

ON A CONJECTURE OF BRAVERMAN-KAZHDAN

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1. INTRODUCTION

This paper is a sequel to [C]. In *loc.cit.* it was shown that a conjecture of Braverman-Kazhdan on acyclicity of ρ -Bessel sheaves on reductive groups follows from a certain vanishing conjecture. The goal of this paper is to give a proof of this vanishing conjecture. In the introduction, we would like to recall the statement of the vanishing conjecture, explain its applications to the conjectures of Braverman-Kazhdan on acyclicity of ρ -Bessel sheaves and non-linear Fourier kernels for finite reductive groups, and outline a proof of the vanishing conjecture.

1.1. The vanishing conjecture. Let k be an algebraic closure of a finite field $\mathbb{F}_q$ with q -element of characteristic $p > 0$ or $k = \mathbb{C}$. We fix a prime number ℓ different from p . We set $E = \overline{\mathbb{Q}}_\ell$ in the case $\text{char } k = p$ and $E = \mathbb{C}$ in the case $k = \mathbb{C}$. We will consider the following two geometric/sheaf-theoretic contexts: (1) ℓ -adic sheaves on schemes over k of characteristic p , (2) holonomic D -modules on schemes over $k = \mathbb{C}$. We refer to context (1) as the ℓ -adic setting and context (2) as the de Rham setting. We will fix a non-trivial character $\psi : \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Depending on the setting, let $\mathcal{L}_\psi$ be the Artin-Schreier sheaf on the additive group $\mathbb{G}_a$ associated to ψ in the ℓ -adic setting or the exponential D -module in the de Rham setting.

The vanishing conjecture proposed in [C] is a generalization of the well-known acyclicity

$$(1.1) \quad H_c^*(\mathbb{G}_a, \mathcal{L}_\psi) = 0$$

of $\mathcal{L}_\psi$ to general reductive groups. The starting point is the observation that (1.1) can be restated as acyclicity of a certain local system on SL_2 over certain affine subspaces. Namely, let $\text{tr} : \text{SL}_2 \rightarrow \mathbb{G}_a$ be the trace map and let U be the unipotent

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radical of the the standard Borel subgroup B of SL_2 . Then (1.1) is equivalent to the following acyclicity of the local system $\Phi = \mathrm{tr}^* \mathcal{L}_\psi$ over U -orbits on the open Bruhat cell: for any $x \in \mathrm{SL}_2 \setminus B$ we have

$$(1.2) \quad H_c^*(xU, i^* \Phi) = 0.$$

Here $i : xU \rightarrow \mathrm{SL}_2$ is the embedding. Indeed, it follows from the fact that for any $x \in \mathrm{SL}_2 \setminus B$, the trace map restricts to an isomorphism $\mathrm{tr} : xU \simeq \mathbb{G}_a$ between the U -orbit through x and $\mathbb{G}_a$.

To state a generalization of (1.2) to general reductive groups, let me first recall some notations and definitions. Let G be a connected reductive group over k . Let T be a maximal torus of G and B be a Borel subgroup containing T with unipotent radical U . Denote by $W = N_G(T)/T$ the Weyl group, where $N_G(T)$ is the normalizer of T in G . Depending on the setting, we denote by $\pi_1^t(T)$ the tame étale fundamental group of T if $\mathrm{char} k > 0$ or the topological fundamental group $\pi_1^t(T) = \pi_1(T(\mathbb{C}))$ of $T(\mathbb{C})$ if $k = \mathbb{C}$. We denote by $\mathcal{C}(T)(E)$ the set of continuous E -valued characters of $\pi_1^t(T)$.

For any character $\chi \in \mathcal{C}(T)(E)$, we write $\mathcal{L}_\chi$ for the corresponding rank one ℓ -adic/de Rham local system on T . The Weyl group W acts naturally on $\mathcal{C}(T)(E)$ and for any $\chi \in \mathcal{C}(T)(E)$, we denote by W'_χ the stabilizer of χ in W and $W_\chi \subset W'_\chi$, the subgroup of W'_χ generated by those reflections s_α such that the pull-back $(\tilde{\alpha})^* \mathcal{L}_\chi$ is isomorphic to the trivial local system, where $\tilde{\alpha} : \mathbb{G}_m \rightarrow T$ is the coroot associated to α .¹

Denote by $\mathcal{D}_W(T)$ the W -equivariant bounded derived category of sheaves on T . For any $\mathcal{F} \in \mathcal{D}_W(T)$ and $\chi \in \mathcal{C}(T)(E)$, the W -equivariant structure on $\mathcal{F}$ together with the natural W'_χ -equivariant structure on $\mathcal{L}_\chi$ give rise to an action of W'_χ on the cohomology groups $H_c^*(T, \mathcal{F} \otimes \mathcal{L}_\chi)$ (resp. $H^*(T, \mathcal{F} \otimes \mathcal{L}_\chi)$). In particular, we get an action of the subgroup $W_\chi \subset W'_\chi$ on the cohomology groups above. Denote by $\mathrm{sign}_W : W \rightarrow \{\pm 1\}$ the sign character of W .

The key in formulating the generalization of (1.2) to general G is the following definition of central complexes on T introduced in [C, Definition 1.1]:

Definition 1.1. A W -equivariant complex $\mathcal{F} \in \mathcal{D}_W(T)$ is called central (resp. $*$ -central) if for any $\chi \in \mathcal{C}(T)(E)$, the group W_χ acts on

$$H_c^*(T, \mathcal{F} \otimes \mathcal{L}_\chi) \quad (\text{resp. } H^*(T, \mathcal{F} \otimes \mathcal{L}_\chi))$$

via the sign character sign_W . It is called strongly central (resp. strongly $*$ -central) if the stabilizer W'_χ acts on the cohomology groups above by the sign character.

Remark 1.1. If the center of G is connected, then it is known that $W_\chi = W'_\chi$ for all $\chi \in \mathcal{C}(T)(E)$ (see, for example, [DL, Theorem 5.13]), thus the notions of central complexes and strongly central complexes are the same. In general, the two notions are different (see the example below).

Example 1.2. Consider the case $G = \mathrm{SL}_2$. The Weyl group $W \simeq \{\pm 1\}$ acts on $T \simeq \mathbb{G}_m$ by the inverse map and it is straightforward to check that $W_\chi \neq \{1\}$ if and only if χ is trivial. Thus a W -equivariant complex $\mathcal{F} \in \mathcal{D}_W(T)$ is central if and only if W acts on $H_c^*(T, \mathcal{F})$ by the sign character, equivalently, $H_c^*(T, \mathcal{F})^W = 0$. Let $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ be a character satisfying $\chi = \chi^{-1}$. The corresponding local system

¹The group W_χ plays an important role in the study of representations of finite reductive groups and character sheaves (see, e.g., [Lu]).

$\mathcal{L}_\chi$ has a canonical equivariant structure $\mathcal{L}_\chi \simeq \mathcal{L}_{\chi^{-1}} \simeq (-1)^* \mathcal{L}_\chi$. If χ is trivial then $H_c^*(T, \mathcal{L}_\chi)^W \simeq H_c^*(T, \overline{\mathbb{Q}}_\ell)^W \neq 0$, and hence $\mathcal{L}_\chi \simeq \overline{\mathbb{Q}}_\ell$ is not central. If χ is non-trivial, then $H_c^*(T, \mathcal{L}_\chi) = 0$ and hence $\mathcal{L}_\chi$ is central. Note that in the latter case we have $W_\chi = \{1\}$ and $W'_\chi = \{\pm 1\}$ and since $H_c^*(T, \mathcal{L}_\chi \otimes \mathcal{L}_\chi)^{W'_\chi} \simeq H_c^*(T, \overline{\mathbb{Q}}_\ell)^W \neq 0$ we see that $\mathcal{L}_\chi$ is not strongly central.

Recall the Lusztig induction functor $\text{Ind}_{T \subset B}^G : \mathcal{D}(T) \rightarrow \mathcal{D}(G)$ between bounded derived category of sheaves on T and G . It is given by pull-push $\text{Ind}_{T \subset B}^G := \tilde{c}_* \circ \tilde{q}^*[\dim G - \dim T]$ along the correspondence $G \xleftarrow{\tilde{c}} \tilde{G} \xrightarrow{\tilde{q}} T$, where $\tilde{G}$ is the Grothendieck-Springer simultaneous resolution and $\tilde{c}$ and $\tilde{q}$ are the natural projection maps, see (3.1).

For $\mathcal{F} \in \mathcal{D}_W(T)$, the W -equivariant structure on $\mathcal{F}$ defines a W -action on $\text{Ind}_{T \subset B}^G(\mathcal{F})$ and we denote by

$$\Phi_{\mathcal{F}} := \text{Ind}_{T \subset B}^G(\mathcal{F})^W$$

the W -invariant factor in $\text{Ind}_{T \subset B}^G(\mathcal{F})$. In [C, Conjecture 1.2], we proposed the Conjecture 1.2 on acyclicity of $\Phi_{\mathcal{F}}$ over certain affine subspaces in G , called the vanishing conjecture:

Conjecture 1.2. *Assume $\mathcal{F} \in \mathcal{D}_W(T)$ is central (resp. $*$ -central). For any $x \in G \setminus B$, we have the following cohomology vanishing*

$$(1.3) \quad H_c^*(xU, i^* \Phi_{\mathcal{F}}) = 0 \quad (\text{resp. } H^*(xU, i^! \Phi_{\mathcal{F}}) = 0),$$

where $i : xU \rightarrow G$ is the natural inclusion map. Equivalently, the derived pushforward $\pi_!(\Phi_{\mathcal{F}})$ (resp. $\pi_*(\Phi_{\mathcal{F}})$) is supported on the closed subset $T = B/U \subset G/U$. Here $\pi : G \rightarrow G/U$ is the quotient map.

Example 1.3. Assume $G = \text{SL}_2$. Consider the trace map $\text{tr} : \text{SL}_2 \rightarrow \mathbb{G}_a$ and let $\text{tr}_T : T \rightarrow \mathbb{G}_a$ be its restriction to the maximal torus. The pullback $\mathcal{F} = \text{tr}_T^* \mathcal{L}_\psi$ together with the canonical W -equivariant structure is central. Moreover, we have

$$\Phi_{\mathcal{F}} \simeq \text{tr}^* \mathcal{L}_\psi[2]$$

(see, e.g., [C, Example 1.2 and 1.7]). It follows that Conjecture 1.2 (in this case) is equivalent to (1.2), and hence is also equivalent to the acyclicity of the Artin-Schreier sheaf (1.1).

1.2. Braverman-Kazhdan conjectures. Assume $k = \overline{\mathbb{F}}_q$ and G is defined over $\mathbb{F}_q$. In [BK1, BK2], Braverman and Kazhdan associated to each representation $\rho : \check{G} \rightarrow \text{GL}(V_\rho)$ of the complex dual group, a $\overline{\mathbb{Q}}_\ell$ -valued function

$$(1.4) \quad \gamma_{G, \rho, \psi} : \text{Irr}(G(\mathbb{F}_q)) \rightarrow \overline{\mathbb{Q}}_\ell$$

on the set of irreducible representation of the finite group $G(\mathbb{F}_q)$, satisfying the following remarkable properties:

- (1) it is constant on Deligne-Lusztig packets, that is, we have $\gamma_{G, \rho, \psi}(\pi) = \gamma_{G, \rho, \psi}(\pi')$ if π and π' appear in the same Deligne-Lusztig representation $R_{T, \theta}$,
- (2) if π appears in $R_{T, \theta}$, then the value $\gamma_{G, \rho, \psi}(\pi)$ is given by a certain explicit Gauss-type sum associated to the character θ .

They called $\gamma_{G,\rho,\psi}$ the γ -function associated to ρ .

The function $\gamma_{G,\rho,\psi}$ on $\text{Irr}(G(\mathbb{F}_q))$ gives rise to a $\overline{\mathbb{Q}}_\ell$ -valued class function $\phi_{G,\rho,\psi}$ on $G(\mathbb{F}_q)$ characterized by the property that the operator

$$\mathcal{F}_\rho : \text{Func}(G(\mathbb{F}_q)) \rightarrow \text{Func}(G(\mathbb{F}_q))$$

on the space of functions on $G(\mathbb{F}_q)$ given by convolution with $\phi_{G,\rho,\psi}$ satisfies

$$\mathcal{F}_\rho(\chi_\pi) = \gamma_{G,\rho,\psi}(\pi)\chi_\pi,$$

where χ_π is the character of $\pi \in \text{Irr}(G(\mathbb{F}_q))$. In the case $G = \text{GL}_n$ and $\rho = \text{std}$ is the standard representation of $\tilde{G} = \text{GL}_n(\mathbb{C})$, the function $\phi_{G,\rho,\psi}$ is given by $\psi \circ \text{tr}$ (up to some power of q) and the operator $\mathcal{F}_\rho$ is the linear Fourier transform on the space of functions on $\text{GL}_n(\mathbb{F}_q)$ (or rather, the restriction of the linear Fourier transform on $\text{Func}(\mathfrak{gl}_n(\mathbb{F}_q))$ to the subspace $\text{Func}(\text{GL}_n(\mathbb{F}_q))$). Thus, one can view $\mathcal{F}_\rho$ as a kind of non-linear Fourier transform and $\phi_{G,\rho,\psi}$ as the corresponding Fourier kernel.

In *loc. cit.* Braverman and Kazhdan proposed a geometric construction of $\phi_{G,\rho,\psi}$ using the theory of ℓ -adic sheaves. To explain their construction, let us fix a F -stable maximal torus $T \subset G$ where $F : G \rightarrow G$ is the geometric Frobenius morphism and consider the restriction of ρ to the dual maximal torus $\tilde{T} \subset \tilde{G}$. Then there exists a collection of weights

$$\underline{\lambda} = \{\lambda_1, \dots, \lambda_r\} \subset \mathbb{X}^\bullet(\tilde{T}) := \text{Hom}(\tilde{T}, \mathbb{C}^\times)$$

such that there is an eigenspace decomposition $V_\rho = \bigoplus_{i=1}^r V_{\lambda_i}$ of V_ρ , where $\tilde{T}$ acts on V_{λ_i} via the character λ_i . One can regard $\underline{\lambda}$ as collection of co-characters of T using the canonical isomorphism $\mathbb{X}^\bullet(\tilde{T}) \simeq \mathbb{X}_\bullet(T)$ and define

$$\Phi_{T,\rho,\psi} = \text{pr}_{\underline{\lambda},!} \text{tr}^* \mathcal{L}_\psi[r] \quad \Phi_{T,\rho,\psi}^* = \text{pr}_{\underline{\lambda},*} \text{tr}^* \mathcal{L}_\psi[r],$$

where

$$\text{pr}_{\underline{\lambda}} := \prod_{i=1}^r \lambda_i : \mathbb{G}_m^r \longrightarrow T, \quad \text{tr} : \mathbb{G}_m^r \longrightarrow \mathbb{G}_a, (x_1, \dots, x_r) \mapsto \sum_{i=1}^r x_i.$$

It is shown in [BK2] that both $\Phi_{T,\rho,\psi}$ and $\Phi_{T,\rho,\psi}^*$ carry natural W -equivariant structures and the resulting objects in $\mathcal{D}_W(T)$, denoted again by $\Phi_{T,\rho,\psi}$ and $\Phi_{T,\rho,\psi}^*$, are called the ρ -Bessel sheaves.² The ρ -Bessel sheaves on G , denoted by $\Phi_{G,\rho,\psi}$ and $\Phi_{G,\rho,\psi}^*$, are defined as

$$\Phi_{G,\rho,\psi} = \text{Ind}_{T \subset B}^G (\Phi_{T,\rho,\psi})^W, \quad \Phi_{G,\rho,\psi}^* = \text{Ind}_{T \subset B}^G (\Phi_{T,\rho,\psi}^*)^W.$$

It is shown in [BK2, Theorem 4.2] and [CN, Appendix B] that, if ρ satisfies certain positivity assumption (see [BK2, Section 1.4]), then the ρ -Bessel sheaves $\Phi_{T,\rho,\psi}$ and $\Phi_{T,\rho,\psi}^*$ on T are in fact local systems on the image of $\text{pr}_{\underline{\lambda}}$, moreover, we have $\Phi_{T,\rho,\psi} \simeq \Phi_{T,\rho,\psi}^*$. This is a generalization of Deligne's theorem on Kloosterman sheaves [De1].

