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ON MULTICOLOR TURÁN NUMBERS*

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Abstract. We address a problem which is a generalization of Turán-type problems recently introduced by Imolay, Karl, Nagy, and Váli. Let F be a fixed graph and let G be the union of k edge-disjoint copies of F, namely $G = \cup_{i=1}^k F_i$, where each F_i is isomorphic to a fixed graph F and $E(F_i) \cap E(F_j) = \emptyset$ for all $i \neq j$. We call a subgraph $H \subseteq G$ *multicolored* if H and F_i share at most one edge for all i. Define $\mathrm{ex}_F(H,n)$ to be the maximum value k such that there exists $G = \cup_{i=1}^k F_i$ on n vertices without a multicolored copy of H. We show that $\mathrm{ex}_{C_5}(C_3,n) \leq n^2/25 + 3n/25 + o(n)$ and that all extremal graphs are close to a blow-up of the 5-cycle. This bound is tight up to the linear error term.

Key words. Turán, extremal graphs, multicolored graphs, rainbow graphs

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1. Introduction.

For a graph G, let $\mathrm{ex}(n,G)$ denote the maximum number of edges in a graph on n vertices that does not contain G as a subgraph. The classical Turán theorem [8] from 1941 states that $\mathrm{ex}(n,K_{t+1}) = e(T_{n,t})$, where K_t denotes the complete graph on t vertices and $T_{n,t}$ denotes the complete t-partite graph on n vertices whose part sizes are as equal as possible. Several multicolored generalizations of Turán-type problems have been studied since then. Keevash, Mubayi, Sudakov, and Verstaëte [5] introduced the concept of the rainbow Turán numbers. For a non-bipartite graph H, they asymptotically determined the maximum number of edges in a graph on n vertices that has a proper edge-coloring with no rainbow H. Another variant was introduced by Conlon and Tyomkyn [2]. For a graph F, they obtained bounds on the minimum number of colors in a proper edge-coloring of K_n that does not contain k vertex-disjoint color isomorphic copies of F.

This paper focuses on a related Turán-type problem recently introduced by Imolay, Karl, Nagy, and Váli [4]. Let F be a fixed graph and let G be the union of k edge-disjoint copies of F. That is, $G = \cup_{i=1}^k F_i$, where each F_i is isomorphic to a fixed graph F and $E(F_i) \cap E(F_j) = \emptyset$ for all $i \neq j$. We call a subgraph $H \subseteq G$ *multicolored* if H and F_i share at most one edge for every i. Define $\mathrm{ex}_F(H,n)$ to be the maximum k such that there exists $G = \cup_{i=1}^k F_i$ on n vertices with no multicolored copy of H.

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Determining $\mathrm{ex}_F(H,n)$ in general can be a very hard problem. For example, $\mathrm{ex}_{C_3}(C_3,n)$ is related to the famous $(6,3)$-problem, introduced by Ruzsa and Szemerédi [7] in 1978. A variant of the $(6,3)$-problem asks for the maximum number of edges in a graph in which each edge belongs to a unique triangle, i.e., asks to determine $\mathrm{ex}_{C_3}(C_3,n)$. The exact asymptotics of $\mathrm{ex}_{C_3}(C_3,n)$ are still unknown.

For pairs of graphs (F,H) for which there is no homomorphism from H to F, Imolay, Karl, Nagy, and Váli [4] showed that the extremal number $\mathrm{ex}_F(H,n)$ is at least $n^2/v(F)^2 - o(n^2)$. The construction is based on packing copies of F into a blow-up of F on n vertices, whose parts have sizes as equal as possible. Imolay, Karl, Nagy, and Váli [4] also proposed the problem of determining the set of pairs (F,H) for which $\mathrm{ex}_F(n,H) = n^2/v(F)^2 + o(n^2)$ and suggested that this should be the case for $F = C_5$ and $H = C_3$. Recently, Kovács and Nagy [6] showed that this is indeed true for $\mathrm{ex}_{C_5}(C_3,n)$. Moreover, from their methods it follows that $\mathrm{ex}_{C_5}(C_3,n) \le n^2/25 + 3n/10$.

It is natural to conjecture that $\mathrm{ex}_{C_5}(C_3,n)$ is precisely equal to the maximum number of edge-disjoint copies of C_5 in a balanced blow-up of C_5 on n vertices. Therefore, depending on the residue of n modulo 5, we expect a linear order term to be present in the bounds on $\mathrm{ex}_{C_5}(C_3,n)$. For more details, see the discussion in section 2.

At the same time that Kovács and Nagy [6] announced their result, we obtained a similar upper bound for $\mathrm{ex}_{C_5}(C_3,n)$. By combining our methods with theirs, we were able to improve the linear order term.

THEOREM 1.1. *For every $\delta > 0$ there exists $n_\delta \in \mathbb{N}$ such that for every $n \ge n_\delta$, we have*

$$ex_{C_5}(C_3,n) \le \frac{n^2}{25} + \frac{3n}{25} + \delta n.$$

Say that a graph G is an extremal multicolored triangle-free graph if the edge set of G can be partitioned into $\mathrm{ex}_{C_5}(C_3,n)$ edge-disjoint copies of C_5 and G does not contain a multicolored copy of C_3. Our second result captures the structure of all extremal multicolored triangle-free graphs. For $i \in [5]$, we denote by $G(A_i, A_{i+1})$ the bipartite subgraph of G on vertex set $A_i \cup A_{i+1}$ and edges $uv \in E(G)$ where $u \in A_i$ and $v \in A_{i+1}$.

THEOREM 1.2. *For every $\delta > 0$ there exists $n_\delta \in \mathbb{N}$ such that the following holds for every $n \ge n_\delta$. Let G be an extremal multicolored triangle-free graph on n vertices. Then there exists a partition $V(G) = A_1 \cup \cdots \cup A_5$ such that all but $2n/5 + \delta n$ edges of G belong to $\bigcup_i G(A_i, A_{i+1})$. Moreover, for every $i \in [5]$ we have*

$$n/5 - 2^4 \le |A_i| \le n/5 + 2^6.$$

Our bound on the number of edges that do not belong to $\bigcup_i E_G(A_i, A_{i+1})$ is best possible. One may create a small perturbation of a blow-up of C_5 to generate another extremal graph with $2n/5$ edges across different parts; we provide the details in section 2.

The paper is organized as follows. In section 2, we deduce an approximate version of Theorem 1.1, but with a worse linear error term. This approximate estimate is used in section 3, where we show that all extremal examples are close to a blow-up of C_5, up to a small quadratic error term. We also deduce some other properties of extremal graphs. In section 4, we prove Theorem 1.2. Finally, in section 5 we prove Theorem 1.1.

