The Spatially Variant Fractional Laplacian

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Abstract We introduce a definition of the fractional Laplacian $(-\Delta)^{s(\cdot)}$ with spatially variable order $s : \Omega \to [0,1]$ and study the solvability of the associated Poisson problem on a bounded domain Ω. The initial motivation arises from the extension results of Caffarelli and Silvestre, and Stinga and Torrea; however the analytical tools and approaches developed here are new. For instance, in some cases we allow the variable order $s(\cdot)$ to attain the values 0 and 1 leading to a framework on weighted Sobolev spaces with non-Muckenhoupt weights. Initially, and under minimal assumptions, the operator $(-\Delta)^{s(\cdot)}$ is identified as the Lagrange multiplier corresponding to an optimization problem; and its domain is determined as a quotient space of weighted Sobolev spaces. The well-posedness of the associated Poisson problem is then obtained for data in the dual of this quotient space. Subsequently, two trace regularity results are established, allowing to partially characterize functions in the aforementioned quotient space whenever a Poincaré type inequality is available. Precise examples are provided where such inequality holds, and in this case the domain of the operator $(-\Delta)^{s(\cdot)}$ is identified with a subset of a weighted Sobolev space with spatially variant smoothness $s(\cdot)$. The latter further allows to prove the well-posedness of the Poisson problem assuming functional regularity of the data.

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1 Introduction

The goal of this work is twofold: (i) introduce the spectral fractional Laplacian $(-\Delta)^{s(\cdot)}$ associated with a homogeneous Dirichlet condition on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, in the case the fractional order $s(\cdot)$ is spatially variable and possibly attains the values 0 and 1; (ii) study the well-posedness of the equation

$$\begin{aligned} (-\Delta)^{s(\cdot)}v = h & \quad \text{in } \Omega, \\ v = 0 & \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

for some classes of data h, and where $v = 0$ is understood in an appropriate sense.

Motivated by the extension approach in $\mathbb{R}^N$ by Caffarelli and Silvestre [7], or in bounded domains by Stinga and Torrea [23], we define $(-\Delta)^{s(\cdot)}$ to be the Lagrange multiplier associated to a suitable variational problem. This particular problem is defined in an extended domain and concerns measurable functions $s(\cdot)$ with range contained in the interval $[0, 1]$. For a general class of functions $s(\cdot)$, the domain of $(-\Delta)^{s(\cdot)}$ can be identified with a quotient space $\mathscr{X}(\Omega, w)$ involving weighted Sobolev spaces,

$$\mathscr{X}(\Omega, w) := \mathscr{L}_{0,L}^{1,2}(\mathcal{C}, w)/\mathscr{L}_0^{1,2}(\mathcal{C}, w), \tag{1.2}$$

where $\mathcal{C} = \Omega \times (0, +\infty)$ is the open semi-infinite cylinder (the extended domain) with base Ω, and w is a specific weight function. Roughly speaking, the spaces $\mathscr{L}_{0,L}^{1,2}(\mathcal{C}, w)$ and $\mathscr{L}_0^{1,2}(\mathcal{C}, w)$ are composed of functions that vanish on the lateral boundary of $\mathcal{C}$, and on the whole boundary (including the base Ω), respectively. Equation (1.1) is then solvable for every h in the dual space of $\mathscr{X}(\Omega, w)$. For a smaller class of possible $s(\cdot)$, the domain can be identified as a subset of a weighted Lebesgue space $L^2(\Omega, \tilde{w})$ for some function $\tilde{w}$, and the equation (1.1) is solvable when the right hand side is in $L^2(\Omega, \tilde{w})$. For an even smaller class of functions $s(\cdot)$, this result is further improved since the domain of $(-\Delta)^{s(\cdot)}$ is identified with a subset of a new weighted Sobolev space of functions with spatially variable smoothness related to $s(\cdot)$. Other variable exponent (weighted) Sobolev-type spaces were introduced in, for example, [22, 17,5,26].

The main application that has motivated this work, in addition to the natural theoretical interest, is the recent paper [2]. There, initial results on an extension approach in Hilbert spaces on an open cylinder with base Ω are given. However, the authors stopped short of defining $(-\Delta)^{s(\cdot)}$ due to the lack of a proper functional framework. The current paper aims to fill this gap. It

is worth mentioning that none of the existing results in the literature are applicable to our case and new PDE and variational analysis tools are needed to study the current situation. For example, the extension approaches in [7, 23] assume $s \in (0,1)$ to be a constant and avoid the extreme cases of 0 and 1. In this setting, the nonlocal problem $(-\Delta)^s v = h$ in Ω, where $(-\Delta)^s$ is the s-power of the realization of $-\Delta$ in $L^2(\Omega)$ with zero Dirichlet boundary conditions, can be equivalently formulated as a local one on a Sobolev space with a Muckenhoupt weight. On the other hand, our $s(\cdot)$ is a function which is allowed to touch the extreme cases 0 and 1 and therefore, the associated weights do not fulfill the Muckenhoupt property [2, Proposition 1]. In particular, fundamental results of type "$H = W$" or Poincaré inequalities are not known in our case, leading to a more complex functional analytic framework.

Possible definitions of $(-\Delta)^{s(\cdot)}$ with non-constant s have been proposed in the recent years from different approaches. From the stochastic processes literature, $(-\Delta)^{s(\cdot)}$ was defined in the unbounded case $\Omega = \mathbb{R}^N$; see the monograph [3] and the references therein. By means of the Lévy-Khintchine representation formula, and the Fourier transform, the operator is determined to be of Lévy type. However, strong additional assumptions on $s(\cdot)$ are required to show that the operator is associated to a Feller or a Markov process. To name a few, these include assuming that $s(\cdot)$ is Lipschitz continuous and satisfies $\varepsilon \leq s(\cdot) \leq 1 - \varepsilon$ for some $\varepsilon \in (0,1)$; see [3, Example 3.5.9] (see also [4]). An alternative definition of $(-\Delta)^{s(\cdot)}$ in $\mathbb{R}^N$ was recently proposed in [10] for radial functions $s(\cdot) \in C^1(\mathbb{R}^N)$ taking values in the open interval $(0, N/2)$. Regarding the definition of $(-\Delta)^{s(\cdot)}$ in bounded domains Ω, the literature is restrictive to the Riesz fractional Laplacian approach [16]; see e.g. [17,28]. In the present work, we assume no regularity on the order $s(\cdot)$ and allow it to take values in the closed interval $[0,1]$.

The paper is further motivated by several applications. The extension approach with spatially varying $s(\cdot)$ has shown remarkable potential in image denoising: A rough choice of $s(\cdot)$ performs better than an optimally selected regularization parameter in total variation approaches; see [2]. This is indeed a game changer, especially the variable $s(\cdot)$ approach can enable one to replace the nonlinear Euler-Lagrange equations in case of total variation by a linear one in the case of the variable fractional. The variable $s(\cdot)$ approach can be also applied in geophysics: Models governed by a fractional Helmholtz equation (with constant fractional order) have shown good qualitative agreement with available magnetotelluric data, see [27]. Given the spatially long-range correlated heterogeneity of the medium, nonlocal models with spatially varying fractional order $s(\cdot)$ appear as an atractive tool to further obtain quantitative agreement.

Outline. The notation and main assumptions we make, specially those for the variable exponent $s(\cdot)$, are specified in Section 2. In Section 3 we provide a succinct idea of the approach that we follow to study the fractional Laplacian with spatially variable order, $(-\Delta)^{s(\cdot)}$, which is motivated by well-known results for the usual spectral fractional Laplacian.

Our main results begin in Section 4 where we introduce a definition of $(-\Delta)^{s(\cdot)}$ on the quotient space $\mathscr{X}(\Omega, w)$. Also in this section, we prove the existence and uniqueness of a solution $v \in \mathscr{X}(\Omega, w)$ to the associated Poisson problem (1.1) for every h in the dual space of $\mathscr{X}(\Omega, h)$. It is worth mentioning that the results in Section 4 require minimal conditions on the function $s(\cdot)$, the weight w, and the domain Ω. The results given in Section 4, however, do not provide conditions for solvability of the Poisson problem when the right hand side of the elliptic equation is a (regular) real valued function defined only on Ω.

In a second approach, we are able to better identify the domain of $(-\Delta)^{s(\cdot)}$ as a quotient space also, now on a Sobolev space $\mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w)$ that consist of functions in $W^{1,2}(\mathcal{C}, w)$ that formally vanish on the lateral boundary of $\mathcal{C}$. Differently from the construction given in Section 4, this second approach requires some extra conditions on both, $s(\cdot)$ and Ω. These conditions are intimately related with the existence of Ω-trace results for functions in $\mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w)$, as well as with the existence of a Poincaré inequality in $\mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w)$; thus, we postpone the second construction until Section 7.

In Section 5, we first study the Ω-traces of functions in $\mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w)$, for $2 \leq p < \infty$. In particular, we are able to characterize $s(\cdot)$-dependent integrability and differential regularity of restrictions of functions in $\mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w)$ to Ω. Subsequently, we are able to prove the existence of a Poincaré inequality for $\mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w)$ in Section 6, for a special class of non-constant $s(\cdot)$ functions.

Our results finish in Section 7, where the details on the second definition of $(-\Delta)^{s(\cdot)}$ are given. Here, we identify the domain of $(-\Delta)^{s(\cdot)}$ with a subset of a weighted Lebesgue space $L^2(\Omega, \tilde{w})$ for some weight $\tilde{w}$, provided $s(\cdot)$ vanishes only on a set of zero measure and a Poincaré inequality holds for functions in $\mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w)$. Further, we improve this result for the case when Ω is the N-dimensional unit square and $s(\cdot)$ satisfies some extra conditions. In this latter case we identify the domain of $(-\Delta)^{s(\cdot)}$ with a subset of a Sobolev space of functions with variable smoothness on Ω. The paper closes with Section 8 that includes, in addition to conclusions, a number of open questions and future research directions.

More general elliptic operators of the form

$$(-\mathrm{div} A \nabla)^{s(\cdot)} v = h,$$

with spatially variable fractional order $s(\cdot)$, can be defined by extending the ideas in this paper in a natural way.

2 Notation and main assumptions

We assume that $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a non-empty bounded open set with a Lipschitz boundary $\partial \Omega$ (except in Section 4, where no condition is imposed on the Ω boundary). We denote by $\mathcal{C}$ the open semi-infinite cylinder with base

Ω, by $\partial_L \mathcal{C}$ the lateral boundary of $\mathcal{C}$, and by $\mathcal{C}_\Omega$ the cylinder $\mathcal{C}$ with the base Ω, that is,

$$\mathcal{C} = \Omega \times (0, \infty), \qquad \partial_L \mathcal{C} = \partial \Omega \times [0, \infty), \qquad \mathcal{C}_\Omega = \mathcal{C} \cup (\Omega \times \{0\}).$$

A generic point X in $\mathbb{R}^{N+1}$ is denoted by (x, y), where $x \in \mathbb{R}^N$ and $y \in \mathbb{R}$.

A function ρ is said to be a weight if ρ is positive and finite almost everywhere. For an open set U, and a weight ρ, we denote by $L^p(U, \rho)$ the space of measurable functions $u : U \to \mathbb{R}$ such that

$$\|u\|_{L^p(U,\rho)} := \left(\int_U |u(x)|^p \rho(x) \, \mathrm{d}x \right)^{1/p} < +\infty.$$

The space $L^p(U, \rho)$ endowed with the norm $\| \cdot \|_{L^p(U,\rho)}$ is a Banach space. Further, given $p \in [2, +\infty)$ we say that a weight ρ satisfies the B_p condition, and write $\rho \in B_p$, if $\rho^{-1/p-1}$ is locally integrable, that is,

$$\rho \in B_p \qquad \Leftrightarrow \qquad \rho^{-1/(p-1)} \in L^1_{\mathrm{loc}}(U).$$

For a weight $\rho \in B_p$, we define the weighted Sobolev space $W^{1,p}(U, \rho)$ as the subset of $L^p(U, \rho)$ of functions u with weak gradients ∇u such that $|\nabla u| \in L^p(U, \rho)$. Endowed with the norm

$$\|u\|_{W^{1,p}(U,\rho)} := \left(\int_U |u(x)|^p \rho(x) \, \mathrm{d}x + \int_U |\nabla u(x)|^p \rho(x) \, \mathrm{d}x \right)^{1/p} < +\infty,$$

$W^{1,p}(U, \rho)$ is a Banach space; see [15]. Notice that B_p is a larger class of weights than the Muckenhoupt weights A_p. The latter is also used to define weighted Sobolev spaces; see [25]. Throughout the paper we assume $p \in [2, \infty)$ and denote the (Hölder) conjugate exponent of p by p'.

The measurable function $s(\cdot) : \Omega \to \mathbb{R}$, which will characterize the spatially variable order of the fractional Laplacian, is assumed to satisfy:

(H1) $s(x) \in [0, 1]$ for almost all $x \in \Omega$.

We use the notation $s(\cdot)$ to emphasize the dependence of the function $s : \Omega \to \mathbb{R}$ on the spatial variable $x \in \Omega$, and use s to denote a constant in the interval $(0, 1)$.

Throughout the paper we consider the function $w : \mathcal{C} \to \mathbb{R}$ defined by

$$w(x, y) = \mathrm{G}_s(x) y^{1-2s(x)},$$

and such that for a given $s(\cdot)$, and p, the function $\mathrm{G}_s : \Omega \to \mathbb{R}$ satisfies that

(H2) $\mathrm{G}_s \in B_p$, and if $s(\cdot) = s \in (0, 1)$ constant, then

$$\mathrm{G}_s(x) = \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)},$$

for all $x \in \Omega$. Here Γ is the standard Euler-Gamma function.

Assumptions (H1) and (H2) imply that $w \in B_p$. However, it is known that (in general) w is not expected to be of Muckenhoupt type, see [2, Proposition 1].

