Combinatorica (2024) 44:949–976
https://doi.org/10.1007/s00493-024-00099-y

ORIGINAL PAPER

Turán Density of Long Tight Cycle Minus One Hyperedge

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Received: 18 March 2023 / Revised: 3 March 2024 / Accepted: 5 March 2024 /
Published online: 17 April 2024

Abstract

Denote by $\mathcal{C}_\ell^-$ the 3-uniform hypergraph obtained by removing one hyperedge from the tight cycle on ℓ vertices. It is conjectured that the Turán density of $\mathcal{C}_5^-$ is $1/4$. In this paper, we make progress toward this conjecture by proving that the Turán density of $\mathcal{C}_\ell^-$ is $1/4$, for every sufficiently large ℓ not divisible by 3. One of the main ingredients of our proof is a forbidden-subhypergraph characterization of the hypergraphs, for which there exists a tournament on the same vertex set such that every hyperedge is a cyclic triangle in this tournament. A byproduct of our method is a human-checkable proof for the upper bound on the maximum number of almost similar triangles in a planar point set, which was recently proved using the method of flag algebras by Balogh, Clemen, and Lidický.

Keywords Hypergraph · Turán number · Tight cycles · Discrete geometry

Mathematics Subject Classification 05C65(primary) · 05C35 · 05C38 · 05D05

1 Introduction

For a collection $\boldsymbol{\mathcal{F}}$ of r-uniform hypergraphs (*r-graphs*), we say that an r-graph $\mathcal{H}$ is $\boldsymbol{\mathcal{F}}$*-free* or *free of* $\boldsymbol{\mathcal{F}}$, if $\mathcal{H}$ contains no $\mathcal{F} \in \boldsymbol{\mathcal{F}}$ as a subhypergraph. The *Turán number* $\mathrm{ex}(n, \boldsymbol{\mathcal{F}})$ is defined to be the maximum number of r-edges an n-vertex $\boldsymbol{\mathcal{F}}$-free r-graph can have. To determine $\mathrm{ex}(n, \boldsymbol{\mathcal{F}})$ is a central problem in Extremal Combinatorics, but also notoriously hard when $r \geq 3$, where even the *Turán density* $\pi(\boldsymbol{\mathcal{F}}) := \lim_{n\to\infty} \mathrm{ex}(n, \boldsymbol{\mathcal{F}})/\binom{n}{r}$ is only known for a few $\boldsymbol{\mathcal{F}}$'s.[1] For example, let $\mathcal{K}_4$ be

[1] When $\boldsymbol{\mathcal{F}} = \{\mathcal{F}\}$, we use $\mathrm{ex}(n, \mathcal{F})$ and $\pi(\mathcal{F})$ for $\mathrm{ex}(n, \{\mathcal{F}\})$ and $\pi(\{\mathcal{F}\})$, respectively.

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the complete 3-graph on 4 vertices and $\mathcal{K}_4^-$ be the 3-graph obtained by removing one hyperedge from $\mathcal{K}_4$. It has been a long-standing open problem to determine $\pi(\mathcal{K}_4)$ and $\pi(\mathcal{K}_4^-)$.

A related family that has also received extensive attention for Turán density problems is the tight cycles. For every integer $\ell \geq 4$, let $\mathcal{C}_\ell$ be the *tight cycle* of size ℓ, i.e., it has vertex set $\{0, 1, \ldots, \ell-1\}$ and hyperedges $\{\{i, i+1, i+2 \pmod{\ell}\} : 0 \leq i \leq \ell - 1\}$, and let $\mathcal{C}_\ell^-$ be the *tight cycle minus one hyperedge* of size ℓ, i.e., it is obtained from $\mathcal{C}_\ell$ by removing the hyperedge $\{\ell-1, 0, 1\}$. Note that $\mathcal{C}_4 = \mathcal{K}_4$ and $\mathcal{C}_4^- = \mathcal{K}_4^-$. When ℓ is a multiple of 3, $\mathcal{C}_\ell$ is tripartite, so, by a classical result of Erdős [6], $\pi(\mathcal{C}_\ell^-) = \pi(\mathcal{C}_\ell) = 0$. It is conjectured that $\pi(\mathcal{C}_5) = 2\sqrt{3} - 3$ and $\pi(\mathcal{C}_5^-) = 1/4$, see [15]. In this paper, we make progress toward the latter conjecture by proving the following theorem.

Theorem 1.1 *There is a constant L such that $\pi(\mathcal{C}_\ell^-) = 1/4$, for every $\ell > L$ not divisible by 3.*

Our key method for proving Theorem 1.1 is to reduce this hypergraph Turán problem to a counting problem in tournaments, which is in general much easier to deal with than hypergraphs. We note that a similar framework is used by Kamčev et al. [11] for the Turán density of $\mathcal{C}_\ell$ for sufficiently large ℓ not divisible by 3. Some of our ideas and lemmas are partially inspired by them. We also note that Piga et al. [16] recently proved that the codegree Turán density of $\mathcal{C}_\ell^-$ is 0, for every $\ell \geq 5$.

We now give an outline of the proof for Theorem 1.1. The lower bound in Theorem 1.1 follows from the following construction, which is usually called the iterated blow-up of a hyperedge. It is also conjectured to be the extremal construction for $\mathcal{C}_5^-$, see [15, Sect. 2.5].

Construction 1.2 Define 3-graphs $\mathcal{E}_n$ by induction. The vertex set of $\mathcal{E}_n$ is $\{1, 2, \ldots, n\}$, which is partitioned into three parts V_1, V_2, V_3 with sizes $\lfloor n/3 \rfloor$, $\lfloor (n+1)/3 \rfloor$, and $\lfloor (n+2)/3 \rfloor$, respectively. $\mathcal{E}_n$ contains no hyperedge when $n = 1$ or 2. For $n \geq 3$, $\mathcal{E}_n$ contains all the hyperedges with exactly one vertex in each V_i, and V_i spans a copy of $\mathcal{E}_{|V_i|}$ for $1 \leq i \leq 3$.

It is easy to check that $\mathcal{C}_\ell^- \nsubseteq \mathcal{E}_n$, when $\ell \geq 4$ and $3 \nmid \ell$. A standard induction shows that $\mathcal{E}_n$ has at least $n^3/24 - Cn \log n$ hyperedges for some constant $C > 0$, see also Section 1 in [3].

For the upper bound in Theorem 1.1, we will first work on the Turán problem of *pseudo-cycles*, which are, roughly speaking, tight cycles with repeated vertices allowed. See Definition 3.2 for a rigorous definition. The key step is to connect this problem with counting the number of cyclic triangles in tournaments. We introduce the following notion.

Definition 1.3 A 3-graph $\mathcal{H}$ is *orientable* if there is a tournament T on the same vertex set such that every hyperedge in $\mathcal{H}$ is a cyclic triangle in T.

For example, it can be checked that $\mathcal{C}_5$ is orientable, but $\mathcal{K}_4^-$ is not. We remark that the connection between 3-graphs and tournaments has been noticed decades ago,

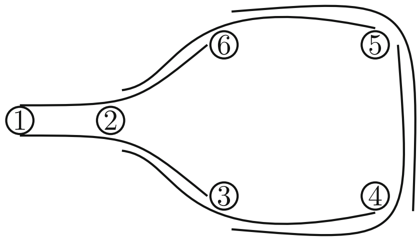


Fig. 1 An example of a bottle of size 8. It has vertex set $\{1, 2, \dots, 6\}$ and hyperedges $\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}, \{5, 6, 2\}, \{6, 2, 1\}$. It can be represented as 12345621

which can be traced back to the work of Erdős and Hajnal [7] in 1972. For example, orientable 3-graphs serve as the constructions for the lower bound of the codegree Turán density of $\mathcal{K}_4^-$, see [9], and the uniform Turán density of $\mathcal{K}_4^-$, see [10, 19]. See also [18] for a generalization of orientable 3-graphs to r-graphs with $r \geq 4$.

We will prove that a 3-graph $\mathcal{H}$ is orientable if and only if it is free of a certain family of hypergraphs, which we call *bottles*, see Proposition 3.3. Using this characterization of orientable 3-graphs, we prove that a 3-graph is orientable if it is free of all the pseudo-cycles minus one hyperedge with length not divisible by 3, see Lemma 3.5. Then, by analyzing tournaments, we are able to prove a stability result for $\mathcal{C}_5^-$-free orientable hypergraphs, which says that the vertex set of an almost maximum $\mathcal{C}_5^-$-free orientable 3-graph can be partitioned into three parts with almost equal sizes such that there are very few *bad* hyperedges, i.e., the hyperedges with two vertices in a part and one vertex in another part, see Proposition 4.1. Building on this structure, a cleaning argument in Sect. 5 shows that the maximum 3-graphs free of the pseudo-cycles minus one hyperedge with length not divisible by 3 and less than a fixed large constant indeed contain no bad hyperedge, from which we can easily prove that such hypergraphs have edge density at most $1/4 + o(1)$. Finally, a standard technique using blow-ups gives Theorem 1.1.

As a direct application of Theorem 1.1, we give a human-checkable answer to the following question about the maximum number of almost similar triangles in a planar point set. For a triangle $\triangle$ with angles $0 < a_1 \leq a_2 \leq a_3 < 180°$, we say that another triangle $\triangle'$ with angles $0 < a_1' \leq a_2' \leq a_3' < 180°$ is ε-*similar* to $\triangle$ if $|a_i - a_i'| \leq \varepsilon$ for $i = 1, 2, 3$. Inspired by the work of Conway et al. [5] about the distribution of angles determined by a planar set, Bárány and Füredi [3] studied $h(n, \triangle, \varepsilon)$, the maximum number of triangles that are ε-similar to $\triangle$ in a planar set of n points. They [3] proved that $h(\triangle, \varepsilon) := \lim_{n\to\infty} h(n, \triangle, \varepsilon)/n^3$ exists and it is at least 1/24 for every triangle $\triangle$ and $\varepsilon > 0$, see their Fig. 1. They also showed that $h(\triangle, \varepsilon) = 1/24$, when $\triangle$ is the equilateral triangle and $\varepsilon \leq 1°$, and $h(\triangle, \varepsilon)$ can be strictly larger for some $\triangle$'s, including all the right-angled triangles. In order to give a general upper bound for $h(n, \triangle, \varepsilon)$, they represented the shape of triangles by points in $S_{tri} := \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1, a_2, a_3 > 0,\ a_1 + a_2 + a_3 = \pi\}$ and considered the Lebesgue measure on S_{tri}. In the same paper, they showed that for almost all triangles $\triangle$, there exists $\varepsilon = \varepsilon(\triangle) > 0$ such that $h(n, \triangle, \varepsilon) \leq 0.25108\binom{n}{3}(1 + o(1))$, and with the further aid of the flag algebra method developed by Razborov [17], they could improve the constant to 0.25072. Their main idea is to reduce this problem to a hypergraph Turán problem, by noticing that there exists a family $\boldsymbol{\mathcal{F}}_{tri}$ of 3-graphs, whose hyperedges cannot be represented by triangles ε-similar to $\triangle$ in any planar set, for almost all triangles $\triangle$. See Definition 6.1 for the rigorous definition of $\mathcal{F}_{tri}$. Extending this idea, Balogh et al. [2] improved this bound to 0.25, which is best possible, by verifying that more 3-graphs are members in $\mathcal{F}_{tri}$ and using flag algebra and the stability method.

Theorem 1.4 ([2, Theorem 1.3]) *For almost all triangles $\triangle$, there exists $\varepsilon = \varepsilon(\triangle) > 0$ such that $h(\triangle, \varepsilon) \leq 1/4$.*

We will show in Sect. 6 that Theorem 1.1 implies Theorem 1.4, by the observation that $\mathcal{C}_\ell^- \in \mathcal{F}_{tri}$ for some large ℓ not divisible by 3.

The rest of this paper is organized as follows. In Sect. 2, we introduce our notation and lemmas used in our proof. In Sect. 3, we give our forbidden-subhypergraph characterization of the orientable 3-graphs and prove several other lemmas about tournaments. In Sect. 4, we prove our stability result. In Sect. 5, we prove Theorem 1.1. In Sect. 6, we give our new proof for Theorem 1.4.

2 Preliminaries

For a positive integer n, we write $[n]$ for the set $\{1, 2, \ldots, n\}$. For a set X and a positive integer k, denote by $\binom{X}{k}$ the collection of all subsets of X of size k. For sets $X_1, X_2, \ldots, X_k$, let $[X_1, X_2, \ldots, X_k] := \{\{x_1, x_2, \ldots, x_k\} : x_i \in X_i \text{ for } 1 \leq i \leq k\}$.

For an r-graph $\mathcal{H}$, we use $V(\mathcal{H})$ for its vertex set and use $\mathcal{H}$ to stand for its r-edges. In particular, $|\mathcal{H}|$ denotes the number of r-edges in $\mathcal{H}$.

