

On nonlinear instability of Prandtl's boundary layers: the case of Rayleigh's stable shear flows

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Abstract

In 1904, Prandtl introduced his famous boundary layer in order to describe the behavior of solutions of Navier Stokes equations near a boundary as the viscosity goes to 0. His Ansatz has later been justified for analytic data by R.E. Caflisch and M. Sammartino. In this paper, we prove that his expansion is false, up to $O(\nu^{1/4})$ order terms in L^∞ norm, in the case of solutions with Sobolev regularity, even in cases where the Prandtl's equation is well posed in Sobolev spaces.

In addition, we also prove that monotonic boundary layer profiles, which are stable when $\nu = 0$, are nonlinearly unstable when $\nu > 0$, provided ν is small enough, up to $O(\nu^{1/4})$ terms in L^∞ norm.

Abstract

En 1905, Prandtl introduit sa célèbre couche limite pour décrire le comportement des solutions des équations de Navier Stokes près d'un bord lorsque la viscosité tend vers 0. Son Ansatz a été justifié dans le cas de données analytiques par R.E. Caflisch et M. Sammartino. Dans cet article nous prouvons que ce développement asymptotique est faux dans L^∞ à l'ordre $O(\nu^{1/4})$ pour des solutions à régularité Sobolev, même dans des cas où l'équation de Prandtl est elle bien posée.

De plus nous prouvons aussi que des profils de cisaillement monotones, qui sont stables à $\nu = 0$, sont nonlinéairement instable quand $\nu > 0$ est assez petit, à l'ordre $O(\nu^{1/4})$ dans L^∞ .

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1 Introduction

In this paper, we are interested in the inviscid limit $\nu \rightarrow 0$ of the Navier-Stokes equations for incompressible fluids, possibly subject to some external forcing f^ν , namely

$$\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla p^\nu = \nu \Delta u^\nu + f^\nu, \quad (1.1)$$

$$\nabla \cdot u^\nu = 0, \quad (1.2)$$

on the half plane $\Omega = \{(x, y) \in \mathbb{T} \times \mathbb{R}^+\}$, with the no-slip boundary condition

$$u^\nu = 0 \quad \text{on} \quad \partial\Omega. \quad (1.3)$$

As ν goes to 0, one would expect the solutions u^ν to converge to solutions of Euler equations for incompressible fluids

$$\partial_t u^0 + (u^0 \cdot \nabla) u^0 + \nabla p^0 = f^0, \quad (1.4)$$

$$\nabla \cdot u^0 = 0, \quad (1.5)$$

with the boundary condition

$$u^0 \cdot n = 0 \quad \text{on} \quad \partial\Omega, \quad (1.6)$$

where n is the unit normal to $\partial\Omega$.

At the beginning of the twentieth century, Prandtl introduced his well known boundary layers in order to describe the transition from Navier-Stokes to Euler equations as the viscosity tends to zero. Formally, we expect that

$$u^\nu(t, x, y) \approx u^0(t, x, y) + u_P\left(t, x, \frac{y}{\sqrt{\nu}}\right) + \mathcal{O}(\sqrt{\nu})_{L^\infty} \quad (1.7)$$

where u^0 solves the Euler equations (1.4)-(1.6), and u_P is the Prandtl boundary layer correction, which is of order one. The size of Prandtl's boundary layer is of order $\sqrt{\nu}$. Formally it is even possible to write an asymptotic expansion for u^ν in terms of powers of $\sqrt{\nu}$. The aim of this paper is to investigate whether (1.7) holds true.

The Prandtl's boundary layer equations on $u_P = (u_{P,1}, u_{P,2})$ read

$$\begin{aligned} \partial_t u_{P,1} + u_P \cdot \nabla u_{P,1} + \partial_x p^0 &= \partial_z^2 u_{P,1} + f^P, \\ \nabla \cdot u_P &= 0, \end{aligned} \quad (1.8)$$

together with appropriate boundary conditions to correct the no-slip boundary conditions of Navier-Stokes solutions. In the above, $\partial_x p^0$ denotes the

pressure gradient of the Euler flow on the boundary, and $u_{P,1}$ is the horizontal component of the velocity.

These equations have been intensively studied in the mathematical literature. Notably, solutions to the Prandtl equations have been constructed for monotonic data [21, 22, 1, 20] or data with Gevrey or analytic regularity [23, 5, 16]. In the case of non-monotonic data with Sobolev regularity, the Prandtl equations are ill-posed [3, 6, 15].

The validity of Prandtl's Ansatz (1.7) has been established in [23, 24] for initial data with analytic regularity, leaving a remainder of order $\sqrt{\nu}$. A similar result is also obtained in [19]. The Ansatz (1.7), with a specific boundary layer profile, has been recently justified for data with Gevrey regularity [4] by deriving sharp linear semigroup bounds near a stationary shear flow via energy estimates.

However these positive results hide a strong instability occurring at high spatial frequencies. For some profiles, there exist instabilities with horizontal wave numbers of order $\nu^{-1/2}$ which grow like $\exp(Ct/\sqrt{\nu})$. Within an analytic framework, these instabilities are initially of order $\exp(-D/\sqrt{\nu})$ and therefore later are of size $\exp((Ct - D)/\sqrt{\nu})$. They remain negligible in bounded time (as long as $t < D/2C$ for instance).

Within Sobolev spaces, these instabilities are predominant. Initially they are of size $C\nu^{-s/2}$ and grow like $C\nu^{-s/2}\exp(Ct/\sqrt{\nu})$. They reach $O(1)$ within vanishing times, of order $\sqrt{\nu}\log\nu$.

To construct instabilities we focus on particular initial data, called shear layer profiles, namely initial data of the form

$$u^\nu(0) = (U_0(\sqrt{\nu}), 0).$$

The function U_0 is called the profile of the shear layer. Two kinds of instability may be described

- An instability with horizontal wave numbers of order $O(\nu^{-1/2})$, growing like $\exp(Ct/\sqrt{\nu})$. This instability occurs when the profile is unstable with respect to the inviscid Euler equations, or, very roughly speaking, when the profile U_0 has a "strong" inflexion point (in the spirit of Rayleigh's criterium of unstability). For such profiles E. Grenier proved in [7] that Prandtl's asymptotic expansion is false, up to a remainder of order $\nu^{1/4}$ in L^∞ norm. More recently, E. Grenier and T. Nguyen managed to replace $O(\nu^{1/4})$ by $O(1)$ and proved that the difference between the genuine solution and the Prandtl's expansion may be of order one in supremum norm, and that this difference does

not vanish as ν goes to 0. These instabilities are driven by inviscid instabilities occurring within the boundary layer.

For such shear layer profiles, Prandtl's Ansatz is false. However, up to now, there is no existence result for Prandtl equation for small Sobolev perturbations of these shear layer profiles. It is therefore not possible to correctly define Prandtl boundary corrector in a neighborhood of $u^\nu(0)$.

- An instability with horizontal wave numbers of order $O(\nu^{-1/8})$, growing like $\exp(Ct/\nu^{1/4})$. These instabilities are much more subtle. Their growth is slower. The instability is driven by the so called "critical layer", which is at a distance $O(\nu^{5/8})$ from the boundary. The current paper is the equivalent of [7] for monotonic profiles. For these profiles we are not able to prove $O(1)$ separation, but only $O(\nu^{1/4})$. However, Prandtl equation is well posed in a Sobolev neighborhood of $u^0(\nu)$. We will discuss this limitation later.

In this paper, we shall prove the nonlinear instability of the Ansatz (1.7) near monotonic profiles. Roughly speaking, given an arbitrary stable boundary layer, the two main results in this paper are

- in the case of time-dependent boundary layers, we construct Navier-Stokes solutions, with arbitrarily small forcing, of order $\mathcal{O}(\nu^P)$, with P as large as we want, so that the Ansatz (1.7) is false near the boundary layer, up to a remainder of order $\nu^{1/4+\epsilon}$ in L^∞ norm, ϵ being arbitrarily small.
- in the case of stationary boundary layers, we construct Navier-Stokes solutions, without forcing term, so that the Ansatz (1.7) is false, up to a remainder of order $\nu^{5/8}$ in L^∞ norm.