Given $\tau > 0$, we denote by $\mathcal{C}^\tau$ the truncated cylinder $\mathcal{C}$ of height τ, that is,

$$\mathcal{C}^\tau = \Omega \times (0, \tau),$$

and define the sets $\partial_L \mathcal{C}^\tau$ and $\mathcal{C}_\Omega^\tau$ accordingly. The restriction of the weight w to $\mathcal{C}^\tau$ is also denoted by w.

Example 1 A possible choice for the function $s(\cdot)$ is given by

$$s(x) = \sigma \min(\operatorname{dist}(x, \mathcal{B}), \varepsilon),$$

where $0 < \varepsilon < 1$, $\mathcal{B} \subset \Omega$ is a closed subset with zero-measure of $\mathbb{R}^N$, $\operatorname{dist}(x, \mathcal{B}) := \inf\{|x - y| : y \in \mathcal{B}\}$ and $\sigma \in (0, 1)$. This type of functions are useful in image processing where the set $\mathcal{B}$ is the approximated set of edges/discontinuities of a certain image that one tries to recover; see [2].

The two examples for G_s that are of relevance to us are defined by

$$G_s^{(1)}(x) = 2^{2\bar{s}-1} \frac{\Gamma(\bar{s})}{\Gamma(1-\bar{s})} \qquad \text{and} \qquad G_s^{(2)}(x) = 2^{2s(x)-1} \frac{\Gamma(s(x))}{\Gamma(1-s(x))}, \quad (2.1)$$

where $\bar{s} = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega s(x)\mathrm{d}x$. It follows that (H2) is satisfied given that $\sigma \in (0, 1)$.

3 The extended domain approach

This section is devoted to review briefly the well-known extension domain approach to define the spectral fractional Laplacian, see for instance [7,23,11]. Throughout this section, we assume that $s \in (0, 1)$ is constant.

We denote by $\{\lambda_n\}$ the sequence of eigenvalues of the Laplace operator with homogeneous Dirichlet boundary conditions, and consider an orthonormal basis $\{\varphi_n\}$ of $L^2(\Omega)$ of associated eigenfunctions. The spectral fractional Laplacian is defined by

$$(-\Delta)^s v = \sum_{n=1}^\infty \lambda_n^s b_n \varphi_n \qquad \text{where} \quad b_n = \int_\Omega v \varphi_n \, \mathrm{d}x, \qquad (3.1)$$

on the space

$$H = \left\{ v = \sum_{n=1}^\infty b_n \varphi_n \in L^2(\Omega) : \|v\|_H^2 = \sum_{n=1}^\infty \lambda_n^s b_n^2 < \infty \right\}.$$

For extensions of (3.1) to non-homogeneous boundary conditions, we refer to [1]. It is worth mentioning that $H = \mathbb{H}_0^s(\Omega)$ if $s \in \left(0, \frac{1}{2}\right)$ or $s \in \left(\frac{1}{2}, 1\right)$ and $H = \mathbb{H}_{00}^s(\Omega)$ for $s = \frac{1}{2}$. Here, $\mathbb{H}_0^s(\Omega)$ is the closure in $\mathbb{H}^s(\Omega)$ of the space of infinitely continuous differentiable functions with compact support in Ω, and

$\mathbb{H}_{00}^s(\Omega)$ is the Lions-Magenes space [24]. Moreover, $\mathbb{H}^s(\Omega)$ is the fractional Sobolev space of order s,

$$\mathbb{H}^s(\Omega) = \left\{ v \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y < \infty \right\},$$

and it is endowed with the norm

$$\|v\|_{\mathbb{H}^s(\Omega)} = \left(\int_\Omega |v|^2 \, \mathrm{d}x + \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}.$$

The extension approach introduced by Caffarelli and Silvestre [8], see [23, 9] for the case of bounded domains, establishes that if $h \in H'$ (dual space of H) then the unique solution to the elliptic equation

$$(-\Delta)^s v = h \quad \text{in } \Omega,$$
$$v = 0 \quad \text{on } \partial\Omega,$$

is given by $v = \operatorname{tr}_\Omega u$, where $u \in H^1_{0,L}(\mathcal{C}, y^{1-2s})$ satisfies

$$\langle h, \operatorname{tr}_\Omega \psi \rangle_{H',H} = \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \int_\mathcal{C} y^{1-2s} \nabla u \cdot \nabla \psi \, \mathrm{d}X, \qquad \forall \psi \in H^1_{0,L}(\mathcal{C}, y^{1-2s}), \tag{3.2}$$

see [9, Lemma 2.2]. Here, $\langle \cdot, \cdot \rangle_{H',H}$ denotes the dual pairing between H' and H. Moreover, tr_Ω is the Ω-trace operator for functions in the space

$$H^1_{0,L}(\mathcal{C}, y^{1-2s}) = \left\{ u \in H^1(\mathcal{C}, y^{1-2s}) : u = 0 \text{ on } \partial_L \mathcal{C} \text{ in the trace sense} \right\}.$$

More precisely,
$$\operatorname{tr}_\Omega : H^1_{0,L}(\mathcal{C}, y^{1-2s}) \to \mathbb{H}_0^s(\Omega),$$

is the unique bounded linear operator that satisfies $\operatorname{tr}_\Omega u = u(\,\cdot\,, 0)$ for every $u \in C^\infty(\bar{\mathcal{C}})$ that vanishes on $\partial_L \mathcal{C}$; which is also onto over H, that is

$$\operatorname{tr}_\Omega H^1_{0,L}(\mathcal{C}, w) = H, \tag{3.3}$$

see [9, Proposition 2.1].

Additionally, since the minimization problem

$$\text{minimize} \quad \frac{1}{2} \int_\mathcal{C} y^{1-2s} |\nabla u|^2 \, \mathrm{d}X \quad \text{over} \quad H^1_{0,L}(\mathcal{C}, y^{1-2s}),$$
$$\text{subject to} \quad \operatorname{tr}_\Omega u = v, \tag{3.4}$$

admits a unique solution $u \in H^1_{0,L}(\mathcal{C}, y^{1-2s})$ for any $v \in \operatorname{tr}_\Omega H^1_{0,L}(\mathcal{C}, w)$, the harmonic extension operator

$$\mathcal{S} : \operatorname{tr}_\Omega H^1_{0,L}(\mathcal{C}, y^{1-2s}) \to H^1_{0,L}(\mathcal{C}, y^{1-2s}), \qquad v \mapsto \mathcal{S}(v) = u,$$

where u is the solution to problem (3.4), is well-defined, linear, and bounded. Then one finds that the spectral fractional Laplacian given by (3.1) satisfies

$$\langle (-\Delta)^s v, \mathrm{tr}_\Omega \psi \rangle_{H',H} = \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \int_{\mathcal{C}} y^{1-2s} \nabla \mathcal{S}(v) \cdot \nabla \psi \; \mathrm{d}X, \qquad (3.5)$$

for all $\psi \in H^1_{0,L}(\mathcal{C}, y^{1-2s})$ and all $v \in H$, which provides an equivalent definition for $(-\Delta)^s$. This second approach is our starting point to study the fractional Laplacian with spatially variable order: We identify a space of traces on which we can define the fractional Laplacian $(-\Delta)^{s(\cdot)}$ by a formula analogous to (3.5).

4 Abstract definition and solution to $(-\Delta)^{s(\cdot)}v = h$

We consider in this section an abstract derivation of the spatially variable fractional Laplacian $(-\Delta)^{s(\cdot)}$. The advantage of this initial approach is that it requires minimal assumptions, namely (H1) and (H2), which are primarily sufficient conditions to have $w \in B_p$; this leads to an appropriate definition of the associated weighted Sobolev spaces. Also, it is worth noticing that the arguments in this section do not require any assumption on the regularity of the Ω boundary $\partial\Omega$. This path starts with the proper derivation of the trace space for the weighted Sobolev spaces in study. For this matter, we consider the space

$$L^{1,2}(\mathcal{C}, w) = \{u : \mathcal{C} \to \mathbb{R} \text{ measurable} : \nabla u \in L^2(\mathcal{C}, w)\},$$

and endow it with the semi-norm

$$\|u\|_{L^{1,2}(\mathcal{C},w)} := \|\nabla u\|_{L^2(\mathcal{C},w)}.$$

Note that $u \mapsto \|u\|_{L^{1,2}(\mathcal{C},w)}$ is a norm on the subset of C^1 functions in $L^{1,2}(\mathcal{C}, w)$ that vanish at $\partial\mathcal{C}$ or $\partial_L\mathcal{C}$. Subsequently, we define $\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$ and $\mathscr{L}^{1,2}_0(\mathcal{C}, w)$ as the completion in $L^{1,2}(\mathcal{C}, w)$ of the infinitely differentiable functions in $L^{1,2}(\mathcal{C}, w)$ with compact support in $\mathcal{C}_\Omega$ and $\mathcal{C}$, respectively, that is:

$$\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w) := \text{completion of } C^\infty_c(\mathcal{C}_\Omega) \cap L^{1,2}(\mathcal{C}, w) \text{ for } \| \cdot \|_{L^{1,2}(\mathcal{C},w)},$$

$$\mathscr{L}^{1,2}_0(\mathcal{C}, w) := \text{completion of } C^\infty_c(\mathcal{C}) \cap L^{1,2}(\mathcal{C}, w) \text{ for } \| \cdot \|_{L^{1,2}(\mathcal{C},w)},$$

where

$$C^\infty_c(\mathcal{C}_\Omega) = \{u \in C^\infty(\bar{\mathcal{C}}) : \mathrm{supp}(u) \cap \partial_L\mathcal{C} = \emptyset\}. \qquad (4.1)$$

The only portion of the boundary where functions in $C^\infty_c(\mathcal{C}_\Omega)$ do not necessarily vanish is the Ω cap. A few words are in order concerning $\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$ and $\mathscr{L}^{1,2}_0(\mathcal{C}, w)$. Note that $C^\infty_c(\mathcal{C}_\Omega) \cap L^{1,2}(\mathcal{C}, w)$ and $C^\infty_c(\mathcal{C}) \cap L^{1,2}(\mathcal{C}, w)$ are both pre-Hilbert spaces when endowed with the inner product

$$(u_1, u_2)_{L^{1,2}(\mathcal{C},w)} = \int_{\mathcal{C}} w \, \nabla u_1 \cdot \nabla u_2 \; \mathrm{d}X.$$

Thus, it follows that their completion, $\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$ and $\mathscr{L}^{1,2}_{0}(\mathcal{C}, w)$, are Hilbert spaces; in particular for $z_1, z_2 \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$ there exist Cauchy sequences $\{z_1^n\}$ and $\{z_2^n\}$ in $C_c^\infty(\mathcal{C}_\Omega) \cap L^{1,2}(\mathcal{C}, w)$ such that

$$(z_1, z_2)_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)} := \lim_{n \to \infty} \int_{\mathcal{C}} w\, \nabla z_1^n \cdot \nabla z_2^n \,\mathrm{d}X.$$

If there is no risk of confusion, and in order to simplify notation, occasionally we simply write

$$(z_1, z_2)_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)} = \int_{\mathcal{C}} w\, \nabla z_1 \cdot \nabla z_2 \,\mathrm{d}X,$$

and analogously we treat $\mathscr{L}^{1,2}_{0}(\mathcal{C}, w)$.

Given that $C_c^\infty(\mathcal{C}) \cap L^{1,2}(\mathcal{C}, w) \subset C_c^\infty(\mathcal{C}_\Omega) \cap L^{1,2}(\mathcal{C}, w)$, then we observe that $\mathscr{L}^{1,2}_{0}(\mathcal{C}, w)$ is a closed subspace of $\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$. Thus, we can define an abstract space of traces on Ω of functions in $\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$ as the quotient space

$$\mathscr{X}(\Omega, w) := \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)/\mathscr{L}^{1,2}_{0}(\mathcal{C}, w).$$

We then define

$$\mathrm{Tr}_\Omega\, u := [u],$$

i.e., the abstract trace on Ω of a function $u \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$ is identified with the equivalence class $[u]$ that contains u. The space $\mathscr{X}(\Omega, w)$ is then endowed with the usual norm

$$\|\mathrm{Tr}_\Omega\, u\|_{\mathscr{X}(\Omega, w)} = \|[u]\|_{\mathscr{X}(\Omega, w)} := \inf\{\|u - z\|_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w))} : z \in \mathscr{L}^{1,2}_{0}(\mathcal{C}, w)\}.$$

Note that

$$\mathrm{Tr}_\Omega : \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w) \to \mathscr{X}(\Omega, w), \tag{4.2}$$

is a linear and bounded operator, and that $\mathscr{X}(\Omega, w)$ is a Hilbert space, given that $\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$ and $\mathscr{L}^{1,2}_{0}(\mathcal{C}, w)$ are also Hilbert spaces. We denote its inner product as $(\cdot, \cdot)_{\mathscr{X}}$. Further notice that, by definition, $\mathrm{Tr}_\Omega\, \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w) = \mathscr{X}(\Omega, w)$. Unless it is not clear from the context, we denote the class $[v] \in \mathscr{X}(\Omega, w)$ simply by v. The following result establishes the existence of the harmonic extension operator in this setting.

Theorem 1 *Let $v \in \mathscr{X}(\Omega, w)$ and $\mu > 0$ be given. The minimization problem:*

$$\text{minimize} \quad J_\mu(u, v) \quad \text{over} \quad u \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w), \tag{$\mathbb{P}_{\mu, v}$}$$

where

$$J_\mu(u, v) := \frac{1}{2}\|u\|^2_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)} + \frac{\mu}{2}\|\mathrm{Tr}_\Omega\, u - v\|^2_{\mathscr{X}(\Omega, w)},$$

admits a unique solution $u_\mu \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$ that, as $\mu \to \infty$, converges strongly to the unique solution to

$$\begin{aligned} &\text{minimize} \quad J(u) \quad \text{over} \quad \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w),\\ &\text{subject to} \quad \mathrm{Tr}_\Omega\, u = v, \end{aligned} \tag{$\mathbb{P}_v$}$$

where

$$J(u) := \frac{1}{2}\|u\|^2_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)}.$$

Proof The existence of a solution $\{u_\mu\}$ to ($\mathbb{P}_{\mu,v}$) follows from arguments of the direct method for calculus of variations: The functional $u \mapsto J_\mu(u,v)$ is non-negative, coercive, and weakly lower semicontinuous; for the latter part note that $\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w) \ni w \mapsto \|\mathrm{Tr}_\Omega w\|_{\mathscr{X}(\Omega,w)}$ is also weakly lower semicontinuous. Uniqueness follows from the strict convexity of $u \mapsto J_\mu(u,v)$.

