

The inviscid limit of Navier-Stokes equations for locally near boundary analytic data on an exterior circular domain

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November 17, 2023

Abstract

In their classical work [34], Caffisch and Sammartino established the inviscid limit and boundary layer expansions of vanishing viscosity solutions to the incompressible Navier-Stokes equations for analytic data on a half-space. The extension to an exterior domain faces a fundamental difficulty that the corresponding linear semigroup may not be contractive in analytic spaces as was the case on the half-space [32]. In this paper, we resolve this open problem for a much larger class of initial data. The resolution is due to the fact that it suffices to propagate solutions that are analytic only near the boundary, following the framework developed in the recent works that involve the boundary vorticity formulation, the analyticity estimates on the Green function, the adapted geodesic coordinates near a boundary, and the Sobolev-analytic iterative scheme.

1 Introduction

In this paper, we consider the Navier-Stokes equations with small viscosity $\nu > 0$

$$\begin{aligned}\partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p^\nu &= \nu \Delta u^\nu, \\ \nabla \cdot u^\nu &= 0, \\ u^\nu|_{\partial\Omega} &= 0,\end{aligned}\tag{1.1}$$

on an exterior circular domain Ω in $\mathbb{R}^2$, modeling the dynamics of an incompressible fluid around a solid body at a sufficiently high Reynolds number. Of great physical and mathematical interest is the asymptotic behavior of solutions to (1.1) in the small viscosity limit. When $\nu = 0$, (1.1) reduces to the Euler equations

$$\partial_t u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = 0, \quad \nabla \cdot u^0 = 0\tag{1.2}$$

with the non-penetration boundary condition $u^0 \cdot n = 0$ on the boundary $\partial\Omega$. Thus, in the limit when $\nu \rightarrow 0$, one would formally expect the solutions of the Navier-Stokes equations to converge to u^0 in $L^2(\Omega)$ uniformly for a short time, however it remains elusive whether this may be the

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case. Boundary layers appear due to the discrepancy between the boundary conditions in (1.1) and in the limiting model (1.2), generating arbitrarily large vorticity near the boundary. Kato in his celebrated work [20] shows that the inviscid limit, i.e. the strong convergence of solutions in the natural energy norm, holds if and only if

$$\nu \int_0^T \int_{\{d(x, \partial\Omega) \lesssim \nu\}} |\nabla u^\nu(t)|^2 dx dt \rightarrow 0 \quad \text{as } \nu \rightarrow 0, \quad (1.3)$$

which implies that the vorticity needs to be controlled quantitatively near the boundary. For general smooth initial data, vorticity can however be very unstable on the boundary that could generate multi-layer solutions at different smaller scales [15, 18], leading to a larger and larger vorticity than expected, and the inviscid limit problem is therefore unlikely to hold. See, for instance, [3, 6, 28] and the references therein for further discussion. In this paper, we consider smooth data that are analytic locally near the boundary.

1.1 Previous results

When Ω is the half-space: In their classical work, Sammartino-Caffisch [34] established the inviscid limit and Prandtl's boundary layer expansions for analytic data: namely,

$$u^\nu(t, x, y) = u^0(t, x, y) + u^P\left(t, x, \frac{y}{\sqrt{\nu}}\right) + o(1)_{L^\infty}, \quad (1.4)$$

where the error term $o(1)_{L^\infty}$ is in fact of order $\sqrt{\nu}$ for such analytic data and thus vanishing in the inviscid limit. The result is extended by Maekawa [26] for Sobolev data whose vorticity is compactly supported away from the boundary. Unlike [34], Maekawa constructed his solution via the vorticity formulation with a nonlocal boundary condition, which reveals more explicitly the localized interaction between boundary layers and interior solutions. It was this vorticity formulation that leads to a more user-friendly direct proof of the inviscid limit given in [32] by the authors of the present work, where we in addition devise analytic boundary layer norms, adapted from those introduced in [18], that capture precisely the unbounded vorticity near the boundary. Building upon [32, 26], Kukavica-Vicol-Wang [22] introduced suitable Sobolev-analytic norms that allow to establish the inviscid limit for data that are analytic only near the boundary; see also [37] for a similar result in 3D, [21] for the validity of (1.4) for such data, and [11, 12] for interesting stability results for data in some Gevrey classes. For Sobolev data, in strong contrast with the analytic case, the Prandtl Ansatz (1.4) is false due to counter-examples given in [15, 18, 17].

When Ω is a bounded domain: There are only few results in the literature that study the inviscid limit problem in fluid domains with a curved boundary. We mention a recent work [14] that studies boundary layers in a suitable linearized flow in a general 3D smooth domain and [36] which establishes a Prandtl asymptotic expansion in domain with a curved boundary. Very recently, building upon the recent advances including the vorticity formulation revived in [26], the direct proof via the Green function approach developed in [32], and the Sobolev-analytic norms introduced in [22], Bardos-Nguyen-Nguyen-Titi [2] prove the inviscid limit for data that are analytic only near the boundary in a 2D bounded domain.

When Ω is an exterior domain: In the polar coordinates, the exterior disk can be written as $\Omega =$

$(r, \theta) \in (1, \infty) \times \mathbb{T}$. Due to the curvature effects and the unbounded values of r , which can be from 1 to ∞ , the inviscid limit of the Navier-Stokes on this domain was a major open problem since 1997, even for analytic data. In [4], Caffisch and Sammartino initiated a program to obtain the inviscid limit for analytic data in an exterior circular domain, expecting that the techniques for the half-space will carry over to the case of the exterior of a disk. In [23], Lombardo, Caffisch and Sammartino were able to perform an asymptotic analysis for the *linearized* Navier-Stokes under the same domain, using Bessel functions and the Weber transform. They also point out that the inviscid limit problem for the fully nonlinear Navier-Stokes remained open. So far, there have been only a few works dealing with the inviscid limit problem with both curvature effects and large $r > 1$, see, for instance, [19], in the stationary setting. We refer the readers to Section 1.4 where we explain the fundamental difficulty and our main strategy to establish the main result.

When Ω is the whole space $\mathbb{R}^2$: For smooth data, the inviscid limit has been addressed in [35]. However, challenges remain with irregular initial data like vortex patches, point vortices, vortex rings and vortex sheets ([7, 8, 29, 5, 9, 33, 10]).

In the presence of symmetric assumptions on the domains or the solutions, we refer the readers to [24, 13, 25, 31, 30].

1.2 Boundary vorticity formulation

We consider the Navier-Stokes equations posed on the following circular exterior domain

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 > 1\},$$

in which for sake of presentation the radius is taken to be one. We shall work with the standard polar coordinates $(x_1, x_2) = (r \cos \theta, r \sin \theta)$ for $(r, \theta) \in [1, \infty) \times \mathbb{T}$. Let $e_r = (\cos \theta, \sin \theta)$ and $e_\theta = (-\sin \theta, \cos \theta)$ be the orthogonal frame, and set $(a, b)^\perp = (b, -a)$. We note that

$$\nabla = e_r \partial_r + \frac{1}{r} e_\theta \partial_\theta, \quad \Delta = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2,$$

Thus, we write

$$u = u_r e_r + u_\theta e_\theta, \quad \omega = \nabla^\perp \cdot u = \frac{1}{r} \partial_\theta u_r - \frac{1}{r} \partial_r (r u_\theta)$$

for velocity and vorticity of the fluid. The Navier-Stokes equations (1.1) can be written in the vorticity formulation as follows:

$$\partial_t \omega - \nu \Delta_{r, \theta} \omega = -u_r \partial_r \omega - \frac{1}{r} u_\theta \partial_\theta \omega \quad (1.5)$$

on $[1, \infty) \times \mathbb{T}$, in which $\Delta_{r, \theta} = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2$. Making use of the incompressibility condition, we introduce the stream function $\psi = \psi(r, \theta)$ defined through $u = \nabla^\perp \psi$, or equivalently

$$u_r = \frac{1}{r} \partial_\theta \psi, \quad u_\theta = -\partial_r \psi. \quad (1.6)$$

By definition, the stream function solves the elliptic problem

$$\begin{cases} \Delta_{r, \theta} \psi = \omega \\ \psi|_{r=1} = 0 \end{cases} \quad (1.7)$$

whose solutions can be constructed explicitly through the Green function; see Section 4

Therefore, the Navier-Stokes equation problem (1.1) reduces to study the scalar vorticity equation (1.5) on $[1, \infty) \times \mathbb{T}$, where the velocity is constructed through the Biot-Savart law (1.6)-(1.7). As for the no-slip boundary condition, $u_r = 0$ follows from the condition $\psi = 0$ on the boundary, while $u_\theta = 0$ is a direct consequence of the following imposed condition

$$\partial_t u_\theta = 0$$

from which we derive the boundary condition on vorticity ω . This formulation was introduced and developed in [1, 26]. See also [32, 2]. Indeed, by construction, we compute

$$0 = \partial_t u_\theta = -\partial_r \Delta^{-1} \partial_t \omega = -\partial_r [\Delta^{-1} (\nu \Delta \omega - u \cdot \nabla \omega)] \quad (1.8)$$

on the boundary. This yields the following boundary condition for vorticity

$$\nu(\partial_r + N)\omega|_{r=1} = [\partial_r \Delta^{-1} (u \cdot \nabla \omega)]|_{r=1} \quad (1.9)$$

where N denotes the Dirichlet-Neumann operator on Ω , which will be detailed in Section 2.1

1.3 Main result

Our main result is to establish a uniform bound on the vorticity and the inviscid limit of solutions to the Navier-Stokes problems for initial data whose vorticity is locally analytic near the boundary $r = 1$. Precisely,

Definition 1.1. Let $\delta_0 > 0$ and $p \geq 1$. An L^p function $f(r)$ defined on $[1, 1 + \delta_0]$ is said to be locally analytic near the boundary $r = 1$ if it can be extended analytically to the pencil-like complex domain

$$R_\rho = \left\{ r \in \mathbb{C} : 1 \leq \Re r \leq 1 + \delta_0, \quad |\Im r| \leq \rho(\Re r - 1) \right\}$$

for some positive analyticity radius ρ with a finite norm $\|f\|_{L_\rho^p} = \sup_{0 \leq \eta < \rho} \|f\|_{L^p(\partial R_\eta)}$.

Note that a locally near boundary analytic function needs not to be analytic on the boundary, but only has bounded derivatives $(r - 1)\partial_r$. Our main result is stated as follows:

Theorem 1.2. Consider the vorticity equation (1.5) on $[1, \infty) \times \mathbb{T}$ with the boundary condition (1.9) and the Biot-Savart law (1.6)-(1.7). Assume that initial vorticity $\omega_0^\nu(r, \theta)$ has Sobolev regularity $r^2 \omega_0^\nu \in H^5([1, \infty) \times \mathbb{T})$, and its Fourier coefficients $\omega_{0,n}^\nu(r)$ with respect to variable θ are locally analytic near the boundary and satisfy

$$\sum_{n \in \mathbb{Z}} e^{\varepsilon_0 |n|} \|\omega_{0,n}^\nu(r)\|_{L_{\rho_0}^1} < \infty \quad (1.10)$$

uniformly in ν , for some positive constants ε_0, ρ_0 . Then, there is a positive time T , independent of ν , so that the Navier-Stokes vorticity satisfies

$$\|\omega^\nu(t)\|_{L^\infty(\partial\Omega)} \leq C_0(\nu t)^{-1/2} \quad (1.11)$$

for $t \in (0, T]$, and the inviscid limit holds: that is, there exists a unique limiting solution u^0 that solves the corresponding solution to Euler equations (1.2) so that

$$\sup_{0 \leq t \leq T} \|u^\nu - u^0\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \nu \rightarrow 0. \quad (1.12)$$

Remark 1.3. If we replace the assumption (1.10) by a stronger assumption

$$\sum_{n \in \mathbb{Z}} e^{\varepsilon_0 |n|} \|\omega_{0,n}^\nu(r)\|_{L_{\rho_0}^\infty} < \infty$$

then (1.11) can be improved to $\sup_{0 \leq t \leq T} \|\omega^\nu(t)\|_{L^\infty(\partial\Omega)} \leq C_0 \nu^{-1/2}$.

The inviscid limit is a direct consequence of the boundary vorticity estimates (1.11), which is optimal in view of the boundary layer expansion (1.4) as predicted by Prandtl and justified for analytic data [34]. The assumption (1.10) holds in particular for data whose vorticity vanishes near the boundary, and the theorem thus recovers the result by Maekawa [26] to the case of exterior circular domains. We stress that the near boundary analyticity assumption (1.10) is necessary for the vorticity bound (1.11) to hold, since otherwise the presence of near boundary high frequency will generate boundary viscous sublayers [18], whose vorticity is proven to reach order $\nu^{-3/4}$, much larger than the Prandtl's classical prediction of order $\nu^{-1/2}$. In general, much worse and more complex structure of boundary vorticity is expected; see [15, 16, 18, 17] for further discussion.

1.4 Difficulties and main ideas

Let us discuss the difficulties in proving the inviscid limit when the domain is an exterior circular disk. In view of the previous works [32, 2], there are several difficulties that one has to overcome in the present setting. Namely, the framework relies on the semigroup of the linear Stokes problem, treating the nonlinearity as a perturbation in the Duhamel representation. For the nonlinear iterative scheme to work, it is crucial that the semigroup is contractive in the function spaces under consideration, namely analytic spaces; see Proposition 3.1* in [32]. However, the contraction in analytic spaces is open for the linear Stokes problem on the exterior domain. Precisely, we are led to study the following Stokes problem

$$\begin{cases} \partial_t \omega - \nu \left(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \right) \omega = 0 \\ (\partial_r + |\partial_\theta|) \omega|_{r=1} = 0 \end{cases} \quad (1.13)$$

whose resolvent kernel and Green kernel can be easily constructed. Deriving the analytic estimates on the Green function and the semigroup uniformly both in time and in the small viscosity limit however appears an impossible task. Indeed, following [32] and working with the Laplace-Fourier transform variables (ζ, n) associated with (t, θ) , the Green kernel for the resolvent problem is of the form

$$G_\zeta(r, r') = \frac{1}{W(I_n, K_n)(\mu r')} \frac{I_{n-1}(\mu)}{K_{n-1}(\mu)} K_n(\mu r) K_n(\mu r') + \begin{cases} \frac{I_n(\mu r) K_n(\mu r')}{W(I_n, K_n)(\mu r')} & \text{if } r < r', \\ \frac{I_n(\mu r') K_n(\mu r)}{W(I_n, K_n)(\mu r')} & \text{if } r > r', \end{cases}$$

*We wish to point out a misprint in [32] Proposition 3.1] where the third estimate on the trace semigroup in the boundary layer norm should read

$$\|\Gamma(\nu(t-s))g\|_{\rho, \sigma, \delta(t), k} \lesssim \sqrt{\frac{t}{t-s}} \|g\|_{\rho, k} + \sqrt{\nu} \|g\|_{\rho, k+1}.$$

Namely, the last term with one loss of derivatives on the boundary was missing! Note however this is harmless in [32], since the estimates were used only to propagate the boundary layer norms *after closing* the nonlinear iteration with L^1 analytic norms where no loss of derivatives is present on the trace estimates; see the analysis in Section 4.2 of that same paper.

with $\mu = \sqrt{\frac{\zeta}{\nu}}$, where the functions $K_n(z)$ and $I_n(z)$ are modified Bessel functions with complex value $z \in \mathbb{C}$ (e.g., [27]), with $W(I_n, K_n)$ being the Wronskian determinant. The temporal Green function is then defined by taking the inverse Laplace transform in t of the kernel $G_\zeta(r, r')$. Unfortunately, the available pointwise bounds and asymptotic expansions of the modified Bessel functions are given only in the regime for

- fixed n , large r
- or fixed r , large n ,

but not when both n, r are sufficiently large and ν is sufficiently small. As a consequence, the propagation of uniform semigroup estimates on analytic spaces remains open, and therefore the pointwise Green function approach developed in [32] does not apply directly.

We overcome the issue by working with functions that are required to be analytic only near the boundary, see Theorem 3.1. Effectively, this only requires analytic estimates of the Green function near the boundary, which is available from the half-space result [32]. Precisely, close to the boundary $r = 1$, we write

$$\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 = (\partial_r^2 + \partial_\theta^2) + \frac{1}{r}\partial_r + \left(\frac{1}{r^2} - 1\right)\partial_\theta^2$$

and using the half-space Green kernel for the operator $\partial_r^2 + \partial_\theta^2$, treating the remaining terms as a perturbation. Importantly, we note that the last term experiences a loss of two derivatives and is thus a perturbation only when r is sufficiently close to 1. See Section 3 where we establish the semigroup estimates for the Stokes problem in Sobolev-analytic spaces.

Unlike the treatment in [2], we need to estimate the solution in the unbounded region and therefore a careful norm with suitable decay is needed. Our vorticity $\omega(r, \theta)$ decays like r^{-2} away from the boundary.

Lastly, we refer the readers to the work by Lombardo-Caffisch-Sammartino [23], where they establish the asymptotic Prandtl expansion for *linearized* Navier-Stokes in Sobolev norms. In the spirit of the main difficulties we mentioned above for the full nonlinear Navier-Stokes equations, their Proposition 3.3 of their work establishes the propagation of the semigroup of Stokes *on the whole exterior domain* only in *Sobolev spaces*. As mentioned above, for the framework of [32, 34] to work, propagation of analytic norms for solutions to the Stokes problem is crucial to treat the loss of derivatives in the nonlinear term. Since Proposition 3.3 in [23] is still open for analytic norms, it is not known to propagate the analyticity of the nonlinear term. Hence, controlling the remainder in the the Prandtl asymptotic expansion in *analytic spaces* is still an open problem. In this work, we resolve this problem by working with data that are analytic only near the boundary. We expect that the Prandtl expansion (1.4) holds for the same class of initial data in our current paper, by combining the analysis of this work and the work on the half-space [21].

