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The Furstenberg–Poisson boundary and CAT(0) cube complexes

TALIA FERNÓS

Department of Mathematics and Statistics, University of North Carolina at Greensboro, 317 College Avenue, Greensboro, NC 27412, USA
(e-mail: t_fernos@uncg.edu)

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Abstract. We show under weak hypotheses that ∂X, the Roller boundary of a finite-dimensional CAT(0) cube complex X is the Furstenberg–Poisson boundary of a sufficiently nice random walk on an acting group Γ. In particular, we show that if Γ admits a non-elementary proper action on X, and μ is a generating probability measure of finite entropy and finite first logarithmic moment, then there is a μ-stationary measure on ∂X making it the Furstenberg–Poisson boundary for the μ-random walk on Γ. We also show that the support is contained in the closure of the regular points. Regular points exhibit strong contracting properties.

Contents

1. *Introduction*

CAT(0) cube complexes are fascinating objects of study, thanks in part to the interplay between two metrics that they naturally admit, the CAT(0) metric, and the median metric. Restricted to each cube, these coincide either with the standard Euclidean metric (ℓ^2) or with the 'taxi-cab' metric (ℓ^1). Somewhat recently, CAT(0) cube complexes played a crucial roll in Agol's proof of the virtual Haken conjecture (an outstanding problem in the theory of 3-manifolds which relied in an essential way on the work of Wise) [**Ago13**, **BW12**, **HW08**, **KM12**, **Wis09**]. The class of CAT(0) cube complexes and groups acting nicely on them include trees, (universal covers of) Salvetti complexes associated with right-angled Artin groups, Coxeter groups, small cancellation groups and is closed under taking finite products.

Associated with a random walk on a group one has the Furstenberg–Poisson boundary. It represents, in some sense, the limits of the trajectories of the random walk. Its existence, as an abstract measure space, for a generating random walk is guaranteed by the seminal result of Furstenberg [**Fur73**]. This important object has since established itself as an integral part in the study of rigidity (see for example [**BF14**]) in particular by realizing it as a geometric boundary of the group in question.

One may associate with any CAT(0) space a visual boundary where each point is an equivalence class of geodesic rays. The visual boundary for a CAT(0) space gives a compactification of the space, at least when the space is locally compact [**BH99**]. For a wide class of hyperbolic groups, and more generally, certain groups acting on CAT(0) spaces, the visual boundary is a Furstenberg–Poisson boundary for suitably chosen random walks [**Kai94**, **KM99**].

The wall metric naturally leads to the Roller compactification of a CAT(0) cube complex. Nevo and Sageev show that the Roller boundary (see §2.3) can be made

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to be a Furstenberg–Poisson boundary for a group Γ when the group admits a non-elementary proper co-compact action on X [**NS13**]. The purpose of this paper is to give a generalization of this result to groups which admit a non-elementary proper action on a finite dimensional CAT(0) cube complex. The complex is not assumed to be locally compact, and in particular, the action is not required to be co-compact. Our approach will be somewhat different to that of Nevo and Sageev and in particular, we shall not address several of the dynamical questions that they consider: for example that the resulting stationary measure is unique, or that the action is minimal or strongly proximal. Such questions will be examined in a forthcoming paper by Lécureux, Mathéus, and the present author.

Let μ be a probability measure on a discrete countable group Γ. Assume that it is generating, i.e. that the semi-group generated by the support of μ is the whole of Γ. Recall that a probability measure μ on Γ is said to have finite entropy if

$$H(\mu) := -\sum_{\gamma\in\Gamma} \mu(\gamma)\log\mu(\gamma) < \infty.$$

Also, if $|\cdot| : \Gamma \to \mathbb{R}$ is a pseudonorm on Γ then μ is said to have finite first logarithmic moment (with respect to $|\cdot|$) if $\sum_{\gamma\in\Gamma}\mu(\gamma)\log|\gamma| < \infty$. (See §8.1 for more details.) If we have an action of Γ on X, then fixing a basepoint $o \in X$ allows us to consider the pseudonorm defined by $|\gamma|_o := d(\gamma o, o)$.

MAIN THEOREM. *Let X be a finite-dimensional CAT(0) cube complex, Γ a discrete countable group, $\Gamma \to \mathrm{Aut}(X)$ a non-elementary proper action by automorphisms on X, and μ a generating probability measure on Γ of finite entropy. If there is a base point $o \in X$ for which μ has finite first logarithmic moment then there exists a probability measure ϑ on the Roller boundary ∂X such that $(\partial X, \vartheta)$ is the Furstenberg–Poisson boundary for the μ-random walk on Γ. Furthermore, ϑ gives full measure to the regular points in ∂X.*

The proof of the Main Theorem follows a standard path. We first show that the Roller boundary is a quotient of the Furstenberg–Poisson boundary (§7) and then apply Kaimanovich's celebrated strip condition to prove maximality (§8).

We note that Karlsson and Margulis show that the visual boundary of a CAT(0) space is the Furstenberg–Poisson boundary for suitable random walks [**KM99**]. They assume very little about the space, but assume that the measure μ has finite first moment and that orbits grow at most exponentially. The Main Theorem above applies to the restricted class of spaces (i.e. CAT(0) cube complexes), which pays off by allowing for significantly weaker hypotheses on the action and the measure μ.

Observe that our Main Theorem applies for example to any non-elementary subgroup of a right-angled Artin group or more generally of a graph product of finitely generated abelian groups [**RW13**].

Furthermore, we remark on the importance of the fact that the regular points are of full measure: they exhibit strong contracting properties. This will be exploited to study random walks on CAT(0) cube complexes in the forthcoming paper of Lécureux, Mathéus, and the present author mentioned above.

An action on a CAT(0) cube complex is said to be Roller *non-elementary* if every orbit in the Roller compactification is infinite (see §2.3). This notion guarantees non-amenability of the closure of the acting group in $\mathrm{Aut}(X)$, and characterizes it for X locally compact. This Tits alternative, is essentially an encapsulation of results of Caprace and Sageev [**CS11**], Caprace [**CFI12**], and Chatterji, Iozzi, and the author [**CFI12**]. It also comes after several versions of Tits alternatives [**CS11**, **SW05**] (see §9 for more details).

THEOREM 1.1. (Tits alternative) *Let X be a finite-dimensional CAT(0) cube complex and $\Gamma \leqslant \mathrm{Aut}(X)$. Either Γ contains a freely acting free group or one of the following equivalent conditions holds:*
(1) *Γ preserves an interval $\mathcal{I} \subset \overline{X}$;*
(2) *the Γ-action is Roller elementary.*
If furthermore X is locally compact then these are equivalent to
(3) *the closure $\overline{\Gamma}$ in $\mathrm{Aut}(X)$ is amenable.*

2. *CAT(0) cube complexes and medians*

We shall say that a metric space is a *Euclidean cube* if there is an $n \in \mathbb{N}$ for which it is isometric to $[0, 1]^n$ with the standard induced Euclidean metric from $\mathbb{R}^n$.

Definition 2.1. A locally countable finite-dimensional simply-connected metric polyhedral complex X is a *CAT(0) cube complex* if the closed cells are Euclidean cubes, the gluing maps are isometries and the link of each vertex is a flag complex.

Recall that a *flag complex* is a simplicial complex in which each complete subgraph on $(k + 1)$-vertices is the 1-skeleton of a k-simplex in the complex. That the link of every vertex is a flag complex is equivalent to the condition of being locally CAT(0), thanks to Gromov's link condition.

We remark that we absorb the condition of finite dimensionality in the definition of a CAT(0) cube complex and, as such, we shall not explicitly mention it in what follows. Furthermore, if the dimension of the CAT(0) cube complex is D, then this is equivalent to the existence of a maximal dimensional cube of dimension D.

A *morphism* between two CAT(0) cube complexes is an isometry that preserves the cubical structures, i.e. it is an isometry $f : X \to Y$ such that $f(C)$ is a cube in Y whenever C is a cube in X. We denote by $\mathrm{Aut}(X)$ the group of automorphisms of X to itself.

2.1. *Walled spaces.*

A *space with walls* or a *walled space* is a set S together with a countable collection of non-empty subsets $\mathfrak{H} \subset 2^S$ called half-spaces with the following properties.

(1) There is a fixed-point free involution $* : \mathfrak{H} \to \mathfrak{H}$
$$h \mapsto h^* := S \backslash h.$$
(2) The collection of half-spaces separating two points of S is finite, i.e. for every $p, q \in S$ the set of half-spaces $h \in \mathfrak{H}$ such that $p \in h$ and $q \in h^*$ is finite.
(3) There is a $D \in \mathbb{N}$ such that for every collection of pairwise transverse half-spaces $\{h_1, \ldots, h_n\}$ we must have that $n \leqslant D$.

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A pair of half-spaces $h, k \in \mathfrak{H}$ is said to be *transverse* if the following four intersections are all non-empty:

$$h \cap k, \quad h \cap k^*, \quad h^* \cap k^*, \quad h^* \cap k.$$

Associated with a walled space is the wall pseudo-metric $d : S \times S \to \mathbb{R}$:

$$d(p, q) = \frac{1}{2}\#(\{h \in \mathfrak{H} : p \in h, q \in h^*\} \cup \{h \in \mathfrak{H} : q \in h, p \in h^*\}).$$

This satisfies the properties of a metric, with the exception that $d(p, q) = 0$ does not necessarily imply that $p = q$.

Let us then consider the associated quotient $S_\sim$ consisting of equivalence classes of points of S whose pseudo-wall distance is 0. Clearly, the wall pseudo-metric descends to a metric on $S_\sim$.

For $h \in \mathfrak{H}$ the *wall* associated with h is the unordered pair $\{h, h^*\}$. This explains the terminology, as well as the factor of $\frac{1}{2}$ in the definition of the (pseudo-)wall metric.

2.2. *CAT(0) cube complexes as walled spaces.* As we shall now see, CAT(0) cube complexes naturally admit a walled (pseudo-)metric and are in some sense the unique examples of such spaces.

Let $[0, 1]^n$ be an n-dimensional cube. The ith coordinate projection is denoted by $\mathrm{pr}_i : [0, 1]^n \to [0, 1]$. A *wall* of a cube $[0, 1]^n$ is the set $\mathrm{pr}_i^{-1}\{1/2\}$. Observe that the complement of each wall in a cube has two connected components.

Definition 2.2. A *wall* of a CAT(0) cube complex X is a convex subset whose intersection with each cube is either a wall of the cube or empty.

The complement of a wall in a CAT(0) cube complex has two connected components [**Sag95**, Theorem 4.10] which we call half-spaces and we denote them by $\mathfrak{H}(X)$. Observe that since X is second countable, there are countably many half-spaces in $\mathfrak{H}(X)$.

The notation and terminology here is purposefully chosen to remind the reader of a walled space. Indeed, in essence, a walled space uniquely generates a CAT(0) cube complex [**CN05**, **Nic04**, **Sag95**]. And it is this walled space structure of the CAT(0) cube complex that we shall ultimately be interested in, if not fascinated by. Since walls separate points in the zero-skeleton of a CAT(0) cube complex, we shall in fact consider the zero-skeleton as our object of study.

Let X_0 denote the vertex set of X and $\mathfrak{H}(X_0) = \{h \cap X_0 : h \in \mathfrak{H}(X)\}$. This yields a fixed-point free involution $* : \mathfrak{H}(X_0) \to \mathfrak{H}(X_0)$,

$$h_0 \mapsto h_0^* := X_0 \backslash h_0. \tag{1}$$

One drawback of passing to the zero-skeleton is that a wall is no longer a subset of X_0. Therefore, for $h_0 \in \mathfrak{H}(X_0)$, we shall denote by $\hat{h}_0$ the pair $\{h_0, h_0^*\}$ and think of it as a wall, as in §2.1.

THEOREM 2.3. [**CN05**, **Nic04**, **Sag95**] *Let $(S, \mathfrak{H})$ be a walled metric space. Then, there exists a CAT(0) cube complex X and an embedding $\iota : S \hookrightarrow X_0$ such that:*

(1) *if S and X_0 are endowed with their respective wall metrics then ι is an isometry onto its image;*

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(2) *the set map induced by ι is a bijection $\mathfrak{H} \to \mathfrak{H}(X_0)$, $h \mapsto k$ such that*

$$k \cap \iota(S) = \iota(h);$$

(3) *if $\gamma : S \to S$ is a wall-isometry then there exists a unique extension to an automorphism $\gamma_0 : X_0 \to X_0$ that agrees with γ on $\iota(S)$.*

Furthermore, if $(X_0, \mathfrak{H}(X_0))$ is the walled space associated with the vertex set of a CAT(0) cube complex X, then the above association applied to $(S, \mathfrak{H}) = (X_0, \mathfrak{H}(X_0))$ yields once more X, and $\iota : X_0 \to X_0$ can be taken to be the identity, and the induced homomorphism $\mathrm{Aut}(X_0) \to \mathrm{Aut}(X_0)$ *is the identity isomorphism.*

When a collection of half-spaces $\mathfrak{H}$ is given, we shall denote the associated CAT(0) cube complex as $X(\mathfrak{H})$, leading to the somewhat abusive formulation of the last part of Theorem 2.3:

$$X(\mathfrak{H}(X_0)) = X.$$

This striking result shows that the combinatorial information of the wall structure completely captures the geometry of the CAT(0) cube complex. This will be exploited in what follows. To this end, we now set $X = X_0$, and $\mathfrak{H} = \mathfrak{H}(X_0)$. Unless otherwise stated, every metric property will be taken with respect to the wall metric.

The first of many beautiful properties of CAT(0) cube complexes is a type of Helly's theorem, as follows.

THEOREM 2.4. [**Rol**] *Let $h_1, \ldots, h_n \in \mathfrak{H}$ be half-spaces. If $h_i \cap h_j \neq \varnothing$ then*

$$\bigcap_{i=1}^{n} h_i \neq \varnothing.$$

Keeping with the terminology of transverse half-spaces introduced in §2.1, if $h_1, \ldots, h_n \in \mathfrak{H}$ are pairwise transverse half-spaces then $n \leqslant D$.

2.3. *Sageev–Roller duality.* Given a subset $\mathfrak{s} \subset \mathfrak{H}$ of halfspaces, we denote by $\mathfrak{s}^*$ the collection $\{h^* : h \in \mathfrak{s}\}$. We say that $\mathfrak{s}$ satisfies:

(1) the *totality* condition if $\mathfrak{s} \cup \mathfrak{s}^* = \mathfrak{H}$;

(2) the *consistency* condition if $\mathfrak{s} \cap \mathfrak{s}^* = \varnothing$ and if $h \in \mathfrak{s}$ and $h \subset k$, then $k \in \mathfrak{s}$.

Remark 2.5. We note that such sets are frequently called *ultrafilters* in the literature. However, as these are not the same as set theoretic ultrafilters, this can at times lead to some confusion. We therefore avoid using the term ultrafilters.

Fix $v \in X$ and consider the collection $U_v = \{h \in \mathfrak{H} : v \in h\}$. It is straightforward to verify that U_v satisfies both totality and consistency as a collection of half-spaces. The *Sageev–Roller duality* is then obtained via the following observation:

$$\bigcap_{h \in U_v} h = \{v\}.$$

This shows that if $w \in X$ then we have that

$$U_v = U_w \iff v = w,$$

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giving an embedding $X \hookrightarrow 2^{\mathfrak{H}}$ obtained by $v \mapsto U_v$. This embedding is made isometric by endowing $2^{\mathfrak{H}}$ with the extended metric†

$$d(A, B) = \frac{1}{2}\#(A \triangle B).$$

For now, let us consider $X \subset 2^{\mathfrak{H}}$. Then, the *Roller compactification* is denoted by $\overline{X}$ and is the closure of X in $2^{\mathfrak{H}}$. The *Roller boundary* is then $\partial X = \overline{X} \setminus X$. Observe that in general, while $\overline{X}$ is a compact space containing X as a dense subset, it is not a compactification in the usual sense. Indeed, unless X is locally compact, the embedding $X \hookrightarrow \overline{X}$ does not have an open image, and ∂X is not closed. This is best exemplified by taking the wedge sum of countably many lines. The limit of any sequence of distinct points in the boundary will be the wedge point. While it is also true that the visual boundary is not a compactification when X is not locally comapact, the Roller boundary does present one significant advantage: the union $X \sqcup \partial X$ is indeed compact.

With this notation in place, the partition $\{h, h^*\}$ extends to a partition of $\overline{X}$ and hence, when we speak of a half-space as a collection of points, we mean

$$h \subset \overline{X} = h \sqcup h^*.$$

Remark 2.6. Given $h \in \mathfrak{H}$, we denote the set $\{h, h^*\}$ by $\hat{h}$. By abuse of notation, for $k \in \mathfrak{H}$, we shall say that $\hat{h} \subset k$ if and only if $h \subsetneq k$ or $h^* \subsetneq k$. This is consistent with the standard notion of the wall corresponding to a *mid-cube*.

We now give characterizations of special types of subsets of $\overline{X}$. To this end, we say that $\mathfrak{s} \in 2^{\mathfrak{H}}$ satisfies the *descending chain condition* if every infinite descending chain of half-spaces is eventually constant.

Facts 1. The following are true for a non-empty $\mathfrak{s} \in 2^{\mathfrak{H}}$.

(1) If $\mathfrak{s}$ satisfies the consistency condition then

$$\varnothing \neq \bigcap_{h \in \mathfrak{s}} h \subset \overline{X}.$$

(2) If $\mathfrak{s}$ satisfies the consistency condition and the descending chain condition then

$$\varnothing \neq \left(\bigcap_{h \in \mathfrak{s}} h \right) \cap X.$$

(3) The collection $\mathfrak{s}$ satisfies both the totality and consistency conditions if and only if there exists $v \in \overline{X}$ such that $\mathfrak{s} = U_v$. Fixing $U_v \in 2^{\mathfrak{H}}$, we have that:
- $v \in X$ if and only if U_v satisfies the descending chain condition;
- $v \in \partial X$ if and only if U_v contains a non-trivial infinite descending chain, i.e. for each n there is an $h_n \in S$ such that $h_{n+1} \subsetneq h_n$.

Let us say a few words about why these facts are true, or where one can find proofs, though likely several proofs are available. In the case of Item (1), this is simple if one can show that the collection has the finite intersection property as $\overline{X}$ is compact. Furthermore, the CAT(0) cube complex version of Helly's Theorem 2.4 allows one to pass

† We note that this extended metric is not continuous unless $\mathfrak{H}$ is finite.

from finite intersections to pairwise intersections, and this last case is easy to verify given the condition of consistency. For the second item, we refer the reader to [**NS13**, Lemma 2.3]. Finally, for the last item, we refer the reader to [**Rol**].

There are also other special sets that will be of interest, as follows.

