J. reine angew. Math. **799** (2023), 109–154
DOI 10.1515/crelle-2023-0014

Journal für die reine und angewandte Mathematik

Free boundary regularity in the multiple membrane problem in the plane

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Abstract. We study the regularity of free boundaries in the multiple elastic membrane problem in the plane. We prove the uniqueness of blow-ups, and that the free boundaries are $C^{1,\log}$-curves near a regular intersection point.

1. Introduction

Given a positive integer N, the N-membrane problem describes the shapes of N elastic membranes under external forces. The membranes cannot penetrate each other, but they can coincide in a priori unknown regions, giving rise to $(N-1)$ free boundaries. The N-membrane problem can be viewed as a coupled system of $(N-1)$ obstacle problems with interacting free boundaries. It is the natural extension of the obstacle problem (which corresponds to the case $N=2$) to the vector valued case, and can be referred to as *the vectorial obstacle problem*.

Mathematically, given a domain $\Omega \subset \mathbb{R}^d$, some positive constants $\{\omega_k\}_{k=1,2,\dots,N}$, and bounded functions $\{f_k\}_{k=1,2,\dots,N}$, we study the minimizer of the following convex functional

$$(1.1) \qquad (u_1, u_2, \dots, u_N) \mapsto \int_\Omega \sum \omega_k \left(\frac{1}{2}|\nabla u_k|^2 + f_k u_k\right) dx$$

over the class of functions with prescribed data on $\partial\Omega$, and subject to the constraint

$$(1.2) \qquad u_1 \geq u_2 \geq \cdots \geq u_N \quad \text{in } \Omega.$$

The function f_k represents the force acting on the kth membrane, whose height is described by the unknown u_k. Each ω_k represents the weight of the kth membrane.

Since the membranes cannot penetrate each other, the functions $\{u_k\}$ are well ordered inside the domain. This leads to the constraint (1.2). On the other hand, consecutive membranes can come in contact with each other. Between the *contact region* $\{u_k = u_{k+1}\}$ and the *non-contact region* $\{u_k > u_{k+1}\}$, we have the kth *free boundary*

$$\Gamma_k := \partial\{u_k > u_{k+1}\}.$$

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Ovidiu Savin is supported by NSF grant DMS-1500438.

We consider the case of *constant* force terms that satisfy a *non-degeneracy condition* specific in obstacle-type problems

$$f_1 > f_2 > \cdots > f_N.$$

The Euler–Lagrange equation is given in the form of the variational inequality

$$\omega_i(v_i - u_i)\triangle u_i \le \omega_i(v_i - u_i) f_i, \tag{1.3}$$

which holds for all $v \in H^1(\Omega)$ that satisfy the constraint (1.2). Since the convex set defined by (1.2) is invariant under addition of the same function and multiplication by the same positive number, we have further

$$\sum \omega_i \triangle u_i = \sum \omega_i f_i, \quad \sum \omega_i u_i \triangle u_i = \sum \omega_i u_i f_i. \tag{1.4}$$

Existence and uniqueness of the minimizer were established by Chipot and Vergara-Caffarelli [3]. They also proved that solutions are $C^{1,\alpha}_{\text{loc}}(\Omega)$ for all $\alpha \in (0,1)$. We obtained the optimal $C^{1,1}$-regularity of solutions and then performed a blow-up analysis in Savin and Yu [8].

The case when $N = 2$ corresponds to the classical obstacle problem. Concerning this problem, there is a large literature, see, for instance, [1,2,4,6,7,13]. For the case when $N = 3$, the free boundary regularity was investigated recently in [9]. The nontrivial analysis occurs near the points where the two free boundaries intersect. Exploiting a maximum principle satisfied by the pair $(u_1, -u_3)$ which is specific to $N = 3$ membranes, we obtained the sharp logarithmic rate of blow-up. With this, we established the $C^{1,\log}$-regularity of the free boundaries near *regular intersections*, and the uniqueness of certain types of blow-up profiles.

In this work, we extend these results in the physical dimension $d = 2$ to an arbitrary number of membranes N, and to all possible blow-up profiles. For arbitrary N, the setting is much more complicated as the complexity of the problem grows exponentially with N. Nevertheless, we are able to prove uniqueness of blow-ups as well as sharp free boundary regularity near a regular intersection point. A consequence of our results is that the free boundaries intersect tangentially if the corresponding coincidence sets have positive densities at the intersection point. This is one of the interesting features of the problem: the $(N-1)$ degrees of freedom of the problem do not usually match the degrees of freedom of the free boundaries when they intersect!

Uniqueness of blow-ups is a central problem in the regularity theory, and it is usually achieved through a differential inequality known as the log-epiperimetric inequality of the type

$$\frac{d}{dr} W(u,r) \le -c W(u,r)^{\gamma}, \quad \gamma < 2.$$

Here W represents the functional that appears in the (Weiss) monotonicity formula, translated so that it tends to 0 as $r \to 0$. For cones with smooth cross sections and when W has analytic structure, a general method to establish the log-epiperimetric inequality is based on the Lojasiewicz–Simon inequality. The method was developed by L. Simon [12] in the setting of minimal surfaces. However, this strategy does not seem to apply in obstacle type problems as the constraint (1.2) is polyhedral. The log-epiperimetric inequality in the standard obstacle problem was established by Colombo-Spolaor-Velichkov [4] by making use of the Fourier decomposition of the traces of u on ∂B_r. The same authors extended their results to cones of even frequency for the thin obstacle problem [5].

Recently in [10, 11], we proposed an ad-hoc strategy to establish the uniqueness of certain blow-up cones in obstacle-type problems, which is inspired by our work for $N = 3$. This is based on introducing approximate solutions, modeled by solutions of the linearized problem. These approximate solutions are so that they minimize the error of the right-hand side in the Euler–Lagrange equation, and are used to approximate the dyadic rescalings of the actual solution u. Their construction usually involves solving appropriate obstacle problems on ∂B_1. The fact that the error cannot be improved reduces to a non-orthogonality condition, which often is given in the form of a nontrivial algebraic statement. The strategy is the following.

Assume the solution u is within an ε error of an approximate solution v in B_1. Then we need to show that in a smaller ball B_ρ, either u has a $\frac{\varepsilon}{2}$-rescaled error with respect to another approximate solution w (which would give a geometric convergence rate for the rescalings of u), or the energy of u in B_ρ decayed at least an ε^2 amount, i.e.,

$$W(u, \rho) \leq W(u, 1) - c\varepsilon^2.$$

This dichotomy is a consequence of the fact that v is "the least error" approximation among functions which project in the same point on the tangent space given by the linearized equation. Then we establish an inequality of the type $W(u, 1) \leq \varepsilon^{1+\mu}$ for some $\mu > 0$, which together with the inequality above gives a discrete version of the log-epiperimetric inequality and leads to the uniqueness of blow-up limits.

In the present work, we follow the same strategy. An important point is that in dimension $d = 2$ all cones are classified, and this plays a key role in the algebra involved, see Section 4. The construction of approximate solutions relies on the solvability of the global problem in one dimension, which we investigate in Section 3. Throughout the paper we use the bold face letter notation for vectors, say

$$\mathbf{u} = (u_1, \dots, u_N).$$

Before we introduce our results a few simplifications are in order. We may assume that all N free boundaries pass through the origin,

$$0 \in \bigcap \Gamma_i,$$

since an intersection point involving fewer free boundaries can be reduced locally to the same problem with fewer membranes. Also, after subtracting the average from all u_k, we may assume that the average of the functions u and f is 0 (see (1.4)):

$$\sum \omega_k u_k = 0, \quad \sum \omega_k f_k = 0.$$

In [8], we showed that the quadratic rescalings

$$\mathbf{u}_r(x) := r^{-2}\mathbf{u}(rx)$$

converge on subsequences as $r \to 0$ to a 2-homogeneous solution $\mathbf{p}$, i.e., a *cone*. Moreover, in dimension $d = 2$, we classified the family $\mathcal{C}_2$ of cones as extensions of one-dimensional cones to two dimensions (see next section for more details).

We state the main results.

Theorem 1.1. *Assume that $d = 2$ and $\mathbf{p} \in \mathcal{C}_2$ is a blow-up limit for $\mathbf{u}$ at the origin. Then $\mathbf{p}$ is unique and*

$$\mathbf{u}(x) = \mathbf{p}(x) + O(|x|^2(-\log|x|)^{-1}).$$

Among the two-dimensional cones, the one of least energy is given by rotations of

$$\mathbf{p_0}(x_2) := \frac{1}{2}(x_2^+)^2\mathbf{f},$$

which represents the situation when all coincidence sets are given by the same half-plane. If $\mathbf{p_0}$ appears as a blow-up limit at the origin, then we say that 0 is a *regular intersection point* for the free boundaries Γ_i. Near these points, the free boundaries enjoy the following regularity:

Theorem 1.2. *Assume $d = 2$ and*

$$|\mathbf{u} - \mathbf{p}_0| \leq \varepsilon_0 \quad \textit{in } B_1$$

for a constant ε_0 depends on N, $\mathbf{f}$ and ω. Then each Γ_i is a $C^{1,\log}$-curve in $B_{1/2}$.

The paper is structured as follows. In Section 2, we introduce the notations, and collect some general facts about the maximum principle and the optimal regularity of solutions. In Section 3, we study the global one-dimensional problem which is crucial to our analysis. In Sections 4 and 5, we prove Theorem 1.1 for those nondegenerate cones (connected cones) $\mathbf{p}$ for which all their coincidence sets have nonempty interiors. In Section 6 we prove Theorem 1.1 for all other degenerate cones. Finally, in Section 7 we prove Theorem 1.2.

We conclude the introduction with a game theoretical interpretation of the N-membrane problem. Suppose there are N players $P_1, \dots, P_N$ which hold N tickets $1, 2, \dots, N$ and a token that moves on a lattice in Ω. Each round the token moves randomly to an neighboring vertex and the players can interchange their tickets according to the following rule: the player with the ticket 1 can choose any ticket he wishes, after that the player with the ticket 2 can choose from the remaining $N - 1$ tickets and so on. Moreover, in order for a player to hold onto the ticket 1 for one round he needs to pay the amount f_1, and for the ticket 2 the amount f_2, etc. When the token exits the domain, the payoff for the ticket k holder is given by the boundary data φ_k. If all players optimize their strategies then the solution u_k to the discrete N-membrane problem (with weights $\omega_k = 1$) represents the expected payoff of the player holding the ticket k, while the coincidence sets give the optimal strategies on the exchange of tickets.

2. Notation and preliminaries

In this section we introduce the notations used through the paper, and collect some basic properties of solutions to the N-membrane problem, such as optimal regularity, maximum principle and introduce the cones in one and two dimensions.

Notation.

- $\mathbf{u} = (u_1, \dots, u_N)$.
- $\mathbf{1} = (1, 1, \dots, 1)$.
- $\mathbf{u} \geq \mathbf{v}$ means $u_i \geq v_i$ for all i.
- For $I \subset \{1, \dots, N\}$, u_I denotes the average of u_i with $i \in I$:

$$u_I := \sum_{i \in I} \frac{\omega_i}{\sum_I \omega_j} u_i. \tag{2.1}$$

- $\mathcal{P}$ denotes the collection of one-dimensional cones, see Definition 2.2.
- $\mathcal{P}^c \subset \mathcal{P}$ are the connected one-dimensional cones, see Definition 2.2.
- $B(\mathbf{p})$ is the space associated to the branches $\mathbf{p} \in \mathcal{P}^c$, see Definition 3.1.
- $\mathbf{h}(x, \mathbf{b})$ is the global one-dimensional solution with linear asymptotics given by $\mathbf{b} \in B(\mathbf{p})$, see Definition 3.2.
- $\tau \in B(\mathbf{p})$ is generated by the 1-translation, see Definition 3.1.
- $\mathbf{e}(\mathbf{b})$ is the error function, see Definition 3.3.
- $\mathbf{p}(x, \mathbf{b})$ the approximate solution generated by $\mathbf{b}$, see Definition 4.1.
- $\mathbf{p}(x, \mathbf{b}_0, \mathbf{b}_1)$, see Definition 4.2.
- $\mathcal{S}(r, \mathbf{p}, \varepsilon)$, see Definition 4.3.
- $W(\mathbf{u}, r)$ the Weiss functional, see Section 5.
- $\mathbf{p}_* \in \mathcal{P} \setminus \mathcal{P}^c$ denotes a degenerate one-dimensional cone.
- $\mathbf{p}_*^i \in \mathcal{P}^c$ are the connected one-dimensional cones which make $\mathbf{p}_*$, see Section 6.
- $\mathcal{S}(r, \mathbf{p}_*, \varepsilon)$, see Definition 6.1.
- σ-connected, see Definition 6.2.
- We denote by c_i, C_i constants depending on N, d, $\mathbf{f}$, ω, and call them universal constants.

If h is a function with $\triangle h = \text{const}$, then $\mathbf{u} - h\mathbf{1}$ solves the N-membrane problem with forces $\mathbf{f} - (\triangle h)\mathbf{1}$, see (1.3)–(1.4). Thus, without loss of generality we assume throughout that the functions f have average 0

$$\sum \omega_i f_i = 0,$$

and by (1.4), $\sum \omega_i u_i$ is harmonic.

Often we subtract the average of the u_i from each function so that we reduce to the case $\sum \omega_i u_i = 0$. When this holds we say that u solves problem P_0.

Definition 2.1. We say that $\mathbf{u}$ solves problem P_0 if it is a solution to the N-membrane problem and also $\sum \omega_i u_i = 0$.

The Euler–Lagrange equation gives that in an open region where l membranes coincide $u_m < u_{m+1} = u_{m+2} = \cdots = u_{m+l} < u_{m+l+1}$, the common function u_{m+1} satisfies

$$\triangle u_{m+1} = f_I, \quad I := \{m+1, \dots, m+l\},$$

i.e., the force acting on each of the l membranes in this coincidence region is the average of the l forces f_i.

Optimal regularity. Existence and uniqueness of solutions in $H^1(\Omega)$ follows easily from the standard methods in the calculus of variations. The optimal $C^{1,1}$ regularity of solutions was obtained in [8]. We sketch the proof for completeness. We show that $u_i \in C^{1,1}_{\text{loc}}$ and

$$\triangle u_i = \sum_{j \le i \le k} f_{A_{jk}} \chi_{A_{jk}}, \quad A_{jk} := \{u_{j-1} < u_j = \cdots = u_k < u_{k+1}\}. \tag{2.2}$$

Lipschitz regularity. If $\mathbf{v} \in H^1(B_1)$ in another solution, then by adding the variational inequalities (1.3) for $\mathbf{u}$ and $\mathbf{v}$ we find

$$\omega_i(v_i - u_i)\triangle(v_i - u_i) \geq 0 \implies \triangle(\omega_i(v_i - u_i)^2) \geq 0,$$

hence $\sum \omega_i(v_i - u_i)^2$ is subharmonic. This shows that

$$\|\mathbf{v} - \mathbf{u}\|_{L^\infty(B_{1/2})} \leq C\|\mathbf{v} - \mathbf{u}\|_{L^2(B_1)}.$$

Taking $\mathbf{v}$ to be a translation of $\mathbf{u}$, we obtain

$$\|\nabla \mathbf{u}\|_{L^\infty(B_{1/2})} \leq C\|\nabla \mathbf{u}\|_{L^2(B_1)}.$$

$\boldsymbol{C^{1,1}}$ **regularity.** We start with the following lemma.

Lemma 2.1. *Assume* $\mathbf{u}$ *solves the* N*-membrane problem in* B_1*. Then*

$$|\triangle u_m| \leq C|\mathbf{f}|, \tag{2.3}$$

$$\|\mathbf{u}\|_{C^{1,1}(B_{1/2})} \leq C(\|\mathbf{u}\|_{L^\infty(B_1)} + |\mathbf{f}|). \tag{2.4}$$

Proof. We use induction on N. The case $N = 1$ is trivial.

For $N > 1$, after subtracting the average, we may assume that $\sum \omega_i u_i = 0$, and say also that $|\mathbf{f}| = \mathbf{1}$. We start with (2.3).

The set where all membranes coincide is

$$K := \{u_i = 0 : \text{for all } i\} = \{u_1 = u_N\}.$$

Inequality (1.3) implies $\triangle u_1 \leq f_1$, $\triangle u_N \geq f_N$, hence $\triangle(u_1 - u_N) \leq f_1 - f_N$. This means that $w := u_1 - u_N \geq 0$, satisfies $\triangle w \leq C$ in Ω and, by the induction hypothesis, $|\triangle w| \leq C$ in the set $\{w > 0\} = \Omega \setminus K$. This shows that w solves a scalar obstacle problem with right-hand side bounded in L^∞, which implies the standard quadratic growth away from its zero set

$$w(x) \leq C d(x, K)^2,$$

where $d(x, K)$ denotes the distance from x to the set K. Then $|u_1|, |u_N| \leq w$ satisfy the same inequality, and it holds for all other $|u_m|$. This shows $|\triangle u_m| \leq C$ on K in the viscosity sense, while outside K the inequality holds (in the viscosity sense) by the induction hypothesis. In conclusion, (2.3) is proved.

As a consequence, $u_m - u_{m+1} \geq 0$ solves an obstacle problem with an L^∞ right-hand side, and it satisfies the standard quadratic growth behavior

$$u_m - u_{m+1} \leq C d(x, \Gamma_k)^2 \quad \text{in } \{u_m > u_{m+1}\}. \tag{2.5}$$

Next we prove (2.4) by showing that each function u_m admits a tangent paraboloid by above/below of opening $1 + \|u\|_{L^\infty}$. For simplicity we prove this at the origin.

Let $r \geq 0$ denote the radius of the smallest ball around the origin B_r which intersects all free boundaries Γ_i. Notice that in B_r the problem decouples into two multi-membranes problems involving fewer membranes than N.

If $r \geq \frac{3}{4}$, then we can apply the induction hypothesis in B_r and get the desired conclusion in $B_{1/2}$. If $r \in (0, \frac{3}{4})$, then by (2.5) we conclude that $u_m - u_{m+1} \leq C(r^2 + |x|^2)$ for all m. Since the average of the functions u is 0, we find

$$|u_m| \leq C(r^2 + |x|^2).$$

In B_r we may apply again the induction hypothesis (for the rescaling $\frac{\mathbf{u}(rx)}{r^2}$). Then we conclude that u_m admits a global tangent polynomial of opening C by above/below at the origin (outside B_r we use the inequality above).

Finally, if $r = 0$, we obtain as above $|u_m| \leq C|x|^2$ which gives again the desired estimate. □

Remark 2.1. Lemma 2.1 implies (2.2) by considering Lebesgue points for A_{jk} where $\mathbf{u}$ is twice differentiable. If we assume that $\mathbf{f}$ satisfies the nondegenerate condition

$$f_1 > f_2 > \cdots > f_N,$$

then the right-hand side for $\triangle(u_m - u_{m+1})$ is positive, and we obtain also the quadratic growth by below

$$\max_{B_r(x_0)} (u_m - u_{m+1}) \geq c r^2 \quad \text{if } x_0 \in \Gamma_m,$$

for some $c > 0$ universal.

Maximum principle. The maximum principle takes the following form in the setting of the N-membrane problem.

Lemma 2.2 (Maximum principle). *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are two solutions with* $\mathbf{u} \geq \mathbf{v}$ *on* $\partial\Omega$*, then* $\mathbf{u} \geq \mathbf{v}$ *in* Ω*. Moreover, if* $u_i(x_0) = v_i(x_0)$ *for some* $x_0 \in \Omega$*, then* $u_i = v_i$*.*

Proof. Let $I \subset \{1, \ldots, N\}$ be the set of indices m for which $u_m(x_0) = u_i(x_0)$ and similarly define J the set of membranes that coincide with v at x_0. We have $\max I \geq \max J$, $\min I \geq \min J$. Then the average function $u_{I\cap J}$ (see (2.1)) satisfies

$$\triangle u_{I\cap J} \leq f_{I\cap J}$$

in a neighborhood of x_0, since we may perturb the membranes u_m with $m \in I \cap J$ upwards by a positive function $\varepsilon\varphi$, $\varphi \in C_0^\infty(B_r(x_0))$ and keep satisfying the constraint (1.2). Similarly,

$$\triangle v_{I\cap J} \geq f_{I\cap J}.$$

Since $u_{I\cap J} \geq v_{I\cap J}$ and they coincide at x_0, we find that they coincide in $B_r(x_0)$. □

One-dimensional and two-dimensional cones.

Definition 2.2. We denote the space of one-dimensional cones by $\mathcal{P}$:

$$\mathcal{P} = \{\mathbf{p} : \mathbf{p} \text{ is a homogeneous of degree 2 solution, and } 0 = \textstyle\bigcap \Gamma_k\}.$$

We denote by $\mathcal{P}^c$ the solutions $\mathbf{p} \in \mathcal{P}$ which are nontrivially *connected* in the sense that each coincidence set $\Lambda_m := \{u_m = u_{m+1}\}$ is a half-line (or equivalently has nonempty interior),

$$\mathcal{P}^c = \{\mathbf{p} \in \mathcal{P} : \operatorname{int} \Lambda_m \neq \emptyset \text{ for all } m \leq N-1\}.$$

There are 3^{N-1} elements in $\mathcal{P}$, since there are three options for each of the coincidence sets Λ_m: $(-\infty, 0]$, $\{0\}$, $[0, \infty)$, and there are 2^{N-1} elements in $\mathcal{P}^c$.

A particular solution in $\mathcal{P}^c$ is $\mathbf{p_0}$ which has the components $p_i = \frac{f_i}{2}(x^+)^2$. It turns out that $\mathbf{p_0}$ and its reflection $\mathbf{p_0}(-\mathbf{x})$ are the least energy solutions among all $\mathbf{p} \in \mathcal{P}$.