Braverman and Kazhdan showed that one can endow the ρ -Bessel sheaf $\Phi_{G,\phi,\psi}$ with a Weil structure $F^* \Phi_{G,\phi,\psi} \simeq \Phi_{G,\phi,\psi}$ and they proposed Conjecture 1.3:

²In [BK1, BK2], the authors called $\Phi_{T,\rho,\psi}$ γ -sheaves on T . However, based on the fact that the classical γ -function is the Mellin transform of the Bessel function, we follow [Ng] and use the term ρ -Bessel sheaves instead of γ -sheaves.

Conjecture 1.3. Let $\mathrm{Tr}(\Phi_{G,\rho,\psi}) : G(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell$ be the function corresponding to $\Phi_{G,\rho}$ via the functions-sheaves correspondence. We have

$$\mathrm{Tr}(\Phi_{G,\rho,\psi}) = \phi_{G,\rho,\psi}$$

Conjecture 1.3 gives a geometric construction of the non-linear Fourier kernel $\phi_{G,\rho,\psi}$. They also showed that Conjecture 1.3 follows from Conjecture 1.4 on acyclicity of ρ -Bessel sheaves:

Conjecture 1.4 ([BK1, Conjecture 9.12]). For any $x \in G \setminus B$, we have the following cohomology vanishing

$$H_c^*(xU, i^* \Phi_{G,\rho,\psi}) = 0 \quad (\text{resp. } H^*(xU, i^! \Phi_{G,\rho,\psi}^*) = 0),$$

where $i : xU \rightarrow G$ is the natural inclusion map. Equivalently, the derived pushforward $\pi_!(\Phi_{G,\rho,\psi})$ (resp. $\pi_*(\Phi_{G,\rho,\psi}^*)$) is supported on the closed subset $T = B/U \subset G/U$. Here $\pi : G \rightarrow G/U$ is the quotient map.

The goal of this paper is to give a proof of Conjecture 1.4, and hence Conjecture 1.3. Note that the construction of ρ -Bessel sheaves and Conjecture 1.4 are entirely geometric and have obvious counterparts in the de Rham setting. Moreover, it is shown in [C] that the ρ -Bessel sheaves on T are in fact strongly central (see Definition 1.1). Thus the vanishing conjecture contains Conjecture 1.4 as a special case and what we actually prove here is the vanishing conjecture.

Remark 1.4. Conjecture 1.3 and Conjecture 1.4 here are (slightly) generalized versions of the original conjectures of Braverman and Kazhdan. The original conjectures require that the representation ρ satisfies the positivity assumption mentioned earlier. In Corollary 1.6 and Corollary 1.7, we will prove that their conjecture holds without any assumption on ρ .

1.3. The main result. Theorem 1.5 is the main result of the paper which confirms the vanishing conjecture for almost all characteristics:

Theorem 1.5 (Theorem 7.1). *There exists a positive integer N depending only on the type of the group G such that the following holds. Assume $k = \mathbb{C}$ or $\mathrm{char} k = p$ is not dividing $N\ell$. Let $\mathcal{F} \in \mathcal{D}_W(T)$ be a central complex (resp. $*$ -central complex) on T and let $\Phi_{\mathcal{F}} = \mathrm{Ind}_{T \subset B}^G(\mathcal{F})^W \in \mathcal{D}(G)$. Then for any $x \in G \setminus B$, we have the following cohomology vanishing*

$$(1.5) \quad H_c^*(xU, i^* \Phi_{\mathcal{F}}) = 0 \quad (\text{resp. } H^*(xU, i^! \Phi_{\mathcal{F}}) = 0).$$

Here $i : xU \rightarrow G$ is the embedding. Equivalently, the derived pushforward $\pi_!(\Phi_{\mathcal{F}})$ (resp. $\pi_*(\Phi_{\mathcal{F}})$) is supported on the closed subset $T = B/U \subset G/U$. Here $\pi : G \rightarrow G/U$ is the quotient map.

Remark 1.5. The assumption on the characteristic of k comes from a spreading out argument used in the proof (see Section 1.5).

Remark 1.6. In [C], we proved the vanishing conjecture in the case $G = \mathrm{GL}_n$ using mirabolic subgroups. The argument in *loc. cit.* was inspired by the work of Cheng and Ngô [CN] on Braverman-Kazhdan conjectures for $G = \mathrm{GL}_n$. The proof of Theorem 1.5 for general G uses different methods (see Section 1.5).

1.4. Applications. In this subsection we assume the characteristic of k is either zero or not dividing $N\ell$, where N is the positive integer in Theorem 1.5.

Corollary 1.6. *Conjecture 1.4 holds.*

Proof. It was shown in [C, Theorem 1.4] that Braverman-Kazhdan's ρ -Bessel sheaf $\Phi_{T,\rho,\psi}$ (resp. $\Phi_{T,\rho,\psi}^*$) on T is strongly central (resp. strongly $*$ -central). Thus Theorem 1.5 immediately implies the corollary. $\square$

Corollary 1.7. *Conjecture 1.3 holds.*

Proof. It was shown in [BK2, Corollary 6.7] that Conjecture 1.4 implies Conjecture 1.3. Thus Corollary 1.6 implies Corollary 1.7. $\square$

Conjecture 1.4 was proved by Braverman and Kazhdan [BK2, Theorem 6.9] in the case when the semi-simple rank of G is less or equal to one, and by Cheng and Ngô [CN, Theorem 2.4] in the case $G = \mathrm{GL}_n$.

Conjecture 1.3 was proved by Braverman and Kazhdan [BK2, Theorem 1.6] when the semi-simple rank of G is less or equal to one or $G = \mathrm{GL}_n$ under some assumption on ρ . In a recent work, G. Laumon and E. Letellier established Conjecture 1.3 via a different method [LL, Theorem 1.0.2]. It is interesting to note that in *loc.cit.* they also made no assumption on the representation ρ .

Remark 1.7. In [LL, Theorem 1.0.1], Laumon-Letellier also proved a formula for the non-linear Fourier kernel $\phi_{G,\rho,\psi}$ in terms of Deligne-Lusztig inductions. It will be interesting to prove a similar result in the de Rham setting, that is, write down an explicit formula for the ρ -Bessel D -module $\Phi_{G,\rho,\psi}$, or rather, the corresponding system of differential equations on G . We expect applications of such a formulation to the Braverman-Kazhdan-Ngô's approach to functional equation of automorphic L -functions [BK1, Ng].

1.5. Outline of the proof. The proof of Theorem 1.5 consists of three steps:

Step 1. We construct for each W -orbit θ in $\mathcal{C}(T)(E)$ a remarkable W -equivariant local system $\mathcal{E}_\theta$ on T and study the equivariant perverse sheaf or D -module $\mathcal{M}_\theta = \mathrm{Ind}_{T \subset B}^G(\mathcal{E}_\theta)^W$ on G .³ A key observation is that to prove Theorem 1.5 it suffices to prove the acyclicity of $\mathcal{M}_\theta$, that is, the cohomology vanishing properties in (1.5) where $\Phi_{\mathcal{F}}$ is replaced by $\mathcal{M}_\theta$. This follows from a computation of the convolution of $\mathcal{M}_\theta$ with $\Phi_{\mathcal{F}}$ (Proposition 7.5), where $\mathcal{F} \in \mathcal{D}_W(T)$ is a $*$ -central complex, and a result of Laumon on the conservativity of the Mellin transform (Lemma 7.7).

Step 2. We use the techniques developed in [BG, BFO] on Drinfeld center of Harish-Chandra bimodules and character D -modules to prove acyclicity of $\mathcal{M}_\theta$ in the de Rham setting. An important point here is that the D -module $\mathcal{M}_\theta$ associated to $\mathcal{E}_\theta$ is a character D -module and hence the results in *loc. cit.* are applicable. A key step in the proof is to show that the Harish-Chandra bimodule corresponding to $\mathcal{E}_\theta$, under the Beilinson-Bernstein localization theorem, has a canonical central structure (Proposition 6.7).

Step 3. We construct a mixed characteristic lifting $\mathcal{M}_{\theta,A}$ of $\mathcal{M}_\theta$ over a strictly henselian discrete valuation ring A with residue field k of characteristic not dividing $N\ell$, where N is a positive integer depending only on the type of G . We prove that

³The author learned the existence of $\mathcal{E}_\theta$ from R. Bezrukavnikov.

$\mathcal{M}_{\theta,A}$ is universally locally acyclic with respect to the quotient map $\pi_A : G_A \rightarrow G_A/U_A$ (here G_A and U_A are models of G and U over A). This allows us to deduce acyclicity of $\mathcal{M}_{\theta}$ in the ℓ -adic setting from the de Rham setting. This completes the proof of Theorem 1.5.

Remark 1.8. The proof of acyclicity of $\mathcal{M}_{\theta}$ in the de Rham setting makes use of Harish-Chandra bimodules, and hence is algebraic. It would be interesting to have a geometric proof which treats the cases of various ground fields and sheaf theories uniformly. Presumably, such a proof will provide an explicit bound on the integer N in Theorem 1.5.

1.6. Related work and further directions. We briefly explain the relationship between the results of the paper and the work of Ginzburg [Gi2], Lonergan [Lo], and Ben-Zvi-Gunningham [BZG] on Whittaker D -modules, Toda lattice, and nil-Hecke algebras, and discuss some generalizations of the results of the paper.

We first recall the definition of Whittaker sheaves on G . Fix a non-degenerate homomorphism $\chi : U \rightarrow \mathbb{G}_a$, that is, the restriction of χ to each root subgroup $U_{\alpha} \subset U$ is non-trivial for each simple root α . Depending on the setting, a Whittaker sheaf on G is a perverse sheaf or a holonomic D -module $\mathcal{F}$ on G together with an isomorphism $a^{\circ}\mathcal{F} \simeq \chi^{\circ}\mathcal{L}_{\psi} \boxtimes \chi^{\circ}\mathcal{L}_{\psi} \boxtimes \mathcal{F}$ satisfying the usual cocycle condition. Here $a : U \times U \times G \rightarrow G$, $a(u_1, u_2, g) = u_1 g u_2^{-1}$. In the de Rham setting, Ginzburg and Lonergan proved that the category of Whittaker D -modules on G is equivalent to the category of holonomic modules over the quantum Toda lattice of G , and is also equivalent to the category of holonomic modules over nil-Hecke algebra of G , see [Gi2, Theorem 1.5.1] and [Lo, Theorem 1.2.2]. It turns out that the results in *loc. cit.* and Theorem 4.2 of the present paper imply that the category of holonomic modules over nil-Hecke algebra of G is equivalent to the category of central D -modules on T . As a corollary, we obtain:

Theorem 1.8. *The category of Whittaker D -modules on G is equivalent to the category of central D -modules on T .*

On the other hand, in the work [BZG], Ben-Zvi and Gunningham constructed a functor from the category of Whittaker D -modules on G to the category of G -conjugation equivariant D -module on G . They called it the Ngô functor and they conjectured that the objects in the essential image of the Ngô functor satisfy the cohomology vanishing property in (1.3), see [BZG, Conjecture 2.9 and 2.14]. We propose the following ℓ -adic counterpart of the results and conjectures discussed above:

Conjecture 1.9. *The following categories are equivalent:*

- (1) *The category of ℓ -adic Whittaker sheaves on G .*
- (2) *The category of ℓ -adic central perverse sheaves on T .*
- (3) *The full subcategory of the category G -conjugation equivariant ℓ -adic perverse sheaves on G whose object Φ satisfy $H_c^*(xU, i^*\Phi) = 0$ for all $x \in G \setminus B$, or equivalently, $\pi_!(\Phi)$ is supported on the closed subset $T = B/U \subset G/U$. Here $\pi : G \rightarrow G/U$ is the quotient map.*

Remark 1.9. We expect the equivalence from (2) to (3) is given by the functor sending a central perverse sheaf $\mathcal{F}$ on T to $\Phi_{\mathcal{F}} = \text{Ind}_{T \subset B}^G(\mathcal{F})^W$. Note that this functor is well-defined thanks to Theorem 1.5.

Remark 1.10. It was mentioned in the introduction of [Gi2] that Drinfeld asked the question of finding a description of an ℓ -adic counterpart of the category of Whittaker D -modules on G in terms of W -equivariant sheaves on T . The conjecture above provides a possible answer to Drinfeld's question.

Remark 1.11. The conjecture above can be viewed as a finite group analogue of the conjectural descriptions of the endomorphism ring of the Whittaker representation of a (quasi-split) p -adic group G in terms of, on one hand, the ring of stable Bernstein center of G (viewed as stable distributions on the group) and, on the other hand, the ring of functions on the moduli stack of Langlands parameters (see, e.g., [FS, Example X.1.6] and [Z2, Conjecture 4.5.1]).

1.7. Organization. We briefly summarize here the main goals of each section. In Section 2 we collect standard notation on algebraic groups, ℓ -adic sheaves, and D -modules. In Section 3 we study induction and restriction functors. In Section 4, we study central and $*$ -central complexes using the Mellin transforms. In Section 5 we study the local systems $\mathcal{E}_\theta$. In Section 6 we establish acyclicity of the character sheaves $\mathcal{M}_\theta$, see Theorem 6.1. In Section 7 we prove the main result of the paper, Theorem 1.5.

2. NOTATIONS

2.1. We denote by $\mathcal{B} = G/B$ the flag variety. We denote by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathfrak{n}$ the Lie algebras of G, B, T, U . We denote by G^{rs} (resp. T^{rs}) the open subset consisting of regular semi-simple elements in G (resp. T). We denote by G^{reg} the open subset consisting of regular elements in G . We denote by $\mathbb{G}_a$ the additive group and $\mathbb{G}_m$ the multiplicative group. We denote by $\check{G}$ the complex dual group of G and $\check{T}$ the dual maximal torus. We denote by $W_a^{\text{ex}} = W \ltimes \Lambda$ the extended affine Weyl group and $W_a = W \ltimes R$ the affine Weyl group of $\check{G}$. Here $\Lambda = \mathbb{X}_\bullet(\check{T}) = \text{Hom}(\mathbb{G}_m, \check{T})$ is the co-character lattice and $R \subset \Lambda$ is the set of co-roots of $\check{G}$.

2.2. For an algebraic stack $\mathcal{X}$ over k , we denote by $\mathcal{D}(\mathcal{X})$ the bounded derived category of ℓ -adic sheaves on $\mathcal{X}$ in the ℓ -adic setting or the bounded derived category of holonomic D -modules on $\mathcal{X}$ in the de Rham setting.

For a smooth scheme X , we will write $1_X \in \mathcal{D}(X)^\heartsuit$ for the constant perverse sheaf $\overline{\mathbb{Q}}_\ell[\dim X]$ on X in the ℓ -adic setting or the structure sheaf $\mathcal{O}_X$ in the de Rham setting.

For a representable morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$, the six functors $f^*, f_*, f_!, g^!, \otimes, \underline{\text{Hom}}$ are understood in the derived sense. For a smooth map $f : \mathcal{X} \rightarrow \mathcal{Y}$ of relative dimension d we write $f^\circ = f^*[d] = f^![-d]$.

For an algebraic group H over k acting on a k -scheme X , we denote by X/H the corresponding quotient stack and $X // H$ the geometric invariant quotient (if exists). We will write $H/\text{ad} H$ for the quotient stack of H with respect to the adjoint action. Consider the case when H is a finite group. Then the pull-back along the quotient map $X \rightarrow X/H$ induces an equivalence between $\mathcal{D}(X/H)$ and the (naive) H -equivariant derived category on X , denoted by $\mathcal{D}_H(X)$, whose objects consist of pair $(\mathcal{F}, \phi)$, where $\mathcal{F} \in \mathcal{D}(X)$ and $\phi : a^*\mathcal{F} \simeq \text{pr}^*\mathcal{F}$ is an isomorphism satisfying the usual compatibility conditions (here a and pr are the action and projection map from $H \times X$ to X respectively).⁴ We will call an object $(\mathcal{F}, \phi)$ in $\mathcal{D}_H(X)$ a

⁴This holds in a more general situation when the neutral component of H is unipotent.