Definition 2.7. The collection of *non-terminating* elements is denoted by $\partial_{\mathrm{NT}} X$ and consists of the elements $v \in \partial X$ such that every finite descending chain can be extended, i.e. given $h \in U_v$ there is a $k \in U_v$ such that

$$k \subset h.$$

In general, it may be the case that $\partial_{\mathrm{NT}} X$ is empty. However, in a case where X admits a non-elementary action (see §3.1) then $\partial_{\mathrm{NT}} X$ is not empty [**CFI12**, **NS13**].

2.4. *The median.* The vertex set of a CAT(0) cube complex with the edge metric (equivalently with the wall metric) is a median space [**CN05**, **Nic04**, **Rol**]. The median structure extends nicely to the Roller compactification.

We define the interval

$$\mathcal{I}(v, w) := \{m \in \overline{X} : U_v \cap U_w \subset U_m\}.$$

In the special case where $v, w \in X$, this is the collection of vertices that are crossed by an edge geodesic connecting v and w.

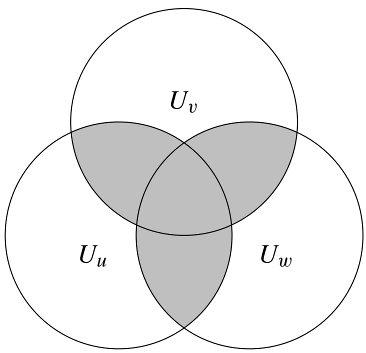
Then, the fact that $\overline{X}$ is a *median space*† is captured by the following: for every $u, v, w \in \overline{X}$ there is a unique $m \in \overline{X}$ such that

$$\{m\} = \mathcal{I}(u, v) \cap \mathcal{I}(v, w) \cap \mathcal{I}(w, u).$$

This unique point is called the *median* of u, v, and w and will sometimes be denoted by $m(u, v, w)$. In terms of half-spaces, we have

$$U_m = (U_u \cap U_v) \cup (U_v \cap U_w) \cup (U_w \cap U_u),$$

which is captured by this beautiful Venn diagram:



† A median space is usually required to satisfy the condition that intervals are finite. However, we weaken this assumption here in order to extend the notion to the Roller compactification.

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While general CAT(0) cube complexes can be quite wild†, the structure of intervals is tamable by the following theorem.

THEOREM 2.8. [**BCG⁺09**, Theorem 1.16] *Let $v, w \in \overline{X}$. Then the vertex interval $\mathcal{I}(v, w)$ isometrically embeds into $\overline{\mathbb{Z}}^D$ (with the standard cubulation) where D is the dimension of X.*

The proof of this employs Dilworth's theorem, which states that a partially ordered set has finite width D if and only if it can be partitioned into D-chains. Here, the partially ordered set is $U_w \backslash U_v$. Set inclusion yields the partial order and an antichain corresponds to a set of pairwise transverse half-spaces. By reversing the chains of half-spaces in $U_w \backslash U_v$ in a consistent way, we may find other pairs $x, y \in \overline{X}$ such that $\mathcal{I}(x, y) = \mathcal{I}(v, w)$. This yields the following corollary.

COROLLARY 2.9. *If X has dimension D, then for any interval $I \subset \overline{X}$, there are at most 2^D elements on which I is an interval.*

2.5. *Projections and lifting decompositions.* It is straightforward, thanks to Theorem 2.3, to deduce that if $\mathfrak{H}' \subset \mathfrak{H}$ is an involution invariant subset then there is a natural quotient map $X(\mathfrak{H}) \to X(\mathfrak{H}')$. Furthermore, if $\mathfrak{H}'$ is Γ-invariant for some acting group Γ, then the quotient is Γ-equivariant as well. One can ask to what extent this can be reversed. Namely, when is it possible to find an embedding $X(\mathfrak{H}') \hookrightarrow X(\mathfrak{H})$? And if $\mathfrak{H}'$ is assumed to be Γ-invariant, can the embedding be made to be Γ-equivariant?

Definition 2.10. Given a subset $\mathfrak{H}' \subset \mathfrak{H}(X)$, a *lifting decomposition* is a choice of a consistent subset $\mathfrak{s} \subset \mathfrak{H}(X)$ such that

$$\mathfrak{H}(X) = \mathfrak{H}' \sqcup (\mathfrak{s} \sqcup \mathfrak{s}^*).$$

We note that a necessary condition for the existence of a lifting decomposition is that $\mathfrak{H}'$ be involution invariant and that it be convex, i.e. if $h, k \in \mathfrak{H}'$, and $h \subset \ell \subset k$ then $\ell \in \mathfrak{H}'$.

Given a consistent set $\mathfrak{s} \subset \mathfrak{H}(X)$, one can associate a set of walls (viewed as an involution invariant set of half-spaces) $\mathfrak{H}_{\mathfrak{s}} := \mathfrak{H}(X) \backslash (\mathfrak{s} \sqcup \mathfrak{s}^*)$ such that $\mathfrak{s}$ is *a* lifting decomposition of $\mathfrak{H}_{\mathfrak{s}}$, though there could of course be others.

The terminology is justified by the following proposition.

PROPOSITION 2.11. [**CFI12**, Lemma 2.6] *The following are true.*

- *Suppose that $\mathfrak{H}' \subset \mathfrak{H}(X)$. If there exists a lifting decomposition $\mathfrak{s}$ for $\mathfrak{H}'$, then there is an isometric embedding $\overline{X}(\mathfrak{H}') \hookrightarrow \overline{X}$ induced from the map $2^{\mathfrak{H}'} \hookrightarrow 2^{\mathfrak{H}(X)}$, where $U \mapsto U \sqcup \mathfrak{s}$ and the image of this embedding is*

$$\bigcap_{h \in \mathfrak{s}} h \subset \overline{X}.$$

† Indeed, if T_∞ is the tree of countably infinite valency, then the stabilizer group stab(v) of any vertex $v \in T_\infty$ contains every discrete countable group.

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- *Conversely, if* $\mathfrak{s} \subset \mathfrak{H}(X)$ *is a consistent set of half-spaces, then, setting* $\mathfrak{H}_{\mathfrak{s}} = \mathfrak{H}(X) \backslash (\mathfrak{s} \sqcup \mathfrak{s}^*)$ *we get an isometric embedding* $\overline{X}(\mathfrak{H}_{\mathfrak{s}}) \hookrightarrow \overline{X}$ *obtained as above, onto*

$$\bigcap_{h \in \mathfrak{s}} h \subset \overline{X}.$$

- *If* $\mathfrak{s}$ *satisfies the descending chain condition, then the image of* $X(\mathfrak{H}_{\mathfrak{s}})$ *is in* X.

Furthermore, if the set $\mathfrak{s}$ *is* Γ*-invariant then, with the restricted action on the image, the above natural embeddings are* Γ*-equivariant.*

Remark 2.12. We note that the projection $X \to X(\mathfrak{H}')$ obtained by forgetting the half-spaces $\mathfrak{H} \backslash \mathfrak{H}'$ is onto. This means that if there is a lifting decomposition $\overline{X}(\mathfrak{H}') \hookrightarrow \overline{X}$ then the relationship between two half-spaces (i.e. facing, transverse, etc.) is equivalent if one considers them as half-spaces in X or in $X(\mathfrak{H}')$.

Let us interpret the significance of Proposition 2.11 in the context of the collection of the involution-invariant set of half-spaces $\mathfrak{H}(v, w) := U_v \triangle U_w$, for $v, w \in \overline{X}$. These are the half-spaces separating v and w. Then, the collection of half-spaces $\mathfrak{H}(v, w)^+ := U_v \cap U_w$, i.e. those that contain both v and w, is a consistent set of half-spaces and it is straightforward to verify that $\mathfrak{H}(v, w)^+$ is a lifting decomposition for $\mathfrak{H}(v, w)$, yielding an isometric embedding of the CAT(0) cube complex associated with $\mathfrak{H}(v, w)$ onto $\mathcal{I}(v, w)$.

3. *Three key notions*

There are three notions that together form a powerful framework within which to study CAT(0) cube complexes. The first is the classical notion of a non-elementary action. Caprace and Sageev showed that this allows one to study the *essential core* of a CAT(0) cube complex [**CS11**], which is the second notion. Finally, Behrstock and Charney introduced the notion of strong separation which allows for the local detection of irreducibility [**BC12**], which was shown by Caprace and Sageev to be available in the non-elementary setting [**CS11**].

3.1. *Non-elementary actions.*

As a CAT(0) space, a CAT(0) cube complex has a visual boundary $\partial_{\triangleleft} X$ which is obtained by considering equivalence classes of geodesic rays, where two rays are equivalent if they are at bounded distance from each other. The topology on $\partial_{\triangleleft} X$ is the cone topology (which coincides with the topology of uniform convergence on compact subsets, when one considers geodesic rays emanating from the same base point) [**BH99**]. While the visual boundary is not well behaved for non-proper spaces in general, the assumption that the space is finite dimensional is sufficient [**CL10**].

Definition 3.1. An isometric action on a CAT(0) space is said to be *elementary* if there is a finite orbit in either the space or the visual boundary.

To exemplify the importance of this property, we have the following theorem.

THEOREM 3.2. [**CS11**] *Suppose* $\Gamma \to \mathrm{Aut}(X)$ *is an action on the CAT(0) cube complex* X. *Then either the action is elementary, or* Γ *contains a freely acting non-abelian free group.*

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3.2. *Essential actions.* Caprace and Sageev [**CS11**] showed that for non-elementary actions, there is a non-empty 'essential core' where the action is well behaved. Let us now develop the necessary terminology and recall the key facts.

Definition 3.3. Fix a group Γ acting by automorphisms on the CAT(0) cube complex X. A half-space $h \in \mathfrak{H}$ is called:

- Γ-*shallow* if for some (and hence all) $x \in X$, the set $\Gamma \cdot x \cap h$ is at bounded distance from h^*, otherwise, it is said to be Γ-*deep*;
- Γ-*trivial* if h and h^* are both shallow;
- Γ-*essential* if h and h^* are both deep;
- Γ-*half-essential* if it is deep and h^* is shallow.

Remark 3.4. Observe that the collections of essential and trivial half-spaces are both closed under involution and that the collection of half-essential half-spaces is consistent. Furthermore, a half-space $h \in \mathfrak{H}$ is Γ-essential if and only if it is Γ_0-essential for any $\Gamma_0 \leqslant \Gamma$ of finite index.

THEOREM 3.5. [**CS11**, Proposition 3.5] *Assume* $\Gamma \to \mathrm{Aut}(X)$ *is a non-elementary action on the CAT(0) cube complex X, then the collection of* Γ*-essential half-spaces is non-empty. Furthermore, if Y is the CAT(0) cube complex associated with the* Γ*-essential half-spaces, then Y is unbounded and there is a* Γ*-equivariant embedding* $Y \hookrightarrow X$.

The image of Y under this embedding is called the Γ-*essential core*. If all half-spaces are essential, then the action is said to be *essential*.

A simple but powerful concept introduced by Caprace and Sageev is that of flipping a half-space. A half-space $h \in \mathfrak{H}$ is said to be Γ-flippable if there is a $\gamma \in \Gamma$ such that $h^* \subset \gamma h$.

LEMMA 3.6. [**CS11**, Flipping lemma] *Assume* $\Gamma \to \mathrm{Aut}(X)$ *is non-elementary. If* $h \in \mathfrak{H}$ *is essential, then h is* Γ*-flippable.*

Recall that a measure λ is said to be *quasi*-Γ-*invariant* whenever the following holds for every $\gamma \in \Gamma$, and every measurable set E: if $\lambda(E) > 0$ then $\lambda(\gamma E) > 0$.

COROLLARY 3.7. [**CFI12**] *Suppose* $\Gamma \to \mathrm{Aut}(X)$ *is a non-elementary and essential action on the CAT(0) cube complex X. If* λ *is a quasi-*Γ*-invariant probability measure on* $\overline{X}$ *then* $\lambda(h) > 0$ *for every half space* $h \in \mathfrak{H}(X)$.

Proof. Let $h \in \mathfrak{H}$. Then $\lambda(h \sqcup h^*) = 1$, which means that either $\lambda(h) > 0$ or $\lambda(h^*) > 0$. If $\lambda(h^*) > 0$ then apply the flipping lemma (Lemma 3.6) and deduce that there is a $\gamma \in \Gamma$ such that $h^* \subsetneq \gamma h$ and hence $\lambda(\gamma h) \geqslant \lambda(h^*) > 0$. But of course, λ is Γ-quasi-invariant so $\lambda(h) > 0$. □

Another very important operation on half-spaces developed by Caprace and Sageev is the notion of double skewering.

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LEMMA 3.8. [**CS11**, Double skewering] *Suppose* $\Gamma \to \mathrm{Aut}(X)$ *is a non-elementary action on the CAT(0) cube complex* X. *If* $h \subsetneq k$ *are two essential half spaces, then there exists a* $\gamma \in \Gamma$ *such that*

$$\gamma k \subsetneq h \subsetneq k.$$

The following is almost a direct consequence of the definitions. The reader will find a more in-depth formulation in [**CS11**, Proposition 3.2].

LEMMA 3.9. *If* Γ *acts on the CAT(0) cube complex* X *and preserves a finite collection of half-spaces, then the* Γ*-action is either elementary or not essential.*

An action of Γ on X is said to be *Roller non-elementary* if there is no finite orbit in the Roller compactification. Of course, having a finite orbit in X is equivalent to having a fixed point, and so what distinguishes Roller non-elementary from visual non-elementary actions is the existence of finite orbits in the corresponding boundaries. Furthermore, (visual) non-elementary actions are necessarily Roller non-elementary, though the converse is false in general. One can take as an example the standard action of $\Gamma = \mathbb{Z} \times F_2$ on $\mathbb{Z} \times T$, where T is the standard Cayley tree. It is straightforward to see that this example is essential and elementary but not Roller elementary. On the other hand, if we set $\partial\mathbb{Z} = \{-\infty, \infty\}$, then both $\{\infty\} \times T$ and $\{-\infty\} \times T$ are Γ-invariant and non-elementary. This phenomenon is captured in the following proposition.

PROPOSITION 3.10. ([**CFI12**, Proposition 2.26], [**CS11**]) *Let* X *be a finite-dimensional CAT(0) cube complex and let* $\Gamma \to \mathrm{Aut}(X)$ *be an action on* X. *One of the following holds.*

(1) *The* Γ*-action is Roller elementary.*
(2) *There is a finite index subgroup* $\Gamma' < \Gamma$ *and a* Γ'*-invariant subcomplex* $\overline{X}' \hookrightarrow \overline{X}$ *associated with a* Γ'*-invariant* $\mathfrak{H}' \subset \mathfrak{H}(X)$ *on which the* Γ'*-action is non-elementary and essential.*

Moreover, if the action of Γ *is non-elementary on* X, *then* $X' \hookrightarrow X$ *and* X' *is the* Γ*-essential core.*

3.3. *Product structures.* A CAT(0) cube complex is said to be *reducible* if it can be expressed as a non-trivial product. Otherwise, it is said to be *irreducible*. A CAT(0) cube complex X with half-spaces $\mathfrak{H}$ admits a product decomposition $X = X_1 \times \cdots \times X_n$ if and only if there is a decomposition

$$\mathfrak{H} = \mathfrak{H}_1 \sqcup \cdots \sqcup \mathfrak{H}_n$$

such that if $i \neq j$ then $h_i \pitchfork h_j$ for every $(h_i, h_j) \in \mathfrak{H}_i \times \mathfrak{H}_j$ and X_i is the CAT(0) cube complex on half-spaces $\mathfrak{H}_i$.

Remark 3.11. This means that an interval in the product is the product of the intervals. Namely if $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \overline{X}_1 \times \cdots \times \overline{X}_n$ then

$$\mathcal{I}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \mathcal{I}(x_1, y_1) \times \cdots \times \mathcal{I}(x_n, y_n).$$

The irreducible decomposition is unique (up to permutation of the factors) and $\mathrm{Aut}(X)$ contains $\mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_n)$ as a finite index subgroup. Therefore, if Γ acts on X

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by automorphisms, then there is a subgroup of finite index which preserves the product decomposition [**CS11**, Proposition 2.6].

We take the opportunity to record here that the Roller boundary is incredibly well behaved when it comes to products:

$$\partial X = \bigcup_{i=1}^{n} \overline{X_1} \times \cdots \times \overline{X_{j-1}} \times \partial X_j \times \overline{X_{j+1}} \times \cdots \times \overline{X_n}.$$

While the definition of (ir)reducibility for a CAT(0) cube complex in terms of its half-space structure is already quite useful, its global character makes it at times difficult to implement. Behrstock and Charney developed an incredibly useful notion for the Salvetti complexes associated with right-angled Artin groups, which was then extended by Caprace and Sageev.

Definition 3.12. [**BC12**] Two half-spaces $h, k \in \mathfrak{H}$ are said to be *strongly separated* if there is no half-space which is simultaneously transverse to both h and k. For a subset $\mathfrak{H}' \subset \mathfrak{H}$ we shall say that $h, k \in \mathfrak{H}'$ are *strongly separated in* $\mathfrak{H}'$ if there is no half-space in $\mathfrak{H}'$ which is simultaneously transverse to both h and k.

The following is proved in [**BC12**] for (the universal cover of) the Salvetti complex of non-abelian RAAGs.

THEOREM 3.13. [**CS11**] *Let X be a finite-dimensional irreducible CAT(0) cube complex such that the action of* $\mathrm{Aut}(X)$ *is essential and non-elementary. Then X is irreducible if and only if there exists a pair of strongly separated half-spaces.*

3.4. *Euclidean complexes.*

Definition 3.14. Let X be a CAT(0) cube complex. We say that X is *Euclidean* if the vertex set with the combinatorial metric embeds isometrically in $\mathbb{Z}^D$ with the ℓ^1-metric, for some $D < \infty$.

Our prime example of a Euclidean CAT(0) cube complex is an interval, which is the content of Theorem 2.8.

Definition 3.15. An n-tuple of half-spaces $h_1, \ldots, h_n \in \mathfrak{H}$ is said to be *facing* if, for each $i \neq j$,

$$h_i^* \cap h_j^* = \varnothing.$$

As an obstruction to when a CAT(0) cube complex is Euclidean, there is the following lemma.

LEMMA 3.16. [**CFI12**, Lemma 2.33] *If X is a Euclidean CAT(0) cube complex that isometrically embeds into $\mathbb{Z}^D$, then any set of pairwise facing halfspaces has cardinality at most $2D$.*

The following is an important characterization of when a complex is Euclidean.

COROLLARY 3.17. ([**CFI12**, Corollary 2.33], [**CS11**]) *Let X be an irreducible finite-dimensional CAT(0) cube complex so that the action of* $\mathrm{Aut}(X)$ *is essential and non-elementary. The following are equivalent:*

(1) *X is an interval;*
(2) *X is Euclidean;*
(3) *$\mathfrak{H}(X)$ does not contain a facing triple of half-spaces.*

Remark 3.18. The statement of [**CFI12**, Corollary 2.33] states that X is Euclidean if and only if $\mathfrak{H}(X)$ does not contain a facing triple. However, the proof actually shows that (2) implies (3) and (3) implies (1). The missing (1) implies (2) is of course provided by Theorem 2.8 [**BCG$^+$09**].