Since $v \in \mathscr{X}(\Omega,w)$, there exists $\tilde{u} \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)$ such that $v = [\tilde{u}] = \mathrm{Tr}_\Omega \tilde{u}$. Thus, given that u_μ is a minimizer of $J_\mu(\cdot,v)$,

$$J_\mu(u_\mu, v) \leq J_\mu(\tilde{u}, v) = J(\tilde{u}), \tag{4.3}$$

for every $\mu > 0$. Then, by basic theory for penalty functions (see [18, Lemma 1 in Chapter 10]) we have that

$$\lim_{\mu\to\infty} \frac{\mu}{2}\|\mathrm{Tr}_\Omega u_\mu - v\|^2_{\mathscr{X}(\Omega,w)} = 0. \tag{4.4}$$

The inequality in (4.3) implies that the sequence $\{u_\mu\}$ is bounded in $\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)$, and thus it admits a weakly convergent subsequence, say

$$u_{\mu'} \rightharpoonup u \qquad \text{in} \quad \mathscr{L}^{1,2}_{0,L}(\mathcal{C},w). \tag{4.5}$$

Further, by (4.4) we observe that $\mathrm{Tr}_\Omega u = v$. Next we show that $J(u_\mu) \to J(u)$ with u being the minimizer to ($\mathbb{P}_v$). By the weak lower semicontinuity of J and (4.4), we observe:

$$J(u) \leq \varliminf_{\mu'\to\infty} J(u_{\mu'}) \leq \varlimsup_{\mu'\to\infty} J(u_{\mu'}) = \varlimsup_{\mu'\to\infty} J_{\mu'}(u_{\mu'},v) \leq \varlimsup_{\mu'\to\infty} J_{\mu'}(u,v) = J(u),$$

that is, we have shown that $J(u_{\mu'}) \to J(u)$. The fact that u is a minimizer to ($\mathbb{P}_v$) follows by selecting an arbitrary $\tilde{u}$ such that $\mathrm{Tr}_\Omega \tilde{u} = v$, then the previous to last inequality above yield

$$J(u) \leq \varlimsup_{\mu'\to\infty} J_{\mu'}(\tilde{u},v) = J(\tilde{u}),$$

i.e., u is a minimizer. Further, by strict convexity, minimizers to ($\mathbb{P}_v$) are unique, so that the entire sequence $\{u_\mu\}$ satisfies

$$u_\mu \rightharpoonup u \qquad \text{in} \quad \mathscr{L}^{1,2}_{0,L}(\mathcal{C},w), \tag{4.6}$$

and also $J(u_\mu) \to J(u)$. Using (4.4), this limit is equivalent to

$$\lim_{\mu\to\infty} \|u_\mu\|_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)} = \|u\|_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)},$$

which together with (4.6) implies that

$$u_\mu \to u \qquad \text{in} \quad \mathscr{L}^{1,2}_{0,L}(\mathcal{C},w), \tag{4.7}$$

see [6, Proposition 3.32].

Theorem 1 ensures the existence of the abstract weighted harmonic extension operator

$$S : \mathrm{Tr}_\Omega \, \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w) \to \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w), \qquad v \mapsto S(v) = u.$$

where u is the solution to ($\mathbb{P}_v$). In addition, the map S is linear and bounded: Linearity follows directly from the examination of the first order optimality conditions. For boundedness, consider (4.3) with $u - z$ instead of $\tilde{u}$, where u solves ($\mathbb{P}_v$) and $z \in \mathscr{L}^{1,2}_0(\mathcal{C}, w)$, to obtain $J_\mu(u_\mu, v) \le J(u - z)$. Then, by taking the limit as $\mu \to \infty$ we observe

$$\|S(v)\|_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)} = \|u\|_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)} \le \|u - z\|_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)}.$$

Then, by considering the infimum over all $z \in \mathscr{L}^{1,2}_0(\mathcal{C}, w)$, we obtain

$$\|S(v)\|_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)} \le \|v\|_{\mathscr{X}(\Omega,w)}.$$

The well-posedness of the map S allows us to establish a definition for the fractional Laplacian with spatially variable order.

Definition 1 Let $\mathscr{X}(\Omega, w)'$ be the dual space of $\mathscr{X}(\Omega, w)$. The operator

$$(-\Delta)^{s(\cdot)} : \mathscr{X}(\Omega, w) \to \mathscr{X}(\Omega, w)'$$

is described as follows: for $v \in \mathscr{X}(\Omega, w)$, then we define $(-\Delta)^{s(\cdot)} v \in \mathscr{X}(\Omega, w)'$ in the following sense

$$\langle (-\Delta)^{s(\cdot)} v, \mathrm{Tr}_\Omega \, \psi \rangle_{\mathscr{X}', \mathscr{X}} = (S(v), \psi)_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C},w)}, \qquad \forall \psi \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w). \tag{4.8}$$

Remark 1 The relation of the above definition with the classical spectral fractional Laplacian (3.5) is straightforward: In light of the abuse of notation disclosed at the beginning of the section, we can write

$$\langle (-\Delta)^{s(\cdot)} v, \mathrm{Tr}_\Omega \, \psi \rangle_{\mathscr{X}', \mathscr{X}} = \int_\mathcal{C} w \, \nabla S(v) \cdot \nabla \psi \, \mathrm{d}X, \qquad \forall \psi \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w).$$

Furthermore, by a formal integration-by-parts formula and using the fact that $S(v)$ is weighted harmonic, we obtain that $(-\Delta)^{s(\cdot)}$ is equal to the generalized Neumann trace of $S(v)$ when restricted to $\Omega \times \{0\}$. Then, as for the cassical case with constant order s, $(-\Delta)^{s(\cdot)}$ can be understood as a Dirichlet-to-Neumann map.

Remark 2 In view of Theorem 1, the expression in (4.8) is equivalent to

$$\langle (-\Delta)^{s(\cdot)} v, \mathrm{Tr}_\Omega \, \psi \rangle_{\mathscr{X}', \mathscr{X}} = \lim_{\mu \to \infty} \mu (\mathrm{Tr}_\Omega \, u_\mu - v, \mathrm{Tr}_\Omega \, \psi)_{\mathscr{X}}, \qquad \forall \psi \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w),$$

where u_μ is the unique solution to ($\mathbb{P}_{\mu,v}$).

The operator $(-\Delta)^{s(\cdot)} : \mathscr{X}(\Omega, w) \to \mathscr{X}(\Omega, w)'$ is well-defined as we see next, and it can be seen as the Lagrange multiplier associated to the harmonic extension problem.

Proposition 1 *For each $v \in \mathscr{X}(\Omega, w)$, there exists a unique $\lambda = \lambda(v) \in \mathscr{X}(\Omega, w)'$ such that*

$$\langle \lambda, \operatorname{Tr}_\Omega \psi \rangle_{\mathscr{X}', \mathscr{X}} = (S(v), \psi)_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)}, \qquad \forall \psi \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w).$$

Proof Initially, note that $S(v)$ is the solution to ($\mathbb{P}_v$). For convenience, we write the constraint in ($\mathbb{P}_v$) as $G(u) = 0$, where $G : \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w) \to \mathscr{X}(\Omega, w)$ is defined by $G(u) = \operatorname{Tr}_\Omega u - v$. Since the operator Tr_Ω is linear and bounded, G is also, and hence $G'(u)h = \operatorname{Tr}_\Omega h$. Thus, $G'(u) : \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w) \to \mathscr{X}(\Omega, w)$ is linear, bounded, and surjective. Therefore, there exists a unique Lagrange multiplier $\lambda \in \mathscr{X}(\Omega, w)'$ such that

$$J'(S(v))\psi = \lambda \circ G'(S(v))\psi, \qquad \forall \psi \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w),$$

which proves the statement.

Remark 3 It follows that $(-\Delta)^{s(\cdot)} : \mathscr{X}(\Omega, w) \to \mathscr{X}(\Omega, w)'$ is a bounded linear operator given that S is linear and bounded.

We are now able to determine existence of solutions to the Poisson problem with spatially variant Laplacian.

Theorem 2 *Let $h \in \mathscr{X}(\Omega, w)'$ be given. The equation*

$$(-\Delta)^{s(\cdot)}v = h, \tag{4.9}$$

admits a unique solution in $\mathscr{X}(\Omega, w)$ that is given by $v = \operatorname{Tr}_\Omega u^*$, where u^* solves

$$\text{minimize} \quad \mathcal{J}(u) \quad \text{over} \quad \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w), \tag{4.10}$$

and $\mathcal{J}$ is defined as

$$\mathcal{J}(u) := \frac{1}{2}\|u\|^2_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)} - \langle h, \operatorname{Tr}_\Omega u \rangle_{\mathscr{X}', \mathscr{X}}.$$

Proof Since Tr_Ω is linear and bounded, we have that

$$u \mapsto \langle h, \operatorname{Tr}_\Omega u \rangle_{\mathscr{X}', \mathscr{X}}$$

is a linear functional over $\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)$. Then, there exists a solution to the problem (4.10) and the solution is unique due to strict convexity of $\mathcal{J}$.

Note that via necessary and sufficient conditions of optimality for (4.10), the unique solution u^* satisfies:

$$(u^*, \psi)_{\mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w)} = \langle h, \operatorname{Tr}_\Omega \psi \rangle_{\mathscr{X}', \mathscr{X}}, \qquad \forall \psi \in \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w), \tag{4.11}$$

and then u^* is identical to its harmonic extension, i.e., $u^* = S(\mathrm{Tr}_\Omega\, u^*)$. To see the latter, we consider $\psi \in C_c^\infty(\mathcal{C}) \cap \mathscr{L}_{0,L}^{1,2}(\mathcal{C}, w)$ in (4.11) and observe that by density

$$(u^*, \psi)_{\mathscr{L}_{0,L}^{1,2}(\mathcal{C}, w)} = 0, \qquad \forall\, \psi \in \mathscr{L}_0^{1,2}(\mathcal{C}, w), \tag{4.12}$$

where we have used the fact that the functions in $C_c^\infty(\mathcal{C})$ vanish on $\Omega \times \{0\}$. Moreover, we also (trivially) have $\mathrm{Tr}_\Omega\, S(\mathrm{Tr}_\Omega\, u^*) = \mathrm{Tr}_\Omega\, u^*$ so that u^* satisfies first order optimality conditions for ($\mathbb{P}_v$) for $v = \mathrm{Tr}_\Omega\, u^*$. Hence, by convexity (uniqueness) $u^* = S(\mathrm{Tr}_\Omega\, u^*)$. Also, by definition of the operator $(-\Delta)^{s(\cdot)}$ and (4.11), we have

$$\langle (-\Delta)^{s(\cdot)} \mathrm{Tr}_\Omega\, u^*, \mathrm{Tr}_\Omega\, \psi \rangle_{\mathscr{X}', \mathscr{X}} = (S(\mathrm{Tr}_\Omega\, u^*), \psi)_{\mathscr{L}_0^{1,2}(\mathcal{C}, w)} = \langle h, \mathrm{Tr}_\Omega\, \psi \rangle_{\mathscr{X}', \mathscr{X}}, \tag{4.13}$$

for all $\psi \in \mathscr{L}_{0,L}^{1,2}(\mathcal{C}, w)$ and hence $\mathrm{Tr}_\Omega\, u^*$ solves (4.9).

To prove uniqueness of the solution, consider a solution v to (4.9) with $h = 0$ and notice that

$$(S(v), \psi)_{\mathscr{L}_0^{1,2}(\mathcal{C}, w)} = 0, \qquad \forall\, \psi \in \mathscr{L}_{0,L}^{1,2}(\mathcal{C}, w).$$

Then, $S(v)$ satisfies first order optimality conditions for

$$\text{minimize} \quad \frac{1}{2} \|u\|_{\mathscr{L}_0^{1,2}(\mathcal{C}, w)}^2 \quad \text{over} \quad \mathscr{L}_{0,L}^{1,2}(\mathcal{C}, w),$$

whose unique minimizer is the zero function. Then, by convexity, $S(v) = 0$, so that $v = \mathrm{Tr}_\Omega\, S(v)$ and hence $v = 0$.

Remark 4 (Truncated cylinder $\mathcal{C}^\tau$) It is worth mentioning that exactly the same construction with $\mathcal{C}$ replaced by the truncated cylinder $\mathcal{C}^\tau$, $\tau > 0$, leads to a definition of $(-\Delta)^{s(\cdot)}$ by means of an extension problem on $\mathcal{C}^\tau$, as well as to the existence and uniqueness of solution to the associated Poisson problem. We care about $\mathcal{C}^\tau$ because it makes the problem tractable from an implementation point of view [2,20].

A few words are in order concerning Theorem 2; although it provides a solvability result for the elliptic problem, it does not establish existence of solutions based on maps defined on Ω. That is, we would like to address the question: Under what conditions on $h : \Omega \to \mathbb{R}$, does the equation $(-\Delta)^{s(\cdot)} v = h$ admit a solution? This question is answered in Section 7 and it is intimately related to the following trace results.

