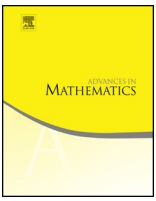
Advances in Mathematics 447 (2024) 109682

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim



An energy model for harmonic functions with junctions

D. De Silva [a,*], O. Savin [b]

[a] *Department of Mathematics, Barnard College, Columbia University, New York, NY 10027, United States of America*
[b] *Department of Mathematics, Columbia University, New York, NY 10027, United States of America*

ARTICLE INFO

Article history:
Received 24 May 2023
Received in revised form 2 February 2024
Accepted 23 March 2024
Available online 6 May 2024
Communicated by YanYan Li

Keywords:
Free boundary problems
Classification of global solutions
Regularity

ABSTRACT

We consider an energy model for harmonic graphs with junctions and study the regularity properties of minimizers and their free boundaries.

© 2024 Elsevier Inc. All rights reserved.

1. Introduction

In this paper we propose an energy model for N harmonic graphs with junctions and study the regularity properties of the minimizers and their free boundaries.

The physical motivation is the following. Let Ω be a bounded domain in $\mathbb{R}^n$ and consider $N \geq 2$ elastic membranes which are represented by the graphs of N real-valued functions

* Corresponding author.
E-mail addresses: desilva@math.columbia.edu (D. De Silva), savin@math.columbia.edu (O. Savin).

https://doi.org/10.1016/j.aim.2024.109682
0001-8708/© 2024 Elsevier Inc. All rights reserved.

$$\{(x, u_i(x))| \quad x \in \Omega\} \quad \subset \mathbb{R}^{n+1}, \qquad i = 1, \dots, N,$$

which are in contact with one-another. In the most simplified form, the elastic membranes are modeled by harmonic graphs. The non-penetration condition implies the functions are ordered in the vertical direction, so we assume that

$$u_1 \geq u_2 \geq \dots \geq u_N.$$

Suppose all membranes coincide initially, say $u_1 = u_2 = \dots = u_N = 0$, and then we move them continuously by pulling apart their boundary data $t\varphi_i$ on $\partial\Omega$ with $t \geq 0$, and $\varphi_1 > \varphi_2 > \dots > \varphi_N$. We consider the physical situation when the membranes do not separate strictly in the whole cylinder $\Omega \times \mathbb{R}$ instantaneously for all small $t > 0$, but rather in a continuous way. In other words, for small t, the strict separation happens only near the boundary while the functions still coincide well in the interior of Ω. This means the membranes *stick* to each other and we need to spend energy to physically separate them (say for example if the membranes are wet, or the elastic material has some adhesive properties).

1.1. The model

In view of the discussion above, a natural energy functional associated to a system of adhesive elastic membranes is given by

$$J_N(U, \Omega) := \int_\Omega (|\nabla U|^2 + W(U))dx, \quad U := (u_1, \dots, u_N), \quad N \geq 2, \tag{1.1}$$

with

$$|\nabla U|^2 := \sum_{i=1}^{N} |\nabla u_i|^2, \quad W(U) = \#\{u_1, \dots, u_N\}, \tag{1.2}$$

and J_N is defined over the class of admissible vector-valued functions

$$\mathcal{A} := \left\{U \in H^1(\Omega)| \quad u_1 \geq u_2 \geq \dots \geq u_N\right\}. \tag{1.3}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, and the potential term $W(U)$ represents the cardinality of the set $\{u_1, \dots, u_N\}$ (that is the number of distinct elements in the set). Clearly $W(U)$ is minimized when all u_i's coincide. On the other hand, the Dirichlet integral is minimized when each u_i is harmonic, and by the maximum principle they belong to the admissible cone $\mathcal{A}$ provided the boundary data is in $\mathcal{A}$ as well. The presence of the potential term has the effect of collapsing some of these graphs that are close to each other in a certain region, and we expect an optimal configuration consisting of piecewise harmonic graphs with junctions.

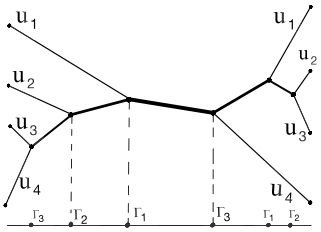


Fig. 1. A minimizer for $N = 4$, $n = 1$.

The functional J_N is lower semicontinuous and the existence of minimizers with given boundary data $\Phi \in \mathcal{A}$ on $\partial\Omega$ follows easily from the direct method of the calculus of variations. We remark that J_N is not convex, and the uniqueness of minimizers can fail.

One of the interesting questions about minimizers U of (1.4) regards the geometry of the graphs near junctions where the membranes separate. Precisely, there are possibly $N-1$ free boundaries originating from this minimization problem which are denoted by

$$\Gamma_i := \partial\{u_i > u_{i+1}\} \cap \Omega, \quad i = 1, \dots, N-1.$$

The sets $\{u_i > u_{i+1}\}$ have no a priori constraints with respect to one-another, which means the Γ_i's can intersect and cross each other and possibly have complicated geometries. The functions u_i are piecewise harmonic, that is they are harmonic in the interior of the regions carved out by the collections of the free boundaries. The problem is interesting even in dimension $n = 1$ in which the Γ_i's consist of points and the u_i's are piecewise linear, and a certain balancing condition needs to hold at the junction points. The physical situation can be described as the equilibrium configurations of N-tapes that stick to one another, see Fig. 1.

A first observation about minimizers U of (1.1) is that the average of the u_i's is harmonic in Ω, see Lemma 2.3. Moreover, minimizers remain invariant under the operation of adding the same harmonic function to each component of U. In view of this, one can reduce the problem to the 0 average situation

$$\sum u_i = 0 \quad \text{in } \Omega,$$

which means that we deal with a system involving only $N-1$ unknowns. Then, the case $N = 2$ corresponds to the scalar minimization problem

$$\min_{u_1 \geq 0} \int_\Omega \left(2|\nabla u_1|^2 + \chi_{\{u_1>0\}}\right) dx,$$

which is the classical one-phase free boundary problem introduced by Alt and Caffarelli in [1]. Here χ_E denotes the characteristic function of a set E.

The one-phase problem appears in cavitation flows in fluid dynamics, flame propagation etc., and it has been extensively studied over the past four decades. We refer to

the books of Caffarelli and Salsa [4] and Velichkov [23] for the mathematical treatment of this problem. Concerning the free boundary regularity of Γ_1, it is an analytic hypersurface outside a closed singular set of dimension at most $n-5$ (see [12]), and there are examples of free boundaries with point singularities in dimension $n = 7$, (see [8]).

For $N \geq 3$ we subtract 1 from W, which does not affect the problem, and rewrite the potential term in the form

$$W(U) = \sum_{i=1}^{N-1} \chi_{\{u_i > u_{i+1}\}}.$$

In this way, the model we propose can be viewed as a system of $N-1$ coupled one-phase free boundary problems that interact in the vertical direction.

1.2. Some related works

There are several works in the literature that involve free boundaries and energy functionals for vector-valued functions similar to the one we propose in (1.1). For example Mazzoleni, Terracini, Velichkov [15,16], Caffarelli, Shahgholian, Yeressian [5], and De Silva, Tortone [9] considered the vectorial one-phase problem with $W(U) = \chi_{\{|U|>0\}}$ and unconstrained U, which is relevant in the study of cooperative systems of species or in optimization problems for spectral functions, see also [13,14]. On the other hand, in [2] the authors studied a vectorial version of the obstacle problem by taking $W(U) = |U|$. In other cases motivated by strongly competitive systems or optimal partition problems, the components $u_i \geq 0$ have disjoint supports which means that the vector U is restricted to the union of the nonnegative coordinate axes. For this situation we refer to [3] where Caffarelli and Lin investigated harmonic maps onto such singular spaces, see also [21].

A minimization problem closely related to our model, which involves the constraint $U \in \mathcal{A}$ as in (1.3) and with the potential $W(U) = F \cdot U$, was introduced by Chipot and Vergara-Caffarelli in [6]. It describes the equilibrium configuration of N elastic membranes interacting with each-other under the action of external forces F. More recently, the second author and H. Yu established the optimal regularity for this problem and studied the free boundary regularity in a series of papers [18–20]. While these results motivated the current work, the two models have quite different qualitative properties that can be seen from simple one-dimensional examples as well as from the theorems in the next section.

Some of our results that involve the junction of $N = 3$ membranes resemble the soap-films like configurations for minimal surfaces in two and three dimensions that were first studied by J. Taylor in [22]. A crucial difference with respect to soap-films is that our energy model counts a common surface where more than two graphs coincide according to its multiplicity, and the collapsing phenomena is only due to the presence of the potential energy W.

1.3. Main results

In this section we state our main results. We recall that

$$J_N(U,\Omega) := \int_\Omega (|\nabla U|^2 + W(U))dx, \quad U = (u_1,\dots,u_N), \quad N \geq 2, \tag{1.4}$$

with

$$|\nabla U|^2 = \sum_{i=1}^N |\nabla u_i|^2, \quad W(U) := \sum_{i=1}^{N-1} \chi_{\{u_i > u_{i+1}\}}, \tag{1.5}$$

is defined over the class of admissible vector-valued functions

$$\mathcal{A} := \left\{ U \in H^1(\Omega) | \quad u_1 \geq u_2 \geq \ldots \geq u_N \right\}. \tag{1.6}$$

Positive constants depending only on N and the ambient dimension n are called universal. The first result gives the optimal regularity of minimizers.

Theorem 1.1 *(Optimal regularity). Let U minimize J_N in B_1. Then, $U \in C^{0,1}$ and*

$$\|U\|_{C^{0,1}(B_{1/2})} \leq C(1 + \|U\|_{L^2(B_1)}),$$

for $C > 0$ universal.

We study the regularity of the $N-1$ free boundaries

$$\Gamma_i := \partial\{u_i > u_{i+1}\}, \qquad i \in \{1,\dots,N-1\},$$

by performing a blow-up analysis at junction points. After rescaling, we may assume that we are in the 0-average situation, and the origin is a junction point where all membranes coincide

$$\sum u_i = 0, \qquad 0 \in \partial\{|U| > 0\}. \tag{1.7}$$

By employing Weiss monotonicity formula Proposition 3.1, and a non-degeneracy property of minimizers, we can deduce that blow-ups are one-homogenous.

Theorem 1.2. *Let U minimize J_N in B_1, and assume (1.7) holds. Then*

$$\max_{B_r} |U| \geq cr,$$

and there exists a sequence of $r_j \to 0$ such that

$$U_{r_j}(x) := \frac{U(r_j x)}{r_j}$$

converges uniformly on compact sets of $\mathbb{R}^n$ *to a cone* $\bar{U}$*, i.e. a homogenous of degree one minimizer with* $0 \in \partial\{|\bar{U}| > 0\}$.

The minimizing cones in dimension $n = 1$ must coincide on one side of 0 and have two branches on the other side. Precisely for each $k \leq N-1$, we define $U_{0,k}$ to be the vector whose components are given by

$$u_i = \left(\frac{1}{k} - \frac{1}{N}\right)^{1/2} x^+ \qquad \text{for } i \leq k, \quad u_i = -\left(\frac{1}{N-k} - \frac{1}{N}\right)^{1/2} x^+ \qquad \text{for } i > k.$$

Proposition 1.3 *(1d cones). The only minimizing cones in dimension* $n = 1$ *that satisfy* (1.7) *are given by* $U_{0,k}$ *(up to a reflection with respect to 0).*

The classification of 1d cones combined with an ϵ-regularity theorem and a dimension reduction argument gives the following general partial regularity result.

Theorem 1.4 *(Partial regularity). Let* U *be a minimizer in* B_1*. The free boundaries* Γ_i *are analytic and disjoint from one another outside a closed set* Σ *of singular points of Hausdorff dimension* $n-2$*, and*

$$\mathcal{H}^{n-1}(\Gamma_i \cap B_{1/2}) \leq C,$$

with C *universal.*

The remaining results focus on the intersection points of two distinct free boundaries which, by Theorem 1.4, has dimension at most $n-2$. For this part we restrict to the simplest case that involves $N = 3$ membranes.

In dimension $n = 2$ we define V_0 as the cone whose components are given by

$$u_1(x) := \frac{1}{\sqrt{6}} \max\left\{x \cdot e_{\frac{\pi}{6}},\, 2x \cdot e_{-\frac{\pi}{6}},\, 0\right\}, \qquad e_\theta := (\cos\theta, \sin\theta),$$
$$u_3(x) := -u_1(x_1, -x_2), \qquad u_2 := -u_1 - u_3.$$

Theorem 1.5. *Let* $N = 3$*. Then, up to rotations and reflections,* V_0 *is the unique minimizing cone in dimension* $n = 2$ *which satisfies* $0 \in \Gamma_1 \cap \Gamma_2$ *and* (1.7).

The proof of Theorem 1.5 is indirect, through an elimination process. It turns out that there are two possible cone candidates, V_0 and V_s, which are critical for the energy J_N. We show however that V_s is not minimizing and also that the set of cones satisfying $0 \in \Gamma_1 \cap \Gamma_2$ and (1.7) is nonempty, see Section 6.

The next result gives the regularity of the free boundaries in two dimensions near an intersection point, see Fig. 2.

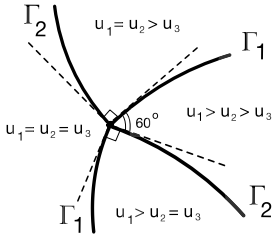


Fig. 2. Free boundary geometry at an intersection point in $\mathbb{R}^2$.

Theorem 1.6. *Let $N = 3$ and let U be a minimizer of J_N in $\Omega \subset \mathbb{R}^2$. Then Γ_1 and Γ_2 are piecewise $C^{1,\alpha}$ curves in a neighborhood of any intersection point $x_0 \in \Gamma_1 \cap \Gamma_2$, with $\alpha \sim 0.36$.*

Theorem 1.6 can be extended to arbitrary dimensions. For this we define regular intersection points

$$x_0 \in Reg(\Gamma_1 \cap \Gamma_2),$$

as those points for which there exists a blow-up cone at x_0 which is a rotation of a two-dimensional cone extended trivially in the remaining variables.

Theorem 1.7. *$Reg(\Gamma_1 \cap \Gamma_2)$ is locally a $C^{1,\alpha}$-smooth submanifold of codimension two. Near such an intersection point, each of the free boundaries Γ_1 and Γ_2 consists of two piecewise $C^{1,\alpha}$ hypersurfaces which intersect on $Reg(\Gamma_1 \cap \Gamma_2)$ at a 120° angle.*

As a consequence we obtain a general partial regularity result for the free boundary. We define $Reg(\Gamma_i)$ as the collection of points $x_0 \in \Gamma_i$ that have a blow-up profile which is a rotation of a one-dimensional cone extended trivially in the remaining variables.

Theorem 1.8 *(Partial regularity). Let $N = 3$ and U be a minimizer of J_N in B_1. Then*

$$\Gamma_1 \cup \Gamma_2 = Reg(\Gamma_1) \cup Reg(\Gamma_2) \cup Reg(\Gamma_1 \cap \Gamma_2) \cup \Sigma',$$

with $Reg(\Gamma_i)$ locally analytic hypersurface, $Reg(\Gamma_1 \cap \Gamma_2)$ locally a $C^{1,\alpha}$ submanifold of codimension two, and Σ' a closed singular set of Hausdorff dimension $n - 3$.

The study of regular intersection points leads to a transmission-type problem which appears in the linearization, which is new and interesting in its own, see (7.2)-(7.7) in Section 7. The value of α in the theorems above is dictated by the spectrum of the linearized problem. The novelty is that the transmission condition does not occur along a hypersurface but it involves a region of full dimension where two functions interact. We refer the reader to Sections 7 and 8 for more details.

The paper is organized as follows. In Section 2 we collect some general facts about minimizers and prove the optimal regularity result Theorem 1.1. In Section 3 we obtain the Weiss monotonicity formula and prove Theorem 1.2. We classify one-dimensional cones in Section 4 and then establish Theorem 1.4 in Section 5. The last three sections are devoted to the study of two dimensional cones and regular intersection points for $N = 3$ membranes.

2. Lipschitz continuity and non-degeneracy of minimizers

2.1. Preliminaries

Recall that, throughout this note, constants depending only on possibly n, N are called universal. Also, whenever this does not create confusion, the dependence of J_N on the domain is omitted.

We start by proving some basic facts about minimizers. Throughout Ω is assumed to be a bounded domain with Lipschitz boundary.

