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A NOTE ON COLOR-BIAS PERFECT MATCHINGS IN HYPERGRAPHS*

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Abstract. A result of Balogh et al. yields the minimum degree threshold that ensures a 2-colored graph contains a perfect matching of significant color-bias (i.e., a perfect matching that contains significantly more than half of its edges in one color). In this note we prove an analogous result for perfect matchings in k-uniform hypergraphs. More precisely, for each $2 \leq \ell < k$ and $r \geq 2$ we determine the minimum ℓ-degree threshold for forcing a perfect matching of significant color-bias in an r-colored k-uniform hypergraph.

Key words. color-bias, discrepancy, perfect matchings

MSC codes. 05C35, 05C65, 05C70

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1. Introduction.

A *perfect matching* in a hypergraph H is a collection of vertex-disjoint edges of H which covers the vertex set $V(H)$ of H. In recent decades there has been significant interest in the problem of establishing *minimum degree* conditions that force a perfect matching in a k-uniform hypergraph. More precisely, given a k-uniform hypergraph H and an ℓ-element vertex set $S \subseteq V(H)$ (where $\ell \in [k-1]$), we define $d_H(S)$ to be the number of edges containing S. The *minimum ℓ-degree* $\delta_\ell(H)$ of H is the minimum of $d_H(S)$ over all ℓ-element sets of vertices in H. We refer to $\delta_1(H)$ as the *minimum vertex degree* of H and to $\delta_{k-1}(H)$ as the *minimum codegree* of H.

Suppose that $\ell, k, n \in \mathbb{N}$ such that $\ell \leq k-1$ and k divides n. Let $m_\ell(k,n)$ denote the smallest integer m such that every k-uniform hypergraph H on n vertices with $\delta_\ell(H) \geq m$ contains a perfect matching.

A simple consequence of Dirac's theorem is that $m_1(2,n) = n/2$ for all even $n \in \mathbb{N}$. Improving earlier asymptotically exact bounds given in [13, 19], Rödl, Ruciński, and Szemerédi [20] determined the minimum codegree threshold for perfect matchings in k-uniform hypergraphs. That is, they showed that if $n \in \mathbb{N}$ is sufficiently large, then $m_{k-1}(k,n) = n/2 - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k.

The value of $m_\ell(k,n)$ is known for various pairs (k,ℓ) when n is sufficiently large. For example, after an earlier asymptotic result of Pikhurko [17], Treglown and Zhao [21] determined the value of $m_\ell(k,n)$ for $\ell \geq k/2$ and n sufficiently large. However, the minimum vertex degree case of the problem is wide open in general, and the only

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case where the asymptotic or exact value of $m_1(k,n)$ is known is when $k = 2,3,4,5$. See, e.g., [18, 23] for discussions on further results in the area.

Given any $1 \leq \ell < k$, it is known that

$$m_\ell(k,n) \geq \max\left\{\frac{1}{2} - o(1), 1 - \left(\frac{k-1}{k}\right)^{k-\ell} - o(1)\right\}\binom{n}{k-\ell}. \tag{1}$$

See, e.g., the introduction of [22] for the two families of hypergraphs that demonstrate (1). It is widely believed that the inequality in (1) is asymptotically sharp for all choices of k, ℓ; see [12, 14]. Moreover, Treglown and Zhao [22] gave a conjecture on the exact value of $m_\ell(k,n)$ for sufficiently large $n \in k\mathbb{N}$.

The aim of this paper is to study the *color-bias* version of this problem. The topic of color-bias structures in graphs was first raised by Erdős in the 1960s (see [5, 6]). Sparked by work of Balogh et al. [1], there has been renewed interest in the topic, particularly in establishing minimum degree conditions that force a color-bias copy of a graph F. More precisely, if a graph G contains a copy of F, then, however, the edges of G are 2-colored, one can clearly ensure that G contains a copy of F with at least $e(F)/2$ edges of the same color. The question then is how large the minimum degree $\delta(G)$ of G needs to be to guarantee that G contains a copy of F with significantly more than $e(F)/2$ edges of the same color, no matter how one 2-colors the edges of G. The following result resolves this problem in the case when F is a Hamilton cycle.