5 Trace theorems

In this section we identify a trace operator that properly relates values of maps on a Sobolev space in $\mathcal{C}$ to their values at Ω. For this matter, in addition to (H1) and (H2), we assume that the measurable function $s(\cdot)$ satisfies:

(H3) The set of points on which $s(\cdot)$ is zero has measure zero, i.e., $|A_0| = 0$ where

$$A_0 := \{x \in \Omega : s(x) = 0\}.$$

We define $\mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w)$ to be the closure in $W^{1,p}(\mathcal{C}, w)$ of the infinitely differentiable functions in $W^{1,p}(\mathcal{C}, w)$ with compact support in $\mathcal{C}_\Omega$, that is,

$$\mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w) = \overline{C_c^\infty(\mathcal{C}_\Omega) \cap W^{1,p}(\mathcal{C}, w)}^{W^{1,p}(\mathcal{C}, w)},$$

where $C_c^\infty(\mathcal{C}_\Omega)$ is given in (4.1). Then, formally speaking, $\mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w)$ is the set of functions in $W^{1,p}(\mathcal{C}, w)$ that vanish on $\partial_L \mathcal{C}$. We now prove the regularity of restrictions of functions in $\mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w)$ on the Ω boundary.

Theorem 3 (Trace theorem) *Provided that* (H1)*,* (H2)*, and* (H3) *hold true, there exists a unique bounded linear operator*

$$\mathrm{tr}_\Omega \,:\, \mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w) \to L^p(\Omega, \tilde{w}),$$

that satisfies $\mathrm{tr}_\Omega u = u(\,\cdot\,, 0)$ *for all* $u \in \mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w) \cap C_c^\infty(\mathcal{C}_\Omega)$*, where the weight* $\tilde{w} : \Omega \to \mathbb{R}$ *is defined as*

$$\tilde{w}(x) = \mathrm{G}_s(x)(p - 2 + 2s(x))^p.$$

The same statement is true if we replace $\mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w)$ *by the space* $\mathscr{H}_{0,L}^{1,p}(\mathcal{C}^\tau, w)$*, for every* $\tau > 0$*.*

Proof For the sake of brevity, we define $\delta(\cdot) := 1 - 2s(\cdot)$ so that

$$\tilde{w}(x) = \mathrm{G}_s(x)(p - 1 - \delta(x))^p.$$

Let $u \in \mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w) \cap C_c^\infty(\mathcal{C}_\Omega)$ and $(x, y) \in \bar{\mathcal{C}}$ be such that $s(x) \neq 0$ and $\mathrm{G}_s(x) \neq 0$. Initially, we write

$$u(x, 0) = u(x, y) - \int_0^1 y D_{N+1} u(x, ty) \, \mathrm{d}t, \qquad (5.1)$$

where $D_{N+1} u$ is the partial derivative of u with respect to the $(N + 1)$ coordinate.

Let $\sigma \in (0, 1)$. Multiplying (5.1) by $w(x, y)^{1/p}$ and then integrating from 0 to σ with respect to y, we find

$$|u(x, 0)| \int_0^\sigma w(x, y)^{1/p} \mathrm{d}y \leq I_1 + I_2,$$

where

$$I_1 := \int_0^\sigma |u(x, y)| w(x, y)^{1/p} \, \mathrm{d}y,$$

$$I_2 := \int_0^1 \int_0^\sigma y |D_{N+1}u(x,ty)| w(x,y)^{1/p} \, \mathrm{d}y \, \mathrm{d}t.$$

Notice that $\int_0^\sigma w(x,y)^{1/p}\mathrm{d}y = \mathrm{G}_s(x)\int_0^\sigma y^{\delta(x)/p}\mathrm{d}y$ and that

$$\int_0^\sigma y^{\delta(x)/p}\mathrm{d}y \geq \int_0^\sigma y^{1/p}\,\mathrm{d}y = \frac{p}{1+p}\sigma^{\frac{1+p}{p}} > 0,$$

since $\sigma \in (0,1)$ and $\delta(x) \leq 1$. Thus,

$$|u(x,0)|\mathrm{G}_s(x)^{1/p} \leq \frac{p+1}{p\,\sigma^{(1+p)/p}}(I_1 + I_2).$$

Multiplying the last expression by $(p - 1 - \delta(x))$, we obtain

$$|u(x,0)|\tilde{w}(x)^{1/p} \leq \frac{p+1}{p\,\sigma^{(1+p)/p}}(p - 1 - \delta(x))(I_1 + I_2). \qquad (5.2)$$

Next, we shall estimate I_1 and I_2. A direct use of the Hölder's inequality yields

$$I_1 \leq \sigma^{1/p'}\left(\int_0^\sigma |u(x,y)|^p w(x,y)\,\mathrm{d}y\right)^{1/p}.$$

We now estimate I_2 in several steps. With the change of variables $y = zt^{-1}$ in the inner integral of I_2, we obtain

$$I_2 \leq \sigma \int_0^1 \int_0^{t\sigma} |D_{N+1}u(x,z)| w(x,z)^{1/p} t^{-1-\delta(x)/p} \, \mathrm{d}z \, \mathrm{d}t. \qquad (5.3)$$

By adding and substracting $(1 + \delta(x))/pp'$ in the exponent of t, we rewrite the r.h.s. of (5.3) as

$$\sigma \int_0^1 t^{-\frac{1+\delta(x)}{pp'}} \int_0^{t\sigma} F(x,z)\, t^{\frac{1-pp'+(1-p')\delta(x)}{pp'}} \, \mathrm{d}z \, \mathrm{d}t,$$

where $F(x,z) = |D_{N+1}u(x,z)| w(x,z)^{1/p}$. Then, by the Hölder's inequality, we observe that

$$I_2 \leq \sigma \left(\int_0^1 t^{-\frac{1+\delta(x)}{p}}\,\mathrm{d}t\right)^{1/p'}\left(\int_0^1 \left(\int_0^{\sigma t} F(x,z)\, t^{\frac{1-pp'+(1-p')\delta(x)}{pp'}}\,\mathrm{d}z\right)^p \mathrm{d}t\right)^{1/p}$$

$$= \sigma \left(\frac{p}{p-1-\delta(x)}\right)^{1/p'}\left(\int_0^1\left(\int_0^{\sigma t} F(x,z)\, t^{\frac{1-pp'+(1-p')\delta(x)}{pp'}}\,\mathrm{d}z\right)^p\mathrm{d}t\right)^{1/p}.$$

Applying the Hölder's inequality again but now on the integral with respect to z, we obtain

$$I_2 \leq \sigma \left(\frac{p}{p-1-\delta(x)} \right)^{1/p'} \left(\int_0^1 (\sigma t)^{p/p'} \int_0^{\sigma t} F(x,z)^p \, t^{\frac{1-pp'+(1-p')\delta(x)}{p'}} \, \mathrm{d}z \, \mathrm{d}t \right)^{1/p}$$

$$\leq \sigma \left(\frac{p}{p-1-\delta(x)} \right)^{1/p'} \left(\int_0^1 (\sigma t)^{p/p'} \int_0^{\sigma} F(x,z)^p \, t^{\frac{1-pp'+(1-p')\delta(x)}{p'}} \, \mathrm{d}z \, \mathrm{d}t \right)^{1/p}$$

$$= \sigma^{1+p/p'} \left(\frac{p}{p-1-\delta(x)} \right)^{1/p'} \left(\int_0^1 t^{-\frac{1+\delta(x)}{p}} \, \mathrm{d}t \right)^{1/p} \left(\int_0^{\sigma} F(x,z)^p \, \mathrm{d}z \right)^{1/p}.$$

Finally, the above implies

$$I_2 \leq \frac{p \, \sigma^{1+p/p'}}{p-1-\delta(x)} \left(\int_0^{\sigma} |D_{N+1}u(x,z)|^p w(x,z) \, \mathrm{d}z \right)^{1/p}.$$

Using the above estimations for I_1 and I_2 in (5.2), and observing that $p-1-\delta(x) \leq p$, it follows that

$$|u(x,0)|\tilde{w}(x)^{1/p} \leq (p+1)\sigma^{-2/p} \left(\int_0^{\sigma} |u(x,y)|^p w(x,y) \, \mathrm{d}y \right)^{1/p}$$

$$+ (p+1)\sigma^{p-1-1/p} \left(\int_0^{\sigma} |D_{N+1}u(x,z)|^p w(x,z) \, \mathrm{d}z \right)^{1/p},$$

from which we obtain

$$|u(x,0)|\tilde{w}(x)^{1/p} \leq (p+1)\sigma^{-2/p'} \left(\int_0^{\sigma} \left(|u(x,y)|^p + |D_{N+1}u(x,y)|^p \right) w(x,y) \, \mathrm{d}y \right)^{1/p}, \tag{5.4}$$

since $\sigma^{p-1-1/p} \leq \sigma^{-2/p}$.

Taking the p power to the inequality (5.4) and then integrating over Ω leads to

$$\int_{\Omega} |u(x,0)|^p \tilde{w}(x) \, \mathrm{d}x \leq (p+1)^p \sigma^{-2p/p'} \left(\|u\|^p_{L^p(\mathcal{C},w)} + \|\nabla u\|^p_{L^p(\mathcal{C},w)} \right).$$

Therefore, $u(\,\cdot\,,0) \in L^p(\Omega, \tilde{w})$ and

$$\|u(\,\cdot\,,0)\|_{L^p(\Omega,\tilde{w})} \leq C(p,\sigma)\|u\|_{W^{1,p}(\mathcal{C},w)},$$

where $C(p,\sigma) = (p+1)\sigma^{-2/p'}$. Notice that σ is an arbitrary, but fixed, number in $(0,1)$, so that in this case we can fix $C(p,\sigma)$ to depend only on p. The operator tr_{Ω} is the unique bounded linear extension of the mapping $u(x,y) \mapsto u(x,0)$ to $\mathscr{H}^{1,p}_{0,L}(\mathcal{C},w)$.

Let us finally see that the same trace result holds true when we replace $\mathscr{H}^{1,p}_{0,L}(\mathcal{C},w)$ by $\mathscr{H}^{1,p}_{0,L}(\mathcal{C}^{\tau},w^{\tau})$, where $\tau > 0$. If $\tau \geq 1$, it follows from (5.4) that

$$|u(x,0)|\tilde{w}(x)^{1/p} \leq (p+1)\sigma^{-2/p'} \left(\int_0^{\tau} \left(|u(x,y)|^p + |D_{N+1}u(x,y)|^p \right) w(x,y) \, \mathrm{d}y \right)^{1/p},$$

from which, exactly as before, we find

$$\|u(\,\cdot\,,0)\|_{L^p(\Omega,\tilde{w})} \leq C(p,\sigma)\|u\|_{W^{1,p}(\mathcal{C}^\tau,w^\tau)}. \qquad (5.5)$$

If, on the contrary, $0 < \tau < 1$, then we select $\sigma = \tau$ in (5.4) and obtain (5.5) in the same way. The trace operator is now obtained as before.

Remark 5 If $s(\cdot) = s \in (0,1)$ is constant, then both $\tilde{w}$ and G_s are also constants. Hence, $L^p(\Omega,\tilde{w}) = L^p(\Omega)$ and $\mathscr{H}^{1,p}_{0,L}(\mathcal{C},w) = \mathscr{H}^{1,p}_{0,L}(\mathcal{C},y^{1-2s})$, so it follows from Theorem 3 that

$$\mathrm{tr}_\Omega : \mathscr{H}^{1,p}_{0,L}(\mathcal{C},y^{1-2s}) \to L^p(\Omega).$$

This is in accordance to the classical case, see [19, Theorem 3.2]. If, additionally, $p = 2$, then we observe that tr_Ω and the trace operator given in [9] (see also Section 3) coincide for functions in $C^\infty_c(\mathcal{C}_\Omega)$. From this, we find that tr_Ω is just given by the restriction to $\mathscr{H}^{1,2}_{0,L}(\mathcal{C},y^{1-2s}) \subset H^1_{0,L}(\mathcal{C},y^{1-2s})$ of the map in [9]. However, a deeper result is true; see Theorem 8.

In Theorem 3, we have characterized the integrability of functions in the trace space of $\mathscr{H}^{1,p}_{0,L}(\mathcal{C},w)$. We aim now to identify the "smoothness" of functions in this trace space. This is a more complicated task since we would like to determine a space with a spatially variable smoothness associated to the function $s(\cdot)$.

For simplicity, from now on we assume that Ω is the N-dimensional unit square $\mathrm{Q}_N = (0,1)^N$. The forthcoming analysis requires one final assumption on the functions $s(\cdot)$ and G_s:

(H4) For almost every $x \in \mathrm{Q}_N, z \in (0,1)$, and $i = 1,\ldots,N$, it holds true that

$$\int_0^1 \left(\mathrm{G}_s(x)|x_i - z|^{1-2s(x)} \right)^{1-p'} \mathrm{d}x_i < \infty,$$

where $x = (x_1,\ldots,x_n)$.

Assumption (H4) enables us to use a Hardy-type inequality (see Lemma 2 below) for two specially chosen weights, which is a key ingredient to prove the subsequent improvement of the trace result in Theorem 5.

Example 2 Let $\Omega = \mathrm{Q}_1$, $p = 2$, and $\mathrm{G}_s = \mathrm{G}_s^{(1)}$ constant; see (2.1) in Example 1. Suppose that $s(\cdot)$ satisfies:

$$s(x) \geq m|x - x_0|^q \quad \text{if} \quad |x - x_0| \leq R, \qquad s(x) > \mu > 0 \quad \text{if} \quad |x - x_0| > R, \qquad (5.6)$$

for some $q, R \in (0,1)$, $m, \mu > 0$, and $x_0 \in (R, 1-R)$. Notice that the only point where s is allowed to be zero is x_0. For this particular setting, although $w \notin A_p(\mathcal{C})$, i.e., w is not a Muckenhoupt weight (see [2]), we find that (H4) holds true as we see next.