LEMMA 3.19. *Let X be an interval on $v, w \in \overline{X}$. Then* $\mathrm{Aut}(X)$ *is elementary.*

Proof. If X is an interval then the collection of points on which it is an interval is finite and bounded above by 2^D by Corollary 2.9. Let Γ_0 be the finite index subgroup which fixes this set pointwise and let v belong to this set. Then, for every finite collection $h_1, \ldots, h_n \in U_v$ the intersection $\bigcap_{i=1}^n h_i$ is not empty. Hence, the intersection of the visual boundaries corresponding to the h_i must be non-empty and its unique circumcenter is fixed for the Γ_0-action by [**CS11**, Proposition 3.6]. □

LEMMA 3.20. [**CFI12**, Lemma 2.28] *Let $\Gamma \to \mathrm{Aut}(X)$ be a non-elementary action. Then the Γ_0-action on the irreducible factors of the essential core is also non-elementary and essential, where Γ_0 is the finite index subgroup preserving this decomposition.*

We immediately deduce the following corollary (see also [**CFI12**, Corollary 2.34]).

COROLLARY 3.21. *If $\Gamma \to \mathrm{Aut}(X)$ is non-elementary then any irreducible factor in the essential core of X is not Euclidean and hence not an interval.*

3.5. *The combinatorial bridge.* Behrstock and Charney showed that the CAT(0) bridge connecting two strongly separated walls is a finite geodesic segment [**BC12**]. In [**CFI12**] this idea is translated to the 'combinatorial', i.e. median, setting. Most of what follows is from or adapted from [**CFI12**], though the notation differs slightly. Recall our convention that $\{k, k^*\}$ is denoted by $\hat{k}$, for a half-space k, and that given another half-space h we shall say that $\hat{k} \subset h$ if either k or k^* is a *proper* subset of h.

Remark 3.22. Observe that for two half-spaces h, k we have that $h \cap k$ and $h^* \cap k$ are both non-empty if and only if $h \pitchfork k$ or $\hat{h} \subset k$.

Definition 3.23. Let $h_1 \subsetneq h_2^*$ be a nested pair of halfspaces and $\beta(h_1, h_2)$ denote the collection of half-spaces $h \in \mathfrak{H}$ such that one of the following conditions holds:
(1) $\hat{h}_1 \subset h$ and $h_2 \pitchfork h$;
(2) $\hat{h}_2 \subset h$ and $h_1 \pitchfork h$;
(3) $\hat{h}_1, \hat{h}_2 \subset h$.
Furthermore, a half-space h will be said to be of types (1), (2), or (3) if it satisfies the corresponding property.

We note that both h_1 and h_2 are not of types (1)–(3). Furthermore, since h_1 and h_2 are disjoint, condition (1) actually means that $h_1 \subset h$ and $h_2 \pitchfork h$ (and analogously for condition (2)).

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LEMMA 3.24. *Given $h_1 \subsetneq h_2^*$, the collection $\beta(h_1, h_2)$ is consistent. Furthermore, $\beta(h_1, h_2)$ satisfies the descending chain condition.*

Proof. We begin by observing that if $h \in \beta(h_1, h_2)$ then we necessarily have that $h^* \notin \beta(h_1, h_2)$.

Now suppose that $h \in \beta(h_1, h_2)$ is of type (3). If $h \subset k$ then clearly k is also of type (3) and hence $k \in \beta(h_1, h_2)$.

Next suppose that h is of type (1) and $h \subset k$. Then $\hat{h}_1 \subset k$. Since $h_2 \pitchfork h$ and $h \subset k$ we have that $h_2 \cap k$ and $h_2^* \cap k$ are both non-empty. By Remark 3.22, either $k \pitchfork h_2$ or $\hat{h}_2 \subset k$, and so $k \in \beta(h_1, h_2)$.

Of course, a symmetric argument shows that if h is of type (2) and $h \subset k$ then $k \in \beta(h_1, h_2)$.

Next we turn to the question of the descending chain condition. Since there are finitely many half-spaces in between any two, an infinite descending chain will eventually fail to satisfy all three conditions (1) through (3). □

Definition 3.25. The (combinatorial) *bridge* between $h_1 \subsetneq h_2^*$ is denoted by $B(h_1, h_2)$ and corresponds to $\bigcap_{h\in\beta(h_1,h_2)} h \subset X$.

LEMMA 3.26. *Assume that $h_1 \subsetneq h_2^*$ and set $\beta = \beta(h_1, h_2)$. The collection $\mathfrak{H}' = \mathfrak{H} \backslash (\beta \sqcup \beta^*)$ consists of half-spaces h such that one of the following holds:*

- *$h \pitchfork h_1$ and $h \pitchfork h_2$;*
- *up to replacing h by h^* we have that*

$$h_1 \subseteq h \subseteq h_2^*.$$

Proof. It is clear that if h is a half-space that is transverse to both h_1 and h_2 then $h \notin \beta \sqcup \beta^*$. It is also clear that if $\hat{h} = \hat{h}_1$ or $\hat{h} = \hat{h}_2$ then $h \notin \beta \sqcup \beta^*$.

Now assume that $h_1 \subsetneq h \subsetneq h_2^*$. Then $h \supset \hat{h}_1$ and h does not contain nor is it transverse to $\hat{h}_2$ and hence $h \notin \beta \sqcup \beta^*$.

Conversely, suppose that $h \notin \beta \sqcup \beta^*$. If $h \pitchfork h_1$ then since h and h^* are not of type (2) we must have that $h \pitchfork h_2$. Assume then that h is not transverse to both h_1 and h_2.

If either $\hat{h} = \hat{h}_1$ or $\hat{h} = \hat{h}_2$ then up to replacing h by h^* we have that $h_1 \subseteq h \subseteq h_2^*$. Therefore, suppose that $\hat{h} \neq \hat{h}_1$ and $\hat{h} \neq \hat{h}_2$. Then, each of $\hat{h}_1$ and $\hat{h}_2$ is contained in h or h^*. Since both h and h^* are not of type (3), we must have, up to replacing h by h^*, that $\hat{h}_1 \subset h$ and $\hat{h}_2 \subset h^*$. Now, of course, since $h_1 \cap h_2 = \varnothing$, we conclude that $h_1 \subset h$ and $h_2 \subset h^*$, i.e.

$$h_1 \subset h \subset h_2^*.$$

□

COROLLARY 3.27. *Assume $h_1 \subset h_2^*$ are strongly separated. With the notation as in Lemma 3.26, β gives a lifting decomposition of $\mathfrak{H}'$. Furthermore, there exists a unique $x_i \in h_i$ such that*

$$B(h_1, h_2) = \mathcal{I}(x_1, x_2).$$

Proof. The fact that β is a lifting decomposition for

$$\mathfrak{H}' = \{h \in \mathfrak{H} : h_1 \subseteq h \subseteq h_2^* \text{ or } h_1 \subseteq h^* \subseteq h_2^*\}$$

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follows from Lemmas 3.24 and 3.26. In particular, $\mathfrak{H}'$ is precisely the set of half-spaces which separate points in $B(h_1, h_2)$.

Let us show that $B(h_1, h_2)$ is an interval. To this end, let $S_i = h_i \cap B(h_1, h_2)$. Since $h_i \in \mathfrak{H}'$ it follows that $S_i \neq \varnothing$.

Fixing i, suppose that $x, y \in S_i$. Then, any wall separating them must belong to $\mathfrak{H}'$. By Lemma 3.26 and the assumption that h_1 and h_2 are strongly separated (again replacing h by h^* if necessary), we see that $h_1 \subset h \subset h_2^*$. This means of course that $h_1 \cap h^* = \varnothing$ and hence $x = y$, i.e. S_i is a singleton, for both $i = 1, 2$.

Set $S_i = \{x_i\}$. Once more, since h_1 and h_2 are strongly separated, the collection $\mathfrak{H}'$ corresponds to half-spaces nested in between h_1 and h_2^* and hence $\mathfrak{H}' \subset \mathfrak{H}(x_1, x_2)$. Conversely, if $h \in \mathfrak{H}(x_1, x_2)$ then h separates the two points $x_1, x_2 \in B(h_1, h_2)$ and hence $h \in \mathfrak{H}'$. □

LEMMA 3.28. *Assume that $h_1 \subset h_2^*$ are strongly separated. If $\xi_i \in h_i \subset \overline{X}$, and $p \in B(h_1, h_2)$, then*

$$m(\xi_1, p, \xi_2) = p.$$

Proof. Let $m = m(\xi_1, p, \xi_2)$. Recall that m is uniquely determined by

$$U_m = (U_{\xi_1} \cap U_p) \cup (U_p \cap U_{\xi_2}) \cup (U_{\xi_2} \cap U_{\xi_1}),$$

and so we must show that if $h \in U_{\xi_2} \cap U_{\xi_1}$ then $h \in U_p$. In fact, we shall show that if $\xi_1, \xi_2 \in h$ then $h \in \beta(h_1, h_2) \subset U_p$.

By assumption $\xi_i \in h \cap h_i \neq \varnothing$. Furthermore, since h_1 and h_2 are strongly separated, h cannot be transverse to both h_1 and h_2. Suppose that h is parallel to h_2. Since $\xi_2 \in h_2 \cap h$ and $\xi_1 \in h_2^* \cap h$, by Remark 3.22 we have that $h \supset \hat{h}_2$. The same argument shows that either h is transverse to h_1 or contains $\hat{h}_1$ and therefore $h \in \beta(h_1, h_2)$. □

3.6. *More consequences.*

LEMMA 3.29. [**CFI12**, Lemma 2.28] *Suppose that $\Gamma \to \mathrm{Aut}(X)$ is a non-elementary and essential action, with X irreducible.*

- *If $h \in \mathfrak{H}$ then there exists $\gamma, \gamma' \in \Gamma$ such that the following are pairwise strongly separated*

$$\gamma h \subset h \subset \gamma' h.$$

- *In each orbit, there are n-tuples of facing and pairwise strongly separated half-spaces.*

LEMMA 3.30. *Let X be an irreducible CAT(0) cube complex with a non-elementary and essential Γ-action. Let $h \in \mathfrak{H}$ and $n \in \mathbb{Z}$ with $n \geqslant 2$. Then, there exists an n-tuple $\{k_1, \ldots, k_n\}$ contained in a single Γ-orbit consisting of facing and pairwise strongly separated half-spaces such that*

$$\hat{h} \in \bigcap_{i=1}^{n} k_i.$$

Proof. Fix $h \in \mathfrak{H}$. For $n = 2$ we take $k_1 = \gamma h^*$ and $k_2 = \gamma' h$ as in Lemma 3.29.

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Now, assume $n > 2$. Let $\{b_1, \ldots, b_{n+1}\}$ be the collection of facing and pairwise strongly separated half-spaces guaranteed by Item (2) of Lemma 3.29. For each $i = 1, \ldots, n+1$ exactly one of the following possibilities holds:

(a) $\hat{h} = \hat{b}_i$;
(b) $h \pitchfork b_i$;
(c) $\hat{h} \subset b_i^*$;
(d) $\hat{h} \subset b_i$.

Furthermore, since the collection is strongly separated and facing, there is at most one i, assume it is $i = n+1$, for which the mutually exclusive items (a) through (c) can occur. Therefore, we have that

$$\hat{h} \subset \bigcap_{i=1}^{n} b_i.$$

Finally, if the constructed set does not belong to the same orbit, one may skewer and flip to assure that they do belong to the same orbit yielding the desired collection. □

LEMMA 3.31. *Suppose that X is an irreducible CAT(0) cube complex with a non-elementary and essential action of the group Γ. Any non-empty subset $\mathfrak{H}' \subset \mathfrak{H}$ verifying the following properties must be equal to $\mathfrak{H}$:*

- (symmetric): $(\mathfrak{H}')^* = \mathfrak{H}'$*;*
- (Γ-invariant): $\Gamma \cdot \mathfrak{H}' = \mathfrak{H}'$*;*
- (convex): *if $h, h' \in \mathfrak{H}'$ with $h \subset h'$ and $k \in \mathfrak{H}$ such that $h \subset k \subset h'$ then $k \in \mathfrak{H}'$.*

Proof. Since X is irreducible, and $\mathfrak{H}'$ is non-empty and Γ-invariant, we can apply Lemmas 3.29 and 3.8 to obtain a bi-infinite sequence of pairwise strongly separated half-spaces $\{h_n : n \in \mathbb{Z}\} \subset \mathfrak{H}'$ with $h_{n+1} \subset h_n$.

Let $k \in \mathfrak{H}$. Then, there is at most one element of $\{h_n : n \in \mathbb{Z}\}$ which is transverse to k. This means that there is an $N \in \mathbb{Z}$ for which $\hat{k} \subset h_{N+2}^* \cap h_N$. Since $\mathfrak{H}'$ is symmetric and convex, we conclude that $k \in \mathfrak{H}'$. □

COROLLARY 3.32. *Assume we have an essential and non-elementary action of Γ on X, and $\Gamma_0 \leqslant \Gamma$ of finite index. If $\mathfrak{H}' \subset \mathfrak{H}$ is a non-empty symmetric convex Γ_0-invariant collection of half-spaces, then either $\mathfrak{H}' = \mathfrak{H}$ or $X \cong X' \times X''$ and $\mathfrak{H}'$ is the half-space structure for X'.*

4. *The Furstenberg–Poisson boundary*

We now assume that Γ is a discrete countable group.

The interested reader should consult the following references for further details [**BF14**, **BS06**, **CFI12**, **Fur02**, **Kai03**]. This exposition follows closely these sources, as well as a nice series of lectures by Uri Bader at CIRM in the winter of 2014.

Definition 4.1. Consider a measurable action $\alpha : \Gamma \times M \to M$ of the group Γ on the measure space (M, m) and μ a measure on Γ. The *convolution* as a measure on M is the pushforward under the action map of the product measure from $\Gamma \times M$:

$$\mu * m = \alpha_*(\mu \otimes m).$$

We shall make use of the following elementary fact.

LEMMA 4.2. *Let* Haar *denote the counting measure on* Γ*,* δ_e *the Dirac measure at the identity* $e \in \Gamma$ *and* $\mu \in \mathcal{P}(\Gamma)$ *be a probability measure. Then* $\mathrm{Haar} * \mu = \mathrm{Haar}$*, and* $\delta_e * \mu = \mu = \mu * \delta_e$*.*

Proof. Let us show that $\mathrm{Haar} * \mu(\gamma) = 1$ for every $\gamma \in \Gamma$. Indeed,

$$\begin{aligned}\mathrm{Haar} * \mu(\gamma) &= \mathrm{Haar} \otimes \mu\{(\gamma_0, \gamma_1) : \gamma_0\gamma_1 = \gamma\} \\ &= \sum_{\gamma_1 \in \Gamma} \mathrm{Haar}(\gamma\gamma_1^{-1})\mu(\gamma_1) \\ &= \sum_{\gamma_1 \in \Gamma} \mu(\gamma_1) = 1.\end{aligned}$$

A similar calculation shows that $\delta_e * \mu = \mu = \mu * \delta_e$. □

Definition 4.3. A probability measure $\mu \in \mathcal{P}(\Gamma)$ is said to be *generating* if for every $\gamma \in \Gamma$ there are $h_i \in \mathrm{supp}(\mu)$ such that $\gamma = h_1 \cdots h_n$, i.e. the support of μ generates Γ as a semigroup.

Given a generating measure μ, we shall associate two spaces with the μ-random walk, the space of increments and the path space. As sets, these two spaces will be the same, but the measures on them will be different.

Let $\Gamma^{\mathbb{N}} = \{\overline{\omega} = (\omega_n)_{n \geqslant 1} : \omega_n \in \Gamma\}$. The measure μ on Γ naturally induces a measure $\mu^{\mathbb{N}}$ on $\Gamma^{\mathbb{N}}$ which assigns measure $\mu(g_1) \cdots \mu(g_n)$ to the cylinder set

$$C_{i_1,\ldots,i_n}(g_1, \ldots, g_n) = \{\overline{\omega} \in \Gamma^{\mathbb{N}} : \omega_{i_j} = g_j \text{ for } j = 1, \ldots, n\}.$$

Let $\Omega := \Gamma \times \Gamma^{\mathbb{N}} = \{(\omega_0, \omega_1, \ldots) : \omega_n \in \Gamma\}$. Given another measure θ on Γ, which is not assumed to be a probability measure, we can consider the associated measure $\theta \otimes \mu^{\mathbb{N}}$ on Ω. This is the *space of increments*, where we see the first factor as where to start the random walk (with distribution θ). We shall consider the action of Γ on Ω which is transitive on the first factor and trivial on the rest.

Next let $\Omega' = \Gamma^{\mathbb{N}}$. We shall consider the diagonal action of Γ on Ω'. Observe that there is a natural map $W : \Omega \to \Omega'$, $(\omega_0, \omega_1, \omega_2, \ldots) \mapsto \overline{\omega}'$ where the nth component of the image is given by

$$\omega'_n = \omega_0\omega_1\omega_2 \cdots \omega_{n-1}.$$

With the actions of Γ defined above on Ω and Ω' we note that W is Γ-equivariant. We think of the image of this map as the space of sample paths. Consider the time shift map

$$S : \Omega \to \Omega;\ (\omega_0, \omega_1, \omega_2, \ldots) \mapsto (\omega_0\omega_1, \omega_2, \omega_3, \ldots),$$

which is just a composition of the standard action map $\Gamma \times \Gamma \to \Gamma$ given by $(\omega_0, \omega_1) \mapsto \omega_0\omega_1$ with the time shift map

$$S' : \Omega' \to \Omega';\ (\omega'_0, \omega'_1, \omega'_2, \ldots) \mapsto (\omega'_1, \omega'_2, \omega'_3, \ldots).$$

With these definitions in place, we observe that $W \circ S = S' \circ W$.

Finally, applying Lemma 4.2, we deduce that

$$S_*(\mathrm{Haar} \otimes \mu^{\mathbb{N}}) = (\mathrm{Haar} * \mu) \otimes \mu^{\mathbb{N}} = \mathrm{Haar} \otimes \mu^{\mathbb{N}}$$

(i.e. that S preserves Haar $\otimes\, \mu^{\mathbb{N}}$) and

$$S_*(\delta_e \otimes \mu^{\mathbb{N}}) = (\delta_e * \mu) \otimes \mu^{\mathbb{N}} = \mu \otimes \mu^{\mathbb{N}}.$$

As it will be important below, we denote by $\mathbf{P} = \delta_e \otimes \mu^{\mathbb{N}}$ and $\mathbf{P}' = W_*\mathbf{P}$ the probability measures on Ω and Ω', respectively.