Lemma 2.1 *(Lower semicontinuity). If $U_m \to U$ in $L^2(\Omega)$, then*

$$\liminf J_N(U_m, \Omega) \geq J_N(U, \Omega).$$

Proof. By working with subsequences we may assume $\limsup J_N(U_m, \Omega) < \infty$, otherwise there is nothing to prove. Then $\nabla U_m \rightharpoonup \nabla U$ in L^2, and $U_m \to U$ a.e. in Ω. The conclusion follows by combining the lower semicontinuity of the Dirichlet energy with the lower semicontinuity of the function $W : \mathbb{R}^N \to \mathbb{R}$. □

As a consequence we obtain the existence of minimizers with boundary data in $H^1(\Omega)$.

Proposition 2.2. *Given $\Phi \in \mathcal{A}$, there exists a minimizer U of J_N in Ω with boundary data Φ on $\partial\Omega$.*

Next we show that the average of the u_i's is harmonic.

Lemma 2.3. *If U is a minimizer of J_N in Ω, then $\sum_{i=1}^N u_i$ is harmonic in Ω.*

Proof. Let $\psi \in C_0^1(\Omega)$, and $\Psi = (\psi, \dots, \psi)$. Then

$$W(U) = W(U + \epsilon\Psi)$$

and

$$\frac{J_N(U + \epsilon\Psi) - J_N(U)}{\epsilon} = \frac{1}{\epsilon} \int_\Omega (|\nabla(U + \epsilon\Psi)|^2 - |\nabla U|^2) dx$$

$$= 2 \int_\Omega \nabla(\sum_{i=1}^n u_i) \cdot \nabla \psi dx + o(\epsilon),$$

from which our claim follows. □

Similarly, the problem remains invariant with respect to the addition of a harmonic function ψ to each component, that is

$$J_N(U + \psi(1, \dots, 1)) = J_N(U) + C(\Psi, \Phi), \quad \Psi = \psi(1, \dots, \psi)$$

for some constant

$$C(\Psi, \Phi) = \int_\Omega |\nabla \Psi|^2 dx + 2 \int_{\partial\Omega} \Phi \cdot \Psi_\nu d\sigma,$$

that depends only on ψ and the boundary data Φ of U. Therefore, at times we may assume that U satisfies,

$$\sum_{i=1}^N u_i = 0. \tag{2.1}$$

Lemma 2.4. *Let U minimize J_N in Ω. Then u_1 is subharmonic and u_N is superharmonic in Ω.*

Proof. We prove the first claim, the second one can be proved similarly. Let v_1 be the harmonic replacement of u_1 in $B_r \subset \Omega$. We wish to prove that $u_1 \le v_1$ in B_r. Indeed, let

$$\tilde{u}_k = \min\{u_k, v_1\}, \quad \tilde{U} = (\tilde{u}_1, \dots, \tilde{u}_N).$$

Clearly,

$$W(U) \ge W(\tilde{U}),$$

hence by minimality

$$\int_{B_r} |\nabla U|^2 dx \le \int_{B_r} |\nabla \tilde{U}|^2 dx. \tag{2.2}$$

On the other hand, since

$$v_1 = u_1 \ge u_k, \quad \forall k = 1, \dots N, \quad \text{on } \partial B_r,$$

we have

$$w_k := (u_k - v_1)^+ \in H_0^1(B_r).$$

After integration by parts and using that v_1 is harmonic, we get

$$\int_{B_r} (|\nabla u_k|^2 - |\nabla \tilde{u}_k|^2) dx = \int_{B_r} (|\nabla (v_1 + w_k)|^2 - |\nabla v_1|^2) dx \geq 0,$$

with strict inequality unless $w_k \equiv 0$. Thus, in view of (2.2), $w_k \equiv 0$ in B_r for all k's, which gives the desired claim that $u_1 \leq v_1$ in B_r.

Since this argument can be repeated for any ball included in Ω, we conclude that u_1 is subharmonic. □

Next, we prove the following obvious yet useful fact.

Lemma 2.5. *Let U minimize J_N in Ω and assume that*

$$u_k > u_{k+1} \quad \text{for some } k \in \{1, \dots, N-1\}.$$

Then, $\underline{U} := (u_1, \dots, u_k)$ minimizes J_k in Ω and $\bar{U} := (u_{k+1}, \dots, u_N)$ minimizes J_{N-k}.

Proof. The proof is straightforward. Assume by contradiction that $\underline{V}$ minimizes J_k in $\Omega' \subset\subset \Omega$ with boundary data $\underline{U}$ and

$$J_k(\underline{V}) < J_k(\underline{U}). \tag{2.3}$$

Let $\bar{V}$ minimize J_{N-k} in Ω' with boundary data $\bar{U}$, and set

$$V = (\underline{V}, \bar{V}).$$

Since, by Lemma 2.4 $\underline{v}_k$ is superharmonic while $\bar{v}_{k+1}$ is subharmonic and one in strictly on top of the other one on the boundary, we conclude that $\underline{v}_k \geq \bar{v}_{k+1}$ in Ω', hence V is an admissible competitor. Furthermore, in view of (2.3) and our assumption,

$$J_N(V) \leq J_k(\underline{V}) + J_{N-k}(\bar{V}) + |\Omega'| < J_k(\underline{U}) + J_{N-k}(\bar{U}) + |\Omega'| = J_N(U)$$

and we reach a contradiction. □

2.2. Lipschitz continuity

We now turn to the proof of Lipschitz continuity. We start first by establishing that minimizers are Hölder continuous, by treating the potential term $W(U)$ as a perturbation. This result can be obtained directly from the theory of almost minimizers of variational integrals, see Section 7 in Giusti's book [11]. For completeness, we provide the details of the proof.

After multiplying U by $\delta^{1/2}$, we may assume that U minimizes

$$J_N^\delta(U) := \int_\Omega (|\nabla U|^2 + \delta W(U))dx,$$

with $\delta > 0$ small. Notice that the Hölder rescaling $\tilde{U}(x) = r^{-\alpha}U(rx)$, with $\alpha \in (0,1)$, minimizes the energy $J_N^{\tilde{\delta}}$ with parameter $\tilde{\delta} = r^{2-2\alpha}\delta$, and $\tilde{\delta} < \delta$ if $r < 1$.

Without loss of generality, we assume that we deal with minimizers in B_1. Hölder continuity follows with standard arguments from the next proposition.

From now on, in the body of the proofs $c, C > 0$ are universal constants possibly changing from step to step.

Proposition 2.6. *Let U be a minimizer of J_N^δ in B_1. For every $\alpha \in (0,1)$, there exist $\delta_0, \rho > 0$ depending on α, such that, if $\delta \le \delta_0$ and*

$$⨍_{B_1} |\nabla U|^2 dx \le 1, \tag{2.4}$$

then

$$\rho^{2-2\alpha} ⨍_{B_\rho} |\nabla U|^2 dx \le 1. \tag{2.5}$$

Proof. Let V be the harmonic replacement of U in B_1. Then, by the maximum principle $v_i \ge v_{i+1}, i = 1, \dots, N$, and V is an admissible competitor for the minimization of J_N^δ. Thus,

$$\begin{aligned}
\int_{B_1} |\nabla(U-V)|^2 dx &\le J_N^\delta(U) + \int_{B_1} (|\nabla V|^2 - 2\nabla U \cdot \nabla V)dx \\
&= J_N^\delta(U) - \int_{B_1} |\nabla V|^2 dx \\
&\le J_N^\delta(V) - \int_{B_1} |\nabla V|^2 dx \\
&= \int \delta W(V) dx \le \delta N.
\end{aligned} \tag{2.6}$$

Hence,

$$⨍_{B_\rho} |\nabla(U-V)|^2 dx \le \delta N \rho^{-n}. \tag{2.7}$$

On the other hand, since $|\nabla V|^2$ is subharmonic in B_1 and by (2.4)

$$\int_{B_1} |\nabla V|^2 dx \leq \int_{B_1} |\nabla U|^2 dx \leq C,$$

we conclude that

$$|\nabla V| \leq C, \quad \text{in } B_{1/2}.$$

In particular,

$$\fint_{B_\rho} |\nabla V|^2 dx \leq C_0,$$

for C_0 universal. Combining this last inequality with (2.7) we conclude that,

$$\fint_{B_\rho} |\nabla U|^2 dx \leq 2\delta N \rho^{-n} + 2C_0 \leq 4C_0,$$

as long as

$$\delta \leq \frac{C_0}{N} \rho^n.$$

Thus,

$$\rho^{2-2\alpha} \fint_{B_\rho} |\nabla U|^2 dx \leq 4C_0 \rho^{2-2\alpha} \leq 1,$$

by choosing ρ such that,

$$\rho^{2-2\alpha} \leq (4C_0)^{-1}. \quad \square$$

The next corollary now follows via standard arguments.

Corollary 2.7. *Let U be a minimizer to J_N in B_1. Then $U \in C^{0,\alpha}_{loc}(B_1)$, for any $\alpha \in (0,1)$ and*

$$[U]_{C^\alpha(B_{1/2})} \leq C(1 + \|\nabla U\|_{L^2(B_1)}),$$

with $C(\alpha, n, N)$.

Prior to proving Lipschitz continuity, we introduce the following definition.

Definition 2.8. We say that U is disconnected in B_ρ if there exists $k \in \{2, \dots, N\}$ such that $u_k > u_{k-1}$ in B_ρ. Otherwise we say that U is connected in B_ρ.

Our main Lipschitz regularity result reads as follows.

Lemma 2.9 *(Uniform Lipschitz estimate). Let U minimize J_N in B_1 and assume that $\sum_{i=1}^N u_i = 0$ and U is connected in $B_{1/2}$. Then*

$$\|U\|_{C^{0,1}(B_{1/2})} \leq C, \tag{2.8}$$

for $C > 0$ universal.

Then, the next theorem is an immediate corollary.

Theorem 2.10. *Let U minimize J_N in B_1. Then, $U \in C^{0,1}(B_{1/2})$ and*

$$\|U\|_{C^{0,1}(B_{1/2})} \leq C(1 + \|U\|_{L^2(B_1)}), \tag{2.9}$$

for $C > 0$ universal.

We use Lemma 2.9 to prove Theorem 2.10 by induction on N.

The case $N = 1$ corresponds to harmonic functions and it is obvious.

Next assume the claim holds for j membranes, for any $j \leq N - 1$. It suffices to prove (2.9) only in the case of N membranes with $\sum u_i = 0$.

Indeed, in the general case we let $h := \sum_i u_i$ which is harmonic, and then $U = H + V$ with $H := (h, \dots, h)$ and $\sum_i v_i = 0$. Since H satisfies (2.9) and

$$\|H + V\|_{L^2} = \|H\|_{L^2} + \|V\|_{L^2},$$

it suffices to show that V satisfies (2.9) as well.

If the N membranes are connected in $B_{1/2}$ then the claim follows from Lemma 2.9. If they are disconnected in $B_{1/2}$, then in view of Lemma 2.5, by the induction hypothesis we have

$$\|U\|_{C^{0,1}(B_{1/4}(x_0))} \leq C(1 + \|U\|_{L^2(B_{1/2})}), \quad \forall x_0 \in B_{1/4}, \tag{2.10}$$

which gives the desired claim.

Proof of Lemma 2.9. We argue by induction. In the case $N = 1$, $u_1 \equiv 0$ and the statement is trivial. Let us assume that (2.8) holds when U minimizes J_k and $k < N$. Then, by the proof above, also the estimate (2.9) holds for minimizers of J_k and $k < N$.

Let V be the harmonic replacement of U in B_1, and by the maximum principle $v_i \geq v_{i+1}$, and moreover

$$\sum_{i=1}^{N} v_i = 0. \tag{2.11}$$

Then, arguing as in (2.6) in Proposition 2.6, and by Poincare's inequality we obtain

$$\|U - V\|_{L^2(B_1)}, \|\nabla(U - V)\|_{L^2(B_1)} \leq C. \tag{2.12}$$

Set

$$\mu := \max_i \{v_i(0)\} = v_1(0).$$

By Harnack inequality,

$$v_i - v_{i+1} \geq c(v_i - v_{i+1})(0) \quad \text{in } B_{7/8}, \quad \forall i = 1, \dots, N-1,$$

and in view of (2.11) simple arithmetic gives that for some $k \in \{1, \dots, N-1\}$,

$$v_k - v_{k+1} \geq c(v_k - v_{k+1})(0) \geq c\mu \quad \text{in } B_{7/8}. \tag{2.13}$$

Furthermore, Harnack inequality for $v_1 \geq 0$ and $-v_N \geq 0$ also guarantees that

$$|V| = \max_i\{|v_i|\} = \max\{v_1, -v_N\} \leq C\mu \quad \text{in } B_{7/8}. \tag{2.14}$$

We now wish to show that μ is bounded by a universal constant. Indeed, if $\mu \gg 1$ then, by the second inequality in (2.12), (2.14), and the fact the V is harmonic, we get

$$\|\nabla U\|_{L^2(B_{7/8})} \leq C(\mu + 1) \leq C\mu.$$

Hence, by Corollary 2.7 and the fact that V is harmonic and satisfies (2.14), we get in $B_{3/4}$,

$$[U]_{C^\alpha} \leq C\mu, \quad [U - V]_{C^\alpha} \leq C\mu.$$

Thus, if at $x_0 \in B_{1/2}$, $|(U - V)(x_0)| = t$, then $|U - V| \geq t/2$ in a small neighborhood of size $r \sim (t/\mu)^{1/\alpha}$. By the first inequality in (2.12) we get that $t \leq C\mu^{n/(2\alpha+n)}$. Hence,

$$|U - V| \leq C\mu^{1-\delta} \quad \text{in } B_{1/2},$$

for some $0 < \delta < 1$. In view of (2.13)

$$u_k - u_{k+1} \geq c\mu - C\mu^{1-\delta} > 0 \quad \text{in} \quad B_{1/2},$$

and we contradict that U is connected in $B_{1/2}$. Thus $\mu \leq C$ as desired, and it follows from the estimates above that

$$\|U\|_{H^1(B_{1/2})} \leq C. \tag{2.15}$$

This proves the statement for the H^1 norm instead of the $C^{0,1}$ norm.

In order to conclude our proof, it suffices to obtain a scale invariant version of this H^1 estimate, i.e.

$$\fint_{B_{r_k}} |\nabla U|^2 \leq C, \quad r_k = 2^{-k}, \quad \forall k \geq 1.$$

In terms of the rescalings

$$\tilde{U}_r(x) := U(rx)/r, \quad x \in B_1,$$

this is equivalent to

$$\fint_{B_1} |\nabla \tilde{U}_{r_k}|^2 \leq C, \quad \forall k \geq 1.$$

Let $m \geq 1$ be the first value for which U is connected in B_{r_m} but it is not connected in $B_{r_{m+1}}$. If no such m exists, then (2.15) holds for all $\tilde{U}_{r_k}$ and the desired statement follows. Otherwise, in view of (2.15),

$$\|U_{r_k}\|_{H^1(B_{1/2})} \leq C, \quad \forall k \leq m. \tag{2.16}$$

Since U is disconnected in $B_{r_{m+1}}$, we can use the induction hypothesis and conclude that

$$\|U_{r_m}\|_{C^{0,1}(B_{1/4})} \leq C(1 + \|U_{r_m}\|_{L^2(B_{1/2})}) \leq C.$$

Then (2.16) holds for all $k \geq m$ and we reached the conclusion. □

Having established the continuity results above, we can obtain the following compactness property of minimizers.

Proposition 2.11 *(Compactness of minimizers). If U_m is a sequence of minimizers in B_1 and $U_m \to \bar{U}$ uniformly on compact sets, then $\bar{U}$ is a minimizer and*

$$J_N(U_m, B_r) \to J_N(\bar{U}, B_r) \qquad \forall r \in (0,1).$$

Proof. Since the U_m's are continuous, then $\bar{U}$ is continuous and the U_m's are uniformly bounded on compact sets. In view of the Lipschitz continuity Theorem 2.10, we conclude that the U_m's and $\bar{U}$ are uniformly Lipschitz on compacts.