To simplify the notation below, we write $\delta(\cdot) := 1 - 2s(\cdot)$. Since $\delta(x)(1 - p') = -\delta(x) > -1$ for all $x \neq x_0$, we have:

$$\int_0^1 |x - z|^{-\delta(x)} \, \mathrm{d}z = \frac{x^{2s(x)} + (1-x)^{2s(x)}}{2s(x)}, \qquad \forall\, x \neq x_0.$$

Then,

$$\int_0^1 \left(\int_0^1 |x - z|^{-\delta(x)} \, \mathrm{d}z \right) \mathrm{d}x \leq \int_0^1 \frac{1}{s(x)} \, \mathrm{d}x.$$

We now observe that, by (5.6), we have:

$$\int_0^1 \frac{1}{s(x)} \, \mathrm{d}x = \int_0^{x_0 - R} \frac{\mathrm{d}x}{s(x)} + \int_{x_0 - R}^{x_0 + R} \frac{\mathrm{d}x}{s(x)} + \int_{x_0 + R}^1 \frac{\mathrm{d}x}{s(x)}$$

$$\leq \frac{2}{\mu}(1 - 2R) + \frac{1}{m} \int_{x_0 - R}^{x_0 + R} |x - x_0|^{-q} \, \mathrm{d}x < \infty.$$

Hence,

$$\int_0^1 \left(\int_0^1 |x - z|^{-\delta(x)} \, \mathrm{d}z \right) \mathrm{d}x < \infty.$$

Therefore, by Tonelli's Theorem, we have that $(x, z) \mapsto |x - z|^{-\delta(x)}$ belongs to $L^1(\mathrm{Q}_2)$, which in turn implies that

$$\int_0^1 |x - z|^{-\delta(x)} \, \mathrm{d}x < \infty,$$

for almost all $z \in (0, 1)$, by Fubini's Theorem.

Minor changes in the above arguments yield the same conclusion for functions s with a finite number of zeros and a local behavior as (5.6) around each of them.

Next, in Definition 2 we present a Sobolev space of functions where smoothness is spatially dependent and related to $s(\cdot)$. First, we introduce the required notation.

For $i = 1, \ldots, N$, let $\varphi_i, \psi_i : \mathrm{Q}_{N+1} \to \mathbb{R}$ be given by

$$\varphi_i(x, \tau) = \Phi_i(x, \tau)^{1-p'} \left(\int_{\min\{x_i, \tau\}}^{\max\{x_i, \tau\}} \Phi_i(x_{t'}^i, \tau)^{1-p'} \, \mathrm{d}t' \right)^{-p},$$

$$\psi_i(x, \tau) = \Phi_i(x, \tau)^{1-p'} \left(\int_{\min\{x_i, \tau\}}^{\max\{x_i, \tau\}} \Phi_i(x, \tau')^{1-p'} \, \mathrm{d}\tau' \right)^{-p},$$

where

$$\Phi_i(x, \tau) = \mathrm{G}_s(x)|x_i - \tau|^{1-2s(x)},$$

and the notation x_a^i for $a \in (0, 1)$ means that the ith-coordinate of $x = (x_1, \ldots, x_N) \in \mathrm{Q}_N$ is replaced by a, that is:

$$x_a^i = (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_N).$$

Definition 2 The space $\mathbb{W}^{s(\cdot),p}(Q_N, \tilde{w}, w_1, \ldots, w_N)$ is defined by

$$\mathbb{W}^{s(\cdot),p}(Q_N, \tilde{w}, w_1, \ldots, w_N) = \{v \in L^p(Q_N, \tilde{w}) : A_i(v) < \infty \text{ for all } i = 1, \ldots, N\}, \tag{5.7}$$

with the norm

$$\|v\|_{\mathbb{W}^{s(\cdot),p}(Q_N, \tilde{w}, w_1, \ldots, w_N)} = \left(\|v\|_{L^p(Q_N, \tilde{w})}^p + \sum_{i=1}^N A_i(v) \right)^{1/p}, \tag{5.8}$$

where

$$A_i(v) = \underbrace{\int_0^1 \ldots \int_0^1}_{(N-1)\text{-fold}} \left(\int_0^1 \int_0^1 w_i(x_t^i, \tau) |v(x_t^i) - v(x_\tau^i)|^p \, d\tau \, dt \right) dx_1 \ldots dx_{i-1} \, dx_{i+1} \ldots dx_N,$$

and

$$w_i = \min\{\varphi_i, \psi_i\} \quad \text{for} \quad i = 1, \ldots, N.$$

In order to address that $s(\cdot)$ controls locally the differential regularity of elements in $\mathbb{W}^{s(\cdot),p}(Q_N, \tilde{w}, w_1, \ldots, w_N)$, consider the following. For $s \in (0,1)$, let $W^{s,p}(Q_N)$ be the fractional Sobolev space of order s, that is,

$$W^{s,p}(Q_N) = \left\{ v \in L^2(Q_N) : \int_{Q_N} \int_{Q_N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}} \, dx \, dy < \infty \right\},$$

equipped with the norm

$$\|v\|_{W^{s,p}(Q_N)} = \left(\|v\|_{L^p(Q_N)}^p + \int_{Q_N} \int_{Q_N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}} \, dx \, dy \right)^{1/p}.$$

If $p = 2$, we have $\mathbb{H}^s(Q_N) = W^{s,2}(Q_N)$. Then, note the following lemma that can be found in [19] (see also [14]).

Lemma 1 *Let $-1 < \varepsilon < p - 1$. There exists a positive constant c such that*

$$\|v\|_{W^{1-\frac{1+\varepsilon}{p},p}(Q_N)}^p \leq c \left(\|v\|_{L^p(Q_N)}^p + \sum_{i=1}^N \mathcal{A}_i(v) \right),$$

for every $v \in L^p(Q_N)$ that satisfies $\mathcal{A}_i(v) < \infty$ for all $i = 1, \ldots, N$, where

$$\mathcal{A}_i(v) := \underbrace{\int_0^1 \ldots \int_0^1}_{(N-1)\text{-fold}} \left(\int_0^1 \int_0^1 \frac{|v(x_t^i) - v(x_\tau^i)|^p}{|t-\tau|^{p-\varepsilon}} \, d\tau \, dt \right) dx_1 \ldots dx_{i-1} \, dx_{i+1} \ldots dx_N.$$

We now can show the relation between $\mathbb{W}^{s(\cdot),p}(Q_N, \tilde{w}, w_1, \ldots, w_N)$ and the classical Sobolev spaces.

Theorem 4 *If $s(\cdot) = s \in (0,1)$ constant, then*

$$\mathbb{W}^{s(\cdot),p}(Q_N, \tilde{w}, w_1, \ldots, w_N) \hookrightarrow W^{1-\frac{2(1-s)}{p},p}(Q_N).$$

Proof Let $\delta := 1 - 2s$ and consider $t, \tau \in (0,1)$. Since $\delta(1 - p') > -1$, we have:

$$\int_{\min\{t,\tau\}}^{\max\{t,\tau\}} |t' - \tau|^{\delta(1-p')} \, \mathrm{d}t' = \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} |t - \tau'|^{\delta(1-p')} \, \mathrm{d}\tau' = \frac{|t - \tau|^{1+\delta(1-p')}}{1 + \delta(1 - p')}.$$

Then, a direct calculation yields:

$$\varphi_i(x,\tau) = \psi_i(x,\tau) = \frac{\mathrm{G}_s(1 + \delta(1 - p'))^p}{|x_i - \tau|^{p-\delta}},$$

for all $(x,\tau) \in \mathrm{Q}_{N+1}$ since G_s is constant by assumption (H2). Therefore,

$$A_i(v) = C(p,s) \underbrace{\int_0^1 \cdots \int_0^1}_{(N-1)\text{-fold}} \left(\int_0^1 \int_0^1 \frac{|v(x_t^i) - v(x_\tau^i)|^p}{|t - \tau|^{p-\delta}} \, \mathrm{d}\tau \, \mathrm{d}t \right) \mathrm{d}x_1 \ldots \mathrm{d}x_{i-1} \, \mathrm{d}x_{i+1} \ldots \mathrm{d}x_N,$$

where $C(p,s) = \mathrm{G}_s(1 + \delta(1 - p'))^p$. In addition, we notice that $L^1(\mathrm{Q}_N, \tilde{w}) = L^1(\mathrm{Q}_N)$ since $\tilde{w}$ is constant. Now the conclusion follows from Lemma 1 with $\varepsilon = \delta$.

Remark 6 In light of the previous result, a more appropriate notation for $\mathbb{W}^{s(\cdot),p}(\mathrm{Q}_N, \tilde{w}, w_1, \ldots, w_N)$ would be $\mathbb{W}^{1 - \frac{2(1 - s(\cdot))}{p}, p}(\mathrm{Q}_N, \tilde{w}, w_1, \ldots, w_N)$. We avoid this for the sake of brevity.

The following lemma is a key tool for the improvement of the result in Theorem 3. The proof can be found in [21, Sect. 2.6].

Lemma 2 (Weighted Hardy-type inequality) *Let ρ be a weight function defined in the interval (a,b). If*

$$\int_a^b \rho(t)^{1-p'} \, \mathrm{d}t < \infty,$$

then

$$\int_a^b \hat{\rho}(t)|f(t)|^p \, \mathrm{d}t \leq C_H(p) \int_a^b \rho(t)|f'(t)|^p \, \mathrm{d}t, \qquad \forall \, x \in (a,b), \qquad (5.9)$$

for all absolutely continuous functions f in (a,b) that satisfy $\lim_{t \to a^+} f(t) = 0$, where

$$\hat{\rho}(t) = \rho(t)^{1-p'} \left(\int_a^t \rho(\xi)^{1-p'} \, \mathrm{d}\xi \right)^{-p},$$

and $C_H(p) = p^p/(p-1)^{p-1}$.

Now we are in shape to prove the improvement of Theorem 3.

Theorem 5 (Improved trace theorem) *Provided that (H1) to (H4) hold true, there exists a unique bounded linear operator*

$$\mathrm{tr}_{Q_N} \; : \; \mathscr{H}^{1,p}_{0,L}(\mathcal{C}, w) \to \mathbb{W}^{s(\cdot),p}(Q_N, \tilde{w}, w_1, \dots, w_N),$$

that satisfies $\mathrm{tr}_{Q_N} u = u(\,\cdot\,, 0)$ *for all* $u \in \mathscr{H}^{1,p}_{0,L}(\mathcal{C}, w) \cap C^\infty_c(\mathcal{C}_{Q_N})$.

The same statement is true if we replace $\mathscr{H}^{1,p}_{0,L}(\mathcal{C}, w)$ *by the space* $\mathscr{H}^{1,p}_{0,L}(\mathcal{C}^\tau, w)$, *for every* $\tau \geq 1$.

Proof For the sake of simplicity, we give the proof only for $N = 1$; with the natural changes, the proof adapts straightforward to the case $N \geq 2$.

Let $u \in \mathscr{H}^{1,p}_{0,L}(\mathcal{C}, w) \cap C^\infty_c(\mathcal{C}_{Q_1})$. Initially, we write:

$$A_1(u(\,\cdot\,, 0)) = I_1 + I_2, \tag{5.10}$$

where:

$$I_1 := \int_0^1 \int_0^t w_1(t, \tau) |u(t, 0) - u(\tau, 0)|^p \; \mathrm{d}\tau \; \mathrm{d}t,$$

$$I_2 := \int_0^1 \int_t^1 w_1(t, \tau) |u(t, 0) - u(\tau, 0)|^p \; \mathrm{d}\tau \; \mathrm{d}t,$$

where $w_1 = \min\{\varphi_1, \psi_1\}$ as in Definition 2. Next, we shall estimate I_1 and I_2 separately. For this, we introduce the auxiliary function $v : Q_2 \to \mathbb{R}$ given by

$$v(t, \tau) = u(t, \max\{t, \tau\} - \min\{t, \tau\}).$$

We have:

$$
\begin{aligned}
I_1 &= \int_0^1 \int_0^t w_1(t, \tau) |v(t, t) - v(\tau, \tau)|^p \, \mathrm{d}\tau \; \mathrm{d}t \\
&= \int_0^1 \int_0^t w_1(t, \tau) \left| \int_\tau^t D_1 v(t', \tau) \, \mathrm{d}t' + \int_\tau^t D_2 v(t, \tau') \, \mathrm{d}\tau' \right|^p \mathrm{d}\tau \; \mathrm{d}t \\
&\leq 2^{p-1} \int_0^1 \int_0^t w_1(t, \tau) \left| \int_\tau^t D_1 v(t', \tau) \, \mathrm{d}t' \right|^p \mathrm{d}\tau \; \mathrm{d}t \\
&\quad + 2^{p-1} \int_0^1 \int_0^t w_1(t, \tau) \left| \int_\tau^t D_2 v(t, \tau') \, \mathrm{d}\tau' \right|^p \mathrm{d}\tau \; \mathrm{d}t,
\end{aligned}
$$

where $D_1 v$ and $D_2 v$ denote the partial derivative of v with respect to the first and second coordinates, respectively.

Interchanging the order of integration in the first term of the right hand side of the above inequality, and introducing the change of variable $\tilde{\tau} = -\tau$ in the second one, we find:

$$I_1 \leq 2^{p-1} \int_0^1 \int_\tau^1 w_1(t, \tau) |f_1(t, \tau)|^p \; \mathrm{d}t \; \mathrm{d}\tau + 2^{p-1} \int_0^1 \int_{-t}^0 w_1(t, -\tilde{\tau}) |f_2(t, \tilde{\tau})|^p \; \mathrm{d}\tilde{\tau} \; \mathrm{d}t, \tag{5.11}$$

where:

$$f_1(t,\tau) = \int_\tau^t D_1 v(t',\tau)\ \mathrm{d}t', \quad \text{and} \quad f_2(t,\tilde\tau) = \int_t^{-\tilde\tau} D_2 v(t,\tau')\ \mathrm{d}\tau'.$$

The function $f_1(\cdot,\tau)$ is absolutely continuous in $(\tau,1)$ and it satisfies $\lim_{t\to\tau^+} f_1(t,\tau) = 0$ for almost all $\tau \in (0,1)$. Additionally, by definition we observe that

$$\varphi_1(t,\tau) = \Phi_1(t,\tau)^{1-p'} \left(\int_\tau^t \Phi_1(t',\tau)^{1-p'}\ \mathrm{d}t' \right)^{-p}, \qquad \forall t \ge \tau,$$

for almost all $\tau \in (0,1)$. Then, by Lemma 2, we have:

$$\int_\tau^1 \varphi_1(t,\tau)|f_1(t,\tau)|^p\ \mathrm{d}t \le C_H(p) \int_\tau^1 \Phi_1(t,\tau)\,|D_1 v(t,\tau)|^p\ \mathrm{d}t, \qquad (5.12)$$

for almost all $\tau \in (0,1)$.