Let $\mathcal{H}$ be a 3-graph. For a vertex $v \in V(\mathcal{H})$ and two (not necessarily disjoint) sets $S_1, S_2 \subseteq V(\mathcal{H})$, let $N^{\mathcal{H}}_{S_1,S_2}(v) := \{\{x, y\} : \{v, x, y\} \in \mathcal{H},\ x \in S_1,\ y \in S_2\}$ be the *link graph* of v between S_1 and S_2 and $d^{\mathcal{H}}_{S_1,S_2}(v) := |N^{\mathcal{H}}_{S_1,S_2}(v)|$ be the *degree* of v between S_1 and S_2. When $S_1 = S_2 = S$, we write $N^{\mathcal{H}}_S(v)$ for $N^{\mathcal{H}}_{S_1,S_2}(v)$ and $d^{\mathcal{H}}_S(v)$ for $d^{\mathcal{H}}_{S_1,S_2}(v)$. Let $N^{\mathcal{H}}(v) := N^{\mathcal{H}}_{V(\mathcal{H})}(v)$ and $d^{\mathcal{H}}(v) := d^{\mathcal{H}}_{V(\mathcal{H})}(v)$. For vertices $u, v \in V(\mathcal{H})$ and a set $S \subseteq V(\mathcal{H})$, let $N^{\mathcal{H}}_S(u, v) := \{w \in S : \{u, v, w\} \in \mathcal{H}\}$ be the set of *neighbors* of u, v in S and $d^{\mathcal{H}}_S(u, v) := |N^{\mathcal{H}}_S(u, v)|$ be the *codegree* of u, v in S. Let $N^{\mathcal{H}}(u, v) := N^{\mathcal{H}}_{V(\mathcal{H})}(u, v)$ and $d^{\mathcal{H}}(u, v) := d^{\mathcal{H}}_{V(\mathcal{H})}(u, v)$. We often omit the superscript $\mathcal{H}$ when it is clear from the context. For vertex sets $S_1, S_2, S_3 \subseteq V(\mathcal{H})$, let $\mathcal{H}[S_1, S_2, S_3] := \mathcal{H} \cap [S_1, S_2, S_3]$ be the set of hyperedges between S_1, S_2, S_3. Let $\mathcal{H}[S_1] := \mathcal{H}[S_1, S_1, S_1]$ and $\bar{\mathcal{H}}[S_1, S_2, S_3] := [S_1, S_2, S_3] \setminus \mathcal{H}[S_1, S_2, S_3]$. For a partition $\pi = (S_1, S_2, S_3)$ of $V(\mathcal{H})$, we write $\mathcal{H}_\pi := \mathcal{H}[S_1, S_2, S_3]$ and $\bar{\mathcal{H}}_\pi := [S_1, S_2, S_3] \setminus \mathcal{H}_\pi$.

For a tournament T and a vertex $v \in V(T)$, let $N^+(v) := \{u \in V(G) : v \to u\}$ and let $N^-(v) := \{u \in V(G) : u \to v\}$ be the sets of *out-neighbors* and *in-neighbors* of v, respectively. Let $d^+(v) := |N^+(v)|$ and $d^-(v) := |N^-(v)|$ be the *out-degree* and *in-degree* of v, respectively. For vertex sets $V_1, V_2 \subseteq V(T)$, we write $V_1 \to V_2$ if $u \to v$ for every $u \in V_1$ and $v \in V_2$.

We use the following two results about triangle-free graphs.

Theorem 2.1 (Mantel [14]) *Every n-vertex triangle-free graph has at most $n^2/4$ edges.*

Theorem 2.2 (Erdős et al. [8, Theorem 1]) *Every triangle-free graph G with n vertices and m edges can be made bipartite by removing at most*

$$\min\left\{\frac{m}{2}-\frac{2m(2m^2-n^3)}{n^2(n^2-2m)}, m-\frac{4m^2}{n^2}\right\},$$

edges.

For a 3-graph $\mathcal{H}$ and a positive integer t, the *t-blow-up* $\mathcal{H}[t]$ is the 3-graph on vertex set $V(\mathcal{H})\times[t]$ with hyperedges $\{\{(v_1,t_1),(v_2,t_2),(v_3,t_3)\}:\{v_1,v_2,v_3\}\in\mathcal{H}, 1\le t_1,t_2,t_3\le t\}$. For a family of 3-graphs $\boldsymbol{\mathcal{H}}=\{\mathcal{H}_1,\ldots,\mathcal{H}_h\}$, let $\boldsymbol{\mathcal{H}}[t]:=\{\mathcal{H}_1[t],\ldots,\mathcal{H}_h[t]\}$.

Theorem 2.3 (See [12, Sect. 2]) *For every family of 3-graphs $\boldsymbol{\mathcal{H}}$ and positive integer t, we have $\pi(\boldsymbol{\mathcal{H}}[t])=\pi(\boldsymbol{\mathcal{H}})$.*

We use the following version of the removal lemma for tournaments by Choi et al. [4] (see their Lemma 5; their original theorem is about oriented graphs, and we can, for example, add the graph consisting of two isolated vertices to the forbidden family to get the following version), which follows from a general theorem by Aroskar and Cummings [1].

Theorem 2.4 *Let $\boldsymbol{T}$ be a (possibly infinite) set of tournaments. For every $\varepsilon>0$, there exist n_0 and $\delta>0$ such that for every tournament T on $n\ge n_0$ vertices, if T contains at most $\delta n^{|V(D)|}$ copies of D for each D in $\boldsymbol{T}$, then there exists T' on the same vertex set such that T' is D-free for every D in $\boldsymbol{T}$ and T' can be obtained from T by reorienting at most εn^2 edges.*

The following is a technical lemma that will be used in our proof of the stability result in Sect. 4. We omit its standard proof.

Lemma 2.5 *For integers $a\ge b>0$, we have*

$$\max\left(\sum_{i=1}^{\infty}x_iy_i\right)\le\left\lfloor\frac{a}{b}\right\rfloor\cdot\frac{b^2}{4}+\frac{1}{4}\left(a-b\left\lfloor\frac{a}{b}\right\rfloor\right)^2,$$

where the maximum is over non-negative integer sequences $(x_1,x_2,\ldots)$ and $(y_1,y_2,\ldots)$ such that $\sum_{i=1}^{\infty}(x_i+y_i)=a$ and $x_j+y_j\le b$ for every $j\ge 1$.

3 Cyclic Triangles in Tournaments

In this section, we provide the first step toward proving Theorem 1.1, by relating it to the number of cyclic triangles in tournaments.

For 3-graphs $\mathcal{H}$ and $\mathcal{F}$, a *surjective homomorphism* from $\mathcal{H}$ to $\mathcal{F}$ is a surjective map from $V(\mathcal{H})$ to $V(\mathcal{F})$ such that $\{f(v_1),f(v_2),f(v_3)\}\in\mathcal{F}$ if $\{v_1,v_2,v_3\}\in\mathcal{H}$ and for every $\{u_1, u_2, u_3\} \in \mathcal{F}$, there is $\{v_1, v_2, v_3\} \in \mathcal{H}$ with $\{f(v_1), f(v_2), f(v_3)\} = \{u_1, u_2, u_3\}$. Following [11], we have the following definitions.

Definition 3.1 For every integer $\ell \geq 3$, let $\mathcal{P}_\ell$ be the path of size ℓ, i.e., it is the 3-graph on vertex set $\{1, 2, \ldots, \ell\}$ with hyperedges $\{\{i, i+1, i+2\} : 1 \leq i \leq \ell - 2\}$. We call a 3-graph $\mathcal{H}$ a *pseudo-path* of size ℓ if there exists a surjective homomorphism from $\mathcal{P}_\ell$ to $\mathcal{H}$.

Hence, pseudo-paths are a generalization of paths, where repeated vertices are allowed. For tight cycles, we have the following similar notion.

Definition 3.2 For every integer $\ell \geq 4$, we call a 3-graph $\mathcal{H}$ a *pseudo-cycle* of size ℓ if there exists a surjective homomorphism from $\mathcal{C}_\ell$ to $\mathcal{H}$, and we call a 3-graph $\mathcal{H}$ a *pseudo-cycle minus one hyperedge* of size ℓ if there exists a surjective homomorphism from $\mathcal{C}_\ell^-$ to $\mathcal{H}$.

For every integer $L \geq 4$, let $\boldsymbol{\mathcal{C}}_{\leq L}^-$ be the set of all the pseudo-cycles minus one hyperedge of size ℓ, where $4 \leq \ell \leq L$ and $3 \nmid \ell$. Let $\boldsymbol{\mathcal{C}}^- := \bigcup_{L \geq 4} \boldsymbol{\mathcal{C}}_{\leq L}^-$. It can be easily checked that $\mathcal{E}_n$, the iterated blow-up of a hyperedge defined in Construction 1.2, is $\boldsymbol{\mathcal{C}}^-$-free.

For a copy $\mathcal{H}$ of pseudo-path, pseudo-cycle minus one hyperedge, or a pseudo-cycle of size ℓ, we often use $v_0 v_2 \ldots v_{\ell-1}$, a sequence of its vertices (with repetition allowed) to stand for it; this means that $\mathcal{H}$ consists of hyperedges $\{v_i, v_{i+1 \pmod{\ell}}, v_{i+2 \pmod{\ell}}\}$ for $0 \leq i \leq \ell - 3$, when $\mathcal{H}$ is a pseudo-path, for $0 \leq i \leq \ell - 2$, when $\mathcal{H}$ is a pseudo-cycle minus one hyperedge, and for $0 \leq i \leq \ell - 1$, when $\mathcal{H}$ is a pseudo-cycle.

For $k \geq 4$, we call a pseudo-path $v_1 v_2 \ldots v_k v_2 v_1$ a *bottle* of size $k + 2$, see Fig. 1. For $L \geq 6$, let $\boldsymbol{\mathcal{B}}_{\leq L}$ be the set of the bottles of size ℓ where $6 \leq \ell \leq L$ and $\boldsymbol{\mathcal{B}} := \bigcup_{L \geq 6} \boldsymbol{\mathcal{B}}_{\leq L}$.

Proposition 3.3 *A 3-graph $\mathcal{H}$ is orientable if and only if it is free of $\boldsymbol{\mathcal{B}}$.*

Proof Assume that there is a bottle $v_1 v_2 \ldots v_k v_2 v_1$ in $\mathcal{H}$, where $k \geq 4$. If $\mathcal{H}$ is orientable, then without loss of generality, assume that in the corresponding tournament T we have $v_1 \to v_2$. Then, by the definition of the bottle, we have $v_{i-1} \to v_i$ for $1 < i \leq k$ and then $v_k \to v_2$ and $v_2 \to v_1$, a contradiction.

Assume that $\mathcal{H}$ is free of $\boldsymbol{\mathcal{B}}$. We say that two pairs of vertices $\{a, b\}$ and $\{c, d\}$ are *tightly connected* if there is a pseudo-path $u_1 u_2 \ldots u_{\ell-1} u_\ell$ such that $\{u_1, u_2\} = \{a, b\}$ and $\{u_{\ell-1}, u_\ell\} = \{c, d\}$. Note that the three pairs of vertices in every hyperedge are always tightly connected to each other. Hence, we can partition the hyperedges of $\mathcal{H}$ into equivalence classes $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_p$, where hyperedges $\{a, b, c\}$ and $\{x, y, z\}$ are in the same class if $\{a, b\}$ and $\{x, y\}$ are tightly connected. Note that for every pair of vertices $\{x, y\}$, there can be at most one i such that $\{x, y\}$ is contained in some hyperedges of $\mathcal{H}_i$, so when trying to orient $\{x, y\}$, we only need to consider hyperedges of $\mathcal{H}_i$ and omit all others. Let P_i be the set of pairs of vertices contained in some hyperedges of $\mathcal{H}_i$.

Construct a tournament T on $V(\mathcal{H})$ as follows. For every i, where $1 \leq i \leq p$, do the following algorithm. Choose an arbitrary pair $\{a, b\}$ in P_i and orient $a \to b$. For every other pair $\{c, d\}$ in P_i, there are two cases.

1. There is a pseudo-path $\mathcal{F}_1 = abw_3 \dots w_{k_1}cd$ or a pseudo-path $\mathcal{F}_2 = bax_3 \dots x_{k_2}dc$.
2. There is a pseudo-path $\mathcal{F}_3 = aby_3 \dots y_{k_3}dc$ or a pseudo-path $\mathcal{F}_4 = baz_3 \dots z_{k_4}cd$.

We claim that exactly one case happens. It is clear that at least one of the cases happens, since $\{a, b\}$ and $\{c, d\}$ are tightly connected. If both cases happen, then we have pseudo-paths $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_2}$, where $1 \le s_1 \le 2$ and $3 \le s_2 \le 4$.

- If $s_1 = 1, s_2 = 3$, then we have a bottle $abw_3 \dots w_{k_1}cdy_{k_3} \dots y_3ba$.
- If $s_1 = 1, s_2 = 4$, then we have a bottle $dcw_{k_1} \dots w_3baz_3 \dots z_{k_4}cd$.
- If $s_1 = 2, s_2 = 3$, then we have a bottle $cdx_{k_2} \dots x_3aby_3 \dots y_{k_3}dc$.
- If $s_1 = 2, s_2 = 4$, then we have a bottle $bax_3 \dots x_{k_2}dcz_{k_4} \dots z_3ab$.

Therefore, exactly one case happens, as claimed. Now, if the first case happens, we orient $c \to d$; otherwise we orient $d \to c$. For a hyperedge $\{x, y, z\} \in \mathcal{H}_i$, if we orient $x \to y$, which means that there is a pseudo-path $ab \dots xy$ or a pseudo-path $ba \dots yx$, then we also have $y \to z$ and $z \to x$, since there is a pseudo-path $ab \dots xyzx$ or a pseudo-path $ba \dots yxzy$. Therefore, every hyperedge in $\mathcal{H}_i$ is a cyclic triangle in this orientation.

Finally, for pairs of vertices not in any P_i, orient them in an arbitrary way. Every hyperedge in $\mathcal{H}$ is a cyclic triangle in T, so $\mathcal{H}$ is orientable. □

For an orientable 3-graph $\mathcal{H}$, let $T(\mathcal{H})$ be a tournament on the same vertex set such that every hyperedge in $\mathcal{H}$ is a cyclic triangle in $T(\mathcal{H})$. For a tournament T, let $\mathcal{H}(T)$ be the 3-graph on the same vertex set whose hyperedges are exactly the cyclic triangles in T. Note that $\mathcal{H} \subseteq \mathcal{H}(T(\mathcal{H}))$ by definition, and strict containment can happen.

For a tournament T, let $t(T)$ be the number of cyclic triangles in T. The following lemma is a well-known upper bound for $t(T)$ by Kendall and Smith [13]. We include its proof.