Let V be a competitor which agrees with $\bar{U}$ in $B_1 \setminus B_r$. Let $0 \leq \varphi \leq 1$ be a radial cutoff function which is 1 in B_r and vanishes outside $B_{r+\delta}$. Then

$$V_m := \varphi V + (1-\varphi)U_m$$

agrees with U_m in $B_1 \setminus B_{r+\delta}$, and it is an admissible competitor as well. Thus, minimality implies that

$$J_N(U_m, B_{r+\delta}) \le J_N(V_m, B_{r+\delta}). \tag{2.17}$$

Since

$$\nabla V_m = (V-U_m)\nabla\varphi + \varphi\nabla V + (1-\varphi)\nabla U_m,$$
$$|\nabla U_m|, |\nabla \bar{U}| \le K, \qquad W(U_m), W(V_m) \le N, \quad |\nabla\varphi| \le C\delta^{-1},$$

we have that in the annulus $\mathcal{A}_\delta := B_{r+\delta} \setminus B_r$,

$$J_N(V_m, \mathcal{A}_\delta) - J_N(U_m, \mathcal{A}_\delta) \le C(1+K^2)|\mathcal{A}_\delta| + C\delta^{-1}\|U_m - \bar{U}\|^2_{L^\infty(\mathcal{A}_\delta)}.$$

The right hand side can be made arbitrarily small by first choosing δ small depending on K and them m large. On the other hand, in view of (2.17) we have

$$J_N(U_m, B_r) \le J_N(V_m, \mathcal{A}_\delta) - J_N(U_m, \mathcal{A}_\delta) + J_N(V, B_r).$$

In conclusion

$$\limsup J_N(U_m, B_r) \le J_N(V, B_r),$$

while the lower semicontinuity of the energy gives

$$J_N(\bar{U}, B_r) \le \liminf J_N(U_m, B_r). \quad \square$$

2.3. Non-degeneracy

Lemma 2.12 *(Caccioppoli's estimate). Let U minimize J_N in B_1. Then,*

$$\int_{B_{1/2}} (|\nabla U|^2 + W(U))\,dx \le C \int_{B_1} |U|^2 dx,$$

for $C > 0$ universal.

Proof. Let $0 \le \phi \le 1$ be a smooth bump function, which equals 1 in $B_{1/2}$ and 0 outside of $B_{3/4}$. Set,

$$\tilde{u}_k := u_k(1-\phi), \quad k = 1, \dots, N.$$

Then $\tilde{U} = (\tilde{u}_1, \dots, \tilde{u}_N)$ is an admissible competitor and by minimality,

$$\int_{B_1} (|\nabla U|^2 + W(U))\, dx \le \int_{B_1} (|\nabla \tilde{U}|^2 + W(\tilde{U}))\, dx,$$

hence

$$W(\tilde{U}) \le W(U) \text{ and } W(\tilde{U}) = 0 \text{ in } B_{1/2},$$

implying that

$$\int_{B_1} (|\nabla U|^2 + W(U)\chi_{B_{1/2}})\, dx \le \int_{B_1} |\nabla \tilde{U}|^2\, dx. \tag{2.18}$$

Moreover,

$$\begin{aligned}\int_{B_1} |\nabla \tilde{U}|^2\, dx &= \int_{B_1} ((1-\phi)^2 |\nabla U|^2 + |U|^2 |\nabla \phi|^2 - (1-\phi)\nabla |U|^2 \cdot \nabla \phi)\ \ dx \\ &\le \int_{B_1 \setminus B_{1/2}} |\nabla U|^2\, dx + \int_{B_1} |U|^2 (1-\phi)\Delta \phi\, dx,\end{aligned}$$

where in the last inequality we integrated by parts. Using this in (2.18) we get,

$$\int_{B_{1/2}} (|\nabla U|^2 + W(U))\, dx \le C \int_{B_1} |U|^2\, dx$$

with $C = C(n)$ as desired. □

A direct consequence of the Caccioppoli's estimate is that $|U|$ cannot be small and strictly positive in B_1.

Corollary 2.13. *If $\sum_{i=1}^N u_i = 0$ and $B_r(x_0) \subset \{|U| > 0\}$ then*

$$\max_{B_r(x_0)} |U| \ge cr,$$

for some $c > 0$ universal.

Indeed, by scaling we may assume that $x_0 = 0$, $r = 1$. Then $|U| > 0$ implies $W(U) \ge 1$ in B_1 and the conclusion follows by the estimate in Lemma 2.12.

We state a stronger version of non-degeneracy for $|U|$ than in Corollary 2.13.

Lemma 2.14 *(Non-degeneracy). Let U be a minimizer in B_1. If $\sum_{i=1}^N u_i = 0$ and $0 \in \partial\{|U| > 0\}$ then*

$$\max_{B_r} |U| \geq cr, \qquad \forall r < 1.$$

Proof. We use the previous corollary and a standard ball sequence construction for the classical one-phase problem. Notice that $\sum_{i=1}^N u_i = 0$ and $u_1 \geq .. \geq u_N$ implies that

$$c_0|U| \leq u_1 \leq |U|,$$

with $c_0 > 0$ depending on N and n, and recall that by Lemma 2.4 u_1 is a subharmonic function.

Assume that $r = 1$. We construct inductively a sequence of points $x_k \in \{|U| > 0\}$ such that x_0 is sufficiently close to the origin and

1) $u_1(x_{k+1}) \geq (1 + c_1)u_1(x_k)$

2)

$$c_2 r_k \leq u_1(x_k) \leq C_2 r_k, \qquad |x_{k+1} - x_k| \leq 4r_k, \qquad r_k := d(x_k, \{|U| = 0\}).$$

The inequality $u_1(x_k) \leq |U(x_k)| \leq C_2 r_k$ is a consequence of Lemma 2.9.

Let's assume that x_1,..,x_k are constructed, and let

$$y_k \in \partial\{|U| = 0\} \cap \partial B_{r_k}(x_k).$$

Since $u_1 \geq 0$ is subharmonic in $B_{r_k}(x_k)$, $u_1(x_k) \geq c_2 r_k$ and, by Lemma 2.9,

$$u_1(x) \leq |U(x)| \leq C_2|x - y_k|,$$

it follows from the mean value inequality that there exists $z_k \in B_{r_k}(x_k)$ such that $u_1(z_k) \geq (1 + c_1)u_1(x_k)$ for some $c_1 > 0$ depending on c_2, C_2 and n. We pick $x_{k+1} \in B_{t_k}(z_k)$ to be the point where the maximum value of u_1 occurs, where t_k represents the distance from z_k to the set $\{|U| = 0\}$. Then

$$\begin{aligned} u_1(x_{k+1}) &= \max_{B_{t_k}(z_k)} u_1 \\ &\geq c_0 \max_{B_{t_k}(z_k)} |U| \\ &\geq c_0\, c\, t_k \\ &\geq c_2\, d(x_{k+1}, \{|U| = 0\}), \end{aligned}$$

where in the second inequality we have used Corollary 2.13, and we chose c_2 small depending on c_0 and c. This proves the existence of the sequence x_k.

Then we have

$$(1+c_1)u_1(x_k) \le u_1(x_{k+1}) \le C_2 r_{k+1} \le C_3 r_k \le C_4 u_1(x_k),$$

hence $u_1(x_k)$ is a sequence that grows at a geometric rate. The conclusion of the lemma follows easily since $u_1(x_k) \sim r_k$. □

Next we show the weak non-degeneracy for the difference of two consecutive membranes. It is not clear whether the strong non-degeneracy holds for two consecutive membranes as well. This lemma will not be used in the remaining of the paper.

Lemma 2.15. *If $B_r \subset \{u_k > u_{k+1}\}$ then $(u_k - u_{k+1})(0) \ge cr$.*

Proof. We only sketch the proof. After rescaling, we can assume that U minimizes J_N in B_1, $\sum u_i = 0$, and

$$u_k - u_{k+1} > 0 \quad \text{in } B_1. \tag{2.19}$$

We wish to show that

$$(u_k - u_{k+1})(0) \ge c_0,$$

for some c_0 universal.

We may also assume that $\bar{U} = \{u_1, .., u_k\}$ and $\underline{U} = \{u_{k+1}, .., u_N\}$ are connected in B_δ for some $\delta > 0$ small to be chosen later (see Definition 2.8). Otherwise we can reduce the problem to one in B_δ with a strictly smaller number of membranes N. Let $\bar{h}$, $\underline{h}$ denote the averages of the membranes in $\bar{U}$, respectively $\underline{U}$.

Then $\bar{h}$, $\underline{h}$ are harmonic in B_1 and by Lemma 2.9 we know that

$$\|u_i - \bar{h}\|_{C^{0,1}(B_{1/2})} \le C, \qquad i \in \{1, .., k\}.$$

Since $\bar{U}$ is connected in B_δ we find that

$$\|u_i - \bar{h}\|_{L^\infty(B_\delta)} \le C\delta,$$

hence

$$u_k \le \bar{h}, \qquad \bar{h}(0) \le u_k(0) + C\delta.$$

Similarly we obtain

$$u_{k+1} \ge \underline{h}, \qquad \underline{h}(0) \ge u_{k+1}(0) - C\delta.$$

Assume by contradiction that

$$(u_k - u_{k+1})(0) \leq \delta. \tag{2.20}$$

Since $\bar{h} > \underline{h}$, by Harnack inequality we find

$$\bar{h} - \underline{h} \leq C\delta \quad \text{in } B_{1/2}.$$

This implies that $u_1 \leq u_N + C\delta$ in $B_{1/2}$ and then by Corollary 2.13 it follows that all membranes coincide in $B_{1/4}$ a contradiction. □

3. Monotonicity formula

In this section we present the Weiss type monotonicity formula associated to our energy functional J_N and prove Theorem 1.2. This is a standard tool necessary for the analysis of the partial regularity of free boundaries arising in a minimization problem, see for example [24].

Proposition 3.1 *(Weiss monotonicity formula). Assume that U is a critical point to J_N in B_1 with respect to continuous deformations in $\mathbb{R}^{n+1}$. Then for $r < 1$,*

$$\Phi(r) := r^{-n} \int_{B_r} \left(|\nabla U|^2 + W(U) \right) dx - r^{-n-1} \int_{\partial B_r} |U|^2 d\sigma,$$

is monotone increasing, and

$$\Phi'(r) = 2\, r^{-n} \int_{\partial B_r} (r^{-1}U - U_\nu)^2 d\sigma \quad \geq 0.$$

The Φ functional is constant in r if and only if U is "a cone" i.e. homogenous of degree one, and then this constant (the energy of U) is given by

$$\Phi(U) = \int_{B_1} W(U) dx.$$

First we specify the notion of U to be critical with respect to continuous deformations in $\mathbb{R}^{n+1}$. Given a smooth diffeomorphism $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with compact support in $\Omega \times \mathbb{R}$, we deform the N-graphs of U by the map

$$X \mapsto X + t\Phi(X), \qquad t \text{ small}, \quad X \in \mathbb{R}^{n+1},$$

onto the graphs of another admissible function that we denote by U_t. Then we say that U is critical if

$$\frac{d}{dt} J_N(U_t, \Omega)\,|_{t=0} = 0.$$

If Φ is independent of the x_{n+1} variable then U becomes critical for J_N with respect to the standard domain deformations in $\mathbb{R}^n$.

We start with the following lemma, in which we compute the first variation for critical points with respect to domain deformations.

Lemma 3.2 *(First variation). If U is a critical point of J_N with respect to domain variations then*

$$\int_{B_1} \left(-2\,\psi_l^k U_k \cdot U_l + (|\nabla U|^2 + W(U))div\,\Psi\right)\, dx = 0, \tag{3.1}$$

for any Lipschitz map $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ with compact support in B_1.

In particular, for a.e. $r \in (0,1)$

$$\frac{d}{dr}(r^{-n} J_N(U, B_r)) = 2r^{-n} \int_{\partial B_r} |U_\nu|^2 d\sigma - 2\,r^{-n-1} \int_{B_r} |\nabla U|^2 dx. \tag{3.2}$$

Proof. Let $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with compact support in B_1, and we consider the domain deformation

$$x \mapsto x + \epsilon\,\Psi(x) =: y.$$

Then

$$D_x y = I + \epsilon\, D\Psi, \qquad D_y x = (D_x y)^{-1} = I - \epsilon D\Psi + O(\epsilon^2),$$
$$dy = 1 + \epsilon\, div\,\Psi + O(\epsilon^2),$$

and

$$J_N(U(y), B_1) = \int_{B_1} |\nabla_y U|^2 + W(U(y))\, dy$$
$$= \int_{B_1} \left(\nabla_x U (I - \epsilon\, D\Psi)(I - \epsilon D\Psi)^T (\nabla_x U)^T + W(U)\right)(1 + \epsilon\, div\,\Psi)\, dx + O(\epsilon^2)$$
$$= J_N(U, B_1) + \epsilon \int_{B_1} \left(-2\psi_l^k U_k \cdot U_l + (|\nabla U|^2 + W(U))div\,\Psi\right)\, dx + O(\epsilon^2).$$

Here $O(\epsilon^2)$ depends on $\|U\|_{H^1}^2$ and $\|D\Psi\|_{L^\infty}$.

We take in (3.1)

$$\Psi(x) = \psi_\delta(|x|)x$$

with ψ_δ a cutoff function which is 1 in $[0, r-\delta]$ and 0 in $[r, \infty)$, and we let $\delta \to 0$ and obtain

$$\int_{B_r} \left((n-2)|\nabla U|^2 + nW(U)\right) dx = r \int_{\partial B_r} \left(|\nabla U|^2 + W(U) - 2|U_\nu|^2\right) d\sigma,$$

which gives (3.2). □

Let us consider continuous deformations of the graph of U in the vertical direction

$$(x, U) \mapsto (x, U + t\varphi(x)U) =: (x, U_t(x))$$

with φ a smooth function with compact support. Notice that these deformations are admissible for all t close to 0, and in addition

$$W(U) = W(U_t).$$

Then, if U is critical for J_N with respect to the family U_t we find

$$0 = \int_{B_1} \nabla U \cdot \nabla(\varphi U) dx, \tag{3.3}$$

which means that

$$U \cdot \triangle U = 0$$

holds in the distribution sense. We take $\varphi(x) = \psi_\delta(|x|)$ in (3.3), with ψ_δ a cutoff function as in the proof above. We let $\delta \to 0$ and obtain that

$$\int_{B_r} |\nabla U|^2 dx = \int_{\partial B_r} U \cdot U_\nu \, d\sigma \qquad \text{for a.e. } r. \tag{3.4}$$

Proof of Proposition 3.1. We compute

$$\frac{d}{dr}\left(r^{-n-1} \int_{\partial B_r} |U|^2 d\sigma\right) = 2r^{-n-1} \int_{\partial B_r} \left(U \cdot U_\nu - r^{-1}|U|^2\right) d\sigma,$$

which together with (3.2) and (3.4) give the formula for $\Phi'(r)$.

If U is a cone, then $\Phi(r)$ is a constant which we denote by $\Phi(U)$, and since

$$\int_{B_1} |\nabla U|^2 dx = \int_{\partial B_1} U \cdot U_\nu d\sigma = \int_{\partial B_1} |U|^2 d\sigma,$$

we find

$$\Phi(U) = \int_{B_1} W(U)dx. \quad \square$$

Having established Weiss' formula, we can now introduce the notion of blow-up cones. Let U be a minimizer in Ω and assume $U(x_0) = 0$. The Lipschitz continuity Theorem 2.10 shows that $\Phi(r)$ is bounded below, hence it has a limit as $r \to 0$. For a sequence of $r_m \to 0$, consider the rescalings,

$$U_{r_m}(x) := \frac{U(x_0 + r_m x)}{r_m}.$$

Then according to Theorem 2.10, up to extracting a subsequence, $U_m \to U_0$ uniformly on compact sets of $\mathbb{R}^n$ and $\Phi(r_m r) = \Phi_m(r) \to \Phi_0(r)$. The limit U_0 is called a blow-up limit, and by our compactness theorem, U_0 is a minimizer. The monotonicity of Φ implies $\Phi_0(r)$ is constant, hence U_0 is homogeneous of degree 1.

This discussion and the non-degeneracy Lemma 2.14 gives Theorem 1.2.

4. One dimensional minimizers

We show that the minimizing cones to J_N in dimension $n = 1$ are given by

$$u_1 = u_2 = .. = u_k \geq u_{k+1} = .. = u_N, \qquad k = 1, \dots, N-1, \tag{4.1}$$

with $u_k = u_{k+1}$ for $x \leq 0$ and $u_k > u_{k+1}$ for $x > 0$.