In [8] we showed that the space of two-dimensional cones $\mathcal{C}_2$ is generated by one-dimensional cones in the following way. If $\mathbf{p} \in \mathcal{P}^c$, then its two-dimensional extension coincides with $\mathbf{p}(x_2)$ up to rotations.

If $\mathbf{p}_* \in \mathcal{P} \setminus \mathcal{P}^c$ (i.e., a *degenerate* cone), then we first decompose $\mathbf{p}_*$ as a union of $m \geq 2$ connected cones in $\mathcal{P}^c$. Each of these cones is extended to two dimensions, and then modified by a harmonic function and an angle of rotation, see Section 6 for more details.

A convergence lemma. We state a lemma about sequences and the convergence of series, which we use in the main result. In out setting w_n will represent the Weiss energy of u in the ball of radius ρ^N, while ε_n the rescaled error between u and an approximate solution.

Lemma 2.3. *Let w_n and ε_n be two sequences of real numbers between* 0 *and* 1. *Suppose that*

$$w_{n+1} \leq C_0 \varepsilon_n^{\frac{3}{2}},$$

and either

$$w_{n+1} \leq w_n \quad \textit{and} \quad \varepsilon_{n+1} = \frac{\varepsilon_n}{2},$$

or

$$w_{n+1} \leq w_n - c\varepsilon_n^2 \quad \textit{and} \quad \varepsilon_{n+1} = C\varepsilon_n.$$

Then

$$\sum_{n \geq k} \varepsilon_n \leq M k^{-1} \tag{2.6}$$

for some M depending only on c, C, C_0.

Proof. We only sketch the proof (see [11] for more details). The sequence

$$a_n := w_n + c'\varepsilon_n^2$$

satisfies $a_{n+1} \leq a_n - c\varepsilon_n^2 \leq a_n - C a_n^{\frac{4}{3}}$ which implies $a_n \leq C n^{-3}$. The conclusion follows by adding the inequalities

$$\varepsilon_n \leq C(a_n - a_{n+1})^{\frac{1}{2}}. \qquad \square$$

3. The one-dimensional problem

In this section, we study the N-membrane problem in one dimension. For each cone $\mathbf{p} \in \mathcal{P}^c$ and vector $\mathbf{b}$ associated to the branches of $\mathbf{p}$, we show that there is a unique global solution with linear asymptotics given by $\mathbf{b}x$ at $\pm\infty$. We also introduce the error function $\mathbf{e}(\mathbf{b})$, which plays an important role in the study of approximate solutions in general dimensions.

In the one-dimensional problem, each component of the solution is piecewise quadratic, and the difference between consecutive membranes is convex. This means that the coincidence set $\{u_m = u_{m+1}\}$ is an interval. Recall that $\mathcal{P}^c$ represents the connected one-dimensional cones, see Definition 2.2. If $\mathbf{p} \in \mathcal{P}^c$, then the graphs of all the components of $\mathbf{p}$ consists of $(N+1)$ disjoint half quadratics starting at the origin, i.e., $a(x^+)^2$ or $a(x^-)^2$. This is because any two consecutive graphs of the p_i have precisely a half quadratic in common. We call these disjoint quadratics the *branches* of $\mathbf{p}$. The right branches of p are the graphs over $[0, \infty)$ and the left branches the ones over $(-\infty, 0]$. The condition $\sum \omega_i p_i = 0$ implies that the right (respectively left) branches average to 0 when counting their weights and multiplicities.

We associate a real number b_k to each of the branches of $\mathbf{p}$ with the compatibility condition that the average of these numbers on the right (respectively left) branches equals 0. The collection of these b_k is denoted by $\mathbf{b} \in B(\mathbf{p})$.

Definition 3.1. For each $\mathbf{p} \in \mathcal{P}^c$, the space $B(\mathbf{p})$ consists of vectors

$$\mathbf{b} = (b_1^-, b_2^-, \ldots, b_N^-, b_1^+, \ldots, b_N^+),$$

with the property that $\sum \omega_i b_i^- = \sum \omega_i b_i^+ = 0$ and

$$\begin{aligned} b_i^- &= b_{i+1}^- \quad \text{if } p_i = p_{i+1} \text{ on } (-\infty, 0], \\ b_i^+ &= b_{i+1}^+ \quad \text{if } p_i = p_{i+1} \text{ on } [0, \infty). \end{aligned}$$

Clearly, $B(\mathbf{p}) \subset \mathbb{R}^{2N}$ is an $(N-1)$-dimensional linear subspace.

We want to solve the N-membrane problem after perturbing the branches of a solution $\mathbf{p} \in \mathcal{P}^{\mathbf{c}}$ by $x\mathbf{b}$.

Proposition 3.1. *Given $\mathbf{b} \in B(\mathbf{p})$, there exists a unique solution u to problem P_0 in $\mathbb{R}$ which satisfies*

$$u_i = p_i + b_i^{\pm} x^{\pm} + o(|x|) \quad \text{as } x \to \pm\infty,$$

where $b_i = b_i^{\pm}$ is the number associated to the branch of p_i.

Proof. We first show the existence.

We solve the problem in the interval $[-R, R]$ with boundary data $u_i = p_i + b_i x$ and obtain a solution $\mathbf{u}^R$, and then let $R \to \infty$. We need some uniform estimates.

Let $M > \max |b_i|$, and let t_0 be the first value as we decrease t for which the inequality

$$\mathbf{p} + (t + M|x|)\mathbf{1} > \mathbf{u}^R \quad \text{on } [-R, +R]$$

fails. When $t = t_0$, we need to replace $>$ with $\geq$ above and equality holds at some x_0 for some i-component.

Notice that $t_0 \geq 0$ which follows from the inequality written at $x = 0$ and

$$\sum \omega_i u_i = \sum \omega_i p_i = 0.$$

The left-hand side is a solution to our problem in each interval $(-R, 0)$, $(0, R)$ and by the strong maximum principle it follows that the first contact point must be $x_0 = 0$, since at the end points $\pm R$ we have strict inequality by the choice of M.

We claim that $t_0 \leq CM^2$ with C a universal constant. We choose $K = \delta^{-1} M$ with $\delta > 0$ the universal constant from Lemma 3.1 below, and then define $\mathbf{v}$ as the translation of $\mathbf{u}$

$$\mathbf{v} := \mathbf{u}^R - (t_0 + MK)\mathbf{1}.$$

We have

$$\mathbf{p} \geq \mathbf{v} \quad \text{in } [-K, K]$$

and

$$v_i(0) = -MK = p_i(0) - \delta K^2.$$

By Lemma 3.1 (rescaled) we find $\mathbf{v}(0) \geq -K^2\mathbf{1}$ which means $u_j^R(0) \geq t_0 + MK - K^2$ and the claim follows from $\sum \omega_i u_i^R(0) = 0$. A symmetric argument gives

$$\mathbf{p} + (CM^2 + M|x|)\mathbf{1} \geq \mathbf{u}^R \geq \mathbf{p} - (CM^2 + M|x|)\mathbf{1}.$$

Since $u_{i+1}^R - u_i^R$ has to grow quadratically away from the free boundary, it follows that if $p_i = p_{i+1}$ say on $[0, \infty)$, then $u_i^R = u_{i+1}^R$ on $[CM, R)$ for some C universal. In particular, u_i^R and $p_i + b_i x$ have the same constant as second derivative on $[CM, R)$. Their difference is at most CM^2 as at the end points of the interval. As $R \to \infty$ we can extract a subsequence which converges uniformly on each compact set and has the asymptotic expansion required.

For the uniqueness, we argue as above and obtain that u_i has the same second derivative as $p_i + b_i x$ in a neighborhood of ∞ (or $-\infty$) and therefore they must differ by a constant. Thus if $\mathbf{v}$ is another solution, $\sum \omega_i (u_i - v_i)^2$ is convex and bounded and therefore it is a constant. In particular, $\nabla(u_i - v_i) = 0$ for each i, thus $u_i - v_i$ is constant for each i. Since the branches of $\mathbf{u}$ and $\mathbf{v}$ are connected we find that these constants are independent of i, and since their average is 0, they all must be 0. □

We give a quantified version of the strong maximum principle for solutions near $\mathbf{p} \in \mathcal{P}^{\mathbf{c}}$.

Lemma 3.1. *Let $\mathbf{p} \in \mathcal{P}^{\mathbf{c}}$ and let $\mathbf{v}$ be a solution of our problem (not necessarily of average* 0*) with $\mathbf{p} \geq \mathbf{v}$ in $[-1, 1]$, $v_i(0) \geq -\delta$ for some i. Then $v_j(0) \geq -1$ for all j, provided that δ is sufficiently small.*

Proof. The inequality is clear if $j \leq i$. It suffices to show that the collection of the graphs of the v_j with $j \geq i$ are all connected in the strip $\{|x| \leq c\}$ for some c small. Assume not, and them let $l \geq i$ be the last membrane connected to v_i in $[-c, c]$. Then $v_1, \dots, v_l$ are uniformly bounded in $[-c, c]$, and solve the l-membrane problem in $[-c, c]$. By compactness (for fixed l), as $\delta \to 0$ we obtain a limiting solution $\tilde{v}$ of the l-membrane problem which is below $(p_1, \dots, p_l)$ and with $\tilde{v}_i(0) = p_i(0) = 0$. Since $l < N$, it follows that $(p_1, \dots, p_l)$ is a strict supersolution to the l-membrane problem, and we contradict the maximum principle between p and $\tilde{v}$. □

Definition 3.2. Given $\mathbf{p} \in \mathcal{P}^c$ and $\mathbf{b} \in B(\mathbf{p})$, we denote by

$$\mathbf{h}(x, \mathbf{b})$$

the unique solution $\mathbf{u}$ from Proposition 3.1 to problem P_0 which has linear coefficients $\mathbf{b}$ in its asymptotic expansion at $\pm\infty$

$$u_i = p_i + b_i x + o(|x|) \quad \text{as } x \to \pm\infty.$$

Remark 3.1. Notice that $\mathbf{p}(x + 1)$ has linear coefficients $\tau_i := \frac{p_i'}{x}$ in its expansion at $\pm\infty$. Hence if $\mathbf{b} = s\boldsymbol{\tau}$, then

$$\mathbf{h}(x, s\boldsymbol{\tau}) = \mathbf{p}(x + s),$$

or more generally

$$\mathbf{h}(x, \mathbf{b} + s\boldsymbol{\tau}) = \mathbf{h}(x + s, \mathbf{b}).$$

With this and the fact that the linear perturbation keeps the contact situation unchanged outside an interval of length $C\|b\|$, we have the following.

Lemma 3.2. *The function $\mathbf{h}(x, \mathbf{b})$ is homogeneous of degree 2 in the variables x and $\mathbf{b}$, and is $C^{1,1}$ and piecewise quadratic in the x-variable. Moreover,*

$$h_i = p_i + b_i x + O(\|\mathbf{b}\|^2),$$

and outside the interval $[-C\|\mathbf{b}\|, C\|\mathbf{b}\|]$ we have

$$h_i = p_i + b_i x + e_i,$$

with e_i a constant which depends only on the branch.

Definition 3.3. We refer to the function $\mathbf{b} \mapsto \mathbf{e}$ which maps $B(\mathbf{p})$ to $B(\mathbf{p})$ as *the error function* (which is a homogeneous of degree 2 map).

It turns out that $\mathbf{h}(x, \mathbf{b})$ is $C^{1,1}$ in the $\mathbf{b}$-variable as well. The proof of this fact is technical and can be skipped on a first reading.

Lemma 3.3. *The function $\mathbf{h}(x, \mathbf{b})$ is piecewise quadratic and of class $C^{1,1}$ in both variables x and $\mathbf{b}$. In particular, the error map $\mathbf{e}(\mathbf{b})$ is piecewise quadratic in $\mathbf{b}$.*

Proof. Each solution $\mathbf{u}$ to problem P_0 which is asymptotic to $\mathbf{p}$ at infinity, in the sense that $R^{-2}\mathbf{u}(Rx) \to \mathbf{p}$ must be of the form $\mathbf{h}(x, \mathbf{b})$ and is uniquely determined by $\mathbf{b}$.

On the other hand each such solution is also uniquely determined by the location of the free boundaries Γ_i. For example u_1 and u_2 coincide on the side of Γ_1 where their corresponding branches agree and they must differ on the other side of Γ_1. So if we know the locations of all the Γ_i, $1 \le i \le N-1$, then we know in each of the corresponding subintervals determined by the Γ_i which membranes coincide, and thus the second derivatives of all the u_i are uniquely determined. In other words if we arrange the free boundary points in increasing order $\Gamma_{i_1} \le \Gamma_{i_2} \le \Gamma_{i_{N-1}}$, then each u_k'' is determined on the interval $[\Gamma_{i_j}, \Gamma_{i_{j+1}}]$ by the permutation $\pi = \{i_1, \dots, i_{N-1}\}$ of $\{1, 2, \dots, N-1\}$. We can then integrate these second derivatives and construct a solution $\mathbf{u}$ to problem P with free boundaries Γ_i. Since the graphs of all the membranes are connected, the solution $\mathbf{u}$ is unique up to a linear function. We explain more in detail how to construct $\mathbf{u}$ inductively in the following way.

Assume that the top membrane p_1 of $\mathbf{p}$ is free on the left and has the common branch with p_2 on the right. Then we construct u_1 on the left of Γ_1 as $\frac{f_1}{2}(x-\Gamma_1)^2$ and then on the right of Γ_1 we need to add to this quadratic a linear combination of terms $[(x-\Gamma_k)^+]^2$ according to values of u_1'' on the subintervals $[\Gamma_{i_j}, \Gamma_{i_{j+1}}]$ to the right of Γ_1. Then we construct u_2 as equal to u_1 on the right side of Γ_1 and on then on the left of Γ_1 we need to adjust it by adding to u_1 a linear combination of terms $[(x-\Gamma_k)^-]^2$ according to the values of u_2'' on the subintervals to the left of Γ_1. Then we define u_3 as equal to u_2 on the side of Γ_2 where the branches of p_2 and p_3 coincide, and modify it on the other side of Γ_2 according to the values of u_3''. We continue this process till u_N. By construction,

$$u_1 \ge u_2 \ge \cdots \ge u_N$$

(since $u_k'' \ge u_{k+1}''$ which is a consequence of non-degeneracy), and the Euler–Lagrange equations are satisfied, hence $\mathbf{u}$ is a solution of problem P with the given free boundaries Γ_k. By construction, each u_i is of the form

$$u_i = \sum_k \mu_{ki}^+[(x-\Gamma_k)^+]^2 + \mu_{ki}^-[(x-\Gamma_k)^-]^2, \tag{3.1}$$

where the coefficients $\mu_{ki}^\pm$ are determined only by the permutation π. We obtain a solution to P_0 after subtracting their total average from each one of them. The corrected u_i have the same form as above. The corresponding vector $\mathbf{b}$ for this solution is obtained from the asymptotic expansion of the u_i at $\pm\infty$, which means that $\mathbf{b}$ is a linear combination of the Γ_i with coefficients depending on the $\mu_{ik}^\pm$. Since $\mathbf{b}$ is uniquely determined by the Γ_i it follows that the map $(\Gamma_1,\dots,\Gamma_{N-1}) \mapsto \mathbf{b}$ is an invertible linear map on each open region of $\mathbb{R}^{N-1}$ where the Γ_i do not change the order. This linear map depends only on the permutation π and in each such region Γ_k is a linear function of $\mathbf{b}$.

We view the function constructed above as a function of N variables $\mathbf{u}(x,\mathbf{\Gamma}) = \mathbf{p}(x,\mathbf{b})$, and notice that $\mathbf{u}(x,\mathbf{\Gamma})$ is purely quadratic in its variable in each of the $N!$ convex polyhedral regions determined by the relative orders between the variables $x, \Gamma_1,\dots,\Gamma_{N-1}$. In each such region $\mathbf{\Gamma} = A_\pi \mathbf{b}$ for an invertible linear map A_π. Thus, when viewed as a function of $(x,\mathbf{b})$, $\mathbf{u}$ is still purely quadratic in its variables in the corresponding $N!$ polyhedral convex regions in the $(x,\mathbf{b})$-variables.

It suffices to show that the normal derivatives of the quadratic polynomials on each side of a common $(N-1)$-dimensional face between two adjacent regions coincide. Then $CI \geq D^2_{(x,\mathbf{b})}\mathbf{p} \geq CI$ except on a set of dimension $N-2$, and this inequality can then be extended by continuity on the remaining lower-dimensional set as well.

Let us consider a point $(x_0,\mathbf{b}_0)$ on a common $(N-1)$-dimensional face between two regions. Let $\mathbf{u}_0(x) = \mathbf{p}(x,\mathbf{b}_0)$ be the corresponding solution for $\mathbf{b}_0$ and let $\mathbf{\Gamma}_0$ be the free boundary vector associated with $\mathbf{u}_0$. In the $(x,\mathbf{\Gamma})$-variables, a common $(N-1)$-dimensional face between two regions corresponds to the case when two of the N coordinates of $(x,\mathbf{\Gamma})$ coincide and all the others are different.

Case 1: x_0 coincides with $\Gamma_{0,k}$. As we let x vary near x_0 and keep $\mathbf{\Gamma}_0$ fixed, the derivatives of $\mathbf{u}_0$ match at $\Gamma_{0,k}$ since $\mathbf{u}_0$ is a $C^{1,1}$ function. This means that the directional derivative with respect to the x-direction at $(x_0,\mathbf{b}_0)$ agree. This direction is transversal to the face $x = \Gamma_k$ (since Γ_k is linear in $\mathbf{b}$ near $(x_0,\mathbf{b}_0)$) and the conclusion follows.

Case 2: $\Gamma_{0,k} = \Gamma_{0,l}$ for some $k < l$. We study the behavior of the solution $\mathbf{u}$ as we vary $\mathbf{\Gamma}$ in an ε-neighborhood near $\mathbf{\Gamma}_0$.

If $u_{0,k}(\Gamma_{0,k}) > u_{0,l}(\Gamma_{0,l})$, then there is no change in the topology of the graph of $\mathbf{u}$ as we vary $\mathbf{\Gamma}$. This means that the right-hand sides for $\mathbf{u}''$ in the subintervals determined by $\mathbf{\Gamma}$ are not affected when Γ_k and Γ_l cross each other. The coefficients $\mu_{ij}^\pm$ in (3.1) remain the same on either side of $\Gamma_k = \Gamma_l$ and the two polynomials coincide.

Next we assume that $u_{0,k}(\Gamma_{0,k}) = u_{0,l}(\Gamma_{0,l})$, and denote by $Z \in \mathbb{R}^2$ the point on the graph of $\mathbf{u}_0$ where kth and lth membranes coincide. We prove our claim by extending the solution given by (3.1) when $\Gamma_k \leq \Gamma_l$ to a whole ε-neighborhood of $\mathbf{b}_0$ and then show that it differs from the exact solution by at most $C\varepsilon^2$.

Let $v(x,\mathbf{b})$ denote the right-hand side of (3.1) corresponding to the permutation π with $\Gamma_k < \Gamma_l$, where Γ_k are viewed as linear functions of $\mathbf{b}$. When $\Gamma_k(\mathbf{b}) \leq \Gamma_l(\mathbf{b})$, then v is the solution to problem P_0 (with asymptote $\mathbf{b}$). However, when $\Gamma_k > \Gamma_l$, then $\mathbf{v}$ might fail to solve our problem near Z. We collect hear the properties of $\mathbf{v}$ in this case:

(1) By construction, $\mathbf{v}$ is a $C^{1,1}$ function and $\mathbf{v}''$ is constant in each of the N subintervals defined by $\mathbf{\Gamma}$.

(2) $|\mathbf{\Gamma} - \mathbf{\Gamma}_0| = O(\varepsilon)$ and $|\mathbf{v} - \mathbf{u}_0| = O(\varepsilon)$ on any compact interval.

(3) The quadratic polynomial expressions in $(x, \mathbf{b})$ that define $\mathbf{v}$ in the open subintervals of Γ remain constant as we exchange the order of Γ_k and Γ_l, except for the ones in the interval between Γ_l and Γ_k. Outside this interval the membranes of $\mathbf{v}$ that coincide when $\Gamma_k \leq \Gamma_l$ continue to coincide, and their right-hand sides remain constant. In particular, $\mathbf{v}$ has the vector $\mathbf{b}$ in its asymptotic expansion at $\pm\infty$, and its average is 0 away from $[\Gamma_l, \Gamma_k]$.

(4) In a neighborhood of the interval $[\Gamma_l, \Gamma_k]$, for the membranes v_i for those i for which Z does not belong to the graph of the ith membrane of $\mathbf{u}_0$, their polynomial expressions remain constant. Indeed, for such i, u_i'' has no discontinuity at Γ_k or Γ_l thus $\mu_{ik}^+ = \mu_{ik}^-$ and $\mu_{il}^+ = \mu_{il}^-$, and the orders of the Γ_k, Γ_l do not affect the polynomial expressions for v_i.

(5) Let J denote the indexes of the membranes of $\mathbf{u}_0$ which pass through Z. If $j \in J$ and $\mathbf{u}$ is a solution near $\mathbf{u}_0$, then near $\Gamma_{0,k}$ we have

$$u_j = \begin{cases} u_k & \text{if } j \leq k, \\ u_{k+1} & \text{if } k+1 \leq j \leq l, \\ u_{l+1} & \text{if } j \geq l+1. \end{cases}$$

The same equalities hold if we replace $\mathbf{u}$ by $\mathbf{v}$. Indeed, by (3) the equalities hold in this neighborhood outside the interval $[\Gamma_l, \Gamma_k]$. They hold also inside this interval which is a consequence of the fact that the difference between two v_j is a $C^{1,1}$ function with constant second derivative.