Lemma 3.4 *For every tournament T on n vertices, we have*

$$t(T) \le \begin{cases} \frac{1}{24}(n^3 - n) & \text{if } n \text{ is odd,} \\ \frac{1}{24}(n^3 - 4n) & \text{if } n \text{ is even.} \end{cases}$$

Proof For every non-cyclic triangle in T, it has exactly one vertex with two out-edges and exactly one vertex with two in-edges. Hence, we have

$$\begin{aligned} t(T) &= \binom{n}{3} - \frac{1}{2} \sum_{v \in V(T)} \left(\binom{d^+(v)}{2} + \binom{d^-(v)}{2} \right) \\ &= \binom{n}{3} - \frac{1}{4} \sum_{v \in V(T)} \left((d^+(v))^2 + (d^-(v))^2 - (n-1) \right) \qquad (1) \\ &\le \binom{n}{3} - \frac{1}{4} \sum_{v \in V(T)} \left(\left(\left\lceil \frac{n-1}{2} \right\rceil \right)^2 + \left(\left\lfloor \frac{n-1}{2} \right\rfloor \right)^2 - (n-1) \right). \end{aligned}$$

The claim then follows via an easy calculation. □

Proposition 3.3 and Lemma 3.4 show that $\pi(\boldsymbol{\mathcal{B}}) \leq 1/4$. Now we convert this bound to $\boldsymbol{\mathcal{C}}^-$.

Lemma 3.5 *If a 3-graph $\mathcal{H}$ is free of $\boldsymbol{\mathcal{C}}^-_{\leq L}$, then $\mathcal{H}$ is free of $\boldsymbol{\mathcal{B}}_{\leq(L+2)}$. In particular, if $\mathcal{H}$ is free of $\boldsymbol{\mathcal{C}}^-$, then $\mathcal{H}$ is orientable.*

Proof Assume for contradiction that $\mathcal{H}$ contains a bottle $v_1v_2v_3\ldots v_kv_2v_1$, where $k \leq L$. We have $k \neq 4$, since otherwise $\mathcal{H}$ contains $\mathcal{K}_4^- = \mathcal{C}_4^- \in \boldsymbol{\mathcal{C}}^-_{\leq L}$. For every $k > 4$, $\mathcal{H}$ contains two pseudo-cycles minus one hyperedge whose sizes differ by one: $v_kv_1v_2v_3\ldots v_{k-2}v_{k-1}$ and $v_2v_3\ldots v_{k-1}v_k$. The sizes of these two cycles are both at most $k \leq L$, and at least one of them is not divisible by 3. Thus, $\mathcal{H}$ cannot be free of $\boldsymbol{\mathcal{C}}^-_{\leq L}$, a contradiction. The second claim then follows from Proposition 3.3. □

By Lemmas 3.4 and 3.5, we have $\pi(\boldsymbol{\mathcal{C}}^-) \leq 1/4$. In order to improve this to get Theorem 1.1, we need to study tournaments more carefully. The following lemmas about tournaments will be used in Sect. 4 to prove our stability result, Proposition 4.1.

Lemma 3.6 *For every $\varepsilon_1, \varepsilon_2 > 0$, there exists $\delta \geq \varepsilon_1\varepsilon_2^2/2$ such that for every tournament T on n vertices with $t(T) > (1/24 - \delta)n^3$, for*

$$V'(T) := \left\{v \in V(T) : \frac{n-1}{2} - \varepsilon_2 n < d^+(v), d^-(v) < \frac{n-1}{2} + \varepsilon_2 n\right\},$$

we have $\left|V'(T)\right| > (1-\varepsilon_1)n$.

Proof Let $\delta = \varepsilon_1\varepsilon_2^2/2$. If the claim is not true, then by (1), we have

$$\begin{aligned} t(T) &< \binom{n}{3} - \frac{1}{4}(\varepsilon_1 n)\left(\left(\frac{n-1}{2} + \varepsilon_2 n\right)^2 + \left(\frac{n-1}{2} - \varepsilon_2 n\right)^2 - (n-1)\right) \\ &\quad - \frac{1}{4}(1-\varepsilon_1)n\left(2 \cdot \left(\frac{n-1}{2}\right)^2 - (n-1)\right) \\ &= \left(\frac{1}{24} - \frac{1}{2}\varepsilon_1\varepsilon_2^2\right)n^3 - \frac{1}{24}n < t(T), \end{aligned}$$

a contradiction. □

Lemma 3.7 *For every n-vertex tournament T, where $n > 8$, we have*

$$\left|\left\{u \in V(T) : d^+(u) \geq \frac{n}{4}\right\}\right| \geq \frac{n}{4} \quad \textit{and} \quad \left|\left\{u \in V(T) : d^-(u) \geq \frac{n}{4}\right\}\right| \geq \frac{n}{4}.$$

Proof If any of these two claims is false, then the number of directed edges in T is at most

$$\frac{n}{4} \cdot n + \frac{3n}{4} \cdot \frac{n}{4} = \frac{7n^2}{16} < \binom{n}{2},$$

a contradiction. □

Lemma 3.8 *For every $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1/24 > \varepsilon_2 > 0$, there exist $\delta > \varepsilon_1^2(\varepsilon_1 - 24\varepsilon_2)/24^3$ and N such that for every tournament T on $n > N$ vertices with $t(T) \geq (1/24 - \delta)n^3$, for*

$$B_T(\varepsilon_2) := \left\{ \{u, v\} \in \binom{V(T)}{2} : \{u, v\} \text{ is in at most } \varepsilon_2 n \text{ cyclic triangles in } T \right\},$$

we have $|B_T(\varepsilon_2)| < \varepsilon_1 n^2$.

Proof Let $\delta > \varepsilon_1^2(\varepsilon_1 - 24\varepsilon_2)/24^3$ be the one obtained from Lemma 3.6 when applying it with $\varepsilon_1' = \varepsilon_1/12$ and $\varepsilon_2' = \varepsilon_1/24 - \varepsilon_2/2$. Let N be sufficiently large such that $\varepsilon_1 N > 24$. Let T be a tournament on $n > N$ vertices with $t(T) \geq (1/24 - \delta)n^3$.

Assume for contradiction that $|B_T(\varepsilon_2)| \geq \varepsilon_1 n^2$. For every vertex $v \in V(T)$, let $N'(v) := \{u \in N^-(v) : \{u, v\} \in B_T(\varepsilon_2)\}$ and $S := \{v \in V(T) : |N'(v)| \geq \varepsilon_1 n/3\}$. If $|S| \leq \varepsilon_1 n/3$, then we have

$$|B_T(\varepsilon_2)| \leq \frac{\varepsilon_1 n}{3} \cdot n + \left(1 - \frac{\varepsilon_1}{3}\right) n \cdot \frac{\varepsilon_1 n}{3} < \frac{2}{3}\varepsilon_1 n^2 < \varepsilon_1 n^2,$$

a contradiction. Therefore, $|S| > \varepsilon_1 n/3$ and then by Lemma 3.6, there exists $v_0 \in S \cap V'(T)$.

Note that for every vertex $u \in N^-(v_0)$ and vertex $w \in N^+(v_0)$, if $w \to u$, then $\{u, v_0, w\}$ forms a cyclic triangle. Hence, for every $u \in N'(v_0)$, we have $|N^-(u) \cap N^+(v_0)| \leq \varepsilon_2 n$, and then,

$$|N^+(u) \cap N^+(v_0)| \geq d^+(v_0) - \varepsilon_2 n.$$

Since $|N'(v_0)| \geq \varepsilon_1 n/3$, we have, by Lemma 3.7, that there are at least $\varepsilon_1 n/12$ vertices $u \in N'(v_0)$ such that $|N^+(u) \cap N'(v_0)| \geq \varepsilon_1 n/12$. However, for every such vertex u, we have

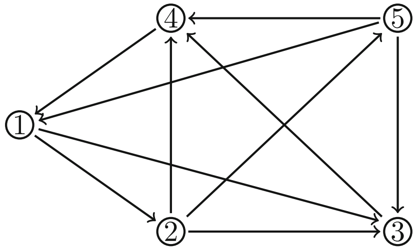
$$\begin{aligned} d^+(u) &\geq |N^+(u) \cap N'(v_0)| + 1 + |N^+(u) \cap N^+(v_0)| > \frac{\varepsilon_1}{12} n + d^+(v_0) - \varepsilon_2 n \\ &= d^+(v_0) + 2\varepsilon_2' n, \end{aligned}$$

so $d^+(u) > (n-1)/2 + \varepsilon_2' n$ and hence $u \notin V'(T)$. Thus, we have $|V'| \leq (1 - \varepsilon_1/12)n = (1 - \varepsilon_1')n$, a contradiction to Lemma 3.6. □

4 Stability Result

In this section, we prove our stability result. Let D_5 be the tournament with vertex set $\{1, 2, 3, 4, 5\}$ and directed edges $1 \to 2$, $1 \to 3$, $1 \leftarrow 4$, $1 \leftarrow 5$, $2 \to 3$, $2 \to 4$, $2 \to 5$, $3 \to 4$, $3 \leftarrow 5$, $4 \leftarrow 5$, see Fig. 2. Let $\boldsymbol{T}_5$ be the set of the tournaments D on 5 vertices such that $\mathcal{H}(D)$ contains a copy of $\mathcal{C}_5^-$ as a subhypergraph. We will apply Theorem 2.4 with these tournaments as $\boldsymbol{T}$, the tournaments to remove, for the proof

Fig. 2 The tournament D_5



of our stability result. For a 3-graph $\mathcal{H}$ and a 3-partition $\pi = (V_1, V_2, V_3)$ of $V(H)$, we write $\mathcal{H}^{\pi}_{bad}$ for $\bigcup_{1\le i\neq j\le 3}\mathcal{H}[V_i, V_i, V_j]$.

Proposition 4.1 (Stability result) *For every $\varepsilon_1, \varepsilon_2 > 0$, there exist $\delta > 0$ and N such that for every $n > N$, the following is true. For every n-vertex $\mathcal{C}_5^-$-free orientable 3-graph $\mathcal{H}$, if $|\mathcal{H}| > (1/24 - \delta)n^3$, then there exists a 3-partition $\pi = (V_1, V_2, V_3)$ of $V(\mathcal{H})$ such that $(1/3 - \varepsilon_2)n < |V_1|, |V_2|, |V_3| < (1/3 + \varepsilon_2)n$ and $|\mathcal{H}^{\pi}_{bad}| < \varepsilon_1 n^3$.*

Proof Given $\varepsilon_1, \varepsilon_2 > 0$, let $\beta \gg \gamma \gg \delta > 0$ be sufficiently small, and let N be sufficiently large. Let $T = T(\mathcal{H})$. By assumption, $t(T) \ge |\mathcal{H}| > (1/24 - \delta)n^3$.

Claim 4.2 *For every $D \in \boldsymbol{T}_5$, the number of induced copies of D in T is at most δn^5.*

Proof Since $\mathcal{H}$ is free of $\mathcal{C}_5^-$, whenever T contains an induced copy of D, this copy must contain a cyclic triangle that is not a hyperedge in $\mathcal{H}$. By Lemma 3.4, we have $t(T) < n^3/24$, so the number of such cyclic triangles is at most δn^3 and hence the number of induced copies of D in T is at most δn^5. □

Claim 4.3 *The number of induced copies of D_5 in T is at most $\delta^{1/4}n^5$.*

Proof Assume that the number of induced copies of D_5 is greater than $\delta^{1/4}n^5$. Define

$$S := \{\{v_1, v_2, v_3, v_4, v_5, u\} \in \binom{V(\mathcal{H})}{6} : v_4 \to u, u \to v_5, \text{ and } v_i \to v_j \text{ in } T \text{ iff } i \to j \text{ in } D_5, \text{for } 1 \le i < j \le 5\},$$

and define $A_T(\delta^{1/4})$ to be

$$\left\{\{v_4, v_5\} : \left|\left\{\{v_1, v_2, v_3\} : v_i \to v_j \text{ in } T \text{ iff } i \to j \text{ in } D_5, \text{ for } 1 \le i < j \le 5\right\}\right| \ge \delta^{1/4}n^3\right\}.$$

If $|A_T(\delta^{1/4})| < \delta^{1/4}n^2/4$, then the number of induced copies of D_5 in T is at most

$$\frac{1}{4}\delta^{\frac{1}{4}}n^2 \cdot \binom{n}{3} + \binom{n}{2} \cdot \frac{1}{4}\delta^{\frac{1}{4}}n^3 < \delta^{\frac{1}{4}}n^5,$$

a contradiction to our assumption, so $|A_T(\delta^{1/4})| \ge \delta^{1/4}n^2/4$. Since $t(T) \ge |\mathcal{H}| > (1/24 - \delta)n^3$, by Lemma 3.8, we have $|B_T(\delta^{1/3})| < 100\delta^{1/3}n^2$, so $|A_T(\delta^{1/4}) \setminus$

$B_T(\delta^{1/3})| \geq \delta^{1/4}n^2/5$. Note that $\{4, 5\}$ is not in any cyclic triangle in D_5, and $\{v_4, v_5, u\}$ forms a cyclic triangle, for every $\{v_1, v_2, v_3, v_4, v_5, u\} \in S$. Hence, every pair in $A_T(\delta^{1/4}) \setminus B_T(\delta^{1/3})$ is in at least $\delta^{1/4}n^3 \cdot \delta^{1/3}n$ sets in S. Trivially, every set in S contains at most $\binom{6}{2} = 15$ pairs in $A_T(\delta^{1/4}) \setminus B_T(\delta^{1/3})$. Therefore, we have

$$\begin{aligned} |S| &\geq \frac{1}{15} \cdot \delta^{\frac{1}{4}}n^3 \cdot \delta^{\frac{1}{3}}n \cdot |A_T(\delta^{1/4}) \setminus B_T(\delta^{1/3})| \\ &\geq \frac{1}{15} \cdot \delta^{\frac{1}{4}}n^3 \cdot \delta^{\frac{1}{3}}n \cdot \frac{1}{5}\delta^{\frac{1}{4}}n^2 = \frac{1}{75}\delta^{\frac{5}{6}}n^6. \end{aligned} \tag{2}$$

For every $F = \{v_1, v_2, v_3, v_4, v_5, u\} \in S$, consider the orientation of edges between v_i and u, for $1 \leq i \leq 3$. There are eight possibilities. We claim that F contains a copy of some $D \in \boldsymbol{T}_5$ in every case.