More precisely, for $k \leq N-1$, define $U_{0,k}$ as

$$u_i = \left(\frac{1}{k} - \frac{1}{N}\right)^{1/2} x^+ \qquad \text{for } i \leq k, \quad u_i = -\left(\frac{1}{N-k} - \frac{1}{N}\right)^{1/2} x^+ \qquad \text{for } i > k.$$

Notice that

$$|\nabla U_{0,k}|^2 = 1 = W(U_{0,k}), \quad x > 0,$$

and

$$\Phi(U_{0,k}) = 1,$$

where we are using the notation

$$\Phi(U) := \int_{-1}^{1} W(U)\, dx.$$

Proposition 4.1. *The only minimizing cones U in dimension $n = 1$ are given by $U(x) = L(x) + U_{0,k}(x)$, with $L(x) = (a \cdot x, \dots, a \cdot x), a \in \mathbb{R}^n$ (up to a reflection with respect to 0). If $\sum u_i = 0$ then $U = U_{0,k}$.*

Toward the proof of Proposition 4.1, we establish the next lemmas. First we obtain an inequality that relates the values of the slopes and the potential term W near a branching point.

Lemma 4.2 *(Inwards perturbations). Let U be a minimizer to J_N in an interval around 0 in dimension $n = 1$. Assume that a number of membranes u_i with indices*

$$i \in I = \{j+1, .., k\},$$

coincide at 0, and on a small interval $(0, \delta)$ to the right of 0 the following two conditions hold:

a) u_i are linear,

b) the family u_i, $i \in I$, is strictly separated from the remaining membranes i.e. $u_{j-1} > u_j$, $u_k > u_{k+1}$.

Then, in $(0, \delta)$ the slopes of the u_i's satisfy the inequality

$$\sum_{i \in I} |\nabla (u_i - u_I)|^2 \geq W(\{u_j, .., u_k\}), \qquad u_I := \frac{1}{k-j} \sum_{i \in I} u_i.$$

Proof. After subtracting a linear function and after a dilation we reduce to the case

$$\delta = 1, \quad u_I = 0.$$

Also, after relabeling the membranes we may assume that

$$I = \{1, 2, .., N\},$$

and U minimizes the energy in $[0, 1]$ among all admissible competitors V with the same boundary data as U in $[0, 1]$ and with $v_1 \leq u_1$ and $v_N \geq u_N$. These inequalities guarantee that V is not interacting with other membranes outside I.

We pick as competitor

$$V_\epsilon(x) := \begin{cases} U(\frac{x-\epsilon}{1-\epsilon}) & \text{if} \quad x \in [\epsilon, 1], \\ \\ 0 \quad \text{if} \quad x \in [0, \epsilon], \end{cases}$$

and find that

$$J_N(V_\epsilon) = \int_0^1 \left(\frac{1}{1-\epsilon} |\nabla U|^2 + (1-\epsilon) W(U) \right) dx.$$

The minimality implies

$$0 \leq \frac{d}{d\epsilon} J_N(V_\epsilon)|_{\epsilon=0} = \int_0^1 (|\nabla U|^2 - W(U))dx,$$

which gives the desired conclusion

$$|\nabla U|^2 \geq W(U). \quad \square$$

Next we obtain the Euler-Lagrange equation at branching points where the membranes coincide on one side.

Lemma 4.3. *In addition to the hypotheses of Lemma 4.2 assume further that the u_i with $i \in I$ coincide in $[-\delta, 0]$, and the strict separation with respect to the remaining membranes holds in $[-\delta, \delta]$, instead of $(0, \delta)$. Then the inequality in Lemma 4.2 becomes an equality:*

$$\sum_{i \in I} |\nabla(u_i - u_I)|^2 = W(\{u_j, .., u_k\}), \qquad in \quad (0, \delta). \tag{4.2}$$

Indeed, in the proof above we can obtain the opposite inequality by replacing ϵ with $-\epsilon$, since the corresponding competitor remains admissible.

The equality (4.2) expresses the equipartition of energy at the branching points. It shows that $U_{0,k}$ is the only homogenous of degree 1 minimizer U that satisfies (4.1).

Next we obtain a bound for the jump in slope of the top membrane u_1 at a branching point.

Lemma 4.4. *Let U be a minimizer to J_N in $(-\delta, \delta)$ in dimension $n = 1$ with*

$$u_1 = u_2 = .. = u_k, \quad \text{for some } k \geq 2.$$

Assume that

a) u_1 is linear in $[0, \delta)$ and $(-\delta, 0]$, and let a^+, respectively a^- denote its slopes;

b) $u_k > u_{k+1}$ in $(0, \delta)$.

Then

$$0 \leq a^+ - a^- \leq \frac{1}{\sqrt{k}}.$$

Proof. As above, after subtracting a linear function we may assume that $\delta = 1$ and $a^- = 0$. Then $a^+ \geq 0$ since u_1 is subharmonic, according to Proposition 2.4.

The top k membranes vanish to the left of 0, and we can use as an admissible competitor for these functions the same one used in the proof of Lemma 4.2 with ϵ replaced by $-\epsilon$. Precisely, let

$$v_{\epsilon,i} := \frac{1}{1+\epsilon} u_i(x+\epsilon)\chi_{[-\epsilon,1]} \quad \text{if } i \le k, \quad \text{and} \quad v_{\epsilon,i} := u_i \quad \text{if } i > k,$$

and notice that

$$J_N(V_\epsilon) - J_N(U) = \int_{-1}^{1} \left(\sum_{i=1}^{k} (|\nabla v_{\epsilon,i}|^2 - |\nabla u_i|^2) + \chi_{\{v_{\epsilon,k} > u_{k+1}\} \cap [-\epsilon,0]} \right) dx$$

$$\le \epsilon - \frac{\epsilon}{1+\epsilon} \int_0^1 \sum_{i=1}^{k} |\nabla u_i|^2 dx,$$

which, as $\epsilon \to 0$, gives the desired inequality

$$k(a^+)^2 \le 1. \quad \square$$

We are now ready to provide the proof of Proposition 4.1.

Proof of Proposition 4.1. Let U be a nonzero minimizing cone of average 0 in dimension $n = 1$ and let

$$a_1^+ > a_2^+ > ... > a_l^+$$

be the slopes of the membranes on $[0, \infty)$ and denote by $k_1^+, .., k_l^+$ their multiplicities that is, k_j^+ is the number of membranes with slope a_j^+. Since U has average 0 we have $a_1^+ \ge 0 \ge a_l^+$. Similarly on $(-\infty, 0]$ we have slopes $-a_1^-, -a_2^-, .., -a_m^-$ with

$$a_1^- > ... > a_m^-, \qquad a_1^- \ge 0 \ge a_m^-,$$

with multiplicities $k_1^-, .., k_m^-$.

We want to show that either $l = 2$, $m = 1$ or $l = 1$, $m = 2$. Then Lemma 4.3 implies that, up to a rotation, $U = U_{0,k}$ for some k.

By symmetry we may assume that

$$k_1^+ \le k_1^-, \tag{4.3}$$

and we show below that $l = 2$, $m = 1$.

We apply Lemma 4.4 to the top k_1^+ membranes and obtain

$$a_1^+ + a_1^- \le \sqrt{\frac{1}{k_1^+}}. \tag{4.4}$$

If $l = 1$ then, $k_1^+ = N$ and therefore $k_1^- = N$ as well. This means $l = m = 1$ hence $U \equiv 0$ which contradicts our hypothesis. In conclusion, $l \ge 2$.

Next we apply Lemma 4.2 for the collection of membranes corresponding to the slopes a_1^+ and a_2^+ (we drop the + superscript for simplicity of notation) and obtain

$$k_1(a_1 - a_I)^2 + k_2(a_2 - a_I)^2 \geq 1, \qquad a_I = \frac{k_1}{k_1 + k_2}a_1 + \frac{k_2}{k_1 + k_2}a_2,$$

hence

$$(a_1 - a_2)^2 \left(k_1 \frac{k_2^2}{(k_1 + k_2)^2} + k_2 \frac{k_2^2}{(k_2 + k_2)^2} \right) \geq 1,$$

or

$$(a_1 - a_2)^2 \geq \frac{1}{k_1} + \frac{1}{k_2}.$$

Thus,

$$a_1^+ - a_2^+ > \sqrt{\frac{1}{k_1^+}},$$

which together with (4.4) gives

$$a_2^+ < -a_1^- \leq 0, \tag{4.5}$$

and since $a_l^+ \leq a_2^+$, we find

$$a_1^- + a_l^+ < 0. \tag{4.6}$$

We claim that

$$k_l^+ < k_m^-. \tag{4.7}$$

Indeed, otherwise $k_l^+ \geq k_m^-$ and we are back at condition (4.3) after a rotation of 180° of the graphs of the membranes. We write (4.6) for this configuration and obtain

$$-a_1^- - a_l^+ < 0.$$

This contradicts (4.6) and the claim (4.7) is proved.

The inequality (4.7) shows that we satisfy condition (4.3) after a reflection of the membranes with respect to the x axis, i.e. $(u_1, .., u_N) \mapsto (-u_N, .., -u_1)$. If $l \geq 3$ then we may apply (4.5) for this configuration and obtain $-a_2^+ < 0$. This contradicts (4.5).

In conclusion $l = 2$, and (4.3), (4.7) imply that $m = 1$.

It remains to show that $U_{0,k}$ is a minimizer. Let V be a perturbation of U which coincides with U outside a compact interval that contains the origin, say $[-1, 1]$. Let ρ

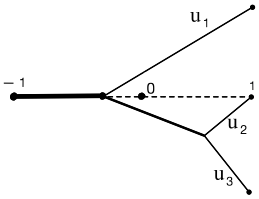


Fig. 3. A minimizer for $N = 3$, $n = 1$.

be such that $u_1 > 0 \in (\rho, 1]$, $u_1(\rho) = 0$ (and therefore $U(\rho) = 0$). Then $W(V) \geq 1$ in $[\rho, 1]$, hence

$$J_N(V, [-1, 1]) \geq J_N(\bar{V}, [-1, 1])$$

where $\bar{V}$ is the harmonic replacement of V in $[\rho, 1]$, with $\bar{V}$ extended to be 0 in $[-1, \rho]$. On the other hand, the right hand side is minimized when $\rho = 0$ as can be seen from Remark 4.3. □

If we consider minimality in the more restrictive class of continuous deformations of the graph of U in $\mathbb{R} \times \mathbb{R}$ (which do not change the topology of the junctions), then there are other cones in 1D. For example, for $N = 3$,

$$u_1 = x^+, \quad u_2 = 0, \quad u_3 = -x^+,$$

is a *minimizing* cone with a triple junction at 0. Notice that the Euler -Lagrange equation (4.2) is satisfied. A minimizer in H^1 with the same boundary data in $[-1, 1]$ of the example above is (see Fig. 3)

$$u_1 = \lambda \left(x - 1 + \frac{1}{\lambda} \right)^+, \quad u_2 = -\frac{1}{2} u_1 + \mu \left(x - 1 + \frac{1}{2\mu} \right)^+, \quad u_3 = -u_1 - u_2,$$
$$\lambda = \frac{\sqrt{2}}{\sqrt{3}}, \quad \mu = \frac{1}{\sqrt{2}}.$$

5. Free boundary regularity near one-dimensional cones

In this section we prove the partial regularity result for the free boundaries Γ_i, in dimension $n \geq 2$. For convenience we restate Theorem 1.4 from the Introduction.

Theorem 5.1 *(Partial regularity). Let U be a minimizer in B_1. The free boundaries Γ_i's are analytic and disjoint from one another outside a closed set Σ of singular points of Hausdorff dimension $n - 2$, and*

$$\mathcal{H}^{n-1}(\Gamma_i \cap B_{1/2}) \leq C.$$

In order to obtain Theorem 5.1, we first establish a series of lemmas. First we write a dimension reduction lemma for minimizers depending on fewer variables.

Lemma 5.2. *$U(x)$ is a minimizer for J_N in $B_1 \subset \mathbb{R}^n$ if and only if its trivial extension in one more variable is a minimizer for J_N in $B_1 \times \mathbb{R} \subset \mathbb{R}^{n+1}$.*

The proof of this lemma is standard and we omit the details.

Next, we obtain a lower bound of the energy of non trivial cones. Recall that cones are 1-homogeneous functions.

Lemma 5.3 *(Least energy cones). Let U be a nonzero cone in $\mathbb{R}^n$, with $\sum u_i = 0$. Then*

$$\Phi(U) \geq \frac{1}{2}|B_1|,$$

with equality if and only if

$$U = U_{0,k}(x \cdot \nu),$$

for some $k < N$, and some unit direction ν.

Proof. We prove this by induction on the dimension n. The case $n = 1$ was addressed in the previous section.

If $\{|U| = 0\}$ does not contain a ray, then $W(U) \geq 1$, and $\Phi(U) \geq |B_1| > \frac{1}{2}|B_1|$. If $|U| = 0$ contains a ray, then we can choose $x_0 \in \partial\{|U| > 0\} \cap \partial B_1$, and by the monotonicity formula obtain

$$\Phi(U) = \lim_{r\to\infty} \Phi_{x_0}(U, r) \geq \Phi_{x_0}(U, 0+),$$

with equality only if U is constant in the x_0 direction. Here $\Phi_{x_0}(U, 0+)$ represents the limit of the Φ functional centered at x_0 and its value coincides with the energy of a blow-up cone at x_0. This cone is constant in the x_0 direction and by the previous lemma it is the extension of an $n-1$ dimensional cone to $\mathbb{R}^n$. Now the conclusion follows easily from the induction hypothesis. □

We state the ϵ-regularity theorem for our problem.

Proposition 5.4. *Assume that U is a minimizer of J_N in B_1 with*

$$0 \in \{|U| > 0\}, \qquad \sum u_i = 0,$$

and

$$\Phi(U, B_1) \leq \frac{1}{2}|B_1| + \delta,$$

for some small δ universal. Then, there exists $k < N$ such that $\Gamma_k \cap B_{1/2}$ is an analytic hypersurface and $\Gamma_i \cap B_{1/2} = \emptyset$ if $i \neq k$.

We start by proving the following result.

Lemma 5.5. *Assume that U satisfies the hypotheses of Proposition 5.4. There exists k such that in $B_{3/4}$ $u_1 = .. = u_k$, $u_{k+1} = .. = u_N$, i.e. $\Gamma_i \cap B_{3/4} = \emptyset$ for all $i \neq k$.*

Proof. First we show that there exists a unit direction ν such that in $B_{7/8}$,

$$|U - U_{0,k}(x \cdot \nu)| \leq \eta(\delta) \qquad \text{with } \eta(\delta) \to 0 \text{ as } \delta \to 0. \tag{5.1}$$

This follows by compactness from Lemma 5.3. Indeed, if U_m is a sequence of minimizers in B_1 which satisfy the hypotheses with $\delta = \delta_m \to 0$, then up to subsequences, $U_m \to \bar{U}$ uniformly on compact sets of $B_{7/8}$ and

$$0 \in \partial\{|\bar{U}| > 0\}, \qquad \Phi(\bar{U}, B_{7/8}) = \lim \Phi(U_m, B_{7/8}) \leq \frac{1}{2}|B_1|.$$

Lemma 5.3 implies that $\bar{U}$ is a cone of least energy, i.e. $\bar{U} = U_{0,k}(x \cdot \nu)$ for some k, and (5.1) is proved.

Conclusion (5.1) holds also for the rescalings $U_r(x) = r^{-1}U(xr)$ with k and ν depending on r. However, the continuity of the U_r's implies that k is in fact independent of r, provided that we choose $\eta(\delta)$ sufficiently small.

On the other hand (5.1) and the non-degeneracy of $|U|$, give that $\{|U| = 0\}$ and the half-space $x \cdot \nu < 0$ are close in the Hausdorff distance sense in $B_{3/4}$. The same argument as above can be applied at any point $x_0 \in \partial\{|U| > 0\} \cap B_{3/4}$ instead of the origin, and the conclusion easily follows. □

Proof of Proposition 5.4. Let us assume that $\sum u_i = 0$. Then, by Lemma 5.5, $\Gamma_i \cap B_{3/4} = \emptyset$ if $i \neq k$, and u_k is the minimizer for the scalar Bernoulli problem

$$\int_{B_{3/4}} \left(\frac{nk}{n-k} |\nabla u|^2 + \chi_{\{u>0\}} \right) dx,$$

with sufficiently flat free boundary in $B_{3/4}$. The result follows from the classical work of Alt and Caffarelli [1]. □

Definition 5.6. We say that x_0 is a regular point of Γ_k if, when restricting to the collections of membranes that coincide at x_0, the corresponding blow-up cone is one-dimensional.

Proof of Theorem 5.1. If at $x_0 \in \Gamma_k$ we have,

$$u_l > u_{l+1} = ... = u_m > u_{m+1},$$

with $l+1 \le k < m$, then

$$\tilde{U} := (u_{l+1}, ..., u_m),$$

minimizes the energy locally for the energy J_{m-l} involving $m-l$ membranes. After subtracting the average from $\tilde{U}$, we reduce to the case $\tilde{U}(x_0) = 0$. If the blow-up cone at x_0 is one-dimensional, then x_0 is a regular point.

