightarrow \pm\infty} \Psi_1(x) = c_0 \mp c_1.$$

Since either  $c_0 + c_1 \neq 0$  or  $c_0 - c_1 \neq 0$ , we conclude that  $\Psi_1 \notin L^2(\mathbb{R})$ .

*Proof of (2).* Define  $\Phi = v_2\Psi$ . Since  $\Psi$  is a distributional solution to (4.23), using  $\mathcal{V} = v_1v_2$ , we have

$$(\mathcal{H}_0 - \mu I)\Psi = v_1\Phi \iff \begin{cases} \Psi_1'' = a\Phi_1 + b\Phi_2, \\ \Psi_2'' - 2\mu\Psi_2 = b\Phi_1 + a\Phi_2. \end{cases}$$

Let  $\eta \in C_0^\infty(\mathbb{R})$  be a non-negative function satisfying  $\eta(x) = 1$  for  $|x| \leq 1$  and  $\eta(x) = 0$  for  $|x| \geq 2$ . Using the first equation from above and integrating by parts, we have for any  $\varepsilon > 0$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}} (a(y)\Phi_1(y) + b(y)\Phi_2(y))\eta(\varepsilon y) dy \right| &= \left| \int_{\mathbb{R}} \Psi_1''(y)\eta(\varepsilon y) dy \right| \\ &= \left| \int_{\mathbb{R}} \Psi_1(y)\varepsilon^2\eta''(\varepsilon y) dy \right| \leq \varepsilon \|\Psi_1\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\eta''(x)| dx. \end{aligned}$$

By taking the limit  $\varepsilon \rightarrow 0$  and using the Lebesgue dominated convergence theorem, we find that  $\langle (a, b)^\top, \Phi \rangle = 0$ . Thus,  $P\Phi = 0$ , i.e.  $\Phi \in Q(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ . Following this fact and using  $\Phi = v_2\Psi$ , we have

$$QTQ\Phi = QT\Phi = Q(I + v_2\mathcal{G}_0v_1)\Phi = Qv_2(\Psi + \mathcal{G}_0(\mathcal{V}\Psi)). \quad (4.38)$$

Now set  $u := \Psi + \mathcal{G}_0(\mathcal{V}\Psi)$ . Since  $u = (u_1, u_2)^\top$  is a distributional solution of  $(\mathcal{H}_0 - \mu I)u = 0$ , i.e.

$$\begin{aligned} -u_1'' &= 0, \\ u_2'' - 2\mu u_2 &= 0, \end{aligned}$$

we find that

$$\begin{aligned} u_1(x) &= \kappa_1 + \kappa_2 x, \\ u_2(x) &= \kappa_3 e^{-\sqrt{2\mu}x} + \kappa_4 e^{\sqrt{2\mu}x}, \end{aligned}$$

for some  $\kappa_i \in \mathbb{C}, i \in \{1, \dots, 4\}$ . By similar arguments from Item (1), we obtain that  $\mathcal{G}_0(\mathcal{V}\Psi) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ . Since  $\Psi \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ , it follows that  $u \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ , which implies that  $\kappa_2 = \kappa_3 = \kappa_4 = 0$ . Thus, we have  $u(x) \equiv (\kappa_1, 0)^\top = \kappa_1 \underline{e}_1$ . Since  $Qv_2 \underline{e}_1 = 0$ , we conclude from (4.38) using the definition of  $u(x)$  that  $QTQ\Phi = 0$ , whence  $\Phi \in S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ .

*Proof of (3).* Suppose there are two linearly independent  $\Phi, \tilde{\Phi} \in S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ . As in the proof of Item (1), for  $x \geq 0$ , we have

$$\begin{aligned} \Psi_1(x) &= c_0 - c_1 + \int_x^\infty (y-x)(V_1(y)\Psi_1(y) + V_2(y)\Psi_2(y)) dy, \\ \Psi_2(x) &= \frac{1}{2\sqrt{2\mu}} \int_x^\infty (e^{-\sqrt{2\mu}(y-x)} - e^{-\sqrt{2\mu}(x-y)})(V_2(y)\Psi_1(y) + V_1(y)\Psi_2(y)) dy, \end{aligned}$$

and

$$\begin{aligned} \tilde{\Psi}_1(x) &= d_0 - d_1 + \int_x^\infty (y-x)(V_1(y)\tilde{\Psi}_1(y) + V_2(y)\tilde{\Psi}_2(y)) dy, \\ \tilde{\Psi}_2(x) &= \frac{1}{2\sqrt{2\mu}} \int_x^\infty (e^{-\sqrt{2\mu}(y-x)} - e^{-\sqrt{2\mu}(x-y)})(V_2(y)\tilde{\Psi}_1(y) + V_1(y)\tilde{\Psi}_2(y)) dy, \end{aligned}$$

where  $d_0$  and  $d_1$  are constants defined from  $\tilde{\Phi}$  which are analogous to  $c_0$  and  $c_1$ . There is some constant  $\theta \in \mathbb{C}$  such that

$$c_0 - c_1 = -\theta(d_0 - d_1),$$

which imply the Volterra integral equation

$$\begin{bmatrix} \Psi_1 + \theta\tilde{\Psi}_1 \\ \Psi_2 + \theta\tilde{\Psi}_2 \end{bmatrix}(x) = \int_x^\infty \begin{bmatrix} y-x & 0 \\ 0 & \frac{e^{-\sqrt{2\mu}(y-x)} - e^{-\sqrt{2\mu}(x-y)}}{2\sqrt{2\mu}} \end{bmatrix} \mathcal{V}(y) \begin{bmatrix} \Psi_1(y) + \theta\tilde{\Psi}_1(y) \\ \Psi_2(y) + \theta\tilde{\Psi}_2(y) \end{bmatrix} dy,$$

for any  $x \geq 0$ . By the same Volterra equation argument used in Item (1), we obtain  $\Psi + \theta\tilde{\Psi} \equiv 0$ , which implies that  $\Phi + \theta\tilde{\Phi} \equiv 0$ , but this contradicts that  $\Phi$  and  $\tilde{\Phi}$  are linearly independent.

Thus, we have shown that  $\dim S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R})) \leq 1$ . Next, we prove (4.29)–(4.31). Write  $S_1 = \|\Phi\|_{L^2 \times L^2}^{-2} \langle \Phi, \cdot \rangle \Phi$ . By (4.33) and the fact that  $P$ ,  $S_1$ , and  $T$  are self-adjoint, we compute for any  $u \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$  that

$$\begin{aligned} S_1 T P T S_1 u &= \|\Phi\|_{L^2 \times L^2}^{-2} \langle \Phi, u \rangle S_1 T P T \Phi = \|\Phi\|_{L^2 \times L^2}^{-2} c_0 \langle \Phi, u \rangle S_1 T \begin{bmatrix} a \\ b \end{bmatrix} \\ &= |c_0|^2 \|\Phi\|_{L^2 \times L^2}^{-2} \|V_1\|_{L^1(\mathbb{R})} S_1 u. \end{aligned}$$

A similar computation reveals

$$P T S_1 T P u = |c_0|^2 \|\Phi\|_{L^2 \times L^2}^{-2} \|V_1\|_{L^1(\mathbb{R})} P u.$$

For the third identity (4.31), in view of (4.11) and (4.17), we write

$$M_1(x, y) = v_2(x) G_1(x, y) v_1(y) = \frac{i|x-y|^2}{4} \begin{bmatrix} a(x) \\ b(x) \end{bmatrix} \begin{bmatrix} a(y) & b(y) \end{bmatrix}.$$

By using the orthogonality

$$\langle \Phi, (a, b)^\top \rangle = \int_{\mathbb{R}} (\Phi_1(x)a(x) + \Phi_2(x)b(x)) dx = 0,$$

and the identity

$$|x-y|^2 = x^2 + y^2 - 2xy,$$

we have

$$\begin{aligned} [S_1 M_1 S_1](x, y) &= \int_{\mathbb{R}^2} S_1(x, x_1) M_1(x_1, y_1) S_1(y_1, y) dx_1 dy_1 \\ &= \frac{i}{4} \frac{\Phi(x)}{\|\Phi\|_{L^2 \times L^2}^2} \int_{\mathbb{R}^2} (|x_1 - y_1|^2 \Phi^*(x_1) \begin{bmatrix} a(x_1) \\ b(x_1) \end{bmatrix} \begin{bmatrix} a(y_1) & b(y_1) \end{bmatrix} \Phi(y_1)) dx_1 dy_1 \frac{\Phi^*(y)}{\|\Phi\|_{L^2 \times L^2}^2} \\ &= -2i \left( \int_{\mathbb{R}} \frac{x_1}{2} \Phi^*(x_1) \begin{bmatrix} a(x_1) \\ b(x_1) \end{bmatrix} dx_1 \right) \left( \int_{\mathbb{R}} \frac{y_1}{2} \begin{bmatrix} a(y_1) & b(y_1) \end{bmatrix} \Phi(y_1) dy_1 \right) \|\Phi\|_{L^2 \times L^2}^{-2} S_1(x, y) \\ &= -2i |c_1|^2 \|\Phi\|_{L^2 \times L^2}^{-2} S_1(x, y). \end{aligned}$$

This proves (4.31) and we are done.  $\square$

**Remark 4.6** By direct computation, the conjugation identity  $\sigma_3 \mathcal{H} = \mathcal{H}^* \sigma_3$  and the identity  $v_1 = -\sigma_3 v_2$  imply that the vector  $\tilde{\Psi} := \sigma_3 \Psi$  solves

$$\mathcal{H}^* \tilde{\Psi} = \mu \tilde{\Psi}, \quad (4.39)$$

where  $\Psi$  is the distribution solution to (4.23). Moreover, one has the identities

$$\sigma_3 \Psi = \mathcal{G}_0(v_2 \Phi) + (c_0, 0)^\top, \quad \Phi = v_2 \Psi = -v_1^\top \tilde{\Psi} \quad (4.40)$$

Similarly, using the conjugation identity  $\sigma_1 \mathcal{H} = -\mathcal{H} \sigma_1$ , we note that the vector  $\Psi_- = \sigma_1 \Psi$  solves the system

$$\mathcal{H} \Psi_- = -\mu \Psi_-. \quad (4.41)$$

Following the preceding discussion, we assume the threshold  $\mu$  is irregular and we derive an expansion for the inverse operator  $M(z)^{-1}$  on a small punctured disk near the origin. We employ the inversion lemma due to Jensen and Nenciu [36, Lemma 2.1].

**Lemma 4.7** *Let  $H$  be a Hilbert space, let  $A$  be a closed operator and  $S$  a projection. Suppose  $A + S$  has a bounded inverse. Then  $A$  has a bounded inverse if and only if*

$$B = S - S(A + S)^{-1}S$$

*has a bounded inverse in  $SH$ , and in this case,*

$$A^{-1} = (A + S)^{-1} + (A + S)^{-1}SB^{-1}S(A + S)^{-1}, \quad \text{on } H.$$

We will now state the inverse operator of  $M(z)$  away from  $z = 0$ .

**Proposition 4.8** *Suppose assumptions (A1)–(A6) hold. Let  $S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R})) = \text{span}(\{\Phi\})$  for some  $\Phi = (\Phi_1, \Phi_2)^T \neq \vec{0}$ . Let  $\kappa := (2i)^{-1}\|V_1\|_{L^1(\mathbb{R})}$ , and let  $d$  be the constant defined by*

$$d := -2i(|c_0|^2 + |c_1|^2)\|\Phi\|_{L^2 \times L^2}^{-2} \neq 0, \quad (4.42)$$

*with  $c_0$  and  $c_1$  defined by (4.21) and (4.26) respectively for this  $\Phi$ . Then, there exists a positive radius  $z_0 > 0$  such that for all  $0 < |z| < z_0$ ,  $M(z)$  is invertible on  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  and*

$$\begin{aligned} M(z)^{-1} &= \frac{1}{d} \left( \frac{1}{z} S_1 - \frac{1}{\kappa} P T S_1 - \frac{1}{\kappa} S_1 T P \right) + \left( \frac{1}{\kappa} + \frac{|c_0|^2 \|\Phi\|_{L^2 \times L^2}^{-2} \|V_1\|_{L^1(\mathbb{R})}}{d\kappa^2} \right) z P \\ &\quad + Q \Lambda_0(z) Q + z Q \Lambda_1(z) + z \Lambda_2(z) Q + z^2 \Lambda_3(z), \end{aligned} \quad (4.43)$$

*where  $\Lambda_j(z)$  are absolutely bounded operators on  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  satisfying the improved bounds*

$$\|\partial_z^k \Lambda_j(z)\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})} \lesssim 1, \quad k = 0, 1, 2, \quad j = 0, 1, 2, 3, \quad (4.44)$$

*uniformly in  $z$  for  $|z| < z_0$ .*

**Proof** Throughout the proof, we will denote by  $\mathcal{E}_j(z)$ , for  $0 \leq j \leq 3$ , as error terms that satisfy the absolute bound

$$|z|^k \left\| \partial_z^k \mathcal{E}_j(z) \right\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R})} \lesssim |z|^j, \quad \forall k = 0, 1, 2, \quad \forall |z| < z_0,$$

for some  $z_0 > 0$  small. This convenient notation will be useful in invoking Neumann series inversion for small values of  $z$ . Since we only need the expansion of  $M(z)^{-1}$  up to a few powers of  $z$ , the exact expressions of  $\mathcal{E}_j(z)$  are insignificant and we allow it to vary from line to line. By Proposition 4.3, we rewrite  $M(z)$  by setting

$$\tilde{M}(z) := \frac{z}{\kappa} M(z) = P + \frac{z}{\kappa} (T + z M_1 + \mathcal{M}_2(z)), \quad (4.45)$$

where  $\mathcal{M}_2(z)$  is the error term in Proposition 4.3. Using  $I = P + Q$ , we write

$$\tilde{M}(z) + Q = I + \frac{z}{\kappa} (T + z M_1 + \mathcal{M}_2(z)), \quad (4.46)$$

and by choosing  $z$  small enough, a Neumann series expansion yields the inverse operator

$$[\tilde{M}(z) + Q]^{-1} = \sum_{n \geq 0} (-1)^n \left( \frac{z}{\kappa} (T + z M_1 + \mathcal{M}_2(z)) \right)^n \quad \text{on } L^2(\mathbb{R}) \times L^2(\mathbb{R}). \quad (4.47)$$