These results prove that there exist no asymptotic expansion of Prandtl's type, even in the case of monotonic profiles. For such profiles, adding viscosity destabilizes the flow, which is counter intuitive. Even if Prandtl boundary layer equation is well posed, it does not describe the limiting behavior as the viscosity goes to 0.

The complete construction of the instability involves a sublayer, of size $\nu^{5/8}$, which was not expected in this context. This sublayer may itself become unstable when it becomes large enough, leading to the creation of a sub - sub - layer.

In this case we are not able to prove that the perturbation reaches $O(1)$ in L^∞ as is the case for Euler unstable profiles, since the linear growth of the

perturbation is much slower. In rescaled variables (see section 3), the linear growth is of order $\nu^{1/4}$. A simple equivalent in terms of ordinary differential equations would be

$$\dot{\phi} = \nu^{1/4}\phi + A\phi^2 \quad (1.9)$$

where $\nu^{1/4}\phi$ models the linear growth and ϕ^2 the nonlinear interaction terms. For (1.9), the nonlinear term is comparable to the linear one when ϕ is of order $\nu^{1/4}$, namely very small. The fate of ϕ then depends on the sign of A (blow up if $A > 0$ and convergence to a $O(\nu^{1/4})$ stable state if $A < 0$). The situation is similar here. The nonlinear term is comparable to the linear one when the perturbation reaches $O(\nu^{1/4})$, preventing further investigations.

In the next sections, we shall introduce the precise notion of Rayleigh's stable boundary layers and present our main results. After a brief recall of the linear instability results [9, 11] in Section 2, we give the proof of the main results in Sections 3 and 4, respectively.

1.1 Stable boundary layer profiles

Throughout this paper, by a boundary layer profile, we mean a shear flow of the form

$$U_{\text{bl}} := \begin{pmatrix} U_{\text{bl}}(t, \frac{y}{\sqrt{\nu}}) \\ 0 \end{pmatrix} \quad (1.10)$$

that solves the Prandtl's boundary layer problem (1.8), with initial data $U_{\text{bl}}(0, z) = U(z)$. Without forcing, U_{bl} is the solution of heat equation

$$\partial_t U_{\text{bl}} - \partial_{YY} U_{\text{bl}} = 0.$$

Boundary layer profiles can also be generated by adding a forcing term f^P , in which case we shall focus precisely on the corresponding stationary boundary layers $U_{\text{bl}} = U(z)$, with $-U''(z) = f^P$. We will consider these two different cases, namely time dependent boundary layers (without forcing) and time independent boundary layers (with given, time independent, forcing).

As mentioned, the Ansatz (1.7) is proven to be false for initial boundary layer profiles $U(z)$ that are spectrally unstable to the Euler equations [7]. *In this paper, we shall thus focus on stable profiles*, those that are spectrally stable to the Euler equations. This includes, for instance, boundary layer profiles without an inflection point by view of the classical Rayleigh's inflection point theorem. In this paper we assume in addition that $U(z)$ is strictly monotonic, real analytic, that $U(0) = 0$ and that $U(z)$ converges exponentially fast at infinity to a finite constant U_+ . By a slight abuse of language, such profiles will be referred to as stable profiles in this paper.

In order to study the instability of such boundary layers, we first analyze the spectrum of the corresponding linearized problem around initial profiles $U(z)$. We first introduce the isotropic boundary layer variables $(t, x, z) = (t, x, y)/\sqrt{\nu}$. This leads to the following linearized problem for vorticity $\omega = \partial_z v_1 - \partial_x v_2$, which reads

$$(\partial_t - L)\omega = 0, \quad L\omega := \sqrt{\nu}\Delta\omega - U\partial_x\omega - v_2U'', \quad (1.11)$$

together with $v = \nabla^\perp\phi$ and $\Delta\phi = \omega$, satisfying the no-slip boundary conditions $\phi = \partial_z\phi = 0$ on $\{z = 0\}$.

We then take the Fourier transform in the x variable only, denoting by α the corresponding wavenumber, which leads to

$$(\partial_t - L_\alpha)\omega = 0, \quad L_\alpha\omega := \sqrt{\nu}\Delta_\alpha\omega - i\alpha U\omega - i\alpha\phi U'', \quad (1.12)$$

where

$$\omega_\alpha = \Delta_\alpha\phi_\alpha,$$

together with the zero boundary conditions $\phi_\alpha = \phi'_\alpha = 0$ on $z = 0$. Here,

$$\Delta_\alpha = \partial_z^2 - \alpha^2.$$

Together with Y. Guo, we proved in [8, 9] that, even for profiles U which are stable as $\nu = 0$, there are unstable eigenvalues to the Navier-Stokes problem (1.12) for sufficiently small viscosity ν and for a range of wavenumber $\alpha \in [\alpha_1, \alpha_2]$, with $\alpha_1 \sim \nu^{1/8}$ and $\alpha_2 \sim \nu^{1/12}$. The unstable eigenvalues λ_* of L_α , found in [9], satisfy

$$\Re\lambda_* \sim \nu^{1/4}. \quad (1.13)$$

Such an instability was first observed by Heisenberg [13, 14], then Tollmien and C. C. Lin [17, 18]; see also Drazin and Reid [2, 25] for a complete account of the physical literature on the subject. See also Theorem 2.1 below for precise details. In coherence with the physical literature [2], we believe that, α being fixed, this eigenvalue is the most unstable one. However, this point is an open question from the mathematical point of view.

Next, we observe that L_α is a compact perturbation of the Laplacian $\sqrt{\nu}\Delta_\alpha$, and hence its unstable spectrum in the usual L^2 space is discrete. Thus, for each α, ν , we can define the maximal unstable eigenvalue $\lambda_{\alpha,\nu}$ so that $\Re\lambda_{\alpha,\nu}$ is maximum. We set $\lambda_{\alpha,\nu} = 0$, if no unstable eigenvalues exist.

In this paper, we assume that the unstable eigenvalues found in the spectral instability result, Theorem 2.1, are maximal eigenvalues. Precisely, we introduce

$$\gamma_0 := \limsup_{\nu \rightarrow 0} \sup_{\alpha \in \mathbb{R}} \nu^{-1/4} \Re\lambda_{\alpha,\nu}. \quad (1.14)$$

The existence of unstable eigenvalues in Theorem 2.1 implies that γ_0 is positive. Our spectral assumption is that γ_0 is finite (that is, the eigenvalues in Theorem 2.1 are maximal).

1.2 Main results

We are ready to state two main results of this paper.

Theorem 1.1. *Let $U_{\text{bl}}(t, z)$ be a time-dependent stable boundary layer profile as described in Section 1.1. Then, for arbitrarily large s, N and arbitrarily small positive ϵ , there exists a sequence of functions u^ν that solves the Navier-Stokes equations (1.1)-(1.3), with some forcing f^ν , so that*

$$\|u^\nu(0) - U_{\text{bl}}(0)\|_{H^s} + \sup_{t \in [0, T^\nu]} \|f^\nu(t)\|_{H^s} \leq \nu^N,$$

but

$$\begin{aligned} \|u^\nu(T^\nu) - U_{\text{bl}}(T^\nu)\|_{L^\infty} &\geq \nu^{\frac{1}{4} + \epsilon}, \\ \|\omega^\nu(T^\nu) - \omega_{\text{bl}}(T^\nu)\|_{L^\infty} &\rightarrow \infty, \end{aligned}$$

for time sequences $T^\nu \rightarrow 0$, as $\nu \rightarrow 0$. Here, $\omega^\nu = \nabla \times u^\nu$ denotes the vorticity of fluids.

This Theorem proves that the Ansatz (1.7) is false, even near stable boundary layers, for data with Sobolev regularity.