H -equivariant complex and ϕ a H -equivariant structure on $\mathcal{F}$. For simplicity, we will write $\mathcal{F} = (\mathcal{F}, \phi)$ for an object in $\mathcal{D}_H(X)$.

We denote by $\tau_{\leq n}, \tau_{\geq n}$ the truncation functors corresponding to the standard t -structure on $\mathcal{D}(\mathcal{X})$. For any $\mathcal{F} \in \mathcal{D}(\mathcal{X})$, we denote by $\mathcal{H}^n(\mathcal{F})$ the n -th cohomology sheaf. In the ℓ -adic setting, we denote by ${}^p\tau_{\leq n}, {}^p\tau_{\geq n}$ the truncation functors corresponding to the perverse t -structure. For any $\mathcal{F} \in \mathcal{D}(\mathcal{X})$, the n -th perverse cohomology sheaf is defined as ${}^p\mathcal{H}^n(\mathcal{F}) = {}^p\tau_{\geq n} {}^p\tau_{\leq n}(\mathcal{F})[n]$.

Depending on the setting, we write $\mathcal{D}(\mathcal{X})^{\heartsuit}$ for the heart corresponding to the perverse t -structure in the ℓ -adic setting and the heart corresponding to the standard t -structure in the de Rham setting.

For any stack $\mathcal{X}$ over k , we denote by $\text{Coh}(\mathcal{X})$ and $\text{QCoh}(\mathcal{X})$ the categories of coherent and quasi-coherent sheaves on $\mathcal{X}$, and $D_{\text{coh}}^b(\mathcal{X})$ and $D_{\text{qcoh}}^b(\mathcal{X})$ the corresponding bounded derived categories.

Let $\mathcal{F}$ be a quasi-coherent sheaf or D -module on a scheme. We will write $\Gamma(\mathcal{F})$ and $R\Gamma(\mathcal{F})$ for the global section and derived global section of $\mathcal{F}$ as an quasi-coherent sheaf. For any scheme X we will write $\mathcal{O}_X$ for the structure sheaf of X and $\mathcal{O}(X) = \Gamma(\mathcal{O}_X)$ the ring of global functions on X .

Assume $k = \mathbb{C}$. For any smooth scheme X we denote by $\mathcal{D}_X$ the sheaf of differential operators on X . Let $f : X \rightarrow Y$ be a principal T -bundle over a smooth scheme Y . A D -module $\mathcal{F}$ on X is called T -monodromic if it is weakly T -equivariant (see [BB, Section 2.5]). A object $\mathcal{F} \in \mathcal{D}(X)$ is called T -monodromic if $\mathcal{H}^i(\mathcal{F})$ is T -monodromic for all i . Let $\mathcal{F} \in \mathcal{D}(X)$ be a T -monodromic object. For any $\mu \in \mathfrak{t}^* \simeq \mathfrak{t}^*$, we denote $\Gamma^{\hat{\mu}}(\mathcal{F})$ (resp. $R\Gamma^{\hat{\mu}}(\mathcal{F})$) the maximal summand of $\Gamma(\mathcal{F})$ (resp. $R\Gamma(\mathcal{F})$) where $\mathfrak{t}$, acting as infinitesimal translations along the action of T , acts with the generalized eigenvalue μ .

3. INDUCTION AND RESTRICTION FUNCTORS

In this section we collect some known facts about induction and restriction functors.

3.1. Recall the Grothendieck-Springer simultaneous resolution of the Steinberg map $c : G \rightarrow T // W$:

$$(3.1) \quad \begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{q}} & T \\ \downarrow \tilde{c} & & \downarrow q \\ G & \xrightarrow{c} & T // W \end{array}$$

where $\tilde{G}$ is the closed subvariety of $G \times G/B$ consisting of pairs (g, xB) such that $x^{-1}gx \in B$. The map $\tilde{c}$ is given by $(g, xB) \rightarrow g$, and the map $\tilde{q}$ is given by $(g, xB) \rightarrow x^{-1}gx \bmod U \in B/U = T$. The induction functor $\text{Ind}_{T \subset B}^G : \mathcal{D}(T) \rightarrow \mathcal{D}(G)$ is given by

$$\text{Ind}_{T \subset B}^G(\mathcal{F}) = \tilde{c}_* \tilde{q}^*(\mathcal{F}).$$

We have the following equivalent constructions of $\text{Ind}_{T \subset B}^G$. Consider the fiber product

$$Z = G \times_{T // W} T.$$

It is known that the map $h : \tilde{G} \rightarrow Z$ induced from (3.1) is a small map and it follows that $\mathrm{IC}(Z) = h_! 1_S \in \mathcal{D}(Z)^\vee$ is the IC-complex of Z . We have

$$(3.2) \quad \mathrm{Ind}_{T \subset B}^G(\mathcal{F}) \simeq (p_G)_*(p_T^*(\mathcal{F}) \otimes \mathrm{IC}(Z))[\dim G - \dim T],$$

where $p_T : Z \rightarrow T$ and $p_G : Z \rightarrow G$ are the natural projection map. Let X be a scheme acted on by an algebraic group H and let $H' \subset H$ be a subgroup. There is an averaging functor $\mathrm{Av}_*^{H/H'} = \pi_* : \mathcal{D}(X/H') \rightarrow \mathcal{D}(X/H)$ (resp. $\mathrm{Av}_!^{H/H'} = \pi_! : \mathcal{D}(X/H') \rightarrow \mathcal{D}(X/H)$) which is the right adjoint (resp. left adjoint) of the forgetful functor $\mathrm{oblv}^{H/H'} : \mathcal{D}(X/H) \rightarrow \mathcal{D}(X/H')$. Here $\pi : X/H' \rightarrow X/H$ is the natural map. If $H' = e$ is trivial we will omit H' and simply write $\mathrm{Av}_*^H = \mathrm{Av}_*^{H/e}$ (resp. $\mathrm{Av}_!^H = \mathrm{Av}_!^{H/e}$) and $\mathrm{oblv}^H = \mathrm{oblv}^{H/e}$.

Note that, for any $\mathcal{F} \in \mathcal{D}(T)$, its $*$ -pull back along $B \rightarrow T = B/U$, denoted by $\mathcal{F}_B$, can be regarded as an object in $\mathcal{D}(G/\mathrm{ad} B)$ and there is a canonical isomorphism

$$\mathrm{Ind}_{T \subset B}^G(\mathcal{F}) \simeq \mathrm{oblv}^{G/e} \circ \mathrm{Av}_*^{G/B}(\mathcal{F}_B).$$

The functor $\mathrm{Ind}_{T \subset B}^G$ admits a right adjoint $\mathrm{Res}_{T \subset B}^G : \mathcal{D}(G) \rightarrow \mathcal{D}(T)$, called the restriction functor, and is given by

$$\mathrm{Res}_{T \subset B}^G(\mathcal{F}) = (q_B)_* i_B^!(\mathcal{F}),$$

where $i_B : B \rightarrow G$ is the natural inclusion and $q_B : B \rightarrow T = B/U$ is the quotient map. More generally, one could define $\mathrm{Res}_{L \subset P}^G : \mathcal{D}(G) \rightarrow \mathcal{D}(L)$, for any pair (L, P) where L is a Levi subgroup of a parabolic subgroup P of G .

We have the following exactness properties of induction and restriction functors:

Proposition 3.1. (1) *The functor $\mathrm{Ind}_{T \subset B}^G$ maps perverse sheaves on T to perverse sheaves on G .* (2) *The functor $\mathrm{Res}_{L \subset P}^G$ maps G -conjugation equivariant perverse sheaves on G to L -conjugation equivariant perverse sheaves on L .*

Proof. This is [BY, Theorem 5.4]. $\square$

3.2. W-action. Let $\mathcal{F} \in \mathcal{D}_W(T)$. Since the map $p_T : Z \rightarrow T$ and the IC-complex $\mathrm{IC}(Z)$ are W -equivariant, it follows from (3.2) that the W -equivariant structure on $\mathcal{F}$ gives rise to a W -action on $\mathrm{Ind}_{T \subset B}^G(\mathcal{F})$. We denote by

$$(3.3) \quad \Phi_{\mathcal{F}} := \mathrm{Ind}_{T \subset B}^G(\mathcal{F})^W$$

the W -invariant factor of $\mathrm{Ind}_{T \subset B}^G(\mathcal{F})$.

In the case when $\mathcal{F}$ is a W -equivariant perverse local system on T , we have the following description of $\Phi_{\mathcal{F}}$: Let $q^{\mathrm{rs}} : T^{\mathrm{rs}} \rightarrow T^{\mathrm{rs}} // W$ and $c^{\mathrm{rs}} : G^{\mathrm{rs}} \rightarrow T^{\mathrm{rs}} // W$ be the restriction of the maps in (3.1) to the regular semi-simple locus and let $j : G^{\mathrm{rs}} \rightarrow G$ be the open embedding. Since q^{rs} is an étale covering and $\tilde{c} : \tilde{G} \rightarrow G$ is a small map, the restriction of $\mathcal{F}$ to T^{rs} descends to a perverse local system $\mathcal{F}'$ on $T^{\mathrm{rs}} // W$ and

$$\mathrm{Ind}_{T \subset B}^G(\mathcal{F}) \simeq j_{!*} j^* \mathrm{Ind}_{T \subset B}^G(\mathcal{F})$$

is isomorphic to the intermediate extension of its restriction to G^{rs} . As a direct factor of $\mathrm{Ind}_{T \subset B}^G(\mathcal{F})$, $\Phi_{\mathcal{F}}$ is also isomorphic to the intermediate extension of its restriction to G^{rs} and it follows that

$$(3.4) \quad \Phi_{\mathcal{F}} \simeq j_{!*} j^* \Phi_{\mathcal{F}} \simeq j_{!*} (c^{\mathrm{rs}})^{\circ}(\mathcal{F}').$$

Let $\mathcal{F} \in \mathcal{D}_W(T)^\vee$ and let $E[W] \rightarrow \text{End}_{\mathcal{D}(G)^\vee}(\text{Ind}_{T \subset B}^G(\mathcal{F}))$ be the map coming from the W -action. By adjunction, we get a map

$$E[W] \rightarrow \text{End}_{\mathcal{D}(G)^\vee}(\text{Ind}_{T \subset B}^G(\mathcal{F})) \simeq \text{Hom}_{\mathcal{D}(G)^\vee}(\mathcal{F}, \text{Res}_{T \subset B}^G \circ \text{Ind}_{T \subset B}^G(\mathcal{F})),$$

which gives rise to

$$(3.5) \quad E[W] \otimes \mathcal{F} \rightarrow \text{Res}_{T \subset B}^G \circ \text{Ind}_{T \subset B}^G(\mathcal{F}).$$

We have the following generalization of [Gu, Theorem 4.6] to the group setting:

Proposition 3.2. *Let $\mathcal{F} \in \mathcal{D}(T)^\vee$. (1) There is a canonical isomorphism $\bigoplus_{w \in W} w^* \mathcal{F} \simeq \text{Res}_{T \subset B}^G \circ \text{Ind}_{T \subset B}^G(\mathcal{F})$. (2) Assume further that $\mathcal{F} \in \mathcal{D}_W(T)^\vee$. Then the composition*

$$(3.6) \quad E[W] \otimes \mathcal{F} \xrightarrow{\phi} \bigoplus_{w \in W} w^* \mathcal{F} \simeq \text{Res}_{T \subset B}^G \circ \text{Ind}_{T \subset B}^G(\mathcal{F})$$

is equal to (3.5). Here ϕ is the W -equivariant structure of $\mathcal{F}$.

Proof. We follow the argument in *loc. cit.*. We shall give a proof in the de Rham setting. The same proof works for the ℓ -adic setting. Consider the product $S_G = \tilde{G} \times_G B$. There are two natural maps $q_S : S_G \rightarrow T$ and $c_S : S_G \rightarrow T$ coming from $\tilde{q} : \tilde{G} \rightarrow T$ and $\tilde{c} : \tilde{G} \rightarrow T$ in (3.1). Let $\mathcal{S}_G := \mathcal{H}^0(q_S \times c_S)_*(\omega_{S_G})$. Using base changes formulas, it is easy to see that the functor $\text{Res}_{T \subset B}^G \circ \text{Ind}_{T \subset B}^G(-)$ is given by the kernel $\mathcal{S}_G$, that is, we have, $\text{Res}_{T \subset B}^G \circ \text{Ind}_{T \subset B}^G(\mathcal{F}) \simeq \text{pr}_{l,*}(\text{pr}_r^* \mathcal{F} \otimes \mathcal{S}_G)$, here $\text{pr}_l : \text{pr}_r : T \times T \rightarrow T$ are the left and right projection maps. Now the same proof as in [Gu, Section 4.3], replacing $\mathfrak{t}$ by T , shows that there is a canonical isomorphism of monads $\mathcal{S}_G \simeq \bigoplus_{w \in W} \mathcal{O}_{\Gamma_w}$, here $\Gamma_w = \{x, wx | x \in T\} \subset T \times T$. It follows that $\text{Res}_{T \subset B}^G \circ \text{Ind}_{T \subset B}^G(\mathcal{F}) \simeq \text{pr}_{l,*}(\text{pr}_r^* \mathcal{F} \otimes \mathcal{S}_G) \simeq \bigoplus_{w \in W} w^* \mathcal{F}$. This completes the proof of (1). Assume $\mathcal{F}$ is W -equivariant. It follows from the construction that the isomorphism in (1) intertwines the W -action on $\text{Res}_{T \subset B}^G \circ \text{Ind}_{T \subset B}^G(\mathcal{F})$ with the one on $\bigoplus_{w \in W} w^* \mathcal{F}$ given by the map

$$a_r : \bigoplus_{w \in W} w^* \mathcal{F} \xrightarrow{w^*(\phi_r)} \bigoplus_{w \in W} w^* r^* \mathcal{F} \simeq \bigoplus_{w \in W} (rw)^* \mathcal{F} = \bigoplus_{w \in W} w^* \mathcal{F},$$

for any $r \in W$. Part (2) follows. $\square$

We will need the following properties of induction functors. Let $m_G : G \times G \rightarrow G$ and $m_T : T \times T \rightarrow T$ be the multiplication maps. For any $\mathcal{M}, \mathcal{M}' \in \mathcal{D}(G)$, we define $\mathcal{M} * \mathcal{M}' := m_{G,*}(\mathcal{M} \boxtimes \mathcal{M}') \in \mathcal{D}(G)$. Similarly, for any $\mathcal{F}, \mathcal{F}' \in \mathcal{D}(T)$, we define $\mathcal{F} * \mathcal{F}' := m_{T,*}(\mathcal{F} \boxtimes \mathcal{F}')$.

Proposition 3.3. *Let $\mathcal{M} \in \mathcal{D}(G/\text{ad}G)^\vee$ and $\mathcal{F} \in \mathcal{D}(T)$. Assume $\text{Av}_*^U(\mathcal{M})$ is supported on $T = B/U \subset G/U$. Then we have a natural isomorphism in $\mathcal{D}(G)$*

$$\text{Ind}_{T \subset B}^G(\mathcal{F}) * \mathcal{M} \simeq \text{Ind}_{T \subset B}^G(\mathcal{F} * \text{Res}_{T \subset B}^G \mathcal{M})$$

which is functorial with respect to $\mathcal{F}$.