Definition 4.4. The space of ergodic components of the semi-group action generated by S' on the space of sample paths Ω' with measure class $W_*(\mathrm{Haar} \otimes \mu^{\mathbb{N}})$ is the *Furstenberg–Poisson boundary* for the μ-random walk on Γ and will be denoted by B. Define the *probability measure* ν on B to be the pushforward of $\mathbf{P}'$ under the natural projection $\Omega' \to B$.

Observing that $W_*(\mathrm{Haar} \otimes \mu^{\mathbb{N}})$ is Γ-invariant and that the action of Γ commutes with the semigroup-action of S', one sees that Γ must preserve the ergodic components of S', and hence the action of Γ descends to B.

Furthermore, S' preserves the measure $W_*(\mathrm{Haar} \otimes \mu^{\mathbb{N}})$ and $W_*(\delta_e \otimes \mu^{\mathbb{N}})$ is absolutely continuous with respect to $W_*(\mathrm{Haar} \otimes \mu^{\mathbb{N}})$ so that ν is well defined on B. Finally, the following calculation shows that $\mu * \nu = \nu$ and hence that ν is μ-stationary:

$$\begin{aligned} \mu * W_*(\delta_e \otimes \mu^{\mathbb{N}}) &= W_*((\mu * \delta_e) \otimes \mu^{\mathbb{N}}) \\ &= W_* S_*(\delta_e \otimes \mu^{\mathbb{N}}) \\ &= S'_* W_*(\delta_e \otimes \mu^{\mathbb{N}}). \end{aligned}$$

Definition 4.5. Let μ be a probability measure on Γ. A Γ-equivariant measurable quotient of the Furstenberg–Poisson boundary (B, ν) is called a (Γ, μ)-*boundary*.

5. *Isometric ergodicity*

Recall that a measurable action of Γ on a measure space is quasi-measure preserving if the image of a measure zero set is always measure zero. Also recall that a (quasi-measure-preserving) action of Γ on a measure space (E, ν) is said to be *ergodic* if any Γ-invariant Borel map $f : E \to \mathbb{R}$ is essentially constant.

The Furstenberg–Poisson boundary has very robust ergodicity properties. Bader and Furman have developed a general and powerful framework within which one can exploit these ergodicity properties in what is part of a great unification (and extension) program of previous super-rigidity results (see [**BF14**]).

Let (E, ν) be a Borel space on which the group Γ acts measurably and quasi-preserves the measure ν. Such a space will be called a *Lebesgue* Γ*-space*. We say that the Γ-action is *isometrically ergodic* if the following holds.

Let (M, d) be a separable metric space and $\Gamma \to \mathrm{Isom}(M, d)$ an action by isometries. If $f : E \to M$ is a Γ-equivariant map, then it is essentially constant.

We remark that, for an isometrically ergodic action, the existence of such a map f is equivalent to the existence of a Γ-fixed point in M.

Let $\mathcal{M}$ and V be standard Borel spaces. We say that a Borel map $q:\mathcal{M}\to V$ is *relatively metrizable* if there is a Borel map on the fibered product $d:\mathcal{M}\times_V\mathcal{M}\to[0,\infty)$ such that the restriction d_v to each fiber $M_v:=q^{-1}(v)$ is a separable metric. (Recall that the fibered product $\mathcal{M}\times_V\mathcal{M}$ is the subset of pairs in $(x, y)\in\mathcal{M}\times\mathcal{M}$ such that $q(x)=q(y)$.) Such a Borel map d is called a *relative metric* on $q:\mathcal{M}\to V$. Furthermore, a *relatively isometric action* of Γ on $q:\mathcal{M}\to V$ is a pair of q-compatible Borel-actions of Γ on $\mathcal{M}$ and V such that, as maps on fibers, each $\gamma\in\Gamma$ is an isometry; that is, if $v\in V$ and $x, y\in M_v$ then

$$d_{\gamma v}(\gamma x, \gamma y)=d_v(x, y).$$

Definition 5.1. Suppose B and B' are Lebesgue Γ-spaces. A Γ-equivariant Borel map $p:B'\to B$ is said to be *relatively isometrically ergodic* if for every relatively isometric action of Γ on $q:\mathcal{M}\to V$, and any (p, q)-compatible maps $u:B'\to\mathcal{M}$ and $\ell:B\to V$, there exists a map $f:B\to\mathcal{M}$ making the following diagram commute:

$$\begin{tikzcd} B' \arrow[r, "u"] \arrow[d, "p"'] & M \arrow[d, "q"] \\ B \arrow[r, "\ell"] \arrow[ur, dashed, "\exists f"] & V \end{tikzcd}$$

Remark 5.2. We note that if one replaces the target spaces B and V with the one point space $\{*\}$ then one will recover the notions defined above in ‘non-relative’ terms. Namely, with respect to the one point projection, relative metrizability is just metrizability; a relatively isometric action is just an isometric action; and relatively isometrically ergodic is just isometrically ergodic.

PROPOSITION 5.3. [**BF14**, Proposition 2.2] *Assume B, B', P, P' are measurable Γ-spaces, $p:B'\to B$ is relatively isometrically ergodic and there are measurable maps $\varphi:B\to P$ and $\varphi':B'\to P'$ such that $\varphi\circ p=q\circ\varphi'$. It follows that $q:P'\to P$ is also relatively isometrically ergodic. That is to say, the property of relative isometric ergodicity is closed under composition of Γ-maps.*

Definition 5.4. Let Γ be a locally compact second countable group. A pair (B_-, B_+) of Γ-Lebesgue spaces forms a *boundary pair* if the Γ-action on both $B_\pm$ are amenable and the projections $B_-\times B_+\to B_\pm$ are relatively isometrically ergodic.

Recall that if a Lebesgue Γ-space B is amenable (in the sense of Zimmer) then given a compact metrizable space on which Γ acts by homeomorphisms, there is a Γ-equivariant map $B\to\mathcal{P}(K)$.

The following is a strengthening of a result of Kaimanovich [**Kai03**]. We state it here for discrete countable groups and note that the same statement holds for a locally compact second countable group under the additional assumption that the measure μ is ‘spread out’.

THEOREM 5.5. [**BF14**, Theorem 2.7, Remark 2.4] *Let Γ be a discrete countable group and $\mu\in\mathcal{P}(\Gamma)$ a generating probability measure. Let (B_-, ν_-) and (B_+, ν_+) be the Furstenberg–Poisson boundaries for (Γ, μ) and $(\Gamma, \check{\mu})$, respectively. Then $B_-\times B_+$ is isometrically ergodic and (B_-, B_+) is a boundary pair for Γ and any of its lattices.*

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Of course, since we have stated the theorem for discrete countable groups, a lattice is necessarily a finite index subgroup.

As a direct consequence of these, we have the following corollary.

COROLLARY 5.6. *Let C be a countable set on which Γ acts by permutations and (P, ϑ) an isometrically ergodic Γ-space. There is a Γ-equivariant map $P \to C$ if and only if there is a Γ-fixed point in C. In particular, this holds for $P = \mathcal{P}_- \times \mathcal{P}_+$ where $(\mathcal{P}_-, \mathcal{P}_+)$ is a Γ-equivariant quotient of a Γ-boundary pair (B_-, B_+).*

Proof. We begin by putting a metric on C by setting $d(x_1, x_2) = 1 - \delta(x_1, x_2)$ for $x_1, x_2 \in C$. Clearly (C, d) is separable.

If there is a Γ-fixed point in C then clearly there is a Γ-equivariant measurable map $P \to C$. Conversely, suppose Γ is acting on C by permutations and hence by isometries. If there is a Γ-equivariant measurable map $P \to C$ then, by isometric ergodicity, there is a Γ-fixed point C. □

6. *Tools*

As we saw in the previous section, a key characteristic of the Furstenberg–Poisson boundary is that it is Zimmer amenable. This connects the study of boundary maps to the study of probability measures on the space of interest. We now develop some background and tools for this purpose. More specifically, we would like to understand a probability measure on the Roller compactification $\overline{X}$ from a geometric perspective.

6.1. *Measures on $\overline{X}$.*

Let $\mathcal{P}(\overline{X})$ denote the space of probability measures on $\overline{X}$. If $m \in \mathcal{P}(\overline{X})$ define

$$\begin{aligned} H_m &:= \{h \in \mathfrak{H}(X) : m(h) = m(h^*)\},\\ H_m^+ &:= \{h \in \mathfrak{H}(X) : m(h) > 1/2\},\\ H_m^- &:= \{h \in \mathfrak{H}(X) : m(h) < 1/2\},\\ H_m^\pm &:= \{h \in \mathfrak{H}(X) : m(h) \neq 1/2\}.\end{aligned}$$

LEMMA 6.1. [**CFI12**, Lemmas 4.6, 4.7] *The maps $\mathcal{P}(\overline{X}) \to 2^{\mathfrak{H}(X)}$ given by $m \mapsto H_m, H_m^+, H_m^-$ are* Aut(X)*-equivariant for the natural actions on $\mathcal{P}(\overline{X})$ and $2^{\mathfrak{H}(X)}$. Furthermore, for $m, m' \in \mathcal{P}(\overline{X})$ the following hold.*

(1) *There are no facing triples in H_m. If X is not Euclidean then H_m^+ has facing triples and in particular $H_m^+ \neq \varnothing$.*
(2) *The collection H_m is convex, involution invariant, and the associated complex $\overline{X}(H_m)$ is an interval.*
(3) *The collection of half-spaces H_m^+ is consistent and yields a lifting decomposition $\mathfrak{H}(X) = H_m \sqcup (H_m^+ \sqcup H_m^-)$ and corresponds to the subcomplex denoted by $\overline{X}_m \subset \overline{X}$. Therefore $\overline{X}_m \cong \overline{X}(H_m)$.*

Remark 6.2. Fix $m \in \mathcal{P}(\overline{X})$. By Proposition 2.11, it follows that $X_m \subset X$ whenever H_m^+ satisfies the descending chain condition. Furthermore, as in Remark 3.11, if X has an irreducible decomposition corresponding to $\mathfrak{H} = \mathfrak{H}_1 \sqcup \cdots \sqcup \mathfrak{H}_n$, then we have that $X_m \subset X$ whenever $H_m \cap \mathfrak{H}_i$ satisfies the descending chain condition for each $i = 1, \ldots, n$.

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6.2. *Strong separation and measures on* $\overline{X}$. As above, consider the product decomposition into irreducible factors $X = X_1 \times \cdots \times X_n$. Let $\mathcal{S}_i \subset \mathfrak{H}_i \times \mathfrak{H}_i$ denote the pairs of disjoint strongly separated half-spaces in X_i and let $\mathcal{S} = \mathcal{S}_1 \sqcup \cdots \sqcup \mathcal{S}_n$. With this notation in place, and recalling that an interval in a product is the product of the corresponding intervals (see Remark 3.11) we have the following easy generalization of [**CFI12**, Lemma 4.18].

LEMMA 6.3. *Let X be a CAT(0) cube complex with product decomposition corresponding to $\mathfrak{H} = \mathfrak{H}_1 \sqcup \cdots \sqcup \mathfrak{H}_n$. If $m \in \mathcal{P}(\overline{X})$ and $(H_m \times H_m) \cap \mathcal{S}_i \neq \varnothing$ for each $i = 1, \ldots, n$, then H_m^+ satisfies the descending chain condition.*

For completeness, we recall the proof from [**CFI12**].

Proof. Observe that H_m^+ satisfies the descending chain condition if and only if its intersection with $\mathfrak{H}_i$ also satisfies the descending chain condition for each i. Consider the pushforward μ_i of μ under the projection of $\overline{X} \to \overline{X}_i$. For $h_i \in \mathfrak{H}_i$, recall that we may consider h_i as a subset of $\overline{X}$ or a subset of $\overline{X}_i$, and so the definition of the pushforward gives $\mu(h_i) = \mu_i(h_i)$. We may therefore without loss of generality assume that X is irreducible.

Let $h, k \in \mathfrak{H}$ be a pair of strongly separated halfspaces in H_μ with $h \subset k$. There is the following decomposition

$$H_\mu^+ = P(h) \cup P(k), \tag{2}$$

where $P(h)$ and $P(k)$ are the subsets of H_μ^+ consisting of halfspaces parallel to h and k, respectively. Notice that, while $P(h)$ and $P(k)$ are not necessarily disjoint, their union is the whole of H_μ^+ since h and k are strongly separated.

Let $h_n \in H_\mu^+$ be a descending chain, i.e. $h_{n+1} \subset h_n$. We must show that the chain terminates. Up to passing to a subsequence if necessary, and relabeling h and k, we may assume that $h_n \in P(h)$ for all $n \in \mathbb{N}$.

Fix $n = 1$. Since $\mu(h_1) > 1/2$ and $\mu(h) = 1/2$, we cannot have that $h_1 \subset h$ or $h \subset h_1^*$. This leaves two cases, namely that $h \subset h_1$ or $h^* \subset h_1$.

Since there are only finitely many halfspaces, nested between any two, we must conclude that H_μ^+ satisfies the descending chain condition. □

LEMMA 6.4. *Assume X is an irreducible CAT(0) cube complex with the action of* $\mathrm{Aut}(X)$ *being non-elementary and essential. For every $m \in \mathcal{P}(\overline{X})$ there exist strongly separated half spaces $h, k \in \mathfrak{H}$ such that $h \subset k$ and $\hat{x} \in h^* \cap k$ for every $x \in H_m$.*

Proof. Fix $x_0 \in H_m$. Applying Lemma 3.30 for $n = 4$, we find $k_1, \ldots, k_4$ pairwise strongly separated facing half-spaces with

$$\hat{x}_0 \subset \bigcap_{i=1}^{4} k_i.$$

By Lemma 6.1, H_m does not contain facing n-tuples for $n \geqslant 3$, and therefore at most two of these belong to H_m, meaning that at least two belong to $H_m^{\pm}$. Up to relabeling, let us assume that $k_1^* \subset x_0 \subset k_2$, and so $k_1, k_2 \in H_m^+$.

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By Lemma 3.29 we find $h, k \in \mathfrak{H}$ such that h and k_1^* are strongly separated, k_2 and k are strongly separated, and

$$h \subset k_1^* \subset x_0 \subset k_2 \subset k.$$

We therefore have that

$$m(h) \leqslant m(k_1^*) < m(x_0) = 1/2 < m(k_2) \leqslant m(k),$$

which in particular means that $h \in H_m^-$ and $k \in H_m^+$.

Now let $x \in H_m$ be an arbitrary element. Since $m(k_2) > 1/2$ and $m(k_1^*) < 1/2$ it follows that $x \not\subset k_1^*$ and $k_2 \not\subset x$. This means that either $x \pitchfork k_1^*$ or $x \pitchfork k_2$ or $\hat{x} \subset k_1 \cap k_2$. Either way, we conclude that $\hat{x} \subset h^* \cap k$. □

LEMMA 6.5. *Let X be an irreducible, essential, and non-elementary CAT(0) cube complex. Let $m \in \mathcal{P}(\overline{X})$ and $E \subset \mathcal{P}(\overline{X})$ be a non-empty subset. Assume that $H_{m'}$ does not contain strongly separated pairs and $H_{m'} \cap H_m \neq \varnothing$ for every $m' \in E$. Then, there exists a strongly separated pair $h, k \in \mathfrak{H}$ such that $h \subset k$, and for every $x \in \bigcup_{m' \in E} H_{m'}$*

$$\hat{x} \subset h^* \cap k.$$

Proof. We begin by applying Lemma 6.4 to the measure m and find a strongly separated pair $h_0, k_0 \in \mathfrak{H}$ such that $h_0 \subset k_0$, and $\hat{x} \in h_0^* \cap k_0$ for every $x \in H_m$.

Now we apply Lemma 3.29, to find the following chain of pairwise strongly separated half-spaces:

$$h_2 \subset h_1 \subset h_0 \subset k_0 \subset k_1 \subset k_2.$$

Now, for each $m' \in E$, and for each $x \in H_m \cap H_{m'} \neq \varnothing$, up to replacing x by x^* if necessary,

$$h_2 \subset h_1 \subset h_0 \subset x \subset k_0 \subset k_1 \subset k_2.$$

This means that $m'(h_i) \leqslant 1/2$ and $m'(k_i) \geqslant 1/2$ for $i = 0, 1, 2$. Furthermore, since h_0 and h_1 are strongly separated, and $H_{m'}$ does not have strongly separated pairs, it follows that $h_1 \in H_{m'}^-$, and hence $h_2 \in H_{m'}^-$. Similarly, we conclude that $k_1, k_2 \in H_{m'}^+$.

Now, let $y \in H_{m'}$ be an arbitrary element. Since $m'(y) = m'(y^*) = 1/2 < m'(k_1)$ it follows that $k_1 \not\subset y$ and $k_1 \not\subset y^*$. Therefore, either $y \pitchfork k_1$ or $\hat{y} \subset k_1$, and by strong separation of k_1 and k_2 we conclude that $\hat{y} \subset k_2$. The same argument applies to $h_1^*, h_2^* \in H_{m'}^+$ and we conclude that $\hat{y} \subset h_2^* \cap k_2$. □

Definition 6.6. Let $H \subset \mathfrak{H}(X)$. An element $h \in H$ is called:

- *minimal in H* if for every $k \in H$ either $k \pitchfork h$, $h \subset k$, or $h \subset k^*$;
- *maximal in H* if for every $k \in H$ either $k \pitchfork h$, $k \subset h$, or $k^* \subset h$, that is to say, h is maximal if h^* is minimal;
- *terminal in $\mathfrak{H}'$* if it is either maximal or minimal.

LEMMA 6.7. [**CFI12**] *The map $\tau : 2^{\mathfrak{H}(X)} \to 2^{\mathfrak{H}(X)}$ taking a collection of half-spaces to its possibly empty collection of terminal elements is measurable and* Aut(X)*-equivariant.*

Let us look at some examples of sets of half-spaces that do and do not have terminal elements. Consider $x \in \overline{X}$ and the associated Dirac mass δ_x. Then U_x, the collection of half-spaces that contain x, corresponds to the heavy half-spaces of δ_x, namely $U_x = H_{\delta_x}^+$. Now, if $x \in X$ then U_x satisfies the descending chain condition. This means exactly that

$\tau(H^+_{\delta_x}) \neq \varnothing$. Furthermore, if $x \in X$ and x belongs to infinitely many cubes (which may be the case if X is not locally finite) then, $\tau(H^+_{\delta_x})$ is in fact infinite. On the other hand, if we set $X = \mathbb{Z}$ with the standard cubulation and take $m = \frac{1}{2}(\delta_{-\infty} + \delta_{+\infty})$, where $\partial\mathbb{Z} = \{-\infty, +\infty\}$, then we see that all half-spaces are balanced for m and that $\tau(H_m) = \varnothing$.