THEOREM 1.1 (Balogh et al. [1]). *Let $0 < c < 1/4$ and $n \in \mathbb{N}$ be sufficiently large. If G is an n-vertex graph with*

$$\delta(G) \geq (3/4 + c)n,$$

then given any 2-coloring of $E(G)$, there is a Hamilton cycle in G with at least $n/2 + cn/32$ edges of the same color. Moreover, if $n \in 4\mathbb{N}$, there is an n-vertex graph G' with $\delta(G') = 3n/4$ and a 2-coloring of $E(G')$ for which every Hamilton cycle in G' has precisely $n/2$ edges in each color.

Note that Theorem 1.1 shows that the minimum degree threshold for forcing a *color-bias* Hamilton cycle in a graph is significantly higher than the threshold for forcing only a Hamilton cycle. Indeed, Dirac's theorem tells us that any n-vertex graph G with $\delta(G) \geq n/2$ contains a Hamilton cycle.

Since a Hamilton cycle on an even number of vertices is the union of two perfect matchings, Theorem 1.1 implies the following result.

THEOREM 1.2 (Balogh et al. [1]). *Let $0 < c < 1/4$ and $n \in 2\mathbb{N}$ be sufficiently large. If G is an n-vertex graph with*

$$\delta(G) \geq (3/4 + c)n,$$

then given any 2-coloring of $E(G)$, there is a perfect matching in G with at least $n/4 + cn/64$ edges of the same color. Moreover, if $n \in 4\mathbb{N}$, there is an n-vertex graph G' with $\delta(G') = 3n/4$ and a 2-coloring of $E(G')$ for which every perfect matching in G' has precisely $n/4$ edges in each color.

Let $n \in 4\mathbb{N}$. We define the graph G' in Theorem 1.2 as follows: $V(G')$ consists of the disjoint union of two vertex classes A and B of sizes $n/4$ and $3n/4$, respectively; $E(G')$ contains all possible red edges whose endpoints are both in B and all possible blue edges with one endpoint in A and one endpoint in B. Thus, $\delta(G') = 3n/4$, and every perfect matching in G' has precisely $n/4$ edges in each color.

Since [1] appeared, a number of analogues of Theorem 1.1 have been established for other types of spanning structures. Given graphs G and F, an *F-factor* in G is a collection of vertex-disjoint copies of F in G that together cover $V(G)$. In [2], the minimum degree threshold for forcing a color-bias K_r-factor was determined.[1] More recently, this result was extended to F-factors for every fixed graph F; see [4]. For $k \geq 2$, the minimum degree threshold for forcing a color-bias kth power of a Hamilton cycle in a graph was established in [3].

Other variants of the problem have also been studied. In [7, 10] an r-color version of Theorem 1.1 was proven: in this setting, now one r-colors $E(G)$ and seeks a Hamilton cycle with significantly more than n/r edges of the same color. Color-bias problems have also been considered for random graphs [9]. Recently, Mansilla Brito [16] gave a minimum codegree result for forcing a color-bias copy of a tight Hamilton cycle in a 3-uniform hypergraph. We remark that all of these color-bias results can be phrased in the equivalent language of *discrepancy*; see, e.g., [1, 2, 3, 4, 10].

Our main result determines the minimum ℓ-degree threshold for forcing a color-bias perfect matching in a k-uniform hypergraph for all $\ell \geq 2$ and $k \geq 3$. To state our result, we need the following definitions: Given integers $1 \leq \ell < k$, let $\mathcal{C}_{k,\ell}$ be the set of all $c > 0$ such that $m_\ell(k,n) \leq c\binom{n}{k-\ell}$ for all sufficiently large $n \in k\mathbb{N}$. Set $c_{k,\ell}$ to be the infimum of $\mathcal{C}_{k,\ell}$. In particular, note that the general conjecture on the asymptotic value of $m_\ell(k,n)$ equivalently states that

$$c_{k,\ell} = \max\left\{\frac{1}{2}, 1 - \left(\frac{k-1}{k}\right)^{k-\ell}\right\}.$$