We collect the terms of power order up to 2 to obtain

$$[\tilde{M}(z) + Q]^{-1} = I - \frac{z}{\kappa} T - z^2 \left( \frac{1}{\kappa} M_1 - \frac{1}{\kappa^2} T^2 \right) + \mathcal{E}_3(z). \quad (4.48)$$

Note that  $z\mathcal{M}_2(z)$  is of the form  $\mathcal{E}_3(z)$ . Recall by Lemma 4.7 that the operator  $\tilde{M}(z)$  is invertible on  $L^2(\mathbb{R}) \times L^2(\mathbb{R})$  if and only if the operator

$$B_1(z) := Q - Q[\tilde{M}(z) + Q]^{-1}Q \quad (4.49)$$

is invertible on the subspace  $QL^2 \equiv Q(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ . Using (4.48), we find that

$$B_1(z) = \frac{z}{\kappa}QTQ + z^2 \left( \frac{1}{\kappa}QM_1Q - \frac{1}{\kappa^2}QT^2Q \right) + Q\mathcal{E}_3(z)Q.$$

We rewrite  $B_1(z)$  by setting

$$\tilde{B}_1(z) := \frac{\kappa}{z}B_1(z) = QTQ + z \left( QM_1Q - \frac{1}{\kappa}QT^2Q \right) + Q\mathcal{E}_2(z)Q. \quad (4.50)$$

Since the threshold  $\mu$  is not regular, the operator  $QTQ$  is not invertible on  $QL^2$  according to Definition 4.4. By considering the operator

$$\tilde{B}_1(z) + S_1 = (QTQ + S_1) + z \left( QM_1Q - \frac{1}{\kappa}QT^2Q \right) + Q\mathcal{E}_2(z)Q,$$

and the fact that we have  $QD_0Q = D_0 = (QTQ + S_1)^{-1}$  on  $QL^2$ , we can pick  $z$  small enough such that

$$\left\| z \left( QM_1Q - \frac{1}{\kappa}QT^2Q \right) + Q\mathcal{E}_2(z)Q \right\|_{L^2 \times L^2 \rightarrow L^2 \times L^2} < \|QD_0Q\|_{L^2 \times L^2 \rightarrow L^2 \times L^2}^{-1}.$$

This allows for the more complicated Neumann series expansion (c.f. Lemma A) on  $QL^2$ :

$$(\tilde{B}_1(z) + S_1)^{-1} = D_0 \sum_{n \geq 0} (-1)^n \left( (z(QM_1Q - \kappa^{-1}QT^2Q) + Q\mathcal{E}_2(z)Q)D_0 \right)^n \text{ on } QL^2. \quad (4.51)$$

We collect the leading order terms in this expansion and write

$$(\tilde{B}_1(z) + S_1)^{-1} = D_0 - zD_0(QM_1Q - \kappa^{-1}QT^2Q)D_0 + Q\mathcal{E}_2(z)Q. \quad (4.52)$$

At this step, it is crucial that the operator  $D_0$  is absolutely bounded to ensure that the remainder term  $Q\mathcal{E}_2(z)Q$  and its derivatives are absolutely bounded. Next, we set

$$B_2(z) := S_1 - S_1(\tilde{B}_1(z) + S_1)^{-1}S_1, \quad \text{on } S_1L^2 \equiv S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R})). \quad (4.53)$$

Using the orthogonality conditions

$$\begin{aligned} S_1D_0 &= D_0S_1 = S_1, \\ S_1Q &= QS_1 = S_1, \\ QT S_1 &= S_1TQ = 0, \end{aligned} \quad (4.54)$$

we obtain

$$B_2(z) = zS_1(M_1 - \kappa^{-1}T^2)S_1 + S_1\mathcal{E}_2(z)S_1.$$

By Lemma 4.5, we note that  $S_1L^2$  is spanned by  $\Phi(x)$  and that  $PT\Phi = T\Phi$  holds (c.f. (4.32)), whence  $S_1T^2S_1 = S_1TPT S_1$ . Using Lemma 4.5 (c.f. (4.29), (4.31)), we obtain that

$$\begin{aligned} d &:= \text{Tr}(S_1(M_1 - \kappa^{-1}T^2)S_1) = \text{Tr}(S_1M_1S_1) - \kappa^{-1} \text{Tr}(S_1TPT S_1) \\ &= -2i(|c_0|^2 + |c_1|^2) \|\Phi\|_{L^2 \times L^2}^{-2} \neq 0. \end{aligned}$$

Hence, we apply another Neumann series expansion to invert the operator  $B_2(z)$  on  $S_1 L^2$  for small  $z$  and write

$$B_2(z)^{-1} = \frac{1}{dz} S_1 + S_1 \mathcal{E}_0(z) S_1 \quad \text{on } S_1 L^2. \quad (4.55)$$

Moreover, by Lemma 4.7, we have

$$\tilde{B}_1(z)^{-1} = (\tilde{B}_1(z) + S_1)^{-1} + (\tilde{B}_1(z) + S_1)^{-1} S_1 B_2(z)^{-1} S_1 (\tilde{B}_1(z) + S_1)^{-1} \quad \text{on } Q L^2.$$

Using (4.52), (4.54), and (4.55), we find that

$$\tilde{B}_1(z)^{-1} = \frac{1}{dz} S_1 + Q \mathcal{E}_0(z) Q \quad \text{on } Q L^2.$$

Hence,

$$B_1(z)^{-1} = \frac{\kappa}{z} \tilde{B}_1(z)^{-1} = \frac{\kappa}{dz^2} S_1 + \frac{\kappa}{z} Q \mathcal{E}_0(z) Q \quad \text{on } Q L^2.$$

We return to the expansion of  $\tilde{M}(z)^{-1}$  by using Lemma 4.7 with (4.48) to obtain that

$$\begin{aligned} \tilde{M}(z)^{-1} &= (\tilde{M}(z) + Q)^{-1} + (\tilde{M}(z) + Q)^{-1} Q B_1(z)^{-1} Q (\tilde{M}(z) + Q)^{-1} \\ &= \left( I - \frac{z}{\kappa} T \right) + \frac{\kappa}{dz^2} S_1 - \frac{1}{dz} T S_1 - \frac{1}{dz} S_1 T + \frac{1}{d\kappa} T S_1 T \\ &\quad + \frac{\kappa}{z} (Q \mathcal{E}_0(z) Q + \mathcal{E}_1(z) Q + Q \mathcal{E}_1(z) + \mathcal{E}_2(z)). \end{aligned}$$

Here, we used the identity  $Q = I Q = Q I$ . By reverting back to  $M(z) = \frac{\kappa}{z} \tilde{M}(z)$ , we have

$$\begin{aligned} M(z)^{-1} &= \frac{z}{\kappa} \tilde{M}(z)^{-1} = \frac{z}{\kappa} I + \frac{1}{dz} S_1 - \frac{1}{d\kappa} T S_1 - \frac{1}{d\kappa} S_1 T + \frac{z}{d\kappa^2} T S_1 T \\ &\quad + Q \mathcal{E}_0(z) Q + \mathcal{E}_1(z) Q + Q \mathcal{E}_1(z) + \mathcal{E}_2(z). \end{aligned}$$

Note that we absorb the  $\frac{z^2}{\kappa^2} T$  term into the error  $\mathcal{E}_2(z)$  above. By using the identities  $I = Q + P$ ,  $Q T S_1 = S_1 T Q = 0$ , and by factoring the powers of  $z$  from the error terms  $\mathcal{E}_j(z)$ , we obtain the expansion of  $M(z)^{-1}$  on  $L^2$ : for  $0 < |z| < z_0$ ,

$$\begin{aligned} M(z)^{-1} &= \frac{z}{\kappa} P + \frac{1}{d} \left( \frac{1}{z} S_1 - \frac{1}{\kappa} P T S_1 - \frac{1}{\kappa} S_1 T P + \frac{1}{\kappa^2} P T S_1 T P \right) \\ &\quad + Q \Lambda_0(z) Q + z Q \Lambda_1(z) + z \Lambda_2(z) Q + z^2 \Lambda_3(z), \end{aligned}$$

where the operators  $\Lambda_j(z)$ ,  $j = 0, \dots, 3$ , satisfy (4.44). Here, we choose  $z_0 > 0$  sufficiently small such that the expansion (4.45) and the Neumann series inversions (4.47), (4.51), (4.55) are valid for all  $0 < |z| < z_0$ . Finally, by Lemma 4.5 (c.f. (4.30)), the term  $P T S_1 T P$  can be simplified to  $|c_0|^2 \|\Phi\|_{L^2 \times L^2}^{-2} \|V_1\|_{L^1(\mathbb{R})} P$ , which finishes the proof.  $\square$

**Remark 4.9** We appeal to the reader that each leading term in the expansion (4.43) plays an important role in revealing the cancellations among the finite rank operators that arise in the local decay estimate (1.14). Such a precise expression was also obtained for the one-dimensional Dirac operators in [20], even though the proof we give here is different. See Remark 3.7 in that paper. For the low-energy unweighted dispersive estimates, it is sufficient to work with the simpler expression

$$M(z)^{-1} = \frac{1}{z} Q \tilde{\Lambda}_0(z) Q + Q \tilde{\Lambda}_1(z) + \tilde{\Lambda}_2(z) Q + z \tilde{\Lambda}_3(z), \quad (4.56)$$

where we absorb the operators  $S_1, S_1 T P, P T S_1, P$  in (4.43) into the operators  $Q \tilde{\Lambda}_0(z) Q, Q \tilde{\Lambda}_1(z), \tilde{\Lambda}_2(z) Q, \tilde{\Lambda}_3(z)$  respectively. The operators  $\tilde{\Lambda}_j(z)$ , for  $j = 0, \dots, 3$ , satisfy the same estimates as (4.44).

## 5 Low energy estimates

In this section, we prove the low energy bounds for the perturbed evolution, following the ideas in Sect. 4 of [20]. We will frequently exploit the crucial orthogonality condition

$$\int_{\mathbb{R}} \underline{e}_{11} v_1(x) Q(x, y) dx = \int_{\mathbb{R}} Q(x, y) v_2(y) \underline{e}_{11} dy = \mathbf{0}_{2 \times 2}. \quad (5.1)$$

The following calculus lemma will be helpful for dealing with the lower entry of the free resolvent kernel.

**Lemma 5.1** *For any  $m > 0$  and  $r \geq 0$ , we define*

$$g_m(x) := \frac{e^{-r\sqrt{x^2+m^2}}}{\sqrt{x^2+m^2}}. \quad (5.2)$$

*Then, there exists  $C_m > 0$  (independent of  $r$ ) such that*

$$\|\partial_x^k g_m\|_{L^\infty(\mathbb{R})} \leq C_m \lesssim 1, \quad \forall k = 0, 1, 2. \quad (5.3)$$

**Proof** First, by rescaling, we set  $g_m(x) = \frac{1}{m} \tilde{g}(x/m)$  where

$$\tilde{g}(x) := \frac{e^{-rm\sqrt{x^2+1}}}{\sqrt{x^2+1}} = \frac{1}{e^{\tilde{r}\langle x \rangle}}, \quad \tilde{r} := rm. \quad (5.4)$$

Hence, it is sufficient to prove the same estimate (5.3) for  $\tilde{g}(x)$ . For  $k = 0$ , it is clear that  $|\tilde{g}(x)| \leq 1$  for all  $x \in \mathbb{R}$ . For  $k = 1, 2$ , direct computation shows that

$$\partial_x \tilde{g}(x) = -\frac{x(1 + \tilde{r}\langle x \rangle)}{e^{\tilde{r}\langle x \rangle} \langle x \rangle^3}, \quad (5.5)$$

and

$$\partial_x^2 \tilde{g}(x) = \frac{3x^2 + 3\tilde{r}x^2\langle x \rangle - \langle x \rangle^2 + \tilde{r}^2x^2\langle x \rangle^2 - \tilde{r}\langle x \rangle^4}{e^{\tilde{r}\langle x \rangle} \langle x \rangle^5}. \quad (5.6)$$

Since  $e^{-\tilde{r}\langle x \rangle} \max\{1, \tilde{r}, \tilde{r}^2\} \leq 1$ , it follows from (5.5), (5.6) that the estimate (5.3) holds for  $\tilde{g}$  and thus for  $g(x)$  too.  $\square$

The next proposition establishes the dispersive estimates for the evolution semigroup  $e^{it\mathcal{H}} P_s^+$  for small energies close to the threshold  $\mu$ .

**Proposition 5.2** *Let the assumptions of Theorem 1.4 hold. Let  $\chi_0(z)$  be a smooth, even, non-negative cut-off function satisfying  $\chi_0(z) = 1$  for  $|z| \leq \frac{z_0}{2}$  and  $\chi_0(z) = 0$  for  $|z| \geq z_0$ , where  $z_0 > 0$  is given by Proposition 4.8. Then, for any  $|t| \geq 1$ , and  $\vec{u} = (u_1, u_2) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ , we have*

$$\|e^{it\mathcal{H}} \chi_0(\mathcal{H} - \mu I) P_s^+ \vec{u}\|_{L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \|\vec{u}\|_{L^1(\mathbb{R}) \times L^1(\mathbb{R})}, \quad (5.7)$$

and

$$\|\langle x \rangle^{-2} (e^{it\mathcal{H}} \chi_0(\mathcal{H} - \mu I) P_s^+ - F_t^+) \vec{u}\|_{L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{3}{2}} \|\langle x \rangle^2 \vec{u}\|_{L^1(\mathbb{R}) \times L^1(\mathbb{R})}, \quad (5.8)$$

where  $F_t^+$  is defined by

$$F_t^+(x, y) = \frac{e^{it\mu}}{\sqrt{-4\pi it}} \tilde{\Psi}(x) [\sigma_3 \tilde{\Psi}(y)]^\top. \quad (5.9)$$

We begin with the proof of the dispersive decay estimate (5.7).