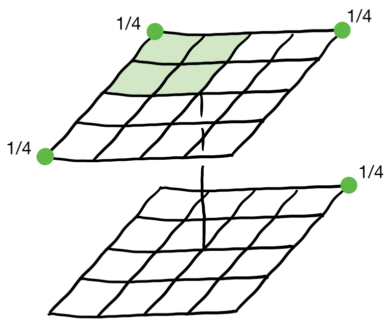
We record the following straightforward but important fact.

Remark 6.8. Recall that to each measure $m \in \mathcal{P}(\overline{X})$ the space associated with H_m is $\overline{X}(H_m)$, which is an interval and therefore Euclidean. This means that, if $H \subset H_m$ is any subset, then it must have finitely many terminal elements.

6.3. *Examples of $\overline{X}_m$.* Let us take the opportunity to consider various examples of measures on a CAT(0) cube complex. More specifically, let X be the universal cover of the Salvetti complex associated with the right-angled Artin group $\mathbb{Z}^2 * \mathbb{Z}$.

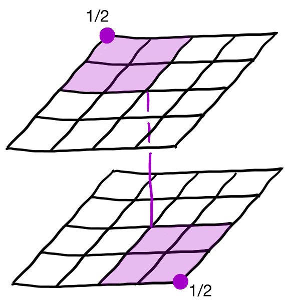
In the examples below, each vertical edge is meant to have length one, all measures are atomic, and the corresponding intervals $\overline{X}_m$ are highlighted in the appropriate color (shading). We shall discuss the various properties of these intervals.

Example 1



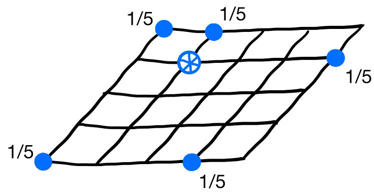
In this example, two points carry equal measure of $1/2$. We see that H_m has strongly separated pairs. An example of such a pair is taking one half-space that crosses the upper green (shaded) region and one half-space crossing the lower green (shaded) region. Therefore, by Lemma 6.3, X_m intersects X. In fact, in this case, X_m is finite and contained in X. Furthermore, the half-spaces closest to the atoms are terminal in H_m. Finally, any two facing half-spaces corresponding to 'vertical' edges that contain this figure will be strongly separated containing every wall associated with H_m.

Example 2



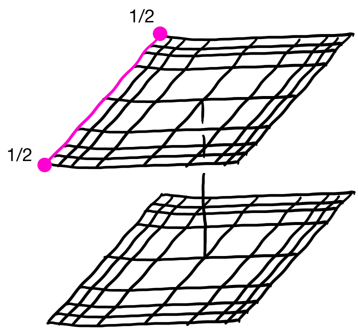
In this example, four points carry equal measure of $1/4$. We see that H_m corresponds to the half-spaces crossing the purple (shaded) region and hence it does not have strongly separated pairs. Nevertheless X_m is finite and contained in X. This time, the terminal elements in H_m are those closest to the unique point of X_m that has positive measure. Finally, any half-space containing the purple (shaded) region in the upper flat together with any half-space in the lower flat (that also contains the purple (shaded) region) will be strongly separated containing every wall associated with H_m.

Example 3



In this example, we note that there are an odd number of points of equal measure. More precisely, five points each of measure $1/5$. Since half-spaces partition with their complements, we cannot have that both a half-space and its complement have an even number of points and hence $H_m = \varnothing$. This means of course that the intersection of the half-spaces in H_m^+ give us a unique point marked with a star. As H_m is empty, there are no terminal elements.

Example 4



In this example, the spacing between the lines is supposed to indicate that distances have been contracted by a factor of $1/2$ as one moves from the central point in each flat. Once more, we have given two points equal measure of $1/2$. However, this time, the points belong to the Roller boundary ∂X, and in fact, are not even at finite distance in the Roller boundary so that the interval $\overline{X}_m$ is infinite and H_m does not contain strongly separated pairs nor terminal elements. Nevertheless, one can take 'vertical' half-spaces (one above and one below) containing this flat and these will be a strongly separated pair containing each wall associated with H_m.

6.4. *Space of intervals.* As we saw above, it is natural to associate with a probability measure $m \in \mathcal{P}(\overline{X})$ an interval $\overline{X}_m \subset \overline{X}$. For this reason, let us now look at intervals from a more global perspective.

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Consider the continuous map $2^{\mathfrak{H}} \times 2^{\mathfrak{H}} \to 2^{\mathfrak{H}} \times 2^{\mathfrak{H}}$ where $(S, T) \mapsto (S \triangle T, S \cap T)$. Consider the restriction of this map to $\overline{X} \times \overline{X} \subset 2^{\mathfrak{H}} \times 2^{\mathfrak{H}}$. For $v, w \in \overline{X}$ we set $\mathfrak{H}(v, w) = U_v \triangle U_w$, and $\mathfrak{H}(v, w)^+ = U_v \cap U_w$, making the first coordinate the half-spaces separating u and v and the second coordinate the half-spaces that contain both u and v. This uniquely determines the interval $\mathcal{I}(u, v)$ and hence this image is called the *space of intervals* and is denoted by $\mathcal{I}(\overline{X})$. Summarizing, we have that

$$\mathcal{I}(\overline{X}) = \{(\mathfrak{H}(v, w), \mathfrak{H}(v, w)^+) : v, w \in \overline{X}\}.$$

Observe that by Corollary 2.9 the map $\overline{X} \times \overline{X} \to \mathcal{I}(\overline{X})$ is finite-to-one.

LEMMA 6.9. *The space of intervals $\mathcal{I}(\overline{X})$ is closed and hence Borel as a subset of $2^{\mathfrak{H}} \times 2^{\mathfrak{H}}$.*

Proof. Let $(S_n, S_n^+) \in \mathcal{I}(\overline{X})$ and assume that $(S_n, S_n^+) \to (S, T) \in 2^{\mathfrak{H}} \times 2^{\mathfrak{H}}$. By assumption, there exists $v_n, w_n \in \overline{X}$ such that $S_n = \mathfrak{H}(v_n, w_n)$ and $S_n^+ = \mathfrak{H}(v_n, w_n)^+$. Since $\overline{X}$ is compact, there is a subsequence such that $(v_{n_j}, w_{n_j}) \to (v, w) \in \overline{X} \times \overline{X}$. We claim that $(S, T) = (\mathfrak{H}(v, w), \mathfrak{H}(v, w)^+)$.

Let $(h, k) \in \mathfrak{H}(v, w) \times \mathfrak{H}(v, w)^+$ and, without loss of generality, assume that $w \in h$ and $v \in h^*$. Then, since $(v_{n_j}, w_{n_j}) \to (v, w)$, for j sufficiently large we have that $w_{n_j} \in h$, $v_{n_j} \in h^*$ and $w_{n_j}, v_{n_j} \in k$. Since $(\mathfrak{H}(v_n, w_n), \mathfrak{H}(v_n, w_n)^+) \to (S, T)$ it follows that $(h, k) \in S \times T$.

Conversely, if $(h, k) \in S \times T$ then, for all n sufficiently large, we have that $(h, k) \in \mathfrak{H}(v_n, w_n) \times \mathfrak{H}(v_n, w_n)^+$. In particular this holds for $n = n_j$ sufficiently large and so $(h, k) \in \mathfrak{H}(v, w) \times \mathfrak{H}(v, w)^+$. □

Let $\mathcal{I}(\overline{X}, *)$ denote the collection of pointed intervals of $\overline{X}$, which is defined to be the collection of elements $2^{\mathfrak{H}} \times 2^{\mathfrak{H}} \times 2^{\mathfrak{H}}$,

$$\{(\mathfrak{H}(v, w), \mathfrak{H}(v, w)^+, U_x) : x \in \mathcal{I}(v, w), v, w \in \overline{X}\}.$$

The proof of Lemma 6.9 easily generalizes to show the following lemma.

LEMMA 6.10. *The subset of pointed intervals $\mathcal{I}(\overline{X}, *) \subset 2^{\mathfrak{H}} \times 2^{\mathfrak{H}} \times 2^{\mathfrak{H}}$ is closed and hence Borel.*

We shall employ $\mathcal{I}(\overline{X}, *)$ in the proof of Theorem 7.1. For the sake of simplicity, we shall think of an element in $\mathcal{I}(\overline{X}, *)$ rather than as triples of sets of half-spaces, as a pair $(\mathcal{I}, x)$ such that $\mathcal{I}$ is an interval of $\overline{X}$ and $x \in \mathcal{I}$.

Remark 6.11. We observe that there is a natural continuous projection $\mathcal{I}(\overline{X}, *) \to \overline{X}$, where $(\mathcal{I}, x) \mapsto x$.

7. *Boundary maps*

In this section we shall prove the existence of measurable equivariant maps between the Furstenberg–Poisson boundary and the Roller boundary. The existence of such maps, when the target has a convergence action of the group in question is guaranteed by [**BF14**, Theorem 3.2]. However, as we shall see in §10, the action on the Roller compactification is rarely a convergence action.

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In this section, we shall prove the existence of such maps, and also study some consequences associated with them.

7.1. *Existence.* The following theorem can be found in [**CFI12**], in the case of a symmetric measure μ. Our proof below however is streamlined thanks to the work of Bader–Furman [**BF14**], which allows us to circumvent several of the cases from the proof in [**CFI12**].

THEOREM 7.1. *Let Γ be a discrete countable group, (B_-, ν_-) and (B_+, ν_+) be a boundary pair for Γ. Assume that $\Gamma \to \mathrm{Aut}(X)$ is an action by automorphisms on a CAT(0) cube complex X. Then there exists a subgroup Γ' of finite index in Γ and Γ'-equivariant measurable maps $\varphi_\pm : B_\pm \to \overline{X}$. Furthermore, if the action of Γ is non-elementary then $\Gamma' = \Gamma$ and $\varphi_\pm(B_\pm) \subset \partial X$.*

Before turning to the proof, let us briefly explain the geometric significance of the several cases that will be considered in the proof. The cases consist of the possible (generic) intersection patterns of H_{m_-} with H_{m_+}. In the case of a tree, these sets uniquely determine $\mathcal{I}_{m_-}$ and $\mathcal{I}_{m_+}$, respectively. Let us expand on the significance of each case when X is a simplicial tree.

(i) $H_{m_-} \cap H_{m_+} = \varnothing$, for $\vartheta_- \otimes \vartheta_+$-almost every (a.e.) (m_-, m_+). This corresponds to the situation that the intervals in question have trivial intersection. There are in fact several geometric cases corresponding to whether the intervals are points, segments, geodesic rays, or geodesic lines.

(ii) $H_{m_-} = H_{m_+} \neq \varnothing$, for $\vartheta_- \otimes \vartheta_+$-a.e. (m_-, m_+). Here we consider when the essential image is a single interval. By Γ-equivariance of the map, this means that there is a Γ-invariant interval. Geometrically, the types of intervals that can arise correspond to the different ways an action can be elementary.

(iii) $H_{m_-} \cap H_{m_+} \neq \varnothing$ and $H_{m_-} \triangle H_{m_+} \neq \varnothing$, for $\vartheta_- \otimes \vartheta_+$-a.e. (m_-, m_+). For a general CAT(0) cube complex, this third case breaks up further in to subcases, which do not arise when we are concerning ourselves with a tree, as we do now. In fact, in the case of a tree, the assumptions mean geometrically that the intervals $\mathcal{I}_{m_-}$ and $\mathcal{I}_{m_+}$ (generically) have non-trivial intersection and are not equal. One may associate with this the finite data of the (oriented) edges in $\mathcal{I}_{m_-} \triangle \mathcal{I}_{m_+}$ that are closest to $\mathcal{I}_{m_-} \cap \mathcal{I}_{m_+} \subset X$.

Proof of Theorem 7.1. If Γ has a finite orbit in $\overline{X}$ then there is a finite index subgroup Γ_0 fixing a point $x_0 \in \overline{X}$ and a Γ_0-equivariant measurable map $B_\pm \to \{x_0\}$.

Now, suppose that Γ does not have a finite orbit in the Roller compactification $\overline{X}$. Then, by Proposition 3.10 there exists a finite index subgroup $\Gamma' \leqslant \Gamma$ and a subcomplex $X' \subset \overline{X}$ on which the Γ'-action is non-elementary and essential, with $\mathfrak{H}(X') \subset \mathfrak{H}$. Furthermore, if the Γ-action was assumed to be non-elementary on X, then we have that $X' \subset X$ is Γ-invaraint and essential.

We record the fact that we have possibly passed to a finite index subgroup and an invariant subcomplex which, *a priori* can belong to the Roller boundary. We also observe that (B_-, B_+) continues to be a boundary pair for Γ' by Theorem 5.5. Finally, we note

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that $\partial X' \subset \partial X$ is invariant under Γ'. And so to conserve notation, we assume now that the Γ-action itself is non-elementary and essential on X.

By amenability of the Γ-action on $B_\pm$ there exist Γ-equivariant maps $B_\pm \to \mathcal{P}(\overline{X})$. Let $\vartheta_\pm$ be the pushforward under these maps of the measures $\nu_\pm$. To avoid confusion, let us denote by $\mathcal{P}_\pm$ the space $\mathcal{P}(\overline{X})$ with the measure $\vartheta_\pm$. Then our goal now is to extract maps $\varphi_\pm : \mathcal{P}_\pm \to \overline{X}$ which are measurable, Γ-equivariant, and defined on a conull set. To this end, observe that by Proposition 5.3 the projection maps $\pi_\pm : \mathcal{P}_- \times \mathcal{P}_+ \to \mathcal{P}_\pm$ are relatively isometrically ergodic.

Let $m \in \mathcal{P}(\overline{X})$ and recall that we have an associated interval denoted by $\overline{X}_m$ whose half-space structure corresponds to the m-balanced half-spaces H_m. As a subset of $\overline{X}$ we have

$$\overline{X}_m = \bigcap_{h \in H_m^+} h.$$

We shall be dealing with various natural maps $\mathcal{P}(\overline{X})$ to $2^{\mathfrak{H}}$ or $\mathbb{R}$. We cite [**CFI12**] for the measurability of all of them and do not address the issue again.

Consider the maps $(m_-, m_+) \mapsto \#(H_{m_-} \cap H_{m_+}), \#(H_{m_-} \triangle H_{m_+})$ which possibly take the value ∞. Since these are Γ-invariant, they must be essentially constant by ergodicity of $\mathcal{P}_- \times \mathcal{P}_+$. Hence, we must be in one of the following cases.

(i) $H_{m_-} \cap H_{m_+} = \varnothing$, for $\vartheta_- \otimes \vartheta_+$-a.e. (m_-, m_+). Fix $(m_-, m_+) \in \mathcal{P}_- \times \mathcal{P}_+$ on a conull set satisfying the hypotheses. Then, we must have that $H_{m_-} \subset H_{m_+}^\pm$ and $H_{m_+} \subset H_{m_-}^\pm$. By Lemma 6.1 both $H_{m_-}^+$ and $H_{m_+}^+$ are consistent sets. This means that the following collections of half-spaces satisfy both consistency and totality, i.e. they correspond to points in the Roller compactification. Indeed, they are the projections onto $\overline{X}_{m_-}$ and $\overline{X}_{m_+}$, respectively, and hence will be denoted by $p_{m_-}(m_+) \in \overline{X}_{m_-}$ and $p_{m_+}(m_-) \in \overline{X}_{m_+}$. They are defined as

$$U_{p_{m_-}(m_+)} = H_{m_-}^+ \cup (H_{m_-} \cap H_{m_+}^+),$$
$$U_{p_{m_+}(m_-)} = H_{m_+}^+ \cup (H_{m_+} \cap H_{m_-}^+).$$

Recall that, as was developed at the end of §2.4, $\mathcal{I}(\overline{X}, *)$, the collection of pointed intervals in $\overline{X}$, is Borel and that there is a natural Borel map from $\mathcal{I}(\overline{X}, *)$ to $\mathcal{I}(\overline{X})$ and $\overline{X}$, obtained by 'forgetting' the additional information of the point, or interval, respectively. This gives rise to the following commutative diagram:

$$\begin{tikzcd}
\mathcal{P}_- \times \mathcal{P}_+ \arrow[r] \arrow[d, "\pi_+"'] & \mathcal{I}(\overline{X}, *) \arrow[r] \arrow[d, "q"] & \overline{X} \\
\mathcal{P}_+ \arrow[r] \arrow[ur, dashed, "\exists \psi_\times"] & \mathcal{I}(\overline{X}) &
\end{tikzcd}$$

The lower horizontal map $\mathcal{P}_+ \to \mathcal{I}(\overline{X})$ corresponds to the map

$$m_+ \mapsto \overline{X}_{m_+},$$

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and the upper horizontal map $\mathcal{P}_- \times \mathcal{P}_+ \to \mathcal{I}(\overline{X}, *)$ is

$$(m_-, m_+) \mapsto (\overline{X}_{m_+}, p_{m_+}(m_-)).$$

We observe that the preimage $q^{-1}(\mathcal{I}) = \{(\mathcal{I}, x) : x \in \mathcal{I}\}$ is a countable set, and therefore the map $d : \mathcal{I}(\overline{X}, *) \times \mathcal{I}(\overline{X}, *) \to [0, \infty)$ defined by

$$d((\mathcal{I}_1, x_1), (\mathcal{I}_2, x_2)) = 1 - \delta(x_1, x_2)$$

clearly makes the preimage $q^{-1}(\mathcal{I})$ into a separable metric space such that Γ acts relatively isometrically on $q : \mathcal{I}(\overline{X}, *) \to \mathcal{I}(\overline{X})$.

Now, since the quotient $\mathcal{P}_- \times \mathcal{P}_+ \to \mathcal{P}_+$ is relatively isometrically ergodic for Γ, we deduce that there is a measurable Γ-equivariant map defined on a conull set $\psi_+ : \mathcal{P}_+ \to \mathcal{I}(\overline{X}, *)$. The same argument with the obvious modifications yields $\psi_- : \mathcal{P}_- \to \mathcal{I}(\overline{X}, *)$. Post composing these with the map $\mathcal{I}(\overline{X}, *) \to \overline{X}$ we obtain

$$\varphi_\pm : \mathcal{P}_\pm \to \overline{X}.$$

Now, since X and ∂X are both measurable Γ-invariant subsets, we must have that the essential image belongs to precisely one of these. Suppose that $\varphi_\pm(\mathcal{P}_\pm) \subset X$. Then since X is countable, by Corollary 5.6 we conclude that there is a Γ-fixed point in X, which contradicts the assumption that the action is non-elementary. Therefore, $\varphi_\pm(\mathcal{P}_\pm) \subset \partial X$.

To finish the proof, we shall show that all other cases lead to a contradiction. Recall that we remain under the assumption that the Γ-action on X is essential and non-elementary.