Proof. This is proved in [BK2, Proposition 2.9]. Let us recall the construction in *loc. cit.*. For any $\mathcal{H} \in \mathcal{D}(G/\text{ad}B)$ and $\mathcal{M} \in \mathcal{D}(G/\text{ad}G)^\vee$, there is a natural isomorphism

$$\text{Av}_*^{G/B}(\mathcal{H} * \text{oblv}^{G/B} \mathcal{M}) \simeq \text{Av}_*^{G/B}(\mathcal{H}) * \mathcal{M}.$$

When $\mathcal{H} = \mathcal{F}_B = q_B^* \mathcal{F}$ is the pull back of $\mathcal{F}$ along $q_B : B \rightarrow T = B/U$, the assumption that $\mathrm{Av}_*^U(\mathcal{M})$ is supported on T implies that

$$\mathcal{F}_B * \mathrm{oblv}^{G/B} \mathcal{M} \simeq (\mathcal{F} * \mathrm{Res}_{T \subset B}^G(\mathcal{M}))_B \in \mathcal{D}(G/\mathrm{ad} B)$$

and it follows that

$$\begin{aligned} \mathrm{Ind}_{T \subset B}^G(\mathcal{F}) * \mathcal{M} &\simeq \mathrm{oblv}^{G/e} \circ (\mathrm{Av}_*^{G/B}(\mathcal{F}_B) * \mathcal{M}) \\ &\simeq \mathrm{oblv}^{G/e} \circ (\mathrm{Av}_*^{G/B}(\mathcal{F}_B * \mathrm{oblv}^{G/B} \mathcal{M})) \\ &\simeq \mathrm{oblv}^{G/e} \circ (\mathrm{Av}_*^{G/B}((\mathcal{F} * \mathrm{Res}_{T \subset B}^G(\mathcal{M}))_B)) \\ &\simeq \mathrm{Ind}_{T \subset B}^G(\mathcal{F} * \mathrm{Res}_{T \subset B}^G(\mathcal{M})). \end{aligned}$$

□

4. MELLIN TRANSFORM, CHARACTERIZATION, AND FUNCTORIALITY

4.1. The scheme of tame characters. We review the construction of the scheme $\mathcal{C}(T)$ parametrizing tame characters of the fundamental group of T .

We first consider the de Rham setting. Let $\mathbb{C}[\pi_1(T(\mathbb{C}))]$ be the group algebra of $\pi_1(T(\mathbb{C}))$ and we set $\mathcal{C}(T) = \mathrm{Spec}(\mathbb{C}[\pi_1(T(\mathbb{C}))])$. Then the $\mathbb{C}$ -points of $\mathcal{C}(T)$ are in bijection with characters of $\pi_1(T(\mathbb{C}))$. Note that, under the isomorphism $\pi_1(T(\mathbb{C})) = \mathbb{X}_\bullet(T)$, characters of $\pi_1(T(\mathbb{C}))$ correspond to elements in the dual torus $\check{T}$.

In the ℓ -adic setting, in [GL], a $\overline{\mathbb{Q}}_\ell$ -scheme $\mathcal{C}(T)$ is defined, whose $\overline{\mathbb{Q}}_\ell$ -points are in bijection with continuous characters of $\pi_1^t(T)$. There is a decomposition

$$\mathcal{C}(T) = \bigsqcup_{\chi \in \mathcal{C}(T)_f} \{\chi\} \times \mathcal{C}(T)_\ell$$

into connected components, where $\mathcal{C}(T)_f \subset \mathcal{C}(T)$ is the subset consisting of tame characters of order prime to ℓ and $\mathcal{C}(T)_\ell$ is the connected component of $\mathcal{C}(T)$ containing the trivial character. It is shown in *loc. cit.* that $\mathcal{C}(T)$ is Noetherian and regular and there is an isomorphism

$$(4.1) \quad \mathcal{C}(T)_\ell \simeq \mathrm{Spec}(\mathbb{Z}_\ell[[\pi_1(T)_\ell]] \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell),$$

where $\mathbb{Z}_\ell[[\pi_1(T)_\ell]]$ is the completed group algebra of the pro- ℓ quotient $\pi_1(T)_\ell$ of $\pi_1^t(T)$. In addition, the $\overline{\mathbb{Q}}_\ell$ -points of $\mathcal{C}(T)_\ell$ are in bijection with characters of $\pi_1(T)_\ell$.

For any $\chi \in \mathcal{C}(T)$, we denote by $\mathcal{L}_\chi$ the corresponding rank one ℓ -adic/de Rham local system on T .

4.2. Mellin transform. We give a review of Mellin transform in both de Rham and ℓ -adic settings and establish some basic facts about them.

4.2.1. We first recall the Mellin transform of D -modules on T . Let $x_i \in \Lambda = \mathbb{X}_\bullet(\check{T}) \simeq \mathrm{Hom}(T, \mathbb{G}_m)$ be a basis and consider the regular function $\mathcal{O}(T) \simeq \mathbb{C}[x_i^{\pm 1}]$ and the algebra of differential operators $\Gamma(\mathcal{D}_T) \simeq \mathbb{C}[x_i^{\pm 1}]\langle v_i \rangle / \{v_i x_j = x_j(\delta_{ij} + v_i)\}$ where $v_i = x_i \partial_{x_i} \in \mathfrak{t}$ are a basis for the T -invariant vector fields. Recall that for any D -module $\mathcal{F}$ on T , the tensor product $\mathcal{F} \otimes \omega_T$ with the canonical line bundle ω_T on T carries a natural right D -module structure. Note also that, if we consider $\Gamma(\mathcal{D}_T)$ as the algebra of difference operators $\mathbb{C}[v_i]\langle x_i^{\pm 1} \rangle / \{v_i x_j = x_j(\delta_{ij} + v_i)\}$, the assignment $N \rightarrow \mathcal{O}_{\mathfrak{t}} \otimes_{\mathcal{O}(\mathfrak{i})} N$ defines a canonical equivalence between the category of

right $\Gamma(\mathcal{D}_T)$ -modules and the category $\mathrm{QCoh}(\check{\mathfrak{t}}/\Lambda)$ of Λ -equivariant quasi-coherent sheaves on $\check{\mathfrak{t}}$.⁵ The Mellin transform functor is defined as

$$\mathfrak{M} : D\text{-mod}(T) \rightarrow \mathrm{QCoh}(\check{\mathfrak{t}}/\Lambda), \quad \mathcal{N} \rightarrow \mathcal{O}_{\check{\mathfrak{t}}} \otimes_{\mathcal{O}(\check{\mathfrak{t}})} \Gamma(\mathcal{N} \otimes \omega_T).$$

We have the following properties:

- (1) The functor $\mathfrak{M}$ is an equivalence.
- (2) Let $\chi \in \check{T}(\mathbb{C}) \simeq \mathcal{C}(T)(\mathbb{C})$ and let $\lambda \in \check{\mathfrak{t}}(\mathbb{C})$ be a lift of χ along the universal covering $\exp : \check{\mathfrak{t}} \rightarrow \check{T}$. Then for any $\mathcal{F} \in D\text{-mod}(T)$ we have

$$(4.2) \quad H^*(T, \mathcal{F} \otimes \mathcal{L}_\chi) \simeq i_\lambda^* \mathfrak{M}(\mathcal{F})$$

here $i_\lambda : \mathrm{pt} \rightarrow \check{\mathfrak{t}}$ is the embedding given by λ .

- (3) Consider the bounded derived category $D_{qcoh}^b(\check{\mathfrak{t}}/\Lambda)$ of Λ -equivariant quasi-coherent sheaves on $\check{\mathfrak{t}}$ with the monoidal structure given by the (derived) tensor product. We have

$$\mathfrak{M}(\mathcal{F} * \mathcal{F}') \simeq \mathfrak{M}(\mathcal{F}) \otimes_{\mathcal{O}_{\check{\mathfrak{t}}}} \mathfrak{M}(\mathcal{F}').$$

- (4) Let $\mathcal{F}$ be a W -equivariant D -module $\mathcal{F}$ on T . Then W -equivariant structure on $\mathcal{F}$ gives rise to a $W_a^{\mathrm{ex}} = W \rtimes \Lambda$ -equivariant structure on $\mathfrak{M}(\mathcal{F})$. Let χ and λ be as in (2) and let $W_{a,\lambda}^{\mathrm{ex}}$ be the stabilizers of λ in W_a^{ex} . Then $H^*(T, \mathcal{F} \otimes \mathcal{L}_\chi)$ (resp. $i_\lambda^* \mathfrak{M}(\mathcal{F})$) carries a natural action of W'_χ (resp. $W_{a,\lambda}^{\mathrm{ex}}$) such that, under the isomorphisms

$$(4.3) \quad W'_\chi \simeq W_{a,\lambda}^{\mathrm{ex}} \quad w \rightarrow (w, w^{-1}\lambda - \lambda)$$

the isomorphism (4.2) intertwines those actions.

Remark 4.1. The restriction of the isomorphism in (4.3) to the subgroup $W_\chi \subset W'_\chi$ gives rise to an isomorphism

$$(4.4) \quad W_\chi \simeq W_{a,\lambda} \quad w \rightarrow (w, w^{-1}\lambda - \lambda),$$

where $W_{a,\lambda}$ be the stabilizers of λ in W_a . Since W_χ is generated by reflections, it follows that $W_{a,\lambda}$ is generated by affine reflections passing through λ (see, e.g., [Lo, Proposition 2.4.3]). This property of $W_{a,\lambda}$ plays an important role in Section 4.3 when we study characterizations of $*$ -central complexes.

Remark 4.2. A non-zero invariant section σ of ω_T gives rise to an isomorphism $\Gamma(\mathfrak{M}(\mathcal{F})) = \Gamma(\mathcal{F} \otimes \omega_T) \xrightarrow{\sigma} \Gamma(\mathcal{F})$, here v_i acts on $\Gamma(\mathcal{F})$ by the formula $v_i \cdot m = -v_i m$. Since the W -action on invariant sections of ω_T is given by the sign character, we obtain an isomorphism of $\mathcal{O}(\check{\mathfrak{t}})$ -modules

$$\Gamma((-1)^* \mathfrak{M}(\mathcal{F} \otimes \mathrm{sign}_W)) \simeq \Gamma(\mathcal{F})$$

compatible with the natural W_a^{ex} -actions. Here $-1 : \check{\mathfrak{t}} \rightarrow \check{\mathfrak{t}}, x \rightarrow -x$.

4.2.2. We now consider the ℓ -adic setting. In [GL], the authors constructed the Mellin transform⁶

$$\mathfrak{M} : \mathcal{D}(T) \rightarrow D_{coh}^b(\mathcal{C}(T))$$

with the following properties:

⁵For any right $\Gamma(\mathcal{D}_T)$ -module N , the action of $\mathbb{C}[v_i] = \mathcal{O}(\check{\mathfrak{t}})$ on N gives a $\mathcal{O}_{\check{\mathfrak{t}}}$ -module structure on $\mathcal{O}_{\check{\mathfrak{t}}} \otimes_{\mathcal{O}(\check{\mathfrak{t}})} N \in \mathrm{QCoh}(\check{\mathfrak{t}})$ and the action of $x_i \in \Lambda$ defines a Λ -equivariant structure.

⁶It is denoted by $\mathcal{M}_*$ in *loc.cit.*

- (1) Let $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ and $i_\chi : \text{pt} \rightarrow \mathcal{C}(T)$ be the embedding given by χ . We have

$$H^*(T, \mathcal{F} \otimes \mathcal{L}_\chi) \simeq i_\chi^* \mathfrak{M}(\mathcal{F}).$$

- (2) For any $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ we have

$$\mathfrak{M}(\mathcal{F} \otimes \mathcal{L}_\chi) \simeq m_\chi^* \mathfrak{M}(\mathcal{F}),$$

where $m_\chi : \mathcal{C}(T) \rightarrow \mathcal{C}(T)$ is the morphism of translation by χ .

- (3) The functor $\mathfrak{M}$ is t -exact with respect to the perverse t -structure on $\mathcal{D}(T)$ and the standard t -structure on $D_{\text{coh}}^b(\mathcal{C}(T))$. Moreover, for any $\mathcal{F} \in \mathcal{D}(T)$, $\mathcal{F}$ is perverse if and only if $\mathfrak{M}(\mathcal{F})$ is a coherent complex in degree zero.
- (4) We have

$$\mathfrak{M}(\mathcal{F} * \mathcal{F}') \simeq \mathfrak{M}(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{C}(T)}} \mathfrak{M}(\mathcal{F}').$$

- (5) The Mellin transform restricts to an equivalence

$$(4.5) \quad \mathfrak{M} : \mathcal{D}(T)_{\text{mon}} \simeq D_{\text{coh}}^b(\mathcal{C}(T))_f$$

between the full subcategory $\mathcal{D}(T)_{\text{mon}}$ of monodromic ℓ -adic complexes on T and the full subcategory $D_{\text{coh}}^b(\mathcal{C}(T))_f$ of coherent complexes on $\mathcal{C}(T)$ with finite support.

- (6) For $\mathcal{F} \in \mathcal{D}_W(T)$, the W -equivariant structure on $\mathcal{F}$ gives rise to a W -equivariant structure on $\mathfrak{M}(\mathcal{F})$ such that for any $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ the isomorphism in (1) above is compatible with the natural W'_χ -actions.

4.2.3. The properties of the Mellin transform above imply the following:

Lemma 4.1. *Let $\mathcal{F} \in \mathcal{D}_W(T)$. (1) Assume $\text{char } k > 0$. Then $\mathcal{F}$ is $*$ -central (resp. strongly $*$ -central) if and only if, for any $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$, the action of W_χ (resp. W'_χ) on $\mathcal{H}^n(i_\chi^* \mathfrak{M}(\mathcal{F} \otimes \text{sign}_W))$ for all $n \in \mathbb{Z}$ is trivial. (2) Assume $k = \mathbb{C}$. Then $\mathcal{F}$ is $*$ -central (resp. strongly $*$ -central) if and only if for any $\lambda \in \check{\mathfrak{t}}(\mathbb{C})$, the action of $W_{a,\lambda}$ (resp. $W_{a,\lambda}^{\text{ex}}$) on $\mathcal{H}^n(i_\lambda^* \mathfrak{M}(\mathcal{F} \otimes \text{sign}_W))$ for all $n \in \mathbb{Z}$ is trivial.*

4.3. Characterization. In [C, Proposition 4.2], we established several characterizations of $*$ -central complexes in the ℓ -adic setting. In this section we extend those results to the de Rham setting.

We first consider the case of $*$ -central D -modules:

Theorem 4.2. *Let $\mathcal{F}$ be a W -equivariant holonomic D -module on T . The following are equivalent*

- (1) $\mathcal{F}$ is $*$ -central.
- (2) for any $\lambda \in \check{\mathfrak{t}}(\mathbb{C})$, the action of $W_{a,\lambda}$ on $\mathcal{H}^n(i_\lambda^* \mathfrak{M}(\mathcal{F} \otimes \text{sign}_W))$, $n = 0, -1$, is trivial.
- (3) for any $\lambda \in \check{\mathfrak{t}}(\mathbb{C})$, the Mellin transform $\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)$, regarded as a $W_{a,\lambda}$ -equivariant quasi-coherent sheaf on $\check{\mathfrak{t}}$, descends to $\check{\mathfrak{t}} // W_{a,\lambda}$.
- (4) for any $\lambda \in \check{\mathfrak{t}}(\mathbb{C})$, the natural morphism $\mathcal{O}(\check{\mathfrak{t}}) \otimes_{\mathcal{O}(\check{\mathfrak{t}})^{W_{a,\lambda}}} \Gamma(\mathcal{F})^{W_{a,\lambda}} \rightarrow \Gamma(\mathcal{F})$ is an isomorphism.

Remark 4.3. The definition of $*$ -central D -modules makes sense for arbitrary W -equivariant D -modules on T (not necessary holonomic) and the proof below shows that Theorem 4.2 remains true in this more general setting. Very similar results were proved in [Gi2, Lo].

We begin with Lemma 4.3.

Lemma 4.3. *Let Γ be a finite reflection group with reflection representation V over $\mathbb{C}$. Let $\mathcal{F}$ be a Γ -equivariant quasi-coherent sheaf on V . Then $\mathcal{F}$ descends to $V // \Gamma$ if and only if for any $\lambda \in V(\mathbb{C})$ the actions of the stabilizer Γ_λ of λ in Γ on $\mathcal{H}^0(i_\lambda^* \mathcal{F})$ and $\mathcal{H}^{-1}(i_\lambda^* \mathcal{F})$ are trivial. Here $i_\lambda : \text{pt} \rightarrow V$ is the embedding given by λ .*

Proof. Assume $\mathcal{F}$ descends to $V // \Gamma$. Then we have $\mathcal{F} \simeq \pi^*(\pi_* \mathcal{F})^\Gamma$ where $\pi : V \rightarrow V // \Gamma$ is the quotient map and it implies $i_\lambda^* \mathcal{F} \simeq i_\lambda^*(\pi_* \mathcal{F})^\Gamma$, where $\bar{\lambda} = \pi(\lambda)$ and $i_{\bar{\lambda}} : \text{pt} \rightarrow V // \Gamma$ is the inclusion. As Γ acts trivially on $\mathcal{H}^n(i_\lambda^*(\pi_* \mathcal{F})^\Gamma)$ for all $n \in \mathbb{Z}$, it follows that Γ acts trivially on $\mathcal{H}^n(i_\lambda^* \mathcal{F})$ for all $n \in \mathbb{Z}$, in particular, for $n = 0, -1$.