2 Scaled equations and locally analytic spaces

2.1 Navier-Stokes equations in the rescaled variables

To take advantage of localization near the boundary, we introduce a change of variables

$$x = \lambda^{-1}\theta, \quad y = \lambda^{-1}(r - 1), \quad \tau = \lambda^{-2}t$$

for some small parameter $\lambda > 0$, and define the function w such that

$$w(\tau, x, y) = \omega(t, \theta, r) = \omega(\lambda^2 \tau, \lambda x, 1 + \lambda y) \quad (2.1)$$

for $x \in \mathbb{T}_{2\pi/\lambda}$ and $y \in \mathbb{R}_+$. By a direct calculation, we have

$$\begin{aligned} \Delta_{r,\theta} &= \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \\ &= \lambda^{-2} (\partial_x^2 + \partial_y^2 + \lambda a(y) \partial_y + \lambda b(y) \partial_x^2) \\ &= \lambda^{-2} (\Delta_{x,y} + \lambda L) \end{aligned}$$

where $\Delta_{x,y} = \partial_x^2 + \partial_y^2$ (and hereafter, we simply write $\Delta = \Delta_{x,y}$),

$$L = a(y) \partial_y + b(y) \partial_x^2, \quad a(y) = \frac{1}{1 + \lambda y}, \quad b(y) = \frac{y(2 + \lambda y)}{(1 + \lambda y)^2}. \quad (2.2)$$

From (1.5), the scaled vorticity w satisfies

$$(\partial_\tau - \nu \Delta)w = \nu \lambda L w + B(\psi, w) \quad (2.3)$$

where

$$B(\psi, w) = -a(y) \nabla^\perp \psi \cdot \nabla w, \quad \nabla^\perp = (\partial_y, -\partial_x). \quad (2.4)$$

Similarly, abusing the same notation, the scaled stream function ψ solves

$$\begin{cases} (\Delta + \lambda L)\psi = \lambda^2 w, \\ \psi|_{y=0} = 0. \end{cases} \quad (2.5)$$

We next derive a boundary condition for w . As mentioned in Section 1.2, we impose $\partial_\tau u_\theta = 0$, which gives $\partial_\tau \partial_y \psi|_{y=0} = 0$ and so

$$\partial_y (\Delta + \lambda L)^{-1} \partial_\tau w|_{y=0} = 0.$$

Using the vorticity equation (2.3), we get

$$\partial_y (\Delta + \lambda L)^{-1} (\nu (\Delta + \lambda L)w + B(\psi, w))|_{y=0} = 0. \quad (2.6)$$

Let w^* solves

$$(\Delta + \lambda L)w^* = 0, \quad w^*|_{y=0} = w|_{y=0}. \quad (2.7)$$

Then (2.6) becomes

$$\nu \partial_y (w - w^*)|_{y=0} = -\partial_y (\Delta + \lambda L)^{-1} (B(\psi, w))|_{y=0}$$

Defining $Nw = -\partial_y w^*|_{y=0}$, which is the classical Dirichlet-to-Neumann operator, we obtain the boundary condition for the vorticity

$$\nu (\partial_y + N)w|_{y=0} = -\partial_y (\Delta + \lambda L)^{-1} (B(\psi, w))|_{y=0} \quad (2.8)$$

In this paper, for any function f depending on $x \in \mathbb{T}_{2\pi/\lambda}$, we denote f_α to be the Fourier coefficient of f in the frequency $\alpha \in \lambda \mathbb{Z}$, and f_n to be the Fourier coefficient of f in the original variable $\theta \in \mathbb{T}_{2\pi}$ where $n \in \mathbb{Z}$. We prove the following lemma regarding the Dirichlet-Neumann operator in the new variables:

Lemma 2.1. *The operator Nw_α can be written as*

$$Nw_\alpha = |\alpha|w_\alpha(0) + \lambda \int_0^\infty \left(w_\alpha(0)L_\alpha(e^{-|\alpha|y}) + L_\alpha\tilde{w}_\alpha^* \right) dy$$

where $\tilde{w}_\alpha^*$ solves the elliptic problem

$$\begin{cases} (\partial_y^2 - \alpha^2)\tilde{w}_\alpha^* = -\lambda L_\alpha(w_\alpha(0)e^{-|\alpha|y}) - \lambda L_\alpha\tilde{w}_\alpha^* \\ \tilde{w}_\alpha^*|_{y=0} = 0 \end{cases} \quad (2.9)$$

and $L_\alpha = a(y)\partial_y - \alpha^2b(y)$ is the linear operator acting on the frequency α of L .

Proof. We recall the definition of w^* in (2.9). Taking Fourier in x , we obtain

$$(\partial_y^2 - \alpha^2)w_\alpha^* + \lambda L_\alpha w_\alpha^* = 0, \quad w_\alpha^* = w_\alpha(0).$$

Let $\tilde{w}_\alpha^* = w_\alpha^*(y) - w_\alpha(0)e^{-|\alpha|y}$, then

$$Nw_\alpha = -\partial_y w_\alpha^*|_{y=0} = -\partial_y \left(\tilde{w}_\alpha^* + w_\alpha(0)e^{-|\alpha|y} \right)|_{y=0} = -\partial_y \tilde{w}_\alpha^*(0) + |\alpha|w_\alpha(0).$$

we have

$$(\partial_y^2 - \alpha^2)\tilde{w}_\alpha^* = -\lambda L_\alpha \left(w_\alpha(0)e^{-|\alpha|y} \right) - \lambda L_\alpha \tilde{w}_\alpha^*, \quad \tilde{w}_\alpha^*|_{y=0} = 0.$$

By a direct calculation, we have

$$\partial_y \tilde{w}_\alpha^*(0) = \int_0^\infty e^{-|\alpha|y} \lambda \left(w_\alpha(0)L_\alpha(e^{-|\alpha|y}) + L_\alpha\tilde{w}_\alpha^* \right) dy.$$

The proof is complete. $\square$

2.2 The half-space problem

To summarize, we have reduced the Navier-Stokes equations on the exterior disk to the following problem on the half-line $y \geq 0$: for each spatial frequency $\alpha \in \lambda\mathbb{Z}$,

$$(\partial_\tau - \nu\Delta_\alpha)w = \nu\lambda L_\alpha w + B_\alpha(\psi, w) \quad (2.10)$$

with notation $\Delta_\alpha = \partial_y^2 - \alpha^2$, $L_\alpha = a(y)\partial_y - \alpha^2b(y)$, and the following nonlocal boundary condition

$$\begin{aligned} \nu(\partial_y + |\alpha|)w_\alpha|_{y=0} = & -\nu\lambda \int_0^\infty e^{-|\alpha|y} \left(w_\alpha(0)L_\alpha(e^{-|\alpha|y}) + L_\alpha\tilde{w}_\alpha^* \right) dy \\ & - \partial_y(\Delta_\alpha + \lambda L_\alpha)^{-1} (B_\alpha(\psi, w))|_{y=0} \end{aligned} \quad (2.11)$$

where $B(\psi, w)$ and ψ are defined as in (2.4)-(2.5).

2.3 Near boundary analytic spaces

In this section, we introduce the near boundary analytic norms to control the near boundary analyticity and the Sobolev regularity of vorticity. These norms are an adaptation from those that were introduced and developed in [34, 32, 2, 22].

Precisely, let $\delta_0 > 0$ be the size of the analytic domain for our solution near the boundary. Throughout the paper, we fix $\rho_0 \geq \delta_0$, and take $\rho \in (0, \rho_0)$. We define the complex domain

$$\Omega_\rho = \left\{ y \in \mathbb{C} : \begin{array}{l} 0 \leq \Re y \leq \delta_0, \quad |\Im y| \leq \rho \Re y \\ \cup \left\{ y \in \mathbb{C} : \delta_0 \leq \Re y \leq \delta_0 + \rho, \quad |\Im y| \leq \delta_0 + \rho - \Re y \right\} \end{array} \right\}.$$

We note that the domain Ω_ρ only contains y with $0 \leq \Re y \leq \delta_0 + \rho$. For a complex valued function f defined on Ω_ρ , let

$$\|f\|_{L_\rho^1} = \sup_{0 \leq \eta < \rho} \|f\|_{L^1(\partial\Omega_\eta)}, \quad \|f\|_{L_\rho^\infty} = \sup_{0 \leq \eta < \rho} \|f\|_{L^\infty(\partial\Omega_\eta)}$$

where the integration is taken over the two directed paths along the boundary of the domain Ω_η . Now for an analytic function $f(x, y)$ defined on $(x, y) \in \mathbb{T}_{2\pi/\lambda} \times \Omega_\rho$, we define

$$\begin{aligned} \|f\|_{\mathcal{L}_\rho^1} &= \sum_{\alpha \in \lambda\mathbb{Z}} \left\| e^{\varepsilon_0(\delta_0 + \rho - \Re y)|\alpha|} f_\alpha \right\|_{L_\rho^1}, \\ \|f\|_{\mathcal{L}_\rho^\infty} &= \sum_{\alpha \in \lambda\mathbb{Z}} \left\| e^{\varepsilon_0(\delta_0 + \rho - \Re y)|\alpha|} f_\alpha \right\|_{L_\rho^\infty}. \end{aligned} \tag{2.12}$$

The function spaces $\mathcal{L}_\rho^1$ and $\mathcal{L}_\rho^\infty$ are to control the scaled vorticity and velocity, respectively. We stress that the analyticity weight is identically zero on $\Re y \geq \delta_0 + \rho$. For convenience, we also introduce the following analytic norms

$$\|f\|_{\mathcal{W}_\rho^{k,p}} = \sum_{i+j \leq k} \|\partial_x^i (y \partial_y)^j f\|_{\mathcal{L}_\rho^p} \tag{2.13}$$

for $k \geq 0$ and $p = 1, \infty$. The above definition also applies for a function g defined on the domain $\mathbb{T}_{2\pi/\lambda}$. Namely,

$$\|g\|_{\mathcal{H}_\rho^k} = \sum_{\alpha \in \lambda\mathbb{Z}} |\alpha|^k e^{\varepsilon_0(\delta_0 + \rho)|\alpha|} |g_\alpha|. \tag{2.14}$$

For convenience, we also write

$$\|D_{x,y}^k f\|_X = \sum_{i+j \leq k} \|\partial_x^i \partial_y^j f\|_X$$

where X is a function space. We recall the following simple algebra.

Lemma 2.2. *There hold*

$$\|fg\|_{\mathcal{L}_\rho^1} \leq \|f\|_{\mathcal{L}_\rho^\infty} \|g\|_{\mathcal{L}_\rho^1} \tag{2.15}$$

and for any $0 < \rho' < \rho$,

$$\|\partial_x f\|_{\mathcal{L}_{\rho'}^1} + \|y \partial_y f\|_{\mathcal{L}_{\rho'}^1} \lesssim \frac{1}{\rho - \rho'} \|f\|_{\mathcal{L}_\rho^1}. \tag{2.16}$$

Proof. The proof is direct; see [2, 32]. $\square$

We also have the following lemma, which will be useful in controlling the velocity in the intermediate region in Section 4.2. We note that in the lemma below, we only give the real pointwise bounds in L^∞ norm on the real line.

Lemma 2.3. *Let $f = f(x, y)$ be analytic in $\mathbb{T}_{2\pi/\lambda} \times \Omega_\rho$ where the analyticity radius $\rho \geq \frac{\delta_0}{4}$. Then for any $\delta_1 < \delta_2 < \delta_0$ and $k \geq 0$, we have*

$$\|D_{x,y}^k f\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} \lesssim \|f\|_{\mathcal{L}_\rho^1}.$$

Proof. We first prove the bound for $\partial_x^k f$. Since $\partial_x^k f = \sum_\alpha e^{i\alpha x} (i\alpha)^k f_\alpha$, we have

$$\|\partial_x^k f\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} \lesssim \sum_\alpha |\alpha|^k \|y f_\alpha(y)\|_{L^\infty(\delta_1 \leq y \leq \delta_2)},$$

noting $y \geq \delta_1$. Now for any $y \leq \delta_2$, we have

$$|y f_\alpha(y)| = \left| \int_0^y \partial_z(z f_\alpha(z)) dz \right| \leq \int_0^{\delta_2} |f_\alpha(z)| dz + \int_0^{\delta_2} |z \partial_z f_\alpha(z)| dz. \quad (2.17)$$

For the first integral, we have

$$\int_0^{\delta_2} |f_\alpha(z)| dz = \int_0^{\delta_2} e^{-\varepsilon_0(\delta_0 + \rho - z)|\alpha|} e^{\varepsilon_0(\delta_0 + \rho - z)|\alpha|} |f_\alpha(z)| dz \leq e^{-\varepsilon_0(\delta_0 - \delta_2)|\alpha|} \|f_\alpha\|_{\mathcal{L}_\rho^1}. \quad (2.18)$$

For the second integral, we first use the estimate (2.16) to get

$$\int_0^{\delta_2} |z \partial_z f_\alpha(z)| dz \lesssim \frac{1}{\delta_0 + \frac{\delta_0}{8} - \delta_2} \|f_\alpha\|_{\mathcal{L}_{\delta_0/8}^1} \lesssim e^{-\varepsilon_0|\alpha|(\delta_0/8)} \|f_\alpha\|_{\mathcal{L}_\rho^1}, \quad (2.19)$$

where in the last inequality, we have used the fact that

$$e^{\varepsilon_0|\alpha|(\delta_0 + \delta_0/8 - \Re z)} \leq e^{-\varepsilon_0|\alpha|\delta_0/8} e^{\varepsilon_0|\alpha|(\delta_0 + \rho - \Re z)} \quad \text{for } \Re z \leq \delta_0 \quad \text{and} \quad \rho \geq \delta_0/4.$$

Combining the inequalities (2.17), (2.18) and (2.19), we get

$$|y f_\alpha(y)| \lesssim \left(e^{-\varepsilon_0(\delta_0 - \delta_2)|\alpha|} + e^{-\varepsilon_0|\alpha|(\delta_0/8)} \right) \|f_\alpha\|_{\mathcal{L}_\rho^1}$$

The proof for $\partial_x^k f$ is complete, by multiplying both sides of the above inequality by $|\alpha|^k$ and summing all over α . Similarly, we compute

$$\begin{aligned} \|\partial_y f\|_{L^\infty(\delta_1 \leq y \leq \delta_0)} &\lesssim \|y^2 \partial_y f\|_{L^\infty(\delta_1 \leq y \leq \delta_0)} \leq \sum_\alpha \|y^2 \partial_y f_\alpha\|_{L^\infty(\delta_1 \leq y \leq \delta_0)} \\ &\lesssim \sum_\alpha \|y \partial_y f_\alpha\|_{L^1(y \leq \delta_0)} + \sum_\alpha \|y^2 \partial_y^2 f_\alpha\|_{L^1(y \leq \delta_0)} \\ &\lesssim \|f\|_{\mathcal{L}_{\delta_0/4}^1} \leq \|f\|_{\mathcal{L}_\rho^1}. \end{aligned}$$

where we use the Cauchy estimate and the fact that $\rho \geq \frac{\delta_0}{4}$. The estimates on higher derivatives follow similarly. $\square$

3 The Stokes problem

In this section, we consider the Stokes problem in the exterior domain, written in the rescaled geodesic coordinates:

$$\begin{aligned} (\partial_\tau - \nu \Delta - \nu \lambda L)w &= f \\ \nu(\partial_z + N)w|_{z=0} &= g \end{aligned} \quad (3.1)$$

where $L = a(z)\partial_z + b(z)\partial_x^2$ is the linear operator defined in (2.2) and N is the Dirichlet-Neumann operator. The main result of this section is to provide uniform estimates on the solution of (3.1) in the Sobolev-analytic spaces. Precisely, we have the following theorem.

Theorem 3.1. *Let $e^{\nu t S}$ be the semigroup of the linear Stokes problem (3.1), and let $\Gamma^S(\nu t)$ be its trace on the boundary. Fix any finite time T . Then, for sufficiently small λ , and for any $0 \leq t \leq T$, $\rho > 0$, and $k \geq 0$, there hold*

$$\begin{aligned} \|e^{\nu t S} w_0\|_{\mathcal{W}_\rho^{k,1}} &\leq C_0 \|w_0\|_{\mathcal{W}_\rho^{k,1}} + \|y^2 D_{x,y}^{k+1} w_0\|_{L^2(y \geq \delta_0 + \rho)} \\ \|\Gamma^S(\nu t) g_b\|_{\mathcal{W}_\rho^{k,1}} &\leq C_0 \|g_b\|_{\mathcal{H}_\rho^k} \end{aligned} \quad (3.2)$$

uniformly in the inviscid limit, where $\|\cdot\|_{\mathcal{W}_\rho^{k,1}}$, $\|\cdot\|_{\mathcal{H}_\rho^k}$ are near boundary analytic norms defined in (2.13) and (2.14), respectively.

The proof relies on the analytic estimates for solutions of the Stokes problem on the half-space. Indeed, in view of Lemma 2.1, we can rewrite the boundary condition in (3.1) as follows:

$$\nu(\partial_z + |\alpha|)w_\alpha|_{z=0} = g_\alpha + h_\alpha$$

where

$$h_\alpha = -\lambda \nu \int_0^\infty \left(w_\alpha(0) L_\alpha(e^{-|\alpha|y}) + L_\alpha \tilde{w}_\alpha^* \right) dy \quad (3.3)$$

and $\tilde{w}_\alpha^*$ solves the elliptic problem (2.9). Therefore, we obtain the following Duhamel principle for solution of (3.1),

$$\begin{aligned} w(\tau) &= e^{\nu \tau B} w_0 + \int_0^\tau e^{\nu(\tau-s)B} (\nu \lambda L w)(s) ds + \int_0^\tau e^{\nu(\tau-s)B} f(s) ds \\ &\quad + \int_0^\tau \Gamma(\nu(\tau-s)) g(s) ds + \int_0^\tau \Gamma(\nu(\tau-s)) h(s) ds, \end{aligned} \quad (3.4)$$

where $e^{\nu \tau B}$ is the Stokes semigroup on the half-space and $\Gamma(\nu \tau)$ denotes its trace on the boundary. To estimate each term on the right hand side, we first recall the following results from [32, 2], which give the Duhamel formula, Green functions estimates, and semigroup bounds for the Stokes problem on the half-space in the vorticity boundary condition.