(ii) $H_{m_-} = H_{m_+} \neq \varnothing$, for $\vartheta_- \otimes \vartheta_+$-a.e. (m_-, m_+). Fix a generic $m_- \in \mathcal{P}_-$ and a ϑ_+-conull and Γ-invariant set $P_+ \subset \mathcal{P}_+$ so that $H_{m_+} = H_{m_-}$ for every $m_+ \in P_+$. Set $\mathfrak{H}' := H_{m_-}$, and observe that this is a non-empty, symmetric, and convex Γ-invariant set of half-spaces. By Corollary 3.32, it follows that either $\mathfrak{H}' = \mathfrak{H}$ or $X \cong X_{m_-} \times X_2$. This contradicts Corollary 3.21: the Γ-action is essential and non-elementary so X cannot have an interval as a factor.

(iii) $H_{m_-} \cap H_{m_+} \neq \varnothing$ and $H_{m_-} \triangle H_{m_+} \neq \varnothing$, for $\vartheta_- \otimes \vartheta_+$-a.e. (m_-, m_+). Let $X = X_1 \times \cdots \times X_n$ be the decomposition of X into irreducible factors, $\mathfrak{H} = \mathfrak{H}_1 \sqcup \cdots \sqcup \mathfrak{H}_n$ be the corresponding decomposition of half-spaces into pairwise transverse collections, and Γ_0 be a normal finite index subgroup whose image is in $\mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_n)$. Recall that $\mathcal{S}_i \subset \mathfrak{H}_i \times \mathfrak{H}_i$ denotes the pairs of disjoint strongly separated half-spaces in the irreducible factor X_i and $\mathcal{S} = \mathcal{S}_1 \sqcup \cdots \sqcup \mathcal{S}_n$. Observe that $\mathcal{S}_i$ is Γ_0-invariant for each $i = 1, \ldots, n$ and that $\mathcal{S}$ is Γ-invariant. We then have that the map $m \mapsto \#((H_m \times H_m) \cap \mathcal{S}_i)$ is measurable, Γ_0-invariant, and hence essentially constant for both ϑ_- and ϑ_+.

Up to changing the roles of ϑ_- and ϑ_+ we have the following two cases corresponding to the essential values of $m \mapsto \#((H_m \times H_m) \cap \mathcal{S}_i)$ for $i \in \{1, \ldots, n\}$: either the ϑ_+-essential value is zero for some $i \in \{1, \ldots, n\}$, or the ϑ_- and ϑ_+ essential values are both non-zero for every $i \in \{1, \ldots, n\}$.

(iii)(a) $(H_{m_+} \times H_{m_+}) \cap \mathcal{S}_i = \varnothing$ for some i and ϑ_+-a.e. $m_+ \in \mathcal{P}_+$. Let us assume that $i = 1$. Fix a ϑ_--generic $m_- \in \mathcal{P}_-$ and a ϑ_+-conull Γ_0-invariant set $P_+ \subset \mathcal{P}_+$ such that the following hold for every $m_+ \in P_+$:

- $H_{m_-} \cap H_{m_+} \neq \varnothing$;
- $H_{m_-} \triangle H_{m_+} \neq \varnothing$;
- $(H_{m_+} \times H_{m_+}) \cap \mathcal{S}_1 = \varnothing$.

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Consider now the Γ_0-equivariant projection $\overline{X} \to \overline{X}_1$ which induces a Γ_0-equivariant map $\mathcal{P}(\overline{X}) \to \mathcal{P}(\overline{X}_1)$. Recall that the Γ_0-action remains essential (Remark 3.4) and non-elementary (Lemma 3.20) on each irreducible factor, in particular on X_1.

Set E and m to be the pushforwards of P_+ and m_-, respectively, under the Γ_0-equivariant projection $\overline{X} \to \overline{X}_1$. Observe that E is Γ_0-invariant and that the above assumptions on P_+ and m_- descend to E and m, respectively. In particular, the hypotheses of Lemma 6.5 are satisfied. This means that there exists a strongly separated pair $(h, k) \in \mathcal{S}_1$ such that $h \subset k$, and for every $x \in \bigcup_{m' \in E} H_{m'}$ we have that

$$\hat{x} \subset h^* \cap k.$$

By Γ_0-invariance of E, it follows that h^* and k are not Γ_0 flippable, which contradicts the flipping lemma (Lemma 3.6), as the action of Γ_0 is essential and non-elementary on X_1.

(iii)(b) For each i we have $(H_m \times H_m) \cap \mathcal{S}_i \neq \varnothing$ for $\vartheta_\pm$-a.e. $m \in \mathcal{P}_\pm$. Fix a generic $(m_-, m_+) \in \mathcal{P}_- \times \mathcal{P}_+$. In this case, by Lemma 6.3 we must have that $H^+_{m_-} \cup H^+_{m_+}$ satisfies the descending chain condition, i.e. every descending chain has a terminal element. Furthermore, our assumption that $H_{m_-} \triangle H_{m_+} \neq \varnothing$ implies that

$$(H_{m_-} \cap H^+_{m_+}) \cup (H_{m_+} \cap H^+_{m_-}) \neq \varnothing.$$

Hence, as a subset of $H_{m_-} \cup H_{m_+}$, as in Remark 6.8, there are finitely many terminal elements in $(H_{m_-} \cap H^+_{m_+}) \cup (H_{m_+} \cap H^+_{m_-})$ and there is at least one because these are non-empty subsets of $H^+_{m_-} \cup H^+_{m_+}$. This yields a Γ-equivariant map from $\mathcal{P}_- \times \mathcal{P}_+$ to the countable collection of finite subsets of $\mathfrak{H}$ and so, by Corollary 5.6, there is a finite set $\mathcal{F} \subset \mathfrak{H}$ which is Γ-invariant. But Lemma 3.9 shows that this is incompatible with our assumption that the action is both essential and non-elementary. $\square$

Recall that the Furstenberg–Poisson boundaries associated with $\check{\mu}$ and μ and (for generating $\mu \in \mathcal{P}(\Gamma)$) give a boundary pair for a group Γ (see Theorem 5.5) and hence by Theorem 7.1 we deduce the following corollary.

COROLLARY 7.2. *Let Γ be a discrete countable group and $\mu \in \mathcal{P}(\Gamma)$ a generating probability measure. Suppose furthermore that $\Gamma \to \mathrm{Aut}(X)$ is a non-elementary and essential action on the CAT(0) cube complex X. Then there exist quasi-Γ-invariant probability measures $\lambda_\pm \in \mathcal{P}(\partial X)$ such that $(\partial X, \lambda_-)$ and $(\partial X, \lambda_+)$ are $(\Gamma, \check{\mu})$- and (Γ, μ)-boundaries, respectively.*

7.2. *The image and regular points.* Nevo and Sageev refined the description of the Furstenberg–Poisson boundary by passing from the full Roller boundary to the closure of the non-terminating elements [**NS13**]. In this section we give a further refinement in terms of the regular points in the Roller boundary, along with some corollaries.

Definition 7.3. Let X be an irreducible CAT(0) cube complex. Define $\partial_{\mathrm{r}} X$, the *regular points*, as the set of $\xi \in \partial X$ such that if $h_1, h_2 \in U_\xi$ then there is a $k \in U_\xi$ such that $k \subset h_1 \cap h_2$ and k is strongly separated from both h_1 and h_2. If X is reducible, define the regular points to be the product of the regular points in each factor, i.e.

$$\partial_{\mathrm{r}} X = \partial_{\mathrm{r}} X_1 \times \cdots \times \partial_{\mathrm{r}} X_n.$$

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Remark 7.4. In the above definition, the reader should consider the particular case where $h_1 \pitchfork h_2$. Indeed, the aim of the definition is to point out that such points do not live on the boundary of any 'piece' of a quarter plane.

Note that $\partial_{\mathrm{r}} X$ could be empty. Consider for example the connected complex obtained by removing the second and fourth quadrants in the plane. This example is admittedly degenerate having only finitely many automorphisms.

PROPOSITION 7.5. *Let X be an* irreducible *CAT(0) cube complex and $\alpha \in \partial X$. The following are equivalent:*

(1) $\alpha \in \partial_{\mathrm{r}} X$;

(2) *there exists an infinite descending chain $\{s_n\}_{n\in\mathbb{N}} \subset U_\alpha$ of pairwise strongly separated half-spaces;*

(3) *there exists a bi-infinite descending chain $\{s_n\}_{n\in\mathbb{Z}} \subset U_\alpha$ of pairwise strongly separated half-spaces.*

Proof. We begin by observing that if $\{s_n\}$ is an infinite descending chain of strongly separated half-spaces and h is a half-space whose intersection with each s_n is non-trivial, then for n sufficiently large we must have that $s_n \subset h$. By strong separation, we may assume that h is parallel to each s_n. Now, if $s_1 \subset h$ then we are done. Otherwise, $s_1 \cap h^* \neq \varnothing$ and since $s_1 \cap h$ is non-empty as well, by Remark 3.22, we have that $\hat{h} \subset s_1$. Since $h \cap s_n \neq \varnothing$ then s_n is not contained in h^* for any n. Finally, since there are finitely many half-spaces in between any two, we conclude that for n sufficiently large, $s_n \subset h$.

(1) $\Longrightarrow$ (2): Let $\alpha \in \partial_{\mathrm{r}} X$. Fix $s_1 \in U_\alpha$. Then there exists $s_2 \in U_\alpha$ such that $s_2 \subset s_1$ and s_2 and s_1 are strongly separated.

Assume that $s_1, \ldots, s_n \in U_\alpha$ are decreasing and pairwise strongly separated. Since $\alpha \in \partial_{\mathrm{r}} X$ there is $s_{n+1} \in U_\alpha$ such that $s_{n+1} \subsetneq s_n \cap s_{n-1}$ with s_{n+1} strongly separated with s_n.

(2) $\Longrightarrow$ (1): This is straightforward. Assume $\{s_n\} \subset U_\alpha$ is an infinite descending sequence of pairwise strongly separated half-spaces and $h_1, h_2 \in U_\alpha$. As was observed in the beginning of this proof, for n sufficiently large,

$$s_{n+1} \subset s_n \subset h_1 \cap h_2.$$

(2) $\Longrightarrow$ (3): Let $\{s_n : n \in \mathbb{N}\} \subset U_\alpha$ be an infinite descending sequence of pairwise strongly separated half-spaces. By the double skewering lemma (Lemma 3.8), there exists γ such that $\gamma s_1 \subset s_2 \subset s_1$, that is, $s_1 \subset \gamma^{-1} s_2 \subset \gamma^{-1} s_1$ and setting $s_{-n} = \gamma^{-n} s_1$ completes the desired sequence.

(3) $\Longrightarrow$ (2): This is trivial. □

We note that conditions (1)–(3) of Proposition 7.5 imply that $\alpha \in \partial_{\mathrm{NT}} X$. That this is true for (1) is immediate from the definition of a regular point for irreducible complexes. That this is true for conditions (2) and (3) follows as well: if $s_n, h \in U_\alpha$ then for n sufficiently large we must have that $s_n \subset h$.

COROLLARY 7.6. *Let X be irreducible. The intersection of any infinite descending chain of strongly separated half-spaces is a singleton. In particular, if $\xi_1, \xi_2 \in \partial_{\mathrm{r}} X$ are distinct, then $\xi_1 \in h_1$ and $\xi_2 \in h_2$ for some strongly separated disjoint pair $h_1, h_2 \in \mathfrak{H}$.*

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Proof. Let us show that if $\{s_n\}$ is a strongly separated descending chain of half-spaces then $\bigcap_{n\in\mathbb{N}} s_n$ is a singleton. Indeed, since every finite intersection of these half-spaces is non-empty, and $\overline{X}$ is compact, $\bigcap_{n\in\mathbb{N}} s_n$ is non-empty. Suppose that $x, y \in \bigcap_{n\in\mathbb{N}} s_n$ are distinct. Then for some h we have that $x \in h$ and $y \in h^*$. This of course means that for each n, $h \cap s_n$ and $h^* \cap s_n$ are both non-empty. By Remark 3.22, we must have that for each n, either $h \pitchfork s_n$ or $\hat{h} \subset s_n$. By strong separation, it follows that $\hat{h} \subset s_n$ for all n sufficiently large. But this is impossible since s_n is descending and there are finitely many half-spaces in between any two. Therefore, no such h exists and $x = y$. □

COROLLARY 7.7. *If X is a CAT(0) cube complex and the action of* $\mathrm{Aut}(X)$ *is essential and non-elementary, then $\partial_{\mathrm{r}} X \neq \varnothing$.*

Proof. Recall that the action of $\mathrm{Aut}(X)$ is essential and non-elementary if and only if the action of $\mathrm{Aut}(X_i)$ is essential and non-elementary for each irreducible factor X_i of X (Lemma 3.20 and Theorem 3.5). Caprace and Sageev's theorem (Theorem 3.13) characterizes such irreducible complexes by the existence of strongly separated pairs $s_1 \subset s_0$ in $\mathfrak{H}_i$. Applying the double skewering lemma (Lemma 3.8), we find $\gamma s_1 \subset \gamma s_0 \subset s_1 \subset s_0$ and γs_1 is strongly separated from s_1. Setting $s_n = \gamma^{n-1} s_1$ we obtain an infinite descending strongly separated chain and, by Proposition 7.5, we have that $\partial_{\mathrm{r}} X_i$ is non-empty and hence $\partial_{\mathrm{r}} X$ is non-empty. □

THEOREM 7.8. *Let Γ be a discrete countable group, and (B_-, ν_-) and (B_+, ν_+) be a boundary pair for Γ. Assume that $\Gamma \to$* $\mathrm{Aut}(X)$ *is an essential and non-elementary action by automorphisms on a CAT(0) cube complex X. Then any Γ-equivariant measurable map $\varphi_\pm : B_\pm \to \partial X$ has an essential target in $\partial_{\mathrm{r}} X$.*

This will follow immediately from Theorem 7.14. The rest of this section is devoted to proving this and other key results. We first establish some notation: for $x, y \in \overline{X}$ the collection of half-spaces containing y and not x will be denoted by $[x, y]$. In terms of U_x and U_y we have $[x, y] = U_y \backslash U_x$. Note that this is an oriented interval!

The following lemma is extremely useful in identifying the $\lambda_- \otimes \lambda_+$-generic relationship between the components of pairs in ∂X^2. More specifically, if a certain (finite) phenomenon happens once, it is guaranteed to happen infinitely many times. It will be applied for example, in the case where S is the collection of strongly separated pairs. Specifically, we shall deduce from the fact that strongly separated half-spaces exist that we must have infinitely many strongly separated half-spaces separating a generic pair $(\xi_-, \xi_+) \in \partial X^2$.

LEMMA 7.9. *Assume $\Gamma \to$* $\mathrm{Aut}(X)$ *is a non-elementary and essential action, $\lambda_\pm$ being quasi-Γ-invariant measures on ∂X such that $(\partial X^2, \lambda_- \otimes \lambda_+)$ is isometrically ergodic. Let $N > 0$ and $S \subset \mathfrak{H}^N \times \mathfrak{H}$ be a Γ-invariant collection of $(N+1)$-tuples. If there is $(h_1, \ldots, h_N, k) \in S$ with*

$$h_1 \subset \cdots \subset h_N \subset k^*,$$

then the map $\partial X^2 \to \mathbb{N} \cup \{\infty\}$ defined by $(\xi_-, \xi_+) \mapsto \#([\xi_+, \xi_-]^N \times [\xi_-, \xi_+] \cap S)$ is $\lambda_- \otimes \lambda_+$-essentially constant with infinite essential value.

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Proof. The measurability of the map in question relies on the Γ-invariance of the non-empty set S. The proof is straightforward and similar to that of [**CFI12**, Corollary A.2].

Since S is Γ-invariant, it follows that the map in question is Γ-invariant and hence essentially constant by (isometric) ergodicity. If the essential value is finite and non-zero, then this gives a Γ-equivariant map from ∂X^2 to the countable collection of finite subsets of $\mathfrak{H}$. By isometric ergodicity and Corollary 5.6 this yields a finite Γ-invariant set in $\mathfrak{H}$ which contradicts the assumption that the action is essential and non-elementary by Lemma 3.9. Therefore the essential value must be 0 or ∞.

Fix $(h_1, \ldots, h_N, k) \in S$ with $h_1 \subset \cdots \subset h_N \subset k^*$. It follows that if $(\xi_-, \xi_+) \in h_1 \times k$ then $(h_1, \ldots, h_N, k) \in [\xi_+, \xi_-]^N \times [\xi_-, \xi_+]$, and since $\lambda_- \otimes \lambda_+(h_1 \times k) > 0$ by Lemma 3.7, we have that the essential value is not zero and hence infinite. □

From this we derive the following important consequences.

LEMMA 7.10. *With the hypotheses as in Lemma 7.9, for $\lambda_- \otimes \lambda_+$ a.e. $(\xi_-, \xi_+) \in \partial X^2$ we have that $\mathcal{I}(\xi_-, \xi_+) \cap X \neq \varnothing$.*

Proof. Let $X = X_1 \times \cdots \times X_n$ be the decomposition of X into irreducible factors and let Γ_0 be the finite index subgroup of Γ which maps to $\mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_n)$. As before, we let $\mathcal{S}_i \subset \mathfrak{H}_i \times \mathfrak{H}_i$ be the collection of disjoint strongly separated pairs of half-spaces in the irreducible factor X_i, and $\mathcal{S} = \mathcal{S}_1 \sqcup \cdots \sqcup \mathcal{S}_n$. Note that $\mathcal{S}_i$ is Γ_0-invariant for each i.

Now, by assumption that the Γ-action is essential and non-elementary on X, it follows that the quotient action of Γ_0 on each X_i is also essential and non-elementary (see Remark 3.4 and Lemma 3.20). By Caprace and Sageev's theorem (Theorem 3.13), $\mathcal{S}_i \neq \varnothing$ and therefore there is an $(h_i, k_i) \in \mathcal{S}_i$ and $h_i \cap k_i = \varnothing$. Observing that all the hypotheses remain true for Γ_0, we may apply Lemma 7.9 to the action of Γ_0 on $S = \mathcal{S}_i$ for each $i = 1, \ldots, n$, and we deduce that the essential value of $(\xi_-, \xi_+) \mapsto \#([\xi_+, \xi_-] \times [\xi_-, \xi_+] \cap \mathcal{S}_i)$ is infinite, for each i.

Next, as observed in Remark 6.2, it follows that $X_m \subset X$, where m is the average of the Dirac masses at ξ_1 and ξ_2. Of course, $\mathcal{I}(\xi_-, \xi_+) = \overline{X}_m$ and hence $\mathcal{I}(\xi_-, \xi_+) \cap X \neq \varnothing$. □

Consider two disjoint half-spaces h and k and define the map

$$\delta(h, k) = \#\{\ell : h \subseteq \ell \subseteq k^*\}.$$

We note that this is not a distance on half-spaces†, although it is true that if $h \subsetneq \ell \subseteq k^*$ then $\delta(h, \ell^*) \leqslant \delta(h, k)$.