Thus there are three different profiles for the functions v_j with $j \in J$ which do not satisfy the correct Euler–Lagrange in $[\Gamma_l, \Gamma_k]$. These three profiles are connected either at Γ_k or Γ_l, since by (3) $v_k = v_{k+1}$ and $v_{l+1} = v_l$ either to the left of Γ_l or the right of Γ_k. The three profiles are uniformly $C^{1,1}$ thus they differ by at most $C\varepsilon^2$ in this interval.

We remark that the v_j with $j \in J$ (the three profiles) might not be monotone with respect to j. However, outside a $C\varepsilon$-neighborhood of $[\Gamma_l, \Gamma_k]$ they become ordered with respect to j due to the non-degeneracy condition that holds outside this interval.

Now we prove that $|\mathbf{v} - \mathbf{p}(x, \mathbf{b})| \leq C\varepsilon^2$. Let t_0 be the first value as we decrease t for which the inequality $\mathbf{p} + t > \mathbf{v}$, fails. Since $\mathbf{p}$ and $\mathbf{v}$ have the same asymptotic expansion at $\pm\infty$, and $\mathbf{v}$ is a solution except on the interval $[\Gamma_l, \Gamma_k]$ for the v_j with $j \in J$, it follows that there exists x_0 in this interval for which $v_j(x_0) = p_j(x_0) + t_0$. Since all v_j and all p_j are connected in this interval and are uniformly $C^{1,1}$ it follows that $|v_j - (p_j + t_0)| \leq C\varepsilon^2$ for all $j \in J$ in an ε-neighborhood of the interval $[\Gamma_l, \Gamma_k]$. However, outside this neighborhood both graphs of $\mathbf{v}$ and $\mathbf{p}$ solve problem P (with the same asymptotic expansion at $\pm\infty$), hence this inequality can be extended everywhere. Now $|t_0| \leq C\varepsilon^2$ is a consequence of the null average of $\mathbf{v}$ and $\mathbf{p}$ outside $[\Gamma_l, \Gamma_k]$. □

4. Approximate solutions

In this section we define the class of the approximate solutions $\mathbf{p}(x,\mathbf{b})$ in $\mathbb{R}^2$ which are perturbations of the one-dimensional profile $\mathbf{p}(x_2)$ with $\mathbf{p}\in\mathcal{P}^c$, and collect some of their properties. We establish the algebraic statement that the error in the Euler–Lagrange equation cannot be improved further unless $\mathbf{p}(x,\mathbf{b})$ is a rotation of $\mathbf{p}$, see Lemma 4.3. In Corollary 4.2, we obtain the convergence of the rescaled errors between $\mathbf{u}$ and an approximate solution $\mathbf{p}(x,\mathbf{b})$.

We begin with the definition of the approximate solution $\mathbf{p}(x,\mathbf{b})$.

Definition 4.1. Given $\mathbf{p}\in\mathcal{P}^c$ and $\mathbf{b}\in B(\mathbf{p})$, we denote by

$$\mathbf{p}(x,\mathbf{b})=\mathbf{h}(x_2,x_1\mathbf{b}).$$

Clearly $\mathbf{p}(x,\mathbf{b})$ is a homogeneous of degree 2 function in its variables.

Lemma 4.1. *The function $\mathbf{v}(x):=\mathbf{p}(x,\mathbf{b})$ satisfies the following:*

(a) *It solves the Euler–Lagrange equations with error $C\|\mathbf{b}\|^2$. Precisely, we have $\mathbf{v}\in C^{1,1}$, $v_1\ge\cdots\ge v_N$ and in an open region where $v_i>v_{i+1}$ and $v_k>v_{k+1}$ we have*

$$|\triangle v_I-f_I|\le C|\mathbf{b}|^2\quad \textit{with } I=\{i+1,\dots,k\}.$$

(b) *We have*

$$v_i(x)=p_i(x_2)+b_ix_1x_2+O(|\mathbf{b}|^2x_1^2),$$

and in the cone $\{|x_2|\ge C\|\mathbf{b}\||x_1|\}$ with C large universal,

$$\triangle v_i=\triangle p_i+2e_i(\mathbf{b})\chi_{\{x_1\ge0\}}+2e_i(-\mathbf{b})\chi_{\{x_1\le0\}},$$

where $\mathbf{e}(\mathbf{b})$ is the error function defined in Definition 3.3.

Proof. By definition, v solves the Euler–Lagrange equations in the x_2-variable hence

$$\triangle v_I-f_I=\partial_{x_1x_1}v_I.$$

Using the homogeneity of $\mathbf{h}$ we find

$$\mathbf{v}_{11}=2\mathbf{h}-2t\mathbf{h}_t+t^2\mathbf{h}_{tt},$$

where h and its derivatives are evaluated at the point $(t,\frac{x_1}{|x_1|}\mathbf{b})$ with $t:=\frac{x_2}{|x_1|}$.

Moreover, by Lemma 3.2, the right-hand side is constant in each of the four connected regions of the set $\{|x_2|>C\|\mathbf{b}\||x_1|\}\setminus\{x_1=0\}$ and equals

$$\mathbf{v}_{11}=2\mathbf{e}(\mathbf{b})\chi_{\{x_1\ge0\}}+2\mathbf{e}(-\mathbf{b})\chi_{\{x_1\le0\}}.\qquad\square$$

Definition 4.2. Similarly we may define the more general class of functions

$$\mathbf{p}(x,\mathbf{b}_0,\mathbf{b}_1)=\mathbf{h}(x_2,\mathbf{b_0}+x_1\mathbf{b}_1).$$

When $\mathbf{b_0}=0$, we are in the situation of Definition 4.1 and then use the simpler notation $\mathbf{p}(x,\mathbf{b}_1)$ for $\mathbf{p}(x,\mathbf{0},\mathbf{b}_1)$ as before.

We give the corresponding lemma for this more general class of solutions.

Lemma 4.2. *The function* $\mathbf{v}(x) := \mathbf{p}(x, \mathbf{b}_0, \mathbf{b}_1)$ *satisfies the following:*

(a) *It solves the Euler–Lagrange equations with error* $C\|\mathbf{b}\|^2$. *Precisely, we have* $\mathbf{v} \in C^{1,1}$, $v_1 \geq \cdots \geq v_N$ *and in an open region where* $v_i > v_{i+1}$ *and* $v_k > v_{k+1}$ *we have*

$$|\triangle v_I - f_I| \leq C|\mathbf{b}_1|^2 \quad \text{with } I = \{i+1, \ldots, k\}.$$

(b) *We have*

$$v_i(x) = p_i(x_2) + b_{0,i}x_2 + b_{1,i}x_1x_2 + O(|\mathbf{b}_0|^2 + |\mathbf{b}_1|^2 x_1^2).$$

Proof. The proof is the same as above, and follows from $|D^2\mathbf{h}| \leq C$ (see Lemma 3.3) and Lemma 3.2. □

Lemma 4.3. *We have* $\mathbf{e}(\mathbf{b}) = \mathbf{e}(-\mathbf{b})$ *if and only if* $\mathbf{b} = s\tau$ *for some* $s \in \mathbb{R}$, *where* τ *is defined in Definition* 3.1.

Notice that $\mathbf{b} = s\tau$ is equivalent to $\mathbf{p}(x, \mathbf{b}) = \mathbf{p}(x_2 + sx_1)$.

As a consequence of the homogeneity of $\mathbf{e}$ we can quantify the difference between $\mathbf{e}(\mathbf{b})$ and $\mathbf{e}(-\mathbf{b})$ in terms of the distance from $\mathbf{b}$ to the line of direction τ.

Corollary 4.1. *There exists a strictly increasing continuous function*

$$\rho : [0, 2] \to [0, \infty) \quad \text{with } \rho(0) = 0$$

such that

$$\frac{|\mathbf{e}(\mathbf{b}) - \mathbf{e}(-\mathbf{b})|}{\|\mathbf{b}\|^2} \geq \rho\left(\operatorname{dist}\left(\frac{\mathbf{b}}{\|\mathbf{b}\|}, \pm\frac{\tau}{\|\tau\|}\right)\right) \quad \text{for all } \mathbf{b} \neq 0.$$

Since $e(\mathbf{b})$ is piecewise quadratic in $\mathbf{b}$, it follows that $\rho(s) \geq cs^2$.

Proof of Lemma 4.3. One implication is trivial.

Due to the homogeneity of $\mathbf{e}$ it suffices to assume that $\mathbf{e}(\mathbf{b}) = \mathbf{e}(-\mathbf{b})$ and $\|\mathbf{b}\| \leq \delta$ for some small δ universal. Let Γ_i^+ denote the free boundaries for the one-dimensional solution $\mathbf{h}(t, \mathbf{b})$ and Γ_i^- the free boundaries of $\mathbf{h}(t, -\mathbf{b})$. We want to show that all Γ_i^+ coincide and that $\Gamma_i^- = -\Gamma_i^+$.

By the lemma above, the function $\mathbf{v}(x) := \mathbf{p}(x, \mathbf{b})$ is a solution to problem P with an error $C\delta^2$, in the sense that

(1) $v \in C^{1,1}$, $v_1 \geq \cdots \geq v_N$,

(2) the free boundaries of $\mathbf{v}$ are given by the rays $x_2 = \Gamma_i^+ x_1$ in $\{x_1 > 0\}$ and $x_2 = -\Gamma_i^- x_1$ in $\{x_1 < 0\}$,

(3) in each of the sectors determined by these rays, the component v_i solves the equation $\triangle v_i = g_I$ with g_I a constant, and $|g_I - f_I| \leq C\delta^2$, where I is the set of j for which $v_j = v_i$ in that sector.

Notice that $\mathbf{e}(\mathbf{b}) = \mathbf{e}(-\mathbf{b})$ is equivalent to the statement that the corresponding right-hand sides g_I agree on either side of the x_2-axis on the two sectors that contain the positive respectively negative x_2-axis. Also, if δ is chosen small, then the non-degeneracy condition holds for the right-hand sides g, i.e., $\triangle v_i > \triangle v_k$ if $v_i > v_k$. Now we can argue as in the classification of homogeneous solutions in two dimensions to conclude that all free boundaries coincide with a single line passing through the origin, which gives the desired conclusion. We provide the details.

We denote by (r, θ) the polar coordinates in $\mathbb{R}^2$. Recall the following elementary lemma from [8]:

Lemma 4.4. *Assume w is homogeneous of degree 2 and defined in the angle $\theta \in [0, \alpha]$ with $w = 0$, $\nabla w = 0$ on the rays $\theta = 0$, $\theta = \alpha$. If*

$$\triangle w = \varphi \geq 0$$

and φ is a step function which is nondecreasing in $[0, \gamma]$, and nonincreasing in $[\gamma, \alpha]$ for some γ, then

$$\alpha \geq \pi.$$

Moreover, if $\alpha = \pi$, then φ must be constant.

We restrict our attention to the values of v_i on the unit circle ∂B_1. We know that each two consecutive membranes v_i and v_{i+1} are connected (agree) at least on an open interval that contains either $(0, 1)$ or $(0, -1)$, and they do not agree on the whole circle.

We focus on those intervals $I \subset \partial B_1$ where $\{v_k > v_{k+1}\}$ and $v_k = v_{k+1}$ at the end points and in addition $\triangle v_k$ is constant in I.

Claim. *Each such interval has length greater than or equal to π.*

Indeed, we look at a minimal such interval and we apply Lemma 4.4 to the difference

$$w_k := v_k - v_{k+1},$$

which vanishes of order two at the end points of I. Moreover,

$$\triangle w_k = \varphi_k := g_k - g_{k+1} > 0 \quad \text{on } I,$$

The minimality of I implies that the nested sets $\{v_{k+1} = v_{k+m}\}$ are connected (intervals) in I, and therefore w_k, φ_k satisfy the hypotheses of Lemma 4.4.

The claim implies that $\{v_1 > v_2\}$ consists of exactly one interval I_1 of length at least π. In the cone generated by I_1, the function v_1 coincides with a quadratic polynomial Q. Denote by $\widetilde{v}_1$ this polynomial Q in the complement of the angle generated by I_1. Here we can apply one more time the argument of the claim above by using the function $\widetilde{w}_1 := \widetilde{v}_1 - v_2$ and conclude that also the complement has length at least π on the unit circle.

In conclusion, I_1 consists exactly of a half-circle. Lemma 4.4 gives in addition that $\triangle v_2$ is in fact constant on I_1 and its complement. This in turn implies that v_2 and v_3 either coincide or are disjoint in each of these two intervals. By arguing as above with v_2 and v_3 instead of v_1 and v_2, we find that also $\triangle v_3$ must be constant in each of these two intervals, which gives that $\{v_3 = v_4\}$ is either I_1 or its complement. We can argue like this inductively and reach that all the free boundaries must coincide. □

Definition 4.3. Given $\mathbf{p} \in \mathcal{P}^c$, we say that a solution $\mathbf{u}$ to problem P_0 is ε-approximated in B_r and write

$$\mathbf{u} \in \mathcal{S}(r, \mathbf{p}, \varepsilon)$$

if, after a rotation around the origin, $\mathbf{u}$ satisfies

$$|\mathbf{u} - \mathbf{p}(\cdot\,, \mathbf{b})| \le \varepsilon r^2 \quad \text{in } B_r$$

for some $\mathbf{b} \in B(\mathbf{p})$ with $|\mathbf{b}| \le \delta \varepsilon^{\frac{1}{2}}$, with δ a small universal constant (to be made precise later).

Lemma 4.5. *Assume that*

$$\mathbf{u} \in \mathcal{S}(1, \mathbf{p}, \varepsilon). \tag{4.1}$$

Then in $B_{3/4}$ *we have* $\Gamma_i \subset \{|x_2| \le C\sqrt{\varepsilon}\}$ *for all* i, *and*

$$|\triangle(u_i - p_i(\cdot\,, \mathbf{b}))| \le \delta\varepsilon \qquad \text{in } \{|x_2| \ge C\sqrt{\varepsilon}\} \cap B_{3/4}. \tag{4.2}$$

Proof. Any two consecutive membranes, say u_i and u_{i+1}, coincide on one side of this strip $\{|x_2| \le C\sqrt{\varepsilon}\}$ and are separated on the opposite side, depending on whether the membranes p_i and p_{i+1} of the one-dimensional solution $\mathbf{p} \in \mathcal{P}^c$ coincide to the right or left of the origin.

Indeed, assume that $p_i = p_{i+1}$ to the left of the origin, and then

$$p_i(x, \mathbf{b}) - p_{i+1}(x, \mathbf{b}) \ge c[(x_2 - C|x_1\mathbf{b}|)^+]^2$$

and

$$p_i(x, \mathbf{b}) = p_{i+1}(x, \mathbf{b}) \quad \text{if } x_2 \le -C|x_1\mathbf{b}|.$$

The bound $|\mathbf{b}| \le \delta\sqrt{\varepsilon}$ from Definition 4.3 and (4.1) implies that

$$u_i > u_{i+1} \quad \text{in } B_1 \cap \{x_2 \ge C\sqrt{\varepsilon}\}$$

and

$$|u_i - u_{i+1}| \le 2\varepsilon \quad \text{in } B_1 \cap \{x_2 \ge C\sqrt{\varepsilon}\}.$$

The claim

$$\Gamma_i \subset \{|x_2| \le C\sqrt{\varepsilon}\} \cap B_{1 - C\sqrt{\varepsilon}}$$

follows since u_i and u_{i+1} separate quadratically away from their free boundary Γ_i.

As a consequence we find that in $\{|x_2| \ge C\sqrt{\varepsilon}\} \cap B_{3/4}$,

$$\triangle u_i = \triangle(p_i(x_2)) = f_I \quad \text{in } \{|x_2| \ge C\sqrt{\varepsilon}\},$$

and, by Lemma 4.1,

$$|\triangle(u_i - p_i(\cdot\,, \mathbf{b}))| \le C|\mathbf{b}|^2 \le C\delta^2\varepsilon \le \delta\varepsilon,$$

provided δ is sufficiently small. □

Lemma 4.6. *Assume that* $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}, \varepsilon)$. *Then in* $B_{1/2}$,

$$|\mathbf{u} - \mathbf{p}(\cdot\,, \mathbf{b})| \le C\varepsilon(|x_2| + \sqrt{\varepsilon})^\alpha$$

for some $\alpha > 0$ *small, universal.*

Proof. We pick a point $Z = (z, 0)$, $|z| \le \frac{1}{2}$ on the x_1-axis. It suffices to show by induction that for $k \ge 0$,

$$|u_i - p_i(\cdot, \mathbf{b})| \le \varepsilon_k := \varepsilon(1-c)^k \quad \text{in } B_{r_k}(Z), \quad r_k := \rho^{k+1},$$

as long as $r_k \ge C'\sqrt{\varepsilon}$, where ρ, c are small, universal constant.

Assume the induction hypothesis holds for k and suppose that $\mathbf{p}$ has at least two branches on the right (in the x_2-direction). We denote by $Y := Z + \frac{1}{2} r_k e_2$, and we claim that if

$$u_j(Y) \ge p_j(Y, \mathbf{b}) \quad \text{for some } j, \tag{4.3}$$

then

$$u_i - p_i(\cdot, \mathbf{b}) \ge (c-1)\varepsilon_k \quad \text{in } B_{\rho r_k}(Z), \quad \text{for all } i. \tag{4.4}$$

By Lemma 4.5, we know that

$$|\triangle(u_i - p_i(\cdot, \mathbf{b}))| \le \delta\varepsilon \le \delta\varepsilon_k r_k^{-2} \quad \text{in } \{|x_2| \ge C\sqrt{\varepsilon}\} \cap B_{r_k}(Z),$$

and

$$u_i - p_i(\cdot, \mathbf{b}) \ge -\varepsilon_k \quad \text{in } B_{r_k}(Z),$$

by the induction hypothesis. We prove (4.4) by comparing $\mathbf{u}$ with an explicit subsolution $\mathbf{v}$ in the rectangle

$$R := \left\{|x_1 - z| \le \frac{r_k}{2}\right\} \times \{|x_2| \le 4\rho r_k\}.$$

The Harnack inequality and (4.3) imply that

$$u_j - p_j(\cdot, \mathbf{b}) \ge (c_0 - 1)\varepsilon_k \quad \text{on } \partial R \cap \{x_2 = 4\rho r_k\} \tag{4.5}$$

for some $c_0 = c_0(\rho)$ universal. This inequality holds for all other membranes which coincide with u_j in the region $\{x_2 \ge C\sqrt{\varepsilon}\}$. We denote by J these indexes l for which $u_l(Y) = u_j(Y)$, and remark that J depends only on the branch configuration of $\mathbf{p}$. We let $\mathbf{t} \in B(\mathbf{p})$ be defined as $t_i^- = 0$ for all i, and

$$t_i^+ = \begin{cases} 1 & \text{if } i \in J, \\ -\mu & \text{otherwise.} \end{cases}$$

The constant $\mu > 0$ is chosen such that the average of all the t_i^+ equals 0, so that $\mathbf{t} \in B(\mathbf{p})$.

We define the barrier (see Definition 4.2)

$$\mathbf{v}(x) := \mathbf{p}(x_2, \mathbf{d}, \mathbf{b}) + \left(c_1 \varepsilon_k q\left(\frac{x - Z}{r_k}\right) - \varepsilon_k\right)\mathbf{1},$$

where

$$\mathbf{d} := c_1 \varepsilon_k r_k^{-1} \mathbf{t}, \quad q(x) := \frac{\mu}{2}(x_2 + 2\rho) + x_2^2 - \frac{1}{2}x_1^2, \tag{4.6}$$

and c_1 is small, depending on the constant c_0 above. The polynomial q and the constant ρ are chosen such that $\triangle q = 1$,

$$q + t_i x_2^+ \ge c_2 := \frac{1}{2}\mu\rho \quad \text{in } B_\rho, \tag{4.7}$$

and on the boundary of the rescaled rectangle

$$R_0 := \left\{|x_1| \le \frac{1}{2}\right\} \times \{|x_2| \le 4\rho\},$$

we have

$$q + t_i x_2^+ \le -c_2 \quad \text{on } \partial R_0 \setminus \{x_2 = 4\rho\}, \quad \text{for all } i,$$

and

$$q + t_i x_2^+ \le -c_2 \quad \text{on } \partial R_0 \quad \text{if } i \notin J. \tag{4.8}$$

We check that $\mathbf{u} \ge \mathbf{v}$ on ∂R, and $\mathbf{v}$ is a subsolution to problem P.

By Lemma 4.2, $\mathbf{p}(x, \mathbf{d}, \mathbf{b})$ solves problem P with an error

$$C|\mathbf{b}|^2 \le C\delta^2\varepsilon \le \delta\varepsilon \le \delta\varepsilon_k r_k^{-2},$$

and since $\triangle q = 1$, it follows that $\mathbf{v}$ is a subsolution to problem P if δ is sufficiently small ($\delta \le c_1$).