Theorem 1.2 (Instability result for stable profiles). *Let $U_{\text{bl}} = U(z)$ be a stable stationary boundary layer profile as described in Section 1.1. Then, for any s, N arbitrarily large, there exists a sequence of solutions u^ν to the Navier-Stokes equations, with forcing $f^\nu = f^P$ (boundary layer forcing), so that u^ν satisfy*

$$\|u^\nu(0) - U_{\text{bl}}\|_{H^s} \leq \nu^N,$$

but

$$\begin{aligned} \|u^\nu(T^\nu) - U_{\text{bl}}\|_{L^\infty} &\gtrsim \nu^{5/8}, \\ \|\omega^\nu(T^\nu) - \omega_{\text{bl}}\|_{L^\infty} &\gtrsim 1, \end{aligned}$$

for some time sequences $T^\nu \rightarrow 0$, as $\nu \rightarrow 0$.

The spectral instability for stable profiles gives rise to sublayers (or critical layers) whose thickness is of order $\nu^{5/8}$. The velocity gradient in this sublayer grows like $\nu^{-5/8} e^{t/\nu^{1/4}}$, and becomes large when t is of order T^ν . As a consequence, they may in turn become unstable after the instability time T^ν obtained in the above theorem. Thus, in order to improve the $\nu^{5/8}$ instability, one needs to further examine the stability properties of this sublayer itself (see [12]).

1.3 Boundary layer norms

We end the introduction by introducing the boundary layer norms to be used throughout the paper. These norms were introduced in [10] to capture the large, but localized, behavior of vorticity near the boundary. Precisely, for each vorticity function $\omega_\alpha = \omega_\alpha(z)$, we introduce the following boundary layer norms

$$\|\omega_\alpha\|_{\beta,\gamma,1} := \sup_{z \geq 0} \left[\left(1 + \delta^{-1} \phi_p(\delta^{-1} z) \right)^{-1} e^{\beta z} |\omega_\alpha(z)| \right], \quad (1.15)$$

where $\beta > 0$, p is a large, fixed number,

$$\phi_p(z) = \frac{1}{1 + z^p},$$

and with the boundary layer thickness

$$\delta = \gamma \nu^{1/8}$$

for some $\gamma > 0$. We introduce the boundary layer space $\mathcal{B}^{\beta,\gamma,1}$ to consist of functions whose $\|\cdot\|_{\beta,\gamma,1}$ norm is finite. We also denote by L_β^∞ the function spaces equipped with the finite norm

$$\|\omega\|_\beta = \sup_{z \geq 0} e^{\beta z} |\omega(z)|.$$

When there is no weight $e^{\beta z}$, we simply write L^∞ for the usual bounded function spaces. Clearly,

$$L_\beta^\infty \subset \mathcal{B}^{\beta,\gamma,1}.$$

In addition, it is straightforward to check that

$$\|fg\|_{\beta,\gamma,1} \leq \|f\|_{L^\infty} \|g\|_{\beta,\gamma,1}. \quad (1.16)$$

Finally, for functions $\omega(x, z)$, we introduce

$$\|\omega\|_{\sigma,\beta,\gamma,1} := \sup_{\alpha \in \mathbb{R}} (1 + |\alpha|)^\sigma \|\omega_\alpha\|_{\beta,\gamma,1},$$

for $\sigma > 1$, in which ω_α is the Fourier transform of ω in the variable x . Combining with (1.16), we have

$$\|fg\|_{\sigma,\beta,\gamma,1} \leq \|f\|_{\sigma,0} \|g\|_{\sigma,\beta,\gamma,1}, \quad (1.17)$$

where $\|f\|_{\sigma,0} = \sup_{\alpha \in \mathbb{R}} (1 + |\alpha|)^\sigma \|f_\alpha\|_{L^\infty}$.

2 Linear instability

In this section, we shall recall the spectral instability of stable boundary layer profiles [9] and the semigroup estimates on the corresponding linearized Navier-Stokes equation [10, 11].

2.1 Spectral instability

The following theorem, proved in [9], provides an unstable eigenvalue of L for generic shear flows.

Theorem 2.1 (Spectral instability; [9]). *Let $U(z)$ be an arbitrary shear profile with $U(0) = 0$ and $U'(0) > 0$ and satisfy*

$$\sup_{z \geq 0} |\partial_z^k (U(z) - U_+) e^{\eta_0 z}| < +\infty, \quad k = 0, \dots, 4,$$

for some constants U_+ and $\eta_0 > 0$. Let $R = \nu^{-1/2}$ be the Reynolds number, and set $\alpha_{\text{low}}(R) \sim R^{-1/4}$ and $\alpha_{\text{up}}(R) \sim R^{-1/6}$ be the lower and upper stability branches.

Then, there is a critical Reynolds number R_c so that for all $R \geq R_c$ and all $\alpha \in (\alpha_{\text{low}}(R), \alpha_{\text{up}}(R))$, there exist a nontrivial triple $c(R), \hat{v}(z; R), \hat{p}(z; R)$, with $\text{Im } c(R) > 0$, such that $v_R := e^{i\alpha(x-ct)} \hat{v}(z; R)$ and $p_R := e^{i\alpha(x-ct)} \hat{p}(z; R)$ solve the linearized Navier-Stokes problem (1.11). Moreover there holds the following estimate for the growth rate of the unstable solutions:

$$\alpha \Im c(R) \approx R^{-1/2}$$

as $R \rightarrow \infty$.

The proof of the previous Theorem, which can be found in [9], gives a detailed description of the unstable mode. In this paper we focus on the lower branch of instability. In this case

$$\alpha_\nu \approx R^{-1/4} = \nu^{1/8}, \quad \Re \lambda_\nu \approx R^{-1/2} = \nu^{1/4},$$

The vorticity of the unstable mode is of the form

$$\omega_0 = e^{\lambda_\nu t} \Delta(e^{i\alpha_\nu x} \phi_0(z)) + \text{complex conjugate} \quad (2.1)$$

The stream function ϕ_0 is constructed through asymptotic expansions, and is of the form

$$\phi_0 := \phi_{in,0}(z) + \delta_{bl} \phi_{bl,0}(\delta_{bl}^{-1} z)$$

for some boundary layer function $\phi_{bl,0}$, where $\delta_{bl} = \nu^{1/8}$.

By construction, derivatives of $\phi_{bl,0}$ satisfy

$$|\partial_z^k \phi_{bl,0}(\delta_{bl}^{-1} z)| \leq C_k \delta_{bl}^{-k} e^{-\eta_0 z / \delta_{bl}}.$$

In addition, it is clear that each x -derivative of ω_0 gains a small factor of $\alpha_\nu \approx \nu^{1/8}$. We therefore have an accurate description of the linear unstable mode.

2.2 Linear estimates

The corresponding semigroup e^{Lt} of the linear problem (1.11) is constructed through the path integral

$$e^{Lt} \omega = \int_{\mathbb{R}} e^{i\alpha x} e^{L_\alpha t} \omega_\alpha d\alpha \quad (2.2)$$

in which ω_α is the Fourier transform of ω in tangential variables and L_α , defined as in (1.12), is the Fourier transform of L . One of the main results proved in [11] is the following theorem.

Theorem 2.2. [11] *Let $\alpha \lesssim \nu^{1/8}$. Let $\omega_\alpha \in \mathcal{B}^{\beta, \gamma, 1}$ for some positive β, γ_0 be defined as in (1.14). Assume that γ_0 is finite. Then, for any $\gamma_1 > \gamma_0$, there is some positive constant C_γ so that*

$$\begin{aligned} \|e^{L_\alpha t} \omega_\alpha\|_{\beta, \gamma, 1} &\leq C_\gamma e^{\gamma_1 \nu^{1/4} t} e^{-\frac{1}{4} \alpha^2 \sqrt{\nu} t} \|\omega_\alpha\|_{\beta, \gamma, 1}, \\ \|\partial_z e^{L_\alpha t} \omega_\alpha\|_{\beta, \gamma, 1} &\leq C_\gamma \left(\nu^{-1/8} + (\sqrt{\nu} t)^{-1/2} \right) e^{\gamma_1 \nu^{1/4} t} e^{-\frac{1}{4} \alpha^2 \sqrt{\nu} t} \|\omega_\alpha\|_{\beta, \gamma, 1}. \end{aligned}$$

3 Approximate solutions

Let us now construct an approximate solution u_{app} , which solves Navier-Stokes equations, up to very small error terms. First, we introduce the rescaled isotropic space time variables

$$\tilde{t} = \frac{t}{\sqrt{\nu}}, \quad \tilde{x} = \frac{x}{\sqrt{\nu}}, \quad \tilde{z} = \frac{z}{\sqrt{\nu}}.$$

Without any confusion, we drop the tildes. The Navier-Stokes equations in these scaled variables read

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + \nabla p &= \sqrt{\nu} \Delta u, \\ \nabla \cdot u &= 0, \end{aligned} \quad (3.1)$$

with the no-slip boundary conditions on $z = 0$. Theorem 1.1 follows at once from the following theorem.