Definition 7.11. Suppose $\mathfrak{H} = \mathfrak{H}_1 \sqcup \cdots \sqcup \mathfrak{H}_n$ corresponds to the irreducible factor decomposition of X. Let $\mathcal{S}_R^{(N)}$ denote the collection of $(h_1, \ldots, h_N, k) \in \mathfrak{H}^{N+1}$ such that $h_1 \subsetneq \cdots \subsetneq h_N \subsetneq k^*$, $\delta(h_1, k) \leqslant R$, and for some $i \in \{1, \ldots, n\}$,

$$(h_1, h_2), \ldots, (h_{N-1}, h_N), (h_N, k) \in \mathcal{S}_i.$$

LEMMA 7.12. *Assume the hypotheses in Lemma 7.9. For each N there is an R such that for each $i = 1, \ldots, n$ and $\lambda_+ \otimes \lambda_-$-a.e. $(\xi_-, \xi_+) \in \partial X \times \partial X$ the cardinality of $[\xi_+, \xi_-]^N \times [\xi_-, \xi_+] \cap \mathcal{S}_R^{(N)} \cap \mathfrak{H}_i^{N+1}$ is infinite.*

† However, if h and k are disjoint and strongly separated, and $x \in h$ and $y \in k$ are as in Corollary 3.27, then $\delta(h, k) = d(x, y)$.

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Proof. Let Γ_0 be the finite index subgroup which preserves the irreducible factor decomposition $X = X_1 \times \cdots \times X_n$ and note that the Γ_0-action is still essential and non-elementary (Remark 3.4 and Lemma 3.20). Then, the result follows by applying Lemma 7.9 to the Γ_0-action on $S = \mathcal{S}_R^{(N)} \cap \mathfrak{H}_i^{N+1}$. We must therefore show that for each N there is an R such that, for each $i = 1, \ldots, n$, this collection is not empty.

Fix i, and $h_i \subset k_i^*$ with $(h_i, k_i) \in \mathcal{S}_i$. Applying the double skewering lemma (Lemma 3.8), we find $\gamma_i \in \Gamma_0$ such that $\gamma_i k_i^* \subset h_i$. In particular, $\gamma_i h_i \subset h_i$ are strongly separated in $\mathfrak{H}_i$. Setting $R = \max_{i=1,\ldots,n} \delta(\gamma_i^{N-1} h_i, k_i)$, we have that $(\gamma_i^{N-1} h_i, \ldots, h_i, k_i) \in \mathcal{S}_R^{(N)} \cap \mathfrak{H}_i^{N+1}$. □

Recall that $\tau : 2^{\mathfrak{H}} \to 2^{\mathfrak{H}}$ measurably assigns to a set its terminal elements (Lemma 6.7). If $H \subset \mathfrak{H}^N$, by abuse of notation, we shall use $\tau(H)$ to denote the terminal elements in the union of the projections of H to each factor. Namely, if the ith projection is $p_i : \mathfrak{H}^N \to \mathfrak{H}$ and $H \subset \mathfrak{H}^N$ then

$$\tau(H) := \tau\left(\bigcup_{i=1}^{N} p_i(H)\right).$$

By Remark 6.8, (and by considering the average of the Dirac masses at two points $x, y \in \overline{X}$) there are finitely many terminal elements in any subset of $[x, y] \cup [y, x]$ and hence, by Corollary 5.6, we deduce the following corollary.

COROLLARY 7.13. *Assume the hypotheses in Lemma 7.9 and let $\mathfrak{H} = \mathfrak{H}_1 \sqcup \cdots \sqcup \mathfrak{H}_n$ be the irreducible factor decomposition. For each N and each $i = 1, \ldots, n$ the following has $\lambda_- \otimes \lambda_+$-essential value zero:*

$$(\xi_-, \xi_+) \mapsto \#(\tau([\xi_+, \xi_-]^N \times [\xi_-, \xi_+] \cap \mathcal{S}_R^{(N)} \cap \mathfrak{H}_i^{N+1})).$$

We shall denote by Δ the *fat diagonal* in $\partial_{\mathrm{r}} X^2$. Namely, if $X = X_1 \times \cdots \times X_n$ is the irreducible decomposition of X then Δ is the collection $((\xi_1^1, \ldots \xi_1^n), (\xi_2^1, \ldots \xi_2^n)) \in \partial_{\mathrm{r}} X^2$ such that $\xi_1^i = \xi_2^i$, for some i.

Theorem 7.8 is a corollary to the following theorem.

THEOREM 7.14. *With the hypotheses in Lemma 7.9, it follows that*

$$\lambda_- \otimes \lambda_+(\partial_{\mathrm{r}} X^2) = 1.$$

and

$$\lambda_- \otimes \lambda_+(\Delta) = 0.$$

Proof. Let $X = X_1 \times \cdots \times X_n$ be the irreducible factor decomposition of X and let Γ_0 be the finite index subgroup of Γ which preserves each factor. Then, applying Lemma 7.12 and Corollary 7.13 to the Γ_0-action, and setting $\mathcal{S}_i(R) = (\mathcal{S}_i)_R^{(3)}$, we deduce that, as maps $\partial X \times \partial X \to \mathbb{N} \cup \{\infty\}$, the essential values of

$$(\xi_-, \xi_+) \mapsto \#([\xi_+, \xi_-] \times [\xi_-, \xi_+] \cap \mathcal{S}_i(R))$$

and

$$(\xi_-, \xi_+) \mapsto \#(\tau([\xi_+, \xi_-] \times [\xi_-, \xi_+] \cap \mathcal{S}_i(R)))$$

are ∞ and 0, respectively.

We claim that if $R > 0$ then $\partial_r X^2$ contains the intersection of

$$\bigcap_{i=1}^{n}\{(\xi_-, \xi_+) \in \partial X \times \partial X : \#([\xi_+, \xi_-] \times [\xi_-, \xi_+] \cap \mathcal{S}_i(R)) = \infty\}$$

with

$$\bigcap_{i=1}^{n}\{(\xi_-, \xi_+) \in \partial X \times \partial X : \#(\tau([\xi_+, \xi_-] \times [\xi_-, \xi_+] \cap \mathcal{S}_i(R))) = 0\},$$

which, for R sufficiently large, has full $\lambda_- \otimes \lambda_+$-measure.

Suppose that (ξ_-, ξ_+) is such that for some $R > 0$ and for each $i = 1, \ldots, n$,

$$\#([\xi_+, \xi_-] \times [\xi_-, \xi_+] \cap \mathcal{S}_i(R)) = \infty,$$

and

$$\#(\tau([\xi_+, \xi_-]^N \times [\xi_-, \xi_+] \cap \mathcal{S}_i(R))) = 0.$$

From these hypotheses, we shall now construct a bi-infinite descending chain $\{s_m : m \in \mathbb{Z}\} \subset [\xi_+, \xi_-] \cap \mathfrak{H}_i$ which, as elements of $\mathfrak{H}_i$, are strongly separated. Proposition 7.5 and the definition of regular points (in the reducible case) complete the proof that $\lambda_- \otimes \lambda_+(\partial_r X^2) = 1$.

Fix i. Suppose that $a_m \subsetneq a_{m-1}$, for $m \in \mathbb{Z}$ is a bi-infinite chain in $[\xi_+, \xi_-]$ such that for each m there exists $b_m, c_m \in [\xi_+, \xi_-]$ such that $a_m \subsetneq b_m \subsetneq c_m$ are pairwise strongly separated in $\mathfrak{H}_i$ with $\delta(a_m, c_m^*) \leqslant R$. We claim that $s_m := a_{mR}$ is a bi-infinite descending chain, pairwise strongly separated in $\mathfrak{H}_i$.

Fix m. Observe that $\delta(a_m, a_{m-R}^*) \geqslant R + 1$. Since $a_m \subset c_m$ and $\delta(a_m, c_m^*) \leqslant R$ it must be that $a_{m-R} \not\subseteq c_m$, i.e. $c_m^* \cap a_{m-R} \neq \varnothing$. Also, as $a_{m-R} \cap c_m$ and $a_{m-R}^* \cap c_m^*$ contain ξ_- and ξ_+, respectively, we deduce that either $c_m \subset a_{m-R}$ or $c_m \pitchfork a_{m-R}$. Either way, $b_m \subset a_{m-R}$ and hence a_m and a_{m-R} are strongly separated.

We now show that $\lambda_- \otimes \lambda_+(\Delta) = 0$. Indeed, by Corollary 7.6, it follows that Δ is contained in the union of the measure 0 set

$$\bigcup_{i=1}^{n}\{(\xi_-, \xi_+) \in \partial X \times \partial X : \#([\xi_+, \xi_-] \times [\xi_-, \xi_+] \cap \mathcal{S}_i(R)) = 0\}. \qquad \square$$

7.3. *Bridge points.* Recall that, as in Lemma 3.28, associated with a strongly separated pair $h \subset k^*$ there is a combinatorial bridge $B(h, k)$ with the property that if $p \in B(h, k)$ then $p = m(x, p, y)$ for every $(x, y) \in h \times k$.

Definition 7.15. Assume that $X = X_1 \times \cdots \times X_n$ is the irreducible decomposition of X into irreducible factors, corresponding to the decomposition $\mathfrak{H} = \mathfrak{H}_1 \sqcup \cdots \sqcup \mathfrak{H}_n$. An element $x \in X$ is called a *bridge point* if, for each i, there exists a pair of disjoint half-spaces h_i, k_i, strongly separated in $\mathfrak{H}_i$ such that $x \in B(h_1, k_1) \times \cdots \times B(h_n, k_n)$.

We note that the property of being a bridge point is Γ-invariant.

LEMMA 7.16. *Assume $\Gamma \to \mathrm{Aut}(X)$ is a non-elementary and essential action, $\lambda_\pm$ being quasi-Γ-invariant measures on ∂X such that $(\partial X^2, \lambda_- \otimes \lambda_+)$ is isometrically* *ergodic. There are disjoint half-spaces h_i, k_i strongly separated in $\mathfrak{H}_i$ such that for every bridge point $x \in B(h_1, k_1) \times \cdots \times B(h_n, k_n)$ and $\lambda_- \otimes \lambda_+$-a.e. $(\xi_-, \xi_+) \in \partial_M X^2$ the map $(\xi_-, \xi_+) \mapsto \#(\mathcal{I}(\xi_-, \xi_+) \cap \Gamma \cdot x)$ is infinite.*

Before proceeding with the proof, we note the straightforward but important fact that an interval in a product is just the product of the corresponding intervals (see Remark 3.11).

Proof of Lemma 7.16. For every $x \in X$, the map $(\xi_-, \xi_+) \mapsto \#(\mathcal{I}(\xi_-, \xi_+) \cap \Gamma \cdot x)$ is Γ-invariant and hence essentially constant by ergodicity. Furthermore, $\Gamma \cdot x \subset X$ is countable, and so the essential value must be 0 or ∞ by Corollary 5.6.

Now, by Proposition 7.5, Corollary 7.6, it follows that

$$\partial_{\mathrm{r}} X^2 \backslash \Delta \subset \bigcup_{\substack{(h_i, k_i) \in \mathcal{S}_i \\ i=1,\ldots,n}} (h_1 \times \cdots \times h_n) \times (k_1 \times \cdots \times k_n).$$

By Theorem 7.14 this union has full measure and hence one of the sets must have positive measure. Fix $(h_1 \times \cdots \times h_n) \times (k_1 \times \cdots \times k_n)$ of positive measure and note that it is chosen precisely to have a positive measure intersection with $\partial_{\mathrm{r}} X^2 \backslash \Delta$. Let $x \in (h_1^* \cap k_1^*) \times \cdots \times (h_n^* \cap k_n^*)$ be a bridge point. Then, by Lemma 3.28, the map $(\xi_-, \xi_+) \mapsto \#(\mathcal{I}(\xi_-, \xi_+) \cap \Gamma \cdot x)$ takes non-zero values on this positive measure set and hence the map has infinite essential value. □

COROLLARY 7.17. *With the hypotheses and notation as in Lemma 7.16, the strip $S : \partial X \times \partial X \to 2^\Gamma$ given by $S(\xi_-, \xi_+) = \{\gamma \in \Gamma : \gamma x \in \mathcal{I}(\xi_-, \xi_+)\}$ is infinite for $\lambda_- \otimes \lambda_+$-a.e. $(\xi_-, \xi_+) \in \partial X \times \partial X$ whenever $x \in B(h_1, k_1) \times \cdots \times B(h_n, k_n)$.*

8. *Maximality*

We shall employ the Kaimanovich strip condition to deduce the maximality of the boundary. This strip condition can be interpreted as follows: if there is a Γ-equivariant assignment $S : B_- \times B_+ \to 2^\Gamma$ that is *well behaved*, then B_+ is the maximal (Γ, μ)-boundary. The sets are called strips and are *well behaved* if they grow subexponentially. For us their growth will be dominated by a polynomial of degree D, where D is the dimension of the complex X.

8.1. *The Kaimanovich strip condition.*

Definition 8.1. A pseudonorm on a group Γ with identity 1_Γ is a map $|\cdot| : \Gamma \to \mathbb{R}_{\geqslant 0}$ such that, for all $\gamma, \gamma' \in \Gamma$, it is:

- normalized: $|1_\Gamma| = 0$;
- symmetric: $|\gamma^{-1}| = |\gamma|$;
- subadditive: $|\gamma\gamma'| \leqslant |\gamma| + |\gamma'|$.

A pseudonorm satisfying the property that $|\gamma| \to \infty$ as $\gamma \to \infty$ is said to be *proper*. Furthermore, if $|\gamma| = 0$ implies that $\gamma = 1_\Gamma$ then it is called a *norm*.

Suppose that Γ acts by isometries on (X, d). Fix a base point $\star \in X$. This allows us to consider the associated pseudonorm $|\gamma|_\star = d(\gamma\star, \star)$ on Γ which in turn yields the following nested increasing subsets which exhaust Γ:

$$\mathcal{B}_k = \mathcal{B}_k(\star) = \{\gamma \in \Gamma : d(\gamma\star, \star) \leqslant k\}.$$

Definition 8.2. For a fixed pseudonorm $|\cdot| : \Gamma \to \mathbb{R}$, a probability measure μ on Γ is said to have:

- *finite first logarithmic moment* (with respect to $|\cdot|$) if
$$\sum_{\gamma\in\Gamma} \mu(\gamma) \log|\gamma| < \infty;$$
- *finite entropy* if $H(\mu) := -\sum_{\gamma\in\Gamma} \mu(\gamma) \log \mu(\gamma) < \infty$.

We should like to know that, under reasonable conditions, the random walk, when translated to an orbit on X, is *transient*†.

LEMMA 8.3. *Let μ be a generating probability measure on the non-amenable group Γ. Then, for any proper pseudonorm $|\cdot| : \Gamma \to \mathbb{R}_+$ and $\mathbf{P}'$-a.e. $\omega' \in \Omega'$ we have that $|\omega_n| \to \infty$.*

Proof. The μ-random walk is transient since Γ is non-amenable and μ is generating [**DG73**, Theorem 2]. Since $|\cdot| : \Gamma \to \mathbb{R}_+$ is a proper function, it follows that $|\gamma| \to \infty$ precisely when $\gamma \to \infty$ in Γ. □

We shall also require the following lemma, the proof of which is straightforward.

LEMMA 8.4. *Let $☼ \in X$ be a point such that the* $\mathrm{stab}_\Gamma(☼)$ *is finite and set* $C = \#\mathrm{stab}_\Gamma(☼)$. *If $S \subset X$ then*
$$\#\{\gamma \in \Gamma : \gamma ☼ \in S\} = C \cdot \#(S \cap \Gamma \cdot ☼).$$

And finally, in the following theorem, we have our main tool for showing maximality.

THEOREM 8.5. [**Kai03**, the strip condition] *Let μ be a probability measure with finite entropy $H(\mu)$ on a countable group Γ, and let (B_-, λ_-) and (B_+, λ_+) be $\check{\mu}$- and μ-boundaries, respectively. If there exists a pseudonorm $|\cdot| : \Gamma \to \mathbb{R}$ and a measurable Γ-equivariant map $S : B_- \times B_+ \to 2^\Gamma$ such that, for all $\gamma \in \Gamma$ and $\lambda_- \otimes \lambda_+$-a.e. $(b_-, b_+) \in B_- \times B_+$,*
$$\frac{1}{n} \log \#[S(b_-, b_+) \cdot \gamma \cap \mathcal{G}_{|\omega'_n|}] \xrightarrow[n\to\infty]{} 0$$
in measure $\mathbf{P}'$ on the space of sample paths $\overline{\omega}' \in \Omega'$, then the boundary (B_+, λ_+) is maximal.

8.2. *Proof of maximality.* By Theorem 7.1, if $\Gamma \to \mathrm{Aut}(X)$ is a non-elementary and essential action on the finite-dimensional CAT(0) cube complex X, then there exist probability measures $\lambda_\pm$ on the Roller boundary ∂X such that $(\partial X, \lambda_-)$ and $(\partial X, \lambda_+)$ are $\check{\mu}$- and μ-boundaries, respectively.

THEOREM 8.6. *Let Γ be a countable discrete group. Assume $\Gamma \to \mathrm{Aut}(X)$ is a non-elementary, essential, and proper action on the finite-dimensional CAT(0) cube complex X. Let μ be a probability measure on Γ with finite entropy $H(\mu)$. If there is a base point $☼ \in X$ for which μ has finite first logarithmic-moment $\sum_{\gamma\in\Gamma} \mu(\gamma) \log|\gamma|_☼ < \infty$ then ∂X*

† Recall that a random walk on a discrete countable group is said to be *transient* if for $\mathbf{P}'$-a.e. $\omega' \in \Omega'$ we have $\omega_n \to \infty$ in Γ.

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admits probability measures λ_- and λ_+ making it the Furstenberg–Poisson boundary for $\check{\mu}$ and μ, respectively.

Proof. Assume $\Gamma \to \mathrm{Aut}(X)$ is a non-elementary, essential, and proper action on the CAT(0) cube complex X. Fix a generating probability measure μ of finite entropy. By Corollary 7.2, there exist $\lambda_\pm \in \mathcal{P}(\partial X)$ such that $(B_-, \lambda_-) := (\partial X, \lambda_-)$ and $(B_+, \lambda_+) := (\partial X, \lambda_+)$ are $(\Gamma, \check{\mu})$ and (Γ, μ) boundaries, respectively. Fix a base point $\star \in X$ for which μ has finite first logarithmic moment with respect to the pseudonorm induced by $\star$; it will be denoted by $|\cdot|$.