Similarly, the function $f_2(t,\cdot)$ is absolutely continuous in $(-t,0)$ and satisfies $\lim_{\tilde\tau\to -t^+} f_2(t,\tilde\tau) = 0$ for almost all $t \in (0,1)$. Since

$$\psi_1(t,-\tilde\tau) = \Phi_1(t,-\tilde\tau)^{1-p'} \left(\int_{-t}^{\tilde\tau} \Phi_1(t,-\tilde\tau')^{1-p'}\ \mathrm{d}\tilde\tau' \right)^{-p}, \qquad \forall \tilde\tau \ge -t,$$

for almost all $t \in (0,1)$, it follows by Lemma 2 that

$$\int_{-t}^0 \psi_1(t,-\tilde\tau)|f_2(t,\tilde\tau)|^p\ \mathrm{d}\tilde\tau \le C_H(p) \int_{-t}^0 \Phi_1(t,-\tilde\tau)\,|D_2 v(t,-\tilde\tau)|^p\ \mathrm{d}\tilde\tau, \quad (5.13)$$

for almost all $t \in (0,1)$.

Then, since $w_1 = \min\{\varphi_1,\psi_1\}$, the estimation (5.11) in conjunction with (5.12) and (5.13) yields:

$$\begin{aligned}
I_1 \le\ & C_H(p)\, 2^{p-1} \int_0^1 \int_\tau^1 \Phi_1(t,\tau)\,|D_1 v(t,\tau)|^p\ \mathrm{d}t\ \mathrm{d}\tau \\
& + C_H(p)\, 2^{p-1} \int_0^1 \int_{-t}^0 \Phi_1(t,-\tilde\tau)\,|D_2 v(t,-\tilde\tau)|^p\ \mathrm{d}\tilde\tau\ \mathrm{d}t.
\end{aligned}$$

Interchanging the order of integration in the first term of the r.h.s., and making the change of variable $\tau = -\tilde\tau$ in the second one, we obtain:

$$I_1 \le C_H(p)\, 2^{p-1} \int_0^1 \int_0^t \Phi_1(t,\tau)\,\big(\,|D_1 v(t,\tau)|^p + |D_2 v(t,\tau)|^p\,\big)\ \mathrm{d}\tau\ \mathrm{d}t. \quad (5.14)$$

Since the function v is given by $v(t,\tau) = u(t,t-\tau)$ for $t > \tau$, we have:

$$\begin{aligned}
D_1 v(t,\tau) &= D_1 u(t,t-\tau) + D_2 u(t,t-\tau), \\
D_2 v(t,\tau) &= -\, D_2 u(t,t-\tau).
\end{aligned}$$

Then,

$$\begin{aligned}
|D_1 v(t,\tau)|^p + |D_2 v(t,\tau)|^p &\leq \big(\,|D_1 u(t,t-\tau)| + |D_2 u(t,t-\tau)|\,\big)^p + |D_2 u(t,t-\tau)|^p \\
&\leq (2^{p-1}+1)\,\big(\,|D_1 u(t,t-\tau)|^p + |D_2 u(t,t-\tau)|^p\,\big) \\
&\leq 2^{p/2}(2^{p-1}+1)|\nabla u(t,t-\tau)|^p.
\end{aligned}$$

Using this estimation in (5.14) and then making the change of variable $y = t-\tau$ in the inner integral, we find:

$$\begin{aligned}
I_1 &\leq C_H(p)\, 2^{p-1} 2^{p/2}(2^{p-1}+1) \int_0^1 \int_0^t \Phi_1(t,\tau)|\nabla u(t,t-\tau)|^p\ \mathrm{d}\tau\ \mathrm{d}t \\
&= C_H(p)\, 2^{p-1} 2^{p/2}(2^{p-1}+1) \int_0^1 \int_0^t \Phi_1(t,t-y)|\nabla u(t,y)|^p\ \mathrm{d}y\ \mathrm{d}t.
\end{aligned}$$

Hence,

$$I_1 \leq C_H(p)\, 2^{p-1} 2^{p/2}(2^{p-1}+1) \int_0^1 \int_0^1 w(t,y)|\nabla u(t,y)|^p\ \mathrm{d}y\ \mathrm{d}t. \qquad (5.15)$$

To estimate I_2, we first write:

$$\begin{aligned}
I_2 &= \int_0^1 \int_t^1 w_1(t,\tau)|v(t,t) - v(\tau,\tau)|^p\ \mathrm{d}\tau\ \mathrm{d}t \\
&= \int_0^1 \int_0^\tau w_1(t,\tau)|v(t,t) - v(\tau,\tau)|^p\ \mathrm{d}t\ \mathrm{d}\tau,
\end{aligned}$$

and notice that, in general, $I_2 \neq I_1$ since $w_1(t,\tau) \neq w_1(\tau,t)$ for $s(\cdot)$ not constant. However, similarly as we obtained (5.15), we identify the same bound for I_2:

$$I_2 \leq C_H(p)\, 2^{p-1} 2^{p/2}(2^{p-1}+1) \int_0^1 \int_0^1 w(t,y)|\nabla u(t,y)|^p\ \mathrm{d}y\ \mathrm{d}t. \qquad (5.16)$$

Using (5.15) and (5.16) in (5.10), we obtain:

$$A_1(u(\cdot,0)) \leq C_H(p)\, 2^p\, 2^{p/2}(2^{p-1}+1)\|\nabla u\|_{L^p(\mathcal{C},w)}^p,$$

hence,

$$A_1(u(\cdot,0)) \leq C_H(p)\, 2^p\, 2^{p/2}(2^{p-1}+1)\|u\|_{W^{1,p}(\mathcal{C},w)}^p.$$

In addition, we know by the Theorem 3 that:

$$\|u(\cdot,0)\|_{L^p(Q_1,\tilde{w})}^p \leq (1+p)^p \sigma^{-2p/p'}\|u\|_{W^{1,p}(\mathcal{C},w)}^p,$$

where σ is some arbitrary, but fixed, number in $(0,1)$.

Therefore, $u(\cdot,0) \in \mathbb{W}^{s(\cdot),p}(Q_1,\tilde{w},w_1)$ and

$$\|u(\cdot,0)\|_{\mathbb{W}^{s(\cdot),p}(Q_1,\tilde{w},w_1)} \leq C(p,\sigma)\|u\|_{W^{1,p}(\mathcal{C},w)},$$

where $C(p,\sigma) = (C_H(p)\, 2^p\, 2^{p/2}(2^{p-1}+1) + (1+p)^p \sigma^{-2p/p'})^{1/p}$.

The operator tr_{Q_1} is the unique bounded linear extension of the map $u \mapsto u(\,\cdot\,,0)$ to $\mathscr{H}_{0,L}^{1,p}(\mathcal{C},w)$.

The proof when $\mathscr{H}_{0,L}^{1,p}(\mathcal{C},w)$ is replaced by $\mathscr{H}_{0,L}^{1,p}(\mathcal{C}^\tau,w)$ where $\tau \geq 1$ is identical.

Remark 7 (Surjectivity of trace operator) Although the previous result represents an improvement on the Ω-trace characterization for functions in $\mathscr{H}_{0,L}^{1,p}(\mathcal{C},w)$, nothing can be said about the surjectivity of the trace operator

$$\mathrm{tr}_\Omega : \mathscr{H}_{0,L}^{1,p}(\mathcal{C},w) \to \mathbb{W}^{s(\cdot),p}(Q_N,w,w_1,\ldots,w_N),$$

for $s(\cdot)$ non-constant.

Remark 8 If $s(\cdot) = s \in (0,1)$ is constant, then it follows by Theorem 4 that $\mathbb{W}^{s,2}(Q_N,\tilde{w},w_1,\ldots,w_N) \hookrightarrow W^{1-\frac{2(1-s)}{p},p}(Q_N)$. Hence, the trace result in Theorem 3 is again in accordance to the classical case, see [19, Theorem 2.8] (see also Remark 5). Moreover, if $p = 2$ and $s \in (0,1/2)$, we observe that

$$\mathbb{W}^{s,2}(Q_N,\tilde{w},w_1,\ldots,w_N) \hookrightarrow H, \tag{5.17}$$

since $\mathbb{H}^s(Q_N) = \mathbb{H}_0^s(Q_N) = H$, so in this case we further partially recover the trace result in [9, Lemma 2.2] given that $\mathscr{H}_{0,L}^{1,p}(\mathcal{C},y^{1-2s}) \subset H_{0,L}^1(\mathcal{C},y^{1-2s})$.

6 Cases where the Poincaré inequality holds

We address now in this section cases and conditions on $s(\cdot)$ not constant that are sufficient for the Poincaré inequality to hold true. Two results are given, one in the entire cylinder and one in the truncated cylinder; see Theorem 6 and Theorem 7 respectively. From now on until the end of the section, we assume that $G_s = G_s^{(1)}$ constant, see (2.1) in Example 1; and $s(\cdot)$ is given by

$$s(\cdot) = \sum_{i=1}^M s_i \mathbb{1}_{\Omega_i}(\cdot), \tag{6.1}$$

where $s_i \in (0,1)$ for $i = 1,\ldots,M$ and $\{\Omega_i : i = 1,\ldots,M\}$ is a finite collection of non-empty open subsets of Ω that satisfies $\bigcup_{i=1}^M \bar{\Omega}_i = \bar{\Omega}$. In other words, we assume that $s(\cdot)$ is a step function in Ω with range contained in the interval $(0,1)$. Our first example is given by the next theorem which basically states that the Poincaré inequality holds provided that all pieces Ω_i of the partition of Ω touch the boundary $\partial\Omega$. The proof is quite direct, thanks to the existence of a Poincaré inequality for functions in $C^\infty(\Omega_i)$ that vanish on a subset of non-zero measure of $\partial\Omega_i$.

Theorem 6 *Assume that* $G_s = G_s^{(1)}$ *constant and* $s(\cdot)$ *is given by* (6.1). *If*

$$|\partial\Omega_i \cap \partial\Omega| > 0, \qquad \forall\, i = 1, \ldots, M, \tag{6.2}$$

then there exists a positive constant $C_P(p, \Omega_1, \ldots, \Omega_M)$ *that satisfies*

$$\|u\|_{L^p(\mathcal{C}, w)} \leq C_P(p, \Omega_1, \ldots, \Omega_M)\|\nabla u\|_{L^p(\mathcal{C}, w)}, \tag{6.3}$$

for all $u \in \mathscr{H}_{0,L}^{1,p}(\mathcal{C}, w)$.

Proof Let $u \in C_c^\infty(\mathcal{C}_\Omega) \cap W^{1,p}(\mathcal{C}, w)$ and $i \in \{1, \ldots, M\}$. For every $y > 0$, the function $u(\cdot, y)$ belongs to $C^\infty(\Omega_i)$ and vanishes on a portion with non-zero measure of $\partial\Omega_i$, by (6.2). Then, by the Poincaré inequality, we have

$$\int_{\Omega_i} |u(x, y)|^p \, \mathrm{d}x \leq c_i \int_{\Omega_i} |\nabla_x u(x, y)|^p \, \mathrm{d}x, \tag{6.4}$$

where c_i is a positive constant that depends only on Ω_i and p, and $\nabla_x u$ is the gradient of u with respect to the first N coordinates. Multiplying (6.4) by y^{1-2s_i}, then integrating for $y \in (0, \infty)$, and finally adding up for $i = 1, \ldots, M$, we obtain

$$\int_{\mathcal{C}} y^{1-2s(x)}|u(x, y)|^p \, \mathrm{d}X \leq c \int_{\mathcal{C}} y^{1-2s(x)}|\nabla_x u(x, y)|^p \, \mathrm{d}X,$$

where $c = c_1 + \ldots + c_M$. Since $|\nabla_x u|^p \leq |\nabla u|^p$ in $\mathcal{C}$ and G_s is constant, we get

$$\int_{\mathcal{C}} w(x, y)|u(x, y)|^p \, \mathrm{d}X \leq c \int_{\mathcal{C}} w(x, y)|\nabla u(x, y)|^p \, \mathrm{d}X,$$

for all $u \in C_c^\infty(\mathcal{C}_\Omega) \cap W^{1,p}(\mathcal{C}, w)$. Now (6.3) follows by density.

Next we prove that the truncated domain allows a much more amenable result than the one in the complete cylinder $\mathcal{C}$. In particular, we prove that (6.1) is a sufficient condition for the Poincaré inequality to hold; the result is given in next Theorem 7. The proof requires the following auxiliary lemma, see [13, Theorem 5.2] for its proof.

Lemma 3 (Classical Hardy inequality) *Let* $\varepsilon > p - 1$ *and let* f *be a differentiable function almost everywhere in* $(0, \infty)$ *that satisfies* $\lim_{t\to\infty} f(t) = 0$. *If*

$$\int_0^\infty t^\varepsilon |f'(t)|^p \, \mathrm{d}t < \infty,$$

then

$$\int_0^\infty t^{\varepsilon-p}|f(t)|^p \, \mathrm{d}t \leq C_H(p, \varepsilon) \int_0^\infty t^\varepsilon |f'(t)|^p \, \mathrm{d}t < \infty,$$

where $C_H(p, \varepsilon) = p^p/(\varepsilon - p + 1)^p$.