Then, by Proposition 5.4, Γ_k is an analytic surface and $\Gamma_i = \emptyset$ near x_0, with $i \in \{l+1, ..., m\}$. Furthermore, Proposition 5.4 implies that if $U_m \to U$ then $\Gamma_{m,k}$ converges to Γ_k in $C^{1,\alpha}$ sense near x_0.

Now the theorem follows from the standard dimension reduction argument of Federer, see [10]. We omit the details. □

6. 2 dimensional cones for $N = 3$ membranes

In this section we study nontrivial cones U which are not one-dimensional. We restrict to the case $N = 3$ membranes in $n = 2$ dimensions.

We introduce two cones V_0 and V_s which turn out to be the only one-homogenous functions which minimize the energy locally at points outside the origin.

We use polar coordinates (r, θ), and view sectors as intervals in the variable θ. We also use the notation

$$e_\theta = (\cos\theta, \sin\theta).$$

Definition 6.1. The cone $V_0 = (v_{0,1}, v_{0,2}, v_{0,3})$ is defined as

$$v_{0,1}(x) = \begin{cases} \sqrt{\frac{2}{3}}\; x \cdot e_{-\frac{\pi}{6}}, & \text{if} \quad \theta \in [-\frac{2}{3}\pi, \frac{1}{6}\pi], \\ \sqrt{\frac{1}{6}}\; x \cdot e_{\frac{\pi}{6}}, & \text{if} \quad \theta \in [\frac{1}{6}\pi, \frac{2}{3}\pi], \\ 0 & \text{otherwise,} \end{cases}$$

$$v_{0,3}(x) = -v_{0,1}(x_1, -x_2), \qquad v_{0,2} = -v_{0,1} - v_{0,2}.$$

The cone $V_s = (v_{s,1}, v_{s,2}, v_{s,3})$ is defined as

$$v_{s,1}(x) = \frac{1}{\sqrt{10}} \max\{|x_1|, 2|x_2|\},$$

$$v_{s,3}(x) = -v_{s,1}(x_2, x_1), \qquad v_{s,2} = -v_{s,1} - v_{s,3}.$$

(Fig. 4.)

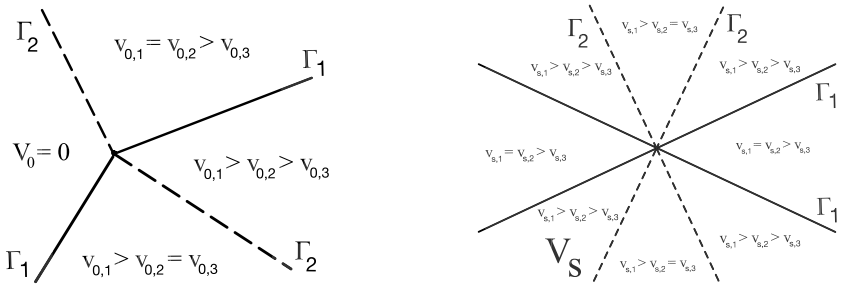


Fig. 4. The free boundaries for the cones V_0 and V_s.

Proposition 6.2. *Let $N = 3$ and U be a cone with $\sum u_i = 0$ which minimizes J_N locally at points on the unit circle ∂B_1, and assume that U is not one-dimensional. Then, up to rotations $U = V_0$, or $U = V_s$.*

Proof. By the partial regularity Theorem 5.1, the free boundaries Γ_1 and Γ_2 of U consist of at most a finite number of distinct rays. Each u_i is linear in the sectors determined by these rays, and on ∂B_1, the membranes behave as in the 1 dimensional case at the points on $\Gamma_i \cap \partial B_1$. More precisely, if at such a point all three membranes coincide, i.e. $U = 0$, then on one side they all coincide and on the other they split with multiplicities 1 and 2, and slopes $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{1}{6}}$ respectively. If at such a point only two membranes coincide, then on one side the two membranes coincide, while on the other they split and the slopes have a jump of size $\pm\frac{1}{\sqrt{2}}$.

Let S be a sector determined by a connected component of $\{u_1 > u_2\}$ which, after a rotation, corresponds to an interval $(0, \theta_0)$ in polar coordinates. Then

$$u_1 = f \cdot x \geq 0 \qquad \text{in } S,$$

for some non-zero vector f. Notice that $\theta_0 \leq \pi$.

We distinguish 2 cases depending on whether or not u_1 vanishes on ∂S.

Case 1: $u_1 > 0$ on $\partial S \setminus \{0\}$.

In this case we show that f bisects the sector S, and $\theta_0 > 2\pi/3$. Moreover, f bisects also the sector $\{u_2 = u_3\} \cap S$.

From the 1 dimensional analysis, in the connected sector $\{\theta \in (0, \theta_1)\} \subset S$ where $u_1 > u_2 > u_3$, we have

$$u_2 = (f - \sqrt{2}\, e_{\frac{\pi}{2}}) \cdot x, \qquad u_3 = (\sqrt{2}\, e_{\frac{\pi}{2}} - 2f) \cdot x.$$

Notice that $\theta_1 < \theta_0$, since otherwise $\theta_0 = \pi$ and we contradict that $u_1 > 0$ on $\partial S \setminus \{0\}$. The two linear functions corresponding to u_2 and u_3 in $[0, \theta_1]$ intersect on the line

$$(3f - 2\sqrt{2}\, e_{\frac{\pi}{2}}) \cdot x = 0, \tag{6.1}$$

hence the ray $\{\theta = \theta_1\} \subset \Gamma_2$ is obtained by intersecting this line with S.

The Euler-Lagrange equation on Γ_2 implies that

$$|3f - 2\sqrt{2}\, e_{\frac{\pi}{2}}| = \sqrt{2}, \tag{6.2}$$

i.e. $\frac{3}{\sqrt{2}} f$ belongs to the unit circle centered at $2e_{\frac{\pi}{2}}$, which means that the angle between f and $e_{\frac{\pi}{2}}$ is at most $\pi/6$.

Similarly, the equality above is satisfied also by the inner normal $e_{\theta_0 - \frac{\pi}{2}}$ to S on the other ray of ∂S. Since (6.2) holds for both $e_{\frac{\pi}{2}}$ and $e_{\theta_0 - \frac{\pi}{2}}$, it means the vector f makes the same angle with these 2 inner directions to S, hence f bisects S. Notice that a vector f determines uniquely the two unit directions which satisfy (6.2) and the lines in (6.1) where the graphs of u_2 and u_3 intersect. By symmetry, f bisects also the sector where $\{u_2 = u_3\}$ included in S. Thus any connected sector S of $\{u_1 > u_2\}$ for which $u_1 > 0$ on $\partial S \setminus \{0\}$ is uniquely determined by the slope of u_1 in that sector.

We remark that the angle between f and e_2 cannot be $\pi/6$. Otherwise, by (6.1), the ray $\theta = \theta_1$ has the direction of f. This means that the set $\{u_2 = u_3\} \cap S$ consists of a single ray which bisects S, and then U cannot be locally minimizing by the 1 dimensional analysis.

Case 2: u_1 vanishes on the ray $\theta = 0$.

In this case we show that U coincides with a $\frac{\pi}{6}$ rotation of V_0.

From the 1 dimensional analysis,

$$u_1 = \sqrt{\frac{2}{3}}\, (x \cdot e_{\frac{\pi}{2}})^+, \quad u_2 = u_3 = -\frac{1}{2} u_1,$$

on a sector near the ray $\theta = 0$. Let $\{\theta \in (0, \theta_1]\}$ be a connected sector in S where $u_2 = u_3$. We claim that

$$\theta_1 < \theta_0.$$

Otherwise $u_2 = u_3 = -\frac{1}{2} u_1$ throughout S, which means that S is a half-space, i.e. $\theta_0 = \pi$. Since u_1 is subharmonic, and vanishes near the endpoints of the interval $[\pi, 2\pi]$ we conclude that it vanishes in the whole interval, i.e. the formulas above hold in $\mathbb{R}^2$ and U is one-dimensional, a contradiction.

We remark that $u_1 > u_2 > u_3$ in the remaining sector $\theta \in (\theta_1, \theta_0)$ of S. Then, in this sector we have

$$u_1 = \sqrt{\frac{2}{3}} (x \cdot e_{\frac{\pi}{2}})^+, \quad u_2 = -\frac{1}{2} u_1 + \frac{1}{\sqrt{2}} (x \cdot e)^+, \quad u_3 = -\frac{1}{2} u_1 - \frac{1}{\sqrt{2}} (x \cdot e)^+,$$

with $e := e_{\theta_1 + \frac{\pi}{2}}$.

The Euler-Lagrange equation at $\theta = \theta_0$ gives

$$\left| \frac{3}{2}\sqrt{\frac{2}{3}} e_{\frac{\pi}{2}} - \frac{1}{\sqrt{2}} e \right| = \sqrt{2}$$

which means that $e = e_\pi$, i.e. $\theta_1 = \pi/2$. Then the ray $\theta = \theta_0$ is given by the direction of

$$\frac{\sqrt{3}}{2} e + \frac{1}{2} e_{\frac{\pi}{2}},$$

along which u_1 and u_2 intersect, i.e. $\theta_0 = 5\pi/6$.

Notice that the direction $\theta = \theta_0$ is also the one of the gradient of $-u_3$ in the connected sector $S' := (\theta_1, \theta_2)$ where $\{u_2 > u_3\}$. We claim that U vanishes on $\theta = \theta_2$. Otherwise we are in case 1) for $-u_3$ in the sector S'. We apply the conclusions for case 1) and deduce that the direction θ_0 bisects the angle of S', and $\theta_2 - \theta_1 > 2\pi/3$. This is a contradiction since $\theta_2 - \theta_1 = 2(\theta_0 - \theta_1) = 2\pi/3$.

Thus we are in case 2) for $-u_3$ and S' which means that $\theta_2 = \theta_0 + \pi/2$. In the remaining sector $\theta = [\theta_2, 2\pi]$, u_1 (and therefore U) must vanish since it is 0 on the boundary and it is subharmonic in the interior. In conclusion U is a $\pi/6$ rotation of V_0.

If u_1 and u_3 do not vanish on any ray, then we are in Case 1) for each sector of U, and then the connected sectors of $\{u_1 > u_2\}$ and $\{u_1 = u_2\}$ occur in a periodic pattern along the circle. Since $\theta_0 \in (\frac{2}{3}\pi, \pi)$, we can only have four such sectors. Then the angle between the bisectors of two such consecutive sectors, f for $\{u_1 > u_2\}$ and $f - \frac{1}{\sqrt{2}} e_{\frac{\pi}{2}}$ for $\{u_1 = u_2\}$, must be $-\pi/2$. This together with (6.2) determines f uniquely as

$$f = \frac{\sqrt{2}}{5}(e_0 + 2e_{\frac{\pi}{2}}),$$

and then it follows that U is a rotation of V_s. □

Next we show that we can lower the energy of V_s by using a compact perturbation near the origin. Our proof does not use the precise form of V_s but only that it has an axis of symmetry. The idea is inspired by [17] where the classification of nonlocal minimal cones in two dimensions was established.

Proposition 6.3. *V_s is not a minimizer of J_N in B_1.*

Proof. Assume by contradiction that V_s is a minimizer of J_N in B_1.

We take the domain deformation

$$x \mapsto y := \Psi(x) = x + \epsilon\varphi(|x|)e_0, \qquad e_0 = (1, 0),$$

with $\varphi(|x|)$ a radial function with compact support in B_1, and with $\|\epsilon D\Psi\|_{L^\infty} < 1$. Then, using the Lipschitz continuity of V_s, we find as in the proof of Lemma 3.2 that

$$U(y) := V_s(x(y)),$$

satisfies

$$J_N(U, B_1) = J_N(V_s, B_1) + \int_{B_1} L(\epsilon D\Psi, V_s) + O(\epsilon^2 |D\Psi|^2) dx,$$

with L linear in the first argument. Since V_s is minimizing, it follows that

$$\int_{B_1} L(\epsilon D\Psi, V_s) dx = 0,$$

hence

$$J_N(U, B_1) - J_N(V_s, B_1) \leq C\epsilon^2 \int_{B_1} |\nabla \varphi|^2 dx = o(\epsilon^2)$$

provided that we choose the logarithmic cutoff

$$\varphi(r) := \frac{\max\{\log r, \log(2\epsilon)\}}{\log(2\epsilon)}.$$

Let

$$U_1 := U(-|x_1|, x_2), \qquad U_2 := U(|x_1|, x_2)$$

and notice that U_1, U_2 have the same boundary data as U, hence

$$J_N(V_s, B_1) \leq J_N(U_1, B_1), \quad J_N(V_s, B_1) \leq J_N(U_2, B_1)$$
$$J_N(U_1, B_1) + J_N(U_2, B_1) = 2J_N(U, B_1) \leq 2J_N(V_s, B_1) + o(\epsilon^2),$$

which gives

$$J_N(U_1, B_1) \leq J_N(V_s, B_1) + o(\epsilon^2).$$

Notice that in $B_{2\epsilon}$, U is a translation of V_s by the vector ϵe_0, and U_1 and U_2 are obtained by reflecting this translation with respect to the line $x_1 = 0$.

We claim that the minimizer $\bar{U}_1$ with the boundary data of U_1 in B_ϵ decreases the energy of $J_N(U_1, B_\epsilon)$ by $c\epsilon^2$, $c > 0$. By the scaling of ϵ^{-1} factor, this is equivalent to prove the claim for $\epsilon = 1$. This means that, we first translate V_s by e_0 and take its values in $B_1 \cap \{x_1 < 0\}$, and then reflect them evenly with respect to $\{x_1 = 0\}$. We need to show that the resulting function is not minimizing J_N in B_1. Indeed, it suffices to look at this function on the x_1 axis. On this line we have $u_1 = u_2 > 0$, $u_3 < 0$ and at the origin u_3 is not smooth, therefore it is not a minimizer. □

Next we prove that V_0 is a minimizing cone in $\mathbb{R}^2$.

Proposition 6.4. *V_0 is a minimizer of J_N.*

First we state a result about the continuity of U near the boundary.

Lemma 6.5. *Assume that Ω is a Lipschitz domain and the boundary data of U is Lipschitz. Then $U \in C^\alpha(\overline{\Omega})$, for some $\alpha > 0$.*

The proof is standard and follows as in the interior case. We omit the details.

Lemma 6.6. *Assume that $0 \in \partial\Omega$ and Ω is a conical domain near the origin. If the boundary data of U is homogenous of degree one near 0, then the rescalings $U_r(x) = U(rx)/r$ of U converge on subsequences as $r \to 0$ to a global minimizing cone (defined in the conical domain).*

This follows from the Weiss monotonicity formula which still applies in this setting. We only use Lemma 6.6 when $\Omega = B_1^+ \subset \mathbb{R}^2$.

Lemma 6.7. *Assume that $N = 3$, $\Omega = B_1^+ = B_1 \cap \{x_n > 0\}$, $\sum u_i = 0$, and U vanishes on $x_n = 0$. If*

$$\max\{u_1, |u_3|\} \leq \frac{1}{\sqrt{6}}\, x_n,$$

then $U \equiv 0$ in a neighborhood of the origin.

Proof. By Lemma 6.6 and non-degeneracy, it suffices to prove the statement when U is a cone. Since $u_1 \geq 0$ is subharmonic, one-homogenous, and vanishes on $x_n = 0$, it must be a linear function in the x_n variable. The same is true for u_3 and therefore also for u_2. By dimension reduction, it suffices to show the statement in dimension $n = 1$. The growth hypothesis implies the slopes of u_1, u_2 and u_3 at the origin are strictly less that $1/\sqrt{6}$, and they need to vanish by Lemma 4.2. □

Proof of Proposition 6.4. We prove that V_0 is a minimizing cone by constructing a minimizer $U : \Omega \to \mathbb{R}$ for which $\Gamma_1 \cap \Gamma_2 \cap \Omega \neq \emptyset$. By the results above, V_0 is the only possible blow-up profile at an intersection point, and it is therefore minimizing.