- If $u \leftarrow v_1$, then $v_4uv_5v_1v_2$ forms a copy of $\mathcal{C}_5^-$ in $\mathcal{H}(T)$.
- If $u \rightarrow v_1$ and $u \leftarrow v_3$, then $v_4v_5uv_3v_1$ forms a copy of $\mathcal{C}_5^-$ in $\mathcal{H}(T)$.
- If $u \rightarrow v_1$, $u \rightarrow v_2$, and $u \rightarrow v_3$, then $v_5v_1v_2v_4u$ forms a copy of $\mathcal{C}_5^-$ in $\mathcal{H}(T)$.
- If $u \rightarrow v_1$, $u \leftarrow v_2$, and $u \rightarrow v_3$, then $uv_2v_1v_4v_3$ forms a copy of $\mathcal{C}_5^-$ in $\mathcal{H}(T)$.

For a fixed $D \in \boldsymbol{T}_5$, by Claim 4.2, the number of copies of D in T is at most δn^5, and for every copy of D, it can be in at most n sets in S. Therefore, we have

$$|S| \leq |\boldsymbol{T}_5| \cdot \delta n^5 \cdot n. \tag{3}$$

By (2) and (3), we have $|\boldsymbol{T}_5| \geq \frac{1}{75}\delta^{-1/6} > 2^{\binom{5}{2}} \geq |\boldsymbol{T}_5|$, a contradiction. □

Applying Theorem 2.4 to T with $\boldsymbol{T} = \{D_5\} \cup \boldsymbol{T}_5$, we get a tournament T' free of D_5, where $\mathcal{H}(T')$ is free of $\mathcal{C}_5^-$ and T' and T differ by at most $\gamma n^2/2$ edges. Since changing the orientation of one edge can remove at most n cyclic triangles, we have $t(T') \geq t(T) - \gamma n^3/2 > (1/24 - \gamma)n^3$. Let $\mathcal{H}' := \mathcal{H}(T')$.

By Lemma 3.6, there exists $V' \subseteq V(T') = V(\mathcal{H})$ with $|V'| = m \geq (1 - \beta/2)n$ such that $(1/2 - \beta/5)n < d^+(v), d^-(v) < (1/2 + \beta/5)n$ in T' for every vertex $v \in V'$. Let T'' be the subtournament of T' induced by V'. Let $\mathcal{H}'' := \mathcal{H}(T'')$. We have

$$t(T'') \geq t(T') - \frac{\beta}{2}n \cdot n^2 > \left(\frac{1}{24} - \gamma - \frac{\beta}{2}\right)n^3 > \left(\frac{1}{24} - \beta\right)n^3 \geq \left(\frac{1}{24} - \beta\right)m^3, \tag{4}$$

and

$$d^+(v), d^-(v) > \left(\frac{1}{2} - \frac{\beta}{5}\right)n - \frac{\beta}{2}n > \left(\frac{1}{2} - \beta\right)n \geq \left(\frac{1}{2} - \beta\right)m, \tag{5}$$

$$d^+(v), d^-(v) < \left(\frac{1}{2} + \frac{\beta}{5}\right)n \leq \left(\frac{\frac{1}{2} + \frac{\beta}{5}}{1 - \frac{\beta}{2}}\right)m < \left(\frac{1}{2} + \beta\right)m, \tag{6}$$

in T'' for every $v \in V'$.

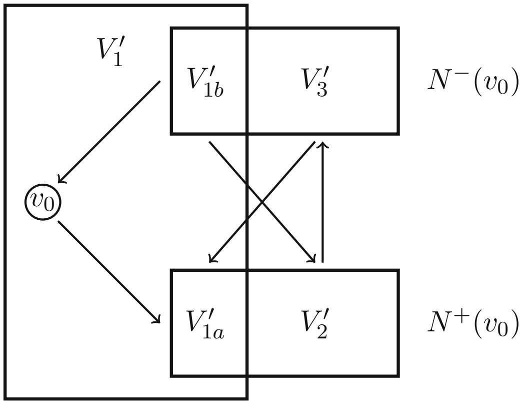


Fig. 3 The subsets V_1', V_2', V_3' of V'

Claim 4.4 *For every vertex $v \in V'$, every component of $N_{V'}^{\mathcal{H}''}(v)$ is a complete bipartite graph.*

Proof For every vertex $v \in V'$, its link graph $N_{V'}^{\mathcal{H}''}(v)$ is a bipartite graph with the two parts $N^+(v)$ and $N^-(v)$ in T''. We first claim that if there are vertices w, x, y, z such that $w, y \in N^+(v)$, $x, z \in N^-(v)$, and $w \to x$, $x \leftarrow y$, $y \to z$ in T'', then we also have $w \to z$ in T''. Equivalently, this is to say that whenever $\{w, x\}, \{x, y\}, \{y, z\} \in N_{V'}^{\mathcal{H}''}(v)$, we also have $\{w, z\} \in N_{V'}^{\mathcal{H}''}(v)$. Assume for contradiction that $w \leftarrow z$. If $w \to y$, then $wxvyz$ forms a copy of $\mathcal{C}_5^-$ in $\mathcal{H}''$. If $x \to z$, then $zyvxw$ forms a copy of $\mathcal{C}_5^-$ in $\mathcal{H}''$. Hence, the only possibility is $w \leftarrow y$ and $x \leftarrow z$, but then $\{v, y, w, x, z\}$ forms a copy of D_5 in T'', still a contradiction.

Now assume for contradiction that there exist vertices $a \in N^+(v)$ and $b \in N^-(v)$ in the same component of $N_{V'}^{\mathcal{H}''}(v)$ but $\{a, b\} \notin N_{V'}^{\mathcal{H}''}(v)$. Let P be the shortest path between a and b in $N_{V'}^{\mathcal{H}''}(v)$. Since $N_{V'}^{\mathcal{H}''}(v)$ is bipartite and $\{a, b\} \notin N_{V'}^{\mathcal{H}''}(v)$, we have that P contains at least 4 vertices, so we can assume that P starts with a, v_1, u_2, v_3. However, by the claim in the last paragraph, we have $\{a, v_3\} \in N_{V'}^{\mathcal{H}''}(v)$, a contradiction to that P is shortest. □

Let $v_0 \in V'$ be a vertex in maximum number of cyclic triangles in T''. Since $t(T'') \geq (1/24 - \beta)m^3$ by (4), we have that v_0 is in at least $(1/8 - 3\beta)m^2$ cyclic triangles and hence

$$d_{V'}^{\mathcal{H}''}(v_0) \geq (1/8 - 3\beta)m^2. \tag{7}$$

Assume that $V_2' \subseteq N^+(v_0)$, $V_3' \subseteq N^-(v_0)$ form the largest component in $N_{V'}^{\mathcal{H}''}(v_0)$. Let $V_{1a}' := N^+(v_0) \setminus V_2'$, $V_{1b}' := N^-(v_0) \setminus V_3'$, and let $V_1' := \{v_0\} \cup V_{1a}' \cup V_{1b}'$. Consider the partition $\pi' := (V_1', V_2', V_3')$ of V'. By Claim 4.4, we have $V_2' \to V_3'$, and by the definition of V_2', V_3', we have $V_2' \leftarrow V_{1b}'$ and $V_{1a}' \leftarrow V_3'$, see Fig. 3.

Claim 4.5 *We have $|V_2'| + |V_3'| \geq (1/2 - 8\beta)m$.*

Proof By Claim 4.4, we can assume that $N_{V'}^{\mathcal{H}''}(v_0)$ is partitioned into complete bipartite graphs $G_1, \ldots, G_p$ and assume that G_i has two parts with sizes x_i and y_i, for $1 \leq$

$i \le k$. Let $a = m-1$ and $b = |V_2'|+|V_3'|$. Then, $\sum_{i=1}^{p}(x_i+y_i) = a$ and $x_j + y_j \le b$ for $1 \le j \le k$, by the definition of V_2', V_3'. Applying Lemma 2.5, we have

$$d_{V'}^{\mathcal{H}''}(v_0) = \sum_{i=1}^{p} x_i y_i \le \left\lfloor \frac{a}{b} \right\rfloor \cdot \frac{b^2}{4} + \frac{1}{4}\left(a - b\left\lfloor \frac{a}{b}\right\rfloor\right)^2.$$

If $b \le (m-1)/3$, then

$$\begin{aligned} d_{V'}^{\mathcal{H}''}(v_0) &\le \frac{1}{4}ab + \frac{1}{4}b^2 \le \frac{m-1}{4}\cdot\frac{m-1}{3} + \frac{1}{4}\left(\frac{m-1}{3}\right)^2 \\ &= \frac{1}{9}(m-1)^2 < \left(\frac{1}{8} - 3\beta\right)m^2. \end{aligned}$$

If $(m-1)/3 < b < (1/2 - 8\beta)m$, then $\lfloor a/b \rfloor = 2$ and

$$\begin{aligned} d_{V'}^{\mathcal{H}''}(v_0) &\le 2\cdot\frac{b^2}{4} + \frac{1}{4}(m-1-2b)^2 = \frac{3}{2}b^2 - (m-1)b + \frac{1}{4}(m-1)^2 \\ &\le \frac{3}{2}\left(\left(\frac{1}{2} - 8\beta\right)m\right)^2 - (m-1)\cdot\left(\frac{1}{2}-8\beta\right)m + \frac{1}{4}(m-1)^2 \\ &= \left(\frac{1}{8} - 4\beta + 96\beta^2\right)m^2 - 8\beta m + \frac{1}{4} < \left(\frac{1}{8} - 3\beta\right)m^2. \end{aligned}$$

Both cases contradict (7). Therefore, $|V_2'| + |V_3'| \ge (1/2 - 8\beta)m$. □

Claim 4.6 *We have* $|V_1'|, |V_2'|, |V_3'| \ge 0.02m$.

Proof Assume for contradiction that $|V_1'| < 0.1m$ and hence $|V_2'| + |V_3'| > 0.9m$. If $|V_2'| \ge |V_3'|$, then $|V_2'| \ge 0.45m$. By (6), we have $|V_3'| > 0.9m - (1/2+\beta)m = (0.4-\beta)m$. By Lemma 3.7, there exists a vertex $v \in V_3'$ such that $|N^-(v) \cap V_3'| \ge |V_3'|/4$. We have

$$\begin{aligned} d^-(v) &\ge |V_2'| + |N^-(v) \cap V_3'| \ge |V_2'| + \frac{|V_3'|}{4} \\ &> 0.45m + \left(0.1 - \frac{\beta}{4}\right)m > \left(\frac{1}{2} + \beta\right)m, \end{aligned}$$

a contradiction to (6). A similar argument holds if $|V_2'| < |V_3'|$ by considering the out-degree of some vertex $v \in V_2'$. Therefore, $|V_1'| \ge 0.1m$.

Assume for contradiction that $|V_2'| < 0.02m$. Then, by (5), we have $|V_{1a}'| \ge (0.48-\beta)m$, and by Claim 4.5, we have $|V_3'| \ge (0.48 - 8\beta)m$. By Lemma 3.7, there exists $v \in V_{1a}'$ such that $|N^-(v) \cap V_{1a}'| \ge (0.12 - \beta/4)m$. Then, we $d^-(v) \ge (0.6 - 9\beta)m$, a contradiction to (6). Therefore, $|V_2'| \ge 0.02m$. Similar arguments give $|V_3'| \ge 0.02m$. □

We define

$$E_{12} := \{\{u, v\} : u \in V'_{1a},\ v \in V'_2,\ u \leftarrow v \text{ in } T''\},$$
$$E_{13} := \{\{u, v\} : u \in V'_{1b},\ v \in V'_3,\ u \rightarrow v \text{ in } T''\}.$$

Claim 4.7 *We have* $|E_{12}|, |E_{13}| \leq 100\beta m^2$.

Proof Let $S_1 := \{u \in V'_{1a} : \{u\} \leftarrow V'_{1b}\}$. For every $u \in S_1$, if $|N^-(u) \cap V'_2| > 2\beta m$, then using (5), we have

$$d^-(u) \geq |N^-(u) \cap V'_2| + |V'_3| + |V'_{1b}| > 2\beta m + d^-(v_0) > \left(\frac{1}{2} + \beta\right) m,$$

a contradiction to (6). Hence $|N^-(u) \cap V'_2| \leq 2\beta m$ for every $u \in S_1$.

Let $S_2 := \{u \in V'_{1a} \setminus S_1 :$ there exists $x \in V'_2$ such that $x \rightarrow u\}$. We claim $|S_2| \leq 60\beta m$. Otherwise, by Lemma 3.7, there exists $u \in S_2$ with $|N^-(u) \cap S_2| \geq 15\beta m$. By the definition of S_2, there exist $x \in V'_2$ with $x \rightarrow u$ and $z \in V'_{1b}$ with $u \rightarrow z$. Let y be an arbitrary vertex in V'_3, which exists since $V'_3 \neq \varnothing$ by Claim 4.6. Now, we have $v_0 \rightarrow \{x, u\}$, $v_0 \leftarrow \{y, z\}$, $x \rightarrow \{y, u\}$, $x \leftarrow z$, $u \leftarrow y$, and $u \rightarrow z$. If $y \rightarrow z$, then v_0uzxy forms a copy of $\mathcal{C}_5^-$ in $\mathcal{H}''$, a contradiction. Hence, $y \leftarrow z$. For every $x' \in V'_2 \setminus \{x\}$, if $x' \leftarrow u$, then $v_0x'yuz$ forms a copy of $\mathcal{C}_5^-$ in $\mathcal{H}''$, a contradiction, so we have $V'_2 \rightarrow \{u\}$. Then, using Claim 4.5, we have

$$d^-(u) \geq |N^-(u) \cap S_2| + |V'_2| + |V'_3| \geq 15\beta m + \left(\frac{1}{2} - 8\beta\right) m > \left(\frac{1}{2} + \beta\right) m,$$

a contradiction to (6). Therefore, $|S_2| \leq 60\beta m$. Thus,

$$|E_{12}| \leq |S_1| \cdot 2\beta m + |S_2| \cdot |V'_2| \leq m \cdot 2\beta m + 60\beta m \cdot m \leq 100\beta m^2.$$

Similar arguments give the upper bound on $|E_{13}|$. □

Claim 4.8 *We have* $|\mathcal{H}^{''\pi'}_{bad}| \leq 200\beta m^3$.