**Proof of (5.7)** We recall the spectral representation from (3.15):

$$e^{it\mathcal{H}} P_s^+ = \frac{e^{it\mu}}{\pi i} \int_{\mathbb{R}} e^{itz^2} z \mathcal{R}_0(z) dz - \frac{e^{it\mu}}{\pi i} \int_{\mathbb{R}} e^{itz^2} z \mathcal{R}_0(z) v_1 (M(z))^{-1} v_2 \mathcal{R}_0(z) dz.$$

Note that the first term on the right is the spectral representation for the free evolution  $e^{it\mathcal{H}_0} P_s^+$  and it satisfies the same estimate as (5.7) thanks to Proposition 2.3. We insert the weaker expansion (4.56) for  $M(z)^{-1}$  following Remark 4.9, and write

$$\begin{aligned} & \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) \mathcal{R}_0(z) v_1 (M(z))^{-1} v_2 \mathcal{R}_0(z) dz \\ &= \int_{\mathbb{R}} e^{itz^2} \chi_0(z^2) \mathcal{R}_0(z) v_1 Q \tilde{\Lambda}_0(z) Q v_2 \mathcal{R}_0(z) dz \\ &+ \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) \mathcal{R}_0(z) v_1 Q \tilde{\Lambda}_1(z) v_2 \mathcal{R}_0(z) dz \\ &+ \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) \mathcal{R}_0(z) v_1 \tilde{\Lambda}_2(z) Q v_2 \mathcal{R}_0(z) dz \\ &+ \int_{\mathbb{R}} e^{itz^2} z^2 \chi_0(z^2) \mathcal{R}_0(z) v_1 \tilde{\Lambda}_3(z) v_2 \mathcal{R}_0(z) dz \\ &=: J_1 + J_2 + J_3 + J_4. \end{aligned}$$

It remains to show for  $k = 1, \dots, 4$  that

$$\|J_k\|_{L^1 \times L^1 \rightarrow L^\infty \times L^\infty} \leq C|t|^{-\frac{1}{2}}. \quad (5.10)$$

In what follows, we will prove the bound for  $J_1$  in detail, and explain the other cases at the end of the proof. First, we recall the kernel of  $\mathcal{R}_0(z)$  from (3.12) and write

$$\mathcal{R}_0(z)(x, y) := \mathcal{R}_1(z)(x, y) + \mathcal{R}_2(z)(x, y) := \frac{ie^{iz|x-y|}}{2z} e_{11} + \frac{-e^{-\sqrt{z^2+2\mu}|x-y|}}{2\sqrt{z^2+2\mu}} e_{22}, \quad (5.11)$$

and we further decompose the integral  $J_1$  as

$$J_1 = J_1^{(1,1)} + J_1^{(1,2)} + J_1^{(2,1)} + J_1^{(2,2)},$$

where

$$J_1^{(i,j)}(x, y) := \int_{\mathbb{R}} e^{itz^2} \chi_0(z^2) [\mathcal{R}_i(z) v_1 Q \tilde{\Lambda}_0(z) Q v_2 \mathcal{R}_j(z)](x, y) dz, \quad i, j \in \{1, 2\}.$$

We begin with the most singular term

$$J_1^{(1,1)}(x, y) = \int_{\mathbb{R}^3} e^{itz^2 + iz(|x-x_1|+|y-y_1|)} \frac{\chi_0(z^2)}{(2iz)^2} [e_{11} v_1 Q \tilde{\Lambda}_0(z) Q v_2 e_{11}](x_1, y_1) dz dx_1 dy_1. \quad (5.12)$$

The orthogonality conditions (5.1) imply that

$$\int_{\mathbb{R}} e^{iz|x|} \underline{e}_{11} v_1(x_1) Q(x_1, x_2) dx_1 = \int_{\mathbb{R}} e^{iz|y|} Q(y_2, y_1) v_2(y_1) \underline{e}_{11} dy_1 = \mathbf{0}. \quad (5.13)$$

Hence, writing

$$e^{iz|x-x_1|} - e^{iz|x|} = iz \int_{|x|}^{|x-x_1|} e^{izs_1} ds_1 \quad \text{and} \quad e^{iz|y-y_1|} - e^{iz|y|} = iz \int_{|y|}^{|y-y_1|} e^{izs_2} ds_2, \quad (5.14)$$

we obtain

$$J_1^{(1,1)}(x, y) = \frac{1}{4} \int_{\mathbb{R}^3} \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} e^{itz^2 + iz(s_1+s_2)} A(z, x_1, y_1) ds_1 ds_2 dx_1 dy_1 dz,$$

where  $A(z, x_1, y_1) = \chi_0(z^2) [\underline{e}_{11} v_1 Q \tilde{\Lambda}_0(z) Q v_2 \underline{e}_{11}] (x_1, y_1)$ , and note that  $A$  is differentiable and compactly supported in  $z$  due to Proposition 4.8 and the compact support of  $\chi_0(z^2)$ . We obtain by Lemma 2.1 and the Fubini theorem that

$$|J_1^{(1,1)}(x, y)| \leq C |t|^{-\frac{1}{2}} \int_{\mathbb{R}^2} \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} \int_{\mathbb{R}} |\partial_z A(z, x_1, x_2)| dz ds_1 ds_2 dx_1 dy_1.$$

Using

$$\int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} 1 ds_1 ds_2 \leq ||x - x_1| - |x|| \cdot ||y - y_1| - |y|| \lesssim \langle x_1 \rangle \langle y_1 \rangle, \quad (5.15)$$

as well as

$$\partial_z A(z, x_1, y_1) = [\underline{e}_{11} v_1 Q \partial_z (\chi_0(z^2) \tilde{\Lambda}_0(z)) Q v_2 \underline{e}_{11}] (x_1, y_1), \quad (5.16)$$

along with the bound (4.44) on  $\tilde{\Lambda}_0$ , we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} \int_{\mathbb{R}} |\partial_z A(z, x_1, x_2)| dz ds_1 ds_2 dx_1 dy_1 \\ & \leq C \|Q\|_{L^2 \rightarrow L^2}^2 \|\langle x_1 \rangle v_1(x_1)\|_{L^2(\mathbb{R})} \|\langle y_1 \rangle v_2(y_1)\|_{L^2(\mathbb{R})} \\ & \quad \cdot \int_{[-z_0, z_0]} (\|\tilde{\Lambda}_0(z)\|_{L^2 \times L^2 \rightarrow L^2 \times L^2} + \|\partial_z \tilde{\Lambda}_0(z)\|_{L^2 \times L^2 \rightarrow L^2 \times L^2}) dz \\ & \lesssim 1. \end{aligned} \quad (5.17)$$

Hence,

$$\|J_1^{(1,1)}\|_{L^1 \times L^1 \rightarrow L^\infty \times L^\infty} \leq C |t|^{-\frac{1}{2}}.$$

Next, we consider the least singular term

$$J_1^{(2,2)}(x, y) = \int_{\mathbb{R}^3} e^{itz^2} B(z, x, y, x_1, y_1) dx_1 dy_1 dz, \quad (5.18)$$

where

$$B(z, x, y, x_1, y_1) := e^{-\sqrt{z^2 + 2\mu}(|x-x_1| + |y-y_1|)} \frac{\chi_0(z^2)}{4(z^2 + 2\mu)} [\underline{e}_{22} v_1 Q \tilde{\Lambda}_0(z) Q v_2 \underline{e}_{22}] (x_1, y_1). \quad (5.19)$$

By Lemma 2.1, we have

$$|J_1^{(2,2)}(x, y)| \leq C|t|^{-\frac{1}{2}}, \quad (5.20)$$

if we can show the uniform estimate

$$\sup_{x, y \in \mathbb{R}} \int_{\mathbb{R}^3} |\partial_z B(z, x, y, x_1, y_1)| \, dz \, dx_1 \, dy_1 \lesssim 1.$$

By Lemma 5.1, we have

$$\sup_{z \in \mathbb{R}} \left| \partial_z^k \left( \frac{e^{-\sqrt{z^2 + 2\mu}(|x-x_1|+|y-y_1|)}}{4(z^2 + 2\mu)} \right) \right| \leq C_\mu \lesssim 1, \quad k = 0, 1,$$

uniformly in the  $x, y, x_1, y_1$  variables. Hence, using the Cauchy-Schwarz inequality in the  $x_1, y_1$  variables and the bound (4.44) on  $\tilde{\Lambda}_0$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^3} |\partial_z B(z, x, y, x_1, y_1)| \, dz \, dx_1 \, dy_1 \\ & \leq C_\mu \int_{\mathbb{R}^3} |(1 + \partial_z) \chi_0(z^2) [e_{22} v_1 Q \tilde{\Lambda}_0(z) Q v_2 e_{22}](x_1, y_1)| \, dz \, dx_1 \, dy_1 \\ & \lesssim \|Q\|_{L^2 \times L^2 \rightarrow L^2 \times L^2}^2 \|v_1\|_{L^2(\mathbb{R})} \|v_2\|_{L^2(\mathbb{R})} \\ & \quad \cdot \int_{[-z_0, z_0]} (\|\tilde{\Lambda}_0(z)\|_{L^2 \times L^2 \rightarrow L^2 \times L^2} + \|\partial_z \tilde{\Lambda}_0(z)\|_{L^2 \times L^2 \rightarrow L^2 \times L^2}) \, dz \\ & \lesssim 1. \end{aligned}$$

Hence, the bound (5.20) is proven. Next, for the term  $J_1^{(1,2)}$ , using (5.14) we write

$$J_1^{(1,2)}(x, y) = \int_{\mathbb{R}^3} \int_{|x|}^{|x-x_1|} e^{itz^2 + izs} C(z, x, y, x_1, y_1) \, ds \, dx_1 \, dy_1 \, dz,$$

where

$$C(z, x, y, x_1, y_1) := e^{-\sqrt{z^2 + 2\mu}|y_1 - y|} \frac{\chi_0(z^2)}{4\sqrt{z^2 + 2\mu}} [e_{11} v_1 Q \tilde{\Lambda}_0(z) Q v_2 e_{22}](x_1, y_1).$$

Note that  $C$  is differentiable and compactly supported in the  $z$  variable due to Proposition 4.8 and Lemma 5.1. Using Fubini theorem, (5.15), and Lemma 5.1, we obtain the bound

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{|x|}^{|x-x_1|} \int_{\mathbb{R}} |\partial_z C(z, x, y, x_1, y_1)| \, dz \, ds \, dx_1 \, dy_1 \\ & \leq C_\mu \|Q\|_{L^2 \times L^2 \rightarrow L^2 \times L^2}^2 \|\langle x_1 \rangle v_1(x_1)\|_{L^2(\mathbb{R})} \|v_2\|_{L^2(\mathbb{R})} \\ & \quad \cdot \int_{[-z_0, z_0]} (\|\tilde{\Lambda}_0(z)\|_{L^2 \times L^2 \rightarrow L^2 \times L^2} + \|\partial_z \tilde{\Lambda}_0(z)\|_{L^2 \times L^2 \rightarrow L^2 \times L^2}) \, dz \\ & \lesssim 1. \end{aligned}$$

Hence, by Lemma 2.1 we obtain the bound

$$\|J_1^{(1,2)}\|_{L^1 \times L^1 \rightarrow L^\infty \times L^\infty} \lesssim |t|^{-\frac{1}{2}},$$

and the analogous bound for the term  $J_1^{(2,1)}$  can be proven similarly. For the remaining cases  $J_2, J_3$ , and  $J_4$ , we may use the additional powers of  $z$  in place of the missing  $Q$  orthogonality to obtain the same bounds (5.10) as the term  $J_1$ . This finishes the proof of (5.7).  $\square$

Next, we turn to the proof of the low-energy weighted estimate (5.8).

**Proof of (5.8)** Recall that the threshold resonance function  $\Psi = (\Psi_1, \Psi_2)^\top$  has been normalized in Theorem 1.4, which means that we need to carefully treat the constants relating to the function  $\Phi$  where  $\Phi := v_2 \Psi$ . By Lemma 4.5, note that  $\Phi$  spans the subspace  $S_1(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ . We define

$$\eta := \|\Phi\|_{L^2(\mathbb{R}) \times L^2(\mathbb{R})}^{-2} \neq 0, \quad (5.21)$$

so that  $S_1(x, y) = \eta \Phi(x) \Phi^*(y)$ , and we fix the constants  $c_0$  and  $c_1$  defined by (4.21) and (4.26) respectively for this  $\Phi$ . By Lemma 4.5, one finds the relation

$$2 = \lim_{x \rightarrow \infty} (|\Psi_1(x)|^2 + |\Psi_1(-x)|^2) = 2(|c_0|^2 + |c_1|^2), \quad (5.22)$$

by the polarization identity (c.f. (4.25)). Thus, the precise expansion (4.43) of  $M(z)^{-1}$  from Proposition 4.8 simplifies to

$$\begin{aligned} M(z)^{-1} &= \frac{i}{2\eta z} S_1 + \frac{1}{\eta \|V_1\|_{L^1(\mathbb{R})}} P T S_1 \\ &+ \frac{1}{\eta \|V_1\|_{L^1(\mathbb{R})}} S_1 T P + \left( \frac{2i}{\|V_1\|_{L^1(\mathbb{R})}} + \frac{2|c_0|^2}{i \|V_1\|_{L^1(\mathbb{R})}} \right) z P \\ &+ Q \Lambda_0(z) Q + z Q \Lambda_1(z) + z \Lambda_2(z) Q + z^2 \Lambda_3(z), \quad 0 < |z| < z_0. \end{aligned} \quad (5.23)$$