2. An approximate asymptotic result. In this section, we prove a version of Theorem 1.1 with a slightly worse linear error term; see Theorem 2.2. From now on we let G be a graph with vertex set $[n]$ whose edge set is written as a union of $\mathrm{ex}_{C_5}(C_3,n)$ edge-disjoint copies of C_5. We denote this decomposition by $G=\cup_{j=1}^{k}F_j$, where each F_j is a copy of C_5 and $k=\mathrm{ex}_{C_5}(C_3,n)$. Moreover, we identify each F_j with the color j and denote the function associated with this coloring by $c:E(G)\to\mathbb{N}$.

2.1. A discussion on lower bounds. By Theorem 3.1 in [4], we have that $\mathrm{ex}_{C_5}(C_3,n)\geq n^2/25-o(n^2)$. Here, we need a more precise estimate. We say that a graph H is a *blow-up* of C_5 if there exists a partition of $V(H)=A_1\cup\cdots\cup A_5$ such that

$$E(H)=\bigcup_{i\in[5]}\{ab:a\in A_i \text{ and } b\in A_{i+1}\},$$

where $[n]:=\{1,\dots,n\}$. Whenever we are dealing with parts $A_1,\dots,A_5$, indices are always interpreted modulo 5. For each $a=(a_1,\dots,a_5)\in\mathbb{N}^5$, define $b(a)=\min_i a_ia_{i+1}$. Observe that if we have a blow-up of C_5 with parts of sizes $a_1,\dots,a_5$, then $b(a)$ is the minimum number of edges between two consecutive parts.

It is not hard to show that if G is a subgraph of the blow-up of C_5 with parts of size $(a_1,\dots,a_5)$, then $k\leq b(a)$ and equality can be attained. Moreover, under the restriction $a_1+\cdots+a_5=n$, $b(a)$ is maximized by a balanced blow-up, that is when the parts have as equal size as possible. Define

$$t(n):=\max\Big\{b(a):a\in\mathbb{N}^5, a_1+\cdots+a_5=n\Big\}.$$

By writing $n=5q+r$, where $r\in\{0,\dots,4\}$, one can check that $t(n)=q^2$ if $r\in\{0,1,2\}$ and $t(n)=q(q+1)$ if $r\in\{3,4\}$. In particular, we have

$$\frac{n^2}{25}-\frac{2n}{5}\leq q^2+q\mathbb{1}_{r\in\{3,4\}}=t(n)\leq \mathrm{ex}_{C_5}(C_3,n). \tag{1}$$

We now show that the term $2n/5$ in Theorem 1.2 is tight by providing graphs on n vertices whose edge sets are decomposed into $t(n)$ monochromatic copies of C_5 without a multicolored triangle (they do contain $2n/5$ (nonmulticolored) triangles). We do so by perturbing a balanced blow-up of C_5. To describe the construction, say that $x_1x_2x_3x_4x_5$ is a copy F of C_5 if $x_1,\dots,x_5$ are the vertices of F, and edges are x_ix_{i+1}, for all $i=1,\dots,5$. For simplicity, assume that n is a multiple of five, let $q=n/5$, and let $A_1,\dots,A_5$ be disjoint sets, each of size q. Let $(v_j^i)_{j\in[q]}$ be a labeling of the vertices of A_i, for each $i\in[5]$. For each $i,j\in[q]$, let $F_{i,j}=v_i^1v_j^2v_i^3v_j^4v_{i+j}^5$ be a copy of C_5. Clearly, this is a collection of q^2 edge-disjoint 5-cycles forming a balanced blow-up of C_5. Now, for every $i\in[q]$, remove the cycle $v_i^1v_i^2v_i^3v_i^4v_{2i}^5$, and replace it by $v_i^1v_i^3v_i^2v_i^4v_{2i}^5$. That is, we remove the edges $v_i^1v_i^2$ and $v_i^3v_i^4$ and replace them by $v_i^1v_i^3$ and $v_i^2v_i^4$. Let us call these new edges *crossing edges.* This switch of edges will not create a multicolored triangle. Indeed, observe that the crossing edges form a matching of size $2q=2n/5$. The only triangles that are created are those with vertices in A_1, A_2, and A_3 or with vertices in A_2, A_3, and A_4. Moreover, the triangles are of the form $v_i^{1+k}v_j^{2+k}v_i^{3+k}$, for some i,j,k, which means that the edges $v_i^{1+k}v_j^{2+k}$ and $v_j^{2+k}v_i^{3+k}$ have the same color. Therefore, the constructed graph does not contain a multicolored triangle.

2.2. Upper bounds. If a graph G is the union of edge-disjoint copies of C_5, each receiving a distinct color, then it is clear that every vertex $v\in V(G)$ has even degree, and that in each color, v is incident to exactly zero or two edges. If G is an extremal multicolored triangle-free graph, then G does not have multicolored triangles. In general, however, G may have many triangles formed by two color classes, as exemplified at the end of the previous subsection. Our next lemma states that for every $v \in V(G)$, the number of edges inside the neighborhood of v is at most $3d(v)/2$; hence G does not have many triangles.

LEMMA 2.1. *For each $v \in V(G)$, there are at most $3d(v)/2$ edges inside $N_G(v)$. Consequently, there are at most $e(G) \leq n^2/2$ triangles in G.*

Proof. Let $v \in V(G)$ and $u_1, \ldots, u_{d(v)}$ be the neighbors of v. Without loss of generality, suppose that $c(vu_i) = c(vu_{i+d(v)/2}) = i$ for all $i \in [d(v)/2]$. Let $\mathcal{M} = \{\{u_i, u_{i+d(v)/2}\} : i \in [d(v)/2]\}$ be the collection of pairs of vertices which form a monochromatic cherry with v. Note that these pairs could be edges in G, as an edge of the form $u_i u_{i+d(v)/2}$ does not create a multicolored triangle with v. Let xy be an edge of G inside $N_G(v)$ which does not belong to $\mathcal{M}$. Then, $c(vx) \neq c(vy)$ and hence we must have $c(xy) = c(vx)$ or $c(xy) = c(vy)$. Therefore, every edge e inside $N_G(v)$ which is not in $\mathcal{M}$ must have an endpoint z such that $c(e) = c(zv)$.

Now, fix some color $j \in [d(v)/2]$ (i.e., a color incident to v). We claim that there are at most two edges in color j inside $N_G(v)$ that are not in $\mathcal{M}$. Indeed, suppose that there are three such edges. These must form a path $u_j z_1 z_2 u_{j+d(v)/2}$ inside $N_G(v)$ contained in the color-j copy of C_5 for some vertices $z_1, z_2 \in N_G(v)$. Then $c(vz_1) = c(vz_2)$, as otherwise there would be a multicolored triangle in G. But this means that $\{z_1 z_2\} \in \mathcal{M}$.