Assume Γ acts trivially on $\mathcal{H}^n(i_\lambda^* \mathcal{F})$ for $n = 0, -1$. We would like to show that $\mathcal{F}$ descends to $V // \Gamma$. By [Lo, Theorem 1.3.2], it suffices to show that $\mathcal{F}$ descends to $V // \langle \sigma \rangle$ for any simple reflection $\sigma \in \Gamma$. So we could assume $\Gamma = \langle \sigma \rangle$ is generated by a reflection σ . Write $F = \Gamma(\mathcal{F})$ for the global sections of $\mathcal{F}$, viewed as a $\mathcal{O}(V) \rtimes \Gamma$ -module, and let $F^\sigma = \{h \in F \mid \sigma(h) = h\}$ and $F^{\sigma=-1} = \{h \in F \mid \sigma(h) = -h\}$. Choose a coordinate $(x_1, \dots, x_n)$ of V such that $\sigma(x_1) = -x_1$ and $\sigma(x_i) = x_i$ for $i \geq 2$. Let $\lambda \in V(\mathbb{C})$ be the origin with coordinate $x_i = 0$.

We claim that if $\Gamma = \Gamma_\lambda$ acts trivially on $\mathcal{H}^0(i_\lambda^* \mathcal{F}) \simeq F/(x_1, \dots, x_n)F$ then the natural map

$$(4.6) \quad \mathcal{O}(V) \otimes_{\mathbb{C}} F^\sigma \rightarrow F$$

is surjective.

Step 1. We first show that if Γ acts trivially on $F/(x_1)F$ then (4.6) is surjective. Indeed, the assumption implies that the image of $f \in F^{\sigma=-1}$ in the quotient $F/(x_1)F$ is zero, that is, $f \in (x_1)F$. Since $\sigma(x_1) = -x_1$ and $F = F^\sigma \oplus F^{\sigma=-1}$, it follows that $f = x_1 f'$ for some $f' \in F^\sigma$ and it implies (4.6) is surjective.

Step 2. We show that if Γ acts trivially on $\mathcal{H}^0(i_\lambda^* \mathcal{F})$ then Γ acts trivially on $F/(x_1)F$. The case $n = 1$ is trivial as $\mathcal{H}^0(i_\lambda^* \mathcal{F}) \simeq F/(x_1)F$. Assume $n > 1$. Consider the exact sequence

$$(4.7) \quad \mathcal{O}(V) \otimes_{\mathbb{C}} F^\sigma \rightarrow F \rightarrow M \rightarrow 0,$$

where M is the cokernel. The quotient $F' = F/(x_2, \dots, x_n)F$ is a $\mathcal{O}(V') \rtimes \Gamma$ -module, $V' = \text{Spec}(\mathbb{C}[x_1])$, such that Γ acts trivially on $F'/(x_1)F' = F/(x_1, \dots, x_n)F$. Thus Step 1 implies the natural map $\mathcal{O}(V') \otimes_{\mathbb{C}} (F')^\sigma \rightarrow F'$ is surjective. On the other hand, we have an exact sequence

$$\mathcal{O}(V') \otimes_{\mathbb{C}} (F')^\sigma \rightarrow F' \rightarrow M/(x_2, \dots, x_n)M \rightarrow 0$$

induced from (4.7) and it follows that $M/(x_2, \dots, x_n)M = 0$. Thus $M/(x_1)M = M/(x_1, x_2, \dots, x_n)M$, which is a quotient of $\mathcal{H}^0(i_\lambda^* \mathcal{F}) = F/(x_1, \dots, x_n)F$, and it implies Γ acts trivially on $M/(x_1)M$. Consider the exact sequence

$$(\mathcal{O}(V)/(x_1)\mathcal{O}(V)) \otimes_{\mathbb{C}} F^\sigma \rightarrow F/(x_1)F \rightarrow M/(x_1)M \rightarrow 0$$

induced from (4.7). As Γ acts both trivially on $(\mathcal{O}(V)/(x_1)\mathcal{O}(V)) \otimes_{\mathbb{C}} F^\sigma$ and $M/(x_1)M$, it also acts trivially on $F/(x_1)F$. This completes the proof of Step 2. The desired claim follows.

To proceed, let us consider the short exact sequence

$$0 \rightarrow N \rightarrow \mathcal{O}(V) \otimes_{\mathbb{C}} F^\sigma \rightarrow F \rightarrow 0,$$

where N is the kernel of (4.6). It gives rise to an exact sequence

$$(4.8) \quad \mathcal{H}^{-1}(i_\lambda^* \mathcal{F}) \rightarrow N/(x_1, \dots, x_n)N \rightarrow (\mathcal{O}(V)/(x_1, \dots, x_n)\mathcal{O}(V)) \otimes_{\mathbb{C}} F^\sigma \\ \rightarrow F/(x_1, \dots, x_n)F \rightarrow 0$$

Since Γ acts both trivially on $\mathcal{H}^{-1}(i_\lambda^* \mathcal{F})$ (by assumption) and $(\mathcal{O}(V)/(x_1, \dots, x_n)\mathcal{O}(V)) \otimes_{\mathbb{C}} F^\sigma$, (4.8) implies that Γ acts trivially on $N/(x_1, \dots, x_n)N$, and the claim above implies that the natural map $\mathcal{O}(V) \otimes_{\mathbb{C}} N^\sigma \rightarrow N$ is surjective. All together, we obtain a Γ -equivariant presentation of F

$$\mathcal{O}(V) \otimes_{\mathbb{C}} N^\sigma \rightarrow \mathcal{O}(V) \otimes_{\mathbb{C}} F^\sigma \rightarrow F \rightarrow 0,$$

where $F_1 = \mathcal{O}(V) \otimes_{\mathbb{C}} N^\sigma$ and $F_0 = \mathcal{O}(V) \otimes_{\mathbb{C}} F^\sigma$ are free $\mathcal{O}(V)$ -modules satisfying $F_i = \mathcal{O}(V) \otimes_{\mathcal{O}(V//\Gamma)} F_i^\sigma$. By [Ne, Lemma 3.1], this implies F also satisfies $F = \mathcal{O}(V) \otimes_{\mathcal{O}(V//\Gamma)} F^\sigma$ and thus the corresponding Γ -equivariant quasi-coherent sheaf $\mathcal{F}$ descends to $V//\Gamma$. $\square$

Proof of Theorem 4.2. (1) implies (2) is clear. (2) implies (3) follows from Lemma 4.3 and the fact that $\Gamma = W_{a,\lambda}$ is generated by affine reflections passing through $\lambda \in \mathfrak{t}(\mathbb{C})$ (see Remark 4.1) and, for any $\mu \in \mathfrak{t}(\mathbb{C})$, the stabilizer $\Gamma_\mu = W_{a,\lambda} \cap W_{a,\mu}^{\text{ex}}$ is a subgroup of $W_{a,\mu}$, and hence acts trivially on $\mathcal{H}^j(i_\mu^* \mathfrak{M}(\mathcal{F} \otimes \text{sign}_W))$, $j = 0, -1$. (3) implies (1) follows from the first paragraph of the proof of Lemma 4.3. The equivalence between (3) and (4) follows from Remark 4.2 and the fact $\mathcal{M}(\mathcal{F} \otimes \text{sign}_W)$ descends to $\mathfrak{t}//W_{a,v}$ if and only if $(-1)^* \mathcal{M}(\mathcal{F} \otimes \text{sign}_W)$ descends to $\mathfrak{t}//W_{a,-v}$. $\square$

We now extend Theorem 4.2 to the case of $*$ -central complexes.

Theorem 4.4. *Let $\mathcal{F} \in \mathcal{D}_W(T)$ be a W -equivariant complex of holonomic D -modules on T . The following are equivalent*

- (1) $\mathcal{F}$ is $*$ -central.
- (2) for any $\lambda \in \mathfrak{t}(\mathbb{C})$, the Mellin transform $\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)$, regarded as an object in $D_{\text{coh}}^b(\mathfrak{t}/W_{a,\lambda})$, descends to $\mathfrak{t}//W_{a,\lambda}$.

Proof. (2) implies (1) follows from the first paragraph of the proof of Lemma 4.3. To show (1) implies (2) let $\mathcal{F}$ be a $*$ -central complex and let $b \in \mathbb{Z}$ be the largest integer such that $\mathcal{H}^b(\mathcal{F}) \neq 0$. We have the following distinguished triangle

$$(4.9) \quad \tau_{\leq b-1} \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{H}^b(\mathcal{F})[-b] \rightarrow \tau_{\leq b-1} \mathcal{F}[1].$$

We claim that both $\mathcal{H}^b(\mathcal{F})$ and $\tau_{\leq b-1} \mathcal{F}$ are $*$ -central. Indeed, for any de Rham local system $\mathcal{L}_\chi$ on T associated to $\chi \in \mathcal{C}(T)(\mathbb{C})$, the distinguished triangle above gives rise to the following exact sequence

$$(4.10) \quad H^j(T, \mathcal{F} \otimes \mathcal{L}_\chi) \rightarrow H^{j-b}(T, \mathcal{H}^b(\mathcal{F}) \otimes \mathcal{L}_\chi) \rightarrow H^{j+1}(T, \tau_{\leq b-1} \mathcal{F} \otimes \mathcal{L}_\chi) \rightarrow H^{j+1}(T, \mathcal{F} \otimes \mathcal{L}_\chi).$$

By the exactness of the Mellin transform and the right-exactness of the tensor product, we have

$$H^{\geq b}(T, \tau_{\leq b-1} \mathcal{F} \otimes \mathcal{L}_\chi) \simeq \mathcal{H}^{\geq b}(i_\lambda^* \mathfrak{M}(\tau_{\leq b-1} \mathcal{F})) = 0$$

(here $\lambda \in \mathfrak{t}$ is a lift of χ) and it follows that the first arrow in (4.10) is surjective for $j \geq b-1$. In particular, for $j = b, b-1$, we have the following surjections

$$H^b(T, \mathcal{F} \otimes \mathcal{L}_\chi) \rightarrow H^0(T, \mathcal{H}^b(\mathcal{F}) \otimes \mathcal{L}_\chi) \rightarrow 0 \\ H^{b-1}(T, \mathcal{F} \otimes \mathcal{L}_\chi) \rightarrow H^{-1}(T, \mathcal{H}^b(\mathcal{F}) \otimes \mathcal{L}_\chi) \rightarrow 0.$$

Since the group W_χ acts on $H^*(T, \mathcal{F} \otimes \mathcal{L}_\chi)$ by the sign character (as $\mathcal{F}$ is $*$ -central by assumption), the above surjections imply that W_χ acts on $H^i(T, \mathcal{H}^b(\mathcal{F}) \otimes \mathcal{L}_\chi)$ by the sign character for $i = 0, -1$, and thus by Theorem 4.2 (2), we conclude that $\mathcal{H}^b(\mathcal{F})$ is $*$ -central. Now the exact sequence (4.10) implies that W_χ acts on $H^*(T, \tau_{\leq b-1} \mathcal{F} \otimes \mathcal{L}_\chi)$ by the sign character and hence, by definition, $\tau_{\leq b-1} \mathcal{F}$ is $*$ -central.

By induction on the (finite) number of non vanishing cohomology sheaves and Theorem 4.2, we could assume both $\tau_{\leq b-1} \mathcal{F}$ and $\mathcal{H}^b(\mathcal{F})$ satisfy statement (2) of the theorem. We shall prove that $\mathcal{F}$ also satisfies statement (2), that is, the Mellin transform $\mathcal{M} := \mathfrak{M}(\mathcal{F} \otimes \text{sign}_W) \in D_{qcoh}^b(\check{\mathfrak{t}}/W_{a,\lambda})$ descends to $\check{\mathfrak{t}} // W_{a,\lambda}$ along the quotient map $\pi_\lambda : \check{\mathfrak{t}} \rightarrow \check{\mathfrak{t}} // W_{a,\lambda}$. Recall the following descent criteria for quasi-coherent complexes (see, e.g., [Ne, Corollary 2.6]): a complex $\mathcal{M} \in D_{qcoh}^b(\check{\mathfrak{t}}/W_{a,\lambda})$ descends to $\check{\mathfrak{t}} // W_{a,\lambda}$ if and only if the canonical morphism

$$(4.11) \quad \pi_\lambda^*((\pi_\lambda)_*(\mathcal{M})^{W_{a,\lambda}}) \rightarrow \mathcal{M}$$

is an isomorphism.⁷

To proceed, let us consider the following distinguished triangle

$$\begin{aligned} \mathcal{M}' := \mathfrak{M}(\tau_{\leq b-1} \mathcal{F} \otimes \text{sign}_W) &\rightarrow \mathcal{M} := \mathfrak{M}(\mathcal{F} \otimes \text{sign}_W) \\ &\rightarrow \mathcal{M}'' := \mathfrak{M}(\mathcal{H}^b(\mathcal{F})[-b] \otimes \text{sign}_W) \rightarrow \mathcal{M}'[1] \end{aligned}$$

obtained by applying the functor $\mathfrak{M}(- \otimes \text{sign}_W)$ to (4.9). There is a morphism between the following distinguished triangles

$$\begin{array}{ccc} \pi_\lambda^*((\pi_\lambda)_*(\mathcal{M}')^{W_{a,\lambda}}) & \xrightarrow{c'} & \mathcal{M}' \\ \downarrow & & \downarrow \\ \pi_\lambda^*((\pi_\lambda)_*(\mathcal{M})^{W_{a,\lambda}}) & \xrightarrow{c} & \mathcal{M} \\ \downarrow & & \downarrow \\ \pi_\lambda^*((\pi_\lambda)_*(\mathcal{M}'')^{W_{a,\lambda}}) & \xrightarrow{c''} & \mathcal{M}'' \\ \downarrow & & \downarrow \\ \pi_\lambda^*((\pi_\lambda)_*(\mathcal{M}')^{W_{a,\lambda}})[1] & \xrightarrow{c'[1]} & \mathcal{M}'[1] \end{array}$$

where the horizontal arrows are the canonical morphisms in (4.11). By assumption and the above descent criteria, both c' and c'' are isomorphisms. It follows that c is also an isomorphism and, by the descent criteria again, we conclude that $\mathcal{M} = \mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)$ descends to $\check{\mathfrak{t}} // W_{a,\lambda}$. This completes the proof of (1) implies (2), and hence the theorem. $\square$

Remark 4.4. The results of this sections were inspired by the work of T. Nevins [Ne]. In *loc. cit.* the author proved similar descent criterions in the case when Γ is a reductive algebraic group and $\mathcal{F}$ is a Γ -equivariant coherent sheaf or a complex Γ -equivariant coherent sheaves. The argument in *loc. cit.* used the coherence assumption of $\mathcal{F}$ and hence cannot be applied directly to the case of quasi-coherent sheaves. The proof above uses some special features of finite reflection groups.

⁷Note that the map π_λ is flat and affine and hence both functor π_λ^* and $(\pi_\lambda)_*$ are exact.

4.4. Functoriality. In this section we show that central complexes (resp. $*$ -central complexes) are stable under pushforward along central isogenies. This result will be used later to reduce the proof the vanishing conjecture for a general connected reductive group to the one with connected center.

We first review the notion of central isogenies between connected reductive groups following [S, Section 2]. Let G and G_1 be two connected reductive groups and let T be a maximal torus of G . Let $f_G : G \rightarrow G_1$ be a central isogeny, that is, f_G is a surjective map with finite kernel such that f_G maps the root subgroup U_α isomorphically onto its image, for all roots α . Denote by $T_1 := f(T)$ the image of T which is a maximal torus of G_1 . The resulting homomorphism $f_T := f_G|_T : T \rightarrow T_1$ induces a map $f_T^* : \mathbb{X}^\bullet(T_1) \rightarrow \mathbb{X}^\bullet(T)$ (resp. $(f_T)_* : \mathbb{X}_\bullet(T) \rightarrow \mathbb{X}_\bullet(T_1)$) between character lattices (resp. co-character lattices), which maps the root lattice R_1 isomorphically onto the root lattice R (resp. the coroot lattice $R_1^\vee$ isomorphically onto the coroot lattice $R^\vee$). Moreover, we have an isomorphism of Weyl groups

$$(4.12) \quad f_W : W = N_{G_1}(T_1)/T_1 \simeq N_G(T)/T = W_1$$

such that the map $f_T : T \rightarrow T_1$ is compatible with the $W \simeq W_1$ -action. Thus the pushforward along f_T induces a functor

$$(f_T)_! : \mathcal{D}_W(T) \rightarrow \mathcal{D}_{W_1}(T_1)$$

between the corresponding equivariant derived categories. Note that as f_T is finite, we have $(f_T)_! \simeq (f_T)_*$.