Theorem 3.2 ([32, 2]). *Consider the Stokes problem*

$$\begin{aligned} (\partial_\tau - \nu \Delta)W &= F \\ \nu(\partial_z + |\partial_x|)W|_{z=0} &= g_b \\ W|_{\tau=0} &= W_0 \end{aligned} \quad (3.5)$$

on the analytic-Sobolev domain $(x, y) \in \mathbb{T}_{2\pi/\lambda} \times \{\Omega_\rho \cup \{y \geq \delta_0 + \rho\}\}$. The solution to (3.5) can be written as

$$W(\tau) = e^{\nu\tau B} W_0 + \int_0^\tau e^{\nu(\tau-s)B} F(s) ds + \int_0^\tau \Gamma(\nu(\tau-s)) g_b(s) ds$$

where $e^{\nu\tau B}$ is the Stokes semigroup of the problem (3.5) and $\Gamma(\nu\tau)$ denotes its trace on the boundary. Moreover, there hold the following semigroup estimates

$$\begin{aligned} \|e^{\nu\tau B} W_0\|_{\mathcal{W}_\rho^{k,1}} &\lesssim \|W_0\|_{\mathcal{W}_\rho^{k,1}} + \|y D_{x,y}^{k+1} W_0\|_{L^2(y \geq \delta_0 + \rho)}, \\ \|e^{\nu(\tau-s)B} F(s)\|_{\mathcal{W}_\rho^{k,1}} &\lesssim \|F(s)\|_{\mathcal{W}_\rho^{k,1}} + \|y D_{x,y}^{k+1} F(s)\|_{L^2(y \geq \delta_0 + \rho)}, \\ \|\Gamma(\nu(\tau-s)) g_b(s)\|_{\mathcal{W}_\rho^{k,1}} &\lesssim \|g_b(s)\|_{\mathcal{H}_\rho^k}, \end{aligned} \quad (3.6)$$

where $\|\cdot\|_{\mathcal{W}_\rho^{k,1}}, \|\cdot\|_{\mathcal{H}_\rho^k}$ are analytic norms defined in (2.13) and (2.14).

In addition, the Fourier coefficients $e^{\nu(\tau-s)B_\alpha}$ of the semigroup $e^{\nu(\tau-s)B}$ have a Green kernel representation $G_\alpha(\tau, y, z)$, in the sense that, for any $\tau \geq 0$, one has

$$e^{\nu\tau B_\alpha} F_\alpha(\tau, y)(z) = \int_0^\infty G_\alpha(\tau, y, z) F_\alpha(\tau, y) dy$$

with the decomposition $G_\alpha(\tau, y, z) = H_\alpha(\tau, y, z) + R_\alpha(\tau, y, z)$, in which $H_\alpha(\tau, y, z)$ is exactly the one-dimensional heat kernel with the homogenous Neumann boundary condition and $R_\alpha(\tau, y, z)$ is the residual kernel due to the boundary condition. Precisely, there hold

$$\begin{aligned} H_\alpha(\tau, y, z) &= \frac{1}{\sqrt{\nu\tau}} \left(e^{-\frac{|y-z|^2}{4\nu\tau}} + e^{-\frac{|y+z|^2}{4\nu\tau}} \right) e^{-\alpha^2 \nu\tau}, \\ |\partial_z^k R_\alpha(\tau, y, z)| &\lesssim \mu_f^{k+1} e^{-\theta_0 \mu_f |y+z|} + (\nu\tau)^{-\frac{k+1}{2}} e^{-\theta_0 \frac{|y+z|^2}{\nu\tau}} e^{-\frac{1}{8} \alpha^2 \nu\tau}, \end{aligned} \quad (3.7)$$

for $y, z \geq 0$, $k \geq 0$, and for some $\theta_0 > 0$ and for $\mu_f = |\alpha| + \frac{1}{\sqrt{\nu}}$.

We now estimate each term on the right of (3.4). The terms involving initial data, f , and g are already estimated using the semigroup estimates in (3.6). We now estimate the second and last terms on the right of (3.4). We start with the linear term $\nu\lambda Lw$.

Proposition 3.3. *Let $L = a(y)\partial_y + b(y)\partial_x^2$. There holds, for $0 \leq k \leq 2$:*

$$\begin{aligned} \left\| \int_0^\tau e^{\nu(\tau-s)B} (\nu\lambda Lw) ds \right\|_{\mathcal{W}_\rho^{k,1}} &\leq C\lambda \sup_{0 \leq s \leq \tau} \left(\|w\|_{\mathcal{W}_\rho^{k,1}} + \|y^2 D_{x,y}^{k+1} w\|_{L^2(y \geq \delta_0 + \rho)} \right) \\ &\quad + C\lambda\nu \int_0^\tau \|w(s)\|_{\mathcal{W}_\rho^{k+1,1}} ds + C\lambda\nu \int_0^\tau \|w(s)|_{\{z=0\}}\|_{\mathcal{H}_\rho^k} ds. \end{aligned}$$

Here, the constant C is independent of λ .

Proof. We focus on the case when $k = 0$; the other cases are similar. Recalling $L = a(y)\partial_y + b(y)\partial_x^2$, we need to estimate

$$\left\| \nu\lambda \int_0^\tau e^{\nu(\tau-s)B} (b(y)\partial_x^2 w(s)) ds \right\|_{\mathcal{L}_\rho^1} + \left\| \nu\lambda \int_0^\tau e^{\nu(\tau-s)B} a(y)\partial_y w(s) ds \right\|_{\mathcal{L}_\rho^1}.$$

Writing the above in Fourier and using the Green kernel decomposition (3.7), we get

$$\begin{cases} (\nu e^{\nu(\tau-s)B} (b(y)\partial_x^2 w))_\alpha = & -\alpha^2 \nu \int_0^\infty H_\alpha(\tau-s, y, z) b(y) w_\alpha(s, y) dy \\ & -\alpha^2 \nu \int_0^\infty R_\alpha(\tau-s, y, z) b(y) w_\alpha(s, y) dy, \\ (\nu e^{\nu(\tau-s)B} (a(y)\partial_y w))_\alpha = & \nu \int_0^\infty H_\alpha(\tau-s, y, z) a(y) \partial_y w_\alpha(s, y) dy \\ & + \nu \int_0^\infty R_\alpha(\tau-s, y, z) a(y) \partial_y w_\alpha(s, y) dy. \end{cases}$$

Treating $\alpha^2 \nu \int_0^\infty H_\alpha(\tau-s, y, z) b(y) w_\alpha(s, y) dy$. In view of $H_\alpha(\tau-s, y, z)$, we need to bound

$$\left\| \alpha^2 \nu \int_0^\tau e^{-\alpha^2 \nu(\tau-s)} \int_0^\infty \frac{1}{\sqrt{\nu(\tau-s)}} e^{-\frac{|y-z|^2}{4\nu(\tau-s)}} b(y) w_\alpha(s, y) dy ds \right\|_{\mathcal{L}_\rho^1}$$

Here we recall that the L_ρ^1 norm is taken in z near the boundary, when $0 \leq \Re z \leq \delta_0 + \rho$. To gain analyticity near the boundary, we use

$$\begin{aligned} e^{\varepsilon_0(\delta_0+\mu-\Re z)|\alpha|} &\leq e^{\varepsilon_0(\delta_0+\mu-\Re y)_+|\alpha|} \cdot e^{\varepsilon_0|\alpha||y-z|} \\ &\leq e^{\varepsilon_0(\delta_0+\mu-\Re y)_+|\alpha|} e^{\varepsilon_0\alpha^2\nu(\tau-s)} \cdot e^{\varepsilon_0\frac{|y-z|^2}{4\nu(\tau-s)}}, \end{aligned} \quad (3.8)$$

where the last two factors can be treated using $e^{-\alpha^2\nu(\tau-s)} e^{-\frac{|y-z|^2}{4\nu(\tau-s)}}$ in the heat kernel. Using this and the fact that the heat kernel is integrable in z , we obtain

$$\begin{aligned} &\left\| \alpha^2 \nu \int_0^\tau e^{-\alpha^2 \nu(\tau-s)} \int_0^\infty \frac{1}{\sqrt{\nu(\tau-s)}} e^{-\frac{|y-z|^2}{4\nu(\tau-s)}} b(y) w_\alpha(s, y) dy ds \right\|_{L_\rho^1} \\ &\lesssim \alpha^2 \nu \int_0^\tau e^{-\alpha^2 \nu(\tau-s)} \int_0^\infty e^{\varepsilon_0(\delta_0+\rho-\Re y)_+|\alpha|} |b(y)| |w_\alpha(s, y)| dy ds. \end{aligned} \quad (3.9)$$

Now since $b(y) = y^{\frac{2+\lambda y}{(1+\lambda y)^2}} \leq 2y$, the above can be bounded by

$$\begin{aligned} &\alpha^2 \nu \int_0^\tau e^{-\alpha^2 \nu(\tau-s)} ds \sup_{0 \leq s \leq \tau} \left(\|y w_\alpha(s)\|_{\mathcal{L}_\rho^1} + \|y w_\alpha(s)\|_{L^1(y \geq \delta_0 + \rho)} \right) \\ &\lesssim \sup_{0 \leq s \leq \tau} \left(\|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y^2 w_\alpha\|_{L^2(y \geq \delta_0 + \rho)} \right). \end{aligned}$$

Treating $\alpha^2 \nu \int_0^\infty R_\alpha(\tau-s, y, z) b(y) w_\alpha(s, y) dy$. Using the bounds of the kernel R_α in Proposition

3.7. we have

$$\begin{aligned}
& e^{\varepsilon_0(\delta_0+\rho-z)|\alpha|} \alpha^2 \nu \int_0^\infty R_\alpha(\tau-s, y, z) b(y) w_\alpha(s, y) dy \\
& \lesssim \alpha^2 \nu \cdot e^{\varepsilon_0(\delta_0+\rho-z)|\alpha|} \int_0^\infty \mu_f e^{-\theta_0 \mu_f (y+z)} y |w_\alpha(s, y)| dy \\
& \lesssim \alpha^2 \nu \int_0^\infty \mu_f e^{-\frac{1}{2} \theta_0 \mu_f (y+z)} e^{\varepsilon_0(\delta_0+\rho-y)+|\alpha|} y |w_\alpha(s, y)| dy \\
& = |\alpha| \nu \cdot (\mu_f e^{-\frac{\theta_0}{2} \mu_f z}) \int_0^\infty (|\alpha| y e^{-\frac{\theta_0}{4} \mu_f y} \cdot e^{-\frac{\theta_0}{4} \mu_f y} \cdot e^{\varepsilon_0(\delta_0+\rho-y)+|\alpha|} |w_\alpha(s, y)| dy \\
& \lesssim |\alpha| \nu \cdot \mu_f e^{-\frac{\theta_0}{2} \mu_f z} \int_0^\infty e^{-\frac{\theta_0}{4} \mu_f y} \cdot e^{\varepsilon_0(\delta_0+\rho-y)+|\alpha|} |w_\alpha(s, y)| dy \\
& = (|\alpha| \nu) (\mu_f e^{-\frac{\theta_0}{2} \mu_f z}) \left(\int_0^{\delta_0+\rho} + \int_{\delta_0+\rho}^\infty \right) e^{-\frac{\theta_0}{4} \mu_f y} \cdot e^{\varepsilon_0(\delta_0+\rho-y)+|\alpha|} |w_\alpha(s, y)| dy \\
& \lesssim |\alpha| \nu \left(\mu_f e^{-\frac{\theta_0}{2} \mu_f z} \right) \|w_\alpha\|_{\mathcal{L}_\rho^1} + |\alpha| \nu \left(\mu_f e^{-\theta_0 \mu_f z} \right) e^{-\frac{\theta_0 \delta_0}{4} (|\alpha| + \nu^{-1/2})} \|y w_\alpha\|_{L^2(y \geq \delta_0 + \rho)}.
\end{aligned}$$

Hence

$$\left\| \alpha^2 \nu \int_0^\infty R_\alpha(\tau-s, y, z) b(y) w_\alpha(s, y) dy \right\|_{\mathcal{L}_\rho^1} \lesssim \nu \|\alpha w_\alpha\|_{\mathcal{L}_\rho^1} + \nu^2 \|y w_\alpha\|_{L^2(y \geq \delta_0 + \rho)}.$$

Treating $\nu \int_0^\infty H_\alpha(\tau-s, y, z) a(y) \partial_y w_\alpha(s, y) dy$. Integrating by parts, we have

$$\begin{aligned}
& \nu \int_0^\infty H_\alpha(\tau-s, y, z) a(y) \partial_y w_\alpha(s, y) dy \\
& = -\nu \int_0^\infty \partial_y (H_\alpha(\tau-s, y, z) a(y)) w_\alpha(s, y) dy - \nu H_\alpha(\tau-s, 0, z) a(0) w_\alpha(s, 0).
\end{aligned}$$

Using the fact that $|\partial_y H_\alpha| \lesssim \frac{1}{\nu(\tau-s)} e^{-\theta_0 \alpha^2 \nu(\tau-s)} e^{-\theta_0 \frac{|y-z|^2}{4\nu(\tau-s)}}$ and $|a'(y)| \lesssim 1$, we have

$$\begin{aligned}
& \left\| \nu \int_0^\infty \partial_y (H_\alpha(\tau-s, y, z) a(y)) w_\alpha(s, y) dy \right\|_{\mathcal{L}_\rho^1} \\
& \lesssim \left(\frac{\sqrt{\nu}}{\sqrt{\tau-s}} + 1 \right) \left(\|w_\alpha(s)\|_{\mathcal{L}_\rho^1} + \|y w_\alpha(s)\|_{L^2(y \geq \delta_0 + \rho)} \right).
\end{aligned}$$

For the boundary term $\nu H_\alpha(\tau-s, 0, z) a(0) w_\alpha(s, 0)$, we have

$$\|\nu H_\alpha(\tau-s, 0, z) a(0) w_\alpha(s, 0)\|_{\mathcal{L}_\rho^1} \lesssim \nu |w_\alpha(s, 0)| e^{\varepsilon_0 |\alpha| (\delta_0 + \rho)}.$$

Hence

$$\begin{aligned}
& \left\| \nu \int_0^\infty H_\alpha(\tau-s, y, z) a(y) \partial_y w_\alpha(s, y) dy \right\|_{\mathcal{L}_\rho^1} \\
& \lesssim \left(\sqrt{\nu} (\tau-s)^{-1/2} + 1 \right) \left(\|w_\alpha(s)\|_{\mathcal{L}_\rho^1} + \|y w_\alpha(s)\|_{L^2(y \geq \delta_0 + \rho)} \right) + \nu |w_\alpha(s, 0)| e^{\varepsilon_0 |\alpha| (\delta_0 + \rho)}.
\end{aligned}$$

Integrating both sides in time $s \in [0, \tau]$, we obtain

$$\begin{aligned} & \left\| \nu \int_0^\infty H_\alpha(\tau - s, y, z) a(y) \partial_y w_\alpha(y, s) dy \right\|_{\mathcal{L}_\rho^1} \\ & \lesssim (\sqrt{\nu\tau} + \tau) \sup_{0 \leq s \leq \tau} \left(\|w_\alpha(s)\|_{\mathcal{L}_\rho^1} + \|y^2 w_\alpha(s)\|_{L^2(y \geq \delta_0 + \rho)} \right) + \int_0^\tau \nu |\omega_\alpha(s, 0)| e^{\varepsilon_0 |\alpha| (\delta_0 + \rho)} ds. \end{aligned}$$

Treating $\nu \int_0^\infty R_\alpha(\tau - s, y, z) a(y) \partial_y w_\alpha(s, y) dy$. Integrating by parts, we get

$$\begin{aligned} & \nu \int_0^\infty R_\alpha(\tau - s, y, z) a(y) \partial_y w_\alpha(s, y) dy \\ & = -\nu \int_0^\infty \partial_y (R_\alpha(\tau - s, y, z) a(y)) w_\alpha(s, y) dy - \nu R_\alpha(\tau - s, 0, z) a(0) w_\alpha(s, 0). \end{aligned}$$

Since $|\partial_y R_\alpha| \lesssim \mu_f^2 e^{-\theta_0 \mu_f (y+z)}$, we have

$$\begin{aligned} \left\| \nu \int_0^\infty \partial_y (R_\alpha(\tau - s, y, z) a(y)) w_\alpha(s, y) dy \right\|_{\mathcal{L}_\rho^1} & \lesssim \nu \mu_f \|w_\alpha\|_{\mathcal{L}_\rho^1} + \nu^2 \|y w_\alpha\|_{L^2(y \geq \delta_0 + \rho)} \\ & \lesssim \sqrt{\nu} \|w_\alpha\|_{\mathcal{L}_\rho^1} + \nu \|\alpha w_\alpha\|_{\mathcal{L}_\rho^1} + \nu^2 \|y w_\alpha\|_{L^2(y \geq \delta_0 + \rho)}. \end{aligned}$$

At the same time, we have

$$\|\nu R_\alpha(\tau - s, 0, z) a(0) w_\alpha(s, 0)\|_{\mathcal{L}_\rho^1} \lesssim \nu |w_\alpha(s, 0)| e^{\varepsilon_0 (\delta_0 + \rho) |\alpha|}.$$

The proof is complete. □

In the next proposition, we estimate the boundary term appearing in (3.4).

Proposition 3.4. *There holds*

$$\left\| \int_0^\tau \Gamma(\nu(\tau - s)) h(s) ds \right\|_{\mathcal{W}_\rho^{k,1}} \lesssim \nu \lambda \int_0^\tau \|w(s)|_{\{z=0\}}\|_{\mathcal{H}_\rho^k} ds.$$

Proof. By the estimate (3.6), we have

$$\left\| \int_0^\tau \Gamma(\nu(\tau - s)) h(s) ds \right\|_{\mathcal{W}_\rho^{k,1}} \lesssim \sum_\alpha |\alpha|^k \int_0^\tau |h_\alpha(s)| e^{\varepsilon_0 (\delta_0 + \rho) |\alpha|} ds.$$

From the identity (3.3), we have $h = h_1 + h_2$ where

$$\begin{cases} h_1 &= -\lambda \nu \int_0^\infty e^{-|\alpha|y} w_\alpha(0, s) L_\alpha(e^{-|\alpha|y}) dy, \\ h_2 &= -\lambda \nu \int_0^\infty e^{-|\alpha|y} L_\alpha \tilde{w}_\alpha^*(y, s) dy. \end{cases}$$

Treating h_1 . Since $L_\alpha = a(y) \partial_y - \alpha^2 b(y)$, by a direct calculation, we have

$$\begin{aligned} h_1 &= \lambda \nu w_\alpha(0, s) \left(\int_0^\infty |\alpha| e^{-2|\alpha|y} a(y) dy + \alpha^2 \int_0^\infty e^{-2|\alpha|y} b(y) dy \right) \\ &\lesssim \lambda \nu |w_\alpha(0, s)|. \end{aligned}$$

Here we use the fact that $a(y) \leq 1, b(y) \leq 2y$ and $\int |\alpha| e^{-|\alpha|y} dy \lesssim 1$.