We are now in a position to present the final result in this section.

Theorem 7 *Assume that* $\mathrm{G}_s = \mathrm{G}_s^{(1)}$ *constant and* $s(\cdot)$ *is given by* (6.1). *For every* $\tau > 0$ *there exists a positive constant* $C_P(\tau, p, \Omega_1, \ldots, \Omega_M)$ *that satisfies*

$$\|u\|_{L^p(\mathcal{C}^\tau, w)} \leq C_P(\tau, p, \Omega_1, \ldots, \Omega_M) \|\nabla u\|_{L^p(\mathcal{C}^\tau, w)}, \tag{6.5}$$

for all $u \in \mathscr{H}_{0,L}^{1,p}(\mathcal{C}^\tau, w)$.

Proof Let $\tau > 0$ and $u \in C_c^\infty(\mathcal{C}_\Omega^\tau) \cap W^{1,p}(\mathcal{C}^\tau, w)$. Initially, we write:

$$\int_{\mathcal{C}^\tau} y^{1-2s(x)} |u(x,y)|^p \, \mathrm{d}X = \sum_{i=1}^{M} I_i, \tag{6.6}$$

where

$$I_i := \int_0^\tau y^{1-2s_i} \int_{\Omega_i} |u(x,y)|^p \, \mathrm{d}x \, \mathrm{d}y.$$

We denote by c a positive constant that may depend only on p and the partition $\{\Omega_i : i = 1, \ldots, M\}$, whose numerical value may be different from one line to another.

Let $i \in \{1, \ldots, M\}$. We define

$$\bar{u}_i(y) = \frac{1}{|\Omega_i|} \int_{\Omega_i} u(x,y) \, \mathrm{d}x,$$

and observe that

$$I_i \leq c \, (I_{i1} + I_{i2}), \tag{6.7}$$

where

$$I_{i1} := \int_0^\tau y^{1-2s_i} \int_{\Omega_i} |u(x,y) - \bar{u}_i(y)|^p \, \mathrm{d}x \, \mathrm{d}y,$$

$$I_{i2} := \int_0^\tau y^{1-2s_i} \int_{\Omega_i} |\bar{u}_i(y)|^p \, \mathrm{d}x \, \mathrm{d}y = |\Omega_i| \int_0^\tau y^{1-2s_i} |\bar{u}_i(y)|^p \, \mathrm{d}y.$$

For each fixed $y \in (0, \tau)$, the function $u(\,\cdot\,, y)$ belongs to $C^\infty(\Omega_i)$. Thus, by the Poincaré-Wirtinger's inequality, we obtain:

$$\int_{\Omega_i} |u(x,y) - \bar{u}_i(y)|^p \, dx \leq c \int_{\Omega_i} |\nabla_x u(x,y)|^p \, \mathrm{d}x.$$

From this, similarly as in the proof of Theorem 6, we find:

$$I_{i1} \leq c \int_0^\tau \int_{\Omega_i} y^{1-2s_i} |\nabla u(x,y)|^p \, \mathrm{d}x \, \mathrm{d}y. \tag{6.8}$$

Let $\bar{u}_{ext}$ be the extension by zero of $\bar{u}$ to $[0, \infty)$. Notice that $\bar{u}_{ext}$ is differentiable almost everywhere in $(0, \infty)$ since $u(x, \cdot) \in C^\infty([0, \tau])$ for all $x \in \Omega_i$, and, trivially, $\bar{u}_{ext}$ satisfies $\lim_{y \to \infty} \bar{u}_{ext}(y) = 0$. Also, observe that

$$\int_0^\infty y^{1+p-2s_i} |\bar{u}_{ext}'(y)|^p \, \mathrm{d}y = \int_0^\tau y^{1+p-2s_i} |\bar{u}'(y)|^p \, \mathrm{d}y \leq c \int_0^\tau y^{1+p-2s_i} \, \mathrm{d}y < \infty,$$

since $1 + p - 2s_i > -1$ and $\bar{u}'$ is bounded in $[0, \tau]$.

Then, by the classical Hardy inequality in Lemma 3 with $\varepsilon = 1 + p - 2s_i$, we have:

$$I_{i2} = |\Omega_i| \int_0^\infty y^{\varepsilon - p} |\bar{u}_{ext}(y)|^p \, dy \leq c \, |\Omega_i| \int_0^\infty y^\varepsilon |\bar{u}'_{ext}(y)|^p \, \mathrm{d}y. \qquad (6.9)$$

We now observe that:

$$\int_0^\infty y^\varepsilon |\bar{u}'_{ext}(y)|^p \, \mathrm{d}y = \frac{1}{|\Omega_i|^p} \int_0^\tau y^\varepsilon \left| \int_{\Omega_i} D_{N+1} u(x, y) \, dx \right|^p \, \mathrm{d}y$$
$$\leq \frac{1}{|\Omega_i|^p} \int_0^\tau y^\varepsilon \left(\int_{\Omega_i} |\nabla u(x, y)| \, dx \right)^p \, \mathrm{d}y,$$

where $D_{N+1} u$ is the partial derivative of u with respect to the $(N + 1)$ coordinate. Then, by the Hölder's inequality on the inner integral, we have:

$$\int_0^\infty y^\varepsilon |\bar{u}'_{ext}(y)|^p \, \mathrm{d}y \leq \frac{1}{|\Omega_i|} \int_0^\tau y^\varepsilon \int_{\Omega_i} |\nabla u(x, y)|^p \, \mathrm{d}x \, \mathrm{d}y.$$

With this estimation in (6.9), we find:

$$I_{i2} \leq c \, \tau^p \int_0^\tau \int_{\Omega_i} y^{1 - 2s_i} |\nabla u(x, y)|^p \, \mathrm{d}x \, \mathrm{d}y. \qquad (6.10)$$

Finally, using (6.8) and (6.10) in (6.7), we obtain:

$$I_i \leq c \, \tau^p \int_0^\tau \int_{\Omega_i} y^{1 - 2s_i} |\nabla u(x, y)|^p \, \mathrm{d}x \, \mathrm{d}y,$$

and hence, by (6.6) and since G_s is constant, we have:

$$\int_{\mathcal{C}^\tau} w(x, y) |u(x, y)|^p \, \mathrm{d}X \leq c \, \tau^p \int_{\mathcal{C}^\tau} w(x, y) |\nabla u(x, y)|^p \, \mathrm{d}X,$$

for all $u \in C_c^\infty(\mathcal{C}_\Omega^\tau) \cap W^{1,p}(\mathcal{C}^\tau, w)$. Then we obtain (6.5) by density.

7 Second definition and solution to $(-\Delta)^{s(\cdot)} v = h$

We are now in a position to give a new definition for the operator $(-\Delta)^{s(\cdot)}$, and to solve the associated Poisson problem for right hand sides defined on Ω. The arguments below are very similar to those developed in Section 4 but now we assume some extra conditions on the function $s(\cdot)$ and the domain Ω, which enable a better characterization of the domain of $(-\Delta)^{s(\cdot)}$. We present the ideas for the semi-infinite cylinder $\mathcal{C}$, but the same arguments are valid for a truncated one $\mathcal{C}^\tau$.

From now on, we assume that the functions $s(\cdot)$ and G_s satisfy hypotheses (H1), (H2), and (H3). Further, we assume that the Poincaré inequality holds true, that is there exists $C > 0$ such that

$$\|u\|_{L^2(\mathcal{C}, w)} \leq C \|\nabla u\|_{L^2(\mathcal{C}, w)}, \qquad \forall\, u \in \mathscr{H}^{1,p}_{0,L}(\mathcal{C}, w).$$

For example, this is satisfied under the assumptions of Theorem 6 (see Theorem 7 for the case of a truncated cylinder). In particular, this implies that

$$\mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w) = \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w),$$

algebraically and topologically. We endow the space $\mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w)$ with the norm $\|v\|_{\mathscr{H}^{1,p}_{0,L}(\mathcal{C}, w)} := \|\nabla v\|_{L^2(\mathcal{C}, w)}$. Under the hypotheses assumed, we have established in Theorem 3 an Ω-trace operator

$$\mathrm{tr}_\Omega : \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w) \to L^2(\Omega, \tilde{w}), \tag{7.1}$$

and proved it is bounded, linear, and such that $\mathrm{tr}_\Omega u = u(\,\cdot\,, 0)$ for all $u \in W^{1,p}(\mathcal{C}, w) \cap C^\infty_c(\mathcal{C}_\Omega)$. Note that this operator is not, however, surjective. Subsequently, consider

$$\mathscr{W}^{1,2}_0(\mathcal{C}, w) := \{u \in \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w) : \mathrm{tr}_\Omega u = 0\},$$

which is a closed subspace of $\mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w)$. Hence, a space of abstract traces on Ω of functions in $\mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w)$ can be defined as the quotient space

$$\mathscr{Y}(\Omega, w) := \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w) / \mathscr{W}^{1,2}_0(\mathcal{C}, w).$$

Remark 9 Due to the absence of density results of the type "$H = W$" for non-Muckenhoupt weights, we are not in a position to assure that the spaces $\mathscr{X}(\Omega, w)$ and $\mathscr{Y}(\Omega, w)$ are actually the same.

Immediately from here, via the isomorphism theorems, we can argue that there is an isomorphism

$$\varphi : \mathscr{Y}(\Omega, w) \xrightarrow{\sim} \mathrm{tr}_\Omega \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w). \tag{7.2}$$

Moreover, one can simply consider φ to be given by $[u] \mapsto \mathrm{tr}_\Omega u$. However, in order to identify $\mathscr{Y}(\Omega, w)$ with a subset of functions defined on Ω, we need further information related with the structure of the function space $\mathrm{tr}_\Omega \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w)$.

Analogously as in Section 4, we define

$$\mathrm{TR}_\Omega : \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w) \to \mathscr{Y}(\Omega, w), \tag{7.3}$$

as $\mathrm{TR}_\Omega\, u := [u]$, and observe that TR_Ω is surjective by definition. In this setting we identify the abstract Ω-trace of $u \in \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w)$ with the equivalence

class $[u]$ that contains u. The space $\mathscr{Y}(\Omega, w)$ is then endowed with the usual quotient norm

$$\|\mathrm{TR}_\Omega\, u\|_{\mathscr{Y}(\Omega, w)} = \|[u]\|_{\mathscr{Y}(\Omega, w)} := \inf\{\|u - z\|_{\mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w))} : z \in \mathscr{W}^{1,2}_0(\mathcal{C}, w))\}.$$

As before, we have $\mathrm{TR}_\Omega\, \mathscr{L}^{1,2}_{0,L}(\mathcal{C}, w) = \mathscr{Y}(\Omega, w)$. Note that $\mathscr{Y}(\Omega, w)$ is a Hilbert space, given that $\mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w)$ and $\mathscr{H}^{1,2}_0(\mathcal{C}, w)$ are also Hilbert spaces.

Identically as in Theorem 1, we argue the existence of the weighted harmonic extension operator

$$S : \mathrm{TR}_\Omega\, \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w) \to \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w), \qquad v \mapsto S(v) = u.$$

where u is the solution to

$$\text{minimize} \quad J(u) \quad \text{over} \quad \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w),$$
$$\text{subject to} \quad \mathrm{TR}_\Omega\, u = v,$$

for

$$J(u) := \frac{1}{2}\|u\|^2_{\mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w)} = \frac{1}{2}\int_\mathcal{C} w|\nabla u|^2 \, \mathrm{d}X.$$

The well-posedness of the map S allows us to establish a definition for the fractional Laplacian with spatially variable order.

Definition 3 Let $\mathscr{Y}(\Omega, w)'$ be the dual space of $\mathscr{Y}(\Omega, w)$. The operator

$$(-\Delta)^{s(\cdot)} : \mathscr{Y}(\Omega, w) \to \mathscr{Y}(\Omega, w)',$$

is determined as follows: for $v \in \mathscr{Y}(\Omega, w)$, then $(-\Delta)^{s(\cdot)}v \in \mathscr{Y}(\Omega, w)'$ is defined by

$$\langle(-\Delta)^{s(\cdot)}v, \mathrm{TR}_\Omega\, \psi\rangle_{\mathscr{Y}', \mathscr{Y}} = \int_\mathcal{C} w\, \nabla S(v) \cdot \nabla \psi \, \mathrm{d}X, \qquad \psi \in \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w).$$

Since Proposition 1 holds true with the usual changes, the operator is then well-defined and Theorem 2 is also proven mutatis mutandis: For a $h \in \mathscr{Y}(\Omega, w)'$, the equation

$$(-\Delta)^{s(\cdot)}v = h \quad \text{in } \Omega, \tag{7.4}$$

admits a unique solution $v \in \mathscr{Y}(\Omega, w)$ that is given by $v = \mathrm{TR}_\Omega\, u$, where u solves

$$\text{minimize} \quad \mathcal{J}(u) \quad \text{over} \quad \mathscr{H}^{1,2}_{0,L}(\mathcal{C}, w), \tag{7.5}$$

for

$$\mathcal{J}(u) := \frac{1}{2}\int_\mathcal{C} w\, |\nabla u|^2 \, \mathrm{d}X - \langle h, \mathrm{Tr}_\Omega\, u\rangle_{\mathscr{Y}', \mathscr{Y}}.$$

Although this approach seems equivalent to the one in Section 4, in this setting we have a more detailed representation of the elements $\mathscr{Y}(\Omega, w)$. In fact, within this approach, there exists an injection

$$I : \mathscr{Y}(\Omega, w) \to L^2(\Omega, \tilde{w}), \qquad u \mapsto I([u]) = \mathrm{tr}_\Omega u,$$

which is linear and bounded. Linearity follows directly, and boundedness follows given that for arbitrary $z \in \mathscr{W}_0^{1,2}(\mathcal{C}, w)$,

$$\|I([u])\|_{L^2(\Omega, \tilde{w})} = \|\text{tr}_\Omega u\|_{L^2(\Omega, \tilde{w})} = \|\text{tr}_\Omega (u - z)\|_{L^2(\Omega, \tilde{w})} \leq C\|u - z\|_{\mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w)},$$

where we have used the linearity of tr_Ω and that $\text{tr}_\Omega z = 0$, and then

$$\|I([u])\|_{L^2(\Omega, \tilde{w})} \leq C \inf_{y \in \mathscr{W}_0^{1,2}(\mathcal{C}, w)} \|u - z\|_{\mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w)} = C\|[u]\|_{\mathscr{Y}(\Omega, w)}.$$