As every edge inside $N_G(v)$ not in $\mathcal{M}$ must have a color in $[d(v)/2]$ and as every such color appears at most two times outside $\mathcal{M}$, we conclude that the number of edges inside $N_G(v)$ is at most $|\mathcal{M}| + 2d(v)/2 = 3d(v)/2$. ∎

Let us briefly argue that Lemma 2.1 is best possible. Consider a 2-coloring of the edges of K_5 in which each color class forms a copy of C_5. In this coloring, every vertex v has $3d(v)/2 = 6$ edges inside its neighborhood. This construction can be extended to larger graphs as follows. Let G be a graph that consists of n copies of K_5, all sharing exactly one vertex. Color each of these copies of K_5 with two colors each, all distinct, in such a way that each monochromatic component is a copy of C_5. The degree of the common vertex is $4n$ and the number of edges in its neighborhood is $6n = 3d(v)/2$.

For a graph G and $v \in V(G)$, define

$$s_v := d(v) - \frac{2e(G)}{n}. \tag{2}$$

The function s_v is the difference between the degree of the vertex v and the average degree of G; in particular, $\sum_v s_v = 0$. Our next theorem, which is the main result in this section, is a version of Theorem 1.1 with a slightly worse linear error term. The upper bound on $\sum_v s_v^2$ shall be used later to bound the number of vertices of small degree in extremal graphs, which is crucial in the proof of Theorem 1.2.

THEOREM 2.2. *Let $q \in \mathbb{N}$, $r \in \{0, \ldots, 4\}$ and set $n = 5q + r$. If $n \geq 26$, then*

$$ex_{C_5}(C_3, n) \leq q^2 + q\left(6 + \frac{8r}{5} - 3 \cdot \mathbb{1}_{r \in \{3,4\}} + o(1)\right) - \frac{1}{n} \sum_{v \in [n]} s_v^2. \tag{3}$$

Furthermore,

$$\sum_{v\in[n]} s_v^2 \le \left(6+\frac{8r}{5}+o(1)\right)qn. \tag{4}$$

Proof. Let G be an extremal multicolored triangle-free graph on n vertices. We find a large bipartite graph $B\subseteq G$ and use the fact that $G\setminus B$ still needs to contain an edge from every F_i, as C_5 is not bipartite. For each $v\in[n]$, define B_v to be the bipartite graph induced by the edges between $N(v)$ and $[n]\setminus N(v)$. For every $v\in[n]$ we have

$$e(B_v)=\sum_{u\in N(v)} d(u)-2e(N(v))\ge \sum_{u\in N(v)} d(u)-3d(v),$$

by Lemma 2.1. Taking the average over every vertex v, we obtain

$$\frac{1}{n}\sum_{v\in[n]} e(B_v)\ge \frac{1}{n}\sum_{v\in[n]}\sum_{u\in N(v)} d(u)-\frac{6e(G)}{n}. \tag{5}$$

Note that for each $u\in[n]$ the degree $d(u)$ appears exactly $d(u)$ times in the double sum. Thus,

$$\frac{1}{n}\sum_{v\in[n]}\sum_{u\in N(v)} d(u)=\frac{1}{n}\sum_{v\in[n]} d(v)^2=\frac{1}{n}\sum_{v\in[n]}\left(\frac{2e(G)}{n}+s_v\right)^2=\frac{4e(G)^2}{n^2}+\frac{1}{n}\sum_{v\in[n]} s_v^2, \tag{6}$$

where we used that $\sum_v s_v=0$. From (5) and (6), it follows that there exists a vertex $v\in[n]$ for which

$$e(B_v)\ge \frac{4e(G)^2}{n^2}+\frac{1}{n}\sum_{v\in[n]} s_v^2-\frac{6e(G)}{n}.$$

That is, there exists a bipartite graph B such that

$$e(G)-e(B)\le e(G)-\frac{4e(G)^2}{n^2}-\frac{1}{n}\sum_{v\in[n]} s_v^2+\frac{6e(G)}{n}. \tag{7}$$

The function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=x-\frac{4x^2}{n^2}+\frac{6x}{n}$ is decreasing in the interval $[\frac{n^2}{8}+\frac{3n}{4},\binom{n}{2}]$. As $n=5q+r$, recall from (1) that $t(n)=q^2+q\mathbb{1}_{r\in\{3,4\}}$. As $e(G)\ge 5t(n)\ge \frac{n^2}{8}+\frac{3n}{4}$ when $n\ge 26$, the right-hand side of (7) is at most $f(5t(n))-\frac{1}{n}\sum_{v\in[n]} s_v^2$ for n sufficiently large. Now, note that

$$\frac{5t(n)}{n}=\frac{5q^2+5q\mathbb{1}_{r\in\{3,4\}}}{5q+r}=q\cdot\frac{1+\frac{\mathbb{1}_{r\in\{3,4\}}}{q}}{1+\frac{r}{5q}}. \tag{8}$$

As $1-\frac{r}{5q}\le(1+\frac{r}{5q})^{-1}\le 1-\frac{r}{10q}$ for every q sufficiently large, it follows from (8) that

$$q\left(1-\frac{r}{5q}+\left(1-\frac{r}{5q}\right)\frac{\mathbb{1}_{r\in\{3,4\}}}{q}\right)\le\frac{5t(n)}{n}\le q+o(q). \tag{9}$$

By plugging (9) into $f(5t(n))$, we obtain

$$f(5t(n)) \leq 5q^2 + 5q\mathbb{1}_{r\in\{3,4\}} - 4q^2\left(1 - \frac{2r}{5q} + \frac{2\cdot\mathbb{1}_{r\in\{3,4\}}}{q}\right) + 6q + o(q)$$

$$\leq q^2 + q\left(6 + \frac{8r}{5} - 3\cdot\mathbb{1}_{r\in\{3,4\}}\right) + o(q). \tag{10}$$

Therefore, it follows from (7) and (10) that

$$e(G) - e(B) \leq q^2 + q\left(6 + \frac{8r}{5} - 3\cdot\mathbb{1}_{r\in\{3,4\}}\right) + o(q) - \frac{1}{n}\sum_{v\in[n]} s_v^2. \tag{11}$$

As B is bipartite but none of the F_i are bipartite, it follows that $G\setminus B$ contains an edge of every F_i, and hence

$$\mathrm{ex}_{C_5}(C_3,n) \leq e(G) - e(B). \tag{12}$$

The upper bound on $\mathrm{ex}_{C_5}(C_3,n)$ follows by combining (11) and (12). The upper bound on $\sum_v s_v^2$ follows from the fact that $\mathrm{ex}_{C_5}(C_3,n) \geq t(n) = q^2 + q\mathbb{1}_{r\in\{3,4\}}$, by (1). ∎

3. The structure of extremal graphs. In this section, we show that G must be close to a blow-up of C_5. Moreover, the vertex partition given by the blow-up structure is approximately balanced. At the end of the section, we refine this structure when restricting our graph to the subgraph spanned by vertices with degree close to the average.