Notice that $\varepsilon_k r_k^{-2}$ is increasing with k, and when $r_k \sim C'\sqrt{\varepsilon}$, then

$$\varepsilon_k r_k^{-2} \le C\varepsilon^{\alpha} \le \delta^2 \quad \text{provided that } \varepsilon \le \varepsilon_0(\delta).$$

Thus,

$$C|\mathbf{d}|^2 \le C\varepsilon_k^2 r_k^{-2} \le \delta\varepsilon_k \quad \text{and} \quad C|\mathbf{b}|^2 x_1^2 \le \delta\varepsilon_k,$$

and by Lemmas 4.2 (b),

$$|p_i(x, \mathbf{d}, \mathbf{b}) - p_i(x, \mathbf{b}) - c_1\varepsilon_k r_k^{-1} t_i x_2^+| \le 3\delta\varepsilon_k \quad \text{in } B_{r_k}(Z). \tag{4.9}$$

Using inequalities (4.8) of q on ∂R_0, we obtain that

$$v_i \le p_i(\cdot, \mathbf{b}) + \varepsilon_k(3\delta - c_1c_2 - 1) \le p_i(x, \mathbf{b}) - \varepsilon_k \le u_i \quad \text{on } \partial R \quad \text{if } i \notin J,$$

and

$$v_i \le p_i(x, \mathbf{b}) - \varepsilon_k \le u_i \quad \text{on } \partial R \setminus \{x_2 = 4\rho r_k\}, \quad \text{for all } i.$$

Finally, on $\partial R \cap \{x_2 = 4\rho r_k\}$ and $i \in J$ we have by (4.5),

$$v_i \le p_i(x, \mathbf{b}) + (C(\mu, \rho)c_1 - 1)\varepsilon_k \le u_i,$$

provided that c_1 is chosen small so that $C(\mu, \rho)c_1 \le c_0$.

In conclusion, $\mathbf{u} \ge \mathbf{v}$ on ∂R, and the inequality holds in the whole R by the maximum principle. In particular, by (4.7) in $B_{\rho r_k}$

$$u_i \ge v_i \ge p_i(\cdot, \mathbf{b}) + (-3\delta + c_1c_2 - 1)\varepsilon_k \ge p_i(\cdot, \mathbf{b}) + (c - 1)\varepsilon_k. \qquad \square$$

Corollary 4.2. *If $\mathbf{u}_m \in \mathcal{S}(1, \mathbf{p}, \varepsilon_m)$, for a sequence of $\varepsilon_m \to 0$, then, up to a subsequence, each of the rescaled error functions*

$$\varepsilon_m^{-1}(u_{m,j} - p_j(\cdot, \mathbf{b}_m))$$

converges uniformly in $B_{1/2}$ to a limit w_j that satisfies

$$\|w_j\|_{L^\infty} \le 1, \quad w_j = 0 \quad \text{on } x_2 = 0,$$

and

$$|\triangle w_j| \le \delta \quad \textit{away from } \{x_2 = 0\}.$$

More precisely, $\triangle w_j$ *is constant in each quadrant*

$$\triangle w_j = -2e_j(\mathbf{b})\chi_{\{x_1>0\}} - 2e_j(-\mathbf{b})\chi_{\{x_1<0\}} \quad \textit{in } \{x_2 < 0\} \cup \{x_2 > 0\},$$

where $\mathbf{b} \in B(\mathbf{p})$ *is the limit of*

$$\mathbf{b} := \lim_{m\to\infty} \varepsilon_m^{-\frac{1}{2}} \mathbf{b}_m, \quad |\mathbf{b}| \le \delta.$$

Proof. The convergence to a limit w_j as above follows directly from Lemmas 4.5 and 4.6. The second part is a consequence of

$$|\mathbf{b}_k| \le \delta \varepsilon_k^{\frac{1}{2}}$$

(see Definition 4.3), and Lemma 4.1 (b), after recalling that the function $\mathbf{e}(\mathbf{b})$ is homogeneous of degree 2 in $\mathbf{b}$ (see Definition 3.3). □

5. Weiss monotonicity

In this section we establish the upper bound for the Weiss energy in Lemma 5.1 and the main dichotomy result Proposition 5.1, which give Theorem 1.1 in the case of nondegenerate cones.

We denote

$$E(\mathbf{u}, r) := r^{-(n+2)} \int_{B_r} \sum \omega_k \left(\frac{1}{2} |\nabla u_k|^2 + f_k u_k \right) dx$$

and

$$F(\mathbf{u}, r) := r^{-(n+3)} \int_{\partial B_r} \sum \omega_k u_k^2 \, d\sigma.$$

The Weiss functional is

$$W(\mathbf{u}, r) := E(\mathbf{u}, r) - F(\mathbf{u}, r).$$

We compute

$$\begin{aligned}
\frac{d}{dr} W(\mathbf{u}, r) &= r^{-(n+2)} \int_{\partial B_r} \sum \omega_k \left(\frac{1}{2} |\nabla u_k|^2 + f_k u_k - 2r^{-1} u_k u_{k,\nu} + 4r^{-2} u_k^2 \right) dx \\
&\quad - (n+2) r^{-1} E(\mathbf{u}, r) \\
&= r^{-(n+2)} \int_{\partial B_r} \sum \frac{\omega_k}{2} \left(u_{k,\nu} - \frac{2}{r} u_k \right)^2 d\sigma + \frac{n+2}{r} (E(\mathbf{u}_h, r) - E(\mathbf{u}, r)) \\
&\ge r^{-(n+2)} \int_{\partial B_r} \sum \frac{\omega_k}{2} \left(u_{k,\nu} - \frac{2}{r} u_k \right)^2 d\sigma,
\end{aligned}$$

where $\mathbf{u}_h$ denotes the homogeneous of degree 2 extension of the boundary data of $\mathbf{u}$ on ∂B_r, and in the last inequality we used the minimality of $\mathbf{u}$ for the energy E in B_r.

Lemma 5.1. *Assume that* $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}, \varepsilon)$. *Then*

$$W(\mathbf{u}, \tfrac{1}{2}) \le W(\mathbf{p}) + C\varepsilon^{\frac{3}{2}}.$$

Proof. We denote by $\mathbf{v} := \mathbf{p}(\cdot, \mathbf{b})$ and we prove the following inequalities:

$$W(\mathbf{u}, \tfrac{1}{2}) \le W(\mathbf{v}) + C\varepsilon^2 \tag{5.1}$$

and

$$W(\mathbf{v}) \le W(\mathbf{p}) + C\varepsilon^{\frac{3}{2}}. \tag{5.2}$$

In order to obtain (5.1) we write

$$\mathbf{v} = \mathbf{u} + \varepsilon\mathbf{w}, \quad |\mathbf{w}| \le 1.$$

By Lemmas 4.5 and 4.6, we know that outside the strip $\{|x_2| \le C\sqrt{\varepsilon}\}$ each component w_k satisfies $|\triangle w_k| \le \delta$, hence

$$|\nabla \mathbf{w}| \le C(|x_2| + \sqrt{\varepsilon})^{\alpha-1} \quad \text{in } \{|x_2| \ge C\sqrt{\varepsilon}\} \cap B_{1/2}. \tag{5.3}$$

Inside the strip, the $C^{1,1}$ norm of w_k is bounded by $C\varepsilon^{-1}$, hence

$$|\nabla \mathbf{w}| \le C\varepsilon^{-\frac{1}{2}} \quad \text{in } \{|x_2| \le C\sqrt{\varepsilon}\} \cap B_{1/2}. \tag{5.4}$$

Then, with $r = \frac{1}{2}$, we write

$$W(\mathbf{v}, r) = W(\mathbf{u}, r) + \varepsilon^2 r^{n-2} I_1 + \varepsilon r^{n-2} I_2,$$

with

$$\begin{aligned}
I_1 &:= \int_{B_r} \sum \frac{\omega_k}{2} |\nabla w_k|^2 \, dx - r^{-1} \int_{\partial B_r} \sum \omega_k w_k^2 \, d\sigma, \\
I_2 &:= \int_{B_r} \sum \omega_k (\nabla u_k \cdot \nabla w_k + f_k w_k) \, dx - \int_{\partial B_r} \sum \omega_k \frac{2}{r} u_k w_k \, d\sigma \\
&= \int_{B_r} \sum \omega_k (f_k - \triangle u_k) w_k \, dx + \int_{\partial B_r} \sum \omega_k \left(u_{k,\nu} - \frac{2}{r} u_k \right) w_k \, d\sigma \\
&\ge \varepsilon \int_{\partial B_r} \sum \omega_k \left(-w_{k,\nu} + \frac{2}{r} w_k \right) w_k \, d\sigma.
\end{aligned}$$

In the last inequality we used (see (1.3))

$$\sum \omega_k (f_k - \triangle u_k) w_k \ge 0, \tag{5.5}$$

and that $\mathbf{v}$ is homogeneous of degree 2. From (5.3)–(5.4) we infer that $I_2 \ge -C\varepsilon$. Since $I_1 \ge -C$, we conclude that (5.1) holds.

For the second inequality (5.2) we argue similarly. We denote

$$\mathbf{p} = \mathbf{v} + \mathbf{g},$$

for some $\mathbf{g}$ that satisfies (see Lemma 4.1 (b))

$$|\mathbf{g}| \le C\sqrt{\varepsilon} \quad \text{in } B_1, \qquad |\mathbf{g}| \le C\varepsilon \quad \text{in } \{|x_2| \le C\sqrt{\varepsilon}\} \cap B_1.$$

We have

$$W(\mathbf{p}) = W(\mathbf{v}) + I_3$$

with

$$I_3 := \int_{B_1} \sum \omega_k \Big(\nabla v_k \cdot \nabla g_k + \frac{1}{2} |\nabla g_k|^2 + f_k g_k \Big)\, dx - \int_{\partial B_1} \sum \omega_k \,(2 v_k g_k + g_k^2)\, d\sigma$$
$$= \int_{B_1} \sum \omega_k \Big(f_k - \triangle v_k - \frac{1}{2} \triangle g_k \Big) g_k \, dx,$$

where we have used that $\mathbf{v}$ and $\mathbf{g}$ are homogeneous of degree 2.

We estimate the last integral. When x belongs to the strip $\{|x_2| \le C\sqrt{\varepsilon}\}$, then

$$|g_k| \le C\varepsilon \quad \text{and} \quad \Big| f_k - \triangle v_k - \frac{1}{2}\triangle g_k \Big| \le C,$$

while outside the strip we have (see Lemma 4.1 (a) and Lemma 4.5)

$$\Big| \sum \omega_k (f_k - \triangle v_k) g_k \Big| \le C\varepsilon^{\frac{3}{2}}, \quad |\triangle g_k| \le \varepsilon.$$

Thus $|I_3| \le C\varepsilon^{\frac{3}{2}}$, and (5.2) is proved. □

Proposition 5.1. *Assume that* $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}, \varepsilon)$, *with* $\varepsilon \le \varepsilon_0$. *Then either*

$$\mathbf{u} \in \mathcal{S}\Big(\rho, \mathbf{p}, \frac{\varepsilon}{2}\Big)$$

or

$$\mathbf{u} \in \mathcal{S}(\rho, \mathbf{p}, C\varepsilon) \quad \textit{and} \quad W(\mathbf{u}, \rho) \le W(\mathbf{u}, 1) - c\varepsilon^2.$$

Moreover, if $\mathbf{v}_1$ *and* $\mathbf{v}_\rho$ *denote the approximate solutions of the type* $\mathbf{p}(\cdot\,, \mathbf{b})$ *in* B_1 *respectively* B_ρ, *then*

$$\|\mathbf{v}_1 - \mathbf{v}_\rho\|_{L^\infty(B_1)} \le C\varepsilon.$$

Here ρ, ε_0, c *(small) and* C *(large) denote universal constants.*

Proof. We remark that the first conclusion of the second alternative $\mathbf{u} \in \mathcal{S}(\rho, \mathbf{p}, C\varepsilon)$ is obvious, by taking $C = \rho^{-2}$.

We prove the statement by compactness. We fix $\rho = \frac{1}{4}$, $C = \rho^{-2}$, and assume that there exists a sequence of $\mathbf{u}_m$, $\mathbf{b}_m$, $\varepsilon_m \to 0$ for which the conclusion does not hold with $c_m = \frac{1}{m} \to 0$. By Corollary 4.2 we may extract a subsequence of the rescaled errors

$$\mathbf{w}_m := \varepsilon_m^{-1} (\mathbf{u}_m - \mathbf{p}(\cdot\,, \mathbf{b}_m))$$

which converges uniformly in $B_{1/2}$ (and in $C^1_{\mathrm{loc}}(B_{1/2} \setminus \{x_2 = 0\})$) to a limit function $\mathbf{w}$ which satisfies

$$w_j = 0 \quad \text{on } \{x_2 = 0\}$$

and

$$\triangle w_j = -2 e_j(\mathbf{b}) \chi_{\{x_1 > 0\}} - 2 e_j(-\mathbf{b}) \chi_{\{x_1 < 0\}} \quad \text{in } \{x_2 < 0\} \cup \{x_2 > 0\},$$

where $\mathbf{b} \in B(\mathbf{p})$ is the limit of

$$\mathbf{b} := \lim_{m \to \infty} \varepsilon_m^{-\frac{1}{2}} \mathbf{b}_m, \quad |\mathbf{b}| \le \delta.$$

Since

$$\varepsilon_m^{-2}(W(\mathbf{u}_m,1)-W(\mathbf{u}_m,\rho)) = \varepsilon_m^{-2}\int_\rho^1 \frac{d}{dr}W(\mathbf{u}_m,r)\,dr \geq \int_{B_1\setminus B_\rho} r^{-(n+2)}\sum \frac{\omega_k}{2}\left(\partial_\nu w_{m,k}-\frac{2}{r}w_{m,k}\right)^2 d\sigma,$$

we may take $m\to\infty$ and conclude that $\mathbf{w}$ is homogeneous of degree 2 in $B_{1/2}$ (first in $B_{1/2}\setminus B_\rho$ by the inequality above, and then in $B_{1/2}$ by unique continuation). This implies that $\mathbf{e}(\mathbf{b})=\mathbf{e}(-\mathbf{b})$ and by Lemma 4.3 we conclude that

$$\mathbf{b}=s\tau \quad \text{for some } s\in[-C\delta,C\delta]. \tag{5.6}$$

Moreover,

$$w_j := \gamma_j x_2^2 + \big(t_j^+\chi_{\{x_2>0\}}+t_j^-\chi_{\{x_2<0\}}\big)x_1x_2,$$

with $\gamma_j=-e_j(\mathbf{b})$, and

$$|\boldsymbol{\gamma}| = |\mathbf{e}(\mathbf{b})| \leq C|\mathbf{b}|^2 \leq C\delta^2 \leq \delta.$$

Moreover, since the average of w_j is 0, we have $\mathbf{t}\in B(\mathbf{p})$, $|\mathbf{t}|\leq C$. Using Lemma 4.1 (b), we find that

$$\mathbf{p}(\,\cdot\,,\mathbf{b}_m+\varepsilon_m\mathbf{t}) = \mathbf{p}(\,\cdot\,,\mathbf{b}_m)+\varepsilon_m\mathbf{w}-\varepsilon_m x_2^2\gamma + O\big((|\mathbf{b}_m|^2+|\mathbf{b}_m+\varepsilon_m\mathbf{t}|^2)x_1^2\big),$$

hence

$$|\mathbf{u}_m-\mathbf{p}(\,\cdot\,,\mathbf{b}_m+\varepsilon_m\mathbf{t})| \leq \varepsilon_m(\delta+C\delta^2)\rho^2 \leq \frac{\varepsilon_m}{4}\rho^2 \quad \text{in } B_\rho. \tag{5.7}$$

We cannot yet conclude that $\mathbf{u}_m\in\mathcal{S}(\rho,\mathbf{p},\frac{\varepsilon_m}{2})$, and reach a contradiction since we do not know that

$$|\mathbf{b}_m+\varepsilon_m\mathbf{t}| \leq \delta\left(\frac{\varepsilon_m}{2}\right)^{\frac{1}{2}}.$$

We achieve this after a rotation of coordinates. We use (5.6) and write

$$\mathbf{b}_m+\varepsilon_m\mathbf{t} = \varepsilon_m^{\frac{1}{2}}(s\tau+\mathbf{d}_m) \quad \text{with } \mathbf{d}_m\to 0,$$

and find (see Definition 3.1)

$$\begin{aligned}\mathbf{p}(x,\mathbf{b}_m+\varepsilon_m\mathbf{t}) &= \mathbf{h}(x_2, x_1\varepsilon_m^{\frac{1}{2}}(s\tau+\mathbf{d}_m))\\ &= \mathbf{h}(x_2+\varepsilon_m^{\frac{1}{2}}sx_1, x_1\varepsilon_m^{\frac{1}{2}}\mathbf{d}_m).\end{aligned}$$

Denote by (y_1,y_2) the new coordinates in the rotated system

$$y_1 := (1+\varepsilon_m s^2)^{-1}(x_1-\varepsilon_m^{\frac{1}{2}}sx_2), \quad y_2 := (1+\varepsilon_m s^2)^{-1}(x_2+\varepsilon_m^{\frac{1}{2}}sx_1),$$

and notice that

$$x_2+\varepsilon_m^{\frac{1}{2}}sx_1 = y_2+O(\varepsilon_m s^2|y|), \quad x_1\varepsilon_m^{\frac{1}{2}}\mathbf{d}_m = y_1\varepsilon_m^{\frac{1}{2}}\mathbf{d}_m+O(\varepsilon_m s|y|).$$

Thus, since $\mathbf{h}$ is homogeneous of degree 2 and has bounded second derivatives,

$$\begin{aligned}\mathbf{p}(x, \mathbf{b}_m + \varepsilon_m \mathbf{t}) &= \mathbf{h}(y_2 + O(\varepsilon_m s^2 |y|), y_1 \varepsilon_m^{\frac{1}{2}} \mathbf{d}_m + O(\varepsilon_m s |y|)) \\ &= \mathbf{h}(y_2, y_1 \varepsilon_m^{\frac{1}{2}} \mathbf{d}_m) + O(\varepsilon_m s |y|^2) \\ &= \mathbf{p}(y, \varepsilon_m^{\frac{1}{2}} \mathbf{d}_m) + O(\varepsilon_m s |y|^2).\end{aligned} \tag{5.8}$$

The error term is bounded by (see (5.6))

$$|O(\varepsilon_m s |y|^2)| \le C \delta \varepsilon_m |y|^2 \le \frac{\varepsilon_m}{4} |y|^2$$

provided that δ is chosen small. Also, for all large m,

$$|\varepsilon_m^{\frac{1}{2}} \mathbf{d}_m| \le \delta \left(\frac{\varepsilon_m}{2} \right)^{\frac{1}{2}},$$

and by (5.7) we conclude $\mathbf{u}_m \in \mathcal{S}(\rho, \mathbf{p}, \frac{\varepsilon_m}{2})$, which is a contradiction. □

Theorem 5.1. *Assume that $d = 2$ and $\mathbf{p} \in \mathcal{P}^c$ is a blow-up limit for $\mathbf{u}$ at the origin. Then $\mathbf{p}$ is unique and*

$$\mathbf{u}(x) = \mathbf{p}(x_2) + O(|x|^2 (-\log |x|)^{-1}).$$

Proof. The theorem follows from Lemma 5.1, Proposition 5.1 and Lemma 2.3. We omit the details. □

6. The degenerate cones

In this section we prove Theorem 1.1 for degenerate two-dimensional cones. The main ideas are similar to the ones of the previous section, however the convergence of the rescaled errors is much more delicate in this case. Also the compactness argument is more involved due to the geometry of singular cones.

We consider one-dimensional cones which do not belong to $\mathcal{P}^c$, and their two-dimensional analogues. Fix such a one-dimensional cone

$$\mathbf{p}_* \in \mathcal{P} \setminus \mathcal{P}^c.$$

We can decompose $\mathbf{p}_*$ as a union of $m \ge 2$ cones in $\mathcal{P}^c$ as follows.

Let $k_1 < k_2 < \cdots < k_{m-1}$ be the indices k with trivial coincidence sets, i.e.,

$$\{p_{*,k} = p_{*,k+1}\} = \{0\}.$$

The consecutive membranes in each of the m groups $\{p_{*,k_i}, p_{*,k_i+1}, \ldots, p_{*,k_{i+1}-1}\}$ are connected nontrivially on a half-line. After subtracting the average $q_{*,i}$ (a quadratic polynomial) from each group we define the corresponding vector

$$\mathbf{p}_*^i := (p_{*,k_{i-1}+1}, \ldots, p_{*,k_i}) - (q_{*,i}, q_{*,i}, \ldots, q_{*,i}) : \mathbb{R} \to \mathbb{R}^{k_i - k_{i-1}},$$

and $\mathbf{p}_*^i$ is a connected cone for the $k_i - k_{i-1}$ membranes. Thus we can write $\mathbf{p}_*$ as a union of m connected cones

$$\mathbf{p}_* = (\mathbf{p}_*^1 + q_{*,1} \mathbf{1}, \ldots, \mathbf{p}_*^m + q_{*,m} \mathbf{1}), \quad \mathbf{p}_*^i \in \mathcal{P}^c. \tag{6.1}$$

The analogue cones in two dimensions corresponding to $\mathbf{p}_*$ have the form

$$\mathbf{p} = (\mathbf{p}^1 + q_1\mathbf{1}, \dots, \mathbf{p}^m + q_m\mathbf{1}), \tag{6.2}$$

with q_i quadratic polynomials such that

$$\triangle q_i = q''_{*,i}, \quad \sum \omega_k q_i = 0,$$

and with $\mathbf{p}^i$ obtained from $\mathbf{p}^i_*$ after a rotation. Here $\mathbf{p}^i_*$ represents the trivial extension from one to two dimensions while the angle of rotation depends on i. The polynomials q_i and rotations $\mathbf{p}^i$ are constrained by the condition $p_k \geq p_{k+1}$ which must hold for all $k \geq 1$. This condition needs to be checked only for consecutive membranes belonging to different connected groups, i.e., when k is one of the k_i, since it is clearly satisfied within each connected group.