Lemma 4.5. *The functor $(f_T)_!$ maps central complexes on T to central complexes on T_1 . The same is true for $*$ -central complexes.*

Proof. Let $\mathcal{F} \in \mathcal{D}_W(T)$ be a central complex and let $\mathcal{F}_1 := (f_T)_! \mathcal{F} \in \mathcal{D}_{W_1}(T_1)$. We need to show that, for any $\chi_1 \in \mathcal{C}(T_1)(E)$, the action of the subgroup W_{χ_1} of the stabilizer of χ_1 in W_1 on $H_c^*(T_1, \mathcal{F}_1 \otimes \mathcal{L}_{\chi_1})$ is given by the sign character (the definition of W_{χ_1} will be recalled below). Let $\chi = f_T^* \chi_1 \in \mathcal{C}(T)(E)$. We claim that the map f_W in (4.12) maps W_χ isomorphically onto W_{χ_1} . Then, by projection formula, we get an isomorphism

$$H_c^*(T_1, \mathcal{F}_1 \otimes \mathcal{L}_{\chi_1}) \simeq H_c^*(T_1, (f_T)_! \mathcal{F} \otimes \mathcal{L}_{\chi_1}) \simeq H_c^*(T, \mathcal{F} \otimes f_T^* \mathcal{L}_{\chi_1}) \simeq H_c^*(T, \mathcal{F} \otimes \mathcal{L}_\chi)$$

compatible with the $W_{\chi_1} \xrightarrow{f_W} W_\chi$ -actions. Since $\mathcal{F}$ is central, the group W_χ acts on $H_c^*(T, \mathcal{F} \otimes \mathcal{L}_\chi)$ via the sign character and the isomorphism above implies that W_{χ_1} also acts on $H_c^*(T_1, \mathcal{F}_1 \otimes \mathcal{L}_{\chi_1})$ by the sign character and hence $\mathcal{F}_1$ is central. Since the Verdier duality $\mathbb{D}$ maps central complexes to $*$ -central complexes and vice versa, and the functor $(f_T)_! \simeq (f_T)_*$ commutes with $\mathbb{D}$, the case of $*$ -central complexes follows from the case of central complexes. This finishes the proof of the lemma.

Proof of the claim. Recall that the map f_T induces an isomorphism $(f_T)_* : R^\vee \simeq R_1^\vee$ between coroot lattices, sending a coroot $\alpha^\vee$ to $\alpha_1^\vee := (f_T)_*(\alpha^\vee) = f_T \circ \alpha^\vee$. Thus for any $\alpha^\vee$ and $\alpha_1^\vee$ as above, we have

$$(4.13) \quad (\alpha^\vee)^* \mathcal{L}_\chi \simeq (\alpha^\vee)^* f_T^* (\mathcal{L}_{\chi_1}) \simeq (f_T \circ \alpha^\vee)^* (\mathcal{L}_{\chi_1}) \simeq (\alpha_1^\vee)^* \mathcal{L}_{\chi_1}.$$

Since the group W_χ (resp. W_{χ_1}) is generated by those reflection s_α (resp. s_{α_1}) such that $(\alpha^\vee)^* \mathcal{L}_\chi$ is trivial (resp. $(\alpha_1^\vee)^* \mathcal{L}_{\chi_1}$), the isomorphism (4.13) implies that $f_W : W \rightarrow W_1$ maps W_χ isomorphically onto W_{χ_1} . The claim follows. $\square$

Example 4.5. Consider the case $G = \mathrm{SL}_2$, $G_1 = \mathrm{PGL}_2$ and the central isogeny $f_G : \mathrm{SL}_2 \rightarrow \mathrm{SL}_2/\{\pm 1\} \simeq \mathrm{PGL}_2$. We have $W \simeq W_1 \simeq \{\pm 1\}$ acting on $T \simeq T_1 \simeq \mathbb{G}_m$ by the inverse map and $f_T : T \rightarrow T_1$ is given by the square map $\mathbb{G}_m \rightarrow \mathbb{G}_m, x \rightarrow x^2$. It is straightforward to check that, in the case of PGL_2 , a complex $\mathcal{F}_1 \in \mathcal{D}_{W_1}(T_1)$ is central if and only if $H_c^*(T_1, \mathcal{F}_1 \otimes \mathcal{L}_{\chi_1})^{W_1} = 0$ for any $\chi_1 \in \mathcal{C}(T_1)(E)$ such that χ_1^2 is trivial. Let $\mathcal{F} \in \mathcal{D}_W(T)$ be a central complex for SL_2 and let χ_1 be as above. We have $H_c^*(T, \mathcal{F})^W = 0$ and $f_T^* \mathcal{L}_{\chi_1} \simeq \mathcal{L}_{\chi_1}^2$ is trivial and it follows that

$$H_c^*(T_1, (f_T)_! \mathcal{F} \otimes \mathcal{L}_{\chi_1})^{W_1} \simeq H_c^*(T, \mathcal{F} \otimes f_T^* \mathcal{L}_{\chi_1})^W \simeq H_c^*(T, \mathcal{F})^W = 0.$$

Thus $(f_T)_! \mathcal{F}$ is indeed a central complex for PGL_2 .

5. LOCAL SYSTEMS $\mathcal{E}_\theta$

5.1. Tame local systems $\mathcal{E}_\theta$. In this subsection we attach each W -orbit $\theta = W_\chi$ in $\mathcal{C}(T)(E)$ a W -equivariant tame local system $\mathcal{E}_\theta$ on T .

We first consider the ℓ -adic setting. Recall the pro- ℓ quotient $\pi_1(T)_\ell$ of the tame fundamental group $\pi_1^t(T)$. Let $R_T = \mathrm{Sym}(\pi_1(T)_\ell \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell)$ be the symmetric algebra of $\pi_1(T)_\ell \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$ and let $R_{T,+}$ be the argumentation ideal. The Weyl group W , and hence the subgroup W_χ , acts naturally on R_T and we define $R_\chi = R_T / \langle R_{T,+}^{W_\chi} \rangle$, where $\langle R_{T,+}^{W_\chi} \rangle$ is the ideal generated by $R_{T,+}^{W_\chi}$. Consider the representation R_χ of $\pi_1(T)_\ell$, on which an element $\gamma \in \pi_1(T)_\ell$ acts as multiplication by $\exp(\gamma)$. It is well-defined since the multiplication action of $\pi_1(T)_\ell$ on R_χ is nilpotent. By pullback along $\pi_1^t(T) \rightarrow \pi_1(T)_\ell$, we obtain a representation of $\pi_1^t(T)$ and we denote by $\mathcal{E}_\chi^{uni}$ the corresponding ℓ -adic local system on T . Since W_χ is normal in W'_χ , the natural action of W'_χ on R_T descends to the quotient R_χ which gives rise to a canonical W'_χ -equivariant structure on $\mathcal{E}_\chi^{uni}$. On the other hand, since $w^* \mathcal{L}_\chi \simeq \mathcal{L}_\chi$ for $w \in W'_\chi$, the above W'_χ -equivariant structure on $\mathcal{E}_\chi^{uni}$ gives rise to a W'_χ -equivariant structure on $\mathcal{E}_\chi = \mathcal{E}_\chi^{uni} \otimes \mathcal{L}_\chi$. Finally, we define $\mathcal{E}_\theta = \mathrm{Ind}_{W'_\chi}^W \mathcal{E}_\chi$ to be the induced W -equivariant ℓ -adic local system on T .

The construction in the de Rham setting is similar to the ℓ -adic setting. Let $R_T = \mathrm{Sym}(\pi_1(T(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{C})$ and let $R_\chi = R_T / \langle R_{T,+}^{W_\chi} \rangle$, where $\langle R_{T,+}^{W_\chi} \rangle$ is the ideal generated by the W_χ -invariants $R_{T,+}^{W_\chi}$ in the argumentation ideal $R_{T,+}$. Define $\mathcal{E}_\chi^{uni}$ to be the de Rham local system on T corresponding to the representation R_χ of $\pi_1(T(\mathbb{C}))$, on which an element $\gamma \in \pi_1(T(\mathbb{C}))$ acts as multiplication by $\exp(\gamma)$. Let $\mathcal{E}_\chi = \mathcal{E}_\chi^{uni} \otimes \mathcal{L}_\chi$. The same reasoning as in the ℓ -adic setting shows that both $\mathcal{E}_\chi^{uni}$ and $\mathcal{E}_\chi$ carry natural W'_χ -equivariant structures and we let $\mathcal{E}_\theta = \mathrm{Ind}_{W'_\chi}^W \mathcal{E}_\chi$ to be the induced W -equivariant de Rham local system on T .

The local systems $\mathcal{E}_\chi$ and $\mathcal{E}_\chi^{uni}$ have rank $|W_\chi|$ and $\mathcal{E}_\theta$ has rank $|W| |W_\chi| / |W'_\chi|$.

Example 5.1. Let $\theta = W_\chi$ be a W -orbit of a character χ . If the character χ is regular, that is, $W'_\chi = e$, then we have $\mathcal{E}_\chi = \mathcal{L}_\chi$ and $\mathcal{E}_\theta = \mathrm{Ind}_e^W \mathcal{L}_\chi \simeq \bigoplus_{w \in W} \mathcal{L}_{w\chi}$ is a local system of rank $|W|$. If χ is quasi-regular, that is, $W_\chi = e$ (e.g., the quadratic character $\chi = \chi^{-1}$ for SL_2 in Example 1.2), then $\mathcal{E}_\chi = \mathcal{L}_\chi$ and $\mathcal{E}_\theta = \mathrm{Ind}_{W'_\chi}^W \mathcal{L}_\chi = \bigoplus_{w \in W/W'_\chi} \mathcal{L}_{w\gamma}$ which is a local system of rank $|W|/|W'_\chi|$. If χ is the trivial character, then $W_\chi = W'_\chi = W$ and $\mathcal{E}_\theta = \mathcal{E}_\chi = \mathcal{E}_\chi^{uni}$ is the local system of rank $|W|$ corresponding to the unipotent representation $R_\chi = R_T / \langle R_{T,+}^W \rangle$ of $\pi_1^t(T)$.

5.2. Mellin transform of $\mathcal{E}_\theta$. In this subsection we study the Mellin transform of $\mathcal{E}_\theta$.

We first consider the ℓ -adic setting. Let $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ be a tame character and let $\theta = W\chi$ be its W -orbit in $\mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$. Consider the quotient map $\pi_\chi : \mathcal{C}(T) \rightarrow \mathcal{C}(T) // W_\chi$. Let $0 \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ be the trivial character and let $D_\chi = \pi_\chi^{-1}(\pi_\chi(0))$ be the scheme theoretic preimage of $\pi_\chi(0) \in \mathcal{C}(T) // W_\chi$ for the map π_χ . Introduce the following coherent sheaves on $\mathcal{C}(T)$

$$(5.1) \quad \mathcal{R}_\chi^{uni} = \mathcal{O}_{D_\chi} \quad \mathcal{R}_\chi = m_\chi^*(\mathcal{O}_{D_\chi}).$$

Since W'_χ stabilizes D_χ and χ , both $\mathcal{R}_\chi^{uni}$ and $\mathcal{R}_\chi$ are canonically W'_χ -equivariant and we let

$$\mathcal{R}_\theta = \text{Ind}_{W'_\chi}^W \mathcal{R}_\chi$$

be the induced W -equivariant coherent sheaf on $\mathcal{C}(T)$. Note that $\mathcal{R}_\theta$ is set theoretically supported on $\theta^{-1} = \{\chi^{-1} | \chi \in \theta\} \subset \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$.

We now consider the de Rham setting. For any $\lambda \in \mathfrak{t}$, let $m_\lambda : \mathfrak{t} \rightarrow \mathfrak{t}$ be the map of translation by λ . For any $\chi \in \mathcal{C}(\mathbb{C})$, we write $\pi_\chi : \mathfrak{t} \rightarrow \mathfrak{t} // W_\chi$ for the quotient map and let $D_\chi = \pi_\chi^{-1}(\pi_\chi(0))$ be the scheme theoretic preimage of $\pi_\chi(0)$ for the map π_χ . Note that the natural identification $R_T \simeq \text{Sym}(\pi_1(T(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{C}) \simeq \text{Sym}(\mathfrak{t}) \simeq \mathcal{O}(\mathfrak{t})$ gives rise to an isomorphism

$$(5.2) \quad \Gamma(\mathcal{O}_{D_\chi}) \simeq \mathcal{O}(\mathfrak{t}) / \langle \mathcal{O}(\mathfrak{t})_+^{W_\chi} \rangle \simeq R_T / \langle R_{T,+}^{W_\chi} \rangle \simeq R_\chi.$$

Introduce the following Λ -equivariant quasi-coherent sheaves on $\mathfrak{t}$

$$(5.3) \quad \mathcal{R}_\chi^{uni} = \bigoplus_{\lambda \in \Lambda} m_\lambda^*(\mathcal{O}_{D_\chi}) \quad \mathcal{R}_\chi = \bigoplus_{\lambda \in \mathfrak{t}, \exp(\lambda) = \chi} m_\lambda^*(\mathcal{O}_{D_\chi}).$$

Since D_χ , Λ , and the set $\{\lambda \in \mathfrak{t} | \exp(\lambda) = \chi\}$ are stable under the natural W'_χ -action, both $\mathcal{R}_\chi^{uni}$ and $\mathcal{R}_\chi$ are canonically W'_χ -equivariant and we define

$$(5.4) \quad \mathcal{R}_\theta = \text{Ind}_{W'_\chi}^W \mathcal{R}_\chi$$

to be the induced $W_a^{\text{ex}} = W \ltimes \Lambda$ -equivariant sheaf on $\mathfrak{t}$. Note that $\mathcal{R}_\theta$ is set theoretically supported on $\{-\lambda | \exp(\lambda) \in \theta\} \subset \mathfrak{t}$.

Proposition 5.1. *There is an isomorphism*

$$\mathfrak{M}(\mathcal{E}_\theta \otimes \text{sign}_W) \simeq \mathcal{R}_\theta$$

compatible with the W -equivariant structures in the ℓ -adic setting, and the W_a^{ex} -equivariant structures in the de Rham setting.