3.1. The approximate structure. To deduce the approximate structure of the extremal graphs, we shall need the triangle removal lemma due to Ruzsa and Szemerédi [7].

LEMMA 3.1 (triangle removal lemma). *For every $\delta > 0$, there exist $n_\delta \in \mathbb{N}$ and $f(\delta) > 0$ such that the following holds for every $n > n_\delta$. If H has at most $f(\delta)n^3$ triangles, then we can make H triangle-free by deleting at most δn^2 edges.*

Another tool that we need is the following stability theorem of Erdős, Győri, and Simonovits [3].

THEOREM 3.2 (Erdős–Győri–Simonovits). *For every $\beta > 0$ there exist $\gamma = \gamma(\beta) > 0$ and n_β such that if H is an n-vertex triangle-free graph with $n > n_\beta$ and $e(H) \geq n^2/5 - \gamma n^2$, then H can be turned into a subgraph of a blow-up of C_5 by deleting at most βn^2 edges.*

Now we deduce our first structural property on extremal multicolored triangle-free graphs, which states that they are close to being a blow-up of C_5.

LEMMA 3.3. *For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n > n_\varepsilon$ and an extremal multicolored n-vertex triangle-free graph H, H can be turned into a subgraph of a blow-up of C_5 by deleting at most εn^2 edges.*

Proof. In the proof we assume that n is sufficiently large in order that all previous lemmas are applicable. For a given $\varepsilon > 0$ set $\beta := \varepsilon/2$ and let $\gamma = \gamma(\beta) > 0$ be given by Theorem 3.2. Set $\delta := \min\{\beta,\gamma\}$. By Lemma 2.1, there are at most $n^2/2 = o(n^3)$ triangles in H; hence we can use the triangle removal lemma (Lemma 3.1). As $\mathrm{ex}_{C_5}(C_3,n) \geq t(n) \geq n^2/25 - 2n/5$ by (1), we have $e(H) \geq n^2/5 - 2n$. Lemma 3.1 applied with $\delta/2$ implies that there exists a triangle-free subgraph $H' \subseteq H$ such that

$$e(H') \geq n^2/5 - 2n - \delta n^2/2 \geq n^2/5 - \gamma n^2$$

for n large enough and by the choice of δ. Thus, H' can be turned into a subgraph of a blow-up of C_5 by deleting at most βn^2 edges, by Theorem 3.2. Together, these imply that there exists $H'' \subseteq G$ which is a subgraph of a blow-up of C_5 and such that $e(H'') \geq e(H) - (\beta + \delta)n^2 \geq e(H) - \varepsilon n^2$. ∎

From now on, we denote by $A_1, \ldots, A_5$ the disjoint sets corresponding to the partition of $V(G)$ given by a subgraph of a blow-up of C_5 as in Lemma 3.3. We denote by $C_5(A_1, \ldots, A_5)$ the graph which is a blow-up of C_5, with parts $A_1, \ldots, A_5$. Moreover, we assume that the intersection of G with the blow-up $C_5(A_1, \ldots, A_5)$ gives a subgraph of a blow-up of C_5 in G with the maximum number of edges. Recall that for $i \in [5]$, we denote by $G(A_i, A_{i+1})$ the bipartite subgraph of G on vertex set $A_i \cup A_{i+1}$ and edges $uv \in E(G)$ where $u \in A_i$ and $v \in A_{i+1}$.

Our next lemma states that $A_1, \ldots, A_5$ is close to being an equipartition and that each $G(A_i, A_{i+1})$ is close to being a complete bipartite graph.

LEMMA 3.4. *For every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$, then the following holds. For every $i \in [5]$ we have*

$$e_G(A_i, A_{i+1}) \geq n^2/25 - \varepsilon n^2 \qquad \text{and} \qquad n/5 - \varepsilon n \leq |A_i| \leq n/5 + \varepsilon n.$$

Proof. For a given $\varepsilon > 0$ let $\delta = \varepsilon/2^{11}$ and assume without loss of generality that n is large enough such that $H \subseteq C_5(A_1, \ldots, A_5)$ is a subgraph of G with at least $e(G) - \delta n^2$ edges (the existence of such H follows from Lemma 3.3). This implies that there are at least $e(G)/5 - \delta n^2 \geq n^2/25 - 5\delta n^2$ monochromatic copies of C_5 in H, and hence

$$|A_i||A_{i+1}| \geq e_G(A_i, A_{i+1}) \geq n^2/25 - 5\delta n^2 \tag{13}$$

for every $i \in [5]$.

Now, suppose that $|A_1| = n/5 - \alpha n$ for some $\alpha \in (2^9\delta, 5^{-1})$. From (13), we obtain

$$\begin{aligned} |A_2|, |A_5| &\geq \frac{n^2/25 - 5\delta n^2}{n/5 - \alpha n} \geq \frac{n^2/25 - 5\delta n^2}{n/5} \cdot (1 + 5\alpha) \\ &= \frac{n}{5} + (\alpha - 25\delta - 125\delta\alpha)n \geq \frac{n}{5} + \frac{2\alpha n}{3}. \end{aligned}$$

Thus, we have

$$|A_3| + |A_4| = n - |A_1| - |A_2| - |A_5| \leq 2n/5 - \alpha n/3.$$

This implies

$$|A_3||A_4| \leq (n/5 - \alpha n/6)^2 = n^2/25 - \alpha n^2/15 + \alpha^2 n^2/36 \leq n^2/25 - \alpha n^2/30,$$

which contradicts (13), as $\alpha/30 > 6\delta$, proving $|A_i| \geq n/5 - 2^9\delta n$ for every $i \in [5]$. From this we conclude that $|A_i| \leq n - 4(n/5 - 2^9\delta n) \leq n/5 + 2^{11}\delta n$ for every $i \in [5]$. The lemma follows by choice of δ. ∎

Define $N_i(v)$ to be the set of neighbors of v in G which are contained in A_i and let $d_i(v) = |N_i(v)|$. We refer to the edges not in $\bigcup_i E_G(A_i, A_{i+1})$ as *unstructured* edges. Observe that, as $G \cap C_5(A_1, \ldots, A_5)$ gives a subgraph of a blow-up of C_5 in G with the maximum number of edges, the number of unstructured edges is minimum over all subgraphs of a blow-up of C_5 in G. Our ultimate goal is to show that the number of unstructured edges is at most linear.

Our next lemma gives an upper bound on the degree and the number of unstructured edges incident to each vertex of G.