We insert the above expression into the spectral representation of  $e^{it\mathcal{H}} \chi_0(\mathcal{H} - \mu I) P_s^+$ , and obtain that

$$\begin{aligned} &e^{it\mathcal{H}} \chi_0(\mathcal{H} - \mu I) P_s^+ \\ &= \frac{e^{it\mu}}{\pi i} \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) \mathcal{R}_0(z) dz - \frac{e^{it\mu}}{\pi i} \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) \mathcal{R}_0(z) v_1(M(z))^{-1} v_2 \mathcal{R}_0(z) dz \\ &= \frac{e^{it\mu}}{\pi i} I_1 - \frac{e^{it\mu}}{\pi i} \left( \frac{i}{2\eta} I_{2,1} + \frac{1}{\eta \|V_1\|_{L^1(\mathbb{R})}} I_{2,2} + \frac{1}{\eta \|V_1\|_{L^1(\mathbb{R})}} I_{2,3} \right. \\ &\quad \left. + \left( \frac{2i}{\|V_1\|_{L^1(\mathbb{R})}} + \frac{2|c_0|^2}{i \|V_1\|_{L^1(\mathbb{R})}} \right) I_{2,4} \right) - \frac{e^{it\mu}}{\pi i} (I_{3,1} + I_{3,2} + I_{3,3} + I_{3,4}), \end{aligned} \quad (5.24)$$

where

$$I_1 := \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) \mathcal{R}_0(z) dz, \quad (5.25)$$

$$I_{2,1} := \int_{\mathbb{R}} e^{itz^2} \chi_0(z^2) [\mathcal{R}_0(z) v_1 S_1 v_2 \mathcal{R}_0(z)] dz, \quad (5.26)$$

$$I_{2,2} := \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) [\mathcal{R}_0(z) v_1 S_1 T P v_2 \mathcal{R}_0(z)] dz, \quad (5.27)$$

$$I_{2,3} := \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) [\mathcal{R}_0(z) v_1 P T S_1 v_2 \mathcal{R}_0(z)] dz, \quad (5.28)$$

$$I_{2,4} := \int_{\mathbb{R}} e^{itz^2} z^2 \chi_0(z^2) [\mathcal{R}_0(z) v_1 P v_2 \mathcal{R}_0(z)] dz, \quad (5.29)$$

and

$$I_{3,1} := \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) [\mathcal{R}_0(z) v_1 Q \Lambda_0(z) Q v_2 \mathcal{R}_0(z)] dz, \quad (5.30)$$

$$I_{3,2} := \int_{\mathbb{R}} e^{itz^2} z^2 \chi_0(z^2) [\mathcal{R}_0(z) v_1 Q \Lambda_1(z) v_2 \mathcal{R}_0(z)] dz, \quad (5.31)$$

$$I_{3,3} := \int_{\mathbb{R}} e^{itz^2} z^2 \chi_0(z^2) [\mathcal{R}_0(z) v_1 \Lambda_2(z) Q v_2 \mathcal{R}_0(z)] dz, \quad (5.32)$$

$$I_{3,4} := \int_{\mathbb{R}} e^{itz^2} z^3 \chi_0(z^2) [\mathcal{R}_0(z) v_1 \Lambda_3(z) v_2 \mathcal{R}_0(z)] dz. \quad (5.33)$$

Now we study the local decay of the terms  $I_1, I_{2,j}, I_{3,\ell}$ , for  $j, \ell \in \{1, \dots, 4\}$  and we will observe in the following propositions that the terms  $I_1, I_{2,1}, \dots, I_{2,4}$  contribute to the leading order for the local decay estimate while the remainder terms  $I_{3,1}, \dots, I_{3,4}$  satisfy the stronger local decay estimate  $\mathcal{O}(|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle)$ . We first handle these remainder terms by Lemma 2.1 in a similar spirit to the proof for the (unweighted) dispersive bound (5.7), exploiting the additional power of  $z$ .  $\square$

**Proposition 5.3** For  $i \in \{1, 2, \dots, 4\}$  and  $|t| \geq 1$ , we have

$$|I_{3,i}(x, y)| \leq C |t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle. \quad (5.34)$$

**Proof** We treat the case for  $I_{3,1}$  as the other cases follow similarly by using the additional powers of  $z$  in place of the missing operators  $Q$ . As before, we consider the decomposition

$$I_{3,1} = I_{3,1}^{(1,1)} + I_{3,1}^{(1,2)} + I_{3,1}^{(2,1)} + I_{3,1}^{(2,2)},$$

where

$$I_{3,1}^{(i,j)} := \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) [\mathcal{R}_i(z) v_1 Q \Lambda_0(z) Q v_2 \mathcal{R}_j(z)] dz, \quad i, j \in \{1, 2\},$$

with  $\mathcal{R}_1$  and  $\mathcal{R}_2$  defined in (5.11). We begin with the term

$$I_{3,1}^{(1,1)}(x, y) = \int_{\mathbb{R}^3} e^{itz^2 + iz(|x-x_1|+|y-y_1|)} \frac{z \chi_0(z^2)}{(2iz)^2} [\underline{e}_{11} v_1 Q \Lambda_0(z) Q v_2 \underline{e}_{11}](x_1, y_1) dz dx_1 dy_1.$$

Using the orthogonality condition (5.1) like in (5.14), we obtain

$$I_{3,1}^{(1,1)}(x, y) = \frac{1}{4} \int_{\mathbb{R}^2} \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} \int_{\mathbb{R}} e^{itz^2 + iz(s_1+s_2)} z A(z, x_1, y_1) dz ds_1 ds_2 dx_1 dy_1,$$

where  $A(z, x_1, y_1) := \chi_0(z^2) [\underline{e}_{11} v_1 Q \Lambda_0(z) v_2 Q \underline{e}_{11}](x_1, y_1)$ . By Lemma 2.1, we obtain that

$$\begin{aligned} & \left| I_{3,1}^{(1,1)}(x, y) \right| \\ & \lesssim |t|^{-\frac{3}{2}} \int_{\mathbb{R}^2} \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} \int_{[-z_0, z_0]} (|\partial_z^2 A| + (s_1 + s_2) |\partial_z A| + |A|) dz ds_1 ds_2 dx_1 dy_1. \end{aligned} \quad (5.35)$$

Using the bounds

$$\begin{aligned} & \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} 1 ds_1 ds_2 \lesssim \langle x_1 \rangle \langle y_1 \rangle, \\ & \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} (s_1 + s_2) ds_1 ds_2 \lesssim \langle x_1 \rangle^2 \langle y_1 \rangle^2 \langle x \rangle \langle y \rangle, \end{aligned} \quad (5.36)$$

we have

$$\left| I_{3,1}^{(1,1)}(x, y) \right| \lesssim |t|^{-\frac{3}{2}} \int_{\mathbb{R}^2} \int_{|z| \leq z_0} \langle x_1 \rangle \langle y_1 \rangle (|\partial_z^2 A| + \langle x_1 \rangle \langle y_1 \rangle \langle x \rangle \langle y \rangle |\partial_z A| + |A|) dz dx_1 dy_1.$$

Noting that  $\langle x \rangle v_1(x_1)$  and  $\langle y_1 \rangle v_2(y_1)$  are in  $L^2$  and that  $\Lambda_0$  satisfies the bound (4.44), we apply Cauchy-Schwarz inequality in  $x_1$  and  $y_1$  variables to obtain the bound

$$\begin{aligned} |I_{3,1}^{(1,1)}(x, y)| &\lesssim |t|^{-\frac{3}{2}} \|Q\|_{L^2 \rightarrow L^2}^2 \|\langle x_1 \rangle v_1\|_{L_{x_1}^2(\mathbb{R})} \|\langle y_1 \rangle v_2\|_{L_{y_1}^2(\mathbb{R})} \\ &\quad \cdot \int_{|z| \leq z_0} (\|\partial_z^2 \Lambda_0(z)\|_{L^2 \times L^2 \rightarrow L^2 \times L^2} + \|\Lambda_0(z)\|_{L^2 \times L^2 \rightarrow L^2 \times L^2}) \, dz \\ &\quad + |t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle \|Q\|_{L^2 \rightarrow L^2}^2 \|\langle x_1 \rangle v_1\|_{L_{x_1}^2(\mathbb{R})} \|\langle y_1 \rangle v_2\|_{L_{y_1}^2(\mathbb{R})} \\ &\quad \cdot \int_{|z| \leq z_0} \|\partial_z \Lambda_0(z)\|_{L^2 \times L^2 \rightarrow L^2 \times L^2} \, dz \\ &\lesssim |t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle. \end{aligned} \quad (5.37)$$

Next, we consider the term

$$\begin{aligned} &I_{3,1}^{(1,2)}(x, y) \\ &= \int_{\mathbb{R}^3} e^{itz^2 + iz|x-x_1| - \sqrt{z^2 + 2\mu}|y-y_1|} \frac{\chi_0(z^2)}{4i\sqrt{z^2 + 2\mu}} [e_{11} v_1 Q \Lambda_0(z) Q v_2 e_{22}](x_1, y_1) \, dz \, dx_1 \, dy_1. \end{aligned}$$

By using the  $Q$  orthogonality (c.f. (5.1)) condition, we write

$$I_{3,1}^{(1,2)}(x, y) = \int_{\mathbb{R}^3} \int_{|x|}^{|x-x_1|} e^{itz^2 + izs_1} z B(z, x_1, y_1, x, y) \, ds_1 \, dz \, dx_1 \, dy_1, \quad (5.38)$$

where

$$B(z, x_1, y_1, x, y) := \frac{e^{-\sqrt{z^2 + 2\mu}|y-y_1|}}{4i\sqrt{z^2 + 2\mu}} \chi_0(z^2) [e_{11} v_1 Q \Lambda_0(z) Q v_2 e_{22}](x_1, y_1). \quad (5.39)$$

Since  $B$  is compactly supported in  $z$ , we can exchange the order of integration and we use Lemma 2.1 to obtain

$$|I_{3,1}^{(1,2)}(x, y)| \leq C |t|^{-\frac{3}{2}} \int_{\mathbb{R}^2} \int_{|x|}^{|x-x_1|} \int_{\mathbb{R}} |[\partial_z^2 + is_1 \partial_z] B(z, x_1, y_1, x, y)| \, dz \, ds_1 \, dx_1 \, dy_1.$$

By Lemma 5.1, we have

$$\sup_{z \in \mathbb{R}} \left| \partial_z^k \left( \frac{e^{-\sqrt{z^2 + 2\mu}|y-y_1|}}{4i\sqrt{z^2 + 2\mu}} \right) \right| \leq C_\mu \lesssim 1, \quad \forall k = 0, 1, 2, \quad (5.40)$$

which implies by Hölder's inequality and Leibniz rule that

$$\begin{aligned} &\int_{\mathbb{R}} |[\partial_z^2 + is_1 \partial_z] B(z, x_1, y_1, x, y)| \, dz \\ &\leq C \langle s_1 \rangle \int_{\mathbb{R}} |e_{11} v_1 Q [1 + \partial_z + \partial_z^2] (\chi_0(z^2) \Lambda_0(z)) Q v_2 e_{22}| \, dz. \end{aligned}$$

Repeating the arguments from (5.35)–(5.37), we obtain

$$|I_{3,1}^{(1,2)}(x, y)| \leq C |t|^{-\frac{3}{2}} \langle x \rangle.$$

Similarly, one has the bounds

$$|I_{3,1}^{(2,1)}(x, y)| \leq C |t|^{-\frac{3}{2}} \langle y \rangle, \quad |I_{3,1}^{(2,2)}(x, y)| \leq C |t|^{-\frac{3}{2}},$$

and we are done.  $\square$

**Proposition 5.4** For all  $|t| \geq 1$ , we have

$$|I_{2,1}(x, y) - F_t^1(x, y)| \leq C|t|^{-\frac{3}{2}} \langle x \rangle^2 \langle y \rangle^2, \quad (5.41)$$

where

$$F_t^1(x, y) := \frac{\eta\sqrt{\pi}}{\sqrt{-it}} [c_0 \underline{e}_1 - \Psi(x)] [\sigma_3 \Psi(y) - c_0 \underline{e}_1]^*. \quad (5.42)$$

**Proof** As in the previous propositions, we decompose  $I_{2,1}$  into the sum

$$I_{2,1} = I_{2,1}^{(1,1)} + I_{2,1}^{(1,2)} + I_{2,1}^{(2,1)} + I_{2,1}^{(2,2)},$$

with

$$I_{2,1}^{(i,j)} := \int_{\mathbb{R}} e^{itz^2} \chi_0(z^2) [\mathcal{R}_i(z) v_1 S_1 v_2 \mathcal{R}_j(z)] dz, \quad i, j \in \{1, 2\}.$$

We start with the most singular term

$$I_{2,1}^{(1,1)}(x, y) = \int_{\mathbb{R}^3} e^{itz^2 + iz(|x-x_1| + |y-y_1|)} \frac{\chi_0(z^2)}{(2iz)^2} [\underline{e}_{11} v_1 S_1 v_2 \underline{e}_{11}](x_1, y_1) dx_1 dy_1 dz.$$

Noting that  $S_1 L^2 \subset QL^2$ , the orthogonality conditions (5.1) imply that

$$\int_{\mathbb{R}} e^{iz|x|} \underline{e}_{11} v_1(x_1) S_1(x_1, x_2) dx_1 = \int_{\mathbb{R}} e^{iz|y|} S_1(y_2, y_1) v_2(y_1) \underline{e}_{11} dy_1 = \mathbf{0}_{2 \times 2}, \quad \forall x, y \in \mathbb{R}. \quad (5.43)$$

Hence, by the Fubini theorem,

$$\begin{aligned} I_{2,1}^{(1,1)}(x, y) &= \frac{1}{4} \int_{\mathbb{R}^2} \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} \int_{\mathbb{R}} \\ &\quad e^{itz^2 + iz(s_1 + s_2)} \chi_0(z^2) [\underline{e}_{11} v_1 S_1 v_2 \underline{e}_{11}](x_1, y_1) dz ds_1 ds_2 dx_1 dy_1 \\ &= \frac{1}{4} \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} G_t(s_1 + s_2) ds_1 ds_2 \int_{\mathbb{R}^2} [\underline{e}_{11} v_1 S_1 v_2 \underline{e}_{11}](x_1, y_1) dx_1 dy_1, \end{aligned}$$

where  $G_t(\cdot)$  is the function defined in Lemma 2.2, which satisfies the estimate

$$\left| G_t(s_1 + s_2) - \frac{\sqrt{\pi}}{\sqrt{-it}} e^{-i\frac{s_1^2}{4t}} e^{-i\frac{s_2^2}{4t}} \right| \leq C|t|^{-\frac{3}{2}} \langle s_1 \rangle \langle s_2 \rangle. \quad (5.44)$$