Theorem 3.1. *Let $U(z)$ be a stable boundary layer profile, and let $U_{\text{bl}}(\sqrt{\nu}t, z)$ be the corresponding Prandtl's boundary layer. Then, there exist an approximate solution $\tilde{u}_{\text{app}}$ that approximately solves (3.1) in the following sense: for arbitrarily large numbers p, M and for any $\epsilon > 0$, the functions $\tilde{u}_{\text{app}}$ solve*

$$\begin{aligned} \partial_t \tilde{u}_{\text{app}} + (\tilde{u}_{\text{app}} \cdot \nabla) \tilde{u}_{\text{app}} + \nabla \tilde{p}_{\text{app}} &= \sqrt{\nu} \Delta \tilde{u}_{\text{app}} + \mathcal{E}_{\text{app}}, \\ \nabla \cdot \tilde{u}_{\text{app}} &= 0, \end{aligned} \quad (3.2)$$

for some remainder $\mathcal{E}_{\text{app}}$ and for time $t \leq T_\nu$, with T_ν being defined through

$$\nu^p e^{\Re \lambda_0 T_\nu} = \nu^{\frac{1}{4} + \epsilon}.$$

In addition, for all $t \in [0, T_\nu]$, there hold

$$\begin{aligned} \|\text{curl}(\tilde{u}_{\text{app}} - U_{\text{bl}}(\sqrt{\nu}t, z))\|_{\beta, \gamma, 1} &\lesssim \nu^{\frac{1}{4} + \epsilon}, \\ \|\text{curl} \mathcal{E}_{\text{app}}(t)\|_{\beta, \gamma, 1} &\lesssim \nu^M. \end{aligned}$$

Furthermore, there are positive constants $\theta_0, \theta_1, \theta_2$ independent of ν so that there holds

$$\theta_1 \nu^p e^{\Re \lambda_0 t} \leq \|(\tilde{u}_{\text{app}} - U_{\text{bl}})(t)\|_{L^\infty} \leq \theta_2 \nu^p e^{\Re \lambda_0 t}$$

for all $t \in [0, T_\nu]$. In particular,

$$\|(\tilde{u}_{\text{app}} - U_{\text{bl}})(T_\nu)\|_{L^\infty} \gtrsim \nu^{\frac{1}{4} + \epsilon}.$$

3.1 Formal construction

The construction is classical, following [7]. Indeed, the approximate solutions are constructed in the following form

$$\tilde{u}_{\text{app}}(t, x, z) = U_{\text{bl}}(\sqrt{\nu}t, z) + \nu^p \sum_{j=0}^M \nu^{j/8} u_j(t, x, z). \quad (3.3)$$

For convenience, let us set $v = u - U_{\text{bl}}$, where u denotes the genuine solution to the Navier-Stokes equations (3.1). Then, the vorticity $\omega = \nabla \times v$ solves

$$\partial_t \omega + (U_{\text{bl}}(\sqrt{\nu}t, y) + v) \cdot \nabla \omega + v_2 \partial_y^2 U_s(\sqrt{\nu}t, y) - \sqrt{\nu} \Delta \omega = 0$$

in which $v = \nabla^\perp \Delta^{-1} \omega$ and v_2 denotes the vertical component of velocity. Here and in what follows, Δ^{-1} is computed with the zero Dirichlet boundary

condition. As U_{bl} depends slowly on time, we can rewrite the vorticity equation as follows:

$$(\partial_t - L)\omega + \nu^{1/8}S\omega + Q(\omega, \omega) = 0. \quad (3.4)$$

In (3.4), L denotes the linearized Navier-Stokes operator around the stationary boundary layer $U = U_s(0, z)$:

$$L\omega := \sqrt{\nu}\Delta\omega - U\partial_x\omega - u_2U'',$$

$Q(\omega, \tilde{\omega})$ denotes the quadratic nonlinear term $u \cdot \nabla \tilde{\omega}$, with $v = \nabla^\perp \Delta^{-1}\omega$, and S denotes the perturbed operator defined by

$$S\omega := \nu^{-1/8}[U_s(\sqrt{\nu}t, z) - U(z)]\partial_x\omega + \nu^{-1/8}u_2[\partial_y^2 U_s(\sqrt{\nu}t, z) - U''(z)].$$

Recalling that U_s solves the heat equation with initial data $U(z)$, we have

$$|U_s(\sqrt{\nu}t, z) - U(z)| \leq C\|U''\|_{L^\infty} e^{-\eta_0 z} \sqrt{\nu}t$$

and

$$|\partial_z^2 U_s(\sqrt{\nu}t, z) - U''(z)| \leq C\|U''\|_{W^{2,\infty}} e^{-\eta_0 z} \sqrt{\nu}t.$$

Hence,

$$S\omega = \nu^{-1/8}\mathcal{O}(\sqrt{\nu}te^{-\eta_0 z})\left[|\partial_x\omega| + |\partial_x\Delta^{-1}\omega|\right] \quad (3.5)$$

in which $\Delta^{-1}\omega$ satisfies the zero boundary condition on $z = 0$. The approximate solutions are then constructed via the asymptotic expansion:

$$\omega_{\text{app}} = \nu^p \sum_{j=0}^M \nu^{j/8} \omega_j, \quad (3.6)$$

in which p is an arbitrarily large integer. Plugging this Ansatz into (3.4) and matching order in ν , we are led to solve

- for $j = 0$:

$$(\partial_t - L)\omega_0 = 0$$

with zero boundary conditions on $v_0 = \nabla^\perp(\Delta)^{-1}\omega_0$ on $z = 0$;

- for $0 < j \leq M$:

$$(\partial_t - L)\omega_j = R_j, \quad \omega_j|_{t=0} = 0, \quad (3.7)$$

with zero boundary condition on $v_j = \nabla^\perp(\Delta)^{-1}\omega_j$ on $z = 0$. Here, the remainders R_j are defined by

$$R_j = S\omega_{j-1} + \sum_{k+\ell+8p=j} Q(\omega_k, \omega_\ell).$$

As a consequence, the approximate vorticity ω_{app} solves (3.4), leaving the error R_{app} defined by

$$R_{\text{app}} = \nu^{p+\frac{M+1}{8}} S\omega_M + \sum_{k+\ell > M+1-8p; 1 \leq k, \ell \leq M} \nu^{2p+\frac{k+\ell}{8}} Q(\omega_k, \omega_\ell) \quad (3.8)$$

which formally is of order $\nu^{p+\frac{M+1}{8}}$, for arbitrary p and M .

3.2 Estimates

We start the construction with ω_0 being the maximal growing mode, constructed in Section 2.1. We recall

$$\omega_0 = e^{\lambda_\nu t} e^{i\alpha_\nu x} \Delta_{\alpha_\nu} \left(\phi_{\text{in},0}(z) + \nu^{1/8} \phi_{\text{bl},0}(\nu^{-1/8} z) \right) + \text{c.c.} \quad (3.9)$$

with $\alpha_\nu \sim \nu^{1/8}$ and $\Re \lambda_\nu \sim \nu^{1/4}$. In what follows, α_ν and λ_ν are fixed. We obtain the following lemma.