LEMMA 3.5. *For every $\delta > 0$ there exists $n_\delta \in \mathbb{N}$ such that if $n > n_\delta$, then the following holds. For every $i \in [5]$, $v \in A_i$, and $j \notin \{i-1, i+1\}$, we have*

$$d(v) \leq 2n/5 + \delta n \qquad \text{and} \qquad d_j(v) \leq \delta n. \tag{14}$$

Moreover, for $t \in \{i-1, i+1\}$ we have

$$d_t(v) \geq d(v) - \frac{n}{5} - \delta n. \tag{15}$$

Proof. Let $\gamma < \delta/4$ be a small enough constant and set $\varepsilon = \gamma/18$. First, apply Lemma 3.4 with parameter $\varepsilon^2/2$ to obtain

$$e_G(A_i, A_{i+1}) \geq n^2/25 - \varepsilon^2 n^2/2 \ \text{ and } \ |A_i| \leq n/5 + \varepsilon^2 n/2 \ \text{ for all } i \in [5]. \tag{16}$$

Let us now argue that for every $i \in [5]$ and $v \in V(G)$, we have

$$\min\left\{d_i(v), d_{i+1}(v)\right\} \leq \varepsilon n \tag{17}$$

for n sufficiently large. Suppose for contradiction that (17) does not hold. Since $e(G[N(v)]) \leq 3n/2$ by Lemma 2.1, there are at most $3n/2$ edges uw with $u \in N_i(v)$ and $w \in N_{i+1}(v)$. This implies that the number of edges between A_i and A_{i+1} is at most

$$|A_i||A_{i+1}| - d_i(v)d_{i+1}(v) + 3n/2 \leq |A_i||A_{i+1}| - \varepsilon^2 n^2 + 3n/2.$$

Thus, using (16) twice, we obtain

$$\begin{aligned}\frac{n^2}{25} - \frac{\varepsilon^2 n^2}{2} \leq e_G(A_i, A_{i+1}) &\leq \left(\frac{n}{5} + \frac{\varepsilon^2 n}{2}\right)^2 - \varepsilon^2 n^2 + \frac{3n}{2} \\ &= \frac{n^2}{25} + \frac{\varepsilon^2 n^2}{5} + \frac{\varepsilon^4 n^2}{4} - \varepsilon^2 n^2 + \frac{3n}{2} < \frac{n^2}{25} - \frac{\varepsilon^2 n^2}{2}\end{aligned}$$

for n sufficiently large, a contradiction. Thus, we conclude that (17) holds.

Now, fix a vertex $v \in V(G)$. If $d_G(v) \leq \gamma n$, then the condition $d_j(v) \leq \delta n$ is trivially satisfied. Thus, let us assume that $d_G(v) > \gamma n$. From (17) it follows that the set

$$S := \left\{j \in [5] : d_j(v) \leq \varepsilon n\right\}$$

has size at least three; in particular there exists $i \in [5]$ such that $\{i, i+2, i+3\} \subseteq S$. Therefore, v can have a large neighborhood only inside the union $A_{i-1} \cup A_{i+1}$. Note that this implies in particular that

$$d(v) \leq |A_{i-1}| + |A_{i+1}| + 3\varepsilon n \leq \frac{2n}{5} + \varepsilon^2 n + 3\varepsilon n \leq 2n/5 + \delta n,$$

by (16), choice of ε, and for n large enough. That is, the first part of (14) is proved.

We claim that if $d_{i-1}(v) > 3\varepsilon n$ and $d_{i+1}(v) > 3\varepsilon n$, then $v \in A_i$. Indeed, as $d_i(v) + d_{i+2}(v) + d_{i+3}(v) \leq 3\varepsilon n$, if we had $v \notin A_i$, then we could move v to A_i and obtain a subgraph of a blow-up of C_5 with more edges, a contradiction. In particular, this implies the second part of (14) in this case.

Now assume that either $d_{i-1}(v) \le 3\varepsilon n$ or $d_{i+1}(v) \le 3\varepsilon n$. As $\{i, i+2, i+3\} \subseteq S$, in both cases we have that there exists an index k for which $d_k(v) \ge d(v) - 6\varepsilon n > 12\varepsilon n$, using our assumption that $d_G(v) > \gamma n = 18\varepsilon n$, and $d_j(v) \le 3\varepsilon n$ for every $j \ne k$. As the set A_k contains most of the neighbors of v, we must have either $v \in A_{k-1}$ or $v \in A_{k+1}$; otherwise we could move v to one of these sets and obtain a subgraph of a blow-up of C_5 with more edges. In both cases, we have that if $v \in A_i$, then $d_j(v) \le \delta n$ for every $j \notin \{i-1, i+1\}$, i.e., (14) holds.

Finally, assuming $v \in A_i$, for some $i \in [5]$, we have for $t \in \{i-1, i+1\}$ that

$$d_t(v) \ge d(v) - \max_j |A_j| - 3\varepsilon n \ge d(v) - n/5 - \delta n,$$

where we used the fact that $|S| \ge 3$, (16), and the choice of ε. This completes our proof. $\square$

3.2. Cleaning the graph. In this section, we obtain a refinement of Lemma 3.5 for "good" vertices in G. Define the set of *good* vertices to be

$$V_g := \big\{v \in [n] : d_G(v) \ge 7n/20\big\}.$$

Observe that in a balanced blow-up of C_5, each vertex has degree approximately $2n/5$, i.e., every vertex is good. Recall that an unstructured edge is an edge of G which is not contained in $\bigcup_i E_G(A_i, A_{i+1})$. Let L be the set of unstructured edges with both endpoints are in V_g. Our next lemma states that L is a matching. Moreover, $V_g \cap A_i$ is an independent set for every $i \in [5]$.

LEMMA 3.6. *For every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n > n_\varepsilon$ we have $G[V_g \cap A_i] = \emptyset$ for every $i \in [5]$ and L is a matching.*

Proof. We may assume that n is large enough so that the assertions of Lemmas 3.4 and 3.5 apply. In particular, we assume for all $i \in [5]$, $j \notin \{i-1, i+1\}$, and $v \in A_i$ that

$$|A_i| \le n/5 + \varepsilon n, \quad d_j(v) \le \varepsilon n, \quad \text{and} \quad d(v) \le 2n/5 + \varepsilon n. \tag{18}$$

Without loss of generality we may assume that $i = 1$. Let $u, v \in A_1 \cap V_g$ and suppose for contradiction that $uv \in E(G)$. For $j \in \{2, 5\}$, let A_j^* be the common neighborhood of u and v in A_j which avoids the vertices of the C_5 of color $c(uv)$. Observe that for each $a \in A_2^* \cup A_5^*$ we have $c(au) = c(av)$; otherwise auv is a multicolored triangle. Moreover, if $a, a' \in A_2^* \cup A_5^*$ are distinct vertices, then $c(au) = c(av) \ne c(a'u) = c(a'v)$; otherwise $aua'v$ is a monochromatic copy of C_4, which cannot exist in G. From this, it follows that there are $|A_2^*| + |A_5^*|$ different colors incident to both u and v, and hence

$$\min\{d(u), d(v)\} \ge 2\big(|A_2^*| + |A_5^*|\big), \tag{19}$$

where the factor of 2 accounts for the fact that every color contributes twice to a degree of a vertex. By the inclusion–exclusion principle (applied separately to each of A_2^* and A_5^*), we have

$$|A_2^*| + |A_5^*| \ge d_2(u) + d_5(u) + d_2(v) + d_5(v) - |A_2| - |A_5| - 3, \tag{20}$$

where the term -3 accounts for neighbors in the copy of C_5 of color $c(uv)$. Now, $d_2(v) + d_5(v) \ge d(v) - 3\varepsilon n$, $d_2(u) + d_5(u) \ge d(u) - 3\varepsilon n$, and $|A_2| + |A_5| \le 2n/5 + 2\varepsilon n$, all by (18). Absorbing the constant term and using that $u, v \in V_g$, we thus obtain

$$|A_2^*| + |A_5^*| \geq 2 \cdot \frac{7n}{20} - \frac{2n}{5} - 9\varepsilon n = \frac{3n}{10} - 9\varepsilon n.$$