Treating h_2 . We have $h_2 = h_{2,1} + h_{2,2}$ where

$$\begin{cases} h_{2,1} &= -\lambda \nu \int_0^\infty e^{-|\alpha|y} a(y) \partial_y \tilde{w}_\alpha^*(y, s) dy, \\ h_{2,2} &= \lambda \nu \alpha^2 \int_0^\infty e^{-|\alpha|y} b(y) \tilde{w}_\alpha^*(s, y) dy. \end{cases}$$

We have

$$\begin{aligned} |h_{2,1}| e^{\varepsilon_0(\delta_0+\rho)|\alpha|} &\lesssim \nu \lambda e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \int_0^\infty e^{-|\alpha|y} a(y) \partial_y \tilde{w}_\alpha^*(y) dy \\ &\lesssim \nu \lambda \int_0^{\delta_0+\rho} e^{\varepsilon_0(\delta_0+\rho-y)|\alpha|} |\partial_y \tilde{w}_\alpha^*(y)| dy \\ &\quad + \nu \lambda e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \int_{\delta_0+\rho}^\infty e^{-\frac{1}{2}|\alpha|(\delta_0+\rho)} a(y) |\partial_y \tilde{w}_\alpha^*(y)| e^{-\frac{1}{2}|\alpha|y} dy \\ &\lesssim \nu \lambda \|\partial_y \tilde{w}_\alpha^*(s)\|_{\mathcal{L}_\rho^1} + \nu \lambda e^{-(1/2-\varepsilon_0)|\alpha|(\delta_0+\rho)} \|a(y) \partial_y \tilde{w}_\alpha^*(s)\|_{L^\infty(y \geq \delta_0+\rho)} \frac{1}{|\alpha|}. \end{aligned}$$

Using the fact that $\lambda \lesssim |\alpha|$, we obtain

$$|h_{2,1}| e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \lesssim \nu \lambda \|\partial_y \tilde{w}_\alpha^*\|_{\mathcal{L}_\rho^1} + \nu e^{-(1/2-\varepsilon_0)|\alpha|\delta_0} \|a(y) \partial_y \tilde{w}_\alpha^*(s)\|_{L^\infty}. \quad (3.10)$$

Now we recall from (2.9) that $\tilde{w}_\alpha^*$ solves the elliptic problem

$$\begin{aligned} (\partial_y^2 - \alpha^2 + \lambda L_\alpha) \tilde{w}_\alpha^* &= -\lambda w_\alpha(0) L_\alpha(e^{-|\alpha|y}) = -\lambda w_\alpha(0) e^{-|\alpha|y} (-|\alpha|a(y) - \alpha^2 b(y)) \\ &= \lambda w_\alpha(0) |\alpha| a(y) e^{-|\alpha|y} + \alpha^2 \lambda w_\alpha(0) b(y) e^{-|\alpha|y} \\ &= \lambda w_\alpha(0) |\alpha| \left(a(y) e^{-|\alpha|y} + |\alpha| e^{-|\alpha|y} b(y) \right) \end{aligned}$$

with the boundary condition $\tilde{w}_\alpha^*|_{y=0} = 0$. Hence by using Lemma 4.3, we get

$$\begin{aligned} \|\partial_y \tilde{w}_\alpha^*\|_{\mathcal{L}_\rho^1} &\lesssim \|\partial_y \tilde{w}_\alpha^*\|_{\mathcal{L}_\rho^\infty} \\ &\lesssim \lambda |w_\alpha(0)| |\alpha| \cdot \left(\|a(y) e^{-|\alpha|y}\|_{\mathcal{L}_\rho^1} + \|\alpha e^{-|\alpha|y} b(y)\|_{\mathcal{L}_\rho^1} \right) \\ &\quad + \lambda |w_\alpha(0)| |\alpha| \left(\|y a(y) e^{-|\alpha|y}\|_{L^2(y \geq \delta_0+\rho)} + \|\alpha e^{-|\alpha|y} b(y)\|_{L^2(y \geq \delta_0+\rho)} \right) \\ &\lesssim e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \cdot \lambda |w_\alpha(0, s)|. \end{aligned}$$

Similarly, for the second term appearing on the right hand side of (3.10), we use Lemma 4.1 to get

$$\begin{aligned} \|a(y) \partial_y \tilde{w}_\alpha^*\|_{L^\infty} &\lesssim \int_0^\infty \lambda |w_\alpha(0)| |\alpha| \cdot \left| a(y) e^{-|\alpha|y} + |\alpha| e^{-|\alpha|y} b(y) \right| dy \\ &\lesssim \lambda |w_\alpha(0, s)|. \end{aligned}$$

The bound for $h_{2,2}$ is nearly the same, as we note that $b(y) = \frac{y(2+\lambda y)}{(1+\lambda y)^2} \lesssim y a(y)$. We skip the details for $h_{2,2}$, and conclude that

$$|h_{2,2}| e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \lesssim \nu |w_\alpha(0, s)|.$$

Hence we get

$$|h_2|e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \lesssim \nu|w_\alpha(0, s)|e^{\varepsilon_0(\delta_0+\rho)|\alpha|},$$

giving the proposition. $\square$

Combining the previous two propositions, we have obtained the following.

Proposition 3.5. *Let w be the solution to the Stokes problem (3.1) with the initial data w_0 . Then for $k \geq 0$ and $\rho > 0$, there hold the coupled semigroup estimates*

$$\begin{aligned} \sup_{0 \leq s \leq \tau} \|w(s)\|_{\mathcal{W}_\rho^{k,1}} &\lesssim \|w_0\|_{\mathcal{W}_\rho^{k,1}} + \|yD_{x,y}^{k+1}w_0\|_{L^2(y \geq \delta_0/2)} \\ &+ \lambda \sup_{0 \leq s \leq \tau} \left(\|w(s)\|_{\mathcal{W}_\rho^{k,1}} + \|y^2D_{x,y}^{k+1}w(s)\|_{L^2(y \geq \delta_0+\rho)} \right) \\ &+ \lambda\nu \int_0^\tau \|w(s)\|_{\mathcal{W}_\rho^{k+1,1}} ds + \lambda\nu \int_0^\tau \|w(s)_{z=0}\|_{\mathcal{H}_\rho^k} ds \\ &+ \int_0^\tau \left(\|f(s)\|_{\mathcal{W}_\rho^{k,1}} + \|yD_{x,y}^{k+1}f(s)\|_{L^2(y \geq \delta_0+\rho)} \right) ds + \int_0^\tau \|g(s)\|_{\mathcal{H}_\rho^k} ds. \end{aligned}$$

Proof. Recall the Duhamel representation (3.4). The proof thus follows directly by combining the semigroup estimates (3.6), the estimates for the perturbation term in Proposition 3.3, and the boundary estimates in Proposition 3.4. $\square$

Finally, we give bounds on $\|w(s)_{z=0}\|_{\mathcal{H}_\rho^k}$ appearing on the right of the previous estimates.

Proposition 3.6. *Let w be the solution to the Stokes problem (3.1). There holds*

$$\begin{aligned} \|w(\tau)_{z=0}\|_{\mathcal{H}_\rho^k} &\lesssim \|y\partial_x w_0\|_{L^2(y \geq \delta_0/2)} + (\nu\tau)^{-1/2} \|w_0\|_{\mathcal{W}_\rho^{k+1,1}} \\ &+ \nu\lambda \int_0^\tau \|w(s)_{z=0}\|_{\mathcal{H}_\rho^{k+1}} ds + \lambda\sqrt{\nu} \sup_{0 \leq s \leq \tau} \|w(s)_{z=0}\|_{\mathcal{H}_\rho^k} \\ &+ \lambda \sup_{0 \leq s \leq \tau} \left(\|w(s)\|_{\mathcal{W}_\rho^{k,1}} + \|y^2D_{x,y}^{k+1}w(s)\|_{L^2(y \geq \delta_0+\rho)} \right) + \nu \int_0^\tau \|\partial_x w(s)\|_{\mathcal{W}_\rho^{k+1,1}} ds \\ &+ \int_0^\tau \|yD_x^1 f(s)\|_{L^2(y \geq \delta_0+\rho)} ds + \nu^{-1/2} \int_0^\tau (\tau-s)^{-1/2} \|f(s)\|_{\mathcal{W}_\rho^{k,1}} ds \\ &+ \int_0^\tau \|\partial_x f(s)\|_{\mathcal{W}_\rho^{k,1}} ds + \int_0^\tau \|g(s)\|_{\mathcal{H}_\rho^k} ds. \end{aligned} \tag{3.11}$$

Proof. We shall bound each term in (3.4), evaluating at $z = 0$. First, we have

$$\begin{aligned} w_\alpha(\tau, 0) &= \int_0^\infty G_\alpha(\tau, y, 0)w_{0,\alpha}(y)dy + \int_0^\tau G_\alpha(\tau-s, y, 0)(\nu\lambda L_\alpha w_\alpha)(s, y)dy \\ &+ \int_0^\tau \int_0^\infty G_\alpha(\tau-s, y, 0)f_\alpha(s, y)dyds + \int_0^\tau \Gamma_\alpha(\nu(\tau-s))(g_\alpha + h_\alpha)(s)ds \\ &= P_1(\tau) + P_2(\tau) + P_3(\tau) + P_4(\tau). \end{aligned}$$

Hence we get

$$|\alpha|^k e^{\varepsilon_0(\delta_0+\rho)|\alpha|} w_\alpha(\tau, 0) = \sum_{i=1}^4 P_i(\tau)$$

where

$$\begin{cases} P_1(\tau) = |\alpha|^k e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \int_0^\infty G_\alpha(\tau, y, 0) w_{0,\alpha}(y) dy \\ P_2(\tau) = |\alpha|^k e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \int_0^\tau G_\alpha(\tau-s, y, 0) (\nu \lambda L_\alpha w_\alpha)(s, y) dy \\ P_3(\tau) = |\alpha|^k e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \int_0^\tau \int_0^\infty G_\alpha(\tau-s, y, 0) f_\alpha(s, y) dy ds \\ P_4(\tau) = |\alpha|^k e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \int_0^\tau \Gamma_\alpha(\nu(\tau-s))(g_\alpha + h_\alpha)(s) ds. \end{cases}$$

We recall the pointwise Green kernel bound:

$$G_\alpha(\tau-s, y, 0) \lesssim (\nu(\tau-s))^{-1/2} e^{-\theta_0 \frac{y^2}{\nu(\tau-s)}} e^{-\theta_0 \alpha^2 \nu(\tau-s)} + \mu_f e^{-\mu_f y}.$$

Let us first bound the term

$$P_3(\tau) = |\alpha|^k e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \left| \int_0^\tau \int_0^\infty G_\alpha(\tau-s, y, 0) f_\alpha(s, y) dy ds \right| \quad (3.12)$$

We will show that

$$P_3(\tau) \lesssim \int_0^\tau \left(\|y f_\alpha(s)\|_{L^2(y \geq \delta_0+\rho)} + \left((\nu(\tau-s))^{-1/2} + |\alpha| + \nu^{-1/2} \right) |\alpha|^k \|f_\alpha(s)\|_{\mathcal{L}_\rho^1} \right) ds. \quad (3.13)$$

To show the above inequality, we split the integral in y in (3.12) into $\int_{\delta_0+\rho}^\infty + \int_0^{\delta_0+\rho}$. We note that if $y \geq \delta_0 + \rho$, then G_α is exponentially decay in α , which is faster than $e^{-\varepsilon_0(\delta_0+\rho)|\alpha|}$ for ε_0 small, giving

$$\begin{aligned} & |\alpha|^k e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \left| \int_0^\tau \int_{\delta_0+\rho}^\infty G_\alpha(\tau-s, y, 0) f_\alpha(s, y) dy ds \right| \\ & \lesssim \int_0^\tau \|y f_\alpha(s)\|_{L^2(y \geq \delta_0+\rho)} ds \end{aligned}$$

Now we consider $y \leq \delta_0 + \rho$. By the Cauchy inequality $\alpha^2 \nu(\tau-s) + \frac{y^2}{\nu(\tau-s)} \geq 4|\alpha|y$ and the fact that θ_0, ε_0 is taken to be small, we obtain

$$\begin{aligned} e^{\varepsilon_0(\delta_0+\rho)|\alpha|} G_\alpha(\tau-s, 0, y) & \lesssim \left((\nu(\tau-s))^{-1/2} + \mu_f \right) e^{-\frac{\theta_0}{2} |\alpha| y} e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \\ & \lesssim \left((\nu(\tau-s))^{-1/2} + \mu_f \right) e^{\varepsilon_0(\delta_0+\rho-y)|\alpha|}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} & |\alpha|^k e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \left| \int_0^\tau \int_0^{\delta_0+\rho} G_\alpha(\tau-s, y, 0) f_\alpha(s, y) dy ds \right| \\ & \lesssim \int_0^\tau \left((\nu(\tau-s))^{-1/2} + |\alpha| + \nu^{-1/2} \right) |\alpha|^k \|f_\alpha(s)\|_{\mathcal{L}_\rho^1} ds. \end{aligned}$$

This concludes the proof for the inequality (3.13). Next, we bound

$$P_2(\tau) = |\alpha|^k e^{\varepsilon_0(\delta_0+\rho)|\alpha|} \int_0^\tau G_\alpha(\tau-s, y, 0) (\nu \lambda L_\alpha w_\alpha)(s, y) dy.$$

The proof for the bound of $P_2(\tau)$ is exactly the same as in the semigroup estimate in Proposition 3.3, except now that we cannot use the L^1 norm in z in this case, as $z = 0$, giving an extra

$\mu_f = |\alpha| + \nu^{-1/2}$ in the estimate involving the kernel $R_\alpha(\tau - s, 0, z)|_{z=0}$. We obtain

$$\begin{aligned} P_2(\tau) &\lesssim \lambda \sup_{0 \leq s \leq \tau} \left(\|w\|_{\mathcal{W}_\rho^{k,1}} + \|y^2 D_{x,y}^{k+1} w\|_{L^2(y \geq \delta_0 + \rho)} \right) + \lambda \nu \int_0^\tau (\nu(\tau - s))^{-1/2} \|w(s)|_{z=0}\|_{\mathcal{H}_\rho^k} ds \\ &\quad + \nu \int_0^\tau \|\partial_x w\|_{\mathcal{W}_\rho^{k+1,1}} + \nu \lambda \sum_\alpha \int_0^\tau |\alpha|^{k+1} |w_\alpha(s, 0)| e^{\varepsilon_0(\delta_0 + \rho)|\alpha|}. \end{aligned}$$

Finally, for the initial data, we obtain

$$P_1(\tau) \lesssim \|y \partial_x^k w_0\|_{L^2(y \geq \delta_0/2)} + (\nu\tau)^{-1/2} \|w_0\|_{\mathcal{W}_\rho^{k+1,1}},$$

giving the proposition. $\square$

Remark 3.7. Note that in the above estimates, the boundary value quantity $\|w(\tau)|_{z=0}\|_{\mathcal{H}_\rho^k}$ has two losses of derivatives compared to the norm $\|w(\tau)\|_{\mathcal{W}_\rho^{k,1}}$. However, it has only one loss of derivative compared to its norm and we are able to close the Sobolev-analytic estimates by introducing an iterative adjusted k -index norms, yielding close estimates on the Stokes semigroup in terms of initial and boundary data f, g given in the problem (3.1).

Proof of Theorem 3.1. Let $w = e^{\nu t S} w_0$ be the solution to (3.1) with $f = 0$ and $g = 0$. In view of the previous propositions, we define the following norm

$$\begin{aligned} \mathcal{A}_k(w(\tau), \rho) &= \left(\|w(\tau)\|_{\mathcal{W}_\rho^{k,1}} + \sqrt{\nu\tau} \|w(\tau)|_{z=0}\|_{\mathcal{H}_\rho^{k-1}} \right) \\ &\quad + \left(\|w(\tau)\|_{\mathcal{W}_\rho^{k+1,1}} + \sqrt{\nu\tau} \|w(\tau)|_{z=0}\|_{\mathcal{H}_\rho^k} \right) (\rho_0 - \rho - \beta\tau)^\gamma \end{aligned} \quad (3.14)$$

and the quantity

$$A(\beta) = \sup_{0 < \tau\beta < \rho_0} \left\{ \sup_{0 < \rho < \rho_0 - \beta\tau} (\mathcal{A}_k(w(\tau), \rho)) \right\} + \sup_{0 < \tau\beta < \rho_0} \|y^2 D_{x,y}^5 w\|_{L^2(y \geq \delta_0/2)}.$$

We claim that

$$A(\beta) \lesssim \|w_0\|_{\mathcal{W}_{\rho_0}^{2,1}} + \|y^2 D_{x,y}^5 w_0\|_{L^2(y \geq \delta_0/4)} + e^{C(1+A(\beta))\beta^{-1}} \|y^2 D_{x,y}^5 w_0\|_{L^2(y \geq \delta_0/4)} \quad (3.15)$$

which would yield the theorem. In fact, in Section 7, using precisely Propositions 3.5 and 3.6 above, we shall prove the claim for the nonlinear solution to (3.1) with f and g being the nonlinear terms inherited from the vorticity formulation of the Navier-Stokes problem. We therefore skip to repeat the details here for the linear problem with zero f and g . $\square$

4 Elliptic estimates

In this section, we prove estimates for velocity near the boundary and away from the boundary in terms of vorticity. In particular, we consider the elliptic problem

$$\begin{cases} (\Delta + \lambda L)\psi = \lambda^2 w, \\ \psi|_{y=0} = 0. \end{cases} \quad (4.1)$$

where $L\psi = a(y)\partial_y\psi + b(y)\partial_x^2\psi$, $a(y) = \frac{1}{1+\lambda y}$, $b(y) = \frac{y(2+\lambda y)}{(1+\lambda y)^2}$. Our main goal in this section is to show the elliptic estimates in the analytic domain near the boundary (see Proposition 4.4 below), in the intermediate region (Proposition 4.6) and the region away from the boundary (Proposition 4.9).