Also recall that Corollary 7.17 guarantees the existence of disjoint half-spaces h_i, k_i strongly separated in $\mathfrak{H}_i$ such that if $\star' \in B(h_1, k_1) \times \cdots \times B(h_n, k_n)$ is a bridge point then for $\lambda_- \otimes \lambda_+$-a.e. (b_-, b_+) these associated strips are infinite:

$$S(b_-, b_+) = \{\gamma \in \Gamma : \gamma\star' \in \mathcal{I}(b_-, b_+)\}.$$

Maximality of the boundary will follow from Kaimanovich's strip condition (Theorem 8.5) if we show that for every $\gamma_0 \in \Gamma$ and $\lambda_- \otimes \lambda_+$-a.e. $(b_-, b_+) \in B_- \times B_+$, the following converges in measure $\mathbf{P}$ for $\overline{\omega} \in \Omega$:

$$\frac{1}{n} \log \#[S(b_-, b_+) \cdot \gamma_0 \cap \mathcal{B}_{|\omega_1 \cdots \omega_n|}] \underset{n\to\infty}{\longrightarrow} 0.$$

To this end, fix $\gamma_0 \in \Gamma$ and a generic $(b_-, b_+) \in B_- \times B_+$ and note that

$$\begin{aligned} S(b_-, b_+)\gamma_0 &= \{\gamma \in \Gamma : \gamma\gamma_0^{-1}\star' \in \mathcal{I}(b_-, b_+)\} \\ &= \{\gamma \in \Gamma : \gamma_0\gamma\gamma_0^{-1}\star' \in \mathcal{I}(\gamma_0 b_-, \gamma_0 b_+)\} \\ &= \gamma_0^{-1} S(\gamma_0 b_-, \gamma_0 b_+)\gamma_0. \end{aligned}$$

Observe that by subadditivity and symmetry $|\gamma_0\gamma\gamma_0^{-1}| \leqslant |\gamma| + 2|\gamma_0|$, and so if $R > 0$ then $\gamma_0 \mathcal{B}_R \gamma_0^{-1} \subset \mathcal{B}_{R+2|\gamma_0|}$. We deduce that

$$\#[S(b_-, b_+) \cdot \gamma_0 \cap \mathcal{B}_R] \leqslant \#[S(\gamma_0 b_-, \gamma_0 b_+) \cap \mathcal{B}_{R+2|\gamma_0|}].$$

Next, we would like to bound $\#[S(b_-, b_+) \cap \mathcal{B}_R]$ as a function of R that is independent of the generic point (b_-, b_+).

By comparing the volume in the ℓ^1-metric in $\mathbb{Z}^D$ to the ℓ^∞-metric we see that the cardinality of any ball of radius R in Euclidean D-space is bounded above by $(2R+1)^D$. Theorem 2.8 guarantees the existence of an isometric embedding $\mathcal{I}(b_-, b_+) \hookrightarrow \overline{\mathbb{R}}^D$, and so it follows that a ball of radius R in an interval also has cardinality bounded above by $(2R+1)^D$.

Fix $x \in \mathcal{I}(b_-, b_+) \cap N_R(\star')$, where $N_R(\star')$ denotes the ball of radius R in X centered at $\star'$. Then applying the triangle inequality we deduce that

$$\mathcal{I}(b_-, b_+) \cap N_R(\star') \subset \mathcal{I}(b_-, b_+) \cap N_{2R}(x)$$

and by the previous paragraph

$$\#(\mathcal{I}(b_-, b_+) \cap N_R(\star')) \leqslant \#(\mathcal{I}(b_-, b_+) \cap N_{2R}(x)) \leqslant (4R+1)^D.$$

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We also have that $d(\gamma\star', \star') \leqslant 2d(\star, \star') + d(\gamma\star, \star)$ and so, setting $\delta = 2d(\star, \star')$ and $C' = \#\mathrm{stab}_\Gamma(\star')$, we apply Lemma 8.4 to deduce that

$$\begin{aligned}\#[S(b_-, b_+) \cap \mathcal{B}_R] &\leqslant \#\{\gamma : d(\gamma\star', \star') \leqslant R + 2d(\star, \star') \text{ and } \gamma\star' \in \mathcal{I}(b_-, b_+)\}\\ &= C' \cdot \#(\mathcal{I}(b_-, b_+) \cap N_{R+\delta}(\star') \cap \Gamma \cdot \star')\\ &\leqslant C' \cdot [4(R+\delta)+1]^D.\end{aligned}$$

By Lemma 8.3 and Corollary 7.17, we have that for $\mathbf{P}$-a.e. $\overline{\omega} \in \Omega$, $\lambda_- \otimes \lambda_+$-a.e. (b_-, b_+), and n sufficiently large, the following is non-empty:

$$S(b_-, b_+) \cdot \gamma_0 \cap \mathcal{B}_{|\omega_1 \cdots \omega_n|}.$$

And so

$$\begin{aligned}0 &\leqslant \frac{1}{n} \log \#(S(b_-, b_+) \cdot \gamma_0 \cap \mathcal{B}_{|\omega_1 \cdots \omega_n|})\\ &\leqslant \frac{1}{n} \log \#(S(\gamma_0 b_-, \gamma_0 b_+) \cap \mathcal{B}_{|\omega_1 \cdots \omega_n| + 2|\gamma_0|})\\ &\leqslant \frac{1}{n} \log(C' \cdot [4(|\omega_1 \cdots \omega_n| + 2|\gamma_0| + \delta) + 1]^D).\end{aligned}$$

Therefore, the quantity on the first line converges in measure to zero if the quantity on the last line converges to zero in measure, if and only if the following converges in measure:

$$\frac{1}{n} \log |\omega_1 \cdots \omega_n| \to 0.$$

To this end, observe that, since the pseudonorm is subadditive, we have that

$$\frac{1}{n} \log |\omega_1 \cdots \omega_n| \leqslant \frac{1}{n} \log\left(\sum_{k=1}^{n} |\omega_k|\right).$$

The right-hand side of this inequality converges in measure to zero since μ has finite first logarithmic moment with respect to $|\cdot|$ (see [**Aar97**, Proposition 2.3.1]). □

9. *Proof of the Tits alternative*

There are by now various Tits alternatives for groups acting on CAT(0) cube complexes. Sageev and Wise showed that groups that admit a (strongly) proper action on a proper CAT(0) cube complex are either virtually free or contain a freely acting free group [**SW05**].

Caprace and Sageev also have two versions of a Tits alternative. They showed that a group acting on a CAT(0) cube complex is either elementary or contains a freely acting free group. They also showed that if the action is proper, then either the group is {locally finite}-by-{virtually abelian} or contains a free group (which may or may not act freely)† [**CS11**].

We have gathered almost all the necessary tools for the proof of our Tits alternative. Here are a few more.

PROPOSITION 9.1. *If $\Gamma \to \mathrm{Aut}(X)$ is Roller elementary, then there exist $v, w \in \overline{X}$ such that $\Gamma \cdot \mathcal{I}(v, w) = \mathcal{I}(v, w)$. Furthermore, if Γ has an orbit whose cardinality is an odd integer, then Γ has a fixed point in $\overline{X}$.*

† In fact, neither the Sageev–Wise result nor the Caprace–Sageev result prove directly that the free groups act freely, but an argument such as the one below should suffice to deduce that these do indeed act freely.

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Proof. Let $o \subset \overline{X}$ be a point whose Γ-orbit is finite. Let m be the average of the Dirac masses on $\Gamma \cdot o$. Then clearly Γ preserves the measure m and therefore $\overline{X}_m$. Furthermore, if o is an orbit whose cardinality is an odd integer, then $m(h) \neq 1/2$ for every $h \in \mathfrak{H}$ and, in particular, H_m^+ satisfies both consistency and totality meaning that $\overline{X}_m$ is a single point. □

From this we obtain the following version of a classical result of Adams and Ballman which states that for a locally compact Hadamard space if Γ is an amenable group acting by isometries then it either fixes a point in the visual boundary or preserves a flat [**AB98**]. This result has been generalized in many contexts such as [**CL10**, **CM13**].

LEMMA 9.2. *Any action of an amenable group on a CAT(0) cube complex is Roller elementary. Furthermore, if there is an odd orbit then there is a fixed point in $\overline{X}$, and otherwise there is an invariant interval.*

Proof. Suppose Γ is an amenable group acting on X. Then, it admits an invariant probability measure m on $\overline{X}$. By invariance, we have that H_m and H_m^+ are invariant as well, and hence $\overline{X}_m$ is Γ-invariant. By Corollary 2.9, there are finitely many elements on which $\overline{X}_m$ is an interval, and therefore that set is Γ-invariant, hence the Γ-action is Roller elementary. □

Recall that a subgroup is said to be X-locally elliptic if every finitely generated subgroup has a fixed point. Also, a subgroup is said to be the X-locally elliptic radical if it is the unique maximal normal subgroup which is X-locally elliptic. We then have the following theorem (see also [**CL11**, Theorem A.5]).

THEOREM 9.3. [**CFI12**, Caprace's Theorem B1] *Consider the stabilizer* $\mathrm{stab}(x) \leqslant \mathrm{Aut}(X)$, *for $x \in \partial X$. Then there is a virtually abelian group A of rank $n \leqslant \dim(X)$ such that if N is the X-locally elliptic radical then we have an exact sequence:*

$$1 \to N \to \mathrm{stab}(x) \to A \to 1.$$

We shall now turn to the proof of the Tits alternative. It has been rephrased from the introduction in the spirit of Pays and Valette [**PV91**].

THEOREM. (Tits alternative) *Let X be a finite-dimensional CAT(0) cube complex and $\Gamma \leqslant \mathrm{Aut}(X)$. The following are equivalent:*

(1) Γ *does not preserve any interval $\mathcal{I} \subset \overline{X}$;*

(2) *the Γ-action is Roller non-elementary;*

(3) Γ *contains a non-abelian free subgroup acting freely on X.*

Furthermore, if X is locally compact, then these are equivalent to

(4) *the closure $\overline{\Gamma}$ in* $\mathrm{Aut}(X)$ *is non-amenable.*

Proof. (1) $\Longrightarrow$ (2): This is Proposition 9.1.

(2) $\Longrightarrow$ (3): If the action is Roller non-elementary then by Proposition 3.10 there is a finite index subgroup $\Gamma' \leqslant \Gamma$ and a subcomplex $X' \subset \overline{X}$ with half-space structure $\mathfrak{H}'$ where the Γ'-action is essential and non-elementary. By Corollary 3.17, there exists a facing triple $h_1, h_2, h_3 \in \mathfrak{H}'$ such that as subsets of X' they are facing. Applying the double skewering lemma (Lemma 3.8), we may assume there is a facing quadruple $h_1, h_2, h_3, h_4 \in \mathfrak{H}' \subset \mathfrak{H}$. Once more, the double skewering lemma (Lemma 3.8) guarantees the existence

of $a, b \in \Gamma'$ such that $ah_1 \subset h_2^* \subset h_1$, and $bh_3 \subset h_4^* \subset h_3$. Letting $A = h_1$, and $B = h_3$, we see that (the complements of) A, aA^*, B, bB^* form a standard ping–pong table and so the obvious homomorphism $F_2 \to \langle a, b\rangle$ is an isomorphism. We note that, by Remark 2.12, these half-spaces as subsets of X have the same relationships and hence we now think of them as subsets of X.

Without loss of generality, we may assume that the F_2 orbits of A, aA^*, B, bB^* are pairwise disjoint. Indeed, we may pass to the cubical subdivision if necessary to assure that $F_2 \cdot \{A\} \cap F_2 \cdot \{aA^*\} = \varnothing = F_2 \cdot \{B\} \cap F_2 \cdot \{bB^*\}$. And up to replacing this copy of F_2 with another (i.e. $\langle a^p, b^q\rangle$ for some $p, q \in \mathbb{N}$), we may assume that $F_2 \cdot \{A\} \cap F_2 \cdot \{bB^*\} = \varnothing = F_2 \cdot \{B\} \cap F_2 \cdot \{A\}$.

We now show that this action is free. To this extent, observe that the proof of the ping–pong lemma shows that if there is a point fixed by an element of $F_2 \backslash \{1\}$, then it does not belong to $\mathcal{F} := A \cap aA^* \cap B \cap bB^*$ or any of its translates. Therefore, we show that if $x \in X$ then $x \in w\mathcal{F}$ for some $w \in F_2$. Let us make an easy observation: if $o \in w\mathcal{F} = wA \cap wB \cap wbB^* \cap waA^*$ with $\#([o, x] \cap F_2 \cdot \{A, aA^*, B, bB^*\}) = 0$ then $x \in w\mathcal{F}$. Therefore, we aim to produce such a pair $w \in F_2$ and $o \in w\mathcal{F}$.

Fix $\star \in \mathcal{F}$. Consider the linearly ordered (by inclusion) set of half-spaces

$$[\star, x] \cap F_2 \cdot \{A, aA^*, B, bB^*\} =: \{w_1 h_1, \ldots, w_n h_n\},$$

where $w_i \in F_2$ and $h_n \in \{A, aA^*, B, bB^*\}$, with $w_i h_i \subset w_{i+1} h_{i+1}$. Let $o = m(\star, w_n\star, x)$. We claim that $o \in w_n\mathcal{F}$. Indeed, $w_n h_n \in U_x \cap U_{w_n\star}$ and so $w_n h_n \in U_o$.

Observe that $w_n \cdot \{A, B, aA^*, bB^*\} \backslash [o, x] = w_n \cdot \{A, B, aA^*, bB^*\} \backslash \{w_n h_n\}$. Indeed, the half-spaces $w_n \cdot \{A, B, aA^*, bB^*\}$ are facing, and $w_n h_n$ is minimal in $[o, x] \cap F_2 \cdot \{A, B, aA^*, bB^*\}$. Therefore, if $h \in w_n \cdot \{A, B, aA^*, bB^*\} \backslash \{w_n h_n\}$ then either both $\star, x \in h$ or both $\star, x \in h^*$. Again, using that these half-spaces are facing, if $\star, x \in h^*$ then $\star \in h^* \cap w_n h_n^* = \varnothing$. Therefore, $\star, x \in h$ and $h \in U_o$.

Observe $o \in w_n\mathcal{F} = wA \cap wB \cap wbB^* \cap waA^*$. By construction we have that

$$\begin{aligned}[o, x] \cap F_2 \cdot \{A, B, aA^*, bB^*\} &\subset ([\star, x] \cap F_2 \cdot \{A, B, aA^*, bB^*\}) \backslash \{w_1 h_1, \ldots, w_n h_n\} \\ &= \varnothing\end{aligned}$$

and hence $x \in w_n\mathcal{F}$.

(3) $\Longrightarrow$ (1): Suppose that Γ preserves the interval $\mathcal{I}$. If $\mathcal{I} \subset X$ then it is finite and Γ has a finite index subgroup fixing each element of $\mathcal{I}$, therefore no free subgroup can act freely.

Suppose now that $\mathcal{I} = \mathcal{I}(x, y)$ with $x \in \partial X$. Once more, Γ has a finite index subgroup Γ_0 by Corollary 2.9, fixing x. Let $F \leqslant \Gamma$ be a non-abelian free subgroup and $F_n = \Gamma_0 \cap F$. Since F_n is finitely generated, by Caprace's Theorem 9.3, the commutator subgroup $[F_n, F_n]$ has a fixed point in X, and hence the action of F is not free.

(3) $\Longrightarrow$ (4): If X is locally compact then so is $\mathrm{Aut}(X)$ and hence the closure $\overline{\Gamma}$. Since a freely acting group is necessarily closed, and $\overline{\Gamma}$ contains such a free subgroup, we concude that $\overline{\Gamma}$ is non-amenable.

(4) $\Longrightarrow$ (1): Once more, assume that X is locally compact. By Corollary 2.9, preserving an interval implies the existence of a finite index subgroup with a fixed point in $\overline{X}$. Therefore, the contrapositive will be shown if we demonstrate that point stabilizers are

amenable. If $x \in X$, the stabilizer $\mathrm{stab}(x)$ is compact and hence amenable. It follows that if $x \in \partial X$ then N_x the X-locally elliptical radical of $\mathrm{stab}(x)$ is also amenable. Indeed, N_x is a union of the compact groups $\mathrm{stab}(y) \cap \mathrm{stab}(x)$, where $y \in X$ and is hence amenable. Applying Caprace's Theorem 9.3 once more finishes the proof. □

10. *Convergence actions*

An action of a countable discrete group Γ by homeomorphisms on a compact metrizable space M is said to be a *convergence action* if the diagonal action on the space of distinct triples is proper. Namely, if $x, y, z \in M$ are pairwise distinct, and $\gamma_n \in \Gamma$ any sequence, then up to passing to a subsequence $\#\{\gamma_n x, \gamma_n y, \gamma_n z\} \to N < 3$. Bader and Furman have shown that if Γ admits a convergence action on M then there is a Γ-equivariant map $\phi : B \to M$, where B is a Furstenberg–Poisson boundary of Γ [**BF14**, Theorem 3.2]. In this section, we show that the action of Γ on the Roller compactification $\overline{X}$ is not a convergence action if there is an interval $\mathcal{I} \subset \overline{X}$ with the following properties.

- The stabilizer of $\mathcal{I}$ in Γ is infinite.
- There exist $x_1, x_2, y_1, y_2 \in \mathcal{I}$ such that $\mathcal{I} = \mathcal{I}(x_1, y_1) = \mathcal{I}(x_2, y_2)$ and

$$\#\{x_1, x_2, y_1, y_2\} \geqslant 3.$$

We observe that these are rather weak conditions on higher dimensional CAT(0) cube complexes, for example they are satisfied if Γ has commuting independent hyperbolic elements, though this is by no means necessary.

Let us now show that, under the above conditions, the diagonal action of Γ on distinct triples is not proper. To this end, recall that since $\mathcal{I}$ is an interval, it embeds into $\overline{\mathbb{Z}}^D$, where D is the dimension of X. Then, the collection of $x \in \mathcal{I}$ for which there is a $y \in \mathcal{I}$ such that $\mathcal{I} = \mathcal{I}(x, y)$ has cardinality bounded above by 2^D by Corollary 2.9 and bounded below by 3, by assumption. Therefore, there is a subgroup of finite index which fixes each of these elements. The assumption that the stabilizer of $\mathcal{I}$ is infinite implies that the point-wise fixator of each of these is also infinite. Therefore, any distinct triple from that set has an infinite stabilizer and hence the action is not a convergence action.

Bowditch showed that if Γ admits a convergence action on a perfect compact metrizable space then it is hyperbolic and the space is homeomorphic to the visual boundary [**Bow98**]. The above discussion shows how the existence of large stabilizers of higher dimensional flats prohibits the action from being a convergence action. However, applying Bowditch's result to the Roller boundary is somewhat problematic as it has many isolated points in general, even without the presence of a higher dimensional flat. Indeed, if we take the cubical subdivision of any complex whose visual boundary is not isomorphic to the Roller boundary, then the vertices in the Roller boundary introduced by this process will all be isolated.

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