This, together with (19), implies that $d(u) \geq 3n/5 - 18\varepsilon n$, which contradicts $d(u) \leq 2n/5 + \varepsilon n$ in (18).

To prove that L is a matching, we split the proof into two cases. For the first case, suppose for contradiction that there exist good vertices $v \in A_1$ and $a, b \in A_3$ such that $va, vb \in E(G)$. The common neighborhood of a, b, and v inside A_2 has size at least $d_2(v) + d_2(a) + d_2(b) - 2|A_2|$. Let B_2^* be the common neighborhood of a, b, and v inside A_2 excluding the vertices of the copies of C_5 with colors $c(av)$ and $c(bv)$, so that

$$|B_2^*| \geq d_2(v) + d_2(a) + d_2(b) - 2|A_2| - 6.$$

Now, v has at least $7n/20 - 3\varepsilon n - |A_5|$ neighbors in A_2, by (18) and since v is a good vertex. Similarly, each of a and b have at least $7n/20 - 3\varepsilon n - |A_4|$ neighbors in A_2. Since $|A_i| \leq n/5 + \varepsilon n$ for all i, by (18), we obtain that

$$|B_2^*| \geq 3\left(\frac{7n}{20} - \frac{n}{5} - 4\varepsilon n\right) - \frac{2n}{5} - 2\varepsilon n - 6 \geq \frac{n}{40}.$$

In particular, B_2^* is nonempty. Let $u \in B_2^*$. The edges uv, ua, and ub have colors different from $c(av)$ and $c(bv)$, by definition of B_2^*. This implies that $c(uv) = c(ua) = c(ub)$, as otherwise there was a multicolored triangle in G. But this is a contradiction since each color class is a copy of a C_5.

For the second case, suppose for contradiction that there exist good vertices $v \in A_1$, $a \in A_4$, and $b \in A_3$ such that $va, vb \in E(G)$. Let D_5^* be the common neighborhood of v and a in A_5, which avoids the vertices of the C_5 of color $c(av)$; and let D_2^* be the common neighborhood of v and b in A_2, which avoids the vertices of the C_5 with color $c(bv)$. Similarly to the first case, we first show that D_5^* and D_2^* are nonempty. Again, we have

$$\begin{aligned} |D_5^*| &\geq d_5(v) + d_5(a) - |A_5| - 6 \\ &\geq \Big(d(v) - |A_2| - 3\varepsilon n\Big) + \Big(d(a) - |A_3| - 3\varepsilon n\Big) - |A_5| - 6 \qquad (21) \\ &\geq 2 \cdot \frac{7n}{20} - \frac{3n}{5} - 9\varepsilon n - 6 \\ &\geq \frac{n}{10} - 10\varepsilon n, \qquad (22) \end{aligned}$$

where we used (18) in the second inequality, and the upper bound on $|A_i|$ from (18), and the fact that a and v are good vertices in the third inequality. Similarly, $|D_2^*| \geq n/10 - 10\varepsilon n$.

Now let $u \in D_5^*$ and $u' \in D_2^*$ be arbitrary vertices. Then we must have $c(vu) = c(au)$ and $c(vu') = c(bu')$ since G does not have a multicolored triangle. Moreover,

$$|\{c(vu) : u \in D_5^*\} \cap \{c(vu') : u' \in D_2^*\}| \leq 1,$$

as otherwise G contained two monochromatic paths of length four, both with endpoints a and b. But this cannot happen since each color class is a copy of a C_5. It follows that v is incident to at least $|D_5^*| + |D_2^*| - 1$ distinct colors, and hence

$$d(v) \geq 2\big(|D_5^*| + |D_2^*|\big) - 2 \geq 2n/5 - 21\varepsilon n, \qquad (23)$$

using (22) and the corresponding bound on $|D_2^*|$. Using this new bound on the degree of v, it follows from (18) that

$$d_5(v) \geq d(v) - |A_2| - 3\varepsilon n \geq d(v) - n/5 - 4\varepsilon n \geq n/5 - 25\varepsilon n.$$

Feeding this new lower bound on $d_5(v)$ into (21), leaving all other bounds in (21)–(22) unchanged, we obtain that

$$|D_5^*| \geq \left(\frac{n}{5} - 25\varepsilon n\right) + \left(\frac{7n}{20} - |A_3| - 3\varepsilon n\right) - |A_5| - 6 \geq \frac{3n}{20} - 30\varepsilon n.$$

Analogously, one obtains that $|D_2^*| \geq 3n/20 - 30\varepsilon n$. With (23) this implies now that $d(v)$ is at least $3n/5 - 121\varepsilon n$, which contradicts the upper bound on $d(v)$ in (18). ◻

4. Proof of Theorem 1.2.

Throughout this section, let $\varepsilon \in (0, 2^{-200})$ be a fixed small constant and n be sufficiently large (and in particular, large enough such that the conclusions of Lemmas 3.4 and 3.5 hold for ε). Recall that we set $A_1, \ldots, A_5$ to be the disjoint sets given by the subgraph of a blow-up of C_5 in Lemma 3.3 such that the intersection of G with the blow-up $C_5(A_1, \ldots, A_5)$ has the maximum number of edges among all subgraphs of G that are also subgraphs of a blow-up of C_5.

We start by bounding the size of the set of vertices whose degree is far from $2n/5$. For $\gamma \in (\varepsilon^{1/4}, 2^{-12})$, define

$$V_\gamma = \left\{v \in V(G) : |d(v) - 2n/5| \leq \gamma n\right\}.$$

Observe that $V_\gamma \subseteq V_g$. In particular, Lemma 3.6 holds with V_g replaced by V_γ. Our first claim bounds the size of $V_\gamma^c := V(G) \setminus V_\gamma$.

CLAIM 4.1. $|V_\gamma^c| \leq 8/\gamma^2$.