Using the bound

$$\int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} \langle s_1 \rangle \langle s_2 \rangle ds_1 ds_2 \lesssim \langle x_1 \rangle^2 \langle y_1 \rangle^2 \langle x \rangle \langle y \rangle, \quad (5.45)$$

the decay assumptions on  $v_1, v_2$ , and the estimate (5.44), we have

$$\begin{aligned} &\left| I_{2,1}^{(1,1)}(x, y) - \frac{\sqrt{\pi}}{4\sqrt{-it}} e^{i\frac{\pi}{4}} \int_{\mathbb{R}^2} H_t(x_1, x) [\underline{e}_{11} v_1 S_1 v_2 \underline{e}_{11}](x_1, y_1) H_t(y_1, y) dx_1 dy_1 \right| \\ &\leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle \|S_1\|_{L^2 \times L^2 \rightarrow L^2 \times L^2} \|\langle x_1 \rangle^2 v_1(x_1)\|_{L^2} \|\langle y_1 \rangle^2 v_2(y_2)\|_{L^2} \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle, \end{aligned}$$

where we set

$$H_t(x_1, x) := \int_{|x|}^{|x_1-x|} e^{-i\frac{s^2}{4t}} ds. \quad (5.46)$$

Since  $S_1(x, y) = \eta \Phi(x) \Phi^*(y)$ , the orthogonality conditions (5.1) imply that

$$\begin{aligned} \int_{\mathbb{R}} |x| \underline{e}_{11} v_1(x_1) S_1(x_1, y_1) dx_1 &= \eta \int_{\mathbb{R}} |x| \underline{e}_{11} v_1(x_1) \Phi(x_1) dx_1 \Phi^*(y_1) = \mathbf{0}_{2 \times 2}, \quad \forall y \in \mathbb{R}, \\ \int_{\mathbb{R}} |y| S_1(x_1, y_1) v_2(y_1) \underline{e}_{11} dy_1 &= \eta \Phi(x_1) \int_{\mathbb{R}} |y| \Phi^*(y_1) v_2(y_1) \underline{e}_{11} dy_1 = \mathbf{0}_{2 \times 2}, \quad \forall x \in \mathbb{R}. \end{aligned}$$

Hence, using the bound

$$|H_t(x_1, x) - (|x - x_1| - |x|)| \leq C|t|^{-1} \langle x \rangle^2 \langle x_1 \rangle^3, \quad (5.47)$$

and the exponential decay of  $v_1, v_2$ , we conclude the estimate

$$\left| I_{2,1}^{(1,1)}(x, y) - \frac{\eta \sqrt{\pi}}{\sqrt{-it}} [G_0(\underline{e}_{11} v_1 \Phi)(x)] [G_0(\underline{e}_{11} v_2 \Phi)(y)]^* \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle^2 \langle y \rangle^2, \quad (5.48)$$

where

$$G_0(x, y) := -\frac{1}{2} |x - y|, \quad (5.49)$$

and

$$\begin{aligned} [G_0(\underline{e}_{11} v_1 \Phi)(x)] &:= -\frac{1}{2} \int_{\mathbb{R}} |x - x_1| \underline{e}_{11} v_1(x_1) \Phi(x_1) dx_1, \\ [G_0(\underline{e}_{11} v_2 \Phi)(y)]^* &:= -\frac{1}{2} \int_{\mathbb{R}} |y - y_1| \Phi^*(y_1) v_2(y_1) \underline{e}_{11} dy_1. \end{aligned}$$

In the preceding definition, we used the identity  $v_2^* = v_2$ . Next, we treat the term

$$\begin{aligned} I_{2,1}^{(2,2)}(x, y) &= \int_{\mathbb{R}^3} e^{itz^2} \chi_0(z^2) \frac{e^{-\sqrt{z^2+2\mu}|x-x_1|}}{-2\sqrt{z^2+2\mu}} [\underline{e}_{22} v_1 S_1 v_2 \underline{e}_{22}](x_1, y_1) \frac{e^{-\sqrt{z^2+2\mu}|y-y_1|}}{-2\sqrt{z^2+2\mu}} dx_1 dy_1 dz. \end{aligned}$$

By Taylor expansion, we have

$$\begin{aligned} I_{2,1}^{(2,2)}(x, y) &= \int_{\mathbb{R}^3} e^{itz^2} \chi_0(z^2) \frac{e^{-\sqrt{2\mu}|x-x_1|}}{-2\sqrt{2\mu}} [\underline{e}_{22} v_1 S_1 v_2 \underline{e}_{22}](x_1, y_1) \frac{e^{-\sqrt{2\mu}|y-y_1|}}{-2\sqrt{2\mu}} dx_1 dy_1 dz \\ &\quad + \int_{\mathbb{R}^3} e^{itz^2} z^2 \chi_0(z^2) [\underline{e}_{22} v_1 S_1 v_2 \underline{e}_{22}](x_1, y_1) \kappa(x, x_1) \kappa(y, y_1) dx_1 dy_1 dz \\ &= \eta \int_{\mathbb{R}} e^{itz^2} \chi_0(z^2) dz [G_2(\underline{e}_{22} v_1 \Phi)(x)] [G_2(\underline{e}_{22} v_2 \Phi)(y)]^* \\ &\quad + \int_{\mathbb{R}^3} e^{itz^2} z^2 \chi_0(z^2) [\underline{e}_{22} v_1 S_1 v_2 \underline{e}_{22}](x_1, y_1) \kappa(x, x_1) \kappa(y, y_1) dx_1 dy_1 dz, \end{aligned} \quad (5.50)$$

where we set

$$G_2(x, y) := \frac{e^{-\sqrt{2\mu}|x-y|}}{-2\sqrt{2\mu}}, \quad (5.51)$$

and where  $\kappa(x, x_1) \kappa(y, y_1)$  is an error term bounded by  $C \langle x \rangle \langle x_1 \rangle \langle y \rangle \langle y_1 \rangle e^{-c(|x-x_1|+|y-y_1|)}$ , for some  $C, c > 0$ , (c.f. (4.7)). The definitions for  $G_2(\underline{e}_{22} v_1 \Phi)(x)$  and  $G_2(\underline{e}_{22} v_2 \Phi)(y)$  are defined analogously to the ones for  $G_0(\underline{e}_{11} v_1 \Phi)(x)$  and  $G_0(\underline{e}_{11} v_2 \Phi)(y)$ . By non-stationary phase, one has the uniform estimate

$$\left| \int_{\mathbb{R}} e^{itz^2} z^2 \chi_0(z^2) dz \right| \leq C|t|^{-\frac{3}{2}}. \quad (5.52)$$

Hence, we can control the remainder term in  $I_{2,1}^{(2,2)}$  by

$$\left| \int_{\mathbb{R}^3} e^{itz^2} z^2 \chi_0(z^2) [\underline{e}_{22} v_1 S_1 v_2 \underline{e}_{22}] (x_1, y_1) \kappa(x, x_1) \kappa(y, y_1) dx_1 dy_1 dz \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle. \quad (5.53)$$

On the other hand, by Lemma 2.2, one has

$$\int_{\mathbb{R}} e^{itz^2} \chi_0(z^2) dz = \frac{\sqrt{\pi}}{\sqrt{-it}} + R_t, \quad |R_t| \leq C|t|^{-\frac{3}{2}}.$$

Hence, the leading contribution of  $I_{2,1}^{(2,2)}$  can be written as

$$\left| \int_{\mathbb{R}} e^{itz^2} \chi_0(z^2) dz [G_2(\underline{e}_{22} v_1 \Phi)(x)] [G_2(\underline{e}_{22} v_2 \Phi)(y)]^* - \frac{\eta\sqrt{\pi}}{\sqrt{-it}} [G_2(\underline{e}_{22} v_1 \Phi)(x)] [G_2(\underline{e}_{22} v_2 \Phi)(y)]^* \right| \leq C|t|^{-\frac{3}{2}}.$$

Thus, one concludes the estimate for  $I_{2,1}^{(2,2)}$ :

$$\left| I_{2,1}^{(2,2)} - \frac{\eta\sqrt{\pi}}{\sqrt{-it}} [G_2(\underline{e}_{22} v_1 \Phi)(x)] [G_2(\underline{e}_{22} v_2 \Phi)(y)]^* \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle. \quad (5.54)$$

Finally, we note that a similar analysis holds for the terms  $I_{2,1}^{(1,2)}$  and  $I_{2,1}^{(2,1)}$  yielding the contributions

$$\begin{aligned} \left| I_{2,1}^{(1,2)} - \frac{\eta\sqrt{\pi}}{\sqrt{-it}} [G_0(\underline{e}_{11} v_1 \Phi)(x)] [G_2(\underline{e}_{22} v_2 \Phi)(y)]^* \right| &\leq C|t|^{-\frac{3}{2}} \langle x \rangle^2 \langle y \rangle, \\ \left| I_{2,1}^{(2,1)} - \frac{\eta\sqrt{\pi}}{\sqrt{-it}} [G_2(\underline{e}_{22} v_1 \Phi)(x)] [G_0(\underline{e}_{11} v_2 \Phi)(y)]^* \right| &\leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle^2. \end{aligned} \quad (5.55)$$

By adding all leading order contributions, we obtain

$$F_t^1(x, y) = \frac{\eta\sqrt{\pi}}{\sqrt{-it}} [(G_0 \underline{e}_{11} + G_2 \underline{e}_{22}) v_1 \Phi](x) [(G_0 \underline{e}_{11} + G_2 \underline{e}_{22}) v_2 \Phi]^*(y).$$

Recalling that  $\mathcal{G}_0 = G_0 \underline{e}_{11} + G_2 \underline{e}_{22}$  from Lemma 4.1, that  $\mathcal{G}_0(v_1 \Phi) = c_0 \underline{e}_1 - \Psi$  from Lemma 4.5, and that  $\mathcal{G}_0(v_2 \Phi) = \sigma_3 \Psi - c_0 \underline{e}_1$  from Remark 4.6 (c.f. (4.40)), we arrive at

$$F_t^1(x, y) = \frac{\eta\sqrt{\pi}}{\sqrt{-it}} [c_0 \underline{e}_1 - \Psi(x)] [\sigma_3 \Psi(y) - c_0 \underline{e}_1]^*,$$

as claimed □

We continue the analysis for the terms involving the operators  $S_1 T P$  and  $P T S_1$ .

**Proposition 5.5** *For all  $|t| \geq 1$ , we have*

$$|I_{2,2}(x, y) - F_t^2(x, y)| \leq C|t|^{-\frac{3}{2}} \langle x \rangle^2 \langle y \rangle^2, \quad (5.56)$$

$$|I_{2,3}(x, y) - F_t^3(x, y)| \leq C|t|^{-\frac{3}{2}} \langle x \rangle^2 \langle y \rangle^2, \quad (5.57)$$

where

$$F_t^2(x, y) := \frac{i\eta \|V_1\|_{L^1(\mathbb{R})}}{2} \frac{\sqrt{\pi}}{\sqrt{-it}} [c_0 \underline{e}_1 - \Psi(x)] [e^{i\frac{y^2}{4t}} c_0 \underline{e}_1]^*, \quad (5.58)$$

$$F_t^3(x, y) := -\frac{i\eta \|V_1\|_{L^1(\mathbb{R})}}{2} \frac{\sqrt{\pi}}{\sqrt{-it}} [e^{-i\frac{x^2}{4t}} c_0 \underline{e}_1] [\sigma_3 \Psi(y) - c_0 \underline{e}_1]^*. \quad (5.59)$$

**Proof** As in the proof of Proposition 5.4, we decompose  $I_{2,2}$  into

$$I_{2,2} = I_{2,2}^{(1,1)} + I_{2,2}^{(1,2)} + I_{2,2}^{(2,1)} + I_{2,2}^{(2,2)},$$

with

$$I_{2,2}^{(i,j)} := \int_{\mathbb{R}} e^{itz^2} z \chi_0(z^2) [\mathcal{R}_i(z) v_1 S_1 T P v_2 \mathcal{R}_j(z)] dz, \quad i, j \in \{1, 2\},$$

where  $\mathcal{R}_1$  and  $\mathcal{R}_2$  were defined in (5.11). We start with

$$I_{2,2}^{(1,1)}(x, y) = \int_{\mathbb{R}^3} e^{itz^2} z \chi_0(z^2) \frac{e^{iz|x-x_1|}}{-2iz} [\underline{e}_{11} v_1 S_1 T P v_2 \underline{e}_{11}](x_1, y_1) \frac{e^{iz|y-y_1|}}{-2iz} dx_1 dy_1 dz.$$

Using the orthogonality (5.43), we have

$$\begin{aligned} I_{2,2}^{(1,1)}(x, y) &= \frac{1}{4} \int_{\mathbb{R}^3} \int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} e^{itz^2+iz(s_1+s_2)} z \chi_0(z^2) [\underline{e}_{11} v_1 S_1 T P v_2 \underline{e}_{11}](x_1, y_1) ds_1 ds_2 dx_1 dy_1 dz \\ &\quad + \frac{1}{4i} \int_{\mathbb{R}^3} \int_{|x|}^{|x-x_1|} e^{itz^2+izs_1} \chi_0(z^2) [\underline{e}_{11} v_1 S_1 T P v_2 \underline{e}_{11}](x_1, y_1) e^{iz|y|} ds_1 dx_1 dy_1 dz \\ &=: I_{2,2;1}^{(1,1)} + I_{2,2;2}^{(1,1)}. \end{aligned}$$

By Lemma 2.1, we have

$$\left| \int_{\mathbb{R}} e^{itz^2+iz(s_1+s_2)} z \chi_0(z^2) dz \right| \leq C |t|^{-\frac{3}{2}} \langle s_1 \rangle \langle s_2 \rangle.$$

Using this estimate, the bound

$$\int_{|x|}^{|x-x_1|} \int_{|y|}^{|y-y_1|} \langle s_1 \rangle \langle s_2 \rangle ds_1 ds_2 \lesssim \langle x_1 \rangle^2 \langle y_2 \rangle^2 \langle x \rangle \langle y \rangle,$$

the absolute boundedness of  $S_1 T P$ , and the exponential decay of  $v_1, v_2$ , we deduce that

$$\begin{aligned} \left| I_{2,2;1}^{(1,1)}(x, y) \right| &\lesssim |t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle \int_{\mathbb{R}^2} |\langle x_1 \rangle^2 \langle y_2 \rangle^2 [\underline{e}_{11} v_1 S_1 T P v_2 \underline{e}_{11}](x_1, y_1)| dx_1 dy_1 \\ &\lesssim |t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle. \end{aligned} \quad (5.60)$$

By Lemma 4.5 and direct computation,

$$\int_{\mathbb{R}} S_1 T P(x_1, y_1) v_2(y_1) \underline{e}_{11} dy_1 = \eta \|V_1\|_{L^1(\mathbb{R})} \Phi(x_1) [c_0 \underline{e}_1]^*. \quad (5.61)$$