THEOREM 1.3. *Let $k, \ell, r \in \mathbb{N}$ where $2 \leq \ell < k$ and $r \geq 2$. Given any $\eta > 0$ where $c_{k,\ell} + \eta < 1$, there exists an $n_0 \in \mathbb{N}$ such that the following holds: Let H be a k-uniform hypergraph on $n \geq n_0$ vertices, where $n \in k\mathbb{N}$. If*

$$\delta_\ell(H) \geq (c_{k,\ell} + \eta)\binom{n}{k-\ell},$$

then given any r-coloring of $E(H)$, there is a perfect matching in H with at least $\frac{n}{rk} + \frac{\eta n}{8r(r-1)k^k(k^2+k)}$ edges of the same color.

We remark that Theorem 1.3 holds even in the cases in which we do not know the value of $c_{k,\ell}$. By definition of $c_{k,\ell}$, the minimum ℓ-degree condition in Theorem 1.3 is essentially best possible. Indeed, for $c < c_{k,\ell}$, a minimum ℓ-degree condition of $\delta_\ell(H) \geq c\binom{n}{k-\ell}$ does not even guarantee a perfect matching, let alone one of significant color-bias. So in this sense the color-bias and "standard" versions of the problem are aligned when $\ell \geq 2$.

In contrast, the same phenomenon does not occur for the minimum vertex degree version of the problem. Indeed, Theorem 1.2 tells us that the minimum degree threshold for a color-bias perfect matching in a *graph* is different from the minimum degree threshold for a perfect matching in a graph. Furthermore, in section 4 we describe a similar phenomenon in the 3-uniform hypergraph setting.

Remark. While finalizing a manuscript that gave the proof of Theorem 1.3 in the case when $\ell = k - 1$ and $r = 2$, we learnt of simultaneous and independent work

[1] Recall K_r denotes the complete graph on r vertices.

of Gishboliner, Glock, and Sgueglia [8]. They determined the *minimum codegree threshold* for forcing a tight Hamilton cycle of significant color-bias in an r-colored k-uniform hypergraph (where $r \geq 2$ and $k \geq 3$). As an immediate consequence of their result, they also established the corresponding *minimum codegree threshold* for perfect matchings. ∎

We therefore decided to seek a generalization of our minimum codegree result to other degree conditions, i.e., Theorem 1.3. In doing so, we found an argument much cleaner than our original approach.

Notation. Let H be a hypergraph. The *neighborhood* $N_H(X)$ *of a set* $X \subseteq V(H)$ is the family of sets $S \subseteq V(H) \setminus X$ such that $S \cup X \in E(H)$. If $X = \{x\}$, we define $N_H(x) := N_H(X)$. Given a vertex $x \in V(H)$ and set $Y \subseteq V(H)$, we sometimes write xY or Yx to denote $\{x\} \cup Y$. Given a coloring c of $E(H)$, we call an edge $e \in E(H)$ a C*-edge* if e is colored C in c. Given a set $X \subseteq V(H)$, we write $H[X]$ for the *induced subhypergraph of* H *with vertex set* X. We define $H \setminus X := H[V(H) \setminus X]$.

Given a hypergraph F with an r-coloring $c: E(F) \to \{C_1, \dots, C_r\}$, its *color profile* is $(x_1, \dots, x_r)$, where x_i is the number of C_i-edges in F for each $i \in [r]$. Two color profiles $(x_1, \dots, x_r)$, $(y_1, \dots, y_r)$ are said to be *different with respect to the color* C_i if $x_i \neq y_i$.

2. Preliminaries and useful results.

2.1. Proof overview and key definitions. Throughout this section, we will suppose that H is a k-uniform hypergraph on n vertices with an r-coloring $c: E(H) \to \{C_1, \dots, C_r\}$.

Our general strategy for the proof of Theorem 1.3 is as follows. Our aim is to find certain *gadgets* inside of H. A gadget is just a subhypergraph of H with some given structure. A gadget G is *good* if G contains two perfect matchings that have different color profiles with respect to the r-coloring c.