Let $\Omega := [-M, M] \times [-1, 1]$, for some M large to be specified later, and let U be a minimizer of J_N with the boundary data given by

$$\varphi(x_1)U_{0,1}(x_2) + (1 - \varphi(x_1))U_{0,2}(x_2),$$

where $U_{0,1}$, $U_{0,2}$ are the one-dimensional solutions in the case $N = 3$, and $\varphi(x_1)$ is a smooth non-decreasing function which is equal to 0 when $x_1 \leq -1$ and equal to 1 when

$x_1 \geq 1$. Thus $U = 0$ on $\partial\Omega \cap \{x_2 \leq 0\}$, $U \neq 0$ on the remaining part of the boundary. Moreover, since u_1, $-u_3$ are subharmonic we have

$$\max\{u_1, -u_3\} \leq \frac{1}{\sqrt{6}}(1 + x_2),$$

and by Lemma 6.7 we find that the boundary points $(-M, M) \times \{-1\}$ are interior to the coincidence set $\{|U| = 0\}$. We have

$$J_N(U, \Omega) \leq (M-1)(J_N(U_{0,1}, [-1,1]) + J_N(U_{0,2}, [-1,1]) + C,$$

and since $U_{0,2}$ is minimizing in one dimension we find

$$J_N(U, \Omega \cap \{x_1 < -1\}) \geq (M-1)J_N(U_{0,2}, [-1,1]),$$

thus

$$J_N(U, \Omega \cap \{x_1 > 1\}) \leq (M-1)J_N(U_{0,1}, [-1,1]) + C.$$

On the other hand for each $y \in (1, M)$ for which

$$\|U(y, \cdot) - U_{0,1}(\cdot)\|_{L^\infty[-1,1]} \geq \delta,$$

we have

$$J_N(U, \mathcal{D}) \geq J_N(U_{0,2}(x_2), \mathcal{D}) + \sigma(\delta), \quad \mathcal{D} := [y-\delta, y+\delta] \times [-1,1],$$

for some $\sigma(\delta) > 0$, hence if $M = M(\delta)$ is large enough we can find a $y \in (M/4, M/2)$ such that

$$|U - U_{0,1}(x_2)| \leq \delta \quad \text{in} \quad \mathcal{R} := [y-1, y+1] \times [-1,1],$$

which, by Proposition 5.4, gives that the coincidence set $\{|U| = 0\}$ has Γ_1 as its boundary in $\mathcal{R}$, while $\Gamma_2 = \emptyset$. The same conclusion holds for some $y \in (-M/2, -M/4)$ with the roles of Γ_1 and Γ_2 interchanged.

Now we can conclude that $\{|U| = 0\}$ has a boundary point in $\Gamma_1 \cap \Gamma_2 \cap \Omega$. Indeed, otherwise $\partial\{|U| = 0\} \cap \Omega$ is a smooth curve which locally is in either Γ_1 or Γ_2 and which does not intersect the lines $x_2 = 1$ or $x_2 = -1$ in the region $|x_1| \leq M/2$. This is topologically impossible and we reached a contradiction.

Finally we show that V_0 is the unique minimizer with its own boundary data. If U is another minimizer with the same energy and boundary data in B_1, then U extended by V_0 outside B_1, is a minimizer in any compact subset of $\mathbb{R}^n$. As above, we find that there exists $x_0 \in \Gamma_1 \cap \Gamma_2$, and since the tangent cone at x_0 has the same energy with the cone at infinity we find that U is a cone with vertex at x_0, i.e. $U = V_0$. $\square$

7. Regularity of cones near V_0

In this section we assume that we are in $n = 2$ dimensions. We establish the uniqueness of the blow-up cone at a point in $\Gamma_1 \cap \Gamma_2$ and obtain the regularity of the two free boundaries near such a point.

Theorem 7.1. *Assume $N = 3$ and U minimizes J_N in B_1. If V_0 is the blow-up profile of U at 0, then Γ_1 and Γ_2 are piecewise $C^{1,\alpha}$ curves.*

Since V_0 is the only possible blow-up profile for $\Gamma_1 \cap \Gamma_2$ in two dimensions, we know that U is well-approximated by rotations of V_0 at all scales. Precisely we may assume that for each $r \in (0, 1]$, there exists a rotation matrix O_r such that

$$\|U - V_0(O_r x)\|_{L^\infty(B_r)} \leq \epsilon r, \tag{7.1}$$

for some ϵ sufficiently small. Our goal is to show that the value of ϵ improves in a C^α fashion with respect to r.

Assume that $O_1 = I$, and set

$$w_1 := \frac{1}{\epsilon}\left(u_1 - \sqrt{\frac{2}{3}}\, x \cdot e_{-\frac{\pi}{6}}\right), \quad \text{defined in } \{u_1 > u_2\},$$

$$w_3 := -\frac{1}{\epsilon}\left(\sqrt{\frac{2}{3}}\, x \cdot e_{\frac{\pi}{6}} + u_3\right), \quad \text{defined in } \{u_2 > u_3\}.$$

We claim that, as $\epsilon \to 0$, the graphs of the functions (w_1, w_3) converge uniformly on compact sets of

$$B_1 \setminus \{0\} \times \mathbb{R} \subset \mathbb{R}^{2+1}$$

to the graphs of a limiting pair $(\bar{w}_1, \bar{w}_3)$ with $\bar{w}_1$ defined in the sector $\mathcal{S}_1$ (Fig. 5) given by

$$\mathcal{S}_1 := \left\{(r, \theta) \in (0, 1) \times (-\frac{2\pi}{3}, \frac{\pi}{6})\right\},$$

and $\bar{w}_1$ satisfies

$$\triangle \bar{w}_1 = 0 \qquad \text{in } \mathcal{S}_1, \tag{7.2}$$

$$\partial_\nu \bar{w}_1 = 0 \qquad \text{on} \quad \left\{r \in (0, 1), \quad \theta = -\frac{2\pi}{3}\right\}, \tag{7.3}$$

$$\partial_\nu \bar{w}_1 = \frac{1}{2}\partial_\nu \bar{w}_3 \quad \text{on} \quad \left\{r \in (0, 1), \quad \theta = \frac{\pi}{6}\right\}, \tag{7.4}$$

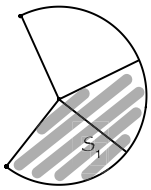


Fig. 5. The sector $\mathcal{S}_1$.

while $\bar{w}_3$ is defined in $\mathcal{S}_3$ with

$$\mathcal{S}_3 := \left\{ (r,\theta) \in (0,1) \times (-\frac{\pi}{6}, \frac{2\pi}{3}) \right\},$$

and

$$\triangle \bar{w}_3 = 0 \qquad \text{in } \mathcal{S}_3, \tag{7.5}$$

$$\partial_\nu \bar{w}_3 = 0 \qquad \text{on} \quad \left\{ r \in (0,1), \quad \theta = \frac{2\pi}{3} \right\}, \tag{7.6}$$

$$\partial_\nu \bar{w}_3 = \frac{1}{2} \partial_\nu \bar{w}_1 \quad \text{on} \quad \left\{ r \in (0,1), \quad \theta = -\frac{\pi}{6} \right\}. \tag{7.7}$$

Here the functions $\bar{w}_1$, $\bar{w}_3$ are bounded by -1 and 1, and their first derivatives are continuous up to the boundary at the points belonging to the lateral sides of the sector. The equations imply that they are in fact smooth up to the boundary at these points.

Moreover, $\bar{w}_1$, $\bar{w}_3$ are continuous at 0,

$$\bar{w}_1(0) = \bar{w}_3(0) = 0, \tag{7.8}$$

and the convergence of $W = (w_1, w_3)$ to $\bar{W} = (\bar{w}_1, \bar{w}_3)$ is uniform also in a neighborhood of 0.

Indeed, by the results of Section 5, u_1 solves a Bernoulli one-phase problem near the line $\theta = -\frac{2}{3}\pi$ which is an ϵ-perturbation of the one-dimensional solution of slope $\sqrt{2/3}$. Now we can apply the results for the scalar one-phase problem in [D], and find that w_1 converges on subsequences as $\epsilon \to 0$ to a limiting function $\bar{w}_1$ which satisfies (7.2)-(7.3). On the other hand

$$u_1 - u_2 = 2u_1 + u_3 = \sqrt{\frac{2}{3}} \, (2e_{-\frac{\pi}{6}} - e_{\frac{\pi}{6}}) \cdot x + \epsilon(2w_1 - w_3)$$

solves a Bernoulli one-phase problem near the line $\theta = \frac{\pi}{6}$ which is an ϵ-perturbation of a one-dimensional solution. The same argument as above gives (7.4). Similarly we obtain the equations (7.5)-(7.7) for $\bar{w}_3$.

Finally, from (7.1) we have $|O_r - O_{r/2}| \leq C\epsilon$, hence $|O_r - I| \leq C\epsilon |\log r|$ which means

$$\|U - V_0(x)\|_{L^\infty(B_r)} \leq C\epsilon r |\log r|.$$

This implies that

$$|w_i| \le C|x|\log|x| \quad \text{if} \quad |x| \in [\epsilon^2, 1],$$

hence $\bar{w}_i$ has a $Cr|\log r|$ modulus of continuity at 0 and satisfies (7.8).

Next we study the linear system (7.2)-(7.7) and notice that it appears as the Euler-Lagrange system for the quadratic functional

$$Q(w_1, w_3) := \frac{3}{2} \int_{\mathcal{S}_1 \setminus \mathcal{S}_3} |\nabla w_1|^2 dx + \frac{3}{2} \int_{\mathcal{S}_3 \setminus \mathcal{S}_1} |\nabla w_3|^2 dx + \int_{\mathcal{S}_1 \cap \mathcal{S}_3} |\nabla w_1|^2 + |\nabla w_3|^2 + |\nabla(w_1 - w_3)|^2 dx,$$

acting on pairs

$$(w_1, w_3) \in H^1(\mathcal{S}_1) \times H^1(\mathcal{S}_3).$$

For simplicity of notation we denote

$$W = (w_1, w_3), \quad \nabla W = (\nabla w_1, \nabla w_3),$$
$$L^2(\mathcal{S}) := \{W | \quad w_i \in L^2(\mathcal{S}_i)\}, \quad H^1(\mathcal{S}) := \{W | \quad w_i \in H^1(\mathcal{S}_i)\},$$

and define the inner product on $L^2(\mathcal{S})$ as

$$\langle W, V \rangle := \frac{3}{2} \int_{\mathcal{S}_1 \setminus \mathcal{S}_3} w_1 v_1 dx + \frac{3}{2} \int_{\mathcal{S}_3 \setminus \mathcal{S}_1} w_3 v_3 dx + \int_{\mathcal{S}_1 \cap \mathcal{S}_3} w_1 v_1 + w_3 v_3 + (w_1 - w_3)(v_1 - v_3) dx.$$

The norm induced by the inner product is equivalent to the standard L^2 norm on $\mathcal{S}$. We also define $\langle \nabla W, \nabla V \rangle$ as above, by replacing the terms $w_i v_j$ by $\nabla w_i \cdot \nabla v_j$. With this notation

$$Q(W) = \langle \nabla W, \nabla W \rangle.$$

We consider minimizers of Q which have fixed boundary data on ∂B_1 or, in other words, we consider *harmonic maps* induced by $\langle \cdot, \cdot \rangle$. We establish the $C^{1,\alpha}$ regularity of minimizers of Q near the origin.

Proposition 7.2. *Assume that $W \in H^1(\mathcal{S})$ is a minimizer of Q among competitors with the same boundary data on $\mathcal{S} \cap \partial B_1$. Then $W \in C^{1,\alpha_0}(\mathcal{S})$ and*

$$W(x) = W(0) + q(e_{\frac{\pi}{3}} \cdot x, -e_{-\frac{\pi}{3}} \cdot x) + O(|x|^{1+\alpha_0}), \quad q \in \mathbb{R},$$

for some explicit $\alpha_0 \in (0,1)$.

The precise formula of α_0 is given by

$$\cos(\frac{\pi}{3}(1+\alpha_0)) = \frac{\sqrt{17}-3}{8},$$

and $\alpha_0 \simeq 0.36$.

First we show that a solution to the system (7.2)-(7.7) which remains bounded near the origin minimizes the energy Q.

Lemma 7.3. *Assume that* $\bar{W}$ *is a bounded solution of* (7.2)-(7.7). *Then it minimizes the energy* Q *with respect to perturbations in* $H_0^1(\mathcal{S})$.

Proof. Indeed, we notice that if $V \in C^1$ vanishes near ∂B_1 and the origin 0, then

$$\begin{aligned}\langle \nabla \bar{W}, \nabla V\rangle &= \langle \triangle \bar{W}, V\rangle \\ &+ \int\limits_{\mathcal{S}_3 \cap \partial \mathcal{S}_1} -\frac{3}{2} v_3 (\bar{w}_3)_\nu + v_1 (\bar{w}_1)_\nu + v_3 (\bar{w}_3)_\nu + (v_1 - v_3)(\bar{w}_1 - \bar{w}_3)_\nu \, d\sigma \\ &+ \int\limits_{\mathcal{S}_1 \cap \partial \mathcal{S}_3} -\frac{3}{2} v_1 (\bar{w}_1)_\nu + v_3 (\bar{w}_3)_\nu + v_1 (\bar{w}_1)_\nu + (v_3 - v_1)(\bar{w}_3 - \bar{w}_1)_\nu \, d\sigma \\ &= 0.\end{aligned}$$

Let W_0 be the minimizer of Q in $\mathcal{S} \cap B_r$ with the same boundary data as $\bar{W}$ on $\mathcal{S} \cap \partial B_r$ for some $r \in (0,1)$. We show that $\bar{W}$ and W_0 coincide in B_r.

The computation above gives

$$\langle \nabla(\bar{W} - W_0), \nabla V\rangle = 0,$$

for any $V \in H^1(\mathcal{S})$ which vanishes on ∂B_1 and in a neighborhood of 0.

We choose $V = \psi^2(\bar{W} - W_0)$, with ψ a radial cutoff function which is 1 outside B_δ and vanishes near 0, and obtain the Caccioppoli inequality

$$\langle \psi \nabla(\bar{W} - W_0), \psi \nabla(\bar{W} - W_0)\rangle \leq C\langle |\nabla\psi|(\bar{W} - W_0), |\nabla\psi|(\bar{W} - W_0)\rangle. \tag{7.9}$$

Since $\bar{W} - W_0$ is bounded near the origin and $\|\nabla\psi\|_{L^2}$ can be made arbitrarily small, we obtain

$$\langle \nabla(\bar{W} - W_0), \nabla(\bar{W} - W_0)\rangle = 0,$$

as $\delta \to 0$, which gives the desired conclusion. □

Theorem 7.1 follows from the estimate of Proposition 7.2 which applies to $\bar{W}$.

Proof of Theorem 7.1. Since $\bar{W}(0) = 0$ then, for all $\rho \leq 1/2$,

$$\|\bar{W} - q(e_{\frac{\pi}{3}} \cdot x, -e_{-\frac{\pi}{3}} \cdot x)\|_{L^\infty(B_\rho)} \leq C\rho^{1+\alpha_0},$$

with $|q| \leq C$, and C universal.

The uniform convergence of the W's to $\bar{W}$ and the inequality above imply

$$\|U - V_0(O_\rho x)\|_{L^\infty(B_\rho)} \leq C\rho^{1+\alpha_0}\epsilon \leq \epsilon\rho^\alpha,$$

with O_ρ the rotation of angle $\epsilon\sqrt{\frac{3}{2}}q$, provided that $\alpha < \alpha_0$ and ρ is chosen sufficiently small depending on α. By iterating this result we find that U is approximated in a $C^{1,\alpha}$ fashion by rotations of V_0. In particular, there is a limiting rotation O_0 such that

$$\|U - V_0(O_0 x)\|_{L^\infty(B_r)} \leq C\epsilon\, r^{1+\alpha} \quad \text{for all } r \in (0,1].$$

This shows that the free boundaries are piecewise $C^{1,\alpha}$ at the origin. The full $C^{1,\alpha}$ regularity of Γ_i's away from the origin follows from the free boundary regularity in the one-phase problem applied in the annuli $B_r \setminus B_{r/2}$, see [1,7]. □

It remains to prove Proposition 7.2.

Proof of Proposition 7.2. We solve the Dirichlet problem for minimizers of Q by the method of Fourier series.

We investigate the eigenvalues and eigenfunctions of the corresponding Q operator on the unit circle ∂B_1. Precisely, let

$$\mathcal{S}_i' := \mathcal{S}_i \cap \partial B_1,$$

and notice that

$$\mathcal{S}_1' = (-b, a), \quad \mathcal{S}_3' = (-a, b),$$

where $a < b$ denote $\frac{\pi}{6}$ and $\frac{2\pi}{3}$.