4.1 Elliptic estimates in the analytic region

We first show the following lemma that gives a L^∞ bound for velocity field:

Lemma 4.1. *There holds*

$$\|a(y)\alpha\psi_\alpha\|_{L^\infty} + \|\partial_y\psi_\alpha\|_{L^\infty} \lesssim \int_0^\infty |w_\alpha(y)|dy \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|yw_\alpha\|_{L^2(y \geq \delta_0 + \rho)}.$$

Proof. We recall the original elliptic problem on the written in the variables $(\theta, r) \in \mathbb{T} \times [1, \infty)$:

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{n^2}{r^2}\right)\psi_n = \omega_n$$

where $\psi(t, r, \theta) = \psi(\lambda^2 t, 1 + \lambda y, \lambda x)$. We note that $\alpha = \lambda n$ where n is the original frequency before making the change of variables. The solution to the above elliptic problem in the original variables is given by

$$-\psi_n(r) = \frac{1}{2|n|} \int_1^r \frac{s^{1+|n|} - s^{1-|n|}}{r^{|n|}} \omega_n(s) ds + \frac{1}{2|n|} \int_r^\infty \left(s^{1-|n|r^{|n|}} - \frac{s^{1-|n|}}{r^{|n|}} \right) \omega_n(s) ds. \quad (4.2)$$

Since the function $s^{1+|n|} - s^{1-|n|}$ is increasing on $[1, r]$ and the function $s^{1-|n|r^{|n|}} - \frac{s^{1-|n|}}{r^{|n|}}$ is decreasing on $[r, \infty)$, we get the pointwise estimate

$$|n\psi_n(r)| \lesssim (r - r^{1-2n}) \int_1^\infty |\omega_n(s)| ds \lesssim r \|\omega_n\|_{L^1(1, \infty)}.$$

Hence we obtain

$$\left\| \frac{n\psi_n(r)}{r} \right\|_{L^\infty} \lesssim \|\omega_n\|_{L^1(1, \infty)}. \quad (4.3)$$

Now in the rescaled variables (α, y) , we get

$$|\alpha| \|a(y)\psi_\alpha\|_{L^\infty} \lesssim \int_0^\infty |w_\alpha(y)| dy \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|yw_\alpha\|_{L^2(y \geq \delta_0 + \rho)},$$

upon noting that $a(y) = \frac{1}{1+\lambda y} = \frac{1}{r}$. Now we show that $\|a(y)\partial_y\psi_\alpha\|_{L^\infty} \lesssim \int_0^\infty |w_\alpha(y)| dy$. By a direct calculation, we get

$$-2\psi'_n(r) = -r^{-|n|-1} \int_1^r (s^{1+|n|} - s^{1-|n|}) \omega_n(s) ds + \left(r^{|n|-1} + r^{-|n|-1} \right) \int_r^\infty s^{1-n} \omega_n(s) ds.$$

Hence

$$|\psi'_n(r)| \lesssim \int_1^\infty |\omega_n(s)| ds.$$

The proof is complete. $\square$

Remark 4.2. It is known from Section 2.2 of [27], that the Biot-Savart law (4.2) defines a unique velocity that decays at infinity, under the decaying assumption $r^{1-|n|}\omega_n \in L^1$. In our current work, the vorticity satisfies the decaying assumption $\|r^2 D_{x,y}^3 \omega\|_{L_{r,\theta}^2} < \infty$, hence the Biot-Savart law (4.2) gives a unique velocity solution for all $|n| \geq 1$. We also note that when $n = 0$, the stream function equation reduces to

$$\partial_r^2 \psi_0 + \frac{1}{r} \partial_r \psi_0 = \omega_0$$

giving $\psi_0'(r) = \frac{1}{r} \int_1^r s \omega_0(s) ds$. This gives the Biot-Savart law $(u_r, u_\theta) = (0, -\frac{1}{r} \int_1^r s \omega_0(s) ds)$ for $n = 0$. We also note that when the frequency $n = \alpha = 0$, the analytic norm in x (or θ) reduces to Sobolev norm.

In the next lemma, we derive the elliptic estimate for velocity in the analytic norm near the boundary:

Lemma 4.3. For λ, δ_0 and ρ small, there holds

$$\|\nabla \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y w_\alpha\|_{L^2(y \geq \delta_0 + \rho)},$$

Proof. We first show that

$$\|\nabla \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y w_\alpha\|_{L^2(y \geq \delta_0 + \rho)}$$

Since ψ_α solves

$$(\partial_y^2 - \alpha^2) \psi_\alpha = \lambda^2 w_\alpha - \lambda a(y) \partial_y \psi_\alpha + \lambda \alpha^2 b(y) \psi_\alpha$$

with the boundary condition $\psi_\alpha|_{y=0} = 0$, we get

$$\begin{aligned} 2\alpha \psi_\alpha(z) &= \lambda^2 \int_0^\infty \left(e^{-\alpha(y+z)} - e^{-\alpha|y-z|} \right) w_\alpha(y) dy - \lambda \int_0^\infty \left(e^{-\alpha(y+z)} - e^{-\alpha|y-z|} \right) a(y) \partial_y \psi_\alpha(y) dy \\ &\quad + \lambda \int_0^\infty \left(e^{-\alpha(y+z)} - e^{-\alpha|y-z|} \right) \alpha^2 b(y) \psi_\alpha(y) dy \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{cases} I_1 &= \lambda^2 \int_0^\infty \left(e^{-\alpha(y+z)} - e^{-\alpha|y-z|} \right) w_\alpha(y) dy, \\ I_2 &= -\lambda \int_0^\infty \left(e^{-\alpha(y+z)} - e^{-\alpha|y-z|} \right) a(y) \partial_y \psi_\alpha(y) dy, \\ I_3 &= \lambda \int_0^\infty \left(e^{-\alpha(y+z)} - e^{-\alpha|y-z|} \right) \alpha^2 b(y) \psi_\alpha(y) dy. \end{cases}$$

Treating I_1 . Using the first estimate in (3.8), we simply bound

$$\begin{aligned} |I_1| e^{\varepsilon_0(\delta_0 + \rho - z)|\alpha|} &\lesssim \int_0^{\delta_0 + \rho} e^{-\frac{|\alpha|}{2}|y-z|} e^{\varepsilon_0(\delta_0 + \rho - y)|\alpha|} |w_\alpha(y)| dy + \int_{\delta_0 + \rho}^\infty |w_\alpha(y)| dy \\ &\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y w_\alpha\|_{L^2(y \geq \delta_0 + \rho)}. \end{aligned}$$

Treating I_2 . We will show that

$$I_2 \lesssim \|\alpha \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \ln(1 + \lambda(\delta_0 + \rho)) + \lambda \|a(y) \psi_\alpha\|_{L^\infty(y \geq \delta_0 + \rho)}$$

Since

$$I_2 = -\lambda \int_0^\infty \left(e^{-\alpha(y+z)} - e^{-\alpha|y-z|} \right) a(y) \partial_y \psi_\alpha(y) dy,$$

we use integration by parts to get

$$\begin{aligned} I_2 &\lesssim \lambda \int_0^\infty |\alpha| e^{-|\alpha||y-z|} a(y) |\psi_\alpha(y)| dy + \lambda \int_0^\infty |a'(y)| e^{-|\alpha||y-z|} |\psi_\alpha(y)| dy \\ &\lesssim \lambda \int_0^\infty |\alpha| e^{-|\alpha||y-z|} a(y) |\psi_\alpha(y)| dy + \lambda^2 \int_0^\infty a(y)^2 e^{-|\alpha||y-z|} |\psi_\alpha(y)| dy \end{aligned}$$

Therefore,

$$\begin{aligned} |I_2| e^{\varepsilon_0(\delta_0+\rho-z)|\alpha|} &\lesssim \lambda \|\alpha \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \int_0^{\delta_0+\rho} \frac{1}{1+\lambda y} dy + \lambda \|a(y) \psi_\alpha\|_{L^\infty(y \geq \delta_0+\rho)} \\ &\quad + \frac{\lambda^2}{|\alpha|} \|\alpha \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \int_0^{\delta_0+\rho} \frac{1}{1+\lambda y} dy + \frac{\lambda^2}{|\alpha|} \|a(y) \psi_\alpha\|_{L^\infty(y \geq \delta_0+\rho)} \\ &\lesssim \|\alpha \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \ln(1 + \lambda(\delta_0 + \rho)) + \lambda \|a(y) \psi_\alpha\|_{L^\infty(y \geq \delta_0+\rho)} \\ &\lesssim \|\alpha \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \ln(1 + \lambda(\delta_0 + \rho)) + \|a(y) \alpha \psi_\alpha\|_{L^\infty(y \geq \delta_0+\rho)}. \end{aligned} \quad (4.4)$$

where we use the fact that $|\alpha| \geq \lambda$ whenever $\alpha \neq 0$.

Treating I_3 . We will show that

$$\|I_3\|_{\mathcal{L}_\rho^\infty} \lesssim \lambda \|\nabla \psi_\alpha\|_{\mathcal{L}_\rho^\infty} + \|a(y) \alpha \psi_\alpha\|_{L^\infty(y \geq \delta_0+\rho)}. \quad (4.5)$$

Indeed, if $y \leq \delta_0 + \rho$, then $b(y) \leq 2y \leq 2(\delta_0 + \rho)$, and hence

$$e^{\varepsilon_0(\delta_0+\rho-z)|\alpha|} \int_0^{\delta_0+\rho} |\alpha| e^{-|\alpha||y-z|} b(y) dy \leq 2(\delta_0 + \rho) \int_0^\infty |\alpha| e^{-\frac{1}{2}|\alpha||y-z|} dy \lesssim 1.$$

If $y \geq \delta_0 + \rho$, then we have

$$\begin{aligned} &e^{\varepsilon_0(\delta_0+\rho-z)|\alpha|} \int_{\delta_0+\rho}^\infty e^{-|\alpha||y-z|} |\alpha| \frac{\lambda y(2+\lambda y)}{(1+\lambda y)^2} (\alpha \psi_\alpha(y)) dy \\ &= \lambda e^{\varepsilon_0(\delta_0+\rho-z)|\alpha|} \int_{\delta_0+\rho}^\infty \frac{|\alpha| |\psi_\alpha(y)|}{1+\lambda y} \left(e^{-|\alpha||y-z|} |\alpha| y \right) \cdot \frac{2+\lambda y}{1+\lambda y} dy \\ &\lesssim \left\| \frac{\alpha \psi_\alpha}{1+\lambda y} \right\|_{L^\infty(y \geq \delta_0+\rho)} \lambda \int_{\delta_0+\rho}^\infty |\alpha| e^{-\frac{1}{2}|\alpha||y-z|} (|y-z|+z) dy \\ &\lesssim \left(\frac{\lambda}{|\alpha|} + \lambda(\delta_0 + \rho) \right) \left\| \frac{\alpha \psi_\alpha}{1+\lambda y} \right\|_{L^\infty(y \geq \delta_0+\rho)} \lesssim \|a(y) \alpha \psi_\alpha\|_{L^\infty(y \geq \delta_0+\rho)}. \end{aligned}$$

since $\lambda \lesssim |\alpha|$. In summary, we get

$$\begin{aligned} \|\nabla \psi_\alpha\|_{\mathcal{L}_\rho^\infty} &\leq C_0 \left(\|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y w_\alpha\|_{L^2(y \geq \delta_0+\rho)} + \|a(y) \alpha \psi_\alpha\|_{L^\infty(y \geq \delta_0+\rho)} \right) \\ &\quad + C_0 \left(\|\nabla \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \right) (\lambda + \ln(1 + \lambda(\delta_0 + \rho))). \end{aligned} \quad (4.6)$$

We note that in the estimate above, the constant C_0 does not depend on α and λ . Taking λ to be small so that

$$\lambda + \ln(1 + \lambda(\delta_0 + \rho)) \leq \frac{1}{2C_0},$$

the last term in the estimate (4.6) can be absorbed to the left hand side, giving

$$\|\nabla \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|yw_\alpha\|_{L^2(y \geq \delta_0 + \rho)} + \|a(y)\alpha\psi_\alpha\|_{L^\infty(y \geq \delta_0 + \rho)}.$$

Finally, using Lemma (4.1) for the last time in the above, we obtain

$$\|\nabla \psi_\alpha\|_{\mathcal{L}_\rho^\infty} \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|yw_\alpha\|_{L^2(y \geq \delta_0 + \rho)}.$$

The proof is complete. $\square$

In order to close the estimate for velocity in terms of vorticity, we need the following lemma

Proposition 4.4. *Let ψ be the solution to the elliptic problem (4.1), and set $\tilde{u} = \nabla^\perp \psi$. For $k \in \{0, 1\}$, there hold*

$$\begin{aligned} \|\partial_x^k \tilde{u}\|_{\mathcal{L}_\rho^\infty} &\lesssim \|w\|_{\mathcal{W}_\rho^{k,1}} + \|yD_{x,y}^{k+1}w\|_{L^2(y \geq \delta_0 + \rho)}, \\ \|(y\partial_y)^k \tilde{u}\|_{\mathcal{L}_\rho^\infty} &\lesssim \|w\|_{\mathcal{W}_\rho^{k,1}} + \|yD_{x,y}^{k+1}w\|_{L^2(y \geq \delta_0 + \rho)}, \\ \|y^{-1}\partial_x \psi\|_{\mathcal{W}_\rho^{k,\infty}} &\lesssim \|w\|_{\mathcal{W}_\rho^{k,1}} + \|\partial_x w\|_{\mathcal{W}_\rho^{k,1}} + \|yD_{x,y}^{k+2}w\|_{L^2(y \geq \delta_0 + \rho)}. \end{aligned}$$

Proof. First, when $k = 0$, from Lemma 4.3, we have

$$\begin{aligned} \|\tilde{u}\|_{\mathcal{L}_\rho^\infty} &\lesssim \|w\|_{\mathcal{L}_\rho^1} + \sum_{\alpha \in \lambda\mathbb{Z}} \|yw_\alpha\|_{L^2(y \geq \delta_0 + \rho)} \\ &\lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_x w\|_{L^2(y \geq \delta_0 + \rho)}. \end{aligned}$$

Now we give the proof for $k = 1$. Since $\partial_x \psi$ solves the same elliptic problem with the condition $\partial_x \psi|_{y=0} = 0$, we obtain

$$\|\partial_x \tilde{u}\|_{\mathcal{L}_\rho^\infty} \lesssim \|\partial_x w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^2 w\|_{L^2(y \geq \delta_0 + \rho)}.$$

Now for $\|y\partial_y \tilde{u}\|_{\mathcal{L}_\rho^\infty}$, we note that

$$\|y\partial_y(\partial_x \psi)\|_{\mathcal{L}_\rho^\infty} \lesssim \|\partial_y(\partial_x \psi)\|_{\mathcal{L}_\rho^\infty} \lesssim \|\partial_x w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^2 w\|_{L^2(y \geq \delta_0 + \rho)}.$$

Now we have

$$y\partial_y(\partial_y \psi) = y\partial_y^2 \psi = y(\lambda^2 w - \lambda a(y)\partial_y \psi - \lambda b(y)\partial_x^2 \psi - \partial_x^2 \psi).$$

Hence we get

$$\begin{aligned} \|y\partial_y^2 \psi\|_{\mathcal{L}_\rho^\infty} &\lesssim \|yw\|_{\mathcal{L}_\rho^\infty} + \|\tilde{u}\|_{\mathcal{L}_\rho^\infty} + \|\partial_x \tilde{u}\|_{\mathcal{L}_\rho^\infty} \\ &\lesssim (\|w\|_{\mathcal{L}_\rho^1} + \|y\partial_y w\|_{\mathcal{L}_\rho^1}) + \|\partial_x w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^2 w\|_{L^2(y \geq \delta_0 + \rho)} \\ &\lesssim \|w\|_{\mathcal{W}_\rho^{1,1}} + \|yD_{x,y}^2 w\|_{L^2(y \geq \delta_0 + \rho)}. \end{aligned}$$

The proof is complete. Now we show the last inequality stated in this proposition. When $k = 0$, we have, for any $y \leq \delta_0 + \rho$:

$$\partial_x \psi = \int_0^y \partial_z (\partial_x \psi)(z) dz.$$

Since $e^{\varepsilon_0 |\alpha| (\delta_0 + \rho - y)} \leq e^{\varepsilon_0 |\alpha| (\delta_0 + \rho - z)}$, we have

$$\|y^{-1} \partial_x \psi\|_{\mathcal{L}_\rho^\infty} \lesssim \|\partial_x \tilde{u}\|_{\mathcal{L}_\rho^1} \lesssim \|\partial_x \tilde{u}\|_{\mathcal{L}_\rho^\infty} \lesssim \|\partial_x w\|_{\mathcal{L}_\rho^1} + \|y D_{x,y}^2 w\|_{L^2(y \geq \delta_0 + \rho)}.$$

For $k = 1$, we note that

$$\begin{cases} \partial_x (y^{-1} \partial_x \psi) = y^{-1} \partial_x (\partial_x \psi), \\ y \partial_y (y^{-1} \partial_x \psi) = -\frac{1}{y} \partial_x \psi + y \partial_y \partial_x \psi. \end{cases}$$

Hence

$$\begin{aligned} \|y \partial_y (y^{-1} \partial_x \psi)\|_{\mathcal{L}_\rho^\infty} &\leq \|y^{-1} \partial_x (\partial_x \psi)\|_{\mathcal{L}_\rho^\infty} + \|\partial_y (\partial_x \psi)\|_{\mathcal{L}_\rho^\infty} \\ &\lesssim \|\partial_x^2 w\|_{\mathcal{L}_\rho^1} + \|y D_{x,y}^3 w\|_{L^2(y \geq \delta_0 + \rho)}. \end{aligned}$$

The proposition follows. $\square$

4.2 Elliptic estimates in the intermediate region

We also need the following elliptic estimates in the intermediate region away from the boundary. We first prove the following elementary lemma.

Lemma 4.5. *Assume $0 < \delta_1 < \delta_2 < \delta_0$ and let $c \in (0, 1)$ be any constant such that $\delta_2 < c\delta_0$. Then for any function $F_\alpha(y)$ and $k \geq 1$, there holds*

$$|\alpha|^k \int_0^\infty e^{-|\alpha||y-z|} |F_\alpha(y)| dy \leq C \left(\|F_\alpha\|_{\mathcal{L}_\rho^1} + \int_{c\delta_0}^\infty e^{-\frac{1}{2}|\alpha||y-z|} |F_\alpha(y)| dy \right)$$

for $z \in [\delta_1, \delta_2]$. The constant C depends only on $\delta_1, \delta_2, \delta_0$ and k .