When $\mathbf{p} \in \mathcal{C}_2$ is a two-dimensional cone extension of $\mathbf{p}_*$ as in (6.2), we write

$$\mathbf{p} \in \mathcal{P}(\mathbf{p}_*).$$

For such a cone $\mathbf{p}$, the free boundaries

$$\Gamma_k := \partial\{p_k > p_{k+1}\}$$

with $k_{i-1} < k < k_i$ coincide with a single line, the line of the rotation of $\mathbf{p}^i_*$ (whenever $\mathbf{p}^i_*$ consists of at least two membranes). When $k = k_i$ then the free boundary Γ_k is the same as the coincidence set $\{p_k = p_{k+1}\}$, and we show that it is either the origin, one ray, or two rays passing through the origin. We make this more precise.

Lemma 6.1. *The set Γ_{k_i} consists of at most two rays that make an angle strictly greater than $\frac{\pi}{2}$.*

Proof. Lemma 4.4 which implies that in each half-plane where $\triangle p_{k_i}$ is constant (or where $\triangle p_{k_i+1}$ is constant), the coincidence set cannot contain two distinct rays, unless they coincide with the boundary of the half-plane and both $\triangle p_{k_i}$, $\triangle p_{k_i+1}$ are constant on either side of the line.

This proves that there are at most two rays in Γ_{k_i}.

Next we denote by φ_j the multiplicity 1 parts of p_{k_i} and p_{k_i+1}:

$$p_{k_i} = \varphi_1 + a_1[(x \cdot \nu_1)^+]^2, \quad p_{k_i+1} = \varphi_2 - a_2[(x \cdot \nu_2)^+]^2,$$

with φ_j homogeneous quadratic polynomials, and the constants $a_j \geq 0$. Moreover, by non-degeneracy

$$\triangle\varphi_1 = f_{k_i} > f_{k_i+1} = \triangle\varphi_2.$$

The coincidence rays are the ones along which $\varphi_2 - \varphi_1$ is tangent by below to the piecewise quadratic function

$$a_1[(x \cdot \nu_1)^+]^2 + a_2[(x \cdot \nu_2)^+]^2 \geq 0.$$

If there are two coincidence rays, then they must belong to the two different components of $\{\varphi_2 - \varphi_1 > 0\}$. The conclusion follows since $\varphi_2 - \varphi_1$ is a strictly superharmonic homogeneous quadratic polynomial. □

We prove Theorem 5.1 for the degenerate cones.

Theorem 6.1. *Assume that* $d = 2$ *and* $\mathbf{p} \in \mathcal{P}(\mathbf{p}_*)$ *is a blow-up limit for* $\mathbf{u}$ *at the origin. Then* $\mathbf{p}$ *is unique and*

$$\mathbf{u}(x) := \mathbf{p}(x) + O(|x|^2(-\log|x|)^{-1}).$$

The strategy of proof is the same as in Sections 3 and 4. First we introduce a family of approximate solutions near cones $\mathbf{p} \in \mathcal{P}(\mathbf{p}_*)$ similar to Definition 4.3. In this case, an approximate solution $\mathbf{v}$ consists of a collection of vector-functions $\mathbf{v}^i$ as in Section 3, with each of them approximating a connected group of $\mathbf{p}$. More precisely, $\mathbf{v}$ has the form

$$\mathbf{v} = (\mathbf{v}^1, \dots, \mathbf{v}^m), \qquad v_k \geq v_{k+1} \quad \text{for all } k, \qquad \sum \omega_k v_k = 0,$$
$$\mathbf{v}^i = \mathbf{p}^i(x, \mathbf{b}_i) + q_i \mathbf{1}, \quad |\mathbf{b}_i| \leq \delta \varepsilon^{\frac{1}{2}}, \tag{6.3}$$

with q_i quadratic polynomials with

$$\triangle q_i = q''_{*,i}, \quad \sum \omega_k q_i = 0$$

and $\mathbf{p}^i(\cdot, \mathbf{b}_i)$ represents an ε-approximation of a rotation of the connected one-dimensional cone $\mathbf{p}^i_*$, as in Definition 4.1.

We make precise the definition of the solutions $\mathbf{u}$ which can be approximated by such $\mathbf{v}$.

Definition 6.1. Given a one-dimensional cone $\mathbf{p}_*$ as in (6.1), we say that a solution $\mathbf{u}$ to problem P_0 is ε-approximated in B_r by $\mathbf{p}_*$ and write

$$\mathbf{u} \in \mathcal{S}(r, \mathbf{p}_*, \varepsilon)$$

if there exists an admissible $\mathbf{v}$ as in (6.3) above such that

$$|\mathbf{u} - \mathbf{v}| \leq \varepsilon r^2 \quad \text{in } B_r, \qquad |\mathbf{b}_i| \leq \delta \varepsilon^{\frac{1}{2}},$$

with δ a small universal constant (to be made precise later).

By definition, $\mathbf{v} \in C^{1,1}$ is homogeneous of degree 2, and the coincidence set between consecutive connected groups, i.e., $\{v_k = v_{k+1}\}$ with $k = k_i$ has empty interior in $\mathbb{R}^2$, since $\triangle(v_k - v_{k+1}) \geq c > 0$. Moreover, on the unit circle this difference grows quadratically away from its minimum points, hence the set where v_k and v_{k+1} are ε close to each other in B_1

$$D_k^\varepsilon := \{v_k - v_{k+1} \leq 2\varepsilon\} \cap B_1$$

is included in a $C\varepsilon^{\frac{1}{2}}$-neighborhood of at most two rays passing through the origin. The upper bound on the number of rays follows by compactness, since $\mathbf{v}$ must converge to an element $\mathbf{p} \in \mathcal{P}(\mathbf{p}_*)$ as $\varepsilon \to 0$.

By Lemma 4.1 (a), $\mathbf{v}$ satisfies the Euler–Lagrange equations with $\delta\varepsilon$-error

$$|\triangle v_I - f_I| \leq C\delta^2 \varepsilon \leq \delta\varepsilon.$$

Moreover, if ν_i denotes the unit direction of rotation for $\mathbf{p}^i$, so that $\mathbf{p}^i(\cdot, \mathbf{b}_i)$ is the ε-approximation of $\mathbf{p}^i_*(x \cdot \nu_i)$, then, by Lemma 4.1 (b), in $B_1 \cap \{|x \cdot \nu_i| \geq \varepsilon^{\frac{1}{2}}\}$ we have

$$\triangle \mathbf{v}^i = q''_{*,i} + \triangle \mathbf{p}^i_* + 2\mathbf{e}(\mathbf{b}_i)\chi_{\{x \cdot \nu_i^\perp \geq 0\}} + 2\mathbf{e}(-\mathbf{b}_i)\chi_{\{x \cdot \nu_i^\perp \leq 0\}}. \tag{6.4}$$

If a solution $\mathbf{u}$ is ε-approximated by $\mathbf{v}$ in B_1, then in $B_{1-C\varepsilon^{\frac{1}{2}}}$ the coincidence sets for $\mathbf{u}$ and $\mathbf{v}$ agree away from the set

$$D^\varepsilon := \bigcup_{k=k_i} D_k^\varepsilon \bigcup_i \{|x \cdot \nu_i| \le C\varepsilon^{\frac{1}{2}}\}, \tag{6.5}$$

with D_k^ε and ν_i as above. The set D^ε lies in a $C\varepsilon^{\frac{1}{2}}$-neighborhood of a finite number of rays. As a consequence, we have the analogue of Lemma 4.5 in our setting.

Lemma 6.2. *Assume that* $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}_*, \varepsilon)$ *is* ε*-approximated by* $\mathbf{v}$ *in* B_1*. Then in* $B_{3/4}$ *we have* $\Gamma_k \subset D^\varepsilon$ *for all* k*, and*

$$|\triangle(u_k - v_k)| \le \delta\varepsilon \quad \text{in } B_{3/4} \setminus D^\varepsilon. \tag{6.6}$$

In the next lemma we establish a Hölder modulus of continuity for the rescaled differences $\frac{u_k - v_k}{\varepsilon}$.

Lemma 6.3. *Assume* $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}_*, \varepsilon)$ *is* ε*-approximated by* $\mathbf{v}$ *in* B_1*. Fix* $z \in B_{3/4} \setminus B_{1/4}$*, and* $r \in [C\varepsilon^{\frac{1}{2}}, c]$*. We have*

$$w_k - 2\varepsilon r^\alpha \le u_k \le w_k + 2\varepsilon r^\alpha \quad \text{in } B_r(z), \quad \text{for some } \alpha > 0,$$

with $\mathbf{w}$ *an admissible function in* $B_r(z)$ *obtained from* $\mathbf{v}$ *by appropriate translating constants* ζ_k *(depending on* r *and* z*),*

$$w_k := v_k + \zeta_k, \quad w_k \ge w_{k+1} \quad \text{for all } k.$$

Moreover, if $B_r(z)$ *intersects* $\{x \cdot \nu_i\} = 0$*, then the constants* ζ_k *are all equal when* k *belongs to the* i*th group* $k \in \{k_{i-1} + 1, \dots, k_i\}$*.*

We postpone the proof of Lemma 6.3 to the end of this section. As a consequence we obtain the following version of Corollary 4.2 in our setting. The difference is that, in the limit, the rescaled errors must agree along the direction of rotation for each of the connected groups of the limiting cone $\mathbf{p}$.

Corollary 6.1. *If* $\mathbf{u}_m \in \mathcal{S}(1, \mathbf{p}_*, \varepsilon_m)$ *are* ε_m*-approximated by* $\mathbf{v}_m$ *for a sequence of* $\varepsilon_m \to 0$*, then, up to a subsequence,* $\mathbf{v}_m \to \mathbf{p} \in \mathcal{P}(\mathbf{p}_*)$ *and each of the rescaled error functions*

$$\varepsilon_m^{-1}(u_{m,j} - v_{m,j})$$

converges uniformly on compact sets of $B_{1/2} \setminus \{0\}$ *to a continuous limit* w_j *that satisfies*

$$\|w_j\|_{L^\infty} \le 1, \qquad w_j = w_l \quad \text{on } \{x \cdot \nu_i = 0\} \text{ whenever } j, l \in \{k_{i-1} + 1, \dots, k_i\},$$

where ν_i *is the direction of rotation for* $\mathbf{p}^i$*.*

Another consequence of Lemma 6.3 is that the corresponding version of Lemma 5.1 holds in the degenerate setting.

Lemma 6.4. *Assume that* $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}_*, \varepsilon)$*. Then*

$$W(\mathbf{u}, \tfrac{1}{2}) \le W(\mathbf{p}) + C\varepsilon^{\frac{3}{2}}.$$

Proof. First we remark that $W(\mathbf{p})$ is the same for all $\mathbf{p} \in \mathcal{P}(\mathbf{p}_*)$.

The quantity

$$J(w) = \int_{B_1} \frac{1}{2}\omega(|\nabla w|^2 + f w)\, dx - \int_{\partial B_1} \omega w^2 \, d\sigma$$

remains invariant if we replace w by $w + q$ with q a homogeneous of degree 2 harmonic polynomial (here f and ω are constants). This follows easily after applying the mean value property for q and then by integration by parts.

From (6.2), we see that each of the connected groups $\mathbf{p}^i + q_i \mathbf{1}$ that form $\mathbf{p}$, is obtained from ith connected group of the trivial extension of $\mathbf{p}^*$ to two dimensions, after a rotation and the addition of a homogeneous of degree 2 harmonic polynomial. The remark above implies $W(\mathbf{p}) = W(\mathbf{p}^*)$.

The proof follows from Lemma 5.1 since t inequalities (5.1)–(5.2), i.e.,

$$W(\mathbf{u}, \tfrac{1}{2}) \le W(\mathbf{v}) + C\varepsilon^2 \tag{6.7}$$

and

$$W(\mathbf{v}) \le W(\mathbf{p}) + C\varepsilon^{\frac{3}{2}}, \tag{6.8}$$

continue to hold, where $\mathbf{v}$ is the ε-approximation of $\mathbf{u}$ given in Definition 6.1.

Indeed, for (6.7) we only used that $\varepsilon^{-1}|\nabla(u_k - v_k)|$ is integrable on $\partial B_{1/2}$ which, as in Section 4, is a consequence of Lemmas 6.2 and 6.3.

The second inequality can be reduced to the one from Section 4 for each of the connected groups. Recall that the ith connected groups of $\mathbf{v}$, and $\mathbf{p}$ are given by

$$\mathbf{p}^i(\,\cdot\,, \mathbf{b}_i) + q_i \mathbf{1} \quad \text{and} \quad \mathbf{p}^i + q_i \mathbf{1}.$$

We claim that

$$W(\mathbf{v}) - W(\mathbf{p}) = \sum_i W^i(\mathbf{p}^i(\,\cdot\,, \mathbf{b}_i)) - W^i(\mathbf{p}^i) \le C\varepsilon^{\frac{3}{2}}, \tag{6.9}$$

where W^i denotes the Weiss energy corresponding to the ith connected group

$$W^i(\mathbf{w}^i) := \sum_{k_{i-1} < k \le k_i} \left(\int_{B_1} \omega_k \left(\frac{1}{2}|\nabla w_k|^2 + f_k^i w_k \right) dx - \int_{\partial B_1} \omega_k w_k^2 \, d\sigma \right),$$

with $f_k^i := f_k - \triangle q_i$. The equality in (6.9) follows easily from the identity

$$J(w + q) - J(v + q) = J(w) - J(v) - \int_{B_1} \omega(\triangle q)(w - v)\, dx,$$

which holds for any homogeneous quadratic polynomial q. □

We are in a position to prove the corresponding version of Proposition 5.1 for degenerate cones $\mathbf{p}_*$.

Proposition 6.1. *Assume that* $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}_*, \varepsilon)$*, with* $\varepsilon \le \varepsilon_0$*. Then either*

$$\mathbf{u} \in \mathcal{S}\left(\rho, \mathbf{p}_*, \frac{\varepsilon}{2}\right)$$

or

$$\mathbf{u} \in \mathcal{S}(\rho, \mathbf{p}_*, C\varepsilon) \quad \textit{and} \quad W(\mathbf{u}, \rho) \le W(\mathbf{u}, 1) - c\varepsilon^2.$$

Proof. As before we prove the statement by compactness.

We fix $\rho = \frac{1}{4}$, $C = \rho^{-2}$, and assume that there exists a sequence of $\mathbf{u}_m$, $\mathbf{v}_m$, $\varepsilon_m \to 0$ for which the conclusion does not hold with $c_m = \frac{1}{m} \to 0$.

By Corollary 6.1 we may extract a subsequence

$$\mathbf{v}_m \to \mathbf{p} \in \mathscr{P}(\mathbf{p}_*),$$

and rescaled errors

$$\mathbf{w}_m := \varepsilon_m^{-1}(\mathbf{u}_m - \mathbf{v}_m)$$

which converge uniformly of compact sets of $B_{1/2} \setminus \{0\}$ to a limit function $\mathbf{w}$.

Denote by ν_i the direction of rotation for the ith connected cone $\mathbf{p}^i$ of $\mathbf{p}$, and by Γ_{k_i} the coincidence set $\{p_k = p_{k+1}\}$ for $k = k_i$, which by Lemma 6.1 consists of at most two rays that form an obtuse angle. The sets D^ε defined in (6.5) converge in the Hausdorff distance to the collection of rays

$$D^0 := \bigcup \Gamma_{k_i} \bigcup_i \{x \cdot \nu_i = 0\},$$

and the convergence of $\mathbf{w}_m$ to $\mathbf{w}$ is in $C^1_{\text{loc}}(B_{1/2} \setminus D^0)$. As in the proof of Proposition 5.1, the inequality

$$W(\mathbf{u}_m, 1) - W(\mathbf{u}_m, \rho) \le c_m \varepsilon_m^2$$

implies that the limit $\mathbf{w}$ is homogeneous of degree 2 in $(B_{1/2} - B_\rho) \setminus D^0$, hence in $B_{1/2} \setminus B_\rho$ by continuity.

Claim. *If k belongs to the ith connected group $J_i := \{k_{i-1} + 1, \dots, k_i\}$, then*

$$w_k = w_l \quad \text{on } \{x \cdot \nu_i = 0\} \quad \text{for all } k, j \in J_i,$$

$$\triangle w_{J_i} = 0, \quad w_{J_i} := \sum_{k \in J_i} \frac{\omega_k}{\sum_{J_i} \omega_j} w_k, \tag{6.10}$$

and on each half space determined by the line $x \cdot \nu_i = 0$,

$$\triangle w_j = -2e_j(\mathbf{b}^i)\chi_{\{x \cdot \nu_i^\perp > 0\}} - 2e_j(-\mathbf{b}^i)\chi_{\{x \cdot \nu_i^\perp < 0\}}, \tag{6.11}$$

where $\mathbf{b}^i \in B(\mathbf{p}^i)$ is the limit of

$$\mathbf{b}^i := \lim_{m \to \infty} \varepsilon_m^{-\frac{1}{2}} \mathbf{b}^i_m, \quad |\mathbf{b}^i| \le \delta.$$

Proof of Claim. Notice that

$$\triangle u_{m,L} \le f_L = \triangle v_{m,L}, \quad L := \{j \le k_j\},$$

which implies that

$$\triangle w_L \le 0.$$

On the other hand outside any small neighborhood of Γ_{k_i}, $p_{k_i} > p_{k_i+1}$ which implies the same inequality for the membranes of $\mathbf{u}_m$. This means that the inequality above is an equality, which gives

$$\triangle w_L = 0 \quad \text{outside } \Gamma_{k_i}.$$

Since w_L is homogeneous of degree two and Γ_{k_i} consists of at most two rays that form an angle different than $\frac{\pi}{2}$, we conclude that w_L must be a harmonic quadratic polynomial. This implies (6.10).

Equality (6.11) follows in $B_{1/2} \setminus D^0$ by equation (6.4). In fact, it can only fail on the rays $\Gamma_{k_{i-1}} \cup \Gamma_{k_i}$ along which the ith connected group can interact with the $i-1$ respectively $i+1$ groups. Indeed, in a compact set outside these rays the graphs of u_k with $k \in J_i$ are disconnected from the ones with $k \notin J_i$, and we are in the situation of Section 4. More precisely, we only need to check (6.11) for those indices $j \in J_i$ and near the rays for which the membrane p_j is either tangent to p_{k_i+1} or $p_{k_{i-1}}$.

It remains to show that if the membrane p_j is tangent to p_{k_i+1}, then $\triangle w_j$ carries no singular part on Γ_{k_i} whenever Γ_{k_i} is not included in $x \cdot \nu_i = 0$. Pick such a ray

$$\ell \in \Gamma_{k_i} \setminus \{x \cdot \nu_i = 0\}$$

and let $J_i' \subset J_i$ denote those indices j in the ith group for which $p_j = p_{k_i}$ along ℓ. Since ℓ is away from the line $x \cdot \nu_i = 0$, we conclude that $p_j = p_{k_i}$ in a neighborhood of ℓ. Using that $\mathbf{v}_m, \mathbf{u}_m$ are small perturbations of $\mathbf{p}$, we find that in an open neighborhood $\mathcal{U}$ of $\ell \cap (B_1 \setminus B_\rho)$,

$$v_j = v_{k_i}, \quad u_j = u_{k_i} \quad \text{if } j \in J_i'.$$

In particular, in this neighborhood $w_j = w_{k_i}$ if $j \in J_i'$, hence

$$\triangle w_j = \triangle w_{J_i'} \quad \text{in } \mathcal{U}.$$

If $J_i' = J_i$, then $\triangle w_j = 0$ by (6.10) which shows that $\triangle w_j$ has no singular part on ℓ. If $J_i' \neq J_i$, then there is strict separation in $\mathcal{U}$ between the membranes p_j with $j \in J_i'$ and $j \in L \setminus J_i'$. This separation holds also for the membranes of $\mathbf{u}_m$, and $\mathbf{v}_m$, hence

$$\triangle u_{m, L\setminus J_i'} = f_{L\setminus J_i'},$$

and since $\mathbf{v}_m$ is an approximate solution with $\delta\varepsilon_m$ error, we find that

$$|\triangle w_{L\setminus J_i'}| \leq \delta \quad \text{in } \mathcal{U}.$$

Using that w_L is harmonic, we find $|\triangle w_{J_i'}| \leq C\delta$. This shows that $\triangle w_j$ has no singular part on ℓ if $j \in J_i'$, and the claim is proved. □

Now we can argue as in the end of the proof of Proposition 5.1. The claim implies that $\mathbf{e}(\mathbf{b}^i) = \mathbf{e}(-\mathbf{b}^i)$, hence, by Lemma 4.3,

$$\mathbf{b}^i = s_i \tau^i \quad \text{for some } s_i \in [-C\delta, C\delta],$$

and τ^i as in Definition 3.1. Moreover, for $j \in J_i$,

$$w_j := \bar{q}_i + \gamma_j (x \cdot \nu)^2 + (t_j^+ \chi_{\{x \cdot \nu > 0\}} + t_j^- \chi_{\{x \cdot \nu_i < 0\}})(x \cdot \nu_i)(x \cdot \nu_i^\perp),$$

with $\gamma_j = -e_j(\mathbf{b}^i)$, $\bar{q}_i = w_{J_i}$ a harmonic quadratic polynomial, and the components $t_j^\pm$ form a vector $\mathbf{t}^i \in B(\mathbf{p}^i)$. Since $|\gamma| \leq C\delta^2$, we infer that

$$|\mathbf{u}_m^i - [\mathbf{p}^i(x, \mathbf{b}_m^i + \varepsilon_m \mathbf{t}^i) + (q_i + \varepsilon_m \bar{q}_i)\mathbf{1}]| \leq C\delta^2 \varepsilon_m \rho^2 \quad \text{in } B_{2\rho} \setminus B_\rho.$$

As in (5.8), we can rotate the axis ν_i of $\mathbf{p}^i$ by an angle $\sim \varepsilon_m^{\frac{1}{2}}$ and rewrite

$$\mathbf{p}^i(x, \mathbf{b}_m^i + \varepsilon_m \mathbf{t}^i) = \mathbf{p}^i(\tilde{x}, \varepsilon_m^{\frac{1}{2}} \mathbf{d}_m^i) + O(\delta \varepsilon_m |x|^2), \quad \mathbf{d}_m^i \to 0,$$

with $\tilde{x}$ representing the coordinates in the rotated system of coordinates. Thus

$$|\mathbf{u}_m^i - \tilde{\mathbf{v}}_m^i| \le C \delta \varepsilon_m \rho^2 \quad \text{in } B_{2\rho} \setminus B_\rho, \tag{6.12}$$

with

$$\tilde{\mathbf{v}}_m^i := \mathbf{p}^i(\tilde{x}, \varepsilon_m^{\frac{1}{2}} \mathbf{d}_m^i) + q_i + \varepsilon_m \bar{q}_i.$$

We do not know yet that the family $\tilde{\mathbf{v}}$ is admissible since the inequality $\tilde{v}_{m,k} \ge \tilde{v}_{m,k+1}$ might fail slightly when $k = k_i$ near Γ_{k_i}. By (6.12), this inequality can fail by at most $C \delta \varepsilon_m |x|^2$. We can modify each group of $\tilde{\mathbf{v}}_m$ by a harmonic quadratic polynomial of size $\delta \varepsilon_m$, and construct an admissible approximate solution $\bar{\mathbf{v}}_m$. Indeed, assume that $\bar{\mathbf{v}}_m^1, \dots, \bar{\mathbf{v}}_m^{i-1}$ were constructed. Then we can add $C_i \delta \varepsilon_m h_i(x)$ to all membranes of $\tilde{\mathbf{v}}_m^i$ with h_i a harmonic quadratic polynomial which is negative on $\Gamma_{k_i} \setminus \{0\}$, which exists in view of Lemma 6.1. We can choose C_i sufficiently large to guarantee that $\bar{\mathbf{v}}_m^i$ lies below $\bar{\mathbf{v}}_m^{i-1}$. After constructing $\bar{\mathbf{v}}_m$, we can subtract its average (a harmonic polynomial) from all of its components, so that $\sum \omega_k \bar{v}_{m,k} = 0$. In conclusion, (6.12) implies that

$$|\mathbf{u}_m^i - \bar{\mathbf{v}}_m^i| \le C' \delta \varepsilon_m \rho^2 \quad \text{in } B_{2\rho} \setminus B_\rho,$$

with $\bar{\mathbf{v}}_m$ satisfying the admissible conditions (6.3) with ε_m replaced by $\varepsilon_m/2$.