Proof. The ℓ -adic setting. Pick a $\chi \in \theta$. Since $\mathcal{E}_\theta = \text{Ind}_{W'_\chi}^W \mathcal{E}_\chi$ and $\mathcal{R}_\theta = \text{Ind}_{W'_\chi}^W \mathcal{R}_\chi$, it suffices to construct an isomorphism $\mathfrak{M}(\mathcal{E}_\chi \otimes \text{sign}_W) \simeq \mathcal{R}_\chi$ compatible with the W'_χ -equivariant structures. Denote by R_θ the completion of the symmetric algebra $R_T = \text{Sym}(\pi_1(T)_\ell \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell)$ with respect to the argumentation ideal $R_{T,+}$. The ring of functions $\mathcal{O}(\mathcal{C}(T)_\ell)$ of $\mathcal{C}(T)_\ell$ is isomorphic to the completed group algebra $\overline{\mathbb{Q}}_\ell[[\pi_1(T)_\ell]]$ and the logarithm defines a natural W -equivariant isomorphism

$$R_\theta \simeq \overline{\mathbb{Q}}_\ell[[\pi_1(T)_\ell]]$$

intertwining the action of multiplication by $\exp(\gamma)$ on R_θ , $\gamma \in \pi_1(T)_\ell$, with the action of multiplication by γ on $\overline{\mathbb{Q}}_\ell[[\pi_1(T)_\ell]]$. The isomorphism above gives rise to

an isomorphism

$$R_\chi \simeq R_{\hat{0}} / \langle R_{\hat{0},+}^{W_\chi} \rangle \simeq \mathcal{O}(\mathcal{C}(T)_\ell) / \langle \mathcal{O}(\mathcal{C}(T)_\ell)_+^{W_\chi} \rangle \simeq \mathcal{O}(D_\chi) \simeq \Gamma(\mathcal{R}_\chi^{uni})$$

and it follows that the $\overline{\mathbb{Q}}_\ell[[\pi_1(T)_\ell]]$ -module corresponding to $\mathcal{E}_\chi^{uni}$ is isomorphic to $\Gamma(\mathcal{R}_\chi^{uni})$. Since $\mathcal{R}_\chi^{uni} \simeq \text{inv}^* \mathcal{R}_\chi^{uni}$ (here inv the inverse map on $\mathcal{C}(T)$), by [GL, Corollary 4.2.2.4], there is an isomorphism

$$\mathfrak{M}(\mathcal{E}_\chi) \simeq m_\chi^* \mathfrak{M}(\mathcal{E}_\chi^{uni}) \simeq m_\chi^* (\mathcal{R}_\chi^{uni} \otimes_{\mathbb{Z}_\ell} \wedge_{\mathbb{Z}_\ell}^{\text{top}} (\pi_1(T)_\ell)^\vee) \simeq \mathcal{R}_\chi \otimes_{\mathbb{Z}_\ell} \wedge_{\mathbb{Z}_\ell}^{\text{top}} (\pi_1(T)_\ell)^\vee,$$

compatible with the W'_χ -actions. By choosing a generator of $\wedge_{\mathbb{Z}_\ell}^{\text{top}} (\pi_1(T)_\ell)^\vee$, we obtain a W -equivariant isomorphism $\wedge_{\mathbb{Z}_\ell}^{\text{top}} (\pi_1(T)_\ell)^\vee \simeq \mathbb{Z}_\ell \otimes \text{sign}_W$. The desired claim follows.

The de Rham setting. It suffices to construct an isomorphism of $W'_\chi \ltimes \Lambda$ -equivariant quasi-coherent sheaves $\mathfrak{M}(\mathcal{E}_\chi \otimes \text{sign}_W) \simeq \mathcal{R}_\chi$. Note that, any lift $\lambda \in \check{\mathfrak{t}}$ of χ gives rise to canonical isomorphisms $\mathfrak{M}(\mathcal{E}_\chi \otimes \text{sign}_W) \simeq m_\lambda^* \mathfrak{M}(\mathcal{E}_\chi^{uni} \otimes \text{sign}_W)$ and $\mathcal{R}_\chi \simeq m_\lambda^* (\mathcal{R}_\chi^{uni})$. Thus we reduce to show that

$$(5.5) \quad \mathfrak{M}(\mathcal{E}_\chi^{uni} \otimes \text{sign}_W) \simeq \mathcal{R}_\chi^{uni}.$$

Since $\mathcal{E}_\chi^{uni}$ is a monodromic D -module on T with unipotent monodromy, the global section $\Gamma(\mathcal{E}_\chi^{uni})$ (viewed as a $\text{Sym}(\mathfrak{t}) \simeq \mathcal{O}(\check{\mathfrak{t}})$ -module) admits a unique decomposition $\Gamma(\mathcal{E}_\chi^{uni}) = \bigoplus_{\lambda \in \Lambda} \Gamma^\lambda(\mathcal{E}_\chi^{uni})$, where the summand $\Gamma^\lambda(\mathcal{E}_\chi^{uni})$ is supported at λ . Moreover, there is a natural identification of $\Gamma^{\hat{0}}(\mathcal{E}_\chi^{uni})$ with the fiber $\mathcal{E}_\chi^{uni}|_1$ at the identity $1 \in T$ such that the action of $\pi_1(T(\mathbb{C}))$ on $\Gamma^{\hat{0}}(\mathcal{E}_\chi^{uni})$, viewed as an element in $\mathfrak{t} = \pi_1(T(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{C}$, becomes the logarithm of the monodromy action of $\pi_1(T(\mathbb{C}))$ on the fiber $\mathcal{E}_\chi^{uni}|_1$. It follows that $\Gamma^{\hat{0}}(\mathcal{E}_\chi^{uni}) \simeq \mathcal{E}_\chi^{uni}|_1 \simeq R_T / \langle R_{T,+}^W \rangle \simeq \Gamma(\mathcal{O}_{D_\chi})$ and, by the Λ -equivariance, we obtain an isomorphism of $\mathcal{O}(\check{\mathfrak{t}})$ -modules

$$(5.6) \quad \Gamma(\mathcal{E}_\chi^{uni}) \simeq \bigoplus_{\lambda \in \Lambda} m_\lambda^* \Gamma^{\hat{0}}(\mathcal{E}_\chi^{uni}) \simeq \bigoplus_{\lambda \in \Lambda} m_\lambda^* \Gamma(\mathcal{O}_{D_\chi}) \stackrel{(5.3)}{\simeq} \Gamma(\mathcal{R}_\chi^{uni}).$$

Since $(-1)^* \mathcal{R}_\chi^{uni} \simeq \mathcal{R}_\chi^{uni}$, Remark 4.2 implies that there is an isomorphism

$$\Gamma(\mathfrak{M}(\mathcal{E}_\chi^{uni} \otimes \text{sign}_W)) \simeq (-1)^* (\Gamma(\mathcal{E}_\chi^{uni})) \stackrel{(5.6)}{\simeq} (-1)^* \Gamma(\mathcal{R}_\chi^{uni}) \simeq \Gamma(\mathcal{R}_\chi^{uni})$$

intertwining the $W'_\chi \ltimes \Lambda$ -actions. The isomorphism (5.5) follows. The proof of the proposition is complete. $\square$

Corollary 5.2. $\mathcal{E}_\theta$ is $*$ -central.

Proof. It is proved in [C, Corollary 5.2] in the ℓ -adic setting. Consider the de Rham setting. By Proposition 5.1, the fiber $i_\lambda^* \mathfrak{M}(\mathcal{E}_\theta \otimes \text{sign}_W)$ at $\lambda \in \check{\mathfrak{t}}$ is equal to $i_\lambda^* \mathfrak{M}(\mathcal{E}_\theta \otimes \text{sign}_W) \simeq i_\lambda^* \mathcal{R}_\theta \simeq i_\lambda^* (m_{-\lambda}^* (\mathcal{O}_{D_\chi}))$ if $\chi = \exp(-\lambda) \in \theta$ and is zero otherwise. We claim that $m_{-\lambda}^* (\mathcal{O}_{D_\chi})$, viewed as a $W_{a,\lambda}$ -equivariant coherent sheaf on $\check{\mathfrak{t}}$, descends along the quotient map $\pi_\lambda : \check{\mathfrak{t}} \rightarrow \check{\mathfrak{t}} // W_{a,\lambda}$. Indeed, the map $m_{-\lambda} : \check{\mathfrak{t}} \rightarrow \check{\mathfrak{t}}$ is $W_{a,\lambda} \stackrel{(4.4)}{\simeq} W_\chi$ -equivariant and we have the following cartesian diagram

$$\begin{array}{ccc} \check{\mathfrak{t}} & \xrightarrow{m_{-\lambda}} & \check{\mathfrak{t}} \\ \downarrow \pi_\lambda & & \downarrow \pi_\chi \\ \check{\mathfrak{t}} // W_{a,\lambda} & \xrightarrow{\bar{m}_{-\lambda}} & \check{\mathfrak{t}} // W_\chi \end{array}$$

where $\bar{m}_{-\lambda}$ is the descent of $m_{-\lambda}$. It follows that $m_{-\lambda}^*(\mathcal{O}_{D_\chi}) \simeq m_{-\lambda}^*(\pi_\chi^* \mathcal{O}_{\pi_\chi(0)}) \simeq \pi_\chi^* \mathcal{O}_{\pi_\chi(\lambda)}$. The claim implies that $W_{a,\lambda}$ acts trivially on $\mathcal{H}^n(i_\lambda^* \mathfrak{M}(\mathcal{E}_\theta \otimes \text{sign}_W)) \simeq \mathcal{H}^n(i_\lambda^*(\pi_\lambda^* \mathcal{O}_{\pi_\lambda(\lambda)}))$ and Theorem 4.2 implies that $\mathcal{E}_\theta$ is $*$ -central. $\square$

5.3. Convolution with $\mathcal{E}_\theta$.

Proposition 5.3. *Assume G has connected center. Let $\mathcal{F}$ be a $*$ -central complex and let $\theta = W_\chi$ be a W -orbit of a tame character χ . There is an isomorphism*

$$\mathcal{F} * (\mathcal{E}_\theta \otimes \text{sign}_W) \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_\theta \in \mathcal{D}_W(T).$$

Proof. We first consider the ℓ -adic setting. Since G has connected center, by Remark 1.1, W_χ is equal to the stabilizer W'_χ of χ in W and it follows that

$$(5.7) \quad \mathfrak{M}(\mathcal{E}_\theta \otimes \text{sign}_W) \stackrel{\text{Lem 5.1}}{\simeq} \mathcal{R}_\theta \simeq \text{Ind}_{W'_\chi}^W(m_\chi^* \mathcal{O}_{D_\chi}) \simeq \text{Ind}_{W_\chi}^W(m_\chi^* \mathcal{O}_{D_\chi}).$$

Thus there is an isomorphism

$$(5.8) \quad \mathfrak{M}(\mathcal{F} * \mathcal{E}_\theta \otimes \text{sign}_W) \simeq \mathfrak{M}(\mathcal{F}) \otimes \mathfrak{M}(\mathcal{E}_\theta \otimes \text{sign}_W) \simeq \text{Ind}_{W_\chi}^W(\mathfrak{M}(\mathcal{F})|_{m_\chi^{-1}(D_\chi)}).$$

On the other hand, since $m_\chi^{-1}(D_\chi)$ is equal to the inverse image $m_\chi^{-1}(D_\chi) = \pi_\chi^{-1}(\pi_\chi(\chi^{-1}))$ of $\pi_\chi(\chi^{-1})$ along $\pi_\chi : \mathcal{C}(T) \rightarrow \mathcal{C}(T) // W_\chi$, the characterization of ℓ -adic $*$ -central complexes in [C, Proposition 4.2] implies that $\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)|_{m_\chi^{-1}(D_\chi)}$ descends to $m_\chi^{-1}(D_\chi) // W_\chi$. It follows that there is an isomorphism

$$(5.9) \quad \mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)|_{m_\chi^{-1}(D_\chi)} \simeq \mathfrak{M}(\mathcal{F})|_{\chi^{-1}} \otimes m_\chi^* \mathcal{O}_{D_\chi} \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes m_\chi^* \mathcal{O}_{D_\chi}$$

compatible with the W_χ -equivariant structure, which induces an isomorphism

$$(5.10) \quad \begin{aligned} \text{Ind}_{W_\chi}^W(\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)|_{m_\chi^{-1}(D_\chi)}) &\simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \text{Ind}_{W_\chi}^W(m_\chi^* \mathcal{O}_{D_\chi}) \\ &\simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{R}_\theta. \end{aligned}$$

All together, we obtain

$$\begin{aligned} \mathfrak{M}(\mathcal{F} * \mathcal{E}_\theta \otimes \text{sign}_W) &\stackrel{(5.8)}{\simeq} \text{Ind}_{W_\chi}^W(\mathfrak{M}(\mathcal{F})|_{m_\chi^{-1}(D_\chi)}) \\ &\stackrel{(5.10)}{\simeq} H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{R}_\theta \otimes \text{sign}_W \\ &\stackrel{(5.7)}{\simeq} H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathfrak{M}(\mathcal{E}_\theta). \end{aligned}$$

Since the Mellin transform restricts to an equivalence on monodromic sheaves (4.5), the isomorphism above comes from an isomorphism

$$(5.11) \quad \mathcal{F} * (\mathcal{E}_\theta \otimes \text{sign}_W) \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_\theta \in \mathcal{D}_W(T).$$

The de Rham setting. By Theorem 4.4, the Mellin transform $\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)$, viewed as an object in $D_{qcoh}^b(\mathfrak{f}/W_{a,\lambda})$, descends to $\mathfrak{f} // W_{a,\lambda}$ and it follows that the restriction $\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)|_{m_\lambda^{-1}(D_\lambda)}$ descends to $m_\lambda^{-1}(D_\lambda) // W_{a,\lambda}$. Now the same argument as in the ℓ -adic setting, replacing D_χ and W_χ by D_λ and $W_{a,\lambda}$, gives the desired isomorphism (5.11). $\square$

Proposition 5.3 can be reformulated as follows:

Proposition 5.4. *Assume G has connected center. Let $\mathcal{F}$ be a $*$ -central complex and let $\theta = W\chi$ be a W -orbit of a tame character χ . For each $w \in W$, there is a canonical isomorphism in $\mathcal{D}(T)$*

$$(5.12) \quad a_w : w^* \mathcal{F} * \mathcal{E}_\theta \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes w^* \mathcal{E}_\theta,$$

such that the following diagram is commutative

$$(5.13) \quad \begin{array}{ccc} w^* \mathcal{F} * \mathcal{E}_\theta & \xrightarrow{a_w} & H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes w^* \mathcal{E}_\theta, \\ \downarrow & & \downarrow \\ \mathcal{F} * \mathcal{E}_\theta & \xrightarrow{a_e} & H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_\theta \end{array}$$

where the vertical arrows are the isomorphism induced from the W -equivariant structures on $\mathcal{F}$ and $\mathcal{E}_\theta$.

6. CHARACTER SHEAVES $\mathcal{M}_\theta$ AND DRINFELD CENTER OF HARISH-CHANDRA BIMODULES

Let $\mathcal{E}_\theta$ be the tame $*$ -central local system in Section 5. Consider the character sheaf $\mathcal{M}_\theta := \Phi_{\mathcal{E}_\theta} = \text{Ind}_{T \subset B}^G(\mathcal{E}_\theta)^W$. Recall the averaging functor $\text{Av}_*^U := \pi_* : \mathcal{D}(G) \rightarrow \mathcal{D}(G/U)$, where $\pi : G \rightarrow G/U$ is the quotient map. Let $\iota : T = B/U \rightarrow G/U$ be the closed embedding. The goal of this section is to prove the following:

Theorem 6.1. *There exists a positive integer N depending only on the type of the group G such that the following holds. Assume $k = \mathbb{C}$ or $\text{char } k = p$ is not dividing $N\ell$. We have*

$$\text{Av}_*^U(\mathcal{M}_\theta) \simeq \iota_* \mathcal{E}_\theta.$$

In particular, $\text{Av}_*^U(\mathcal{M}_\theta)$ is supported on $T = B/U$.

We will first establish Theorem 6.1 in the case $k = \mathbb{C}$ using the techniques developed in [BFO] and [BG] on Drinfeld center of Harish-Chandra bimodules and character D -modules. Then we construct a mixed characteristic lifting $\mathcal{M}_{\theta,A}$ of $\mathcal{M}_\theta$ over a strictly henselian discrete valuation ring A with residue field k of characteristic p not dividing $N\ell$, which is universally locally acyclic with respect to the quotient map $G_A \rightarrow G_A/U_A$. This allows us to use spreading out arguments to prove Theorem 6.1 in the positive characteristic case.

We will assume $k = \mathbb{C}$ until Section 6.8.

6.1. Hecke categories. Consider the left G and right $T \times T$ actions on $Y = G/U \times G/U$. To every $\chi, \chi' \in \check{T}$ we denote by $M_{\chi, \chi'}$ the category of G -equivariant D -modules on $G/U \times G/U$ which are $T \times T$ -monodromic with generalized monodromy (χ, χ') , that is, $U(\mathfrak{t}) \otimes U(\mathfrak{t})$ (acting as infinitesimal translations along the right action of $T \times T$) acts locally finite with generalized eigenvalues in (χ, χ') . Consider the quotient Y/T where T acts diagonally from the right. The group T acts on Y/T via the formula $t(xU, yU) \bmod T = (xtU, yU) \bmod T$. To every $\chi \in \check{T}$ we denote by M_χ the category of G -equivariant T -monodromic D -modules on Y/T with generalized monodromy χ . We write $\mathcal{D}(M_{\chi, \chi'})$ and $\mathcal{D}(M_\chi)$ for the corresponding G -equivariant monodromic derived categories.