Hence, integrating in  $y_1$ , we have

$$\begin{aligned} I_{2,2;2}^{(1,1)}(x, y) &= \frac{\eta \|V_1\|_{L^1(\mathbb{R})}}{4i} \left( \int_{\mathbb{R}} \int_{|x|}^{|x-x_1|} \int_{\mathbb{R}} e^{itz^2+iz(s_1+|y|)} \chi_0(z^2) \underline{e}_{11} v_1(x_1) \Phi(x_1) dz ds_1 dx_1 \right) [c_0 \underline{e}_1]^* \\ &= \frac{\eta \|V_1\|_{L^1(\mathbb{R})}}{4i} \left( \int_{\mathbb{R}} \int_{|x|}^{|x-x_1|} G_t(s_1 + |y|) ds_1 \underline{e}_{11} v_1(x_1) \Phi(x_1) dx_1 \right) [c_0 \underline{e}_1]^*, \end{aligned}$$

where  $G_t$  is the function defined in Lemma 2.2. By Lemma 2.2 (c.f. (5.44)–(5.48) for similar computations), we have

$$\left| I_{2,2;2}^{(1,1)}(x, y) - \frac{i\eta \|V_1\|_{L^1(\mathbb{R})}}{2} [G_0(\underline{e}_{11} v_1 \Phi)(x)] [e^{i\frac{y^2}{4t}} c_0 \underline{e}_1]^* \right| \leq C |t|^{-\frac{3}{2}} \langle x \rangle^2 \langle y \rangle^2,$$

where  $G_0$  is the operator defined in (5.49). This completes the analysis of the term  $I_{2,2}^{(1,1)}$ . Next, we treat the term

$$\begin{aligned} I_{2,2}^{(2,1)}(x, y) &= \int_{\mathbb{R}^3} e^{itz^2} z \chi_0(z^2) \frac{e^{-\sqrt{z^2+2\mu}|x-x_1|}}{-2\sqrt{z^2+2\mu}} [\underline{e}_{22} v_1 S_1 T P v_2 \underline{e}_{11}](x_1, y_1) \frac{e^{iz|y-y_1|}}{-2iz} dx_1 dy_1 dz. \end{aligned} \quad (5.62)$$

By inserting  $e^{iz|y|}$ , we write

$$\begin{aligned} I_{2,2}^{(2,1)}(x, y) &= -\frac{1}{2} \int_{\mathbb{R}^3} e^{itz^2} z \chi_0(z^2) \frac{e^{-\sqrt{z^2+2\mu}|x-x_1|}}{-2\sqrt{z^2+2\mu}} [\underline{e}_{22} v_1 S_1 T P v_2 \underline{e}_{11}](x_1, y_1) \int_{|y|}^{|y-y_1|} e^{izs_2} ds_2 dx_1 dy_1 dz \\ &\quad + \int_{\mathbb{R}^3} e^{itz^2} z \chi_0(z^2) \frac{e^{-\sqrt{z^2+2\mu}|x-x_1|}}{-2\sqrt{z^2+2\mu}} [\underline{e}_{22} v_1 S_1 T P v_2 \underline{e}_{11}](x_1, y_1) \frac{e^{iz|y|}}{-2iz} dx_1 dy_1 dz \\ &=: I_{2,2;1}^{(2,1)}(x, y) + I_{2,2;2}^{(2,1)}(x, y), \end{aligned}$$

where  $I_{2,2;2}^{(2,1)}$  is the leading term. By Lemma 2.1 and Lemma 5.1,

$$\left| \int_{\mathbb{R}} e^{itz^2+izs_2} z \chi_0(z^2) \frac{e^{-\sqrt{z^2+2\mu}|x-x_1|}}{-2\sqrt{z^2+2\mu}} dz \right| \leq C|t|^{-\frac{3}{2}} \langle s_2 \rangle. \quad (5.63)$$

Hence, using the absolute boundedness of  $S_1 T P$  and the bound (5.45), we have

$$\left| I_{2,2;1}^{(2,1)}(x, y) \right| \lesssim |t|^{-\frac{3}{2}} \int_{\mathbb{R}^2} \langle y_1 \rangle^2 \langle y \rangle [\underline{e}_{22} v_1 S_1 T P v_2 \underline{e}_{11}](x_1, y_1) dx_1 dy_1 \lesssim |t|^{-\frac{3}{2}} \langle y \rangle.$$

On the other hand, we treat  $I_{2,2;1}^{(2,1)}$  similarly as in (5.50)–(5.53) and find that

$$\begin{aligned} \left| I_{2,2;2}^{(2,1)}(x, y) - \frac{i}{2} \int_{\mathbb{R}^3} e^{itz^2+iz|y|} \chi_0(z^2) G_2(x, x_1) [\underline{e}_{22} v_1 S_1 T P v_2 \underline{e}_{11}](x_1, y_1) dx_1 dy_1 dz \right| \\ \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle, \end{aligned}$$

where  $G_2$  is defined in (5.51). Hence, by Lemma 2.2 and (5.61), we conclude that

$$\left| I_{2,2}^{(2,1)}(x, y) - \frac{i\eta \|V_1\|_{L^1(\mathbb{R})}}{2} \frac{\sqrt{\pi}}{\sqrt{-it}} [G_2(\underline{e}_{22} v_1 \Phi)(x)] [e^{i\frac{y^2}{4t}} c_0 \underline{e}_1]^* \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle. \quad (5.64)$$

Finally, we show that the terms  $I_{2,2}^{(1,2)}$  and  $I_{2,2}^{(2,2)}$  satisfy the better decay rates of  $\mathcal{O}(|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle)$ . By orthogonality (c.f. (5.43)),

$$\begin{aligned} I_{2,2}^{(1,2)}(x, y) &= \frac{1}{-2} \int_{\mathbb{R}^3} e^{itz^2} z \chi_0(z^2) \int_{|x|}^{|x-x_1|} e^{izs_1} ds_1 [\underline{e}_{11} v_1 S_1 T P v_2 \underline{e}_{22}](x_1, y_1) \frac{e^{-\sqrt{z^2+2\mu}|y-y_1|}}{-2\sqrt{z^2+2\mu}} dx_1 dy_1 dz. \end{aligned}$$

By Lemma 2.1 and Lemma 5.1, we note that the  $z$ -integral satisfy the bound

$$\left| \int_{\mathbb{R}} e^{itz^2+izs_1} z \chi_0(z^2) \frac{e^{-\sqrt{z^2+2\mu}|y-y_1|}}{-2\sqrt{z^2+2\mu}} dz \right| \leq C|t|^{-\frac{3}{2}} \langle s_1 \rangle.$$

Hence, by the absolute boundedness of  $S_1 T P$  and decay of  $v_1, v_2$ , we conclude that

$$\left| I_{2,2}^{(1,2)}(x, y) \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle.$$

The analysis of  $I_{2,2}^{(2,2)}$  is analogous to the preceding one, yielding the bound

$$\left| I_{2,2}^{(2,2)}(x, y) \right| \leq C|t|^{-\frac{3}{2}} \langle y \rangle.$$

Thus, using  $\mathcal{G}_0 = G_0 \underline{e}_{11} + G_2 \underline{e}_{22}$ , and  $\mathcal{G}_0(v_1 \Phi) = c_0 \underline{e}_1 - \Psi$  from Lemma 4.5, we conclude (5.56) and (5.58). For the estimate (5.57) involving  $I_{2,3}$ , one should instead use the identity

$$\int_{\mathbb{R}} \underline{e}_{11} v_1(x_1) P T S_1(x_1, y_1) dx_1 = -\eta \|V_1\|_{L^1(\mathbb{R})} c_0 \underline{e}_1 \Phi(y_1)^*, \quad (5.65)$$

and we leave the remaining details to the reader.  $\square$

Next, we remark that the analysis for  $I_{2,4}$  involving the operator  $P$  leads to a similar estimate as the free evolution in Proposition 2.3.

**Proposition 5.6** *For all  $|t| \geq 1$ , we have*

$$\left| I_{2,4}(x, y) - F_t^4(x, y) \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle^2 \langle y \rangle^2, \quad (5.66)$$

where

$$F_t^4(x, y) := \frac{\|V_1\|_{L^1(\mathbb{R})}}{4} \frac{\sqrt{\pi}}{\sqrt{-it}} e^{-i\frac{x^2}{4t}} \underline{e}_1 e^{-i\frac{y^2}{4t}} \underline{e}_1^\top. \quad (5.67)$$

**Proof** As before, we write

$$I_{2,4} = I_{2,4}^{(1,1)} + I_{2,4}^{(1,2)} + I_{2,4}^{(2,1)} + I_{2,4}^{(2,2)},$$

with

$$I_{2,4}^{(i,j)} := \int_{\mathbb{R}} e^{itz^2} z^2 \chi_0(z^2) [\mathcal{R}_i(z) v_1 P v_2 \mathcal{R}_j(z)] dz, \quad i, j \in \{1, 2\},$$

where  $\mathcal{R}_1$  and  $\mathcal{R}_2$  were defined in (5.11). We first treat the leading term

$$I_{2,4}^{(1,1)}(x, y) = \int_{\mathbb{R}} e^{itz^2} z^2 \chi_0(z^2) \frac{e^{iz|x-x_1|}}{2iz} [\underline{e}_{11} v_1 P v_2 \underline{e}_{11}](x_1, y_1) \frac{e^{iz|y-y_1|}}{2iz} dx_1 dy_1 dz. \quad (5.68)$$

By adding and subtracting  $e^{iz|x|}$  and  $e^{iz|y|}$  twice, we further consider

$$\begin{aligned} I_{2,4}^{(1,1)}(x, y) &= \int_{\mathbb{R}^3} e^{itz^2} z^2 \chi_0(z^2) \frac{e^{iz|x|}}{2iz} [\underline{e}_{11} v_1 P v_2 \underline{e}_{11}](x_1, y_1) \frac{e^{iz|y|}}{2iz} dx_1 dy_1 dz \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} e^{itz^2} z^2 \chi_0(z^2) \frac{e^{iz|x|}}{2iz} [\underline{e}_{11} v_1 P v_2 \underline{e}_{11}](x_1, y_1) \int_{|y|}^{|y-y_1|} e^{izs_2} ds_2 dx_1 dy_1 dz \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} e^{itz^2} z^2 \chi_0(z^2) \int_{|x|}^{|x-x_1|} e^{izs_1} ds_1 [\underline{e}_{11} v_1 P v_2 \underline{e}_{11}](x_1, y_1) \frac{e^{iz|y|}}{2iz} dx_1 dy_1 dz \\ &\quad + \frac{1}{4} \int_{\mathbb{R}^3} e^{itz^2} z^2 \chi_0(z^2) \int_{|x|}^{|x-x_1|} e^{izs_1} ds_1 [\underline{e}_{11} v_1 P v_2 \underline{e}_{11}](x_1, y_1) \int_{|y|}^{|y-y_1|} e^{izs_2} ds_2 dx_1 dy_1 dz \\ &=: I_{2,4;1}^{(1,1)}(x, y) + I_{2,4;2}^{(1,1)}(x, y) + I_{2,4;3}^{(1,1)}(x, y) + I_{2,4;4}^{(1,1)}(x, y). \end{aligned}$$

By direct computation,

$$\int_{\mathbb{R}^2} [e_{11} v_1 P v_2 e_{11}](x_1, y_1) dx_1 dy_1 = -\|V_1\|_{L^1(\mathbb{R})} e_1 e_1^\top.$$

Hence, by Lemma 2.2,

$$\left| I_{2,4;1}^{(1,1)}(x, y) - \frac{\|V_1\|_{L^1(\mathbb{R})}}{4} \frac{\sqrt{\pi}}{\sqrt{-it}} e^{-i\frac{x^2}{4t}} e_1 e^{-i\frac{y^2}{4t}} e_1^\top \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle.$$

For the terms  $I_{2,4;2}^{(1,1)}$ ,  $I_{2,4;3}^{(1,1)}$ , the additional factor of  $z$  allows to invoke Lemma 2.1,

$$\begin{aligned} \left| \int_{\mathbb{R}} e^{itz^2 + iz(|x|+s_2)} z \chi_0(z^2) dz \right| &\leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle s_2 \rangle, \\ \left| \int_{\mathbb{R}} e^{itz^2 + iz(s_1+|y|)} z \chi_0(z^2) dz \right| &\leq C|t|^{-\frac{3}{2}} \langle y \rangle \langle s_1 \rangle. \end{aligned}$$

Thus, we infer from the exponential decay of  $v_1$  and  $v_2$  that

$$\left| I_{2,4;2}^{(1,1)}(x, y) \right| + \left| I_{2,4;3}^{(1,1)}(x, y) \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle. \quad (5.69)$$

For the term  $I_{2,4;4}^{(1,1)}$ , we can use non-stationary phase to conclude the same bound. Hence, we have

$$\left| I_{2,4}^{(1,1)}(x, y) - \frac{\|V_1\|_{L^1(\mathbb{R})}}{4} \frac{\sqrt{\pi}}{\sqrt{-it}} e^{-i\frac{x^2}{4t}} e_1 e^{-i\frac{y^2}{4t}} e_1^\top \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle.$$

Thus, it remains to prove that the other terms  $I_{2,4}^{(1,2)}$ ,  $I_{2,4}^{(2,1)}$ ,  $I_{2,4}^{(2,2)}$  have the better  $\mathcal{O}(|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle)$  weighted decay estimate to finish the proposition. We first treat the term

$$\begin{aligned} I_{2,4}^{(1,2)}(x, y) &= \frac{1}{2i} \int_{\mathbb{R}^3} e^{itz^2} z \chi_0(z^2) \\ &\quad e^{iz|x-x_1|} [e_{11} v_1 P v_2 e_{22}](x_1, y_1) \frac{e^{-\sqrt{z^2+2\mu}|y-y_1|}}{-2\sqrt{z^2+2\mu}} dx_1 dy_1 dz. \end{aligned} \quad (5.70)$$

By Lemma 2.1 and Lemma 5.1,

$$\left| \int_{\mathbb{R}} e^{itz^2 + iz(|x-x_1|)} z \chi_0(z^2) \frac{e^{-\sqrt{z^2+2\mu}|y-y_1|}}{-2\sqrt{z^2+2\mu}} dz \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle x_1 \rangle.$$