Finally, since $\bar{\mathbf{v}}_m$ solves the system with error $\delta \varepsilon_m$, it follows by maximum principle that the inequality above can be extended to B_ρ after relabeling the constant C'. Thus

$$|\mathbf{u}_m^i - \bar{\mathbf{v}}_m^i| \le C'' \delta \varepsilon_m \rho^2 \le \frac{\varepsilon_m}{2} \rho^2 \quad \text{in } B_\rho,$$

provided δ is chosen small. We obtain $\mathbf{u}_m \in \mathcal{S}(\mathbf{p}_*, \rho, \varepsilon_m/2)$ and reached a contradiction. □

The remaining of the section is devoted to the proof of Lemma 6.3 which relies on a version of the Harnack inequality for one-dimensional membranes.

Lemma 6.5. *Assume that* $\mathbf{u} \ge \mathbf{v}$ *are one-dimensional solutions to the* N *membrane problem in* $[-1, 1]$ *and*

$$u_k(0) \le v_k(0) + \sigma, \textit{ for some } k \textit{ and } \sigma \ge 0.$$

Then

$$u_k \le v_k + C\sigma \quad \textit{in } [-1, 1]$$

for some C *depending only on* N*, and the weights* ω_i*.*

Proof. We prove the statement by induction on the cardinality of the complement of the set of indices I defined as

$$I := \{ j : u_j(0) \le v_j(0) + a \}.$$

Precisely, we show that there exists a constant $C(|I|)$ depending only on the cardinality $|I|$ of the set I such that in $[-1, 1]$,

$$u_j \le v_j + C(|I|)\sigma \quad \text{for all } j \in I.$$

If $|I| = N$, then $I = \{1, \dots, N\}$. We have $v_I(0) + \sigma \ge u_I(0) \ge v_I(0)$ and since $u_I - v_I \ge 0$ is harmonic in $[-1, 1]$, we conclude that $u_I \le v_I + 2\sigma$ which gives the desired conclusion.

Assume that $|I| < N$, and denote $I = \{j_0, \dots, j_0 + m\}$. Let (a, b) be the largest interval containing 0 on which the inequalities

$$u_{j_0-1} > u_{j_0} \quad \text{and} \quad v_{j_0+m} > v_{j_0+m+1}$$

hold. Notice that the origin is interior to this interval, since otherwise either $j_0 - 1$ or $j_0 + m + 1$ would belong to I as well.

Assume that $|a| \le |b|$ and say that $u_{j_0-1}(a) = u_{j_0}(a)$.

In the interval (a, b) the same argument as above applies. Indeed, in this interval the membranes u_j, and respectively v_j, with $j \in I$, can be perturbed upwards, and respectively downwards. We find $\triangle u_I \le f_I \le \triangle v_I$ hence $u_I - v_I \ge 0$ is a concave function in (a, b). We conclude that

$$u_j \le v_j + C_1\sigma \quad \text{in } [a, |a|], \quad \text{for all } j \in I. \tag{6.13}$$

In particular, at $x = a$ we have

$$u_k = u_{k+1} \le v_{k+1} + C_1\sigma \le v_k + C_1\sigma, \quad k = j_0 - 1.$$

We can apply the induction hypothesis on the largest interval L_a centered at a which is included in $[-1, 1]$ with $\widetilde{\sigma} = C_1\sigma$, and then the corresponding set of indices $\widetilde{I}$ contains I and $j_0 - 1$. We find that

$$u_j \le v_j + C_2\sigma \quad \text{in } L_a \quad \text{for all } j \in I \cup \{j_0 - 1\}. \tag{6.14}$$

If L_a contains the origin, then we can apply one more time the induction hypothesis at the origin and obtain the desired conclusion in the whole interval $[-1, 1]$. Otherwise, inequality (6.13) is valid in $[a, b]$ after relabeling C_1 if necessary. We can argue as above at the other end point b and obtain a similar inequality as (6.14) in the largest interval $L_b \subset [-1, 1]$ centered at b. Since $[-1, 1]$ is covered by L_a, $[a, b]$ and L_b, we obtain the inductive conclusion for I. □

We introduce the notion of σ-connectedness in $B_r \subset \mathbb{R}^n$ for membranes whose collection of σ-neighborhood of their graphs form a connected set.

Definition 6.2. We say that the membranes v_j and v_{j+m} are σ-connected in B_r if we can find points $x_i \in B_r$ with $j + 1 \le i \le j + m$ such that $v_{i-1}(x_i) \le v_i(x_i) + \sigma$.

Remark 6.1. After relabeling the constant C, the conclusion of Lemma 6.5 holds for all indices $j \le k$ for which u_j is σ-connected to u_k in the half-interval

$$I := \left[-\frac{1}{2}, \frac{1}{2}\right]$$

or $j \ge k$ for which v_j is σ-connected to v_k in I.

An equivalent statement is the following.

Corollary 6.2. *Assume that* $\mathbf{u} \ge \mathbf{v}$ *are one-dimensional solutions to the* N *membrane problem in* $[-1, 1]$ *and*

$$u_k(1) \ge v_k(1) + \sigma \quad \textit{for some } k \textit{ and } \sigma \ge 0.$$

Then

$$u_j \geq v_j + c\sigma \quad \text{in } I$$

for all $j \leq k$ for which v_j is $c\sigma$-connected to v_k in I, and all $j \geq k$ for which u_k is $c\sigma$-connected to u_k in I. Here $c = C^{-1}$ depends only on N and ω_i.

We now consider the case when $\mathbf{u}$ is defined in the cylindrical domain

$$\mathcal{R} := B'_{C_n} \times [-1, 1] \subset \mathbb{R}^n,$$

with C_n a large constant that depends only on n and $\mathbf{v}$ is one-dimensional and

(6.15) $\quad$ $\mathbf{v}$ solves the Euler–Lagrange equation in $[-1, 1]$ with a $c_0\sigma$-error,

for some c_0 sufficiently small.

Lemma 6.6. *Assume that $\mathbf{u}$ is a solution in $\mathcal{R}$ and $\mathbf{v}$ satisfies* (6.15) *and*

$$\mathbf{u}(x', x_n) \geq \mathbf{v}(x_n) \quad \text{in } \mathcal{R},$$

and

$$u_k(x', l) \geq v_k(l) + \sigma \quad \text{for some } l \in [-1, 1].$$

for some $\sigma \leq \sigma_0$ universal. Then

$$u_j \geq v_j + c_0\sigma \quad \text{in } \frac{1}{2}\mathcal{R},$$

for all $j \in J_k$ which consists of the indices j such that

(a) *either $j \leq k$ and v_j is $c_0\sigma$-connected to v_k in I,*

(b) *or $j > k$ and the coincidence sets $\{v_k = v_{k+1}\}, \{v_{k+1} = v_{k+2}\}, \dots, \{v_{j-1} = v_j\}$ have length more than $\frac{1}{10}$ in I.*

Remark 6.2. Notice that the collection of functions v_j when $j \notin J_k$ and $v_j + c_0\sigma$ when $j \in J_k$, which bounds u_j by below, is admissible in $\frac{1}{2}\mathcal{R}$.

Proof. We assume first that $l = 1$ and then explain how to deduce the more general statement from this case.

Let $\mathbf{w}$ be the one-dimensional solution in $[-1, 1]$ with the boundary data given by $\mathbf{v}$.

We compare $\mathbf{w}$ with $\mathbf{v} \pm c_0\sigma(|x|^2 - 1)\mathbf{1}$ in $[-1, 1]$ and find

(6.16) $$|w_j - v_j| \leq c_0\sigma \quad \text{for all } j.$$

In particular, w_j are $3c_0\sigma$-connected in I if $j \leq k$ and $j \in J_k$.

Let $\bar{\mathbf{w}}$ be the one-dimensional solution with boundary data $\mathbf{w}$ at -1 and

$$\bar{w}_j(1) = \begin{cases} w_j(1) & \text{if } j > k, \\ \max\{w_j(1), w_k(1) + \sigma\} & \text{if } j \leq k. \end{cases}$$

Clearly,

$$|\bar{w}_j - w_j| \leq \sigma \quad \text{in } [-1, 1], \quad \text{for all } j,$$

which together with (6.16) and $\sigma \leq \sigma_0$ implies that the $\bar{w}_j$ are 0-connected in I if $j > k$ and $j \in J_k$.

By Corollary 6.2 applied to $\bar{\mathbf{w}}$, $\mathbf{w}$, we can find $c_1 = c_1(N, \omega_i)$ such that

$$\bar{w}_j \geq w_j + 4c_1\sigma \geq v_j + 3c_1\sigma \quad \text{in } I, \quad \text{for all } j \in J_k, \tag{6.17}$$

provided we choose $c_0 \leq c_1$.

Next we compare $\mathbf{u}$ with the subsolution

$$\bar{\mathbf{w}} + c_1\sigma(x_n^2 - 4C_n^{-2}|x'|^2 - 1)\mathbf{1}$$

in $\mathcal{R}$ and obtain

$$u_j \geq \bar{w}_j - 2c_1\sigma \quad \text{in } \tfrac{1}{2}\mathcal{R}, \quad \text{for all } j,$$

which, by (6.17), gives the conclusion $u_j \geq v_j + c_1\sigma$ for all $j \in J_k$.

It suffices to check the claim on $\partial\mathcal{R}$. On $\partial\mathcal{R} \setminus \{x_n = 1\}$ the test function is below $\mathbf{v}$ and therefore below $\mathbf{u}$. This inequality holds also on $\partial\mathcal{R} \cap \{x_n = 1\}$ by hypothesis. This completes the case $l = 1$.

Next we discuss the case when l is arbitrary. The same proof applies if $|l| \geq \frac{3}{4}$. In the case when, say $l \in [0, \frac{3}{4})$, then the arguments above show that an inequality of the form

$$u_k\left(x', -\frac{3}{4}\right) \geq v_k\left(-\frac{3}{4}\right) + c_1'\sigma \quad \text{if} |x'| \leq \frac{3}{4}C_n,$$

holds for the index $j = k$ at $-\frac{3}{4}$. Again we may repeat that proof above with $\tilde{l} = -\frac{3}{4}$ and $\tilde{\sigma} = c_1'\sigma$, and obtain the conclusion by choosing c_0 much smaller if necessary. □

We provide a version of Lemma 6.6 when $\mathbf{v}$ is a homogeneous of degree 2 approximate solution in a rectangular domain in polar coordinates $\mathcal{R}_\tau \subset \mathbb{R}^2$ defined as

$$\mathcal{R}_\tau := \{(r, \theta) : |\theta| \leq \tau, \ |r - 1| \leq C\tau\} \quad \text{with } \tau < \tau_0. \tag{6.18}$$

Lemma 6.7. *Assume that* $\mathbf{u}$ *is a solution to the* N*-membrane problem in* $\mathcal{R}_\tau$*, and* $\mathbf{v}$ *is a* $C^{1,1}$ *homogeneous of degree* 2 *function which solves the Euler–Lagrange equation in* $\mathcal{R}_\tau$ *with* $c_0\sigma\tau^{-2}$ *error. If* $\mathbf{u} \geq \mathbf{v}$ *in* $\mathcal{R}_\tau$*, and* $u_k \geq v_k + \sigma|x|^2$ *on a ray* $\mathcal{R}_\tau \cap \{\theta = l\}$*, then*

$$u_j \geq v_j + c_0\sigma|x|^2 \quad \text{in } \mathcal{R}_{\tau/2}$$

for all $j \in J_k$ *for which either* $j \leq k$ *and* v_j *is* $c_0\sigma$*-connected to* v_k*, or* $j > k$ *and the coincidence sets* $\{v_k = v_{k+1}\}$*,* $\{v_{k+1} = v_{k+2}\}, \dots, \{v_{j-1} = v_j\}$ *have length more than* $\frac{\tau}{10}$ *in the interval* $\theta \in [-\frac{\tau}{2}, \frac{\tau}{2}]$.

The proof of Lemma 6.7 follows as the one of Lemma 6.6 after we establish a version of the one-dimensional lemma, Lemma 6.5, on the unit circle. We omit the details but point out some of the changes in this setting.

We consider functions $\mathbf{v}$ on small intervals $[-\tau, \tau]$ on the unit circle which solve the N-membrane problem for the operator $-\partial_{\theta\theta} - 4$ which is positive definite if $\tau < \frac{\pi}{4}$. Then the homogeneous 2 extension of $\mathbf{v}$ solves the N-membrane problem in the corresponding sector in $\mathbb{R}^2$. The energy corresponding to the new operator has the form

$$\int_{-\rho}^{\rho} \sum \omega_k \left(\frac{1}{2}|v_k'|^2 - 2v_k^2 + f_k v_k\right) d\theta$$

and the existence of solutions follows in the same way as before. The proof of Lemma 6.5 is identical since the following Harnack inequality continues to hold:

$$\partial_{\theta\theta} w + 4w \leq 0 \text{ and } w \geq 0 \implies w \leq C w(0) \text{ in } [-\tau, \tau].$$

We are ready to prove Lemma 6.3 by comparing $\mathbf{u}$ with appropriate translations of $\mathbf{v}$ that are homogeneous of degree 2, and make use of Lemma 6.7 above.

Proof of Lemma 6.3. Assume for simplicity that $z = \frac{1}{2}e_1$, and choose ρ universal such that (see (6.18)) $\mathcal{R}_{4\rho r} \subset B_r(e_1)$.

We prove by induction on $m \geq 0$ that in $B_r(z)$ with $r = \bar{c}\rho^m$, for some $\bar{c}$ small to be specified later, as long as $r \geq C\varepsilon^{\frac{1}{2}}$ we have

$$v_k + \zeta^-_{k,m}|x|^2 \leq u_k \leq v_k + \zeta^+_{k,m}|x|^2, \quad \zeta^{\pm}_{k,m} = \zeta_{k,m} \pm \varepsilon_m,\ \varepsilon_m := 8(1-c)^m\varepsilon, \tag{6.19}$$

for some $c > 0$ small universal, and constants $\zeta_{k,m}$ for which $v_k + \zeta_{k,m}|x|^2$ is admissible. Moreover, $\zeta_{k,m}$ are all equal when k belongs to the ith group $k \in \{k_{i-1}+1, \dots, k_i\}$ and

$$\text{the line } \{x \cdot \nu_i\} = 0 \text{ intersects } B_r(z). \tag{6.20}$$

Notice that our hypothesis $|\mathbf{u} - \mathbf{v}| \leq \varepsilon$ implies that $\zeta_{k,m} \in [-16\varepsilon, 16\varepsilon]$.

When $m = 0$, we can take $\zeta_{k,0} = 0$ by hypothesis.

Assume the induction hypothesis holds for $r = r_m$. We want to show that (6.19) holds in $B_{\rho r}(z)$ for some constants $\xi^{\pm}_k$ with

$$\zeta^-_{k,m} \leq \xi^-_k \leq \xi^+_k \leq \zeta^+_{k,m}, \quad \xi^+_k - \xi^-_k \leq (1-c)(\zeta^+_{k,m} - \zeta^-_{k,m}),$$

and $v_k + \xi^{\pm}_k|x|^2$ are admissible, and with $\xi^{\pm}_{k+1} = \xi^{\pm}_k$ whenever condition (6.20) holds for $B_{\rho r}$. We define $\zeta_{k,m+1}$ as the averages of $\xi^{\pm}_k$ and the conclusion follows for $m+1$.

We pick a unit direction $\bar{\nu}$ close to the direction e_1 of z

$$|\bar{\nu} - e_1| \leq \rho r$$

such that a cr-neighborhood of the ray of direction $\bar{\nu}$ does not intersect the set D^{ε} (defined in (6.5)) in $B_r(z)$. This is possible since $r \geq C\varepsilon^{\frac{1}{2}}$. Assume that at $\frac{1}{2}\bar{\nu}$, u_k is closer to the upper bound in (6.19), i.e.,

$$u_k(\frac{1}{2}\bar{\nu}) \geq (v_k + (\zeta^-_{k,m} + \varepsilon_m)|x|^2)\left(\frac{1}{2}\bar{\nu}\right). \tag{6.21}$$

By Lemma 6.2, outside D^{ε},

$$|\triangle(u_k - (v_k + \zeta^-_{k,m}|x|^2))| \leq \delta\varepsilon + 2|\zeta^-_{k,m}| \leq 40\varepsilon \leq \bar{c}\varepsilon_m r^{-2}.$$

By the Harnack inequality applied to the difference

$$u_k - (v_k + \zeta^-_{k,m}|x|^2) \geq 0$$

we find that (6.21) can be extended to

$$u_k \geq v_k + \zeta^-_{k,m}|x|^2 + c'\varepsilon_m \geq v_k + (\zeta^-_{k,m} + c'\varepsilon_m)|x|^2$$

for some c' universal on the whole ray

$$B_{r/2}(z) \cap \{t\bar{\nu} : t \geq 0\},$$

provided that $\bar{c}$ is sufficiently small. Now we can apply Lemma 6.7 to $u_k(x)$ (in fact the quadratic rescalings $4u_k(\frac{x}{2})$) and $v_k + \zeta^-_{k,m}|x|^2$ in $\mathcal{R}_{8\rho r}$ with $\sigma := c'\varepsilon_m$, since the error for the approximate solutions is bounded by

$$40\varepsilon \leq \bar{c}\varepsilon_m r^{-2} \leq c_0\sigma(8\rho r)^{-2},$$

and obtain

$$u_j \geq v_j + (\zeta^-_{j,m} + c''\varepsilon_m)|x|^2,$$

in $B_{\rho r}(z)$ for all $j \in J_k$, for some c'' small, universal. As in Remark 6.2, the righthand sides correspond to an admissible family in $B_{\rho r}(z)$. Moreover, they change by the same amount on a set of indices j that belong to an ith group $\{k_{i-1}+1, \dots, k_i\}$ for which $\{x \cdot \nu_i\}$ intersects $B_{\rho r}(z)$, since in this case the coincidence sets $\{v_{j-1} = v_j\}$ cover more than $\frac{1}{10}$ of the interval $\theta \in [-4\rho r, 4\rho r]$ on the unit circle ∂B_1. This means that we can choose $\xi^\pm_k$ accordingly in $B_{\rho r}$ and the lemma is proved. □

7. Regular intersection points

In this section we study the regularity of the free boundaries for solutions $\mathbf{u}$ that stay close to the blow-up cone

$$\mathbf{p_0}(x) := \frac{1}{2}(x_2^+)^2\mathbf{f},$$

and prove Theorem 1.2 which we recall.

Theorem 7.1. *Assume $d = 2$ and*

$$|\mathbf{u} - \mathbf{p_0}| \leq \varepsilon_0 \quad \textit{in } B_1.$$

Then each Γ_i is a $C^{1,\log}$ curve in $B_{1/2}$.