Lemma 3.2. *Let ω_0 be the maximal growing mode (3.9), and let ω_j be inductively constructed by (3.7). Then, there hold the following uniform bounds:*

$$\|\partial_x^a \partial_z^b \omega_j\|_{\sigma, \beta, \gamma, 1} \leq C_0 \nu^{a/8} \nu^{-b/8} \nu^{-\frac{1}{4}[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}t} \quad (3.10)$$

for all $a, j \geq 0$ and for $b = 0, 1$. In addition, the approximate solution ω_{app} defined as in (3.6) satisfies

$$\|\partial_x^a \partial_z^b \omega_{\text{app}}\|_{\sigma, \beta, \gamma, 1} \lesssim \nu^{a/8} \nu^{-b/8} \sum_{j=0}^M \nu^{-\frac{1}{4}[\frac{j}{8p}]} \left(\nu^p e^{\gamma_0 \nu^{1/4}t} \right)^{1+\frac{j}{8p}}, \quad (3.11)$$

for $a \geq 0$ and $b = 0, 1$. Here, $[k]$ denotes the largest integer so that $[k] \leq k$.

Proof. For $j \geq 1$, we construct ω_j having the form

$$\omega_j = \sum_{n \in \mathbb{Z}} e^{in\alpha_\nu x} \omega_{j,n}$$

It follows that $\omega_{j,n}$ solves

$$(\partial_t - L_{\alpha_n})\omega_{j,n} = R_{j,n}, \quad \omega_{j,n}|_{t=0} = 0$$

with $\alpha_n = n\alpha_\nu$ and $R_{j,n}$ the Fourier transform of R_j evaluated at the Fourier frequency α_n . Precisely, we have

$$R_{j,n} = S_{\alpha_n} \omega_{j-1,n} + \sum_{k+\ell+8p=j} \sum_{n_1+n_2=n} Q_{\alpha_n}(\omega_{k,n_1}, \omega_{\ell,n_2}),$$

in which S_{α_n} and Q_{α_n} denote the corresponding operator S and Q in the Fourier space. The Duhamel's integral reads

$$\omega_{j,n}(t) = \int_0^t e^{L_{\alpha_n}(t-s)} R_{j,n}(s) ds \quad (3.12)$$

for all $j \geq 1$ and $n \in \mathbb{Z}$.

It follows directly from an inductive argument and the quadratic non-linearity of $Q(\cdot, \cdot)$ that for all $0 \leq j \leq M$, $\omega_{j,n} = 0$ for all $|n| \geq 2^{j+1}$. This proves that $|\alpha_n| \leq 2^{M+1} \alpha_\nu \lesssim \nu^{1/8}$, for all $|n| \leq 2^{M+1}$. Since $\alpha_n \lesssim \nu^{1/8}$, the semigroup bounds from Theorem 2.2 read

$$\begin{aligned} \|e^{L_{\alpha} t} \omega_{\alpha}\|_{\beta, \gamma, 1} &\lesssim e^{\gamma_1 \nu^{1/4} t} e^{-\frac{1}{4} \alpha^2 \sqrt{\nu} t} \|\omega_{\alpha}\|_{\beta, \gamma, 1}, \\ \|\partial_z e^{L_{\alpha} t} \omega_{\alpha}\|_{\beta, \gamma, 1} &\lesssim \left(\nu^{-1/8} + (\sqrt{\nu} t)^{-1/2} \right) e^{\gamma_1 \nu^{1/4} t} e^{-\frac{1}{4} \alpha^2 \sqrt{\nu} t} \|\omega_{\alpha}\|_{\beta, \gamma, 1}. \end{aligned} \quad (3.13)$$

In addition, since $\alpha_n \lesssim \nu^{1/8}$, from (3.5), we compute

$$S_{\alpha_n} \omega_{j-1,n} = \mathcal{O}(\sqrt{\nu} t e^{-\eta_0 z}) \left[|\omega_{j-1,n}| + |\Delta_{\alpha_n}^{-1} \omega_{j-1,n}| \right]$$

and hence by induction we obtain

$$\begin{aligned} \|S_{\alpha_n} \omega_{j-1,n}\|_{\beta, \gamma, 1} &\lesssim \sqrt{\nu} t \left[\|\omega_{j-1,n}\|_{\beta, \gamma, 1} + \|e^{-\eta_0 z} \Delta_{\alpha_n}^{-1} \omega_{j-1,n}\|_{\beta, \gamma, 1} \right] \\ &\lesssim \sqrt{\nu} t \left[\|\omega_{j-1,n}\|_{\beta, \gamma, 1} + \|\Delta_{\alpha_n}^{-1} \omega_{j-1,n}\|_{L^\infty} \right] \\ &\lesssim \sqrt{\nu} t \nu^{-\frac{1}{4} [\frac{j-1}{8p}]} e^{\gamma_0 (1 + \frac{j-1}{8p}) \nu^{1/4} t}, \end{aligned} \quad (3.14)$$

where we used $\|e^{-\eta_0 z} \cdot\|_{\beta, \gamma, 1} \leq \|\cdot\|_{L^\infty}$ for $\beta < \eta_0$, and

$$\|\Delta_{\alpha}^{-1} \omega\|_{L^\infty} \leq C \|\omega\|_{\beta, \gamma, 1},$$

uniformly in small α ; we shall prove this inequality in the Appendix. Let us first consider the case when $1 \leq j \leq 8p-1$, for which $R_{j,n} = S_{\alpha_n} \omega_{j-1,n}$. That is, there is no nonlinearity in the remainder. Using the above estimate on S_{α_n} and the semigroup estimate (3.13) into (3.12), we obtain, for $1 \leq j \leq 8p-1$,

$$\begin{aligned} \|\omega_{j,n}(t)\|_{\beta, \gamma, 1} &\leq \int_0^t \|e^{L_{\alpha_n}(t-s)} S_{\alpha_n} \omega_{j-1,n}(s)\|_{\beta, \gamma, 1} ds \\ &\leq C \int_0^t e^{\gamma_1 \nu^{1/4}(t-s)} \|S_{\alpha_n} \omega_{j-1,n}(s)\|_{\beta, \gamma, 1} ds \\ &\leq C \int_0^t e^{\gamma_1 \nu^{1/4}(t-s)} \sqrt{\nu} s e^{\gamma_0 (1 + \frac{j-1}{8p}) \nu^{1/4} s} ds. \end{aligned}$$

We choose

$$\gamma_1 = \gamma_0 \left(1 + \frac{j-1}{8p} + \frac{1}{16p}\right)$$

in (3.13) and use the inequality

$$\nu^{1/4}t \leq C e^{\frac{\gamma_0}{16p}\nu^{1/4}t}.$$

and obtain

$$\begin{aligned} \|\omega_{j,n}(t)\|_{\beta,\gamma,1} &\leq C \int_0^t e^{\gamma_1 \nu^{1/4}(t-s)} \nu^{1/4} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4}s} ds \\ &\leq C \nu^{1/4} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4}t} \int_0^t ds \\ &\leq C \nu^{1/4} t e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4}t} \\ &\leq C e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}t}. \end{aligned} \tag{3.15}$$

Similarly, as for derivatives, we obtain

$$\begin{aligned} &\|\partial_z \omega_{j,n}(t)\|_{\beta,\gamma,1} \\ &\leq \int_0^t \|e^{L_{\alpha_n}(t-s)} S_{\alpha_n} \omega_{j-1,n}(s)\|_{\beta,\gamma,1} ds \\ &\leq C \int_0^t \left(\nu^{-1/8} + (\sqrt{\nu}(t-s))^{-1/2} \right) e^{\gamma_1 \nu^{1/4}(t-s)} \|S_{\alpha_n} \omega_{j-1,n}(s)\|_{\beta,\gamma,1} ds \\ &\leq C \int_0^t \left(\nu^{-1/8} + (\sqrt{\nu}(t-s))^{-1/2} \right) e^{\gamma_1 \nu^{1/4}(t-s)} \sqrt{\nu} s e^{\gamma_0(1+\frac{j-1}{8p})\nu^{1/4}s} ds, \end{aligned}$$

in which the integral involving $\nu^{-1/8}$ is already treated in (3.15) and bounded by $C \nu^{-1/8} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}t}$. As for the second integral, we estimate

$$\begin{aligned} &\int_0^t (\sqrt{\nu}(t-s))^{-1/2} e^{\gamma_1 \nu^{1/4}(t-s)} \sqrt{\nu} s e^{\gamma_0(1+\frac{j-1}{8p})\nu^{1/4}s} ds \\ &\leq \int_0^t (\sqrt{\nu}(t-s))^{-1/2} e^{\gamma_1 \nu^{1/4}(t-s)} \nu^{1/4} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4}s} ds \\ &\leq \nu^{1/4} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4}t} \int_0^t (\sqrt{\nu}(t-s))^{-1/2} ds \\ &\leq C \sqrt{t} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4}t} \\ &\leq C \nu^{-1/8} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}t}. \end{aligned} \tag{3.16}$$

Thus,

$$\|\partial_z \omega_{j,n}(t)\|_{\beta,\gamma,1} \leq C \nu^{-1/8} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}t}.$$

This and (3.15) prove the inductive bound (3.10) for $j \leq 8p-1$.