We define the corresponding spaces $L^2(\mathcal{S}')$, $H^1(\mathcal{S}')$, and the inner product $\langle W, V\rangle$ as above. The eigenfunctions Φ_k and eigenvalues λ_k are defined inductively through the Rayleigh quotient formula

$$\lambda_k := \min_{W \in (span\{\Phi_1,..,\Phi_{k-1}\})^\perp} \frac{\langle \nabla W, \nabla W\rangle}{\langle W, W\rangle},$$

and $\Phi_k = W$ is the element of $H^1(\mathcal{S}')$ which has unit norm in $L^2(\mathcal{S}')$, where the minimum is realized. Then $\{\Phi_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(\mathcal{S}')$.

The minimizer of the Q functional with boundary data $\Phi_k(\theta)$ on ∂B_1 is

$$r^{\sqrt{\lambda_k}}\Phi_k(\theta).$$

We write the Euler-Lagrange equations for an eigenfunction $\Phi_k = W$ of eigenvalue $\lambda_k = \lambda$ and obtain

$$w_1'' + \lambda w_1 = 0 \quad \text{in} \quad (-b,-a)\cup(-a,a), \tag{7.10}$$

$$w_1'(-b)_+ = 0,$$

$$\frac{3}{2}w_1'(-a)_- + w_3'(-a)_+ - 2w_1'(-a)_+ = 0, \quad w_1'(-a)_+ - 2w_3'(-a)_+ = 0, \tag{7.11}$$

together with similar equations for w_3. Here $w_1'(-a)_\pm$ denote the left and right limits of the derivative of w_1' at $-a$. From (7.11) we find

$$w_1'(-a)_- = w_1'(-a)_+ = 2w_3'(-a)_+,$$

which means that equation (7.10) holds in the full interval $(-b,a)$. In conclusion

$$w_1(\theta) = \mu_1\cos(\sqrt{\lambda}(\theta+b)), \quad w_3(\theta) = \mu_3\cos(\sqrt{\lambda}(\theta-b)),$$

and

$$w_1'(-a) = 2w_3'(-a), \quad w_3'(a) = 2w_1'(a).$$

This implies that

$$\sin(\sqrt{\lambda}(b-a)) = \pm 2\sin(\sqrt{\lambda}(b+a))$$

or

$$|\sin(3t)| = 2|\sin(5t)|, \quad t := \frac{\sqrt{\lambda}}{6}\pi. \tag{7.12}$$

This equation has periodic solutions in t of period π. In the interval $[0,\pi)$ we have 9 solutions, two in each interval of length $\pi/5$. The first one is $t_0 = 0$, then

$$t_1 = \frac{\pi}{6}, \quad t_2 \in (\frac{\pi}{5},\frac{\pi}{3}), \quad t_3 \in (\frac{\pi}{3},\frac{2\pi}{5}) \qquad \text{etc.}$$

The solutions can be computed explicitly since after dividing by $\sin t$ in (7.12), we end up with two quadratic equations in $4\cos(2t)$

$$\beta^2 + 2\beta - 4 = \pm(\beta+2), \qquad \beta := 4\cos(2t).$$

The corresponding eigenvalues and eigenfunctions are

$$\begin{aligned}
&\lambda_1 = 0, && W = (1,0),\\
&\lambda_2 = 0, && W = (0,1),\\
&\lambda_3 = 1, && W = (\cos(\theta+b), -\cos(\theta-b)),\\
&\sqrt{\lambda_4} \in (\frac{6}{5}, 2), && W = \left(\cos\left(\sqrt{\lambda_4}(\theta+b)\right), \cos\left(\sqrt{\lambda_4}(\theta-b)\right)\right),
\end{aligned}$$

and $\sqrt{\lambda_3} \in (2, \frac{12}{5})$ etc. The eigenvalues corresponding to $t/\pi \in \mathbb{Z}$ have multiplicity 2, while the others are simple, and notice that $\sqrt{\lambda_k} \sim \frac{3}{5}k$ for k large.

If $\Phi(\theta)$ is the boundary data of W, then we decompose it in $L^2(\mathcal{S}')$ as

$$\Phi = \sum \sigma_k \Phi_k$$

with $\sum \sigma_k^2 < \infty$, and write the solution W in $\mathcal{S}$ in polar coordinates as the series

$$W = \sum \sigma_k r^{\sqrt{\lambda_k}} \Phi_k(\theta),$$

which converges uniformly in compact intervals of $r \in [0,1)$.

This gives the desired conclusion with

$$1 + \alpha = \sqrt{\lambda_4},$$

and

$$4\cos(\frac{\pi}{3}\sqrt{\lambda_4}) = \frac{\sqrt{17}-3}{2}.$$

We remark that Φ is the trace of a function in $H^1(\mathcal{S})$ is equivalent to

$$Q(W) = \langle \nabla W, \nabla W \rangle = \sum \sqrt{\lambda_k}\ \sigma_k^2 < \infty. \quad \square$$

8. Regularity for $\Gamma_1 \cap \Gamma_2$ in higher dimensions

In this section we prove a version of Theorem 7.1 in arbitrary dimensions. First we recall the following definitions.

Definition 8.1. Let U be a minimizer of J_N with $N = 3$. We say that

$$x_0 \in Reg(\Gamma_1 \cap \Gamma_2)$$

if there exists a blow-up profile at x_0 which is a rotation of a two-dimensional cone V_0 extended trivially in the remaining variables.

We say that

$$x_0 \in Reg(\Gamma_k)$$

if there exists a blow-up cone $x_0 \in \Gamma_k$ which is one-dimensional.

For convenience we restate Theorems 1.7 and 1.8 from the Introduction which will be proved in this section.

Theorem 8.2. *$Reg(\Gamma_1 \cap \Gamma_2)$ is locally a $C^{1,\alpha}$-smooth manifold of codimension two. Near such an intersection point, each of the free boundaries Γ_1 and Γ_2 consists of two piecewise $C^{1,\alpha}$ hypersurfaces which intersect on $Reg(\Gamma_1 \cap \Gamma_2)$.*

As a consequence we obtain the partial regularity result.

Theorem 8.3 *(Partial regularity). Let $N = 3$ and U be a minimizer of J_N in B_1. Then*

$$\partial\{|U| > 0\} = Reg(\Gamma_1) \cup Reg(\Gamma_2) \cup Reg(\Gamma_1 \cap \Gamma_2) \cup \Sigma',$$

with Σ' a closed singular set of Hausdorff dimension $n-3$ and

$$\mathcal{H}^{n-1}(Reg(\Gamma_i) \cap B_{1/2}) \leq C, \qquad \mathcal{H}^{n-2}(Reg(\Gamma_1 \cap \Gamma_2) \cap B_{1/2}) \leq C.$$

Theorems 8.2 and 8.3 are deduced from the next proposition which will be proved later in the section. It states that a minimizer U which is approximated in each ball B_r with $r \in (0,1]$, by a rotation of V_0, must be a $C^{1,\alpha}$ deformation of V_0.

Proposition 8.4. *Let U be a minimizer in B_1. Assume that for each $r \in (0,1]$ there exists a pair of orthonormal vectors ν_r^1, ν_r^2 such that*

$$\|U - V_0(\nu_r^1 \cdot x, \nu_r^2 \cdot x)\|_{L^\infty(B_r)} \leq \epsilon r,$$

for some $\epsilon \leq \epsilon_0(n)$ small universal. Then there exists ν_0^1, ν_0^2 such that

$$\|U - V_0(\nu_0^1 \cdot x, \nu_0^2 \cdot x)\|_{L^\infty(B_r)} \leq C\epsilon r^{1+\alpha}.$$

A direct consequence of this result is that there are no other cones that are sufficiently close to the set of rotations of V_0.

Corollary 8.5. *Assume that U is a minimal cone and $\|U - V_0\|_{L^\infty(B_1)} \leq \epsilon_0$. Then U is a rotation of V_0.*

We show that the hypotheses of Proposition 8.4 can be relaxed to require that U is approximated by V_0 only in B_1 and the energy of the blow-up cones at the origin does not go below the energy of V_0.

Lemma 8.6. *Let U be a minimizer in B_1 with $0 \in \partial\{|U| > 0\}$. Assume that*

$$\|U - V_0\|_{L^\infty(B_1)} \leq \epsilon, \quad \text{and} \quad \Phi_U(0+) \geq \Phi(V_0).$$

Then the only possible blow-up cones for U at 0 are rotations of V_0.

Proof. First we notice that the hypotheses imply that

$$\Phi_U(\frac{1}{2}) \leq \Phi(V_0) + C\epsilon,$$

and then

$$0 \leq \Phi_U(r) - \Phi(V_0) \leq C\epsilon, \quad \forall r \in (0, \frac{1}{2}). \tag{8.1}$$

Assume by contradiction that the conclusion does not hold for a sequence U_n, and $\epsilon = \epsilon_n \to 0$. Then, by Proposition 8.4, we can find appropriate dilations $\tilde{U}_n := r_n^{-1} U_n(r_n x)$ such that

$$dist(\tilde{U}_n, \mathcal{V}_0) = \epsilon_0,$$

where $\mathcal{V}_0$ represents the collections of cones obtained by rotations of V_0 and the distance between $\tilde{U}_n$ and the elements of $\mathcal{V}_0$ is measured in $L^\infty(B_1)$. As $n \to \infty$ we may extract a convergent subsequence in $L^\infty(B_1)$ of the $\tilde{U}_n$'s to a limiting function $\bar{U}$. Then $\bar{U}$ must be a cone, since its $\Phi_{\bar{U}}$ energy is a constant in the radial variable by (8.1). The distance from $\bar{U}$ to $\mathcal{V}_0$ is ϵ_0, and we contradict Corollary 8.5. □

Next we use the dimension reduction argument to show that the set of free boundary points $\partial\{|U| > 0\}$ whose tangent cones have energy strictly between the one dimensional solutions $U_{0,k}$ and the two-dimensional solution V_0, has Hausdorff dimension $n - 3$.

Lemma 8.7. *The set*

$$\mathcal{A} := \{x \in \partial\{|U| > 0\} \cap B_1 | \quad \Phi_{U,x}(0+) \in (\Phi(U_{0,k}), \Phi(V_0))\},$$

has Hausdorff dimension $n - 3$.

Here $\Phi_{U,x}(r)$ denotes the Weiss energy of U in a ball of radius r centered at x, and $\Phi_{U,x}(0+)$ its limit as $r \to 0$. The continuity of $\Phi_{U,x}(r)$ with respect to x, and r fixed, shows that

$$\Phi_{U,x}(r) < \Phi(V_0),$$

for all $x \in \partial\{|U| > 0\}$ near a point in $\mathcal{A}$ and r sufficiently small. Thus, it suffices to prove the following lemma.

Lemma 8.8. *Fix $\delta > 0$, and assume that U is a minimizer in B_3 and*

$$\Phi_{U,x}(r) \leq \Phi(V_0) - \delta, \qquad \forall x \in \partial\{|U| > 0\} \cap \overline{B}_1, \quad \forall r \leq 1.$$

Then

$$\mathcal{H}^{n-3+\delta}(\mathcal{A} \cap \overline{B}_1) = 0.$$

We remark that $\mathcal{A} \cap \overline{B}_1$ is a closed set by the regularity result in Proposition 5.4.

Proof. The proof follows the standard dimension reduction argument of Federer [10]. Notice that in dimension $n = 3$ the set $\mathcal{A} \cap \overline{B}_1$ is discrete by Proposition 5.4, and the statement is obvious.

We prove the statement by induction on the dimension $n \geq 3$, in two steps. We only sketch the main ideas and leave the details to the interested reader.

Step 1: Assume the result holds in dimension n, then it holds for any $n+1$-dimensional cone U with $\Phi(U) \leq \Phi(V_0) - \delta$.

Step 2: Assume the result holds for cones in dimension n, then it holds for a minimizer in dimension n.

For Step 1, assume U is a cone in $\mathbb{R}^{n+1}$. We take its restriction to a ball $B_r(x_0)$ with $x_0 \in \mathcal{A} \cap \partial B_1$ and then normalize it to the unit ball after a translation and dilation. The resulting function is uniformly-well approximated as $r \to 0$ by a minimizer that is constant in the x_0-direction, for which the induction hypothesis holds. By compactness, this means that, there exists $r_0(n) > 0$ small such that if $r \leq r_0$ then $\mathcal{A} \cap \partial B_1 \cap B_r(x_0)$ can be covered by a finite collection of balls of radii r_i and centers on $\mathcal{A} \cap \partial B_1$ with

$$\sum r_i^{n-3+\delta} \leq \frac{1}{2} r^{n-3+\delta}. \tag{8.2}$$

Step 1 follows by iterating this result a number of times.

Step 2 is a consequence of the fact that around each point in $\mathcal{A}$, U is well-approximated by cones at all small scales, and the conclusion holds for these cones by Step 1. Precisely, by compactness, for each $x_0 \in \mathcal{A} \cap \overline{B}_1$ there exists $\delta(x_0) > 0$ such that if $r \leq \delta(x_0)$ the set $\mathcal{A} \cap B_r(x_0)$ can be covered by a finite collection of balls of radii r_i that satisfy inequality (8.2).

Let $\mathcal{A}_k$ denote the set of points x_0 in $\mathcal{A}$ with the property that $\delta(x_0) \geq \frac{1}{k}$, and notice that $\mathcal{A} \subset \cup \mathcal{A}_k$. On the other hand $\mathcal{H}^{n-3+\delta}(\mathcal{A}_k \cap \overline{B}_1) = 0$ since, as in Step 1, we can iterate (8.2) for $\mathcal{A}_k$. Thus, the desired conclusion holds for $\mathcal{A}$ as well. □

In view of Lemma 8.6 and Lemma 8.7 we obtain a stronger version of Proposition 8.4 in which the only hypothesis is that U is approximated by V_0 in B_1.

Proposition 8.9. *Assume that U is a minimizer in B_1, $0 \in \partial\{|U| > 0\}$, and*

$$\|U - V_0(x_1, x_2)\|_{L^\infty(B_1)} \leq \epsilon_0,$$

for some ϵ_0 small universal. Then there exist ν_0^1, ν_0^2 such that

$$\|U - V_0(\nu_0^1 \cdot x, \nu_0^2 \cdot x)\|_{L^\infty(B_r)} \leq r^{1+\alpha}.$$

In particular, $\Gamma_1 \cap \Gamma_2 \cap B_{1/2}$ consists only of regular intersection points.

Proof. For each

$$x \in \mathcal{D} := \{x_1 = x_2 = 0\} \cap B_{1/2},$$

we look at the two dimensional plane generated by the first two coordinates of x. By topological considerations in this plane, the set $\Gamma_1 \cap \Gamma_2$ contains at least one point $\bar{x}$ in the disk of radius $C\epsilon_0$ around the origin.

Indeed, our hypothesis and Proposition 5.4 imply that in the two-dimensional annulus $C\epsilon_0 \leq r \leq 2C\epsilon_0$ the open sets $\{u_1 > u_2\}$ and $\{u_2 > u_3\}$ are C^1 perturbations of the sectors $\mathcal{S}_1$ and $\mathcal{S}_3$ defined in Section 7. By two-dimensional topology, the boundaries of these two open sets must intersect in the disk of radius $C\epsilon_0$.

Since $dim(\mathcal{A}) \leq n-3$, we find that

$$\bar{x} \notin \mathcal{A} \quad \text{for } \mathcal{H}^{n-2} \text{ a.e. } x \in \mathcal{D},$$

hence $\Phi_{U,\bar{x}}(0+) \geq \Phi(V_0)$ and we can apply Proposition 8.4 at $\bar{x}$.

The conclusion follows since the set of such x's is dense in $\mathcal{D}$. □

Theorem 8.2 follows easily from Proposition 8.9 and we omit the details.

Regarding Theorem 8.3, we notice that

$$\Sigma' := \partial\{|U| > 0\} \setminus (Reg(\Gamma_1) \cup Reg(\Gamma_2) \cup Reg(\Gamma_1 \cap \Gamma_2)),$$

is a closed set according to Proposition 8.9 and Theorem 5.1. The dimension reduction argument as in Lemma 8.7 implies that $dim(\Sigma') \leq n-3$, and rest of Theorem 8.3 follows by standard techniques.

It remains to prove Proposition 8.4. The considerations at the beginning the previous section remain valid, and they reduce the proof of Proposition 8.4 to the validity of $C^{1,\alpha}$ estimates for bounded solutions of the elliptic system (7.2)-(7.7).