Proof. Splitting the integral in y into $y \leq c\delta_0$ and $y \geq c\delta_0$, we have two cases:

Case 1. $y \leq c\delta_0$. In this case, we get $y \leq \delta_0 + \rho$, and moreover

$$e^{-\varepsilon_0 |\alpha| (\delta_0 + \rho - y)} \leq e^{-\varepsilon_0 (1-c) |\alpha| \delta_0}.$$

Hence

$$\begin{aligned} \int_0^{c\delta_0} |\alpha|^k e^{-|\alpha||y-z|} |F_\alpha(y)| dy &\leq \int_0^{c\delta_0} e^{-|\alpha||y-z|} |\alpha|^k e^{-\varepsilon_0 (1-c) |\alpha| \delta_0} e^{\varepsilon_0 |\alpha| (\delta_0 + \rho - y)} |F_\alpha(y)| dy \\ &\lesssim \|F_\alpha\|_{\mathcal{L}_\rho^1}. \end{aligned}$$

Case 2. $y \geq c\delta_0$. In this case we have $|y - z| \geq c\delta_0 - \delta_2$. And hence $e^{-\frac{1}{2}|\alpha||y-z|} \leq e^{-\frac{1}{2}(c\delta_0 - \delta_2)|\alpha|}$.

$$\begin{aligned} \int_{c\delta_0}^\infty |\alpha|^k e^{-|\alpha||y-z|} |F_\alpha(y)| dy &\leq \int_{c\delta_0}^\infty |\alpha|^k e^{-\frac{1}{2}(c\delta_0 - \delta_2)|\alpha|} e^{-\frac{1}{2}|\alpha||y-z|} |F_\alpha(y)| dy \\ &\lesssim \int_0^\infty e^{-\frac{1}{2}|\alpha||y-z|} |F_\alpha(y)| dy. \end{aligned}$$

The proof is complete. $\square$

Proposition 4.6. *Let ψ be the solution to the elliptic problem (4.1), and set $\tilde{u} = \nabla^\perp \psi$. Then for any $\delta_1 < \delta_2 < \delta_0$, we have*

$$\|D_{x,y}^k \tilde{u}\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} \lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_x w\|_{L^2(y \geq c\delta_0)}$$

where $c \in (0, 1)$ is any constant such that $c\delta_0 \in (\delta_2, \delta_0)$.

Proof. We give the proof for $\partial_x^3 \tilde{u}$ and $\partial_y^3 \tilde{u}$ only. The other cases are similar. Since ψ solves

$$\Delta \psi = \lambda^2 w - \lambda L \psi, \quad \psi|_{y=0} = 0,$$

we use the Green kernel for the Laplacian $(\partial_y^2 - \alpha^2)$ and integrating by parts for the term $a(y)\partial_y \psi$, to get

$$|\alpha|^3 |\tilde{u}_\alpha(z)| \lesssim |\alpha|^3 \int_0^\infty e^{-|\alpha||y-z|} (\lambda^2 |w_\alpha(y)| + \lambda |\alpha| a(y) |\psi_\alpha(y)| + \lambda |a'(y)| |\psi_\alpha(y)| + \lambda \alpha^2 b(y) |\psi_\alpha(y)|) dy.$$

Applying Lemma 4.5 for three terms on the right hand side in the above, we get

$$\begin{aligned} |\alpha|^3 |\tilde{u}_\alpha(z)| &\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|\partial_y \psi_\alpha\|_{\mathcal{L}_\rho^1} + \|\alpha \psi_\alpha\|_{\mathcal{L}_\rho^1} + \int_{c\delta_0}^\infty e^{-\frac{1}{2}|\alpha||y-z|} |w_\alpha(y)| dy \\ &\quad + \lambda \int_{c\delta_0}^\infty e^{-\frac{1}{2}|\alpha||y-z|} |a(y)| |\alpha| |\psi_\alpha(y)| dy + \lambda \int_{c\delta_0}^\infty e^{-\frac{1}{2}|\alpha||y-z|} |a'(y)| |\psi_\alpha(y)| dy \\ &\quad + \lambda \int_{c\delta_0}^\infty e^{-\frac{1}{2}|\alpha||y-z|} \alpha^2 b(y) |\psi_\alpha(y)| dy. \end{aligned} \quad (4.7)$$

Now we will bound each term appearing on the right hand side of the above inequality. Using Proposition 4.4, we have

$$\begin{aligned} \|\partial_y \psi_\alpha\|_{\mathcal{L}_\rho^1} + \|\alpha \psi_\alpha\|_{\mathcal{L}_\rho^1} &\lesssim \|\tilde{u}_\alpha\|_{\mathcal{L}_\rho^1} \lesssim \|\tilde{u}_\alpha\|_{\mathcal{L}_\rho^\infty} \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|yw_\alpha\|_{L^2(y \geq \delta_0 + \rho)} \\ &\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|yw_\alpha\|_{L^2(y \geq c\delta_0)}. \end{aligned}$$

Also, it is obvious that

$$\int_{c\delta_0}^\infty e^{-\frac{1}{2}|\alpha||y-z|} |w_\alpha(y)| dy \lesssim \|yw_\alpha\|_{L^2(y \geq c\delta_0)}.$$

Now for the terms involving $a(y)$ on the right hand side of (4.7), we recall from the proof of (4.4) that this term can be bounded by

$$\|\tilde{u}_\alpha\|_{\mathcal{L}_\rho^\infty} + \|a(y)\psi_\alpha\|_{L^\infty(y \geq \delta_0 + \rho)} \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|yw_\alpha\|_{L^2(y \geq \delta_0 + \rho)}$$

thanks to Proposition 4.4 and Lemma 4.1.

Now for the last term on the right hand side of (4.7), we bound this term by

$$\begin{aligned} \left\| \lambda \int_{c\delta_0}^\infty e^{-\frac{1}{2}|\alpha||y-z|} \alpha^2 b(y) |\psi_\alpha(y)| dy \right\|_{\mathcal{L}_\rho^\infty} &\lesssim \lambda \|\nabla \psi_\alpha\|_{\mathcal{L}_\rho^\infty} + \left\| \frac{\alpha \psi_\alpha}{1 + \lambda y} \right\|_{L^\infty(y \geq \delta_0 + \rho)} \\ &\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|yw_\alpha\|_{L^2(y \geq \delta_0 + \rho)}. \end{aligned}$$

Here, we use the inequality (4.5) and Lemma 4.1. The bound for the last term appearing in (4.7) is complete.

Finally, combining the the bounds for all of the terms on the right hand side of (4.7), we get

$$|\alpha|^3 |\tilde{u}_\alpha(z)| \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|yw_\alpha\|_{L^2(y \geq c\delta_0)}.$$

Summing all $\alpha \in \lambda\mathbb{Z}$, we get

$$\begin{aligned} \|\partial_x^3 \tilde{u}\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} &\leq \sum_{\alpha} |\alpha|^3 \|\tilde{u}_\alpha\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} \\ &\lesssim \sum_{\alpha} |\alpha|^3 \|\tilde{u}_\alpha\|_{L^\infty} \lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_x w\|_{L^2(y \geq c\delta_0)}. \end{aligned} \quad (4.8)$$

On the other hand, for $\partial_y^3 \tilde{u}$, we use $\partial_y^2 \psi = -\partial_x^2 \psi - \lambda L\psi + \lambda^2 w$ to compute

$$\begin{aligned} \|\partial_y^3 \tilde{u}\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} &\lesssim \|D_{x,y}^2 w\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} + \|\partial_x^3 \tilde{u}\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} \\ &\quad + \lambda \|\partial_y L(\partial_x \psi) + L(\partial_x^2 \psi) + \partial_y^2(L\psi)\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} \\ &\lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yw\|_{L^2(y \geq c\delta_0)} \\ &\quad + \lambda \|\partial_y L(\partial_x \psi) + L(\partial_x^2 \psi) + \partial_y^2(L\psi)\|_{L^\infty(\delta_1 \leq y \leq \delta_2)}. \end{aligned}$$

Using $L = a(y)\partial_y + b(y)\partial_x^2$, we thus obtain

$$\sum_{k \leq 3} \|\partial_x^k \tilde{u}\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} + \|D_{x,y}^2 w\|_{L^\infty(\delta_1 \leq y \leq \delta_2)} \lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_x w\|_{L^2(y \geq c\delta_0)}.$$

The proof is complete. $\square$

4.3 Elliptic estimates away from the boundary

We first show the following simple lemma that will be used in the next proposition.

Lemma 4.7. *Let $f(r), \xi(r)$ be smooth functions on $r \geq 1$, and $\xi(r) = 0$ on $[1, R]$. Let ϕ solves the elliptic problem*

$$\left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{n^2}{r^2} \right) \phi = \xi(r) \partial_r f(r)$$

with the boundary condition $\phi|_{r=1} = 0$. There holds

$$|n| \left\| \frac{\phi_n(r)}{r} \right\|_{L^\infty} \lesssim \|\xi f\|_{L^\infty} + \|\xi' f\|_{L^1}$$

Proof. As in (4.2), we get, for $n > 0$

$$-2n\phi_n(r) = \int_1^r \frac{s^{1+n} - s^{1-n}}{r^n} \xi(s) f'(s) ds + \int_r^\infty \left(s^{1-n} r^n - \frac{s^{1-n}}{r^n} \right) \xi(s) f'(s) ds.$$

By integrating by parts, we get

$$\begin{aligned} -2n\phi_n(r) &= -\int_1^r \frac{(1+n)s^n - (1-n)s^{-n}}{r^n} \xi(s)f(s)ds - \int_1^r \frac{s^{1+n} - s^{1-n}}{r^n} \xi'(s)f(s)ds \\ &\quad + \frac{(r^{1+n} - r^{1-n})\xi(r)f(r)}{r^n} - \int_r^\infty ((1-n)s^{-n}r^n - (1-n)s^{-n}r^{-n}) \xi(s)f(s)ds \\ &\quad - \int_r^\infty \left(s^{1-n}r^n - \frac{s^{1-n}}{r^n} \right) \xi'(s)f(s)ds. \end{aligned}$$

Hence

$$n|\phi_n(r)| \lesssim r \left(\|\xi f\|_{L^\infty} + \|\xi' f\|_{L^1} \right).$$

The proof is complete. $\square$

Finally, we state the main Proposition for this section:

Proposition 4.8. *Let ψ be the solution to the elliptic problem (4.1), and set $\tilde{u} = \nabla^\perp \psi$. For any $\delta \in (0, \delta_0)$, $k \geq 0$ and $\rho \in (\delta_0/4, \delta_0)$, one has*

$$\begin{aligned} \|a(y)D_{x,y}^k \tilde{u}\|_{L^\infty(y \geq \delta)} &\lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^{k+1}w\|_{L^2(y \geq \delta_0/2)}, \\ \|D_{x,y}^k(a(y)\tilde{u})\|_{L^2(y \geq \delta)} &\lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^k w\|_{L^2(y \geq \delta_0/2)}. \end{aligned} \quad (4.9)$$

for $k \geq 0$, where $a(y) = \frac{1}{1+\lambda y}$.

Proof. We first give the proof for $\|D_{x,y}^k \tilde{u}\|_{L^\infty(y \geq \delta)}$. When $k = 0$, the inequality

$$\|a(y)\tilde{u}\|_{L^\infty} \lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_x w\|_{L^2(y \geq \delta_0/2)} \quad (4.10)$$

follows from Lemma 4.1. Moreover, we also have

$$\begin{aligned} \|a(y)\partial_x^k \tilde{u}\|_{L^\infty} &\lesssim \|\partial_x^k w\|_{\mathcal{L}_{\delta_0/8}^1} + \|yD_x^{k+1}w\|_{L^2(y \geq \delta_0/2)} \\ &\lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_x^{k+1}w\|_{L^2(y \geq \delta_0/2)} \end{aligned}$$

where we use the fact that $\rho \geq \frac{\delta_0}{8}$. We can now assume that $D_{x,y}^k = \partial_y^k$ and we will use induction on $k \geq 0$. We first give a proof for $k = 1$, which is ∂_y . We have

$$\begin{cases} \partial_y(\partial_x \psi) = \partial_x(\partial_y \psi) \\ \partial_y(\partial_y \psi) = \partial_y^2 \psi = \lambda^2 w - \partial_x^2 \psi - \lambda a(y)\partial_y \psi - \lambda b(y)\partial_x^2 \psi. \end{cases} \quad (4.11)$$

For the first term $\partial_y(\partial_x \psi)$, we simply bound

$$\|\partial_x \partial_y \psi\|_{L^\infty(y \geq \delta)} \leq \|\partial_x \tilde{u}\|_{L^\infty(y \geq \delta)} \lesssim \|\partial_x w\|_{\mathcal{L}_\rho^1} + \|yD_x^2 w\|_{L^2(y \geq \delta_0/2)}$$

For the second term $\partial_y^2 \psi$ in (4.11) we get, for any $y \geq \delta$:

$$\begin{aligned} |a(y)\partial_y^2 \psi_\alpha(y)| &\leq a(y)|w_\alpha(y)| + a(y)|\alpha|^2 |\psi_\alpha(y)| + a(y)|\partial_y \psi_\alpha(y)| + \lambda a(y)b(y)|\alpha|^2 |\psi_\alpha(y)| \\ &\lesssim \|w_\alpha\|_{L^\infty(y \geq \delta)} + \|a(y)\alpha \tilde{u}_\alpha\|_{L^\infty} + \|a(y)\tilde{u}_\alpha\|_{L^\infty} \\ &\lesssim \|w_\alpha\|_{L^\infty(y \geq \delta)} + \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y\alpha w_\alpha\|_{L^2(y \geq \delta_0/2)}. \end{aligned}$$

Let ζ be a cut-off function so that

$$\zeta(y) = \begin{cases} 0 & y \leq \delta/2, \\ 1 & y \geq \delta. \end{cases} \quad (4.12)$$

Then we have

$$\|a(y)w_\alpha\|_{L^\infty(y \geq \delta)} \lesssim \|w_\alpha\|_{L^\infty(y \geq \delta)} \leq \|\zeta(z)w_\alpha(z)\|_{L^\infty}.$$

We have

$$\zeta(z)w_\alpha(z) = \int_0^z \zeta'(y)w_\alpha(y)dy + \int_0^z \zeta(y)\partial_y w_\alpha(y)dy.$$

Hence, for every $z \geq 0$, we bound

$$\begin{aligned} |\zeta(z)w_\alpha(z)| &\leq \|w_\alpha\|_{L^\infty(\delta/2 \leq y \leq \delta)} + \int_{\delta/2}^\infty |\partial_y w_\alpha(z)|dz \\ &\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|\partial_y w_\alpha\|_{L^\infty(\delta/2 \leq y \leq \delta_0/2)} + \int_{\delta_0/2}^\infty |\partial_y w_\alpha(z)|dz \\ &\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y\partial_y w_\alpha\|_{L^2(y \geq \delta_0/2)}. \end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned} \|a(y)\partial_y^2 \psi_\alpha\|_{L^\infty(y \geq \delta)} &\lesssim \|w\|_{\mathcal{L}_\rho^1} + \sum_{\alpha \in \lambda\mathbb{Z}} (\|y\partial_y w_\alpha\|_{L^2(y \geq \delta_0/2)} + \|y\alpha w_\alpha\|_{L^2(y \geq \delta_0/2)}) \\ &\lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^2 w\|_{L^2(y \geq \delta_0/2)}. \end{aligned}$$

This finishes the proof for $k = 1$. Now we assume $k \geq 1$. We proceed by induction on the number of derivatives of y . Assume that the inequality is true for $k - 1$, we show that it is also true for k . We recall that ζ be a cut-off function defined in (4.12). Then $\zeta\partial_y^k \psi$ solves the elliptic problem

$$\begin{aligned} (\Delta + \lambda L)(\zeta\partial_y^k \psi) &= \lambda^2(\zeta\partial_y^k w_\alpha) + 2\zeta'(y)\partial_y \partial_y^k \psi + \zeta''(y)\partial_y^k \psi \\ &\quad + \lambda L(\zeta\partial_y^k \psi) - \lambda\zeta\partial_y^k (L\psi) \end{aligned} \quad (4.13)$$

with the boundary condition $\zeta\partial_y^k \psi|_{y=0} = 0$. In Fourier frequency α , the right hand side in the above can be decomposed into $F_1 + F_2 + F_3$ where

$$\begin{cases} F_1 &= \lambda^2(\zeta\partial_y^k w_\alpha) + 2\zeta'(y)\partial_y \partial_y^k \psi + \zeta''(y)\partial_y^k \psi_\alpha, \\ F_2 &= \lambda a(y)\partial_y(\zeta\partial_y^k \psi_\alpha) - \lambda\zeta\partial_y^k (a(y)\partial_y \psi_\alpha), \\ F_3 &= -\lambda b(y)\alpha^2(\zeta\partial_y^k \psi_\alpha) + \lambda\zeta\partial_y^k (b(y)\alpha^2 \psi_\alpha). \end{cases}$$

From the equation (4.13), we get

$$\zeta\partial_y^k \psi = \Psi_1 + \Psi_2 + \Psi_3$$

where $(\Delta + \lambda L)\Psi_i = F_i$ for $1 \leq i \leq 3$ with the boundary condition $\Psi_i|_{y=0} = 0$ (this can also be seen from the formula (4.2)). We also denote

$$U_i = \nabla^\perp \Psi_i. \quad (4.14)$$

Treating U_1 . Using the same argument as in Lemma 4.1 for every $z \geq 0$, we get

$$\begin{aligned}
|a(z)U_1(z)| &\lesssim \int_0^\infty \left(|\zeta \partial_y^k w_\alpha(y)| + |\zeta'(y) \partial_y^k \tilde{u}_\alpha(y)| + |\zeta''(y)| |\partial_y^{k-1} \tilde{u}_\alpha(y)| \right) dy \\
&\lesssim \|y \zeta \partial_y^k w_\alpha\|_{L^2} + \|\zeta'(y) D_{x,y}^k \tilde{u}_\alpha\|_{L^1} \lesssim \|y D_{x,y}^k w_\alpha\|_{L^2(y \geq \delta/2)} + \|D_{x,y}^k \tilde{u}_\alpha\|_{L^1(\delta/2 \leq y \leq \delta)} \\
&\lesssim \|y D_{x,y}^k w_\alpha\|_{L^2(\delta/2 \leq y \leq \delta_0/2)} + \|y D_{x,y}^k w_\alpha\|_{L^2(y \geq \delta_0/2)} + \left(\|w\|_{\mathcal{L}_\rho^1} + \|y w_\alpha\|_{L^2(y \geq \delta_0/2)} \right) \\
&\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y D_{x,y}^k w_\alpha\|_{L^2(y \geq \delta_0/2)}.
\end{aligned}$$

Treating U_2 . We have

$$\begin{aligned}
F_2 &= -\lambda \sum_{1 \leq i \leq k} \binom{k}{i} \zeta(y) \partial_y^i a(y) \partial_y^{k+1-i} \psi_\alpha + \lambda a(y) \zeta'(y) \partial_y^k \psi_\alpha \\
&= -\lambda \sum_{2 \leq i \leq k} \binom{k}{i} \partial_y^i a(y) \zeta(y) \partial_y^{k+1-i} \psi_\alpha - \lambda a'(y) \zeta(y) \partial_y^k \psi_\alpha + \lambda a(y) \zeta'(y) \partial_y^k \psi_\alpha \\
&= F_{2,1} + F_{2,2} + F_{2,3}.
\end{aligned}$$

Hence we get $U_2 = \sum_{i=1}^3 (\Delta + \lambda L)^{-1} F_{2,i} = \sum_{i=1}^3 U_{2,i}$. Arguing as in Lemma 4.1, for every $z \geq 0$, we get

$$\begin{aligned}
a(z)U_{2,1}(z) &\lesssim \max_{2 \leq i \leq k} \int_0^\infty |\partial_y^i a(y) \zeta(y) \partial_y^{k-1} \tilde{u}_\alpha(y)| dy \\
&\lesssim \max_{2 \leq i \leq k} \int_0^\infty a(y)^{i+1} \zeta(y) |\partial_y^{k-1} \tilde{u}_\alpha(y)| dy \\
&\lesssim \|a(y) \zeta(y) \partial_y^{k-1} \tilde{u}_\alpha(y)\|_{L^\infty} \max_{2 \leq i \leq k} \int_0^\infty a(y)^i dy \\
&\lesssim \|a(y) \zeta(y) \partial_y^{k-1} \tilde{u}_\alpha\|_{L^\infty} \lesssim \|w\|_{\mathcal{L}_\rho^1} + \|y D_{x,y}^{k-1} w_\alpha\|_{L^2(y \geq \delta_0/2)}.
\end{aligned}$$

Here, we have used the fact that $\partial_y^i a(y) \lesssim a(y)^{i+1}$, $\int_0^\infty a(y)^i dy \lesssim 1$ for all $i \geq 2$, and the induction hypothesis in the last inequality.