Hence, using the decay assumptions on  $v_1$  and  $v_2$ , we conclude that

$$\left| I_{2,4}^{(1,2)}(x, y) \right| \leq C|t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle.$$

The same bound holds for the term  $I_{2,4}^{(2,1)}$  and we will skip the details. Finally, we are left with

$$\begin{aligned} I_{2,4}^{(2,2)}(x, y) &= \int_{\mathbb{R}^3} e^{itz^2} z^2 \chi_0(z^2) \frac{e^{-\sqrt{z^2+2\mu}|x-x_1|}}{-2\sqrt{z^2+2\mu}} [e_{22} v_1 P v_2 e_{22}](x_1, y_1) \frac{e^{-\sqrt{z^2+2\mu}|y-y_1|}}{-2\sqrt{z^2+2\mu}} dx_1 dy_1 dz. \end{aligned}$$

By direct computation using (3.2),

$$[e_{22} v_1 P v_2 e_{22}](x_1, y_1) = \frac{1}{\|V_1\|_{L^1(\mathbb{R})}} [V_2 e_2](x_1) [V_2 e_2]^\top(y_1),$$

and by Lemmas 2.1 and 5.1, we have the uniform estimate

$$\left| \int_{\mathbb{R}} e^{itz^2} z^2 \chi_0(z^2) \frac{e^{-\sqrt{z^2+2\mu}|x-x_1|}}{-2\sqrt{z^2+2\mu}} \frac{e^{-\sqrt{z^2+2\mu}|y-y_1|}}{-2\sqrt{z^2+2\mu}} dz \right| \leq C_{\mu} |t|^{-\frac{3}{2}}.$$

Hence, by exchanging the order of integration, we conclude that

$$|I_{2,4}^{(2,2)}(x, y)| \leq C |t|^{-\frac{3}{2}}. \quad (5.71)$$

Thus, we conclude (5.66) by summing over the four terms.  $\square$

Finally, we are ready to complete the proof of the local decay estimate (5.8). We sum the leading contributions of the spectral representation of  $e^{it\mathcal{H}}\chi_0(\mathcal{H} - \mu I)P_s^+$  in (5.24) by invoking Propositions 2.3, Proposition 5.4, Proposition 5.5, and Proposition 5.6 to obtain

$$\begin{aligned} F_t^0 - \frac{e^{it\mu}}{\pi i} & \left( \frac{i}{2\eta} F_t^1 + \frac{1}{\eta \|V_1\|_{L^1(\mathbb{R})}} F_t^2 + \frac{1}{\eta \|V_1\|_{L^1(\mathbb{R})}} F_t^3 + \left( \frac{2i}{\|V_1\|_{L^1(\mathbb{R})}} + \frac{2|c_0|^2}{i \|V_1\|_{L^1(\mathbb{R})}} \right) F_t^4 \right) \\ &= \frac{e^{it\mu}}{\sqrt{-4\pi i t}} \left( -[c_0 \underline{e}_1 - \Psi(x)][\sigma_3 \Psi(y) - c_0 \underline{e}_1]^* - [c_0 \underline{e}_1 - \Psi(x)][e^{i\frac{x^2}{4t}} c_0 \underline{e}_1]^* \right. \\ & \quad \left. + [e^{-i\frac{x^2}{4t}} c_0 \underline{e}_1][\sigma_3 \Psi(y) - c_0 \underline{e}_1]^* + |c_0|^2 e^{-i\frac{x^2}{4t}} e^{-i\frac{y^2}{4t}} \underline{e}_1 \underline{e}_1^\top \right) \\ &= \frac{e^{it\mu}}{\sqrt{-4\pi i t}} \left( \Psi(x)[\sigma_3 \Psi(y)]^* + (e^{-i\frac{x^2}{4t}} - 1) c_0 [\sigma_3 \Psi(y)]^* + (e^{-i\frac{y^2}{4t}} - 1) \Psi(x)[c_0 \underline{e}_1]^* \right. \\ & \quad \left. + (1 - e^{-i\frac{x^2}{4t}} - e^{-i\frac{y^2}{4t}} + e^{-i\frac{x^2}{4t}} e^{-i\frac{y^2}{4t}}) |c_0|^2 \underline{e}_1 \underline{e}_1^\top \right), \end{aligned}$$

where we use the cancellation  $F_t^0 - \frac{e^{it\mu}}{\pi i} \frac{2i}{\|V_1\|_{L^1(\mathbb{R})}} F_t^4 = 0$  in the first equality. We note that the first term gives us the finite rank operator

$$F_t^+(x, y) = \frac{e^{it\mu}}{\sqrt{-4\pi i t}} \Psi(x)[\sigma_3 \Psi(y)]^*, \quad (5.72)$$

and we show that the last three terms satisfy the better decay rate. Using,

$$|1 - e^{-i\frac{x^2}{4t}}| \leq \frac{x^2}{4|t|}, \quad (5.73)$$

and the fact that  $\Psi \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ , we have

$$\left| \frac{e^{it\mu} e^{i\frac{\pi}{4}}}{2\sqrt{\pi}\sqrt{t}} (e^{-i\frac{x^2}{4t}} - 1) c_0 \underline{e}_1 [\sigma_3 \Psi(y)]^* \right| \lesssim |t|^{-\frac{3}{2}} \langle x \rangle^2,$$

and similarly

$$\left| \frac{e^{it\mu} e^{i\frac{\pi}{4}}}{2\sqrt{\pi}\sqrt{t}} (e^{-i\frac{y^2}{4t}} - 1) \overline{c_0} \Psi(x) \underline{e}_1^\top \right| \lesssim |t|^{-\frac{3}{2}} \langle y \rangle^2.$$

For the last term, we have

$$\left| 1 - e^{-i\frac{x^2}{4t}} - e^{-i\frac{y^2}{4t}} + e^{-i\frac{x^2}{4t}} e^{-i\frac{y^2}{4t}} \right| = \left| 1 - e^{-i\frac{x^2}{4t}} \right| \left| 1 - e^{-i\frac{y^2}{4t}} \right| \lesssim |t|^{-2} \langle x \rangle^2 \langle y \rangle^2. \quad (5.74)$$

Thus, the leading contribution to  $e^{it\mathcal{H}}\chi_0(\mathcal{H} - \mu I)P_s^+$  is  $F_t^+$ .  $\square$

## 6 Intermediate and high energy estimates

In order to complete the proof of Theorem 1.4, we also need to prove the dispersive estimates when the spectral variable is bounded away from the thresholds  $\pm\mu$ . As usual, we focus on the positive semi-axis  $[\mu, \infty)$  of the essential spectrum and prove the dispersive estimates for energies  $\lambda > \mu$ . The negative semi-axis  $(-\infty, -\mu]$  can be treated by symmetry of  $\mathcal{H}$ . We recall from Sect. 2 that the kernel of the limiting resolvent operator for  $\mathcal{H}_0$  has the formula

$$\mathcal{R}_0^\pm(z)(x, y) := (\mathcal{H}_0 - (z^2 + \mu \pm i0))^{-1} = \begin{bmatrix} \pm \frac{ie^{\pm iz|x-y|}}{2z} & 0 \\ 0 & -\frac{e^{-\sqrt{z^2+2\mu}|x-y|}}{2\sqrt{z^2+2\mu}} \end{bmatrix}, \quad \forall 0 < z < \infty. \quad (6.1)$$

From this, we have the following bound

$$\|\mathcal{R}_0^\pm(z)\|_{L^1 \times L^1 \rightarrow L^\infty \times L^\infty} \leq C|z|^{-1}.$$

Hence, for sufficiently large  $z$ , the perturbed resolvent  $\mathcal{R}^\pm(z)$  can be expanded into the infinite Born series

$$\mathcal{R}^\pm(z) = \sum_{n=0}^{\infty} \mathcal{R}_0^\pm(z)(-\mathcal{V}\mathcal{R}_0^\pm(z))^n. \quad (6.2)$$

More precisely, since the operator norm  $L^1 \times L^1 \rightarrow L^\infty \times L^\infty$  in the  $n$ -th summand above is bounded by  $C|z|^{-1}(C\|\mathcal{V}\|_1|z|^{-1})^n$ , the Born series converges in the operator norm whenever  $|z| > z_1 := 2C\|\mathcal{V}\|_{L^1 \times L^1}$ . We define the high-energy cut-off by

$$\chi_h(z) := 1 - \chi(z), \quad (6.3)$$

where  $\chi(z)$  is a standard smooth even cut-off supported on  $[-z_1, z_1]$  satisfying  $\chi(z) = 1$  for  $|z| \leq \frac{z_1}{2}$  and  $\chi(z) = 0$  for  $|z| \geq z_1$ . We insert the cut-off and the Born series expansion into the spectral representation  $e^{it\mathcal{H}}\chi_h(\mathcal{H} - \mu I)P_s^+$  and look to bound the following

$$\begin{aligned} |(e^{it\mathcal{H}}\chi_h(\mathcal{H} - \mu I)P_s^+\vec{u}, \vec{v})| &= \left| \int_0^\infty e^{itz^2} z\chi_h(z^2) \langle [\mathcal{R}^+(z) - \mathcal{R}^-(z)]\vec{u}, \vec{v} \rangle dz \right| \\ &\leq C \sum_{\pm} \sum_{n=0}^{\infty} \left| \int_0^\infty e^{itz^2} z\chi_h(z^2) \langle \mathcal{R}_0^\pm(z)(\mathcal{V}\mathcal{R}_0^\pm(z))^n \vec{u}, \vec{v} \rangle dz \right|, \end{aligned} \quad (6.4)$$

where  $\vec{u}, \vec{v} \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ . From [40], we have the following dispersive estimates:

**Proposition 6.1** *Under the same hypothesis as Theorem 1.4, we have*

$$\left\| e^{it\mathcal{H}}\chi_h(\mathcal{H} - \mu I)P_s^+\vec{u} \right\|_{L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \|\vec{u}\|_{L^1(\mathbb{R}) \times L^1(\mathbb{R})}, \quad (6.5)$$

and

$$\left\| \langle x \rangle^{-1} e^{it\mathcal{H}}\chi_h(\mathcal{H} - \mu I)P_s^+\vec{u} \right\|_{L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{3}{2}} \|\langle x \rangle \vec{u}\|_{L^1(\mathbb{R}) \times L^1(\mathbb{R})}, \quad (6.6)$$

for any  $|t| \geq 1$ .

**Proof** For (6.5), see the proof of [40, Proposition 7.1], and for (6.6), see the proof of [40, Proposition 8.1]. Note that the high-energy dispersive estimate holds irrespective of the regularity of the thresholds  $\pm\mu$ .  $\square$

Let  $z_0 > 0$  be the constant from Proposition 4.8. It may happen that  $z_1$  is strictly larger than  $z_0$ . In this case, we need to derive estimates analogous to the above proposition in the remaining intermediate energy regime  $[-z_1, -z_0] \cup [z_0, z_1]$ . To this end, we set  $\chi_m(z)$  to be the intermediate energy cut-off given by

$$\chi_m(z) := 1 - \chi_0(z) - \chi_h(z), \quad (6.7)$$

where  $\chi_0(z)$  was the cut-off defined in the previous section in Proposition 5.2.

**Proposition 6.2** *For any  $|t| \geq 1$ , we have*

$$\left\| e^{it\mathcal{H}} \chi_m(\mathcal{H} - \mu I) P_s^+ \vec{u} \right\|_{L_x^\infty(\mathbb{R}) \times L_x^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \|\vec{u}\|_{L_x^1(\mathbb{R}) \times L_x^1(\mathbb{R})}, \quad (6.8)$$

and

$$\left\| \langle x \rangle^{-1} e^{it\mathcal{H}} \chi_m(\mathcal{H} - \mu I) P_s^+ \vec{u} \right\|_{L_x^\infty(\mathbb{R}) \times L_x^\infty(\mathbb{R})} \lesssim |t|^{-\frac{3}{2}} \|\langle x \rangle \vec{u}\|_{L_x^1(\mathbb{R}) \times L_x^1(\mathbb{R})}. \quad (6.9)$$

Before proving the above proposition, we need the following lemmas for pointwise bounds and operator norm bounds on the resolvent operators and its derivatives. The first lemma follows immediately from the expression (6.1) and the triangle inequality  $||x - x_1| - |x|| \leq |x_1|$ .

**Lemma 6.3** *Let  $\gamma_0 > 0$ . For every  $z > \gamma_0$ , and  $k \in \{0, 1, 2\}$ , we have*

$$\left| \partial_z^k \mathcal{R}_0^\pm(z)(x, y) \right| \leq C \gamma_0^{-1-k} \langle x - y \rangle^k, \quad (6.10)$$

and hence

$$\left\| \partial_z^k \mathcal{R}_0^\pm(z)(x, \cdot) \right\|_{X_{-(\frac{1}{2}+k)-}} \leq C \gamma_0^{-1-k} \langle x \rangle^k. \quad (6.11)$$

Moreover, define

$$\mathcal{G}_\pm(z)(x, x_1) = \begin{bmatrix} e^{\mp i z |x|} & 0 \\ 0 & 1 \end{bmatrix} \mathcal{R}_0^\pm(z)(x, x_1) = \begin{bmatrix} \pm \frac{ie^{\pm i z(|x-x_1|-|x|)}}{2z} & 0 \\ 0 & -\frac{e^{-\sqrt{z^2+2\mu}|x-x_1|}}{2\sqrt{z^2+2\mu}} \end{bmatrix}. \quad (6.12)$$

Then, for any  $k \geq 0$ ,

$$\sup_{x \in \mathbb{R}} \left| \partial_z^k \mathcal{G}^\pm(z)(x, x_1) \right| \leq C \gamma_0^{-1-k} |x_1|. \quad (6.13)$$

With these bounds, we are able to give operator norm bounds on the perturbed resolvent via the resolvent identity.