Proof. Let $n = 5q + r$, where $q \in \mathbb{N}$ and $r \in \{0, \ldots, 4\}$. It follows from (1) and (3) that

$$\frac{n^2}{5} - 2n \leq e(G) \leq \frac{n^2}{5} + 12n.$$

Thus, the average degree of G is $2n/5 + O(1)$. Recall that for each vertex $v \in V(G)$, we defined $s_v = d(v) - 2e(G)/n$. If $v \in V_\gamma^c$, then

$$|s_v| = |d(v) - 2e(G)/n| \geq \gamma n/2.$$

Therefore, by (4) (and observing $(6 + 4r/5)q \leq 2n$) we have

$$|V_\gamma^c| \cdot \frac{\gamma^2 n^2}{4} \leq 2n^2,$$

which implies

$$|V_\gamma^c| \leq \frac{8}{\gamma^2}.$$ ◻

We say that a 5-cycle C in G is *great* if all of its edges have the same color and all of its vertices are in V_γ. For each pair of vertices a and b, let g_{ab} be the number of great 5-cycles containing a and b.

LEMMA 4.2. *Let $i \in [5]$ and $ab \in E(G)$, with $a \in A_i \cap V_\gamma$ and $b \in A_{i+2} \cap V_\gamma$. Then, we have $g_{ab} \geq n/5 - 4\gamma n$.*

Proof. Without loss of generality, suppose that $i = 1$ and let $a \in A_1 \cap V_\gamma$ and $b \in A_3 \cap V_\gamma$. By Lemma 3.5, we have

$$\min\{d_2(a), d_2(b)\} \geq \frac{2n}{5} - \gamma n - \frac{n}{5} - 4\varepsilon n = \frac{n}{5} - \gamma n - 4\varepsilon n. \tag{24}$$

Thus, the vertices a and b are incident to most vertices in A_2.

Recall that $c : E(G) \to \mathbb{N}$ is the coloring associated to the partition of $E(G)$ into copies of C_5. Define $R_a = \{c(av) : av \in E(G),\ v \in A_1 \cup A_3 \cup A_4\}$ and $R_b = \{c(bv) : bv \in E(G),\ v \in A_1 \cup A_3 \cup A_5\}$. These sets correspond to the set of colors incident to a and b, respectively, which appear at an edge which is not contained in $G \cap C_5(A_1, \dots, A_5)$. By Lemma 3.5, both R_a and R_b have size at most $3\varepsilon n$. In particular, $|R_a \cup R_b| \leq 6\varepsilon n$. Let A_2^* be the common neighborhood of a and b inside A_2 excluding the vertices incident to some color in $R_a \cup R_b$. Then

$$|A_2^*| \geq d_2(a) + d_2(b) - |A_2| - 5|R_a \cup R_b| \geq \frac{n}{5} - 2\gamma n - 9\varepsilon n - 30\varepsilon n \geq \frac{n}{5} - 3\gamma n, \tag{25}$$

where we use (24), $|A_2| \leq n/5 + \varepsilon n$ (cf. Lemma 3.4) and $|R_a \cup R_b| \leq 6\varepsilon n$ in the second inequality, and $\gamma > 39\varepsilon$ by choice of γ in the last inequality.

As $c(ab) \in R_a \cup R_b$, for every $v \in A_2^*$ we have $c(av) \neq c(ab)$ and $c(bv) \neq c(ab)$. This implies that $c(av) = c(bv)$ for every $v \in A_2^*$. In particular, the number of monochromatic 5-cycles containing a and b is at least $n/5 - 3\gamma n$. At most $8/\gamma^2$ of these cycles contain a vertex in V_γ^c, by Claim 4.1. Therefore, it follows that there are at least $n/5 - 3\gamma n - 8/\gamma^2$ great cycles containing a and b. ∎

Recall that a vertex v is good if $d(v) \geq 7n/20$. Recall that an edge $e \in E(G)$ is unstructured if $e \notin \bigcup_i E_G(A_i, A_{i+1})$. Let M be the set of unstructured edges. By Lemma 3.6, we know that M is a matching when restricted to good vertices, and in particular when restricted to V_γ. With this in mind, one would hope to prove that $|M| \leq n/2 + o(n)$. It turns out that we can prove a much better bound, which is even close to optimal (as discussed in section 2). Recall that $\varepsilon \in (0, 2^{-200})$ and $\gamma \in (\varepsilon^{1/4}, 2^{-12})$.

LEMMA 4.3. *Let $q \in \mathbb{N}$ and $r \in \{0, \dots, 4\}$ be such that $n = 5q + r$. Then, we have*

$$|M| \leq 2q + 2^6\gamma q.$$

Proof. First note that the number of unstructured edges with at least one of its endpoints in V_γ^c is at most

$$8/\gamma^2 \cdot 3\varepsilon n \leq 24\gamma^2 n \leq \gamma q, \tag{26}$$

by Lemma 3.5 and Claim 4.1. It remains to bound the number of unstructured edges with both endpoints in V_γ.

Let $M_\gamma \subseteq M$ be the set of such unstructured edges ab with $a, b \in V_\gamma$, and let P be the set of ordered pairs (e, C) such that C is a great 5-cycle and e is an unstructured edge with both endpoints in $V(C)$. Observe that if $(e, C) \in P$, then $e \in M_\gamma$. In particular, $\sum_{ab \in M_\gamma} g_{ab} \leq |P|$, where we recall that g_{ab} denotes the number of great 5-cycles containing a and b. For $ab \in M_\gamma$ we must have $a \in A_i$ and $b \in A_{i+2}$ for some $i \in [5]$ (or vice versa), by definition of an unstructured edge and since $G[A_i \cap V_\gamma]$ is empty, by Lemma 3.6. Using Lemma 4.2, we thus obtain

$$|P| \geq |M_\gamma| \cdot \left(\frac{n}{5} - 4\gamma n\right). \tag{27}$$

Now, let s_C be the number of unstructured edges with both endpoints in $V(C)$, for each great 5-cycle C. The set of unstructured edges spanned by V_γ is a matching, by Lemma 3.6, so we must have $s_C \le 2$ for all C. It follows that

$$|P| = \sum_{C \text{ great}} s_C \le \frac{2e(G)}{5}.$$

Combining this bound with (27), we thus obtain

$$|M_\gamma| \cdot \left(\frac{n}{5} - 4\gamma n\right) \le \frac{2\,e(G)}{5} \le 2\,\mathrm{ex}_{C_5}(C_3, n) \le 2q^2 + 2q\left(6 + \frac{8r}{5} - 3\mathbb{1}_{r\in\{3,4\}} + o(1)\right) \tag{28}$$

by Theorem 2.2, where we recall that q is defined by $n = 5q + r$ and $r \in \{0, \ldots, 4\}$. Now,

$$\frac{1}{n/5 - 4\gamma n} \le \frac{1}{q - 4\gamma n} \le \frac{1}{q}\left(1 + 24\gamma + 2\cdot(24)^2\gamma^2\right) \le \frac{1}{q}(1 + 25\gamma), \tag{29}$$

where we use $n \le 6q$, $(1-x)^{-1} \le 1 + x + 2x^2$ for every $x \in (0, 1/2)$, and that γ is small enough. Combining this last estimate with (28), we obtain

$$|M_\gamma| \le 2q + 50\gamma q + O(1).$$

This, together with the bound in (26), proves our lemma. ∎

From the constructions given in section 2.1, note that we can have $2n/5$ unstructured edges, and hence our bound of $2n/5 + o(n)$ on the number of unstructured edges given by Lemma 4.3 is tight. Our next lemma improves the bounds on the size of each A_i from an additive linear error term as in Lemma 3.4 to an additive error of constant size.