We prove Theorem 7.1 by induction on the number of membranes N. One of the technical points is that we need a lower bound for the Weiss energy, see Lemma 7.7, which is not obvious since we no longer assume $0 \in \bigcap \Gamma_i$.

Similar to Definition 4.3, we approximate solutions $\mathbf{u}$ by the slightly more general functions from Definition 4.2

$$\mathbf{p}(x, \mathbf{b}_0, \mathbf{b}_1) = \mathbf{h}(x_2, \mathbf{b}_0 + x_1\mathbf{b}_1), \quad \mathbf{b}_i \in \mathbf{B}(\mathbf{p_0}).$$

Proposition 7.1. *Assume that a solution $\mathbf{u}$ to problem P_0 satisfies*

$$|\mathbf{u} - \mathbf{p}(\cdot, \mathbf{b}_0, \mathbf{b}_1)| \leq \varepsilon r^2 \quad \textit{in } B_r, \tag{7.1}$$

for some $\mathbf{b}_i \in B(\mathbf{p_0})$ with $|\mathbf{b}_0| \leq \varepsilon^{\frac{1}{2}} r$, $|\mathbf{b}_1| \leq 2\delta\varepsilon^{\frac{1}{2}}$. Then

$$|\mathbf{u} - \mathbf{p}(\cdot, \mathbf{b}'_0, \mathbf{b}'_1)| \leq \frac{\varepsilon}{2}(\rho r)^2 \quad \textit{in } B_{\rho r} \tag{7.2}$$

with $\mathbf{b}_i' \in B(\mathbf{p}_0)$ *and*

$$|\mathbf{b}_0' - \mathbf{b}_0| \le C_0 \varepsilon r, \quad |\mathbf{b}_1' - \mathbf{b}_1| \le C_0 \varepsilon. \tag{7.3}$$

The constant C_0 *depends only on the dimension* $d = 2$, $\rho \le \rho_0$ *universal,* $\delta \le \delta(\rho)$ *depending on* ρ, *and* $\varepsilon \le \varepsilon_0(\delta, \rho)$ *sufficiently small.*

After rescaling it suffices to prove the proposition for $r = 1$. First we estimate the change in $\mathbf{h}(x, \mathbf{b})$ as we vary $\mathbf{b}$.

Lemma 7.1. *We have*

$$|\mathbf{h}(x, \mathbf{b} + \mathbf{d}) - (\mathbf{h}(x, \mathbf{b}) + x\mathbf{d})| \le C|\mathbf{d}|(|\mathbf{b}| + |\mathbf{d}|)$$

Proof. By the homogeneity of $\mathbf{h}$ we may assume that $|\mathbf{b}| + |\mathbf{d}| = 1$. Then by Lemma 3.2 we know that the left-hand side is constant when x is outside the interval $[-C, C]$. So it suffices to prove the inequality when $|x| \le C$. Now the inequality follows from the Lipschitz continuity of $\mathbf{h}$ in its second variable. □

Next we establish in the context of Proposition 7.1 the estimate for the rescaled error of $\mathbf{u} - \mathbf{p}$ in terms of the distance to the x_2-axis, as we did in Lemma 4.6.

Lemma 7.2. *Assume that* $\mathbf{u}$ *satisfies* (7.1) *with* $r = 1$. *Then in* $B_{1/2}$,

$$|\mathbf{u} - \mathbf{p}(\cdot, \mathbf{b}_0, \mathbf{b}_1)| \le C\varepsilon(|x_2| + \sqrt{\varepsilon})^\alpha,$$

for some $\alpha > 0$ *small, universal.*

Proof. The proof is essentially the same with the one of Lemma 4.6, after replacing $\mathbf{p}(\cdot, \mathbf{b})$ by $\mathbf{p}(\cdot, \mathbf{b}_0, \mathbf{b}_1)$. A few comments are in order.

First we remark that the approximate solution solves the Euler–Lagrange equations with error $C|\mathbf{b}_1|^2 \le \delta\varepsilon$ as before, and is not affected by the presence of $\mathbf{b}_0$, see Lemma 4.2.

The comparison function $\mathbf{v}$ in $B_{r_k}(Z)$ is defined as before

$$\mathbf{v}(x) := \mathbf{p}(x_2, \mathbf{b}_0 + \mathbf{d}, \mathbf{b}_1) + \left(c_1 \varepsilon_k q\left(\frac{x - Z}{r_k}\right) - \varepsilon_k\right)\mathbf{1},$$

with $\mathbf{d}$, q as in (4.6). Inequality (4.9) is then replaced by

$$|\mathbf{p}(x, \mathbf{b}_0 + \mathbf{d}, \mathbf{b}_1) - \mathbf{p}(x, \mathbf{b}_0, \mathbf{b}_1) - x_2\mathbf{d}| \le \frac{C}{C_1'}\varepsilon_k \quad \text{in } B_{r_k}(Z), \tag{7.4}$$

and the rest of the proof remains the same, by choosing C_1' sufficiently large depending on the other constants c_1, c_2 and μ. We no longer use Lemma 4.2 to establish (7.4), but Lemma 7.2 above with $\mathbf{b} = \mathbf{b}_0 + x_1\mathbf{b}_1$. Then $|\mathbf{b}| \le 2\varepsilon^{\frac{1}{2}}$ and, since $|\mathbf{d}| \le C\varepsilon_k r_k^{-1}$ and $r_k \ge C_1'\varepsilon^{\frac{1}{2}}$, the left-hand side in (7.4) is bounded by

$$C\varepsilon^{\frac{1}{2}}\varepsilon_k r_k^{-1} \le \frac{C}{C_1'}\varepsilon_k.$$ □

Remark 7.1. As a consequence of Lemma 7.2 and of the quadratic separation of consecutive membranes from their common free boundary, we find that in $B_{1/2}$ the free boundaries $\Gamma_i(\mathbf{u})$ of $\mathbf{u}$ lie in a $\varepsilon^{\frac{1}{2}+\frac{\alpha}{4}}$-neighborhood of the corresponding free boundaries of the approximate solution

$$\mathbf{p}(x, \mathbf{b}_0, \mathbf{b}_1) = \mathbf{h}(x_2, \mathbf{b}_0 + x_1 \mathbf{b}_1).$$

In particular, $\Gamma_i(\mathbf{u})$ lie in an $C\delta\varepsilon^{\frac{1}{2}}$-neighborhood of the free boundaries $x_2 = \Gamma_i(\mathbf{b}_0)$ of the exact solution $\mathbf{p}(x, \mathbf{b}_0, 0) = \mathbf{h}(x_2, \mathbf{b}_0)$.

Assume that the free boundaries of $\mathbf{h}(x_2, \mathbf{b}_0)$ separate of order $\varepsilon^{\frac{1}{2}}$, i.e., there exists an interval $[a - c_0\varepsilon^{\frac{1}{2}}, a + c_0\varepsilon^{\frac{1}{2}}]$ for some c_0 small, which does not intersect the $\Gamma_i(\mathbf{b}_0)$, but at least one of these points falls to the left of this interval and at least one to the right. Assume $\delta \ll c_0$ is sufficiently small. Then the free boundaries $\Gamma_i(\mathbf{u})$ do not intersect the strip

$$S := \left\{ |x_2 - a| \le \frac{c_0}{2}\varepsilon^{\frac{1}{2}} \right\},$$

and the N-membrane problem decouples into several multi-membrane problems in $B_{1/2}$ involving fewer membranes.

Indeed, for each set of indices $j \in J$ for which u_j agree in the strip S, we replace u_j by u_J to the right of the strip (we think x_2 is the horizontal direction). If there are $J_1, \dots, J_l$ such sets, then we obtain a multi-membrane problem involving l-membranes. The free boundaries of the new problem coincide with the free boundaries of $\mathbf{u}$ that were on the left of the strip S. On the other hand, for each set J, $u_j - u_J$ solves a multi-membrane problem which has $\Gamma_j(\mathbf{u})$ with $j \in J$ as free boundaries, which lie to the right of the strip S. The same decoupling procedure can be performed to the approximate solution $\mathbf{p}(x, \mathbf{b}_0, \mathbf{b}_1)$, hence the decoupled multi-membrane problems in $B_{1/2}$ are still ε-approximated by corresponding functions of the type $\mathbf{p}(\cdot, \mathbf{b}_0, \mathbf{b}_1)$.

Also Lemma 7.2 implies the uniform convergence of the rescaled errors.

Corollary 7.1. *If* $|\mathbf{u}_m - \mathbf{p}(\cdot, \mathbf{b}_0^m, \mathbf{b}_1^m)| \le \varepsilon_m$ *in* B_1, *with* $|\mathbf{b}_0^m| \le \varepsilon_m^{\frac{1}{2}}$, $|\mathbf{b}_1^m| \le 2\delta\varepsilon_m^{\frac{1}{2}}$, *for a sequence of* $\varepsilon_m \to 0$, *then, up to a subsequence, each of the rescaled error functions*

$$\varepsilon_m^{-1}(u_{m,j} - p_j(\cdot, \mathbf{b}_0^m, \mathbf{b}_1^m))$$

converges uniformly in $B_{1/2}$ *to a limit* w_j *that satisfies*

$$\|w_j\|_{L^\infty} \le 1, \qquad w_j = 0 \quad \text{on } x_2 = 0$$

and

$$|\triangle w_j| \le \delta \quad \text{away from } \{x_2 = 0\}.$$

Proof of Proposition 7.1. The rescaled error functions

$$\varepsilon^{-1}(u_j - p_j(\cdot, \mathbf{b}_0, \mathbf{b}_1))$$

are well approximated in $B_{1/2}$ by continuous functions w_j which vanish on $x_2 \le 0$ and satisfy $|\triangle w_j| \le \delta$ in $\{x_2 > 0\}$. Denote by $\mathbf{d}_0, \mathbf{d}_1 \in \mathbf{B}(\mathbf{p}_0)$ as

$$d_{0,j}^+ = \partial_{x_2} w_j(0), \quad d_{0,j}^- = 0, \quad d_{1,j}^+ = \partial_{x_1 x_2} w_j(0), \quad d_{1,j}^- = 0.$$

Then $|\mathbf{d}_i| \le C_0$, and

$$|\mathbf{w} - x_2(\mathbf{d}_0 + x_1\mathbf{d}_1)| \le C_0(\rho^3 + \delta) \quad \text{in } B_\rho,$$

for a constant C_0 that depends only on the dimension $d = 2$. If $\rho \le \rho_0$ universal, and $\delta \le \delta(\rho)$ depending on ρ, then the right-hand side is less than $\frac{1}{4}\rho^2$.

By Lemma 7.1,

$$\mathbf{p}(x, \mathbf{b}_0 + \varepsilon\mathbf{d}_0, \mathbf{b}_1 + \varepsilon\mathbf{d}_1) - \mathbf{p}(x, \mathbf{b}_0, \mathbf{b}_1) = \varepsilon x_2(\mathbf{d}_0 + x_1\mathbf{d}_1) + O(\varepsilon^{\frac{3}{2}}),$$

and we obtain the desired result by choosing $\mathbf{b}_0' = \mathbf{b}_0 + \varepsilon\mathbf{d}_0$, $\mathbf{b}_1' = \mathbf{b}_1 + \varepsilon\mathbf{d}_1$. □

Remark 7.2. Assume that in B_1 we satisfy (7.1) and in addition $\mathbf{b}_0 = 0$. We have the following dichotomy depending on the size of $\mathbf{d}_0$ in the proof above.

(a) If

$$|\mathbf{d}_0| \le c(\rho_0) =: c_1 \tag{7.5}$$

then we may choose $\mathbf{b}_0' = 0$ and satisfy the conclusion

$$|\mathbf{u} - \mathbf{p}(\cdot, 0, \mathbf{b}_1')| \le \frac{\varepsilon}{2}\rho_0^2 \quad \text{in } B_{\rho_0}, \quad |\mathbf{b}_1' - \mathbf{b}_1| \le C_0\varepsilon.$$

Moreover, a similar analysis as in Proposition 5.1 can be performed. If $\mathbf{b}_1/\delta\varepsilon^{\frac{1}{2}}$ is at distance at most μ_0 (with μ_0 small universal) away from the line $\{s\tau : s \in \mathbb{R}\}$, then, as in the last part of the proof of Proposition 5.1, after a rotation of coordinates as in (5.8) we may reduce to the case when $\mathbf{b}_1$ satisfies the improved bound $|\mathbf{b}_1| \le \frac{1}{4}\delta\varepsilon^{\frac{1}{2}}$. Then $\mathbf{u} \in \mathcal{S}(\rho_0, \mathbf{p}_0, \frac{\varepsilon}{2})$ and the approximate solutions $\mathbf{v}_1, \mathbf{v}_{\rho_0}$ for $\mathbf{u}$ in B_1 respectively B_{ρ_0} satisfy $|\mathbf{v}_1 - \mathbf{v}_{\rho_0}| \le C\varepsilon$.

Assume now that $\mathbf{b}_1/\delta\varepsilon^{\frac{1}{2}}$ is at distance greater than $\frac{1}{2}\mu_0$ away from the line $\{s\tau | s \in \mathbb{R}\}$. Then in the proof of Proposition 7.1, by Corollary 4.1, the right-hand side of $\triangle\mathbf{w}$ is constant in each quadrant in $\{x_2 > 0\}$ but has a discontinuity jump greater than $c(\delta, \mu_0) > 0$ across $\{x_1 = 0\}$. This implies that $\mathbf{w}$ cannot be homogeneous of degree 2 in the annulus $B_{1/2} \setminus B_{1/4}$ which, as in Proposition 5.1 implies the energy inequality

$$W(\mathbf{u}, \rho_0) \le W(\mathbf{u}, 1) - c\varepsilon^2, \tag{7.6}$$

for some c small depending on δ and μ_0.

(b) If $|\mathbf{d}_0| \ge c_1$, then we satisfy the conclusion

$$|\mathbf{u} - \mathbf{p}(\cdot, \mathbf{b}_0', \mathbf{b}_1')| \le \varepsilon\rho_1^2 \quad \text{in } B_{\rho_1}, \quad |\mathbf{b}_0'| \ge c_1\varepsilon,$$

for some small ρ_1, provided that δ is chosen small, depending on ρ_1.

Next we show that when we end up in situation (b), then the N-membrane problem near the origin can be reduced to one involving fewer membranes. For this we need to iterate Proposition 7.1 from scale 1 to scale $\varepsilon^{\frac{1}{2}}$. Precisely, let us assume that, as a starting point we have

$$|\mathbf{u} - \mathbf{p}(\cdot, \mathbf{b}_0, \mathbf{b}_1)| \le \varepsilon\rho_1^2 \quad \text{in } B_{\rho_1},$$

with

$$|\mathbf{b}_0| \le \frac{\varepsilon}{2}, \qquad |\mathbf{b}_1| \le \delta\varepsilon^{\frac{1}{2}}.$$

We can iterate the proposition with $r = \rho_1^m$ till $r \sim \varepsilon^{\frac{1}{2}}$ and obtain

$$|\mathbf{u} - \mathbf{p}(\cdot, \bar{\mathbf{b}}_0, \bar{\mathbf{b}}_1)| \leq \varepsilon r^2 \quad \text{in } B_r, \quad \text{with } r = \varepsilon^{\frac{1}{2}}, \tag{7.7}$$

with

$$|\bar{\mathbf{b}}_0 - \mathbf{b}_0| \leq 2C_0\rho_1\varepsilon, \quad |\bar{\mathbf{b}}_1 - \mathbf{b}_1| \leq C|\log\varepsilon|\varepsilon \tag{7.8}$$

(in the last step of the iteration we applied the proposition for some $\rho \in [\rho_1, \rho_1^2]$). Here ρ_1 is chosen small such that $4C_0\rho_1 \leq c_1 \leq 1$ (see (7.5)) and throughout the iteration the inequalities

$$|\bar{\mathbf{b}}_0| \leq \varepsilon, \quad |\bar{\mathbf{b}}_1| \leq 2\delta\varepsilon^{\frac{1}{2}}$$

are satisfied. Moreover, if $|\mathbf{b}_0| \geq c_1\varepsilon$ then $|\bar{\mathbf{b}}_0| \geq \frac{c_1}{2}\varepsilon$.

We rescale (7.7) to the unit ball and obtain that

$$|r^{-2}\mathbf{u}(rx) - \mathbf{p}(x, r^{-1}\bar{\mathbf{b}}_0, \bar{\mathbf{b}}_1)| \leq \varepsilon \quad \text{if } x \in B_1, \quad r = \varepsilon^{\frac{1}{2}}.$$

If 0 belongs to one of the free boundaries of $\mathbf{u}$, say $0 \in \Gamma_{i_0}$, and $|\mathbf{b}_0| \geq c_1\varepsilon$, then we are in the setting of Remark 7.1. Precisely, we find that in B_1, $r^{-1}\Gamma_{i_0}$ is the free boundary of a solution $\tilde{u}_r$ to a multiple membrane problem involving fewer membranes, which satisfies back hypothesis (7.1) with the same value of ε. We summarize the above discussion in the next lemma.

Lemma 7.3. *Assume that $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}_0, \varepsilon)$ for some $\varepsilon \leq \varepsilon_0$, i.e.,*

$$|\mathbf{u} - \mathbf{p}(\cdot, 0, \mathbf{b}_1)| \leq \varepsilon \quad \text{in } B_1, \quad |\mathbf{b}_1| \leq \delta\varepsilon^{\frac{1}{2}},$$

and $0 \in \Gamma_{i_0}(\mathbf{u})$, for some i_0. Then one of the following alternative hold:

(a) *We have*

$$|\mathbf{u} - \mathbf{p}(\cdot, 0, \mathbf{b}_1')| \leq \frac{\varepsilon}{2}\rho_0^2 \quad \text{in } B_{\rho_0}, \quad |\mathbf{b}_1' - \mathbf{b}_1| \leq C_0\varepsilon,$$

(b) *We have*

$$\Gamma_{i_0} \cap B_r \subset \{|x_n| \leq C\varepsilon^{\frac{1}{2}}r\} \quad \text{if } r \in [\varepsilon^{\frac{1}{2}}, 1].$$

When $r = \varepsilon^{\frac{1}{2}}$, Γ_{i_0} is a free boundary to a solution $\tilde{\mathbf{u}}$ to the multiple membrane problem in B_r involving fewer membranes than N. Moreover, $\tilde{\mathbf{u}}$ satisfies

$$|\tilde{\mathbf{u}} - \tilde{\mathbf{p}}(\cdot, \tilde{\mathbf{b}}_0, \tilde{\mathbf{b}}_1)| \leq 2\varepsilon r^2 \quad \text{in } B_r, \quad |\tilde{\mathbf{b}}_0| \leq (2\varepsilon)^{\frac{1}{2}}r, \quad |\mathbf{b}_1| \leq \delta(2\varepsilon)^{\frac{1}{2}}.$$

Also $0 \notin \bigcap \Gamma_i$.

Alternative (b) reduces the situation to one involving fewer membranes.

It remains to investigate alternative (a). While $\mathbf{u}$ improves at a $C^{2,\alpha}$ rate as we zoom in B_{ρ_0}, the bound on the size of $\mathbf{b}_1$ can deteriorate. Part (a) implies that

$$\mathbf{u} \in \mathcal{S}(\rho_0, \mathbf{p}_0, \varepsilon') \quad \text{with } \varepsilon' = \varepsilon + C(\delta)\varepsilon^{\frac{3}{2}}. \tag{7.9}$$

As we iterate part (a) we want to show that the approximating polynomials converge. It suffices to prove the following lemma.

Lemma 7.4. *Assume that the hypothesis of Lemma* 7.3 *hold and* $\mathbf{u}$ *satisfies alternative* (a). *Then either* (a1) *or* (a2) *below hold:*

(a1) *We have*

$$\mathbf{u} \in \mathcal{S}\left(\rho_0, \mathbf{p}_0, \frac{\varepsilon}{2}\right). \tag{7.10}$$

(a2) *We have*

$$\mathbf{u} \in \mathcal{S}(\rho_0, \mathbf{p}_0, 2\varepsilon) \quad \text{and} \quad W(\mathbf{p}_0) + c\varepsilon^{\frac{3}{2}} \leq W(\mathbf{u}, \rho_0) \leq W(\mathbf{u}, 1) - c\varepsilon^2. \tag{7.11}$$

In both cases $|\mathbf{v}_1 - \mathbf{v}_{\rho_0}|_{L^\infty(B_1)} \leq C\varepsilon$, *where* $\mathbf{v}_1$, $\mathbf{v}_{\rho_0}$ *denote the approximate solutions for* $\mathbf{u}$ *in* B_1 *respectively* B_{ρ_0}.

The lemma is essentially included in Proposition 5.1 except the crucial lower bound on $W(\mathbf{u}, \rho_0)$. The statement that $W(\mathbf{p}_0) \leq W(\mathbf{u}, \rho_0)$ allows one to prove the convergence of $\sum \varepsilon_k$ as in Section 4. The inequality follows easily when $0 \in \bigcap \Gamma_i$ by the Weiss monotonicity formula and the fact that $\mathbf{p}_0$ is the least energy solution. However, for the general case we need to establish a lower bound on the energy of approximate solutions the type

$$W(\mathbf{p}(\cdot, \mathbf{b})) \geq W(\mathbf{p}_0) - C\varepsilon^2.$$

First we establish the opposite inequality in (5.1) of Lemma 5.1.