For $j \geq 8p$, the quadratic nonlinearity starts to play a role. For $k + \ell = j - 8p$, we compute

$$Q_{\alpha_n}(\omega_{k,n_1}, \omega_{\ell,n_2}) = i\alpha_n \left(n_2 \partial_z \Delta_{\alpha_n}^{-1} \omega_{k,n_1} \omega_{\ell,n_2} - n_1 \Delta_{\alpha_n}^{-1} \omega_{k,n_1} \partial_z \omega_{\ell,n_2} \right). \quad (3.17)$$

Using the algebra structure of the boundary layer norm (see (1.16)), we have

$$\begin{aligned} \alpha_n \|\partial_z \Delta_{\alpha_n}^{-1} \omega_{k,n_1} \omega_{\ell,n_2}\|_{\beta,\gamma,1} &\lesssim \nu^{1/8} \|\partial_z \Delta_{\alpha_n}^{-1} \omega_{k,n_1}\|_{L^\infty} \|\omega_{\ell,n_2}\|_{\beta,\gamma,1} \\ &\lesssim \nu^{1/8} \|\omega_{k,n_1}\|_{\beta,\gamma,1} \|\omega_{\ell,n_2}\|_{\beta,\gamma,1} \\ &\lesssim \nu^{1/8} \nu^{-\frac{1}{4}[\frac{k}{8p}]} \nu^{-\frac{1}{4}[\frac{\ell}{8p}]} e^{\gamma_0(2+\frac{k+\ell}{8p})\nu^{1/4}t} \end{aligned}$$

where we used

$$\|\partial_z \Delta_{\alpha_n}^{-1} \omega_{k,n_1}\|_{L^\infty} \leq C \|\omega_{k,n_1}\|_{\beta,\gamma,1},$$

an inequality which is proven in the Appendix. Moreover,

$$\begin{aligned} \alpha_n \|\Delta_{\alpha_n}^{-1} \omega_{k,n_1} \partial_z \omega_{\ell,n_2}\|_{\beta,\gamma,1} &\lesssim \nu^{1/8} \|\Delta_{\alpha_n}^{-1} \omega_{k,n_1}\|_{L^\infty} \|\partial_z \omega_{\ell,n_2}\|_{\beta,\gamma,1} \\ &\lesssim \nu^{1/8} \|\omega_{k,n_1}\|_{\beta,\gamma,1} \|\partial_z \omega_{\ell,n_2}\|_{\beta,\gamma,1} \\ &\lesssim \nu^{-\frac{1}{4}[\frac{k}{8p}]} \nu^{-\frac{1}{4}[\frac{\ell}{8p}]} e^{\gamma_0(2+\frac{k+\ell}{8p})\nu^{1/4}t}, \end{aligned}$$

in which the derivative estimate (3.10) was used. We note that

$$[\frac{k}{8p}] + [\frac{\ell}{8p}] \leq [\frac{k+\ell}{8p}] = [\frac{j}{8p}] - 1.$$

This proves

$$\|Q_{\alpha_n}(\omega_{k,n_1}, \omega_{\ell,n_2})\|_{\beta,\gamma,1} \lesssim \nu^{1/4} \nu^{-\frac{1}{4}[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}t}$$

for all $k + \ell = j - 8p$. This, together with the estimate (3.14) on S_{α_n} , yields

$$\begin{aligned} \|R_{j,n}(t)\|_{\beta,\gamma,1} &\lesssim \sqrt{\nu} \nu^{-\frac{1}{4}[\frac{j-1}{8p}]} e^{\gamma_0(1+\frac{j-1}{8p})\nu^{1/4}t} + \nu^{1/4} \nu^{-\frac{1}{4}[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}t} \\ &\lesssim \nu^{1/4} \nu^{-\frac{1}{4}[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}t}, \end{aligned}$$

for all $j \geq 8p$ and $n \in \mathbb{Z}$, in which we used $\nu^{1/4}t \leq e^{\gamma_0 t/8p}$.

Putting these estimates into the Duhamel's integral formula (3.12), we obtain, for $j \geq 8p$,

$$\begin{aligned} \|\omega_{j,n}(t)\|_{\beta,\gamma,1} &\leq C \int_0^t e^{\gamma_1 \nu^{1/4}(t-s)} \|R_{j,n}(s)\|_{\beta,\gamma,1} ds \\ &\leq C \int_0^t e^{\gamma_1 \nu^{1/4}(t-s)} \nu^{1/4} \nu^{-\frac{1}{4}[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}s} ds \\ &\lesssim \nu^{-\frac{1}{4}[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}s} \end{aligned}$$

and

$$\begin{aligned} &\|\partial_z \omega_{j,n}(t)\|_{\beta,\gamma,1} \\ &\leq C \int_0^t \left(\nu^{-1/8} + (\sqrt{\nu}(t-s))^{-1/2} \right) e^{\gamma_1 \nu^{1/4}(t-s)} \|R_{j,n}(s)\|_{\beta,\gamma,1} ds \\ &\leq C \int_0^t \left(\nu^{-1/8} + (\sqrt{\nu}(t-s))^{-1/2} \right) e^{\gamma_1 \nu^{1/4}(t-s)} \nu^{1/4} \nu^{-\frac{1}{4}[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}s} ds. \end{aligned}$$

Using (3.16), we obtain

$$\|\partial_z \omega_{j,n}(t)\|_{\beta,\gamma,1} \lesssim \nu^{-1/8} \nu^{-\frac{1}{4}[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}s},$$

which completes the proof of (3.10). The lemma follows. $\square$

3.3 The remainder

We recall that the approximate vorticity ω_{app} , constructed as in (3.6), approximately solves (3.4), leaving the error R_{app} defined by

$$R_{\text{app}} = \nu^{p+\frac{M+1}{8}} S\omega_M + \sum_{k+\ell > M+1-8p; 1 \leq k, \ell \leq M} \nu^{2p+\frac{k+\ell}{8}} Q(\omega_k, \omega_\ell).$$

Using the estimates in Lemma 3.2, we obtain

$$\begin{aligned} \|S\omega_M\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{1/4} \nu^{-\frac{1}{4}[\frac{M+1}{8p}]} e^{\gamma_0(1+\frac{M+1}{8p})\nu^{1/4}t} \\ \|Q(\omega_k, \omega_\ell)\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{1/4} \nu^{-\frac{1}{4}[\frac{k+\ell}{8p}]} e^{\gamma_0(2+\frac{k+\ell}{8p})\nu^{1/4}t}. \end{aligned}$$

This yields

$$\|R_{\text{app}}\|_{\sigma,\beta,\gamma,1} \lesssim \nu^{1/4} \sum_{j=M+1}^{2M} \nu^{-\frac{1}{4}[\frac{j}{8p}]} \left(\nu^p e^{\gamma_0 \nu^{1/4}t} \right)^{1+\frac{j}{8p}}. \quad (3.18)$$

3.4 Proof of Theorem 3.1

The proof of the Theorem now straightforwardly follows from the estimates from Lemma 3.2 and the estimate (3.18) on the remainder. Indeed, we choose the time T_* so that

$$\nu^p e^{\gamma_0 \nu^{1/4} T_*} = \nu^\tau \quad (3.19)$$

for some fixed $\tau > \frac{1}{4}$. It then follows that for all $t \leq T_*$ and $j \geq 0$, there holds

$$\nu^{-\frac{1}{4}[\frac{j}{8p}]} \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right)^{1+\frac{j}{8p}} \lesssim \nu^\tau \nu^{(\tau-\frac{1}{4})\frac{j}{8p}}.$$

Using this into the estimates (3.11) and (3.18), respectively, we obtain

$$\begin{aligned} \|\partial_z^b \omega_{\text{app}}(t)\|_{\sigma, \beta, \gamma, 1} &\lesssim \nu^{p-b/8} e^{\gamma_0 \nu^{1/4} t} \lesssim \nu^{\tau-b/8}, \\ \|R_{\text{app}}(t)\|_{\sigma, \beta, \gamma, 1} &\lesssim \nu^{1/4} \nu^{-\frac{1}{4}[\frac{M}{8p}]} \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right)^{1+\frac{M}{8p}} \\ &\lesssim \nu^{\tau+1/4} \nu^{(\tau-\frac{1}{4})\frac{M}{8p}}, \end{aligned} \quad (3.20)$$

for all $t \leq T_*$. Since $\tau > \frac{1}{4}$ and M is arbitrarily large (and fixed), the remainder is of order ν^P for arbitrarily large number P . The theorem is proved.