In order to see that I is an injection, suppose that $I([u]) = 0$, then $\text{tr}_\Omega u = 0$ so that $u \in \mathscr{W}_0^{1,2}(\mathcal{C}, w)$, and the class $\mathscr{W}_0^{1,2}(\mathcal{C}, w)$ is the zero element of $\mathscr{Y}(\Omega, w)$. This identification allows us to consider I to be the identity, and identify the continuous embedding

$$\mathscr{Y}(\Omega, w) \hookrightarrow L^2(\Omega, \tilde{w}).$$

For a schematic relationship between the trace operators tr_Ω, Tr_Ω, the isomorphism φ and the embedding I, see Figure 7. An amenable consequence of this identification is given in Theorem 9, however in first place we address the reduction to case where $s(\cdot) = s \in (0, 1)$, a constant, where we obtain that H is recovered as the domain of $(-\Delta)^s$.
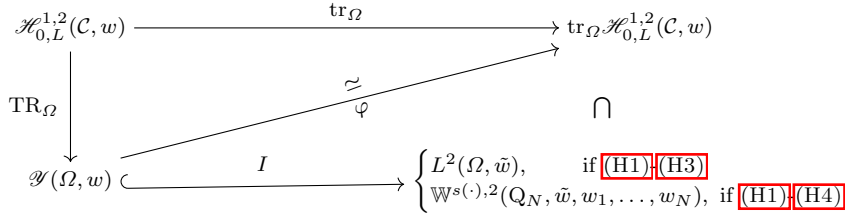
Fig. 1 Diagram relating the operators $\text{tr}_\Omega, \text{TR}_\Omega$, the isomorphism φ, and the operator I.

Theorem 8 *Let $s(\cdot) = s \in (0, 1)$ be constant and suppose that functions in $\mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w)$ satisfy a Poincaré inequality, then*

$$\text{tr}_\Omega \mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w) = H,$$

and therefore,

$$\mathscr{Y}(\Omega, w) \simeq H.$$

Proof Given that $s(\cdot) = s \in (0, 1)$ is constant, we have that G is constant, and hence $\mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w) = \mathscr{H}_{0,L}^{1,2}(\mathcal{C}, y^{1-2s})$. Additionally, by Remark 5, we have that $\text{tr}_\Omega \mathscr{H}_{0,L}^{1,2}(\mathcal{C}, y^{1-2s}) \subset H$. Then, there is only left to prove that for each $v \in H$ there exists a sequence $\{u_n\}$ in $C_f^\infty(\mathcal{C}_\Omega) \cap W^{1,2}(\mathcal{C}, y^{1-2s})$ convergent in the sense of $W^{1,2}(\mathcal{C}, y^{1-2s})$ to a $u \in \mathscr{H}_{0,L}^{1,2}(\mathcal{C}, y^{1-2s})$, and such that $\text{tr}_\Omega u = v$. We divide the proof into four steps for the sake of clarity.

Step 1: Let $v \in H$ be arbitrary. Since $H = \mathbb{H}_0^s(\Omega)$ for $s \in (0, 1/2)$ or $s \in (1/2, 1)$, and $H = \mathbb{H}_{00}^s(\Omega)$ for $s = 1/2$, it follows that $C_c^\infty(\Omega)$ is dense in H. Then, there exists a sequence $\{v_k\}$ in $C_c^\infty(\Omega)$ such that

$$v_k \to v \qquad \text{in } H,$$

as $k \to \infty$. We denote $v = \sum_{n=1}^\infty b_n \varphi_n$ and $v_k = \sum_{n=1}^\infty b_n^k \varphi_n$ to their spectral decomposition where $b_n^k \to b_n$ as $k \to \infty$. Further, define $u, u_k : \mathcal{C} \to \mathbb{R}$ by

$$u(x, y) = \sum_{n=1}^\infty b_n \varphi_n(x) g_n(y) \quad \text{and} \quad u_k(x, y) = \sum_{n=1}^\infty b_n^k \varphi_n(x) g_n(y),$$

where each g_n satisfies the Bessel equation:

$$g_n'' + \frac{1 - 2s}{y} g_n' - \lambda_n g_n = 0 \qquad \text{in} \quad (0, +\infty),$$
$$g_n(0) = 1,$$
$$g_n(+\infty) = 0.$$

By the construction of the proof in [9, Proposition 2.1], we have that $u, u_n \in W^{1,2}(\mathcal{C}, y^{1-2s})$, and

$$\int_0^\infty \int_\Omega y^{1-2s} |\nabla u(x, y) - \nabla u_k(x, y)|^2 \mathrm{d}x \mathrm{d}y = c_{N,s} \sum_{k=1}^\infty (b_n - b_n^k)^2 \lambda_n^s = c_{N,s} \|v - v_k\|_H^2,$$

and thus

$$u_k \to u \qquad \text{in } W^{1,2}(\mathcal{C}, y^{1-2s}), \tag{7.6}$$

as $k \to \infty$. Note that since v_k has compact support, the support of u_k is uniformly away from $\partial_L \mathcal{C}$.

Step 2: For $\tau \geq 1$ and $0 < \sigma < 1$, we consider a smooth non-increasing function $\eta_\tau : \mathbb{R}^+ \to [0, 1]$ such that:

$$\eta_\tau(y) = 1 \quad \text{if} \quad 0 < y < \tau - \sigma, \qquad\qquad \eta_\tau(y) = 0 \quad \text{if} \quad y > \tau,$$

and notice that the function $u_{k,\tau}(x, y) := \eta_\tau(y) u_k(x, y)$ belongs to $W^{1,2}(\mathcal{C}, y^{1-2s})$. By direct calculation we have that

$$u_{k,\tau} \to u_n \qquad \text{in } W^{1,2}(\mathcal{C}, y^{1-2s}), \tag{7.7}$$

as $\tau \to \infty$.

Step 3: For $0 < \varepsilon \ll 1$ and $\tau' > \tau + \varepsilon$, we consider the shifted cylinder

$$\mathcal{C}_\varepsilon^{\tau'} := \{(x, y - \varepsilon) : (x, y) \in \mathcal{C}^{\tau'}\},$$

and the weighted space $W^{1,2}(\mathcal{C}_\epsilon^{\tau'}, \rho)$, where

$$\rho(x, y) = \begin{cases} y^{1-2s} & \text{if} \quad 0 < y < \tau' - \varepsilon, \\ (-y)^{1-2s} & \text{if} \ -\varepsilon < y < 0. \end{cases}$$

Further, let $\hat{u}_{k,\tau} \in W^{1,2}(\mathcal{C}_\varepsilon^{\tau'}, \rho)$ defined by reflection as

$$\hat{u}_{k,\tau}(x,y) = \begin{cases} u_{k,\tau}(x,y) & \text{if} \quad 0 < y < \tau' - \varepsilon, \\ u_{k,\tau}(x,-y) & \text{if} \quad -\varepsilon < y \le 0, \end{cases}$$

and note that $\rho \in A_2(\mathcal{C}_\varepsilon^{\tau'})$, i.e.,

$$\sup_{B \subset \mathcal{C}_\varepsilon^{\tau'}} \left(\frac{1}{|B|} \int_B \rho \, \mathrm{d}X \right) \left(\frac{1}{|B|} \int_B \rho^{-1} \, \mathrm{d}X \right) < +\infty,$$

for all squares $B \subset \mathcal{C}_\varepsilon^{\tau'}$.

Let L_r be the usual mollifier operator, i.e.,

$$(L_r f)(x) = \frac{1}{r^{N+1}} \int_{\mathbb{R}^{N+1}} \omega\left(\frac{x-z}{r} \right) f(z) \, \mathrm{d}z,$$

where $\omega : \mathbb{R}^{N+1} \to [0,+\infty)$ belongs to $C^\infty(\mathbb{R}^{N+1})$, $\operatorname{supp}\omega \subset \overline{B(0,1)}$, and $\int_{\mathbb{R}^{N+1}} \omega = 1$. Since $\rho \in A_2(\mathcal{C}_\varepsilon^{\tau'})$ and since $\mathcal{C}_\varepsilon^{\tau'}$ is bounded and with Lipschitz boundary, it follows that for $f \in W^{1,2}(\mathcal{C}_\varepsilon^{\tau'}, \rho)$,

$$L_r f \to f \qquad \text{in} \quad W^{1,2}(D, y^{1-2s}),$$

for any $D \subset\subset \mathcal{C}_\varepsilon^{\tau'}$; see [12]. Given that the support of $u_{k,\tau}$ is uniformly away from $\partial_L \mathcal{C}$, and that $u_{k,\tau} = 0$ if $\tau < y$, it follows that

$$L_r \hat{u}_{k,\tau} \to u_{k,\tau} \qquad \text{in} \quad W^{1,2}(\mathcal{C}, y^{1-2s}), \tag{7.8}$$

as $r \to \infty$. Note in addition, that for sufficiently large $r > 0$, we have

$$L_r \hat{u}_{k,\tau} \in C_c^\infty(\mathcal{C}_\Omega) \subset \mathscr{H}_{0,L}^{1,2}(\mathcal{C}, y^{1-2s}).$$

Step 4: In view of (7.6), (7.7), and (7.8), by appropriately selecting a sequence $\{(r_i, k_i, \tau_i)\}_{i=1}^\infty$, we observe

$$\tilde{u}_i := L_{r_i} \hat{u}_{k_i,\tau_i} \to u \qquad \text{in} \quad W^{1,2}(\mathcal{C}, y^{1-2s}),$$

as $i \to \infty$, so that $u \in \mathscr{H}_{0,L}^{1,2}(\mathcal{C}, y^{1-2s})$. In particular,

$$\operatorname{tr}_\Omega \tilde{u}_i \to v \qquad \text{in} \quad L^2(\Omega),$$

and $v = \operatorname{tr}_\Omega u$; the result is then proven.

Next we can establish the well-posedness of the elliptic equation of interest.

Theorem 9 *Assume that* (H1)*,* (H2)*, and* (H3) *hold true, and functions in* $\mathscr{H}_{0,L}^{1,2}(\mathcal{C}, w)$ *satisfy the Poincaré inequality. For every* $h \in L^2(\Omega, \tilde{w})$*, the equation*

$$(-\Delta)^{s(\cdot)} v = h, \tag{7.9}$$

admits a unique solution $v \in \mathscr{Y}(\Omega, w) \subset L^2(\Omega, \tilde{w})$.

Proof The conclusion follows from the existence and uniqueness of solution to the same problem with right hand side in $\mathscr{Y}(\Omega, w)$ since we identify $L^2(\Omega, \tilde{w})' \simeq L^2(\Omega, \tilde{w})$, so that $L^2(\Omega, \tilde{w}) \subset \mathscr{Y}(\Omega, w)$ by means of $I' : L^2(\Omega, \tilde{w}) \to \mathscr{Y}(\Omega, w)'$.

The result above can be refined in terms of regularity if in addition we observe (H4), and consider $\Omega = Q_N$. In this case, the injection I is given as

$$I : \mathscr{Y}(Q_N, w) \to \mathbb{W}^{s(\cdot), 2}(Q_N, \tilde{w}, w_1, \ldots, w_N),$$

leading to our last theorem, whose proof is obtained as for Theorem 9.

Theorem 10 *In addition to the hypotheses of Theorem 9, consider $\Omega = Q_N$ and assume that* (H4) *holds true. Then, for every*

$$h \in \mathbb{W}^{s(\cdot), 2}(Q_N, \tilde{w}, w_1, \ldots, w_N)',$$

the equation (7.9) *admits a unique solution*

$$v \in \mathscr{Y}(\Omega, w) \subset \mathbb{W}^{s, 2}(Q_N, \tilde{w}, w_1, \ldots, w_N).$$

The problem in the truncated cylinder $\mathcal{C}^\tau$ is treated identically, and Theorem 9 and Theorem 10 still hold true under the obvious changes.

8 Conclusions and open questions

This paper continues the program initiated in [2] and provides a rigorous definition of the variable order fractional Laplacian. The proposed theoretical framework enables solutions to the Poisson equation on bounded Lipschitz domains Ω. The techniques introduced in the paper are new and none of the existing works applies to our setting. However, the existing setting, where $s(\cdot)$ is a constant, can be recovered from our proofs as a special case.

The following are open questions and topics for future research:

- The study of $-\Delta^{s(\cdot)}$ as regularizer in optimization problems, i.e.,

$$\min_u J(u) + \gamma R(u) \qquad \text{with} \quad R(u) = \langle (-\Delta)^{s(\cdot)} u, u \rangle_{\mathscr{Y}', \mathscr{Y}},$$

  and the optimal selection of $s(\cdot)$ in a bilevel framework.
- The extension to more general settings of the Poincaré inequality type result presented in Section 6.
- The surjectivity of the new trace operator is still open (cf. Remark 7).
- We have introduced Sobolev spaces with $s(\cdot)$-dependent weights for the extension problem and $s(\cdot)$-dependent differentiability for the space on Ω. New approaches need to be established to prove additional regularity of solutions to $(-\Delta)^{s(\cdot)} u = h$ in these Sobolev spaces.
- Extensions to parabolic, semilinear and obstacle type problems are of interest.

- The authors in [2] proposed a numerical method for the truncated problem. But the numerical analysis of this problem is completely open. Also, convergence of the truncated solution to the full solution is of interest as well.
- Optimal control problems with variable order PDEs as constraints.

Conflict of interest

The authors declare that they have no conflict of interest.

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