The rest of the section is devoted to establish Proposition 7.2 in arbitrary dimensions, see Proposition 8.10 below.

We introduce some notation. We denote by

$$x = (x', x''), \quad x' = (x_1, x_2), \quad x'' = (x_3, .., x_n),$$

$$(r, \theta) \quad \text{the polar coordinates for } x',$$

$$\mathcal{S}_1 := \{\theta \in (-4a, a)\} \cap B_1, \quad \mathcal{S}_3 := \{\theta \in (-a, 4a)\} \cap B_1, \quad a := \frac{\pi}{6},$$

and recall the definitions of $L^2(\mathcal{S})$, $H^1(\mathcal{S})$, $\langle W, V\rangle$, $\langle \nabla W, \nabla V\rangle$, and Q from the previous section. We establish the $C^{1,\alpha}$ regularity of minimizers of Q.

Proposition 8.10. *Assume that $W \in H^1(\mathcal{S})$ is a minimizer of Q among functions with the same trace on ∂B_1. Then $W \in C^{1,\alpha}(\mathcal{S})$ and*

$$W(x) = W(0) + \left((qe_{\frac{\pi}{3}}, \nu_1'') \cdot x, (-qe_{-\frac{\pi}{3}}, \nu_3'') \cdot x\right) + O(|x|^{1+\alpha}),$$

for some $\alpha \in (0,1)$.

As in the previous section, Proposition 8.4 follows from Proposition 8.10 provided we show that bounded solutions of the system (7.2)-(7.7) do minimize the energy, as in Lemma 7.3 for $n = 2$.

Lemma 8.11. *Assume that $\bar{W} \in L^\infty$ solves the system (7.2)-(7.7) in the classical sense in the domain*

$$\overline{\mathcal{S}} \setminus (\{x' = 0\} \cup \partial B_1).$$

Then it minimizes the energy Q with respect to perturbations in $H^1(\mathcal{S})$ which vanish near ∂B_1.

Proof. The proof is essentially the same as the one of Lemma 7.3. However, in order to justify the existence of a minimizer W_0 with the same boundary data as $\bar{W}$ on ∂B_r we need to show first that $\bar{W} \in H^1(\mathcal{S} \cap B_r)$.

Notice that for any $V \in C^1$ that vanishes near $\{x' = 0\} \cup \partial B_1$, we have

$$\langle \nabla \bar{W}, \nabla V\rangle = 0.$$

Then the Caccioppoli inequality

$$\langle \varphi \nabla \bar{W}, \varphi \nabla \bar{W}\rangle \leq C\langle |\nabla \varphi| \bar{W}, |\nabla \varphi| \bar{W}\rangle,$$

holds if φ vanishes near $\{x' = 0\} \cup \partial B_1$. We choose

$$\varphi(x) = \psi(x')\eta(x),$$

with ψ a radial cutoff function which vanishes near the origin and $\psi = 1$ when $|x'| \geq \delta$, and η a cutoff function which vanishes near ∂B_1. Since $\bar{W} \in L^\infty$, it follows that we can take $\psi \equiv 1$ in the limit as $\delta \to 0$, i.e.

$$\bar{W} \in H^1(\mathcal{S} \cap B_r) \quad \text{for any } r < 1.$$

We define W_0 to be the minimizer of Q in $\mathcal{S} \cap B_r$ with the same boundary data as $\bar{W}$ on ∂B_r. Now the proof of Lemma 7.3 applies, by taking $\psi = \psi(x')$ depending only on the variable x'. □

There are several ways to prove Proposition 8.10. Here we take advantage of the product structure of the problem and reduce it back to the two-dimensional case. The system is invariant under translations in the x'' variable, and then we can estimate higher order derivatives $D^{\beta}_{x''}W$ through successive iterations. Then

$$\triangle W = 0 \quad \Longrightarrow \quad \triangle_{x'} W = -\triangle_{x''} W,$$

and the right hand side is well behaved.

We start with some preliminary estimates.

Lemma 8.12. *Let W be a minimizer of Q in $H^1(\mathcal{S})$. Then,*

a) (Caccioppoli inequality) If $\varphi \in C^1_0(B_1)$ then

$$\langle \varphi \nabla W, \varphi \nabla W \rangle \leq C \langle |\nabla \varphi| W, |\nabla \varphi| W \rangle,$$

b) W is smooth up to the boundary of S away from $\partial B_1 \cup \{x' = 0\}$, and the Euler-Lagrange equations (7.2)-(7.7) are satisfied in the classical sense,

c) $W \in L^{\infty}(\mathcal{S} \cap B_{1/2})$.

Proof. Part a) is standard and we skip the details.

For part b) we remark that in a ball $B_\delta(x_0)$ near a point $x_0 \in \mathcal{S}_1 \cap \partial \mathcal{S}_3$ the energy can be written as

$$\frac{3}{2} \int\limits_{B_\delta(x_0)} |\nabla w_1|^2 dx + \frac{1}{2} \int\limits_{B_\delta(x_0) \cap S_3} |\nabla (2w_2 - w_1)|^2 dx.$$

This shows that w_1 is harmonic near x_0, and $2w_2 - w_1$ can be extended harmonically in the whole $B_\delta(x_0)$ by the even reflection across $\partial \mathcal{S}_3$. Hence W and its derivatives can be bounded in $\mathcal{S} \cap B_{1/2}$ away from the codimension two edge $\{x' = 0\}$ in terms of the $L^2(\mathcal{S})$ norm of W.

For part c) we first show that

$$\fint\limits_{\mathcal{S} \cap \partial B_r} |W|^2 dx, \tag{8.3}$$

remains bounded for all r small. For this we prove a mean value inequality with respect to the $L^2(\mathcal{S})$ norm:

$$\langle W(rx), W(rx) \rangle_{\mathcal{S} \cap \partial B_1}, \tag{8.4}$$

is monotone increasing in r, where $\langle \cdot, \cdot \rangle_{S \cap \partial B_r}$ denotes the inner product induced on the sphere ∂B_r.

Let $\varphi \in C_0^1(B_1)$, $\varphi \geq 0$, and then

$$\langle \nabla W, \nabla(\varphi W) \rangle = 0,$$

or

$$0 \leq \langle \nabla W, \varphi \nabla W \rangle \leq \langle -\nabla \varphi \cdot \nabla W, W \rangle.$$

We take $\varphi \to \chi_{B_r}$ and obtain

$$0 \leq \langle W_\nu, W \rangle_{\mathcal{S} \cap \partial B_r}.$$

This means that the derivative in r of the expression in (8.4) is nonnegative, and the claim (8.3) is proved.

From (8.3) and part b) applied in $\mathcal{S} \cap B_r$, we deduce that $|W|$ is bounded in $B_{1/2} \cap \mathcal{S} \cap \{x'' = 0\}$ by a multiple of its $L^2(\mathcal{S})$ norm. The conclusion follows by translating the origin at the other points in $\{x' = 0\} \cap B_{1/2}$. □

Lemma 8.13. *Assume that $W \in H^1(\mathcal{S})$ is a minimizer of Q. Then*

$$\|D^\beta_{x''} W\|_{L^\infty(\mathcal{S} \cap B_{1/2})} \leq C(\beta) \|W\|_{L^2(\mathcal{S})},$$

and, for each fixed x'', the function $W(\cdot, x'')$ minimizes the two-dimensional energy

$$\langle \nabla W, \nabla W \rangle_{x'} + \int_{\mathcal{S}_1} f_1 w_1 dx' + \int_{\mathcal{S}_3} f_3 w_3 dx',$$

for some bounded functions f_1, f_3.

Proof. The discrete differences of W in the x''-directions are minimizers of Q. By iterating Caccioppoli inequality we obtain that $D^\beta_{x''} W$ minimizes Q and

$$\|D^\beta_{x''} W\|_{H^1(\mathcal{S} \cap B_{1/2})} \leq C(\beta) \|W\|_{L^2(\mathcal{S})}.$$

The L^∞ bound follows from Lemma 8.12, part c). This means that $W(x', 0)$ satisfies in the two-dimensions

$$\triangle_{x'} W \in L^\infty,$$

and the boundary conditions in (7.2)-(7.7) hold in the classical sense on $\partial \mathcal{S} \setminus \{0\}$. The conclusion follows as in Lemma 7.3 since $W \in L^\infty$. □

The results of the previous section imply the $C^{1,\alpha}$ estimates in this inhomogeneous setting.

Lemma 8.14. *Assume that $W \in H^1(\mathcal{S})$ is a minimizer of the 2 dimensional energy*

$$\langle \nabla W, \nabla W \rangle + \int_{\mathcal{S}_1} f_1 w_1 dx + \int_{\mathcal{S}_3} f_3 w_3 dx,$$

for some bounded functions f_i. Then $W \in C^{1,\alpha}(\mathcal{S})$ and

$$|W(x) - W(0) + q(e_{\frac{\pi}{3}} \cdot x, -e_{-\frac{\pi}{3}} \cdot x)| \leq C(1 + \|f_1\|_{L^\infty} + \|f_3\|_{L^\infty})\|W\|_{L^2} \, |x|^{1+\alpha},$$

in $\mathcal{S} \cap B_{1/2}$, for some universal constant C.

Now we can prove the $C^{1,\alpha}$ estimate in arbitrary dimensions.

Proof of Proposition 8.10. By Lemma 8.13 and Lemma 8.14 (applied to W and $D_{x''}W$) we find that

$$\begin{aligned} W(x', 0) &= W(0) + q(e_{\frac{\pi}{3}} \cdot x', -e_{-\frac{\pi}{3}} \cdot x') + O(|x'|^{1+\alpha}), \\ D_{x''}W(x', 0) &= (\nu_1'', \nu_3'') + O(|x'|), \qquad |D^2_{x''}W| \leq C, \end{aligned}$$

which gives the desired conclusion. □

Finally we prove the Schauder estimates in the two-dimensional inhomogeneous setting.

Proof of Lemma 8.14. The proof is standard and uses Campanato iterations. We sketch some of the details.

Assume that $\|f_i\|_{L^\infty} \leq \delta$, $\|W\|_{H^1} \leq \delta$. It suffices to show inductively in k that for each $r = \rho^k$ there exist *linear* functions at 0 of the type

$$L_r = \left(d_1 + q e_{\frac{\pi}{3}} \cdot x, d_3 - q e_{-\frac{\pi}{3}} \cdot x\right),$$

depending on r which approximate W in B_r such that

$$\left(\fint_{\mathcal{S} \cap B_r} |W - L_r|^2 dx \right)^{1/2} \leq r^{1+\alpha}.$$

Indeed, the rescaled function

$$\tilde{W}(x) = \frac{1}{r^{1+\alpha}} (W - L_r)(rx),$$

satisfies

$$\fint_{S\cap B_1} |\tilde{W}|^2 dx \leq 1,$$

and $\tilde{W}$ is a minimizer for a functional as above with

$$\|\tilde{f}_i\|_{L^\infty} \leq \delta r^{1-\alpha} \leq \delta.$$

The Caccioppoli inequality for $\tilde{W}$ gives

$$\|\tilde{W}\|_{H^1(\mathcal{S}\cap B_{1/2})} \leq C.$$

Let W_0 be the minimizer of the Q functional with the same boundary data as $\tilde{W}$ on $\mathcal{S} \cap \partial B_{1/2}$, and notice that

$$\|W_0\|_{H^1(\mathcal{S}\cap B_{1/2})} \leq C.$$

Then in $\mathcal{S} \cap B_{1/2}$ we have

$$\begin{aligned}\langle \nabla(\tilde{W} - W_0), \nabla(\tilde{W} - W_0)\rangle &= \langle \nabla\tilde{W}, \nabla(\tilde{W} - W_0)\rangle - \langle \nabla\tilde{W}_0, \nabla(\tilde{W} - W_0)\rangle \\ &= \frac{1}{2}\int_{\mathcal{S}_1} \tilde{f}_1(\tilde{w}_1 - w_{0,1})dx + \frac{1}{2}\int_{\mathcal{S}_3} \tilde{f}_3(\tilde{w}_3 - w_{0,3})dx \\ &\leq C\delta \|\tilde{W} - W_0\|_{L^2}^{\frac{1}{2}}.\end{aligned}$$

By Poincaré inequality it follows that

$$\|\tilde{W} - W_0\|_{L^2(\mathcal{S}\cap B_{1/2})} \leq C'\delta^2.$$

The C^{1,α_0} regularity of W_0 (see Proposition 7.2) gives

$$\left(\fint_{\mathcal{S}\cap B_\rho} |W_0 - L_0|^2 dx\right)^{1/2} \leq C\rho^{1+\alpha_0},$$

for some *linear* function L_0 at 0. The last two inequalities imply the inductive result for W with $r = \rho^{k+1}$ by first choosing ρ sufficiently small depending on $\alpha < \alpha_0$, and then δ small depending on ρ. □

References

[1] H.W. Alt, L.A. Caffarelli, Existence and regularity for a minimum problem with free boundary, J. Reine Angew. Math. 325 (1981) 105–144.
[2] J. Andersson, H. Shahgholian, N. Uraltseva, G. Weiss, Equilibrium points of a singular cooperative system with free boundary, Adv. Math. 280 (2015) 743–771.
[3] L.A. Caffarelli, F.H. Lin, Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries, J. Am. Math. Soc. 21 (3) (2008) 847–862.
[4] L.A. Caffarelli, S. Salsa, A Geometric Approach to Free Boundary Problems, Graduate Studies in Mathematics, vol. 68, American Mathematical Society, Providence, RI, ISBN 0-8218-3784-2, 2005, x+270 pp.
[5] L.A. Caffarelli, H. Shahgholian, K. Yeressian, A minimization problem with free boundary related to a cooperative system, Duke Math. J. 167 (10) (2018) 1825–1882.
[6] M. Chipot, G. Vergara-Caffarelli, The N-membrane problem, Appl. Math. Optim. 13 (3) (1985) 231–249.
[7] D. De Silva, Free boundary regularity for a problem with right hand side, Interfaces Free Bound. 13 (2011) 223–238.
[8] D. De Silva, D. Jerison, A singular energy minimizing free boundary, J. Reine Angew. Math. 635 (2009) 1–22.
[9] D. De Silva, G. Tortone, Improvement of flatness for vector valued free boundary problems, Math. Eng. 2 (4) (2020) 598–613.
[10] H. Federer, The singular set of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension, Bull. Am. Math. Soc. 76 (1970) 767–771.
[11] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific Publishing Co., 2003.
[12] D. Jerison, O. Savin, Some remarks on stability of cones for the one-phase free boundary problem, Geom. Funct. Anal. 25 (4) (2015) 1240–1257.
[13] D. Kriventsov, F.H. Lin, Regularity for shape optimizers: the nondegenerate case, Commun. Pure Appl. Math. 71 (8) (2018) 1535–1596.
[14] D. Kriventsov, F.H. Lin, Regularity for shape optimizers: the degenerate case, Commun. Pure Appl. Math. 72 (8) (2019) 1678–1721.
[15] D. Mazzoleni, S. Terracini, B. Velichkov, Regularity of the optimal sets for some spectral functionals, Geom. Funct. Anal. 27 (2) (2017) 373–426.
[16] D. Mazzoleni, S. Terracini, B. Velichkov, Regularity of the free boundary for the vectorial Bernoulli problem, Anal. PDE 13 (3) (2020) 741–764.
[17] O. Savin, E. Valdinoci, Regularity of nonlocal minimal cones in dimension 2, Calc. Var. Partial Differ. Equ. 48 (1–2) (2013) 33–39.
[18] O. Savin, H. Yu, On the multiple membranes problem, J. Funct. Anal. 277 (6) (2019) 1581–1602.
[19] O. Savin, H. Yu, Free boundary regularity in the triple membrane problem, Ars Inveniendi Anal. (2021).
[20] O. Savin, H. Yu, Free boundary regularity in the multiple membrane problem in the plane, J. Reine Angew. Math. 799 (2023) 109–154.
[21] H. Tavares, S. Terracini, Regularity of the nodal set of segregated critical configurations under a weak reflection law, Calc. Var. Partial Differ. Equ. 45 (3–4) (2012) 273–317.
[22] J. Taylor, The structure of singularities in soap-bubble and soap-film-like minimal surfaces, Ann. Math. 103 (1976) 489–539.
[23] B. Velichkov, Regularity of the One-Phase Free Boundaries, Lecture Notes of the Unione Matematica Italiana, Springer, 2023, https://library.oapen.org/handle/20.500.12657/61877.
[24] G. Weiss, Partial regularity for a minimum problem with free boundary, J. Geom. Anal. 9 (1999) 317–326.