For a certain well-chosen $t \in \mathbb{N}$, we will prove that there are t vertex-disjoint good gadgets $G_1, \dots, G_t$ in H and a $j \in [r]$ so that, for each good gadget G_i, the two perfect matchings M_i and M_i' in G_i have color profiles that are different with respect to the color C_j.

We will then be able to easily find a perfect matching in H of significant color-bias. Indeed, removing the vertices of $G_1, \dots, G_t$ from H will result in a k-uniform hypergraph H' that contains a perfect matching M. The flexibility of the good gadgets then allows us to extend M into a perfect matching in H with significant color-bias, whatever the colour profile of M may be.

We next state the definitions required to formally introduce the notion of a good gadget.

Definition 2.1. *Let* $u, v \in V(H)$ *be distinct and* $T \in N_H(u) \cap N_H(v)$*. We say*
- $\boldsymbol{uTv}$ *is* $\boldsymbol{S}$ *if* $c(T \cup \{u\}) = c(T \cup \{v\})$,
- $\boldsymbol{uTv}$ *is* $\boldsymbol{C_iC_j}$ *if* $c(T \cup \{u\}) = C_i$ *and* $c(T \cup \{v\}) = C_j$.

Let $C_iC_j(uv)$ *denote the collection of sets* $T \in N_H(u) \cap N_H(v)$ *for which* uTv *is* C_iC_j*. Define* $S(uv)$ *analogously.*

Note that $C_iC_j(uv) = C_jC_i(vu)$ for all distinct $u, v \in V(H)$.

Definition 2.2. *Let* $D > 0$*, and let* $u, v \in V(H)$ *be distinct. We say that* $N_H(u) \cap N_H(v)$ *is*
- ***type*** $\boldsymbol{S(D)}$ *if* $|S(uv)| \geq Dn^{k-2}$,
- ***type*** $\boldsymbol{C_iC_j(D)}$ *if* $i \neq j$ *and* $|C_iC_j(uv)| \geq Dn^{k-2}$.

We remark that it may be the case that $N_H(u) \cap N_H(v)$ has more than one type.

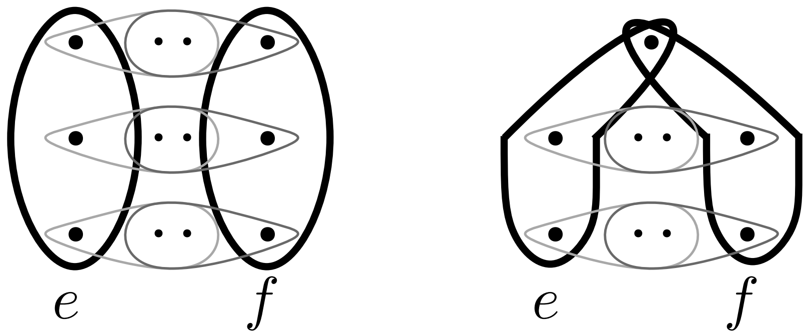

FIG. 1. *On the left, a* $(12, e, f)$*-gadget. On the right, a* $(9, e, f)$*-gadget.*

DEFINITION 2.3. *Let* $e = \{e_1, \ldots, e_k\}$ *and* $f = \{f_1, \ldots, f_k\}$ *be two edges in* H. *A* $\boldsymbol{(k^2+k, e, f)}$**-gadget** G *is a subhypergraph of* H *on* $k^2 + k$ *vertices so that*

- $V(G)$ *is the disjoint union of* e, f, *and* $T_1, \ldots, T_k$, *where* $T_i \in N_H(e_i) \cap N_H(f_i)$ *for each* $i \in [k]$;
- $e, f \in E(G)$;
- $e_iT_i, f_iT_i \in E(G)$ *for all* $i \in [k]$.

A $(k^2 + k, e, f)$*-gadget in which every* $e_iT_if_i$ *is* S *will be called an* $\boldsymbol{S}$**-**$\boldsymbol{(k^2+k, e, f)}$**-*gadget***.