Now we turn to $F_{2,2} = -\lambda a'(y) \zeta(y) \partial_y^k \psi_\alpha$. Applying Lemma 4.7 for $\xi(y) = -\lambda a'(y) \zeta(y)$ and $f(y) = \partial_y^{k-1} \psi_\alpha$, for every $z \geq 0$, we get

$$\begin{aligned}
a(z)U_{2,2}(z) &\lesssim \|a'(y) \zeta(y) \partial_y^{k-1} \psi_\alpha\|_{L^\infty} + \|\partial_y(a'(y) \zeta(y)) \partial_y^{k-1} \psi_\alpha\|_{L^1} \\
&\lesssim \|a(y) \zeta(y) \partial_y^{k-1} \psi_\alpha\|_{L^\infty} + \|a''(y) \zeta(y) \partial_y^{k-1} \psi_\alpha\|_{L^1} + \|a'(y) \zeta'(y) \partial_y^{k-1} \psi_\alpha\|_{L^1} \\
&\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y D_{x,y}^{k-1} w_\alpha\|_{L^2(y \geq \delta_0/2)} + \|a''(y)\|_{L^1} \|a(y) \zeta(y) \partial_y^{k-1} \psi_\alpha\|_{L^\infty} + \|\partial_y^{k-1} \psi_\alpha\|_{L^\infty(\delta/2 \leq y \leq \delta)} \\
&\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y D_{x,y}^{k-1} w_\alpha\|_{L^2(y \geq \delta_0/2)}.
\end{aligned}$$

Finally, for $U_{2,3}$ which solves $(\Delta + \lambda L)U_{2,3} = F_{2,3} = \lambda a(y) \zeta'(y) \partial_y^k \psi_\alpha$, we use Lemma 4.1 again, for every $z \geq 0$, to get

$$a(z)U_{2,3}(z) \lesssim \int_0^\infty \lambda a(y) |\zeta'(y) \partial_y^k \psi_\alpha(y)| dy \lesssim \|\partial_y^k \psi_\alpha\|_{L^\infty(\delta/2 \leq y \leq \delta)} \lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y w_\alpha\|_{L^2(y \geq \delta_0/2)}.$$

The proof for U_2 is complete.

Treating U_3 . We recall that

$$\begin{aligned} F_3 &= -\lambda b(y) \alpha^2 (\zeta \partial_y^k \psi_\alpha) + \lambda \zeta \partial_y^k (b(y) \alpha^2 \psi_\alpha) \\ &= \lambda \alpha^2 \sum_{i=1}^k \binom{k}{i} \zeta(y) \partial_y^i b(y) \partial_y^{k-i} \psi_\alpha(y). \end{aligned}$$

Using Lemma [4.1](#) for the equation $(\Delta + \lambda L)\Psi_3 = F_3$, for $z \geq 0$, we get

$$\begin{aligned} |a(z)U_3(z)| &\lesssim \max_{1 \leq i \leq k} \lambda \alpha^2 \int_0^\infty \zeta(y) |\partial_y^i b(y)| |\partial_y^{k-i} \psi_\alpha(y)| dy \\ &\lesssim \max_{1 \leq i \leq k} |\alpha| \int_0^\infty a(y)^{i+2} \zeta(y) |\alpha \partial_y^{k-i} \psi_\alpha(y)| dy \\ &\lesssim \|a(y) \zeta(y) D_{x,y}^{k-1} \tilde{u}_\alpha\|_{L^\infty} \max_{1 \leq i \leq k} \int_0^\infty a(y)^{i+1} dy \\ &\lesssim \|w_\alpha\|_{\mathcal{L}_\rho^1} + \|y D_{x,y}^{k-1} w_\alpha\|_{L^2(y \geq \delta_0/2)}. \end{aligned}$$

where we use the induction hypothesis in the last inequality, and the fact that $\partial_y^i b(y) \lesssim a(y)^{i+2}$ for all $i \geq 1$. The proof is complete for the $\|\cdot\|_{L^\infty(y \geq \delta)}$ norm of the velocity. The estimates in L^2 norm follow similarly. $\square$

5 Bilinear estimates

In this section, we recall the bilinear estimates for the nonlinear terms. We define the nonlinear quantity for w as follows:

$$\begin{aligned} N_\rho(w, k) &= \|w\|_{\mathcal{W}_\rho^{k+1,1}} \left(\|w\|_{\mathcal{W}_\rho^{k,1}} + \|y D_{x,y}^k w\|_{L^2(z \geq \delta_0 + \rho)} \right) \\ &\quad + \|w\|_{\mathcal{W}_\rho^{k,1}} \|y D_{x,y}^{k+2} w\|_{L^2(z \geq \delta_0 + \rho)}. \end{aligned} \tag{5.1}$$

Proposition 5.1. *Let $N_\rho(w, k)$ be the nonlinear quantity defined in [\(5.1\)](#), and $\psi = (\Delta + \lambda L)^{-1}(\lambda^2 w)$ be the corresponding stream function defined in the elliptic problem [\(4.1\)](#). For $k \in \{0, 1\}$, there hold*

$$\|\tilde{u} \cdot \nabla w\|_{\mathcal{W}_\rho^{k,1}} \lesssim N_\rho(w, k)$$

where $\tilde{u} = \nabla^\perp \psi$.

Proof. For $k = 0$, we have

$$\begin{aligned} \|\partial_x \psi \partial_y w\|_{\mathcal{L}_\rho^1} &\leq \|y^{-1} \partial_x \psi\|_{\mathcal{L}_\rho^\infty} \|y \partial_y w\|_{\mathcal{L}_\rho^1} \\ &\lesssim \left(\|w\|_{\mathcal{L}_\rho^1} + \|\partial_x w\|_{\mathcal{L}_\rho^1} + \|y D_{x,y}^2 w\|_{L^2(y \geq \delta_0 + \rho)} \right) \|y \partial_y w\|_{\mathcal{L}_\rho^1}. \end{aligned}$$

upon using Proposition [4.4](#). Similarly, for $k = 1$, we compute

$$\begin{cases} \partial_x(\partial_x \psi \partial_y w) = y^{-1} \partial_x^2 \psi \cdot y \partial_y w + y^{-1} \partial_x \psi \cdot \partial_x(y \partial_y w) \\ y \partial_y(\partial_x \psi \partial_y w) = \partial_x(\partial_y \psi) \cdot y \partial_y w + y^{-1} \partial_x \psi \cdot \{(y \partial_y)^2 w - y \partial_y w\}. \end{cases} \tag{5.2}$$

This implies

$$\begin{aligned}
\|\partial_x(\partial_x\psi\partial_yw)\|_{\mathcal{L}_\rho^1} &\lesssim \|y^{-1}\partial_x^2\psi\|_{\mathcal{L}_\rho^\infty}\|y\partial_yw\|_{\mathcal{L}_\rho^1} + \|y^{-1}\partial_x\psi\|_{\mathcal{L}_\rho^\infty}\|\partial_x(y\partial_yw)\|_{\mathcal{L}_\rho^1} \\
&\lesssim \left(\|\partial_xw\|_{\mathcal{L}_\rho^1} + \|\partial_x^2w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^3w\|_{L^2(y\geq\delta_0+\rho)}\right)\|y\partial_yw\|_{\mathcal{L}_\rho^1} \\
&\quad + \left(\|w\|_{\mathcal{L}_\rho^1} + \|\partial_xw\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^2w\|_{L^2(y\geq\delta_0+\rho)}\right)\|\partial_x(y\partial_yw)\|_{\mathcal{L}_\rho^1} \\
&\lesssim \|w\|_{\mathcal{W}_\rho^{2,1}}\|w\|_{\mathcal{W}_\rho^{1,1}} + \|yD_{x,y}^3w\|_{L^2(y\geq\delta_0+\rho)}\|w\|_{\mathcal{W}_\rho^{1,1}} + \|yD_{x,y}^2w\|_{L^2(y\geq\delta_0+\rho)}\|w\|_{\mathcal{W}_\rho^{2,1}}.
\end{aligned}$$

Similarly, from the calculation in (5.2), we have

$$\begin{aligned}
&\|y\partial_y(\partial_x\psi\partial_yw)\|_{\mathcal{L}_\rho^1} \\
&\lesssim \|\partial_x\tilde{u}\|_{\mathcal{L}_\rho^\infty}\|y\partial_yw\|_{\mathcal{L}_\rho^1} + \|y^{-1}\partial_x\psi\|_{\mathcal{L}_\rho^\infty}\|w\|_{\mathcal{W}_\rho^{2,1}} \\
&\lesssim \left(\|w\|_{\mathcal{W}_\rho^{1,1}} + \|yD_{x,y}^2w\|_{L^2(y\geq\delta_0+\rho)}\right)\|w\|_{\mathcal{W}_\rho^{1,1}} + \left(\|w\|_{\mathcal{W}_\rho^{1,1}} + \|yD_{x,y}^2w\|_{L^2(y\geq\delta_0+\rho)}\right)\|w\|_{\mathcal{W}_\rho^{2,1}},
\end{aligned}$$

giving the proposition. $\square$

Next we show the nonlinear estimate away from the boundary:

Lemma 5.2. *There holds*

$$\|yD_{x,y}^2(a(y)\tilde{u} \cdot \nabla w)\|_{L^2(y\geq\delta_0+\rho)} \lesssim \|yD_{x,y}^3w\|_{L^2(y\geq\delta_0/2)} \left(\|w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^3w\|_{L^2(y\geq\delta_0/2)}\right)$$

Proof. We give the proof for the case when there is no derivative only. The other cases are treated similarly. We have

$$\begin{aligned}
\|ya(y)\tilde{u}\partial_yw\|_{L^2(y\geq\delta_0+\rho)} &= \|a(y)\tilde{u}_2\|_{L^\infty(y\geq\delta_0+\rho)}\|y\partial_yw\|_{L^2(y\geq\delta_0+\rho)} + \|a(y)\tilde{u}_1\|_{L^\infty(y\geq\delta_0+\rho)}\|\partial_xw\|_{L^2(y\geq\delta_0+\rho)} \\
&\lesssim \left(\|w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^1w\|_{L^2(y\geq\delta_0/2)}\right)\|yD_{x,y}^1w\|_{L^2(y\geq\delta_0/2)}.
\end{aligned}$$

where we used (4.9). The proof is complete. $\square$

6 Estimates for vorticity away from the boundary

In this section, we estimate

$$\|y^2D_{x,y}^5w\|_{L^2(y\geq\delta_0/2)} = \sum_{i+j\leq 5} \|y^2\partial_x^i\partial_y^jw\|_{L^2(y\geq\delta_0/2)}$$

for the scaled vorticity w solving (2.3). We take a cut off function $\eta : [0, \infty) \rightarrow [0, \infty)$ such that

$$\eta(y) = \begin{cases} 0 & \text{if } y \leq \delta_0/4 \\ y^2 & \text{if } y \geq \delta_0/2. \end{cases} \quad (6.1)$$

We define

$$\mathcal{E}(t) = \sum_{i+j\leq 5} \frac{1}{2} \int_0^\infty \eta(y) |\partial_x^i \partial_y^j w(t)|^2 dy$$

to be the main control for the norm $\|y^2 D_{x,y}^5 w\|_{L^2(y \geq \delta_0/2)}$, and

$$\mathcal{D}(t) = \sum_{i+j \leq 5} \nu \left\{ \int (1 + \lambda b(y)) \eta(y) |\partial_x^{i+1} \partial_y^j w|^2 + \frac{1}{2} \int \eta'(y) |\partial_x^i \partial_y^{j+1} w|^2 \right\}$$

coming from the dissipation term in the energy estimate. We note that $\eta'(y) > 0$, so all the terms in $\mathcal{D}(t)$ are non-negative. Moreover, we define the following quantity away from the boundary that is needed to bound $\mathcal{E}(t)$:

$$N_a(\tilde{u}, w) = \|D_{x,y}^4(a(y)\tilde{u})\|_{L^\infty(y \geq \delta_0/4)} + \|D_{x,y}^5(a(y)\tilde{u})\|_{L^2(y \geq \delta_0/4)} + \|D_{x,y}^5 w\|_{L^\infty(\delta_0/4 \leq y \leq \delta_0/2)}. \quad (6.2)$$

We obtain the following proposition.

Proposition 6.1. *Let (w, ψ) solve (2.3)-(2.5), and set $\tilde{u} = \nabla^\perp \psi$. For λ sufficiently small, there holds*

$$\mathcal{E}'(t) + c_0 \mathcal{D}(t) \leq C_0 \left(\mathcal{E}(t) + N_a(\tilde{u}, w) \mathcal{E}(t) + N_a(\tilde{u}, w)^2 + N_a(\tilde{u}, w)^2 \mathcal{E}(t)^{1/2} \right)$$

for some constants $c_0, C_0 > 0$.

Proof. Using (2.3), we compute

$$\begin{aligned} \mathcal{E}'(t) &= \sum_{i+j \leq 5} \int_0^\infty \eta(y) \partial_x^i \partial_y^j \partial_t w \cdot \partial_x^i \partial_y^j w \\ &= \sum_{i+j \leq 5} \nu \int \eta \Delta(\partial_x^i \partial_y^j w) \cdot \partial_x^i \partial_y^j w + \nu \lambda \int \eta(y) \partial_x^i \partial_y^j (a(y) \partial_y w) \cdot \partial_x^i \partial_y^j w \\ &\quad + \nu \lambda \int \eta(y) \partial_x^i \partial_y^j (b(y) \partial_x^2 w) \cdot \partial_x^i \partial_y^j w + \int \eta(y) \partial_x^i \partial_y^j (a(y) \tilde{u} \cdot \nabla w) \cdot \partial_x^i \partial_y^j w \\ &= \sum_{k=1}^7 \mathcal{I}_k \end{aligned}$$

where

$$\begin{cases} \mathcal{I}_1 &= \sum_{i+j \leq 5} \nu \int \eta \Delta(\partial_x^i \partial_y^j w) \cdot \partial_x^i \partial_y^j w \\ \mathcal{I}_2 &= \sum_{i+j \leq 5} \nu \lambda \int \eta(y) (a(y) \partial_x^i \partial_y^{j+1} w) \cdot \partial_x^i \partial_y^j w \\ \mathcal{I}_3 &= \sum_{i+j \leq 5} \nu \lambda \int \eta(y) \left\{ \partial_x^i \partial_y^j (a(y) \partial_y w) - (a(y) \partial_x^i \partial_y^{j+1} w) \right\} \cdot \partial_x^i \partial_y^j w \\ \mathcal{I}_4 &= \sum_{i+j \leq 5} \nu \lambda \int \eta(y) b(y) \partial_x^{i+2} \partial_y^j w \cdot \partial_x^i \partial_y^j w \\ \mathcal{I}_5 &= \sum_{i+j \leq 5} \nu \lambda \int \eta(y) \left\{ \partial_x^i \partial_y^j (b(y) \partial_x^2 w) - b(y) \partial_x^{i+2} \partial_y^j w \right\} \cdot \partial_x^i \partial_y^j w \\ \mathcal{I}_6 &= \sum_{i+j \leq 5} \int \eta(y) \left(a(y) \tilde{u} \cdot \nabla \partial_x^i \partial_y^j w \right) \cdot \partial_x^i \partial_y^j w \\ \mathcal{I}_7 &= \sum_{i+j \leq 5} \int \eta(y) \left(\partial_x^i \partial_y^j (a(y) \tilde{u} \cdot \nabla w) - a(y) \tilde{u} \cdot \nabla \partial_x^i \partial_y^j w \right) \cdot \partial_x^i \partial_y^j w \end{cases}$$

Below, we sometimes skip writing $\sum_{i+j \leq 5}$, without any confusion.