4 Nonlinear instability

We are now ready to give the proof of Theorem 1.2. Let $\tilde{u}_{\text{app}}$ be the approximate solution constructed in Theorem 3.1 and let

$$v = u - \tilde{u}_{\text{app}},$$

with u being the genuine solution to the nonlinear Navier-Stokes equations. The corresponding vorticity $\omega = \nabla \times v$ solves

$$\partial_t \omega + (\tilde{u}_{\text{app}} + v) \cdot \nabla \omega + v \cdot \nabla \tilde{\omega}_{\text{app}} = \sqrt{\nu} \Delta \omega + R_{\text{app}}$$

for the remainder $R_{\text{app}} = \text{curl } \mathcal{E}_{\text{app}}$ satisfying the estimate (3.18). Let us write

$$u_{\text{app}} = \tilde{u}_{\text{app}} - U_{\text{bl}}.$$

To make use of the semigroup bound for the linearized operator $\partial_t - L$, we rewrite the vorticity equation as

$$(\partial_t - L)\omega + (u_{\text{app}} + v) \cdot \nabla \omega + v \cdot \nabla \omega_{\text{app}} = R_{\text{app}}$$

with $\omega|_{t=0} = 0$. We note that since the boundary layer profile is stationary, the perturbative operator S defined as in (3.5) is in fact zero. The Duhamel's principle then yields

$$\omega(t) = \int_0^t e^{L(t-s)} \left(R_{\text{app}} - (u_{\text{app}} + v) \cdot \nabla \omega - v \cdot \nabla \omega_{\text{app}} \right) ds. \quad (4.1)$$

Using the representation (4.1), we shall prove the existence and give estimates on ω . We shall work with the following norm

$$|||\omega(t)||| := \|\omega(t)\|_{\sigma,\beta,\gamma,1} + \nu^{1/8} \|\partial_x \omega(t)\|_{\sigma,\beta,\gamma,1} + \nu^{1/8} \|\partial_z \omega(t)\|_{\sigma,\beta,\gamma,1} \quad (4.2)$$

in which the factor $\nu^{1/8}$ added in the norm is to overcome the loss of $\nu^{-1/8}$ for derivatives (see (4.4) for more details).

Let p be an arbitrary large number. We introduce the maximal time T_ν of existence, defined by

$$T_\nu := \max \left\{ t \in [0, T_*] : \sup_{0 \leq s \leq t} |||\omega(s)||| \leq \nu^p e^{\gamma_0 \nu^{1/4} t} \right\} \quad (4.3)$$

in which T_* is defined as in (3.19). By the short time existence theory, with zero initial data, T_ν exists and is positive. It remains to give a lower bound estimate on T_ν . First, we obtain the following lemmas.

Lemma 4.1. *For $t \in [0, T_*]$, there hold*

$$\begin{aligned} \|\partial_x^a \partial_z^b \omega_{\text{app}}(t)\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{a/8-b/8} \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right) \\ \|\partial_x^a \partial_z^b R_{\text{app}}(t)\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{1/4+a/8-b/8} \nu^{-\frac{1}{4}[\frac{M}{8p}]} \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right)^{1+\frac{M}{8p}}. \end{aligned}$$

Proof. This follows directly from Lemma 3.2 and the estimate (3.18) on the remainder R_{app} , upon noting the fact that for $t \in [0, T_*]$, $\nu^p e^{\gamma_0 \nu^{1/4} t}$ remains sufficiently small. $\square$

Lemma 4.2. *There holds*

$$\left\| (u_{\text{app}} + v) \cdot \nabla \omega + v \cdot \nabla \omega_{\text{app}} \right\|_{\sigma,\beta,\gamma,1} \lesssim \nu^{-\frac{1}{8}} \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right)^2.$$

for $t \in [0, T_\nu]$.

Proof. We first recall the elliptic estimate

$$\|u\|_{\sigma,0} \lesssim \|\omega\|_{\sigma,\beta,\gamma,1}$$

which is proven in the Appendix [A](#), and the following uniform bounds (see [\(1.17\)](#))

$$\begin{aligned}\|u \cdot \nabla \tilde{\omega}\|_{\sigma, \beta, \gamma, 1} &\leq \|u\|_{\sigma, 0} \|\nabla \tilde{\omega}\|_{\sigma, \beta, \gamma, 1} \\ &\leq \|\omega\|_{\sigma, \beta, \gamma, 1} \|\nabla \tilde{\omega}\|_{\sigma, \beta, \gamma, 1}.\end{aligned}$$

Using this and the bounds on ω_{app} , we obtain

$$\|v \cdot \nabla \omega_{\text{app}}\|_{\sigma, \beta, \gamma, 1} \lesssim \nu^{-1/8} \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right) \|\omega\|_{\sigma, \beta, \gamma, 1} \lesssim \nu^{-\frac{1}{8}} \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right)^2$$

and

$$\begin{aligned}\|(u_{\text{app}} + v) \cdot \nabla \omega\|_{\sigma, \beta, \gamma, 1} &\lesssim \left(\nu^p e^{\gamma_0 \nu^{1/4} t} + \|\omega\|_{\sigma, \beta, \gamma, 1} \right) \|\nabla \omega\|_{\sigma, \beta, \gamma, 1} \\ &\lesssim \nu^{-\frac{1}{8}} \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right)^2.\end{aligned}$$

This proves the lemma. $\square$

Next, using Theorem [2.2](#) and noting that $\alpha e^{-\alpha^2 \nu t} \lesssim 1 + (\nu t)^{-1/2}$, we obtain the following uniform semigroup bounds:

$$\begin{aligned}\|e^{Lt} \omega\|_{\sigma, \beta, \gamma, 1} &\leq C_0 \nu^{-1/4} e^{\gamma_1 \nu^{1/4} t} \|\omega\|_{\sigma, \beta, \gamma, 1} \\ \|\partial_x e^{Lt} \omega\|_{\sigma, \beta, \gamma, 1} &\leq C_0 \nu^{-1/4} \left(1 + (\sqrt{\nu} t)^{-1/2} \right) e^{\gamma_1 \nu^{1/4} t} \|\omega\|_{\sigma, \beta, \gamma, 1} \\ \|\partial_z e^{Lt} \omega\|_{\sigma, \beta, \gamma, 1} &\leq C_0 \nu^{-1/4} \left(\nu^{-1/8} + (\sqrt{\nu} t)^{-1/2} \right) e^{\gamma_1 \nu^{1/4} t} \|\omega\|_{\sigma, \beta, \gamma, 1}.\end{aligned}\tag{4.4}$$