A $\boldsymbol{(3k, e, f)}$**-gadget** G *is a subhypergraph of* H *on* $3k$ *vertices so that*

- $e_i = f_i$, *for all* $i \in \{3, \ldots, k\}$;
- $V(G)$ *is the disjoint union of* e, f_1, f_2, T_1, *and* T_2, *where* $T_i \in N_H(e_i) \cap N_H(f_i)$ *for each* $i \in [2]$;
- $e, f \in E(G)$;
- $e_1T_1, f_1T_1, e_2T_2, f_2T_2 \in E(G)$.

Given $t \in \{3k, k^2+k\}$, *we say that a* (t, e, f)*-gadget* G *is* good *if it contains two perfect matchings with different color profiles (with respect to the* r*-coloring of* G *induced by the* r*-coloring* c *of* H*).*

Note that e and f are vertex-disjoint in a $(k^2 + k, e, f)$-gadget but intersect in $k - 2$ vertices in a $(3k, e, f)$-gadget; see Figure 1.

2.2. Tools for the proof of Theorem 1.3. The following well-known result allows one to deduce a lower bound on $\delta_\ell(H)$ given a lower bound on $\delta_{\ell'}(H)$ for any $\ell \le \ell'$.

PROPOSITION 2.4. *Let* $1 \le \ell \le \ell' < k$, *and let* H *be a* k*-uniform hypergraph on* n *vertices. If* $\delta_{\ell'}(H) \ge x\binom{n-\ell'}{k-\ell'}$ *for some* $0 \le x \le 1$, *then* $\delta_\ell(H) \ge x\binom{n-\ell}{k-\ell}$.

The next result gives a sufficient condition for finding a good $(3k, e, f)$-gadget in a k-uniform hypergraph of large minimum 2-degree.

LEMMA 2.5. *Let* $k \ge 3$ *and* $D := 3k$. *Let* H *be a* k*-uniform hypergraph on* n *vertices with an* r*-coloring* $c : E(H) \to \{C_1, \ldots, C_r\}$. *Suppose there exists* $i \ne j \in [r]$ *and distinct* $v_1, v_2, v_3, v_4 \in V(H)$ *such that* $N_H(v_1) \cap N_H(v_2)$ *and* $N_H(v_3) \cap N_H(v_4)$ *are both type* $C_iC_j(D)$. *If*

$$\delta_2(H) > \frac{1}{2}\binom{n}{k-2},$$

then there exists a good $(3k, e, f)$*-gadget in* H *for some* $e, f \in E(H)$.

Proof. By the minimum 2-degree condition, there exists a set $X \subseteq V(H)$ of size $k-2$ such that $A = X \cup \{v_1, v_3\}$ and $B = X \cup \{v_2, v_4\}$ are both in $E(H)$. We show that we can construct a $(3k, A, B)$-gadget, and then we prove that it is good.

Given that $N_H(v_1) \cap N_H(v_2)$ is type $C_iC_j(D)$, there are at least $3kn^{k-2}$ sets $T_{1,2} \in N_H(v_1) \cap N_H(v_2)$ such that $c(v_1T_{1,2}) = C_i$ and $c(v_2T_{1,2}) = C_j$. As $|A \cup B| = k+2 < 3k$, we may choose such a set $T_{1,2}$ so that it is also vertex-disjoint from $A \cup B$. Similarly, there is a set $T_{3,4} \in N_H(v_3) \cap N_H(v_4)$ such that $c(v_3T_{3,4}) = C_i$, $c(v_4T_{3,4}) = C_j$ and $T_{3,4}$ is vertex-disjoint from A, B, and $T_{1,2}$.

Then, define a gadget G as follows:

- $V(G)$ is the union of A, B, $T_{1,2}$, and $T_{3,4}$;
- A, B, $v_1T_{1,2}$, $v_2T_{1,2}$, $v_3T_{3,4}$, and $v_4T_{3,4}$ are in $E(G)$.

By definition, G is a $(3k, A, B)$-gadget.