Proof Recall that we have $V'_2 \rightarrow V'_3$. For vertices $x \in V'_1$, $y \in V'_2$, we have $x \rightarrow y$ unless $\{x, y\} \in E_{12}$. For vertices $x \in V'_1$, $z \in V'_3$, we have $x \leftarrow z$ unless $\{x, z\} \in E_{13}$. Therefore, every hyperedge in $\mathcal{H}^{''\pi'}_{bad}$ contains a pair in $E_{12} \cup E_{13}$, so $|\mathcal{H}^{''\pi'}_{bad}| \leq 200\beta m^3$ by Claim 4.7. □

Claim 4.9 *We have* $(1/3 - \varepsilon_2/2)m < |V'_1|, |V'_2|, |V'_3| < (1/3 + \varepsilon_2/4)m$.

Proof By Claim 4.6, we can assume that $0.02m \leq |V'_i| \leq |V'_j| \leq |V'_k|$, where $\{i, j, k\} = \{1, 2, 3\}$. Assume for contradiction that $|V'_k| \geq (1/3 + \varepsilon_2/4)m$. Then, $|V'_i| \leq (m - |V'_k|)/2 \leq (1/3 - \varepsilon_2/8)m$. If $j \equiv i + 1 \pmod 3$, we consider the in-degree of vertices in V'_j. Let

$$d^-(V'_j) := \left|\{(u, v) : u \in V', v \in V'_j, u \rightarrow v\}\right|.$$

By (5), we have $d^-(V'_j) > |V'_j|(1/2 - \beta)m$. On the other hand, by the definition of E_{12}, E_{13} and Claim 4.7, we have

$$d^-(V'_j) \le |V'_i||V'_j| + \binom{|V'_j|}{2} + |E_{12}| + |E_{13}| \le \left(|V'_i| + \frac{|V'_j|}{2}\right)|V'_j| + 200\beta m^2$$
$$= \left(\frac{m - |V'_k|}{2} + \frac{|V'_i|}{2}\right)|V'_j| + 200\beta m^2 \le |V'_j|\left(\frac{1}{2} - \frac{3\varepsilon_2}{16}\right)m$$
$$+200\beta m^2 < |V'_j|\left(\frac{1}{2} - \beta\right)m,$$

a contradiction, where the last inequality is because $|V'_j| \ge 0.02m$ and $\varepsilon_2 \gg \beta$. If $j \equiv i - 1 \pmod 3$, a similar argument holds by considering the out-degree of vertices in V'_j. Thus, we get $|V'_k| < (1/3 + \varepsilon_2/4)m$ and hence $|V'_i| \ge m - |V'_j| - |V'_k| > (1/3 - \varepsilon_2/2)m$. □

Finally, let $V_1 := V'_1 \cup \{V(\mathcal{H}) \setminus V'\}$, $V_2 := V'_2$, and $V_3 := V'_3$. Consider the partition $\pi := (V_1, V_2, V_3)$ of $V(\mathcal{H})$. Recall that $|V'| = m \ge (1 - \beta/2)n$. By Claim 4.9, we have

$$\left(\frac{1}{3} - \varepsilon_2\right)n < |V_1|, |V_2|, |V_3| < \left(\frac{1}{3} + \varepsilon_2\right)n.$$

Recall that T'' is a subtournament of T' induced by V' and $\mathcal{H}' = \mathcal{H}(T')$. For every hyperedge e in $\mathcal{H}'^{\pi}_{bad}$, either $e \cap (V(\mathcal{H}) \setminus V') \ne \varnothing$ or $e \in \mathcal{H}''^{\pi'}_{bad}$. Hence, by Claim 4.8, we have

$$|\mathcal{H}'^{\pi}_{bad}| \le |V(\mathcal{H}) \setminus V'| \cdot n^2 + |\mathcal{H}''^{\pi'}_{bad}| \le \frac{\beta}{2}n^3 + 200\beta m^3 \le \frac{\beta}{2}n^3 + 200\beta n^3 \le 300\beta n^3.$$

Recall that T and T' differ by at most $\gamma n^2/2$ edges, so $\mathcal{H}(T)$ and $\mathcal{H}'$ differ by at most $\gamma n^3/2$ hyperedges. Also recall that $\mathcal{H} \subseteq \mathcal{H}(T)$. Thus,

$$\left|\mathcal{H}^{\pi}_{bad}\right| \le |\mathcal{H}(T)^{\pi}_{bad}| \le |\mathcal{H}'^{\pi}_{bad}| + \frac{\gamma}{2}n^3 \le 300\beta n^3 + \frac{\gamma}{2}n^3 < \varepsilon_1 n^3.$$

□

5 Proof of Theorem 1.1

In this section, we provide the proof for Theorem 1.1. Recall that we defined in Sect. 3 that $\mathcal{C}^-_{\le L}$ is the set of all the pseudo-cycles minus one hyperedge of size ℓ, where $4 \le \ell \le L$ and $3 \nmid \ell$, and $\mathcal{C}^- = \bigcup_{L \ge 4} \mathcal{C}^-_{\le L}$. We will first prove that $\pi(\mathcal{C}^-_{\le L}) = 1/4$ for sufficiently large L, see Proposition 5.6, and then convert it to Theorem 1.1 using Theorem 2.3.

If we could prove that for some $L \geq 5$, the maximum $\mathcal{C}^-_{\leq L}$-free 3-graphs can be made free of $\mathcal{C}^-$ by removing $o(n^3)$ hyperedges, then we get $\pi(\mathcal{C}^-_{\leq L}) \leq 1/4$ immediately by Lemmas 3.4 and 3.5. We are not able to show this. The following lemma is what we can achieve. Its proof is parallel to Propositions 6.3 and 6.4 in [11].

Lemma 5.1 *If a 3-graph $\mathcal{H}$ is free of $\mathcal{C}^-_{\leq L}$, where $L \geq 27$, then we can delete at most $\frac{1}{2}\sqrt{\frac{21}{L-26}}n^3$ hyperedges from $\mathcal{H}$ to make it free of $\mathcal{C}^-$.*

Proof Let $\delta = \sqrt{\frac{21}{L-26}}$. Do the following algorithm for $\mathcal{H}$. Check whether there is a pair of vertices $\{u, v\}$ with codegree in $(0, \delta n)$. If so, delete all the hyperedges containing $\{u, v\}$ and check again; otherwise end the algorithm. Call the remaining hypergraph $\mathcal{H}'$. In $\mathcal{H}'$, every pair of vertices has codegree either 0 or at least δn. Note that $\mathcal{H}'$ is $\mathcal{C}^-_{\leq L}$-free and $|\mathcal{H}'| \geq |\mathcal{H}| - \delta n^3/2$, since for every pair of vertices $\{u, v\}$, the algorithm does the removal at most once. We will prove that $\mathcal{H}'$ is free of $\mathcal{C}^-$.

We first show that whenever $\mathcal{H}'$ contains a pseudo-path $abu_1 \dots u_t cd$, for some $t \geq 0$, then $\mathcal{H}'$ also contains a pseudo-path $abv_1 \dots v_k cd$, where $k < (L-8)/3$. Let $\mathcal{P} = abv_1 \dots v_k cd$ be the shortest pseudo-path starting from a, b and ending with c, d. For vertices v, v', let $M(v, v') := \{(u, u') : \{v, v', u\}, \{v', u, u'\} \in \mathcal{H}'\}$. For $1 \leq i < j \leq k$, where both i, j are divisible by 7, we claim that $M(v_i, v_{i+1}) \cap M(v_j, v_{j+1}) = \varnothing$. Otherwise, let $(w, w') \in M(v_i, v_{i+1}) \cap M(v_j, v_{j+1})$. Then,

$$abv_1 \dots v_{i-1} v_i v_{i+1} w w' v_{j+1} w v_j v_{j+1} v_{j+2} \dots v_k cd,$$

is a shorter pseudo-path between a, b and c, d than $\mathcal{P}$, a contradiction. By the codegree condition of $\mathcal{H}'$, we have $|M(v_i, v_{i+1})| \geq \delta^2 n^2$, for every $1 \leq i < k$. Therefore,

$$\begin{aligned} n^2 > \left| \bigcup_{i:\, 1 \leq i < k} M(v_i, v_{i+1}) \right| &\geq \sum_{i:\, 1 \leq i < k,\, 7 \mid i} |M(v_i, v_{i+1})| \\ &\geq \left\lfloor \frac{k}{7} \right\rfloor \cdot \delta^2 n^2 \geq \left(\frac{k}{7} - \frac{6}{7} \right) \cdot \delta^2 n^2, \end{aligned}$$

implying $k < 7/\delta^2 + 6 = (L-8)/3$, as desired.

Now, assume for contradiction that $u_1 u_2 \dots u_{\ell-1} u_\ell$ is the shortest pseudo-cycle minus one hyperedge in $\mathcal{H}'$ with size ℓ not divisible by 3. Note that $\ell > L$. Let $s = \lfloor (L+1)/3 \rfloor + 1$. Since there is a pseudo-path $u_1 u_2 \dots u_s u_{s+1}$ in $\mathcal{H}'$, by the claim in the last paragraph, there is a pseudo-path $\mathcal{P} = u_1 u_2 v_1 v_2 \dots v_k u_s u_{s+1}$, where $k < (L-8)/3$. Then, $\mathcal{P} u_{s+2} \dots u_{\ell-1} u_\ell$ is a pseudo-cycle minus one hyperedge of size $k + 4 + (\ell - s - 1) < \ell$ in $\mathcal{H}'$, so $3 \mid k + 4 + (\ell - s - 1)$ and hence $3 \mid k + 2s + \ell$. Let $\mathcal{P}'$ be the pseudo-path $u_s u_{s+1} u_{s-1} u_s u_{s-2} u_{s-1} \dots u_4 u_2 u_3 u_1 u_2$. Then, $v_1 v_2 \dots v_k \mathcal{P}'$ is a pseudo-cycle of size $k + 2s < L$ in $\mathcal{H}'$, so $3 \mid k + 2s$. However, we get $3 \mid \ell$, a contradiction. Thus, $\mathcal{H}'$ is free of $\mathcal{C}^-$. □

Lemma 5.2 *For every integer $L \geq 27$, we have*

$$\mathrm{ex}(n, \mathcal{C}^{-}_{\leq L}) \leq \left(\frac{1}{24} + \frac{1}{2}\sqrt{\frac{21}{L-26}}\right) n^3.$$

Proof This follows immediately from Lemmas 3.4, 3.5, and 5.1. □

We use a standard symmetrization method in the following lemmas to bound the degrees in a maximum $\mathcal{C}^{-}_{\leq L}$-free 3-graph. For a 3-graph $\mathcal{H}$, a vertex set $S \subseteq V(\mathcal{H})$, and a vertex $v \in V(\mathcal{H}) \setminus S$, let $T_{S,v} = \{v_1, \ldots, v_{|S|}\}$ be a set such that $T_{S,v} \cap V(\mathcal{H}) = \varnothing$, and let $\mathcal{H}_{S,v}$ be the 3-graph on vertex set $(V(\mathcal{H}) \setminus S) \cup T_{S,v}$ with hyperedges

$$(\mathcal{H} \setminus \{e \in \mathcal{H} : e \cap S \neq \varnothing\}) \bigcup \left(\bigcup_{i=1}^{|S|} \{\{v_i, x, y\} : \{v, x, y\} \in \mathcal{H}, \{x, y\} \cap S = \varnothing\}\right).$$

We write $\mathcal{H}_{u,v}$ for $\mathcal{H}_{\{u\},v}$.

Lemma 5.3 *If a 3-graph $\mathcal{H}$ is $\mathcal{C}^{-}_{\leq L}$-free for some $L \geq 4$, then $\mathcal{H}_{S,v}$ is also $\mathcal{C}^{-}_{\leq L}$-free, for every vertex set $S \subseteq V(\mathcal{H})$ and vertex $v \in V(\mathcal{H}) \setminus S$.*

Proof Assume for contradiction that $\mathcal{H}_{S,v}$ is not $\mathcal{C}^{-}_{\leq L}$-free. Let $v_1 v_2 \ldots v_\ell$ be a copy of some $\mathcal{C}^{-} \in \mathcal{C}^{-}_{\leq L}$ in $\mathcal{H}_{S,v}$. Note that the codegree of v, u' and the codegree of u', u'' are zero in $\mathcal{H}_{S,v}$, for every $u' \neq u'' \in T_{S,v}$. Replacing all appearances of the new vertices in $T_{S,v}$ with v in $v_1 v_2 \ldots v_\ell$, we get a copy of $\mathcal{C}^{-} \in \mathcal{C}^{-}_{\leq L}$ in $\mathcal{H}$, a contradiction to that $\mathcal{H}$ is $\mathcal{C}^{-}_{\leq L}$-free. □

Lemma 5.4 *There exists a constant $N > 0$ such that the following is true for every $L \geq 4$ and $n > N$. Let $\mathcal{H}$ be a maximum n-vertex $\mathcal{C}^{-}_{\leq L}$-free 3-graph. For every vertex $v \in V(\mathcal{H})$, we have $d(v) \geq n^2/8 - 2n$.*

Proof Assume for contradiction that there exists $v \in V(H)$ with $d(v) < n^2/8 - 2n$. By Construction 1.2, $|\mathcal{H}| \geq n^3/24 - Cn \log n$ for some constant $C > 0$. Hence, there is $x \in V(\mathcal{H})$ such that $d(x) \geq n^2/8 - 3C \log n$. By Lemma 5.3, $\mathcal{H}_{v,x}$ is also $\mathcal{C}^{-}_{\leq L}$-free. However, we have

$$|\mathcal{H}_{v,x}| > |\mathcal{H}| - (n^2/8 - 2n) + (n^2/8 - 3C \log n) - n > |\mathcal{H}|,$$

a contradiction to the maximality of $\mathcal{H}$. □

Lemma 5.5 *For every $\varepsilon > 0$, there exist $\delta \geq \varepsilon^2/100$, L_0, and N such that the following is true for every $L > L_0$ and $n > N$. Let $\mathcal{H}$ be an n-vertex $\mathcal{C}^{-}_{\leq L}$-free 3-graph with $|\mathcal{H}| \geq (1/24 - \delta)n^3$. For every vertex $v \in V(\mathcal{H})$, we have $d(v) \leq (1/8 + \varepsilon)n^2$.*

Proof Without loss of generality, we can assume ε is sufficiently small. Let $\delta = \varepsilon^2/100$, and let $L_0 > 21 \cdot 50^2 \cdot (1/\varepsilon)^4 + 26$ and N be sufficiently large. Let $L > L_0$. Note that for every $\mathcal{C}^{-}_{\leq L}$-free 3-graph $\mathcal{F}$, by Lemma 5.2, we have $|\mathcal{F}| \leq (1/24 + \delta)n^3$.