Lemma 7.5. *Assume that* $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}_0, \varepsilon)$ *is* ε*-approximated in* B_1 *by* $\mathbf{v} := \mathbf{p}(\cdot, \mathbf{b})$. *Then*

$$W(\mathbf{u}, r) \geq W(\mathbf{v}) - C(r)\varepsilon^2.$$

Proof. The proof is essentially the same as (5.1) in Lemma 5.1 after reverting the roles of $\mathbf{u}$ and $\mathbf{v}$. We write $\mathbf{u} = \mathbf{v} + \varepsilon\mathbf{w}$, with $|\mathbf{w}| \leq 1$. Then we write

$$W(\mathbf{u}, r) = W(\mathbf{v}, r) + \varepsilon^2 r^{n-2} I_1 + \varepsilon r^{n-2} I_2,$$

with

$$\begin{aligned} I_1 &:= \int_{B_r} \sum \frac{\omega_k}{2} |\nabla w_k|^2 \, dx - r^{-1} \int_{\partial B_r} \sum \omega_k w_k^2 \, d\sigma, \\ I_2 &:= \int_{B_r} \sum \omega_k (\nabla v_k \cdot \nabla w_k + f_k w_k) \, dx - \int_{\partial B_r} \sum \omega_k \frac{2}{r} v_k w_k \, d\sigma \\ &= \int_{B_r} \sum \omega_k (f_k - \triangle v_k) w_k \, dx. \end{aligned}$$

Now we use the fact that $\mathbf{v}$ is a solution in the x_2-variable and find (see (1.3))

$$\omega_k (f_k - \partial_{x_2 x_2} v_k) w_k \geq 0.$$

Since $|\partial_{x_1 x_1} v_k| \leq \delta\varepsilon$, we find

$$\omega_k (f_k - \triangle v_k) w_k \geq \omega_k (f_k - \partial_{x_2 x_2} v_k) w_k - C |\partial_{x_1 x_1} v_k| \geq -C\varepsilon,$$

which together with $I_1 \geq -C$ gives the desired conclusion. □

In the next lemma we show that each $\mathbf{p}(\cdot\,,\mathbf{b})$ ε-approximates at leat one solution for which all the free boundaries intersect at the origin.

Lemma 7.6. *Given a vector $\mathbf{b}\in\mathbf{B}(\mathbf{p_0})$ with $|\mathbf{b}|\le\delta^{\frac{1}{2}}\varepsilon$, there exists $\mathbf{u_b}\in\mathcal{S}(1,\mathbf{p_0},\varepsilon)$ with $0\in\bigcap\Gamma_i$ which is ε-approximated in B_1 by $\mathbf{p}(\cdot\,,\mathbf{b})$.*

Proof. For each solution $\mathbf{u}$ we associate the vector $\mathbf{z}\in\mathbb{R}^{n-1}$ given by

$$z_i := \operatorname{dist}(0,\Gamma_i)\chi_{\{u_i=u_{i+1}\}}(0)-\sqrt{(u_i-u_{i+1})(0)}\,\chi_{\{u_i>u_{i+1}\}}(0).$$

The quadratic growth of u_i-u_{i+1} away from its zero set implies that $\mathbf{u}\mapsto\mathbf{z}(\mathbf{u})$ is a continuous map, and $0\in\Gamma_i(\mathbf{u})$ if and only if $z_i=0$. Moreover, if we consider the solutions $\mathbf{h}(x_2,\mathbf{b_0})$ with free boundaries $x_2=\Gamma_i(\mathbf{b_0})$, then the corresponding z_i satisfies

$$c\le\frac{z_i}{\Gamma_i(\mathbf{b_0})}\le C. \tag{7.12}$$

For any $\Gamma\in\mathbb{R}^{n-1}$ with $|\Gamma|\le c'$ we associate the corresponding vector $\mathbf{b}_0(\Gamma)\in\mathbf{B}(\mathbf{p_0})$ for which $h(x_2,\mathbf{b}_0)$ has free boundaries Γ. Recall from Section 2 that $\Gamma\mapsto\mathbf{b}_0(\Gamma)$ is a bi-Lipschitz map. We choose c' small universal such that $|\mathbf{b}_0|\le\frac{1}{2}$.

We consider the solutions $\mathbf{u}_\Gamma$ in B_1 with boundary data $\mathbf{p}(x,\varepsilon\mathbf{b}_0(\Gamma),\mathbf{b})$. We claim that one of these functions satisfies the conditions of the lemma.

Notice that since $\mathbf{p}(x,\varepsilon\mathbf{b}_0,\mathbf{b})$ solves the Euler–Lagrange equations with error $\delta\varepsilon$ we know that

$$|\mathbf{u}_\Gamma-\mathbf{p}(x,\varepsilon\mathbf{b}_0,\mathbf{b})|\le\delta\varepsilon\quad\text{in } B_1.$$

On the other hand, by Lemma 7.1,

$$\mathbf{p}(x,\varepsilon\mathbf{b}_0,\mathbf{b})=\mathbf{p}(x,\mathbf{b})+\varepsilon x\mathbf{b}_0+O(\varepsilon^{\frac{3}{2}}),$$

which imply that $\mathbf{u}_\Gamma$ is ε-approximated in B_1 by $\mathbf{p}(\cdot\,,\mathbf{b})$.

If δ is sufficiently small, then

$$|\mathbf{u}_\Gamma-\mathbf{p}(x,\varepsilon\mathbf{b}_0,\mathbf{b})|\le\varepsilon\rho_1^2\quad\text{in } B_{\rho_1}.$$

and the arguments before Lemma 7.3 applies. In particular, the free boundaries of the rescaling

$$\tilde{\mathbf{u}}_\Gamma(x):=r^{-2}\mathbf{u}_\Gamma(rx)\quad\text{with } r=\varepsilon^{\frac{1}{2}}$$

are in $B_{1/2}$ in a $C\delta\varepsilon^{\frac{1}{2}}$-neighborhood of the free boundaries of $\mathbf{h}(x_2,r\bar{\mathbf{b}}_0)$ for some $\bar{\mathbf{b}}_0$ that satisfies

$$|\bar{\mathbf{b}}_0-\mathbf{b}_0|\le 2C_0\rho_1,$$

(see Remark 7.1 and equations (7.7)–(7.8) with $\mathbf{b}_0$, $\bar{\mathbf{b}}_0$ replaced by $\varepsilon\mathbf{b}_0$ and $\varepsilon\bar{\mathbf{b}}_0$).

Thus the free boundaries of $\tilde{\mathbf{u}}_\Gamma$ are in a $c(\rho_1,\delta)\varepsilon^{\frac{1}{2}}$-neighborhood of the free boundaries of $\mathbf{h}(x_2,\varepsilon^{\frac{1}{2}}\mathbf{b}_0)$ with $c(\rho_1,\delta)\to 0$ as $\rho_1,\delta\to 0$.

This means that the vector

$$\mathbf{y}_\Gamma:=\varepsilon^{-\frac{1}{2}}\mathbf{z}(\tilde{\mathbf{u}}_\Gamma)$$

associated to the rescaled solution $\tilde{\mathbf{u}}_\Gamma$ above is in a $c(\rho_1,\delta)$-neighborhood of the vector

$$\mathbf{z}_\Gamma:=\mathbf{z}(\mathbf{h}(x_2,\mathbf{b}_0))$$

corresponding to $\mathbf{h}(x_2,\mathbf{b}_0)$.

We can find the desired solution to $\mathbf{y}_\Gamma = 0$ by a standard topological argument. Indeed, by (7.12) we know that $\Gamma \cdot \mathbf{z}_\Gamma \sim |\Gamma|^2$ hence $\Gamma \cdot \mathbf{z}_\Gamma \geq c_1 > 0$ when $|\Gamma| = c'$. Then $\Gamma \cdot \mathbf{y}_\Gamma > 0$ when $\Gamma \in \partial B_{c'}$ provided that $c_1(\rho_1, \delta)$ is sufficiently small. This implies that we can find $\Gamma \in B_{c'}$ such that $\mathbf{y}_\Gamma = 0$. □

As a corollary of Lemma 7.6 we obtain by (5.1) that if $|\mathbf{b}| \leq \delta\varepsilon^{\frac{1}{2}}$, then

$$W(\mathbf{p}(\,\cdot\,,\mathbf{b})) \geq W\left(\mathbf{u_b}, \frac{1}{2}\right) - C\varepsilon^2 \geq W(\mathbf{p}_0) - C\varepsilon^2, \tag{7.13}$$

where $\mathbf{u_b}$ is the solution provided by Lemma 7.6.

The lower bound on $W(\mathbf{p}(\,\cdot\,,\mathbf{b}))$ can be improved when $\mathbf{b}/\delta\varepsilon^{\frac{1}{2}}$ is at distance greater than μ_0 away from the line $\{s\tau : s \in \mathbb{R}\}$. For this we apply inductively Proposition 7.1 from scale 1 to scale $r = \varepsilon^{\frac{1}{2}}$ to the function $\mathbf{u_b}$ of Lemma 7.6. Notice that we cannot end up in alternative (b) of Remark 7.2 (or Lemma 7.3) since $0 \in \bigcap \Gamma_i$. The iteration requires $m_0 \sim |\log\varepsilon|$ steps and the distance from the corresponding sequence of $\mathbf{b}_1$ to the $s\tau$-line remains greater than $\frac{\mu_0}{2}$ throughout. From Remark 7.2 (a) we obtain that (see (7.6))

$$W(\mathbf{p}_0) \leq W(\mathbf{u_b}, \rho_0^m) \leq W(\mathbf{u_b}, \rho_0) - (m-1)c\varepsilon^2,$$

hence

$$W(\mathbf{u_b}, \rho_0) \geq W(\mathbf{p}_0) + c|\log\varepsilon|\varepsilon^2.$$

Then, by the first inequality in (7.13) we find

$$W(\mathbf{p}(\,\cdot\,,\mathbf{b})) \geq W(\mathbf{p}_0) + c|\log\varepsilon|\varepsilon^2. \tag{7.14}$$

In the next lemma we show that the right-hand side can be improved further, and obtain the reversed inequality to (5.2) in Lemma 5.1.

Lemma 7.7. *We have*

$$W(\mathbf{p}(\,\cdot\,,\mathbf{b})) \geq W(\mathbf{p}_0) + c\varepsilon^{\frac{3}{2}}$$

if $\mathbf{b}/\delta\varepsilon^{\frac{1}{2}}$ *is at distance greater than* μ_0 *away from the line* $\{s\tau : s \in \mathbb{R}\}$.

Proof. We claim that if $\mathbf{v} := \mathbf{p}(\,\cdot\,,\mathbf{b})$, with $\mathbf{b} = \varepsilon^{\frac{1}{2}}\mathbf{d}$ for some $\mathbf{d}$ with $|\mathbf{d}| \leq 1$, then

$$W(\mathbf{v}) = \varepsilon^{\frac{3}{2}} g(\mathbf{d}) + O(\varepsilon^2), \tag{7.15}$$

for some continuous function $g(\mathbf{d})$. Inequality (7.14) implies that if $\mathbf{d}$ is at distance greater than $\delta\mu_0$ away from the line $\{s\tau : s \in \mathbb{R}\}$, then $g(\mathbf{d}) > 0$ and the lemma easily follows. It remains to prove the claim (7.15).

Since $\mathbf{v}$ is homogeneous of degree 2, we find

$$W(\mathbf{v}) = \int_{B_1} \left(v_i f_i - \frac{1}{2} v_i \triangle v_i\right)\omega_i \, dx.$$

Using the same formula for $\mathbf{p}_0$ and that

$$\int_{B_1} (v_i \triangle p_{0,i} - p_{0,i} \triangle v_i)\omega_i \, dx = 0,$$

we get

$$W(\mathbf{v}) - W(\mathbf{p}_0) = \int_{B_1} (v_i - p_{0,i})\left(f_i - \frac{1}{2}\triangle v_i - \frac{1}{2}\triangle p_{0,i}\right)\omega_i \, dx.$$

We split the integral on the right-hand side into three angular regions:

$$A_1 := \{|x_2| \le C\varepsilon^{\frac{1}{2}}|x_1|\}, \quad A_2 := \{x_2 > C\varepsilon^{\frac{1}{2}}|x_1|\}, \quad A_3 := \{x_2 < C\varepsilon^{\frac{1}{2}}|x_1|\}.$$

In A_3, $\mathbf{v} = \mathbf{p_0} = 0$ and the integral is 0. We show that the integrals in $A_1 \cap B_1$ and $A_2 \cap B_1$ have the same form as the right-hand side of (7.15).

In $A_2 \cap B_1$, this follows easily from Lemma 4.1 which gives

$$v_i - p_{0,i} = \varepsilon^{\frac{1}{2}} d_i x_1 x_2 + O(\varepsilon)$$

and

$$f_i - \frac{1}{2}\triangle v_i - \frac{1}{2}\triangle p_{0,i} = -\varepsilon(e_i(\mathbf{d})\chi_{\{x_1>0\}} + e_i(-\mathbf{d})\chi_{\{x_1<0\}}).$$

In $A_1 \cap B_1$ we use that $|\mathbf{v}|, |\mathbf{p_0}| \le C\varepsilon$, and we replace the integral in $A_1 \cap B_1$ by the integral in $T_\varepsilon := A_1 \cap \{|x_1| < 1\}$ since their difference is $O(\varepsilon^{\frac{5}{2}})$. Also we may replace our function by

$$w_\varepsilon := (v_i - p_{0,i})\Big(f_i - \frac{1}{2}\partial_{22} v_i - \frac{1}{2}\triangle p_{0,i}\Big)\omega_i$$

which differs from the original function by $O(\varepsilon^2)$, and we integrate them in a domain of measure $\sim \varepsilon^{\frac{1}{2}}$. However, the function w_ε is obtained from w_1 by the quadratic rescaling in the second variable $w_\varepsilon(x_1, x_2) = \varepsilon w_1(x_1, x_2/\varepsilon^{\frac{1}{2}})$ which means that

$$\int_{T_\varepsilon} w_\varepsilon\, dx = \varepsilon^{\frac{3}{2}} \int_{T_1} w_1\, dx.$$

The claim follows since the right-hand side depends (continuously) only on $\mathbf{d}$. □

Proof of Lemma 7.4. We distinguish two cases as in Remark 7.2 (a) depending on the case whether or not $\mathbf{b}_1/\delta\varepsilon^{\frac{1}{2}}$, with $\mathbf{b}_1$ as in Lemma 7.3, is μ_0 close to the $s\tau$-line. If $\mathbf{b}_1/\delta\varepsilon^{\frac{1}{2}}$ is μ_0 close to this line, then we already showed in Remark 7.2 that alternative (7.10) holds. Otherwise the alternative (7.11) holds since, by Lemma 7.5 and Lemma 7.7

$$W(\mathbf{u}, \rho_0) \ge W(\mathbf{p}(\,\cdot\,, \mathbf{b})) - C\varepsilon^2 \ge W(\mathbf{p_0}) + c\varepsilon^{\frac{3}{2}}. \qquad \square$$

The proof of Theorem 7.1 follows from the following lemma.

Lemma 7.8. *Assume that* $0 \in \Gamma_{i_0}$ *and for some* $\varepsilon \le \varepsilon_0$ *small, and with* $r = 1$,

$$|\mathbf{u} - \mathbf{p}(\,\cdot\,, \mathbf{b_0}, \mathbf{b_1})| \le \varepsilon r^2 \quad \text{in } B_r, \quad \text{for some } |\mathbf{b_0}| \le \varepsilon^{\frac{1}{2}} r,\ |\mathbf{b_1}| \le \delta\varepsilon^{\frac{1}{2}}. \tag{7.16}$$

Then there exists a unit direction ν *with* $|\nu - e_2| \le C\varepsilon^{\frac{1}{2}}$ *such that*

$$\Gamma_{i_0} \subset \big\{|x \cdot \nu| \le C|x|\big(\varepsilon^{-\frac{1}{2}} + |\log|x||\big)^{-1}\big\}.$$

Proof. We prove the statement by induction depending on the number N of membranes.

We iterate Proposition 7.1 in $B_{\rho_0^m}$ as long as the hypotheses are satisfied. We want to show that

$$\Gamma_{i_0} \cap B_{\rho_0^m} \subset \big\{|x \cdot \nu| \le C\rho_0^m\big(\varepsilon^{-\frac{1}{2}} + m\big)^{-1}\big\}.$$

We distinguish two cases.

Case 1: $|\mathbf{b_0}| \geq 3C_0\varepsilon$. We apply Proposition 7.1 by keeping ε fixed through the iteration (by replacing $\frac{\varepsilon}{2}$ by ε in (7.2)). Denote by $\mathbf{b}_0^m$, $\mathbf{b}_1^m$ the corresponding vectors in $B_{\rho_0^m}$, and we stop the iteration when

$$\mathbf{b}_0^m > \varepsilon^{\frac{1}{2}}\rho_0^m.$$

By (7.3), throughout the iteration $|\mathbf{b}_0 - \mathbf{b}_0^m| \leq 2C_0\varepsilon$ (provided that ρ_0 is chosen small) hence the iteration stops when

$$r_m = \rho_0^m \sim |\mathbf{b}_0|\varepsilon^{-\frac{1}{2}} \geq \varepsilon^{\frac{1}{2}}.$$

Then we end up in the situation of alternative (b) in Lemma 7.3 with $r = r_m$. We may apply the induction hypothesis in B_r (with ε replaced by 2ε) to the problem involving fewer membranes, and reach the desired result.

Case 2: $|\mathbf{b_0}| \leq 3C_0\varepsilon$. We may replace $\mathbf{b}_0$ by 0 and ε into $C\varepsilon$. After relabeling ε we reduce to the situation $\mathbf{u} \in \mathcal{S}(1, \mathbf{p}, \varepsilon)$ of Lemma 7.3.

We iterate Lemmas 7.3 and 7.4 accordingly in $B_{\rho_0^m}$.

We discuss the estimates as long as we remain in alternative (a). By Lemma 7.4, we obtain that $\mathbf{u} \in \mathcal{S}(\rho_0^m, \mathbf{p}, \varepsilon_m)$ for a sequence ε_m, and the approximating solutions $\mathbf{v}_m := \mathbf{p}(\cdot, \mathbf{b}_1^m)$ satisfy $\|\mathbf{v}_m - \mathbf{v}_{m+1}\|_{L^\infty(B_1)} \leq C\varepsilon_m$.

Moreover, up to the last value of m, $m = m_0$ (possibly infinite) for which alternative (a2) applies, we know that

$$w_m := W(\mathbf{u}, \rho_0^m) - W(\mathbf{p}_0) \geq c\varepsilon_m^{\frac{3}{2}} \quad \text{for all } m \leq m_0,$$

hence since, by Lemma 5.1, $w_m \leq C\varepsilon_m^{\frac{3}{2}}$ we find that for some c_1, c_1' small

$$a_{m+1} \leq a_m - c_1\varepsilon_m^2 \leq a_m - c_1' a_m^{\frac{4}{3}}, \quad a_m := w_m + 2c_1\varepsilon_m^2 \geq 0.$$

This implies that $a_{m+1}^{-\frac{1}{3}} \geq a_m^{-\frac{1}{3}} + c$, hence

$$a_m \leq (a_0^{-\frac{1}{3}} + c(m-k))^{-3}, \quad m \leq m_0.$$

Using that $a_0 \sim \varepsilon^{\frac{3}{2}}$, $a_m \sim \varepsilon_m^{\frac{3}{2}}$, we find

$$\varepsilon_m \leq C(\varepsilon^{-\frac{1}{2}} + m)^{-2}.$$

This inequality remains valid if we replace m_0 by $m_1 \geq m_0$ with m_1 denoting the first value of m (possibly infinite) for which alternative (b) holds, since by (a1), the values of ε_m decay geometrically when m goes from m_0 to m_1. We find

$$\sum_k^{m_1} \varepsilon_m \leq C(\varepsilon^{-\frac{1}{2}} + k)^{-1}.$$

This implies that

$$\|\mathbf{v}_m - \mathbf{v}_k\|_{L^\infty(B_1)} \leq C(\varepsilon^{-\frac{1}{2}} + k)^{-1} \quad \text{if } k \leq m \leq m_1.$$

Then the angle between the rotation directions ν_m and ν_k of $\mathbf{v}_m$, $\mathbf{v}_k$ satisfy the same inequality, and we can use the inequality

$$\varepsilon_k^{\frac{1}{2}} \leq C(\varepsilon^{-\frac{1}{2}} + k)^{-1}$$

to deduce that

$$\Gamma_{i_0} \cap B_r \subset \left\{|x \cdot \nu_{m_1}| \le Cr\left(\varepsilon^{-\frac{1}{2}} + k\right)^{-1}\right\} \quad \text{if } r \ge \rho_0^k,\, k \le m_1.$$

By Lemma 7.3 (b), the inclusion holds also when

$$\rho_0^{m_1} \ge r \ge \varepsilon_{m_1}^{\frac{1}{2}} \rho_0^{m_1}$$

with k replaced by m_1. In the ball of radius $\varepsilon_{m_1}^{\frac{1}{2}} \rho_0^{m_1}$ we can apply the induction hypothesis to obtain that

$$\Gamma_{i_0} \cap B_r \subset \left\{|x \cdot \bar{\nu}| \le Cr\left(\varepsilon_{m_1}^{-\frac{1}{2}} + m - m_1\right)^{-1}\right\} \quad \text{if } r = \rho_0^m \le \varepsilon_{m_1}^{\frac{1}{2}} \rho_0^{m_1},$$

for some direction $\bar{\nu}$ with $|\bar{\nu} - \nu_{m_1}| \le C\varepsilon_{m_1}^{\frac{1}{2}}$. We obtain the desired conclusion with unit direction given by $\bar{\nu}$ since

$$\varepsilon_{m_1}^{\frac{1}{2}} \le C(\varepsilon^{-\frac{1}{2}} + m_1)^{-1}. \qquad \square$$

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Eingegangen 17. November 2021, in revidierter Fassung 25. Februar 2023