To prove that G is good, we need to find two perfect matchings in G with different color profiles. Define $M_A := \{A, v_2T_{1,2}, v_4T_{3,4}\}$ and $M_B := \{B, v_1T_{1,2}, v_3T_{3,4}\}$. Both M_A and M_B are perfect matchings in G. While M_A has at least two C_j-edges ($v_2T_{1,2}$ and $v_4T_{3,4}$), M_B has at least two C_i-edges ($v_1T_{1,2}$ and $v_3T_{3,4}$). Thus, M_A and M_B have different color profiles, as desired. ∎

The next lemma ensures that a hypergraph H as in Theorem 1.3 contains a good gadget or a perfect matching of huge color-bias.

LEMMA 2.6. *Let $2 \le \ell < k$ and $\eta > 0$. There exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \ge n_0$ with $n \in k\mathbb{N}$: Let H be a k-uniform hypergraph on n vertices with an r-coloring $c\colon E(H) \to \{C_1, \dots, C_r\}$ and*

$$\delta_\ell(H) \ge (c_{k,\ell} + \eta)\binom{n}{k-\ell}.$$

Suppose that H does not have a perfect matching containing at least $n/k - \binom{r}{2}$ edges of the same color. Then

- *there exists a good $(3k, e, f)$-gadget in H for some $e, f \in E(H)$; or*
- *there exists a good (k^2+k, e, f)-gadget in H for some $e, f \in E(H)$.*

Proof. Let H and c be as in the lemma, and suppose n is sufficiently large. Let $D := k^2 + k \ge 3k$. Note that, given our minimum ℓ-degree condition, Proposition 2.4 implies that

$$\begin{aligned} &\delta_1(H) \ge (c_{k,\ell} + \eta)\binom{n-1}{k-1} > \left(\frac{1}{2} + \frac{\eta}{2}\right)\binom{n}{k-1} \quad \text{and} \\ &\delta_2(H) \ge (c_{k,\ell} + \eta)\binom{n-2}{k-2} > \frac{1}{2}\binom{n}{k-2}. \end{aligned} \tag{2}$$

Here the inequalities follow as $c_{k,\ell} \ge 1/2$ by (1).

As n is sufficiently large, and by definition of $c_{k,\ell}$, the minimum ℓ-degree condition ensures a perfect matching M in H.

Let $L := \binom{r}{2} + 1$. By the hypothesis of the lemma, M does not contain $n/k - \binom{r}{2}$ edges of the same color; so, there exist distinct edges $e_1, \dots, e_L, f_1, \dots, f_L \in M$ such that $c(e_i) \ne c(f_i)$ for each $i \in [L]$.

Given any distinct $x, y \in V(H)$, (2) implies that $|N_H(x) \cap N_H(y)| \ge \eta\binom{n}{k-1}$. In particular, this means that $N_H(x) \cap N_H(y)$ is of type $S(D)$ or of type $C_iC_j(D)$ for some distinct $i, j \in [r]$.

Suppose there exists $i \ne j \in [r]$ and distinct $x, y, z, w \in V(H)$ such that $N_H(x) \cap N_H(y)$ and $N_H(z) \cap N_H(w)$ are both type $C_iC_j(D)$. Then by Lemma 2.5, there exists a good $(3k, e, f)$-gadget in H for some $e, f \in E(H)$.

So, we may assume no such $i \neq j \in [r]$ and $x, y, z, w \in V(H)$ exist. In particular, for each of the $\binom{r}{2} = L - 1$ choices for $i \neq j \in [r]$, there is at most one pair (e_s, f_s) such that there exist $u \in e_s$ and $v \in f_s$ so that either $N_H(u) \cap N_H(v)$ or $N_H(v) \cap N_H(u)$ is type $C_iC_j(D)$. Thus, the following claim holds.

CLAIM 2.7. *There is a pair (e_s, f_s) such that for each $u \in e_s$ and $v \in f_s$, we have that $N_H(u) \cap N_H(v)$ is type $S(D)$.*

Let $e_s = \{u_1, \ldots, u_k\}$ and $f_s = \{v_1, \ldots, v_k\}$. For each $i \in [k]$, we choose a set T_i so that

(i) $T_i \in S(u_iv_i)$;
(ii) $T_1, \ldots, T_k, e_s, f_s$ are all vertex-disjoint.