LEMMA 4.4. *Let $q \in \mathbb{N}$ and $r \in \{0, \ldots, 4\}$ be such that $n = 5q + r$. For every $j \in [5]$, we have*

$$q - 15 < |A_j| \le q + 64.$$

Proof. The proof is similar to the proof of Lemma 3.4. Without loss of generality, suppose that A_1 is the part of smallest size. Suppose for contradiction that $|A_1| = q - i$, for some $i \ge 15$.

Let $C_1, \ldots, C_k$ be the colored 5-cycles given by the edge-partition of G. Note that $k \ge q^2$ since we assume that G is extremal, by (1). At most $8/\gamma^2$ of these cycles contain a vertex in V_γ^c, by Claim 4.1, and at most $2q + 2^6\gamma q$ contain an unstructured edge by Lemma 4.3. Observe that a C_j with all vertices in V_γ that does not contain an unstructured edge is a great cycle $v_1v_2v_3v_4v_5$ such that $v_i \in A_i$ for each $i \in [5]$. Therefore, for each $i \in [5]$,

$$e(A_i, A_{i+1}) \ge q^2 - 2q - 2^6\gamma q - 8/\gamma^2. \tag{30}$$

Similarly to the proof of Lemma 3.4, this implies that

$$|A_2|, |A_5| \ge \frac{q^2 - 2q - 2^6\gamma q - 8/\gamma^2}{q - i} \ge \frac{q^2 - 3q}{q - i} \ge q + i - 3 \tag{31}$$

since γ is small and n is large. Thus, we have

$$|A_3| + |A_4| = n - |A_1| - |A_2| - |A_5| \le 2q - i + r + 6 \le 2q - i + 10.$$

Using this for the upper bound and (30) for the lower bound, we obtain

$$(q - i/2 + 5)^2 \ge |A_3||A_4| \ge q^2 - 2q - 2^6\gamma q - 8/\gamma^2,$$

which is a contradiction, as $i \ge 15$. As every set $A_1, \dots, A_5$ has size at least $q - 15$, we conclude that $|A_j| \le q + 64$ for any $j \in [5]$. This proves the lemma. ∎

Finally, note that Lemmas 4.4 and 4.3 imply Theorem 1.2.

5. Proof of Theorem 1.1. Let $\varepsilon \in (0, 2^{-200})$, $\gamma \in (\varepsilon^{1/4}, 2^{-12})$ and set $\delta = 2^8\gamma$. We start by bounding the number of triangles in G.

LEMMA 5.1. *The number of triangles in G is at most $2n^2/25 + 2\delta n^2$.*

Proof. Let Δ_γ be the number of triangles where all three vertices belong to V_γ. By Lemma 3.6, if $a, b, c \in V_\gamma$ and they form a triangle, then we must have $a \in A_i$, $b \in A_{i+1}$, and $c \in A_{i+2}$ for some $i \in [5]$. Let M be the set of unstructured edges between pairs of good vertices. By Lemma 4.3 we have $|M| \le 2n/5 + \delta n$ and by Lemma 3.4 we have $\max_i |A_i| \le n/5 + \delta n$. As every triangle contains an unstructured edge,

$$\Delta_\gamma \le |M| \cdot \max_i |A_i| \le 2n^2/25 + \delta n^2.$$

Let Δ_γ^c be the number of triangles containing at least one vertex outside V_γ. By Lemma 2.1 and Claim 4.1, we have

$$\Delta_\gamma^c \le \sum_{v \in V_\gamma^c} \frac{3d(v)}{2} \le \frac{3n}{2}|V_\gamma^c| \le \frac{12n}{\gamma^2} \le \delta n^2.$$ ∎

Below, we denote by $C_1, C_2, \dots, C_k$ the set of monochromatic 5-cycles in G. The next lemma is due to Kovács and Nagy [6]. For completeness, we provide their proof here.

LEMMA 5.2 (Kovács–Nagy). *Let $i \in [k]$ and denote by Δ_i^1 and Δ_i^2 the number of triangles with exactly one and two edges colored i, respectively. Then,*

$$\sum_{v \in C_i} d(v) \le 2n + 2\Delta_i^2 + \Delta_i^1.$$

Proof. Note that $e(G[V(C_i)]) = 5 + \Delta_i^2$. Observe that each vertex in $[n] \setminus C_i$ sends at most two colors to C_i; otherwise a multicolored triangle is created. For the same reason, each vertex in $[n] \setminus C_i$ sends at most three edges to C_i, and if exactly three are sent, then two of them with the same color must go to adjacent vertices in C_i. Thus, we have

$$\sum_{v \in C_i} d(v) \le 2e(G[V(C_i)]) + \Delta_i^1 + 2(n-5) \le 2n + 2\Delta_i^2 + \Delta_i^1.$$ ∎

Proof of Theorem 1.1. Following Kovács and Nagy [6], we estimate the double sum

$$S := \sum_{i=1}^{k} \sum_{v \in C_i} d(v).$$

Every vertex v is contained in $d(v)/2$ monochromatic cycles; hence $d(v)$ is counted $d(v)/2$ times in the double sum above. Therefore,

$$S = \sum_{v \in [n]} \frac{d(v)^2}{2} \geq \frac{n}{2} \cdot \left(\frac{2e}{n}\right)^2 = \frac{2e^2}{n}. \tag{32}$$

On the other hand, by Lemma 5.2, we have

$$S \leq 2nk + \sum_{i=1}^{k} \left(2\Delta_i^2 + \Delta_i^1\right) \leq 2nk + \frac{6n^2}{25} + 6\delta n^2, \tag{33}$$

using that each triangle is counted three times and Lemma 5.1. As $e(G) = 5k$, from (32) and (33), it follows that

$$\frac{50k^2}{n} \leq 2kn + \frac{6n^2}{25} + 6\delta n^2;$$

hence

$$k \leq \frac{n^2}{25} + \frac{3n^3}{625k} + \frac{\delta n^3}{k}.$$

As $k \geq n^2/25 - 2n$, it follows that $k \leq n^2/25 + (3/25 + 50\delta)n$. As ϵ (and hence δ) can be chosen to be arbitrarily small, the theorem follows. $\square$

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