Let $\mathcal{H}$ be an n-vertex $\mathcal{C}^-_{\le L}$-free 3-graph with $|\mathcal{H}| \ge (1/24 - \delta)n^3$. Assume for contradiction that a vertex $v \in V(\mathcal{H})$ has $d(v) > (1/8 + \varepsilon)n^2$. Let $S_0 = \{u \in V(\mathcal{H}) : d(u) \le (1/8 + 3\sqrt{\delta})n^2\}$. We have

$$|V(\mathcal{H}) \setminus S_0| \cdot \left(\frac{1}{8} + 3\sqrt{\delta}\right) n^2 \cdot \frac{1}{3} \le |\mathcal{H}| < \left(\frac{1}{24} + \delta\right) n^3,$$

so

$$|S_0| > \frac{\sqrt{\delta} - \delta}{\frac{1}{24} + \sqrt{\delta}} n > \sqrt{\delta} n.$$

Let S be a subset of S_0 with size $\lfloor \sqrt{\delta} n \rfloor$. Note that $v \notin S_0 \supseteq S$. By Lemma 5.3, $\mathcal{H}_{S,v}$ is $\mathcal{C}^-_{\le L}$-free. However, we have

$$\begin{aligned}
|\mathcal{H}_{S,v}| &> |\mathcal{H}| - |S| \cdot \left(\frac{1}{8} + 3\sqrt{\delta}\right) n^2 + |S| \cdot \left(\left(\frac{1}{8} + \varepsilon\right) n^2 - |S| \cdot n\right) \\
&\ge \left(\frac{1}{24} - \delta\right) n^3 + |S| \left(\varepsilon - 4\sqrt{\delta}\right) n^2 \\
&\ge \left(\frac{1}{24} - \delta\right) n^3 + \left(\sqrt{\delta} n - 1\right) \left(\varepsilon - 4\sqrt{\delta}\right) n^2 \\
&\ge \left(\frac{1}{24} - \delta + \sqrt{\delta}\left(\varepsilon - 4\sqrt{\delta}\right)\right) n^3 - \varepsilon n^2 > \left(\frac{1}{24} + \delta\right) n^3,
\end{aligned}$$

a contradiction. Therefore, $d(v) \le (1/8 + \varepsilon)n^2$ for every vertex $v \in V(\mathcal{H})$. □

Proposition 5.6 *For every sufficiently large L, we have $\pi(\mathcal{C}^-_{\le L}) = 1/4$.*

Proof For every integer $L \ge 4$, by Construction 1.2, we have $\mathrm{ex}(n, \mathcal{C}^-_{\le L}) \ge n^3/24 - Cn \log n$ for some absolute constant $C > 0$.

For the upper bound, fix ε to be 10^{-10000} and let L, M be sufficiently large integers. We will prove that for every $n \ge 1$, we have

$$\mathrm{ex}(n, \mathcal{C}^-_{\le L}) \le n^3/24 + M^2 n. \tag{8}$$

When $n \le M$, (8) is trivial, since $M^2 n > \binom{n}{3}$. Now assume that $n > M$ and (8) is true for every positive integer less than n. Recall that for a 3-graph $\mathcal{H}$ and a partition $\pi = (V_1, V_2, V_3)$ of $V(\mathcal{H})$, we defined $\mathcal{H}_\pi := \mathcal{H}[V_1, V_2, V_3]$ and $\overline{\mathcal{H}}_\pi := [V_1, V_2, V_3] \setminus \mathcal{H}_\pi$ in Sect. 2 and $\mathcal{H}^\pi_{bad} := \bigcup_{1 \le i \ne j \le 3} \mathcal{H}[V_i, V_i, V_j]$ in Sect. 4.

Let $\mathcal{H}$ be a maximum n-vertex $\mathcal{C}^-_{\le L}$-free 3-graph. By Construction 1.2, $|\mathcal{H}| \ge n^3/24 - Cn \log n$. By Lemmas 5.4 and 5.5, we have for every vertex $v \in V(\mathcal{H})$,

$$(1/8 - \varepsilon)\, n^2 < d(v) < (1/8 + \varepsilon)\, n^2. \tag{9}$$

By Lemma 5.1, there exists a $\mathcal{C}^-$-free subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ with $|\mathcal{H} \setminus \mathcal{H}'| \leq \frac{1}{2}\sqrt{\frac{21}{L-26}}n^3$, so $|\mathcal{H}'| \geq \left(\frac{1}{24} - \frac{1}{2}\sqrt{\frac{21}{L-26}}\right)n^3 - Cn\log n$. By Lemma 3.5, $\mathcal{H}'$ is orientable. Hence, we can apply Proposition 4.1 to $\mathcal{H}'$ and get a 3-partition $\pi = (V_1, V_2, V_3)$ of $V(\mathcal{H}') = V(\mathcal{H})$ such that

$$(1/3 - \varepsilon)\, n < |V_1|, |V_2|, |V_3| < (1/3 + \varepsilon)\, n, \tag{10}$$

and $|\mathcal{H}'^{\pi}_{bad}| < \varepsilon n^3/2$. Then, $|\mathcal{H}^{\pi}_{bad}| < \varepsilon n^3/2 + \frac{1}{2}\sqrt{\frac{21}{L-26}}n^3 < \varepsilon n^3$. Note that $(\mathcal{H} \setminus \mathcal{H}^{\pi}_{bad}) \cup \bar{\mathcal{H}}_\pi$ is still $\mathcal{C}^-_{\leq L}$-free. By the maximality of $\mathcal{H}$, we have

$$|\bar{\mathcal{H}}_\pi| \leq |\mathcal{H}^{\pi}_{bad}| < \varepsilon n^3. \tag{11}$$

For $\{i, j, k\} = \{1, 2, 3\}$, we define

$$A_i := \{x \in V_i \mid d_{V_j,V_k}(x) \geq |V_j||V_k| - \varepsilon^{1/2}n^2\}, \quad B_i := V_i \setminus A_i,$$
$$A := A_1 \cup A_2 \cup A_3, \quad \text{and} \quad B := B_1 \cup B_2 \cup B_3.$$

Claim 5.7 *We have $|B_i| \leq \varepsilon^{1/2}n$ for $i \in [3]$.*

Proof If $|B_i| > \varepsilon^{1/2}n$ for some $i \in [3]$, then $|\bar{\mathcal{H}}_\pi| \geq |B_i| \cdot \varepsilon^{1/2}n^2 > \varepsilon n^3$, contradicting (11). □

Claim 5.8 *Let $\{i, j, k\} = [3]$. For vertices $v_1, v_2 \in V(\mathcal{H})$ with $d_{V_j,V_k}(v_1) \geq 2\varepsilon^{1/4}n^2$, we have*

$$d_{V_i}(v_1, v_2) \leq \frac{3n}{d_{V_j,V_k}(v_1)}\left(|V_i||V_k| - d_{V_i,V_k}(v_2)\right) + 2\varepsilon^{1/2}n. \tag{12}$$

Proof Assume for contradiction that (12) is false for some $v_1, v_2 \in V(\mathcal{H})$. We will prove that $\mathcal{H}$ contains a copy of C_5^-. Let $Z = \{z \in V_k : d_{V_j}(v_1, z) \geq d_{V_j,V_k}(v_1)/n\}$. Then, by the upper bounds on $|V_j|, |V_k|$ in (10), we have

$$\begin{aligned} d_{V_j,V_k}(v_1) &\leq |Z| \cdot |V_j| + |V_k \setminus Z| \cdot \frac{d_{V_j,V_k}(v_1)}{n} \\ &\leq |Z| \cdot \left(\frac{1}{3} + \varepsilon\right)n + \left(\frac{1}{3} + \varepsilon\right)d_{V_j,V_k}(v_1), \end{aligned}$$

so

$$|Z| \geq \frac{\frac{2}{3} - \varepsilon}{\left(\frac{1}{3} + \varepsilon\right)n} \cdot d_{V_j,V_k}(v_1) \geq \frac{3}{2n}d_{V_j,V_k}(v_1). \tag{13}$$

Let $X = \{x \in N_{V_i}(v_1, v_2) : d_Z(x, v_2) \geq |Z|/2\}$. If $|X| \leq d_{V_i}(v_1, v_2)/2$, then by (13) and our assumptions that (12) is false and $d_{V_j,V_k}(v_1) \geq 2\varepsilon^{1/4}n^2$, we have

$$|V_i||V_k| - d_{V_i,V_k}(v_2) \geq |N_{V_i}(v_1, v_2) \setminus X| \cdot \left(|Z| - \frac{|Z|}{2}\right) \geq \frac{1}{2}d_{V_i}(v_1, v_2) \cdot \frac{3}{4n}d_{V_j,V_k}(v_1)$$

$$\geq \frac{3}{8n} d_{V_j,V_k}(v_1) \left(\frac{3n}{d_{V_j,V_k}(v_1)} \left(|V_i||V_k| - d_{V_i,V_k}(v_2) \right) + 2\varepsilon^{1/2} n \right)$$

$$d_{V_j,V_k}(v_1) \geq \frac{9}{8} \left(|V_i||V_k| - d_{V_i,V_k}(v_2) \right) + \frac{3}{2}\varepsilon^{3/4} n^2 > |V_i||V_k| - d_{V_i,V_k}(v_2),$$

a contradiction, so $|X| > d_{V_i}(v_1, v_2)/2 \geq \varepsilon^{1/2} n^2$. By Claim 5.7, $|B_i| \leq \varepsilon^{1/2} n$. Hence, we are able to choose and fix a vertex $x \in X \setminus B_i \subseteq A_i$. We have $\{v_1, v_2, x\} \in \mathcal{H}$.

Let $Z' = N_Z(x, v_2)$. By the definition of X, we have $|Z'| \geq |Z|/2$. Then, by the definition of Z and (13), we have

$$d_{V_j,Z'}(v_1) \geq \frac{d_{V_j,V_k}(v_1)}{n} \cdot |Z'| \geq \frac{d_{V_j,V_k}(v_1)}{n} \cdot \frac{|Z|}{2} \geq \frac{3}{4n^2} d^2_{V_j,V_k}(v_1),$$

and hence, by the assumption that $d_{V_j,V_k}(v_1) \geq 2\varepsilon^{1/4} n^2$, we have

$$\begin{aligned} d_{V_j \setminus \{v_2\},Z'}(v_1) &\geq \frac{3}{4n^2} d^2_{V_j,V_k}(v_1) - |Z'| \geq \frac{3}{4n^2} \cdot 4\varepsilon^{1/2} n^4 - n \\ &= 3\varepsilon^{1/2} n^2 - n > 2\varepsilon^{1/2} n^2. \end{aligned} \tag{14}$$

We can choose and fix $y \in V_j \setminus \{v_2\}$ and $z \in Z'$ such that $\{y, z\} \in N(v_1) \cap N(x)$, since otherwise, by (14), we have

$$d_{V_j,V_k}(x) \leq |V_j||V_k| - d_{V_j \setminus \{v_2\},Z'}(v_1) < |V_j||V_k| - 2\varepsilon^{1/2} n^2,$$

a contradiction to that $x \in A_i$. Note that $y \neq v_2$. We have $\{x, v_2, z\} \in \mathcal{H}$ by the definition of Z'. We also have $\{v_1, y, z\}, \{x, y, z\} \in \mathcal{H}$.

Now, we have hyperedges $\{v_1, v_2, x\}, \{x, v_2, z\}, \{x, y, z\}$, and $\{v_1, y, z\}$, so $v_1 v_2 x z y$ is a copy of $\mathcal{C}_5^-$ in $\mathcal{H}$, a contradiction. Thus, we have (12). □

Claim 5.9 *Let $\{i, j, k\} = [3]$. For every vertex $v \in V(\mathcal{H})$ with $d_{V_j,V_k}(v) \geq 2\varepsilon^{1/4} n^2$, we have $d_{V_i,V_j}(v) \leq \varepsilon^{1/4} n^2$.*

Proof For every vertex $y \in A_j$, by Claim 5.8, we have

$$d_{V_i}(v, y) \leq \frac{3n}{2\varepsilon^{1/4} n^2} \cdot \varepsilon^{1/2} n^2 + 2\varepsilon^{1/2} n \leq 2\varepsilon^{1/4} n.$$

Therefore, by (10) and Claim 5.7, we have

$$d_{V_i,V_j}(v) \leq |A_j| \cdot 2\varepsilon^{1/4} n + |B_j| \cdot |V_i| \leq \left(\frac{1}{3} + \varepsilon \right) n \cdot \left(2\varepsilon^{1/4} n + \varepsilon^{1/2} n \right) \leq \varepsilon^{1/4} n^2.$$

□

Claim 5.10 *We have $|\mathcal{H}[V_i]| \geq (1/24 - 27\varepsilon^{1/4})(n/3)^3$ for $i \in [3]$.*

Proof Let $\{i, j, k\} = [3]$. Let v be an arbitrary vertex in A_i. By the definition of A_i and Claim 5.9, we have $d_{V_i,V_j}(v), d_{V_i,V_k}(v) \leq \varepsilon^{1/4}n^2$. Since $\mathcal{K}_4^- \in \mathcal{C}_{\leq L}^-$, the link graph of v is triangle-free. By Theorem 2.1 and (10), we have

$$d_{V_j \cup V_k, V_j \cup V_k}(v) \leq \frac{1}{4}\left(|V_j| + |V_k|\right)^2 \leq \frac{1}{4}\left(2 \cdot \left(\frac{1}{3} + \varepsilon\right)n\right)^2 \leq \left(\frac{1}{9} + \varepsilon\right)n^2.$$