By integrating by parts, we obtain

$$\begin{aligned} \mathcal{I}_1 &= -\nu \int \eta |\partial_x^{i+1} \partial_y^j w|^2 - \nu \int \partial_x^i \partial_y^{j+1} w \partial_y (\eta \partial_x^i \partial_y^{j+1} w) \\ &= -\nu \int \eta |\partial_x^{i+1} \partial_y^j w|^2 - \nu \int \eta'(y) |\partial_x^i \partial_y^{j+1} w|^2 - \nu \int \eta(y) \partial_x^i \partial_y^{j+1} w \cdot \partial_x^i \partial_y^{j+2} w, \end{aligned}$$

which yields

$$\mathcal{I}_1 = -\nu \int \eta |\partial_x^{i+1} \partial_y^j w|^2 - \frac{1}{2} \nu \int \eta'(y) |\partial_x^i \partial_y^{j+1} w|^2.$$

Similarly, we get

$$\mathcal{I}_4 = -\nu \lambda \int \eta(y) b(y) |\partial_x^{i+1} \partial_y^j w|^2.$$

On the other hand, we will now show that $\mathcal{I}_2 + \mathcal{I}_3 \lesssim \mathcal{E}(t)$. Indeed, by integrating by parts, we have

$$\mathcal{I}_2 = -\nu \lambda \frac{1}{2} \sum_{i+j \leq 5} \int |\partial_x^i \partial_y^j w|^2 \partial_y(\eta(y) a(y)) \lesssim \sum_{i+j \leq 5} \int \eta(y) |\partial_x^i \partial_y^j w|^2,$$

and for $\mathcal{I}_3$, we have

$$\mathcal{I}_3 \lesssim \nu \lambda \sum_{i+j \leq 5} \int \eta(y) |\partial_x^i \partial_y^j w|^2 \lesssim \mathcal{E}(t).$$

For $\mathcal{I}_5$, we use integration by parts in x to get

$$\mathcal{I}_5 \lesssim \nu \lambda \sum_{i+j \leq 5} \int \eta(y) |\partial_x^{i+1} \partial_y^j w|^2 \lesssim \lambda \mathcal{D}(t).$$

For $\mathcal{I}_6$, we have

$$\begin{aligned} \mathcal{I}_6 &= \sum_{i+j \leq 5} \frac{1}{2} \int \eta(y) a(y) \tilde{u} \cdot \nabla (|\partial_x^i \partial_y^j w|^2) = - \sum_{i+j \leq 5} \frac{1}{2} \int \operatorname{div}(\eta(y) a(y) \tilde{u}) |\partial_x^i \partial_y^j w|^2 \\ &= - \sum_{i+j \leq 5} \frac{1}{2} \int \tilde{u} \cdot \nabla(\eta(y) a(y)) |\partial_x^i \partial_y^j w|^2 = - \frac{1}{2} \sum_{i+j \leq 5} \int \tilde{u}_2 \partial_y(\eta(y) a(y)) |\partial_x^i \partial_y^j w|^2. \end{aligned}$$

If $y \geq \delta_0/2$ then we have

$$|\eta'(y)| = 2y \lesssim y^2 = \eta(y).$$

Hence

$$- \sum_{i+j \leq 4} \frac{1}{2} \int_{\delta_0}^{\infty} \tilde{u}_2 \partial_y(\eta(y) a(y)) |\partial_x^i \partial_y^j w|^2 \lesssim \|a(y) \tilde{u}_2\|_{L^\infty(y \geq \delta_0/4)} \mathcal{E}(t).$$

When $\frac{\delta_0}{4} \leq y \leq \frac{\delta_0}{2}$, we get

$$- \sum_{i+j \leq 5} \frac{1}{2} \int_{\delta_0/4}^{\delta_0/2} \tilde{u}_2 \partial_y(\eta(y) a(y)) |\partial_x^i \partial_y^j w|^2 \lesssim \|\tilde{u}_2\|_{L^\infty(\delta_0/4 \leq y \leq \delta_0/2)} \left(\mathcal{E}(t) + \sum_{i+j \leq 5} \|\partial_x^i \partial_y^j w\|_{L^2(\delta_0/4 \leq y \leq \delta_0/2)} \right).$$

This implies that

$$\mathcal{I}_6 \lesssim N_a(\tilde{u}, w) \mathcal{E}(t) + N_a(\tilde{u}, w)^2.$$

Lastly, we have

$$\mathcal{I}_7 \lesssim \|D_{x,y}^4(a(y) \tilde{u})\|_{L^\infty(y \geq \delta_0/4)} \mathcal{E}(t) + \|D_{x,y}^5(a(y) \tilde{u})\|_{L^2(y \geq \delta_0/4)} \|\eta(y)^{1/2} \nabla w\|_{L^\infty} \mathcal{E}(t)^{1/2}.$$

Using the Sobolev embedding $L^\infty(\mathbb{T} \times \mathbb{R}) \subset H^4(\mathbb{T} \times \mathbb{R})$, we have

$$\|\eta(y)^{1/2} \nabla w\|_{L^\infty} \lesssim \|D_{x,y}^4(\eta^{1/2} \nabla w)\|_{L^2} \lesssim \mathcal{E}(t)^{1/2} + \|D_{x,y}^5 w\|_{L^\infty(\delta_0/4 \leq y \leq \delta_0/2)}.$$

The proof is complete. $\square$

Proposition 6.2. *There holds*

$$N_a(\tilde{u}, w) \lesssim \|w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^5 w\|_{L^2(y \geq \delta_0/2)}$$

Proof. This is a direct consequence of the inequality (4.9) and Lemma 2.3. The proof is complete. $\square$

7 Nonlinear analysis

Our goal in this section is to combine all the estimates in analytic norm and Sobolev norms in the previous sections. We recall that w is the solution to the problem

$$\begin{aligned} (\partial_\tau - \nu \Delta - \nu \lambda L)w &= f, \\ \nu(\partial_y + N)w|_{y=0} &= g, \end{aligned}$$

where

$$\begin{aligned} f &= a(y)\tilde{u} \cdot \nabla w, \quad \tilde{u} = -\nabla^\perp \psi, \\ g &= -\partial_y(\Delta + \lambda L)^{-1}f|_{y=0}. \end{aligned}$$

We will use the coupled semigroup estimate for the exterior domain 3.5. We also recall the quantity defined in (5.1):

$$\begin{aligned} N_\rho(w, k) &= \|w\|_{\mathcal{W}_\rho^{k+1,1}} \left(\|w\|_{\mathcal{W}_\rho^{k,1}} + \|yD_{x,y}^k w\|_{L^2(y \geq \delta_0 + \rho)} \right) \\ &\quad + \|w\|_{\mathcal{W}_\rho^{k,1}} \|yD_{x,y}^{k+2} w\|_{L^2(y \geq \delta_0 + \rho)}, \end{aligned}$$

First we show the semigroup estimates.

Proposition 7.1. *Let $0 \leq k \leq 2$, there holds*

$$\|g(s)\|_{\mathcal{H}_\rho^k} \lesssim N_\rho(w(s), k) + \|yD_{x,y}^{k+2} w(s)\|_{L^2(y \geq \delta_0/2)}^2.$$

Proof. We define the function p solving the elliptic problem $(\Delta + \lambda L)p = a(y)\tilde{u} \cdot \nabla w$ with the boundary condition $p|_{y=0} = 0$. We have

$$\begin{aligned} \sum_\alpha e^{\varepsilon_0(\delta_0 + \rho)|\alpha|} |g| &= \sum_\alpha e^{\varepsilon_0(\delta_0 + \rho)|\alpha|} |\partial_y p_\alpha(0)| \lesssim \|\partial_y p\|_{\mathcal{L}_\rho^\infty} \\ &\lesssim \|a(y)\tilde{u} \cdot \nabla w\|_{\mathcal{L}_\rho^1} + \|ya(y)D_x(\tilde{u} \cdot \nabla w)\|_{L^2(y \geq \delta_0 + \rho)} \\ &\lesssim N_\rho(w, 0) + \|a(y)\tilde{u}\|_{L^\infty} \|yD_{x,y}^2 w\|_{L^2(y \geq \delta_0 + \rho)} + \|a(y)D_x \tilde{u}\|_{L^\infty(y \geq \delta_0 + \rho)} \|yD_{x,y}^1 w\|_{L^2(y \geq \delta_0/2)} \\ &\lesssim N_\rho(w, 0) + \left(\|w\|_{\mathcal{L}_\rho^1} + \|yD_{x,y}^1 w\|_{L^2(y \geq \delta_0/2)} \right) \|yD_{x,y}^2 w\|_{L^2(y \geq \delta_0/2)} \end{aligned}$$

where we use Proposition 5.1. The proof is complete. $\square$

Now we give the proof for our main theorem. Using the coupled semigroup estimates 3.5, we define the norm for $1 \leq k \leq 3$ (we can take $k = 1$).

$$\begin{aligned} \mathcal{A}_k(w(\tau), \rho) &= \left(\|w(\tau)\|_{\mathcal{W}_\rho^{k,1}} + \sqrt{\nu\tau} \|w(\tau)|_{z=0}\|_{\mathcal{H}_\rho^{k-1}} \right) \\ &\quad + \left(\|w(\tau)\|_{\mathcal{W}_\rho^{k+1,1}} + \sqrt{\nu\tau} \|w(\tau)|_{z=0}\|_{\mathcal{H}_\rho^k} \right) (\rho_0 - \rho - \beta\tau)^\gamma \end{aligned} \tag{7.1}$$

and the quantity

$$A(\beta) = \sup_{0 < \tau \beta < \rho_0} \left\{ \sup_{0 < \rho < \rho_0 - \beta \tau} (\mathcal{A}_k(w(\tau), \rho)) \right\} + \sup_{0 < \tau \beta < \rho_0} \|y D_{x,y}^3 w\|_{L^2(y \geq \delta_0/2)}.$$

Proposition 7.2. *There holds*

$$\begin{aligned} A(\beta) &\lesssim \|w_0\|_{\mathcal{W}_{\rho_0}^{2,1}} + \|y D_{x,y}^5 w_0\|_{L^2(y \geq \delta_0/4)} + \beta^{-1} A(\beta)^2 \\ &\quad + e^{C(1+A(\beta))\beta^{-1}} (\|y D_{x,y}^5 w_0\|_{L^2(y \geq \delta_0/4)} + \beta^{-1} A(\beta)^2). \end{aligned}$$

Proof. For simplicity, we let

$$M_0 = \|w_0\|_{\mathcal{W}_{\rho_0}^{k+1,1}} + \|y D_{x,y}^5 w_0\|_{L^2(y \geq \delta_0/2)}$$

First we bound $\|w(\tau)\|_{\mathcal{W}_{\rho}^k}$. By Theorem 3.5 and Proposition 7.1, we get

$$\begin{aligned} \|w(\tau)\|_{\mathcal{W}_{\rho}^k} &\lesssim M_0 + \lambda A(\beta) + \lambda \nu \int_0^{\tau} A(\beta) (\rho_0 - \rho - \beta s)^{-\gamma} ds \\ &\quad + \sqrt{\lambda \nu} A(\beta) \int_0^{\tau} (\rho_0 - \rho - \beta s)^{-\gamma} ds + A(\beta)^2 \int_0^{\tau} (\rho_0 - \rho - \beta s)^{-\gamma} ds + \beta^{-1} A(\beta) \\ &\lesssim M_0 + \lambda A(\beta) + \beta^{-1} A(\beta) + \beta^{-1} A(\beta)^2. \end{aligned}$$

Next, we bound $\sqrt{\nu \tau} \|w(\tau)|_{z=0}\|_{\mathcal{H}_{\rho}^{k-1}}$. From Propositions 3.6 and 7.1, we get

$$\begin{aligned} \sqrt{\nu \tau} \|w(\tau)|_{z=0}\|_{\mathcal{H}_{\rho}^{k-1}} &\lesssim M_0 + \lambda \nu \sqrt{\tau} A(\beta) \int_0^{\tau} s^{-1/2} (\rho_0 - \rho - \beta s)^{-\gamma} ds \\ &\quad + \lambda \sqrt{\nu \tau} A(\beta) + \nu \sqrt{\nu \tau} A(\beta) \int_0^{\tau} (\rho_0 - \rho - \gamma s)^{-\gamma} ds + \sqrt{\nu \tau} \beta^{-1} A(\beta) \\ &\quad + A(\beta)^2 \int_0^{\tau} \sqrt{\tau} (\tau - s)^{-1/2} ds + \sqrt{\nu \tau} A(\beta)^2 \int_0^{\tau} (1 + (\rho_0 - \rho - \beta s)^{-\gamma}) ds \\ &\lesssim M_0 + \lambda A(\beta) + \beta^{-1} A(\beta)^2. \end{aligned}$$

Next we bound $\|w(\tau)\|_{\mathcal{W}_{\rho}^{k+1,1}}$. Again using Propositions 3.5 and 7.1, we get

$$\begin{aligned} \|w(\tau)\|_{\mathcal{W}_{\rho}^{k+1,1}} &\lesssim M_0 + \lambda A(\beta) + \lambda \nu A(\beta) \int_0^{\tau} (\rho_0 - \rho - \beta s)^{-1-\gamma} ds \\ &\quad + \lambda \sqrt{\nu} A(\beta) \int_0^{\tau} s^{-1/2} (\rho_0 - \rho - \beta s)^{-\gamma-1} ds \\ &\quad + A(\beta)^2 \int_0^{\tau} (\rho_0 - \rho - \beta s)^{-1-\gamma} ds + \beta^{-1} A(\beta) \\ &\lesssim M_0 + \lambda A(\beta) + (\beta^{-1} A(\beta) + \beta^{-1} A(\beta)^2) (\rho_0 - \rho - \beta \tau)^{-\gamma}. \end{aligned}$$

Finally, we bound $\sqrt{\nu\tau}\|w(\tau)|_{z=0}\|_{\mathcal{H}_\rho^k}$. From Propositions [3.6](#) and [7.1](#), we get

$$\begin{aligned}\sqrt{\nu\tau}\|w(\tau)|_{z=0}\|_{\mathcal{H}_\rho^k} &\lesssim M_0 + \lambda\nu\sqrt{\nu\tau}A(\beta) \int_0^\tau s^{-1/2}(\rho_0 - \rho - \beta s)^{-\gamma-1} ds \\ &\quad + \lambda A(\beta) + \nu^{3/2}\sqrt{\tau}A(\beta) \int_0^\tau (\rho_0 - \rho - \beta s)^{-\gamma-1} ds \\ &\quad + \sqrt{\nu\tau}\beta^{-1}A(\beta)^2 + A(\beta)^2 \int_0^\tau \frac{\sqrt{\tau}}{\sqrt{\tau-s}}(\rho_0 - \rho - \beta s)^{-\gamma} ds \\ &\quad + \sqrt{\nu\tau}A(\beta)^2 \int_0^\tau (\rho_0 - \rho - \beta s)^{-1-\gamma} ds + \sqrt{\nu\tau} \int_0^\tau A(\beta)^2(1 + (\rho_0 - \rho - \beta s)^{-\gamma}) ds \\ &\lesssim M_0 + \lambda A(\beta) + \beta^{-1}(A(\beta) + A(\beta)^2)(\rho_0 - \rho - \beta\tau)^{-\gamma}.\end{aligned}$$

Finally, for $\|y^2 D_{x,y}^5 w\|_{L^2(y \geq \delta_0/2)}$, this is bounded by the functional energy $\mathcal{E}(t)$ in section [6](#). From Proposition [6.1](#) and Proposition [6.2](#), we get

$$\mathcal{E}'(\tau) \leq C_0 \left(\mathcal{E}(\tau) + A(\beta)\mathcal{E}(\tau) + A(\beta)^2 + A(\beta)^2 \mathcal{E}(\tau)^{1/2} \right).$$

By Gronwall lemma, we get

$$\mathcal{E}(\tau) \leq e^{C_0(1+A(\beta))\tau} \left(\mathcal{E}(0) + C_0 \int_0^\tau A(\beta)^2 ds \right)$$

Hence

$$\|y^2 D_{x,y}^5 w\|_{L^2(y \geq \delta_0/2)} \leq e^{C_0(1+A(\beta))\tau} \left(\|y^2 D_{x,y}^5 w_0\|_{L^2(y \geq \delta_0/4)} + C_0 \beta^{-1} A(\beta)^2 \right).$$

This completes the proof. $\square$

8 Proof of the main theorem

Taking β sufficiently large in Proposition [7.2](#), we have $A(\beta) \leq C_0$ for some constant C_0 that only depends on the size of the initial data. This implies

$$\|w(\tau)\|_{\mathcal{W}_\rho^{k,1}} + \sqrt{\nu\tau}\|w(\tau)\|_{\mathcal{H}_\rho^k} \leq C_0$$

uniformly in the time interval $\tau \in \left[0, \frac{\rho_0}{2\beta}\right]$. This implies

$$\sup_{0 \leq \tau \leq \frac{\rho_0}{2\beta}} \sum_{\alpha \in \mathbb{Z}} e^{\varepsilon_0 \delta_0 |\alpha|} |w_\alpha(\tau)|_{z=0} \leq C_0.$$

To show the uniform bound [\(1.11\)](#) on the vorticity, it is natural to switch back to the original variables (t, θ, r) . Using the relation [\(2.1\)](#), we obtain

$$\sup_{0 \leq t \leq \frac{\lambda^2 \rho_0}{2\beta}} \sqrt{\nu t} \sum_{n \in \mathbb{Z}} e^{\delta_0 \varepsilon_0 \lambda |n|} |\omega_n(t)|_{z=0} \leq C_0$$

Let $T = \frac{\lambda^2 \rho_0}{2\beta}$. We have, for any $\theta \in \mathbb{T}$ and $t \in [0, T]$:

$$|\omega^\nu(t, \theta, 1)| \leq \sum_{n \in \mathbb{Z}} |\omega_n(t, 1)| \leq C_0(\nu t)^{-1/2} \sum_{n \in \mathbb{Z}} e^{-\delta_0 \varepsilon_0 \lambda |n|}.$$

Hence we obtain, for some constant $C_0 > 0$:

$$\|\omega^\nu(t, \theta, r = 1)\|_{L^\infty(\mathbb{T})} \leq C_0(\nu t)^{-1/2} \quad (8.1)$$

for all $0 \leq t \leq T$. The proof of (1.11) is complete. To justify the inviscid limit (1.12), we check the condition

$$\nu \int_0^T |\omega^\nu(t, \theta, 1)| dt \rightarrow 0 \quad \text{as } \nu \rightarrow 0.$$

This is direct from the bound (8.1). The proof of Theorem 1.2 is complete.

Data Availability Statements. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Statements and Declarations.

- Competing Interests: There are no competing interests that are directly or indirectly related to the work submitted for publication.

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