Note that we can guarantee (ii) since $|S(u_iv_i)| \geq Dn^{k-2} = (k^2 + k)n^{k-2}$ for each $i \in [k]$.

We construct a $(k^2 + k, e_s, f_s)$-gadget G as follows:

- $V(G)$ is the union of e_s, f_s, $T_1, \ldots, T_k$;
- e_s and f_s are edges in G;
- u_iT_i, v_iT_i are edges in G for all $i \in [k]$.

By definition, G is an S-$(k^2 + k, e_s, f_s)$-gadget with $c(e_s) \neq c(f_s)$. This implies that G is a good $(k^2 + k, e_s, f_s)$-gadget. Indeed, $M_e := \{e_s, v_1T_1, \ldots, v_kT_k\}$ and $M_f := \{f_s, u_1T_1, \ldots, u_kT_k\}$ are perfect matchings in G with different color profiles.

3. Proof of Theorem 1.3.

Let H be a sufficiently large n-vertex k-uniform hypergraph as in the statement of the theorem. Let $c : E(H) \to \{C_1, \ldots, C_r\}$ be an r-coloring of $E(H)$. If H contains a perfect matching with at least $n/k - \binom{r}{2}$ edges of the same color, then we are done.

So, suppose no perfect matching in H contains at least $n/k - \binom{r}{2}$ edges of the same color. By Lemma 2.6, we can find either a good $(3k, e, f)$-gadget or a good $(k^2 + k, e, f)$-gadget in H. Call this gadget G_1.

Next, consider $H_1 := H \setminus V(G_1)$. Clearly $\delta_\ell(H_1) \geq (c_{k,\ell} + \eta/2)\binom{n}{k-\ell}$. Suppose H_1 contains a perfect matching M_1 with at least $|H_1|/k - \binom{r}{2}$ edges of the same color. Thus, by taking any perfect matching in G_1 and adding it to M_1, we obtain a perfect matching in H containing at least $|H_1|/k - \binom{r}{2} \geq n/k - |G_1|/k - \binom{r}{2} \geq n/k - k - 1 - \binom{r}{2}$ edges of the same color, as desired.

Hence, we may assume H_1 does not contain such a perfect matching M_1. By Lemma 2.6, we can find either a good $(3k, e, f)$-gadget or a good $(k^2 + k, e, f)$-gadget in H_1. Call this gadget G_2, and set $H_2 := H_1 \setminus V(G_2)$.

Repeating this argument, we obtain either a perfect matching in H of significant color-bias, or a collection of $t := \frac{\eta n}{4k^k(k^2+k)}$ vertex-disjoint gadgets $G_1, \ldots, G_t$ where, given any $i \in [t]$, G_i is either a good $(3k, e, f)$-gadget or a good $(k^2 + k, e, f)$-gadget in H. In particular, note that each gadget we select has size at most $k^2 + k$, and if we remove $t(k^2 + k)$ vertices from H, we still have that $\delta_\ell(H) \geq (1/2 + \eta)\binom{n}{k-\ell} - t(k^2 + k)n^{k-\ell-1} \geq (1/2 + \eta/2)\binom{n}{k-\ell}$. Thus, we can indeed repeatedly apply Lemma 2.6 to obtain these gadgets $G_1, \ldots, G_t$.

Set $\mathcal{G} := \{G_1, \ldots, G_t\}$. For each color C_i, consider the set $\mathcal{G}_i$ of all the gadgets in $\mathcal{G}$ that contain two perfect matchings with different color profiles with respect to the color C_i. Clearly, there exists some $j \in [r]$ such that $\mathcal{G}_j$ contains at least t/r gadgets.

For each gadget G_i in $\mathcal{G}_j$, consider the perfect matching M_i in G_i with the largest possible number of edges colored C_j; let M_i' be the perfect matching in G_i with the fewest possible edges colored C_j. So, M_i has at least one more C_j-edge than M_i'.

Let M^+ denote the union of all these M_i, and let M^- denote the union of all these M_i'. So, M^+ contains at least $t/r = \frac{\eta n}{4rk^k(k^2+k)}$ more C_j-edges than M^-.