Then, by the bound $d(v) \geq (1/8 - \varepsilon)n^2$ in (9), we have

$$d_{V_i,V_i}(v) = d(v) - d_{V_i,V_j}(v) - d_{V_i,V_k}(v) - d_{V_j \cup V_k, V_j \cup V_k}(v) \geq \left(\frac{1}{72} - 2\varepsilon^{1/4} - 2\varepsilon\right)n^2.$$

Thus, by (10) and Claim 5.7, we have

$$\begin{aligned}|\mathcal{H}[V_i]| &\geq \frac{1}{3}\sum_{v \in A_i} d_{V_i,V_i}(v) \geq \frac{1}{3} \cdot (|V_i| - |B_i|) \cdot \left(\frac{1}{72} - 2\varepsilon^{1/4} - 2\varepsilon\right)n^2 \\ &\geq \frac{1}{3} \cdot \left(\frac{1}{3} - \varepsilon - \varepsilon^{1/2}\right)n \cdot \left(\frac{1}{72} - 3\varepsilon^{1/4}\right)n^2 \geq \left(\frac{1}{24} - 27\varepsilon^{1/4}\right)\left(\frac{n}{3}\right)^3.\end{aligned}$$

□

Claim 5.11 *For every vertex $v \in B$, there exists a unique pair $\{j, k\} \subset [3]$ such that $d_{V_j,V_k}(v) \geq |V_j||V_k| - 5\varepsilon^{1/9}n^2$.*

Proof For every $i \in [3]$, by Claim 5.10 and (10), we have

$$\begin{aligned}|\mathcal{H}[V_i \cup \{v\}]| \geq |\mathcal{H}[V_i]| &\geq \left(\frac{1}{24} - 27\varepsilon^{1/4}\right)\left(\frac{n}{3}\right)^3 \\ &\geq \left(\frac{1}{24} - 36\varepsilon^{1/4}\right)\left(\left(\frac{1}{3} + \varepsilon\right)n + 1\right)^3 \geq \left(\frac{1}{24} - 36\varepsilon^{1/4}\right)|V_i \cup \{v\}|^3.\end{aligned}$$

Note that, by (10), we have $|V_i \cup \{v\}| \geq |V_i| \geq (1/3 - \varepsilon)n$, so $|V_i \cup \{v\}|$ is sufficiently large. We can apply Lemma 5.5 to $\mathcal{H}[V_i \cup \{v\}]$ and get

$$\begin{aligned}d_{V_i,V_i}(v) &\leq \left(\frac{1}{8} + 60\varepsilon^{1/8}\right)|V_i \cup \{v\}|^2 \leq \left(\frac{1}{8} + 60\varepsilon^{1/8}\right)\left(\left(\frac{1}{3} + \varepsilon\right)n + 1\right)^2 \\ &\leq \left(\frac{1}{72} + 8\varepsilon^{1/8}\right)n^2. \qquad (15)\end{aligned}$$

If $d_{V_1,V_2}(v), d_{V_1,V_3}(v), d_{V_2,V_3}(v) < 2\varepsilon^{1/4}n$, then we have

$$d(v) \leq \sum_{i=1}^{3} d_{V_i,V_i}(v) + \sum_{1 \leq i < j \leq 3} d_{V_i,V_j}(v) \leq 3 \cdot \left(\frac{1}{72} + 8\varepsilon^{1/8}\right)n^2 + 3 \cdot 2\varepsilon^{1/4}n^2 < \frac{n^2}{23},$$

a contradiction to (9). Hence, we can fix $\{j, k\} \subset [3]$ such that $d_{V_j,V_k}(v) \geq 2\varepsilon^{1/4}n^2$. Let $\{i\} = [3] \setminus \{j, k\}$. By Claim 5.9, we have

$$d_{V_i,V_j}(v), d_{V_i,V_k}(v) \leq \varepsilon^{1/4}n^2. \tag{16}$$

It remains to prove that $d_{V_j,V_k}(v) \geq |V_j||V_k| - 5\varepsilon^{1/9}n^2$.

Let $G = N_{V_j \cup V_k, V_j \cup V_k}(v)$. We view G as a graph on vertex set $(V_j \cup V_k) \setminus \{v\}$. Note that G is triangle-free, since $\mathcal{H}$ is $\mathcal{K}_4^-$-free. By (10), we have

$$|V(G)| \leq |V_j| + |V_k| \leq \left(\frac{2}{3} + 2\varepsilon\right)n. \tag{17}$$

By (9), (15) and (16), we have

$$\begin{aligned}|G| &= d_{V_j,V_j}(v) + d_{V_j,V_k}(v) + d_{V_k,V_k}(v) = d(v) - d_{V_i,V_i}(x) - d_{V_i,V_j}(x) - d_{V_i,V_k}(x)\\ &\geq \left(\frac{1}{8} - \varepsilon\right)n^2 - \left(\frac{1}{72} + 8\varepsilon^{1/8}\right)n^2 - 2 \cdot \varepsilon^{1/4}n^2 \geq \left(\frac{1}{9} - 9\varepsilon^{1/8}\right)n^2. \end{aligned} \tag{18}$$

For a bipartition (S_1, S_2) of $V_j \cup V_k$, let $G[S_1, S_2]$ be the set of edges in G between S_1 and S_2. Fix (S_1, S_2) to be the bipartition of $V_j \cup V_k$ which minimizes $|G \setminus G[S_1, S_2]|$. By (17), (18), and Theorem 2.2, we have

$$\begin{aligned}|G \setminus G[S_1, S_2]| &\leq |G| - \frac{4|G|^2}{|V(G)|^2} \leq |G| - \frac{4|G|^2}{\left(\frac{4}{9} + \frac{8}{3}\varepsilon + 4\varepsilon^2\right)n^2}\\ &\leq \left(\frac{1}{9} - 9\varepsilon^{1/8}\right)n^2 - \frac{4\left(\left(\frac{1}{9} - 9\varepsilon^{1/8}\right)n^2\right)^2}{\left(\frac{4}{9} + \frac{8}{3}\varepsilon + 4\varepsilon^2\right)n^2} \leq \left(\frac{1}{9} - 9\varepsilon^{1/8}\right)n^2\\ &\quad - \left(\frac{1}{9} - 20\varepsilon^{1/8}\right)n^2 = 11\varepsilon^{1/8}n^2. \end{aligned} \tag{19}$$

By (18) and (19), We have $|G[S_1, S_2]| \geq (1/9 - 20\varepsilon^{1/8})n^2$. Then, by (10),

$$\begin{aligned}|[S_1, S_2] \setminus G[S_1, S_2]| &= |S_1||S_2| - |G[S_1, S_2]| \leq \left(\frac{|V_j| + |V_k|}{2}\right)^2 - |G[S_1, S_2]|\\ &\leq \left(\left(\frac{1}{3} + \varepsilon\right)n\right)^2 - \left(\frac{1}{9} - 20\varepsilon^{1/8}\right)n^2 \leq 21\varepsilon^{1/8}n^2. \end{aligned} \tag{20}$$

Note that if $\{u, w_1\}, \{u, w_2\} \in G$, then $\{u, w_1, w_2\} \notin \mathcal{H}$, since otherwise, vw_1uw_2 forms a copy of $\mathcal{K}_4^-$ in $\mathcal{H}$. Hence, every hyperedge in $\mathcal{H}[S_1, S_1, S_2] \cup \mathcal{H}[S_2, S_2, S_1]$ contains at least one pair of vertices in $[S_1, S_2] \setminus G[S_1, S_2]$. Therefore, by (20),

$$|\mathcal{H}[S_1, S_1, S_2]| + |\mathcal{H}[S_2, S_2, S_1]| \leq 21\varepsilon^{1/8}n^3. \tag{21}$$

Now, assume for contradiction that $d_{V_j,V_k}(v) < |V_j||V_k| - 5\varepsilon^{1/9}n^2$. By (18) and (10),

$$d_{V_j,V_j}(v) + d_{V_k,V_k}(v) \geq \left(\frac{1}{9} - \varepsilon^{1/8}\right)n^2 - \left(|V_j||V_k| - 5\varepsilon^{1/9}n^2\right) \geq 4\varepsilon^{1/9}n^2,$$

so we can assume without loss of generality that $d_{V_j,V_j}(v) \geq 2\varepsilon^{1/9}n^2$. Let $V_{j1} = V_j \cap S_1$ and $V_{j2} = V_j \cap S_2$. Then, by (19), we have

$$\begin{aligned}2\varepsilon^{1/9}n^2 \leq d_{V_j,V_j}(v) &= d_{V_{j1},V_{j1}}(v) + d_{V_{j2},V_{j2}}(v) + d_{V_{j1},V_{j2}}(v)\\ &\leq |G \setminus G[S_1, S_2]| + |V_{j1}||V_{j2}| \leq 11\varepsilon^{1/8}n^2 + |V_{j_1}||V_{j_2}|,\end{aligned}$$

so

$$|V_{j1}|, |V_{j2}| \geq \varepsilon^{1/9}n. \tag{22}$$

By Lemma 5.2, we have

$$|\mathcal{H}[V_{ja}]| \leq \left(\frac{1}{24} + \frac{1}{2}\sqrt{\frac{21}{L-26}}\right)|V_{ja}|^3 \leq \left(\frac{1}{24} + \varepsilon\right)|V_{ja}|^3, \quad \text{for } a = 1, 2. \tag{23}$$

Then, by (23), (21), (10), and (22), we have

$$\begin{aligned}|\mathcal{H}[V_j]| &\leq |\mathcal{H}[V_{j1}]| + |\mathcal{H}[V_{j2}]| + |\mathcal{H}[S_1, S_1, S_2]| + |\mathcal{H}[S_2, S_2, S_1]|\\ &\leq \left(\frac{1}{24} + \varepsilon\right)\left(|V_{j1}|^3 + |V_{j2}|^3\right) + 21\varepsilon^{1/8}n^3\\ &\leq \left(\frac{1}{24} + \varepsilon\right)\left(|V_{j1}|^3 + \left(|V_j| - |V_{j1}|\right)^3\right) + 21\varepsilon^{1/8}n^3\\ &\leq \left(\frac{1}{24} + \varepsilon\right)\left(\left(\varepsilon^{1/9}n\right)^3 + \left(\left(\frac{1}{3} + \varepsilon\right)n - \varepsilon^{1/9}n\right)^3\right) + 21\varepsilon^{1/8}n^3\\ &\leq \left(\frac{1}{24} - \frac{1}{10}\varepsilon^{1/9}\right)\left(\frac{n}{3}\right)^3,\end{aligned}$$

a contradiction to Claim 5.10, where the second-to-last inequality is due to the fact that the function $f_c(x) = x^3 + (c-x)^3$, where c is a positive constant, is decreasing on $[0, c/2]$ and increasing on $[c/2, c]$. Thus, we have $d_{V_j,V_k}(v) \geq |V_j||V_k| - 5\varepsilon^{1/9}n^2$. □

For $\{i, j, k\} = [3]$, let $B'_i := \{v \in B : d_{V_j,V_k}(v) \geq |V_j||V_k| - 5\varepsilon^{1/9}n^2\}$ and $V'_i := A_i \cup B'_i$. Let $\pi' := (V'_1, V'_2, V'_3)$ be a new partition of $V(\mathcal{H})$. By (10) and Claim 5.7, we have

$$\left(\frac{1}{3} - 2\varepsilon^{1/2}\right)n < |V'_1|, |V'_2|, |V'_3| < \left(\frac{1}{3} + 3\varepsilon^{1/2}\right)n. \tag{24}$$

Claim 5.12 *For $\{i, j, k\} = [3]$ and vertices $x \in V_i'$, $y \in V_j'$, we have $d_{V_i'}(x, y) \leq \varepsilon^{1/10}n$.*

Proof By Claim 5.8 and (10), we have

$$d_{A_i}(x, y) \leq d_{V_i}(x, y) \leq \frac{3n}{|V_j||V_k| - 5\varepsilon^{1/9}n^2} \cdot 5\varepsilon^{1/9}n^2 + 2\varepsilon^{1/2}n \leq \frac{1}{2}\varepsilon^{1/10}n.$$

By Claim 5.7, we have $d_{B_i'}(x, y) \leq |B_i'| \leq |B| \leq 3\varepsilon^{1/2}n$. Therefore, $d_{V_i'}(x, y) \leq \varepsilon^{1/10}n$. □

Claim 5.13 *We have $\mathcal{H}_{bad}^{\pi'} = \varnothing$.*

Proof For $\{i, j, k\} = \{3\}$, let

$$Q_{ij} := \{ (\{x_1, x_2, y\}, z) : \{x_1, x_2, y\} \in \mathcal{H}[V_i', V_i', V_j'],\ x_1, x_2 \in V_i',\ y \in V_j',\ z \in V_k' \},$$

and $Q := \bigcup_{1 \leq i \neq j \leq 3} Q_{ij}$. Note that for every $(\{x_1, x_2, y\}, z) \in Q_{ij}$, we have $\{x_1, y, z\} \in \bar{\mathcal{H}}_{\pi'}$ or $\{x_2, y, z\} \in \bar{\mathcal{H}}_{\pi'}$, since otherwise x_1zyx_2 forms a copy of $\mathcal{K}_4^-$. For every $\{x, y, z\} \in \bar{\mathcal{H}}_{\pi'}$, by Claim 5.12, there can be at most $\varepsilon^{1/10}n$ such $x' \in V_i'$ that $\{x, x', y\} \in \mathcal{H}[V_i', V_i', V_j']$, so we have

$$|Q| = \sum_{1 \leq i \neq j \leq 3} |Q_{ij}| \leq 6 \cdot \varepsilon^{1/10}n \cdot |\bar{\mathcal{H}}_{\pi'}|. \tag{25}$$

On the other hand, fix $i \neq j \in [3]$ which maximize $|\mathcal{H}[V_i', V_i', V_j']|$. Let $k = [3] \setminus \{i, j\}$. By (24),

$$|Q| \geq |Q_{ij}| = |\mathcal{H}[V_i', V_i', V_j']| \cdot |V_k'| \geq |\mathcal{H}[V_i', V_i', V_j']| \left(\frac{1}{3} - 2\varepsilon^{1/2}\right) n. \tag{26}$$

Combining (25) and (26), we have $|\mathcal{H}[V_i', V_i', V_j']| \leq 0.01|\bar{\mathcal{H}}_{\pi'}|$, so

$$|\mathcal{H}_{bad}^{\pi'}| \leq 6|\mathcal{H}[V_i', V_i', V_j']| \leq \frac{1}{2}\left|\bar{\mathcal{H}}_{\pi'}\right|.$$

If $|\mathcal{H}_{bad}^{\pi'}| > 0$, then $(\mathcal{H} \setminus \mathcal{H}_{bad}^{\pi'}) \cup \bar{\mathcal{H}}_{\pi'}$ is $\boldsymbol{\mathcal{C}}_{\leq L}^-$-free and has strictly more hyperedges than $\mathcal{H}$, which is a contradiction to the maximality of $\mathcal{H}$. Therefore, $\mathcal{H}_{bad}^{\pi'} = \varnothing$. □

Note that by (24), we have $|V_i'| < n$ for $i \in [3]$. By induction, we now have

$$\begin{aligned}\mathrm{ex}(n, \boldsymbol{\mathcal{C}}_{\leq L}^-) = |\mathcal{H}| = |\mathcal{H}_{\pi'}| + \sum_{i=1}^{3} |\mathcal{H}[V_i']| \leq |V_1'||V_2'||V_3'| + \sum_{i=1}^{3} \left(\frac{1}{24}|V_i'|^3 + M^2 \cdot |V_i'|\right)\\ = |V_1'||V_2'||V_3'| + \frac{1}{24}\sum_{i=1}^{3} |V_i'|^3 + M^2 n \leq \frac{1}{24}n^3 + M^2 n,\end{aligned}$$

where for the last inequality, we use (24) and the fact that the function $g(x_1, x_2, x_3) = x_1x_2x_3 + \sum_{i=1}^{3} x_i^3/24$ defined on the domain $\{(x_1, x_2, x_3) \in [0.32, 0.34]^3 : x_1 + x_2 + x_3 = 1\}$ has maximum value $1/24$. This proves (8).