We are now ready to apply the above estimates into the Duhamel's integral formula [\(4.1\)](#). We obtain

$$\begin{aligned}\|\omega(t)\|_{\sigma, \beta, \gamma, 1} &\lesssim \nu^{-1/4} \int_0^t e^{\gamma_1 \nu^{1/4} (t-s)} \nu^{-\frac{1}{8}} \left(\nu^p e^{\gamma_0 \nu^{1/4} s} \right)^2 ds \\ &\quad + \nu^{-1/4} \int_0^t e^{\gamma_1 \nu^{1/4} (t-s)} \nu^{1/4} \nu^{-\frac{1}{4} \lceil \frac{M}{8p} \rceil} \left(\nu^p e^{\gamma_0 \nu^{1/4} s} \right)^{1 + \frac{M}{8p}} ds \\ &\lesssim \nu^{-5/8} \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right)^2 + \nu^P \left(\nu^p e^{\gamma_0 \nu^{1/4} t} \right),\end{aligned}$$

upon taking γ_1 sufficiently close to γ_0 . Set T_1 so that

$$\nu^p e^{\gamma_0 \nu^{1/4} T_1} = \theta_0 \nu^{\frac{5}{8}},\tag{4.5}$$

for some sufficiently small and positive constant θ_0 . Then, for all $t \leq T_1$, there holds

$$\|\omega(t)\|_{\sigma, \beta, \gamma, 1} \lesssim \nu^p e^{\gamma_0 \nu^{1/4} t} \left[\theta_0 + \nu^P \right]$$

Similarly, we estimate the derivatives of ω . The Duhamel integral and the semigroup bounds yield

$$\begin{aligned} \|\nabla\omega(t)\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{-1/4} \int_0^t e^{\gamma_1\nu^{1/4}(t-s)} \left(\nu^{-1/8} + (\sqrt{\nu}(t-s))^{-1/2} \right) \\ &\quad \times \left[\nu^{-\frac{1}{8}} \left(\nu^p e^{\gamma_0\nu^{1/4}s} \right)^2 + \nu^{-\frac{1}{4}[\frac{M}{8p}]} \left(\nu^p e^{\gamma_0\nu^{1/4}s} \right)^{1+\frac{M}{8p}} \right] ds \\ &\lesssim \nu^{-5/8} \left[\nu^{-\frac{1}{8}} \left(\nu^p e^{\gamma_0\nu^{1/4}s} \right)^2 + \nu^{-\frac{1}{4}[\frac{M}{8p}]} \left(\nu^p e^{\gamma_0\nu^{1/4}s} \right)^{1+\frac{M}{8p}} \right] \end{aligned}$$

By view of (4.5) and the estimate (3.16), the above yields

$$\|\nabla\omega(t)\|_{\sigma,\beta,\gamma,1} \lesssim \nu^{p-\frac{1}{8}} e^{\gamma_0\nu^{1/4}t} [\theta_0 + \nu^P].$$

To summarize, for $t \leq \min\{T_*, T_1, T_\nu\}$, with the times defined as in (3.19), (4.3), and (4.5), we obtain

$$|||w(t)||| \lesssim \nu^p e^{\gamma_0\nu^{1/4}t} [\theta_0 + \nu^P].$$

Taking θ_0 sufficiently small, we obtain

$$|||w(t)||| \ll \nu^p e^{\gamma_0\nu^{1/4}t}$$

for all time $t \leq \min\{T_*, T_1, T_\nu\}$. In particular, this proves that the maximal time of existence T_ν is greater than T_1 , defined as in (4.5). This proves that at the time $t = T_*$, the approximate solution grows to order of $\nu^{5/8}$ in the L^∞ norm. Theorem 1.2 is proved.

A Elliptic estimates

In this section, for sake of completeness, we recall the elliptic estimates with respect to the boundary layer norms. These estimates are proven in [10, Section 3].

First, we consider the classical one-dimensional Laplace equation

$$\Delta_\alpha \phi = \partial_z^2 \phi - \alpha^2 \phi = f \tag{A.1}$$

on the half line $z \geq 0$, with the Dirichlet boundary condition $\phi(0) = 0$. We recall the function space L_β^∞ defined by the finite norm $\|f\|_\beta = \sup_{z \geq 0} |f(z)|e^{\beta z}$. We will prove

Proposition A.1. *If $f \in L^\infty_\beta$ for some $\beta > 0$, then $\phi \in L^\infty$. In addition, there holds*

$$(1 + \alpha^2)\|\phi\|_{L^\infty} + (1 + |\alpha|)\|\partial_z \phi\|_{L^\infty} + \|\partial_z^2 \phi\|_{L^\infty} \leq C\|f\|_\beta, \quad (\text{A.2})$$

where the constant C is independent of $\alpha \in \mathbb{R}$.

Proof. The solution ϕ of (A.1) is explicitly given by

$$\phi(z) = \int_0^\infty G_\alpha(x, z) f(x) dx \quad (\text{A.3})$$

where $G_\alpha(x, z) = -\frac{1}{2\alpha} \left(e^{-\alpha|x-z|} - e^{-\alpha|x+z|} \right)$. A direct bound leads to

$$\|\phi\|_{L^\infty} \leq \frac{C}{\alpha^2} \|f\|_\beta$$

in which the extra factor of α^{-1} is due to the x -integration. Differentiating the integral formula, we get

$$\|\partial_z \phi\|_{L^\infty} \leq \frac{C}{\alpha} \|f\|_\beta.$$

The estimate for $\partial_z^2 \phi$ follows by using directly the equation $\partial_z^2 \phi = \alpha^2 \phi + f$. This yields the lemma for the case when α is bounded away from zero.

As for small α , we note that $G_\alpha(0, z) = 0$ and $|\partial_x G_\alpha(x, z)| \leq 1$. Hence, $|G_\alpha(x, z)| \leq |x|$ and so

$$|\phi(z)| \leq \int_0^\infty |G_\alpha(x, z) f(x)| dx \leq \|f\|_\beta \int_0^\infty |x| e^{-\beta x} dx \leq C\|f\|_\beta.$$

Similarly, since $|\partial_z G_\alpha(x, z)| \leq 1$, we get

$$|\partial_z \phi(z)| \leq \int_0^\infty |\partial_z G_\alpha(x, z) f(x)| dx \leq \|f\|_\beta \int_0^\infty e^{-\beta x} dx \leq C\|f\|_\beta.$$

The lemma follows. $\square$

We now establish a similar property for $\mathcal{B}^{\beta, \gamma, 1}$ norms:

Proposition A.2. *If $f \in \mathcal{B}^{\beta, \gamma, 1}$ for some $\beta > 0$, then $\phi \in L^\infty$. In addition, there holds*

$$(1 + |\alpha|)\|\phi\|_{L^\infty} + \|\partial_z \phi\|_{L^\infty} \leq C\|f\|_{\beta, \gamma, 1}, \quad (\text{A.4})$$

where the constant C is independent of $\alpha \in \mathbb{R}$.

Proof. We will only consider the case $\alpha > 0$, the opposite case being similar. As above, since $G_\alpha(x, z)$ is bounded by α^{-1} , using (A.3), we have

$$\begin{aligned} |\phi(z)| &\leq \alpha^{-1} \|f\|_{\beta, \gamma, 1} \int_0^\infty e^{-\alpha|z-x|} e^{-\beta x} \left(1 + \delta^{-1} \phi_P(\delta^{-1}x)\right) dx \\ &\leq \alpha^{-1} \|f\|_{\beta, \gamma, 1} \left(\alpha^{-1} + \delta^{-1} \int_0^\infty \phi_P(\delta^{-1}x) dx \right) \end{aligned}$$

which yields the claimed bound for ϕ since $P > 1$. A similar proof applies for $\partial_z \phi$. $\square$

Next, let us now turn to the two dimensional Laplace operator.

Proposition A.3. *Let ϕ be the solution of*

$$-\Delta \phi = \omega$$

with the zero Dirichlet boundary condition, and let

$$v = \nabla^\perp \phi.$$

If $\omega \in \mathcal{B}^{\sigma, \beta, \gamma, 1}$, then $\phi \in \mathcal{B}^{\sigma, 0}$ and $v = (v_1, v_2) \in \mathcal{B}^{\sigma, 0}$. Moreover, there hold the following elliptic estimates

$$\|\phi\|_{\sigma, 0} + \|v_1\|_{\sigma, 0} + \|v_2\|_{\sigma, 0} \leq C \|\omega\|_{\sigma, \beta, \gamma, 1}, \quad (\text{A.5})$$

Proof. The proof follows directly from taking the Fourier transform in the x variable, with dual integer Fourier component α , and using Proposition (A.2). $\square$

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