Let $V(\mathcal{G}_j)$ denote the set of vertices in H that lie in one of the gadgets in $\mathcal{G}_j$. Note that $\delta_\ell(H \setminus V(\mathcal{G}_j)) \geq (c_{k,\ell} + \eta/2)\binom{n}{k-\ell}$, so there exists a perfect matching M in $H \setminus V(\mathcal{G}_j)$. Thus, $M \cup M^+$ and $M \cup M^-$ are both perfect matchings in H.

If $M \cup M^-$ contains at least $\frac{n}{rk} + \frac{\eta n}{8r(r-1)k^k(k^2+k)}$ edges of the same color, then the theorem holds. Thus, we may assume this is not the case. This immediately implies the following claim.

Claim 3.1. *For every $i \in [r]$, the number of C_i-edges in $M \cup M^-$ is at least $\frac{n}{rk} - \frac{\eta n}{8rk^k(k^2+k)}$.*

In particular, $M \cup M^-$ contains at least $\frac{n}{rk} - \frac{\eta n}{8rk^k(k^2+k)}$ C_j-edges. Since there are at least $\frac{\eta n}{4rk^k(k^2+k)}$ more C_j-edges in M^+ than in M^-, we obtain that $M \cup M^+$ contains at least $\frac{n}{rk} + \frac{\eta n}{8rk^k(k^2+k)}$ C_j-edges, as desired.

4. Concluding remarks. In this paper we have determined the minimum ℓ-degree threshold for forcing a color-bias perfect matching in a k-uniform hypergraph for all $2 \leq \ell < k$. The only remaining open case of the problem is the minimum *vertex* degree version.

A result of Hàn, Person, and Schacht [12] yields that $m_1(3,n) = (5/9 + o(1))\binom{n-1}{2}$. The following example shows that the corresponding color-bias problem has a significantly higher minimum vertex degree threshold.

Example 4.1. Given any $n \in 6\mathbb{N}$, there exists an n-vertex 3-uniform hypergraph H with

$$\delta_1(H) \geq \frac{3}{4}\binom{n-1}{2}$$

and with a 2-coloring of $E(H)$ so that every perfect matching in H has precisely $n/6$ edges in each color.

Proof. Define H so that (i) $V(H)$ is the disjoint union of two vertex classes A and B, both of size $n/2$; and (ii) $E(H)$ consists of all those 3-uniform edges containing at least one vertex from each of A and B. Thus,

$$\delta_1(H) = \binom{n/2}{2} + \frac{n}{2}\left(\frac{n}{2} - 1\right) \geq \frac{3}{4}\binom{n-1}{2}.$$

Color each edge containing two vertices from A red and each edge containing two vertices from B blue. It is easy to see that every perfect matching in H uses the same number of red and blue edges. ∎

We suspect that this example is extremal for the minimum vertex degree problem in 3-uniform hypergraphs.

Question 4.2. *Given any $\eta > 0$, does there exist a $\gamma > 0$ so that the following holds for all sufficiently large $n \in 3\mathbb{N}$? Suppose that H is an n-vertex 3-uniform hypergraph with*

$$\delta_1(H) \geq \left(\frac{3}{4} + \eta\right)\binom{n-1}{2}.$$

Then given any 2-coloring of $E(H)$, there is a perfect matching in H with at least $n/6 + \gamma n$ edges of the same color.

Remark. Since this paper has been accepted, Question 4.2 has been answered in the affirmative; see [11, 15]. In fact, this new work resolves the minimum vertex degree problem fully (i.e., for all choices of the uniformity $k \geq 3$ and number of colors $r \geq 2$). ☐

By tweaking the proof of Theorem 1.3, one can show that given any $k \geq 3$ and $r \geq 2$, there is a constant C such that every sufficiently large r-colored n-vertex k-uniform hypergraph H with $\delta_{k-1}(H) \geq n/2 + C$ contains a perfect matching with at least $(n/rk) + 1$ edges of the same color. Moreover, the lower bound on the color-bias grows linearly as one increases the minimum codegree further. The Ph.D. thesis of the third author will contain a rigorous proof of this.

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