Combining the lower bound and the upper bound, we have $\pi(\boldsymbol{\mathcal{C}}^-_{\le L}) = 1/4$. □

Finally, we are able to present the proof of Theorem 1.1.

Proof of Theorem 1.1 For every $\ell \ge 4$ not divisible by 3, by Construction 1.2, we have $\pi(\mathcal{C}^-_\ell) \ge 1/4$. For the upper bound, we have the following claims. Recall that for a 3-graph $\mathcal{H}$, we write $\mathcal{H}[t]$ for its t-blow-up.

Claim 5.14 *For integers $\ell_1, \ell_2 \ge 4$, where $\ell_1 \ge 2\ell_2 - 3$ and $3 \nmid \ell_2$, there exists a positive integer t such that $\mathcal{C}^-_{\ell_1} \subseteq \mathcal{C}^-_{\ell_2}[t]$.*

Proof Let $v_1v_2 \dots v_{\ell_2}$ be a copy of $\mathcal{C}^-_{\ell_2}$. Let t be sufficiently large, and let $\{v_i^1, \dots, v_i^t\}$ be the copies of vertex v_i in $\mathcal{C}^-_{\ell_2}[t]$. For $1 \le k \le t-2$, we write $(v_{i_1}v_{i_2} \dots v_{i_{j-1}}v_{i_j})^k$ for the sequence

$$v_{i_1}^3 v_{i_2}^3 \dots v_{i_{j-1}}^3 v_{i_j}^3 v_{i_1}^4 v_{i_2}^4 \dots v_{i_{j-1}}^4 v_{i_j}^4 \dots v_{i_1}^{k+2} v_{i_2}^{k+2} \dots v_{i_{j-1}}^{k+2} v_{i_j}^{k+2}.$$

- If $\ell_1 \equiv 0 \pmod 3$, then $(v_1v_2v_3)^{\frac{\ell_1}{3}}$ forms a copy of $\mathcal{C}_{\ell_1} \supseteq \mathcal{C}^-_{\ell_1}$ in $\mathcal{C}^-_{\ell_2}[t]$.
- If $\ell_1 \equiv \ell_2 \pmod 3$, then $(v_1v_2v_3)^{\frac{\ell_1-\ell_2}{3}} v_1^1v_2^1v_3^1v_4^1 \dots v_{\ell_2-1}^1 v_{\ell_2}^1$ forms a copy of $\mathcal{C}^-_{\ell_1}$ in $\mathcal{C}^-_{\ell_2}[t]$.
- If $\ell_1 \equiv 2\ell_2 \pmod 3$, then

$$(v_1v_3v_2)^{\frac{\ell_1-2\ell_2+3}{3}} v_1^1v_3^1v_2^1v_4^1v_3^2v_5^1 \dots v_{\ell_2-2}^1 v_{\ell_2-3}^2 v_{\ell_2-1}^1 v_{\ell_2-2}^2 v_{\ell_2}^1 v_{\ell_2-1}^2,$$

  forms a copy of $\mathcal{C}^-_{\ell_1}$ in $\mathcal{C}^-_{\ell_2}[t]$.

□

Claim 5.15 *For integers $\ell, L \ge 4$, where $\ell \ge 2L - 3$, there exists a positive integer t such that $\mathcal{C}^-_\ell \subseteq \mathcal{F}[t]$ for every $\mathcal{F} \in \boldsymbol{\mathcal{C}}^-_{\le L}$.*

Proof Let t be sufficiently large. Note that for every pseudo-cycle minus one hyperedge of size ℓ', its ℓ'-blow-up contains a copy of $\mathcal{C}^-_{\ell'}$. Hence, for every pseudo-cycle minus one hyperedge $\mathcal{F} \in \boldsymbol{\mathcal{C}}^-_{\le L}$ with size $\ell' \le L$, by Claim 5.14, we have $\mathcal{C}^-_\ell \subseteq \mathcal{C}^-_{\ell'}[t/L] \subseteq \mathcal{F}[t]$. □

Let ℓ and L be sufficiently large integers, where $\ell \ge 2L - 3$. Let t be sufficiently large. By Theorem 2.3 and Proposition 5.6, we have $\pi(\boldsymbol{\mathcal{C}}^-_{\le L}[t]) = \pi(\boldsymbol{\mathcal{C}}^-_{\le L}) = 1/4$. By Claim 5.15, we have $\pi(\mathcal{C}^-_\ell) \le \pi(\boldsymbol{\mathcal{C}}^-_{\le L}[t]) = 1/4$.

Thus, we have $\pi(\mathcal{C}^-_\ell) = 1/4$, for every sufficiently large ℓ not divisible by 3. □

6 Maximum Number of Almost Similar Triangles in the Plane

In this section, we provide our new proof for Theorem 1.4.

For a triangle $\triangle$, a real number $\varepsilon > 0$, and a finite set of points $P \in \mathbb{R}^2$, let $\mathcal{H}(P, \triangle, \varepsilon)$ be the 3-graph on vertex set P, where $\{a, b, c\}$ is a hyperedge if a, b, c form a triangle in $\mathbb{R}^2$ that is ε-similar to $\triangle$, and let $\mathcal{H}(P, \triangle)$ be the 3-graph on vertex set P, where $\{a, b, c\}$ is a hyperedge if a, b, c form a triangle in $\mathbb{R}^2$ that is similar to $\triangle$.

Definition 6.1 Fix C_{tri} to be an absolute constant. The *forbidden family* $\boldsymbol{\mathcal{F}}_{tri}$ is the collection of 3-graphs $\mathcal{F}$ with at most C_{tri} vertices, where for almost all triangles $\triangle$, there exists $\varepsilon = \varepsilon(\triangle) > 0$ such that $\mathcal{H}(P, \triangle, \varepsilon)$ is $\mathcal{F}$-free, for every point set $P \subseteq \mathbb{R}^2$.

The C_{tri} in Definition 6.1 can be chosen arbitrarily and is to make $\boldsymbol{\mathcal{F}}_{tri}$ a finite set to avoid some technical problems about infinity. We let $C_{tri} := L + 3$, where L is the constant in Theorem 1.1. By definition, we have $h(n, \triangle, \varepsilon) = \max_{P\subseteq\mathbb{R}^2,|P|=n} |\mathcal{H}(P, \triangle, \varepsilon)|$. Hence, $\text{ex}(n, \boldsymbol{\mathcal{F}}_{tri}) \geq h(n, \triangle, \varepsilon)$ for almost all triangles $\triangle$ and small enough $\varepsilon = \varepsilon(\triangle)$. Bárány and Füredi [3] proved that several 3-graphs, including $\mathcal{K}_4^-$ and $\mathcal{C}_5^-$, are in $\boldsymbol{\mathcal{F}}_{tri}$ and then gave an upper bound for $\text{ex}(n, \boldsymbol{\mathcal{F}}_{tri})$ using flag algebra. Balogh et al. [2] provided more members of $\boldsymbol{\mathcal{F}}_{tri}$ and then used a combination of flag algebra and stability method to obtain Theorem 1.4. We will prove that every tight cycle minus one hyperedge of size $4 \leq \ell \leq C_{tri}$ not divisible by 3 is in $\boldsymbol{\mathcal{F}}_{tri}$, and then Theorem 1.4 follows immediately from Theorem 1.1.

Proposition 6.2 *For every integer ℓ, where $4 \leq \ell \leq C_{tri}$ and $3 \nmid \ell$, we have $\mathcal{C}_\ell^- \in \boldsymbol{\mathcal{F}}_{tri}$.*

Proof Using the fact that every algebraic set, which is not the whole space, has measure 0, Bárány and Füredi [3] showed that in order to prove that a 3-graph $\mathcal{H}$ is in $\boldsymbol{\mathcal{F}}_{tri}$, we only need to prove that there exists *one* triangle $\triangle$ such that $\mathcal{H}(P, \triangle)$ is $\mathcal{H}$-free, for every point set $P \subseteq \mathbb{R}^2$, see their proof of Lemma 9.2. See also the proof of Lemma 2.3 in [2]. Therefore, denoting by $\triangle_0$ the equilateral triangle, we only need to prove that $\mathcal{H}(P, \triangle_0)$ is $\mathcal{C}_\ell^-$-free, for every integer ℓ, where $4 \leq \ell \leq C_{tri}$ and ℓ is not divisible by 3, and point set $P \in \mathbb{R}^2$. This follows from the following simple coloring argument.

Assume for contradiction that there are a point set $P \subseteq \mathbb{R}^2$ and an integer $\ell \geq 4$ not divisible by 3 such that $\mathcal{H}(P, \triangle_0)$ contains $v_0 v_1 \dots v_{\ell-1}$ as a copy of $\mathcal{C}_\ell^-$. Without loss of generality, we can assume that $v_0 = (0, 0)$, $v_1 = (1, 0)$, and $v_2 = (1/2, \sqrt{3}/2)$. Let P_0 be the point set $\{x v_1 + y v_2 : x, y \in \mathbb{Z}\}$. Color P_0 with colors in $\{0, 1, 2\}$ as follows. For every point $x v_1 + y v_2 \in P_0$, color it with color $c \in \{0, 1, 2\}$, where $c \equiv x + 2y \pmod 3$. Note that every equilateral triangle with side length one formed by points in P_0 is a rainbow, i.e., its vertices have all three colors. Now, $v_0 = (0, 0)$ has color 0 and $v_1 = (1, 0)$ has color 1. Since $\{v_i, v_{i+1}, v_{i+2}\} \in \mathcal{H}(P, \triangle_0)$ for $0 \leq i \leq \ell - 3$, we have, by induction, $v_i \in P_0$, v_i, v_{i+1}, v_{i+2} form an equilateral triangle with side length one, and v_i has color c, where $c \equiv i \pmod 3$. Then, $v_{\ell-2}, v_{\ell-1}, v_0$ also need to form an equilateral triangle with side length one, so it is a rainbow. However, since $3 \nmid \ell$, we have that one of $v_{\ell-2}, v_{\ell-1}$ has color 0, the same color as vertex v_0, a contradiction. □

7 Concluding Remarks

The constant L in Theorem 1.1 given by the current proof can be large, due to the following two reasons.

For the stability result in Sect. 4, we use a regularity lemma, Theorem 2.4, which can make the dependence between $\varepsilon_1, \varepsilon_2$ and δ very poor in Proposition 4.1. We remark that using the regularity lemma is not really necessary: we can instead use a similar averaging argument as in the proof of Claim 4.2. This would make the proof of Proposition 4.1 much longer and more technical, and we still cannot make L close to 5 (due to the reason in the next paragraph).

As mentioned at the beginning of Sect. 5, the bottleneck in our proof is about the following problem.

Problem 7.1 For a maximum n-vertex $\mathcal{C}^-_{\leq L}$-free 3-graph $\mathcal{H}$, how many hyperedges do we need to remove to make $\mathcal{H}$ free of $\mathcal{C}^-$?

In Lemma 5.1, our bound is $O(n^3/\sqrt{L})$. Any improvement to this can lead to a significant improvement for the constant L in Theorem 1.1. We note that Lemma 5.1 is the only place where we need L to be large; for all other proofs, we actually only use that the forbidden family includes $\mathcal{K}_4^-$ and $\mathcal{C}_5^-$. If it can be shown that at most cn^3 hyperedges are needed to be removed from every maximum $\mathcal{C}_5^-$-free 3-graph $\mathcal{H}$ to make $\mathcal{H}$ free of $\mathcal{C}^-$, where c is small enough, then the same proof gives $\pi(\mathcal{C}_5^-) = 1/4$.

Acknowledgements The authors are grateful to Bernard Lidický and Zoltán Füredi for helpful communication. They would also like to thank the anonymous referees for their helpful comments.

Funding József Balogh: Research is partially supported by NSF Grants DMS-1764123, RTG DMS-1937241 and FRG DMS-2152488, the Arnold O. Beckman Research Award (UIUC Campus Research Board RB 22000), and the Langan Scholar Fund (UIUC). Haoran Luo: Research is partially supported by FRG DMS-2152488, the Arnold O. Beckman Research Award (UIUC Campus Research Board RB 22000).

Data availability Not applicable.

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