**Lemma 6.4** *Let  $\gamma_0 > 0$ . We have*

$$\sup_{|z| > \gamma_0} \left\| \partial_z \mathcal{R}^\pm(z) \right\|_{X_{\frac{3}{2}+} \rightarrow X_{-\frac{3}{2}-}} \lesssim 1, \quad (6.14)$$

$$\sup_{|z| > \gamma_0} \left\| \partial_z^2 \mathcal{R}^\pm(z) \right\|_{X_{\frac{5}{2}+} \rightarrow X_{-\frac{5}{2}-}} \lesssim 1. \quad (6.15)$$

**Proof** By Lemma 3.1, for any  $|z| > \gamma_0$ , we have

$$\mathcal{R}^\pm(z) = (I + \mathcal{R}_0^\pm(z)\mathcal{V})^{-1}\mathcal{R}_0^\pm(z) =: S^\pm(z)^{-1}\mathcal{R}_0^\pm(z), \quad (6.16)$$

as a bounded operator from  $X_{\frac{1}{2}+}$  to  $X_{-\frac{1}{2}-}$ . Note that  $S^\pm(z)$  is boundedly invertible on  $X_{-\sigma}$  for any  $\sigma > 0$ . By differentiation, we have

$$\partial_z \mathcal{R}^\pm(z) = -S^\pm(z)^{-1} \partial_z \mathcal{R}_0^\pm(z) \mathcal{V} S^\pm(z)^{-1} \mathcal{R}_0^\pm(z) + S^\pm(z)^{-1} \partial_z \mathcal{R}_0^\pm(z). \quad (6.17)$$

Moreover, as a multiplication operator,  $\mathcal{V} : X_{-\sigma} \rightarrow X_\sigma$  is bounded for any  $\sigma > 0$  due to the exponential decay of  $\mathcal{V}$ . By Lemma 6.3,  $\partial_z \mathcal{R}_0^\pm(z) : X_{\frac{3}{2}+} \rightarrow X_{-\frac{3}{2}-}$  is bounded and since the embedding  $X_{-\frac{1}{2}-} \subset X_{-\frac{3}{2}-}$  is continuous, we infer the bound (6.14) by taking composition. By a similar argument,

$$\|\partial_z^2 \mathcal{R}^\pm(z)\|_{X_{\frac{5}{2}+} \rightarrow X_{-\frac{5}{2}-}} \lesssim 1. \quad (6.18)$$

□

**Proof of Proposition 6.2** By iterating the second resolvent identity, we write the perturbed resolvent as a finite sum

$$\mathcal{R}^\pm(z) = \mathcal{R}_0^\pm(z) - \mathcal{R}_0^\pm(z)\mathcal{V}\mathcal{R}_0^\pm(z) + \mathcal{R}_0^\pm(z)\mathcal{V}\mathcal{R}^\pm(z)\mathcal{V}\mathcal{R}_0^\pm(z), \quad (6.19)$$

and we write

$$e^{it\mathcal{H}}\chi_m(\mathcal{H} - \mu I)P_s^+(x, y) = \sum_{j=1}^3 \int_0^\infty e^{itz^2} z \chi_m(z^2) (-1)^{j+1} (\mathcal{E}_j^+(z) - \mathcal{E}_j^-(z))(x, y) dz, \quad (6.20)$$

with

$$\mathcal{E}_1^\pm(z) = \mathcal{R}_0^\pm(z), \quad \mathcal{E}_2^\pm(z) = \mathcal{R}_0^\pm(z)\mathcal{V}\mathcal{R}_0^\pm(z), \quad \mathcal{E}_3^\pm(z) = \mathcal{R}_0^\pm(z)\mathcal{V}\mathcal{R}^\pm(z)\mathcal{V}\mathcal{R}_0^\pm(z). \quad (6.21)$$

Hence, to prove (6.8) and (6.9), it is sufficient to establish the estimates

$$\sup_{\pm} \sup_{j=1,2,3} \left| \int_0^\infty e^{itz^2} z \chi_m(z^2) \mathcal{E}_j^\pm(z)(x, y) dz \right| \lesssim \min\{|t|^{-\frac{1}{2}}, |t|^{-\frac{3}{2}} \langle x \rangle \langle y \rangle\}. \quad (6.22)$$

The term involving  $\mathcal{E}_1^\pm$  is handled by the earlier Proposition 2.3, while the second term involving  $\mathcal{E}_2^\pm$  can be treated analogously as in Proposition 6.1. We refer the reader to [31, Lemma 3] and [26, Proposition 3] for similar computations. For the term involving  $\mathcal{E}_3^\pm$ , we first write

$$\mathcal{R}_0^\pm(z)(s_1, s_2) = \begin{bmatrix} e^{\pm iz|s_1|} & 0 \\ 0 & 1 \end{bmatrix} \mathcal{G}_\pm(z)(s_1, s_2),$$

where the operator  $\mathcal{G}_\pm(z)$  was defined in (6.12). Then, using that the kernel  $\mathcal{R}_0^\pm(z)(x, y)$  is symmetric in  $x$  and  $y$  variables, and using the matrix identity

$$e_{jj} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} e_{kk} = a_{jk} e_j e_k^\top, \quad j, k \in \{1, 2\}, \quad (6.23)$$

we compute the following kernel identity

$$\begin{aligned}
 \mathcal{E}_3^\pm(z)(x, y) &= \int_{\mathbb{R}^2} \mathcal{R}_0^\pm(x, x_1) [\mathcal{V} \mathcal{R}^\pm(z) \mathcal{V}](x_1, y_1) \mathcal{R}_0^\pm(y, y_1) dx_1 dy_1 \\
 &= \begin{bmatrix} e^{\pm iz|x|} & 0 \\ 0 & 1 \end{bmatrix} \int_{\mathbb{R}^2} \mathcal{G}^\pm(x, x_1) [\mathcal{V} \mathcal{R}^\pm(z) \mathcal{V}](x_1, y_1) \mathcal{G}^\pm(y, y_1) dx_1 dy_1 \begin{bmatrix} e^{\pm iz|y|} & 0 \\ 0 & 1 \end{bmatrix} \\
 &= e^{\pm iz(|x|+|y|)} \langle (\mathcal{G}^\pm)^*(z)(x, \cdot) \underline{e}_1, \mathcal{V} \mathcal{R}^\pm(z) \mathcal{V} \mathcal{G}^\pm(z)(y, \cdot) \underline{e}_1 \rangle \underline{e}_1 \underline{e}_1^\top \\
 &\quad + e^{\pm iz|x|} \langle (\mathcal{G}^\pm)^*(z)(x, \cdot) \underline{e}_2, \mathcal{V} \mathcal{R}^\pm(z) \mathcal{V} \mathcal{G}^\pm(z)(y, \cdot) \underline{e}_1 \rangle \underline{e}_1 \underline{e}_2^\top \\
 &\quad + e^{\pm iz|y|} \langle (\mathcal{G}^\pm)^*(z)(x, \cdot) \underline{e}_1, \mathcal{V} \mathcal{R}^\pm(z) \mathcal{V} \mathcal{G}^\pm(z)(y, \cdot) \underline{e}_2 \rangle \underline{e}_2 \underline{e}_1^\top \\
 &\quad + \langle (\mathcal{G}^\pm)^*(z)(x, \cdot) \underline{e}_2, \mathcal{V} \mathcal{R}^\pm(z) \mathcal{V} \mathcal{G}^\pm(z)(y, \cdot) \underline{e}_2 \rangle \underline{e}_2 \underline{e}_2^\top \\
 &=: e^{\pm iz(|x|+|y|)} A_1^\pm(z, x, y) + e^{\pm iz|x|} A_2^\pm(z, x, y) + e^{\pm iz|y|} A_3^\pm(z, x, y) \\
 &\quad + A_4^\pm(z, x, y).
 \end{aligned}$$

We plug this identity into the left hand side of (6.22), and hence it will be sufficient to provide the bounds

$$\left| \int_0^\infty e^{itz^2 \pm izr} z \chi_m(z^2) A_k^\pm(z, x, y) dz \right| \lesssim \min\{|t|^{-\frac{1}{2}}, |t|^{-\frac{3}{2}} \langle r \rangle\}, \quad k \in \{1, \dots, 4\}, \quad (6.24)$$

where  $r$  can represent 0 or  $|x|$ ,  $|y|$ , or the sum of both variables. For the case  $k = 1$ , by Lemma 2.1, we have that

$$\left| \int_0^\infty e^{itz^2 \pm iz(|x|+|y|)} z \chi_m(z^2) A_1^\pm(z, x, y) dz \right| \leq C |t|^{-\frac{1}{2}} \|\partial_z (z \chi_m(z^2) A_1^\pm(z, x, y))\|_{L_z^1(\mathbb{R})}.$$

Since the term  $z \chi_m(z^2)$  is smooth and has compact support, we only need to track the derivatives when they fall onto either  $\mathcal{G}^\pm(z)$  or  $\mathcal{R}^\pm(z)$ . In any case, thanks to the exponential decay of  $\mathcal{V}$ , and the bounds (6.13), (6.14) from the previous lemmas, we have the following uniform bound

$$\begin{aligned}
 &\sup_{\pm} \sup_{z \in \text{supp}(\chi_m)} \sup_{j, k=1, 2} |\partial_z \langle (\mathcal{G}^\pm)^*(z)(y, \cdot) \underline{e}_j, \mathcal{V} \mathcal{R}^\pm(z) \mathcal{V} \mathcal{G}^\pm(z)(x, \cdot) \underline{e}_k \rangle| \\
 &\lesssim \sup_{\pm} \sup_{z \in \text{supp}(\chi_m)} \sup_{j, k=1, 2} \left\| \sqrt{|\mathcal{V}|}(x_1) (|\mathcal{R}^\pm(z)(x_1, x_2)| + |\partial_z \mathcal{R}^\pm(z)(x_1, x_2)|) \sqrt{|\mathcal{V}|}(x_2) \right\|_{L_{x_2}^2 \rightarrow L_{x_1}^2} \\
 &\quad \cdot \|\sqrt{|\mathcal{V}|}(x_1) (|\mathcal{G}^\pm(z)(x, x_1)| + |\partial_z \mathcal{G}^\pm(z)(x, x_1)|) \underline{e}_j\|_{L_{x_1}^2} \\
 &\quad \cdot \|\sqrt{|\mathcal{V}|}(x_2) (|\mathcal{G}^\pm(z)(x_2, y)| + |\partial_z \mathcal{G}^\pm(z)(x_2, y)|) \underline{e}_k\|_{L_{x_2}^2} \\
 &\lesssim 1,
 \end{aligned} \quad (6.25)$$

for all  $x, y \in \mathbb{R}$ .

To prove the weighted dispersive estimate, we invoke the stronger estimate in Lemma 2.1:

$$\begin{aligned}
 &\left| \int_0^\infty e^{itz^2 \pm iz(|x|+|y|)} z \chi_m(z^2) A_1^\pm(z, x, y) dz \right| \\
 &\leq C |t|^{-\frac{3}{2}} \left\| [\partial_z^2 \pm i(|x| + |y|) \partial_z] (\chi_m(z^2) A_1^\pm(z, x, y)) \right\|_{L_z^1(\mathbb{R})}
 \end{aligned}$$

Here, we can apply the same argument as in (6.25) for the two derivatives bound on  $A_1^\pm$  using the estimates (6.13) and (6.15), whereas the bound on one derivative for  $A_1^\pm$  leads to

the weights  $\langle x \rangle \langle y \rangle$ . Thus, we prove (6.24) for  $k = 1$ . The other cases follow by the same argument and we are done.  $\square$

Finally, we conclude with the proof of Theorem 1.4.

**Proof of Theorem 1.4** By combining the estimates from Propositions 5.2, 6.1, and 6.2, we have established the bounds

$$\left\| e^{it\mathcal{H}} P_s^+ \vec{u} \right\|_{L_x^\infty(\mathbb{R}) \times L_x^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \|\vec{u}\|_{L_x^1(\mathbb{R}) \times L_x^1(\mathbb{R})},$$

as well as

$$\left\| \langle x \rangle^{-2} (e^{it\mathcal{H}} P_s^+ - F_t^+) \vec{u} \right\|_{L_x^\infty(\mathbb{R}) \times L_x^\infty(\mathbb{R})} \lesssim |t|^{-\frac{3}{2}} \|\vec{u}\|_{L_x^1(\mathbb{R}) \times L_x^1(\mathbb{R})},$$

for any  $\vec{u} := (u_1, u_2)^\top \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$  and  $|t| \geq 1$ , with  $F_t^+$  given by (5.9). By Remark 3.3, we can similarly deduce that the unweighted dispersive estimate for the evolution  $e^{it\mathcal{H}} P_s^-$  using the identity (3.17). On the other hand, for the weighted estimate, we find that the leading contribution to  $e^{it\mathcal{H}} P_s^-$  is given by

$$F_t^-(x, y) = \sigma_1 F_{-t}^+(x, y) \sigma_1 = -\frac{e^{-it\mu}}{\sqrt{4\pi it}} [\sigma_1 \Psi(x)] [\sigma_3 \sigma_1 \Psi(y)]^*, \quad (6.26)$$

where we used the anti-commutation identity  $\sigma_3 \sigma_1 = -\sigma_1 \sigma_3$ . Thus, we conclude the local decay estimate (1.14) and the formula (1.15) by setting  $F_t := F_t^+ + F_t^-$ .  $\square$

## Appendix A: Neumann series

**Lemma A** *Let  $A$  be an invertible operator and  $B$  be a bounded operator satisfying  $\|B\| < \|A^{-1}\|^{-1}$ . Then,  $A - B$  is invertible with*

$$(A - B)^{-1} = A^{-1} \sum_{n=0}^{\infty} (BA^{-1})^n = A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} + \cdots, \quad (\text{A.1})$$

and

$$\|(A - B)^{-1}\| \leq (\|A^{-1}\|^{-1} - \|B\|)^{-1}. \quad (\text{A.2})$$

**Proof** By the hypothesis  $\|B\| < \|A^{-1}\|^{-1}$ , we have  $\|A^{-1}B\| < 1$ . Consider the identity

$$(A - B)^{-1} = (I - A^{-1}B)^{-1} A^{-1}.$$

The term on the right hand side can be written in the usual Neumann series

$$(I - A^{-1}B)^{-1} = \sum_{n=0}^{\infty} (A^{-1}B)^n.$$

Thus, by multiplying  $A^{-1}$ , we deduce (A.1). Note that the argument also holds true for  $(A - B)^{-1} = A^{-1}(I - BA^{-1})^{-1}$ . Now, since we have the estimate

$$\|(I - A^{-1}B)^{-1}\| \leq (1 - \|A^{-1}B\|)^{-1},$$

we deduce (A.2) by the sub-multiplicative property for operator norms.  $\square$

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