The groups B and $T \times T$ act on $X = G/U$ by the formula $b(xU) = bxb^{-1}U$, $(t, t')(xU) = txt'U$. For any $(\chi_1, \chi_2) \in \check{T} \times \check{T}$ we write H_{χ_1, χ_2} for the category of U -equivariant $T \times T$ -monodromic D -modules on X with generalized monodromy (χ_1, χ_2) . For any $\chi \in \check{T}$ we write H_χ for the category of B -equivariant T -monodromic D -modules on X with generalized monodromy χ , where B acts on X by the same formula as before and T acts on X by the formula $t(xU) = txU$. We denote by $\mathcal{D}(H_\chi)$ (resp. $\mathcal{D}(H_{\chi_1, \chi_2})$) the corresponding B -equivariant (resp. U -equivariant) monodromic derived category.

6.2. The Harish-Chandra functor. Consider the following correspondence

$$G \xleftarrow{p} G \times G/B \xrightarrow{q} Y/T = (G/U \times G/U)/T,$$

where $p(g, xB) = g$ and $q(g, xB) = (gxU, xU) \bmod T$. The group G acts on G , $G \times G/B$ and Y/T by the formulas $a \cdot g = aga^{-1}$, $a \cdot (g, xB) = (aga^{-1}, axB)$, $a(xU, yU) = (axU, ayU)$. One can check that p and q are compatible with those G -actions.

Following [Gi1, MV], we consider the functor

$$(6.1) \quad \text{HC} = q_* p^\circ : \mathcal{D}(G) \rightarrow \mathcal{D}(Y/T).$$

The functor above admits a right adjoint $\text{CH} = p_* q^\circ : \mathcal{D}(Y/T) \rightarrow \mathcal{D}(G)$. We use the same notations for the corresponding functors between G -equivariant derived categories $\mathcal{D}(G/\text{ad}G)$ and $\mathcal{D}(G \backslash Y/T)$.

Consider the embedding $i : X \rightarrow Y, gU \rightarrow (gU, eU)$ and the projection map $\pi : G \rightarrow X = G/U$.

Lemma 6.2 ([MV, Gi1]). (1) *The functor $i^0 = i^![\dim X] : \mathcal{D}(G \backslash Y) \rightarrow \mathcal{D}(U \backslash X)$ is an equivalence of categories with the inverse given by $(i^0)^{-1} := \text{Ind}_{T \subset B}^G \circ i_*[\dim G - \dim B]$.* (2) *We have $i^0 \circ \text{HC} \simeq \pi_*$.* (3) *The identity functor is a direct summand of $\text{CH} \circ \text{HC}$.*

We have the convolution product $\mathcal{D}(G \backslash Y) \times \mathcal{D}(G \backslash Y) \rightarrow \mathcal{D}(G \backslash Y)$ given by $(\mathcal{F}, \mathcal{F}') \rightarrow (p_{13})_*(p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{F}')$. Here $p_{ij} : G \backslash (G/U \times G/U \times G/U) \rightarrow G \backslash Y = G \backslash (G/U \times G/U)$ is the projection on the (i, j) -factors. The convolution product on $\mathcal{D}(G \backslash Y)$ restricts to a convolution product on $\mathcal{D}(M_{\chi, \chi^{-1}})$. The equivalence $i^0 : \mathcal{D}(G \backslash Y) \simeq \mathcal{D}(U \backslash X)$ above induces convolution products on $\mathcal{D}(U \backslash X)$ and $\mathcal{D}(H_{\chi, \chi})$. In addition, there is an action of $\mathcal{D}(U \backslash X)$ on $\mathcal{D}(X)$ by right convolution. The convolution operation will be denoted by $*$.

We will need Lemma 6.3. Let X be an algebraic variety with an action of an affine algebraic group G . Denote the action map by $a : G \times X \rightarrow X$.

Lemma 6.3 ([BFO, Lemma 2.1]). *For any $\mathcal{A} \in \mathcal{D}(G)$, $\mathcal{F} \in \mathcal{D}(X)$, we have a canonical isomorphism*

$$R\Gamma(a_*(\mathcal{A} \boxtimes \mathcal{F})) \simeq R\Gamma(\mathcal{A}) \otimes_{U(\mathfrak{g})}^L R\Gamma(\mathcal{F}).$$

Example 6.1. Consider the action map $a : G \times G/U \rightarrow G/U, a(x, gU) = xgU$. Let $\delta \in \mathcal{D}(G/U)$ be the delta D -module supported at the base point $eU \in G/U$. For any D -module $\mathcal{F}$ on G , there is a canonical isomorphism $\text{Av}_*^U(\mathcal{F}) \simeq a_*(\mathcal{F} \boxtimes \delta)$ and lemma above implies that

$$R\Gamma(\text{Av}_*^U(\mathcal{F})) \simeq R\Gamma(a_*(\mathcal{F} \boxtimes \delta)) \simeq R\Gamma(\mathcal{F}) \otimes_{U(\mathfrak{g})}^L R\Gamma(\delta).$$

Note that $R\Gamma(\mathcal{F}) = \Gamma(\mathcal{F})$ (since G is affine) and $R\Gamma(\delta) \simeq U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}$, and it follows that

$$R\Gamma(\mathrm{Av}_*^U(\mathcal{F})) \simeq \Gamma(\mathcal{F}) \otimes_{U(\mathfrak{g})}^L U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}.$$

6.3. Character D -modules. We denote by $\mathcal{CS}(G)$ the category of finitely generated G -equivariant D -modules on G such that the action of the center $Z \subset U(\mathfrak{g})$, embedding as left invariant differential operators, is locally finite. To every W -orbit $\theta \subset \check{T}$, we denote by $\mathcal{CS}(G)_\theta$ the category of finitely generated G -equivariant D -modules on G such that the action of the center $Z(U(\mathfrak{g})) \subset U(\mathfrak{g})$ is locally finite and has generalized eigenvalues in $\{\lambda \in \check{\mathfrak{t}} \mid \exp(\lambda) \in \theta\}$. We denote by $\mathcal{D}(\mathcal{CS}(G))$ (resp. $\mathcal{D}(\mathcal{CS}(G)_\theta)$) the minimal triangulated full subcategory of $\mathcal{D}(G/\mathrm{ad}G)$ containing all objects $\mathcal{M} \in \mathcal{D}(G/\mathrm{ad}G)$ such that $\mathcal{H}^i(\mathcal{M}) \in \mathcal{CS}(G)$ (resp. $\mathcal{H}^i(\mathcal{M}) \in \mathcal{CS}(G)_\theta$). We call $\mathcal{CS}(G)$ and $\mathcal{CS}(G)_\theta$ (resp. $\mathcal{D}(\mathcal{CS}(G))$ and $\mathcal{D}(\mathcal{CS}(G)_\theta)$) the category (resp. derived category) of character D -modules on G and character D -modules on G with generalized central character θ .

Proposition 6.4. *We have the following:*

- (1) Let $\mathcal{G} \in \mathcal{CS}(G)_\theta$. Then $\mathrm{HC}(\mathcal{G}) \in \bigoplus_{\chi \in \theta} \mathcal{D}(M_\chi)$ and $\mathrm{Av}_*^U(\mathcal{G}) \in \bigoplus_{\chi \in \theta} \mathcal{D}(H_\chi)$.
- (2) The functors $\mathrm{Ind}_{T \subset B}^G$ and $\mathrm{Res}_{T \subset B}^G$ preserve the derived categories of character D -modules. The resulting functors $\mathrm{Ind}_{T \subset B}^G : \mathcal{D}(\mathcal{CS}(T)) \rightarrow \mathcal{D}(\mathcal{CS}(G))$, $\mathrm{Res}_{T \subset B}^G : \mathcal{D}(\mathcal{CS}(G)) \rightarrow \mathcal{D}(\mathcal{CS}(T))$ are independent of the choice of the Borel subgroup B and t -exact with respect to the natural t -structures on $\mathcal{D}(\mathcal{CS}(G))$ and $\mathcal{D}(\mathcal{CS}(T))$. Moreover, for any $\mathcal{G} \in \mathcal{CS}(G)$ we have $\mathrm{Res}_{T \subset B}^G(\mathcal{G}) \simeq (j_T)_!^*(\mathcal{G}|_{T^{\mathrm{rs}}})$, here $j_T : T^{\mathrm{rs}} \rightarrow T$ is the embedding.
- (3) Let $\mathcal{G} \in \mathcal{CS}(G)$. There is a canonical W -equivariant structure on $\mathrm{Res}_{T \subset B}^G(\mathcal{G})$. Let $j : G^{\mathrm{rs}} \rightarrow G$ be the open embedding. If $\mathcal{G} = j_! j^* \mathcal{G}$, then we have $\mathcal{G} \simeq \mathrm{Ind}_{T \subset B}^G(\mathrm{Res}_{T \subset B}^G(\mathcal{G}))^W$.

Proof. Part (1) and (2) are proved in [Gi1, Lu]. We now prove part (3). We first show that $\mathcal{F} = \mathrm{Res}_{T \subset B}^G(\mathcal{G})$ is canonically W -equivariant. Let $x \in N(T)$ and $w \in N(T)/T = W$ be its image in the Weyl group. Denote $B_x := \mathrm{Ad}_x B$. Consider the following commutative diagram

$$\begin{array}{ccccc} T & \longleftarrow & B & \longrightarrow & G \\ \downarrow w & & \downarrow \mathrm{Ad}_x & & \downarrow \mathrm{Ad}_x \\ T & \longleftarrow & B_x & \longrightarrow & G \end{array}$$

where $w : T \rightarrow T$ the natural action of $w \in W$ on T and the horizontal arrows are the natural inclusion and projection maps. The base change theorems and the fact that the functors $\mathrm{Res}_{T \subset B}^G$ and $\mathrm{Res}_{T \subset B_x}^G$ are canonical isomorphic (see part (2)) imply

$$(6.2) \quad \mathrm{Res}_{T \subset B}^G(\mathrm{Ad}_x^* \mathcal{G}) \simeq w^* \mathrm{Res}_{T \subset B_x}^G(\mathcal{G}) \simeq w^* \mathrm{Res}_{T \subset B}^G(\mathcal{G}).$$

Since $\mathcal{G}$ is G -conjugation equivariant, we have a canonical isomorphism $c_x : \mathcal{G} \simeq \mathrm{Ad}_x^* \mathcal{G}$. Applying $\mathrm{Res}_{T \subset B}^G$ to c_x and using (6.2) we get

$$(6.3) \quad \mathcal{F} = \mathrm{Res}_{T \subset B}^G(\mathcal{G}) \simeq \mathrm{Res}_{T \subset B}^G(\mathrm{Ad}_x^* \mathcal{G}) \simeq w^* \mathrm{Res}_{T \subset B}^G(\mathcal{G}) = w^* \mathcal{F}.$$

We claim that the isomorphism above depends only the image w and we denote it by

$$(6.4) \quad c_w : \mathcal{F} \simeq w^* \mathcal{F}.$$

To prove the claim it is enough to check that for $x \in T$ the restriction of the isomorphism (6.3) to T^{rs} is equal to the identity map. By [Gi1], the restriction $\mathcal{F}|_{T^{\text{rs}}}$ is canonically isomorphic to $\mathcal{G}|_{T^{\text{rs}}}$ and the map in (6.3) is equal to the restriction of c_x to T^{rs} . Since the adjoint action $\text{Ad}_x : G \rightarrow G$ is trivial on T , the claim follows from the fact that any T -equivariant structure of a local system on T is trivial. The G -conjugation equivariant structure on $\mathcal{G}$ implies $\{c_w\}_{w \in W}$ satisfies the required co-cycle condition and hence the data $(\mathcal{F}, \{c_w\}_{w \in W})$ defines a W -equivariant structure on $\mathcal{F} = \text{Res}_{T \subset B}^G(\mathcal{G})$. We shall prove $\text{Ind}_{T \subset B}^G(\mathcal{F})^W \simeq \mathcal{G}$. Let $c^{\text{rs}} : G^{\text{rs}} \rightarrow T^{\text{rs}} // W$ be the restriction of the Chevalley map $c : G \rightarrow T // W$ to G^{rs} . Since $\mathcal{G} \simeq j_{!*}(\mathcal{G}|_{G^{\text{rs}}})$ and $\text{Ind}_{T \subset B}^G(\mathcal{F})^W \simeq j_{!*}(c^{\text{rs}})^{\circ}(\tilde{\mathcal{F}})$, where $\tilde{\mathcal{F}} \in \mathcal{D}(T^{\text{rs}} // W)^{\vee}$ is the descent of $\mathcal{F}|_{T^{\text{rs}}}$ along the map $q^{\text{rs}} : T^{\text{rs}} \rightarrow T^{\text{rs}} // W$, it suffices to show $\mathcal{G}|_{G^{\text{rs}}} \simeq (c^{\text{rs}})^{\circ}(\tilde{\mathcal{F}}) \in \mathcal{D}(G^{\text{rs}}/G)^{\vee}$. This follows again from the fact that $\mathcal{G}|_{T^{\text{rs}}} \simeq \mathcal{F}|_{T^{\text{rs}}} \simeq (c^{\text{rs}})^{\circ}(\tilde{\mathcal{F}})|_{T^{\text{rs}}} \in \mathcal{D}(T^{\text{rs}}/W)^{\vee}$. $\square$

6.4. Drinfeld center of Harish-Chandra bimodules and character D -modules. We give a review of the work [BFO, BG] on Drinfeld center of Harish-Chandra bimodules and character D -modules.

Write $U = U(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$ and $Z = Z(U)$ for the center of U . Consider the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ on $\mathfrak{t}$ where ρ is the half-sum of positive roots. Denote by $\dot{W}$ the group W acting via the dot action $\dot{\mathfrak{t}}$. We have the Harish-Chandra isomorphism $hc : Z \simeq \mathcal{O}(\dot{\mathfrak{t}})^{\dot{W}}$ such that for any $\lambda \in \mathfrak{t}$ the center Z acts on the Verma module associated to λ via $z \rightarrow hc(z)(\lambda)$. For any $\lambda \in \mathfrak{t}$ we write $\mathfrak{m}_{\lambda}$ for the corresponding maximal ideal and denote by I_{λ} the maximal ideal of Z corresponding to $\mathfrak{m}_{\lambda}$ under the Harish-Chandra isomorphism. Consider the extended universal enveloping algebra $\tilde{U} = U \otimes_Z \mathcal{O}(\dot{\mathfrak{t}})$, where Z acts on $\mathcal{O}(\dot{\mathfrak{t}})$ via the Harish-Chandra isomorphism. We denote by $U_{\lambda} = U / U I_{\lambda}$, $U_{\hat{\lambda}} = \varprojlim (U / U I_{\lambda}^n)$, $\tilde{U}_{\lambda} = \tilde{U} / \tilde{U} \mathfrak{m}_{\lambda}$, and $\tilde{U}_{\hat{\lambda}} = \varprojlim (\tilde{U} / \tilde{U} \mathfrak{m}_{\lambda}^n)$. The action of $\dot{W}$ on $\mathcal{O}(\dot{\mathfrak{t}})$ gives rise to an action of $\dot{W}$ on $\tilde{U}$ such that $\tilde{U}^{\dot{W}} = U$. In addition, the stabilizer $\dot{W}_{\lambda}$ of $\lambda \in \mathfrak{t}$ in $\dot{W}$ acts naturally on $\tilde{U}_{\hat{\lambda}}$ and the natural inclusion $U \rightarrow \tilde{U}$ induces an isomorphism $U_{\hat{\lambda}} \simeq \tilde{U}_{\hat{\lambda}}^{\dot{W}_{\lambda}}$ (see, e.g., [BG, Section 1]).

We denote by $\mathcal{HC}_{\hat{\lambda}}$ the category of finitely generated Harish-Chandra bimodules over $U_{\hat{\lambda}}$, that is, finitely generated continuous $U_{\hat{\lambda}}$ -bimodules such that the diagonal action of $\mathfrak{g}$ is locally finite. We denote by $\mathcal{D}(\mathcal{HC}_{\hat{\lambda}})$ the corresponding derived category. The tensor product $\mathcal{M} \otimes_U \mathcal{M}', \mathcal{M}, \mathcal{M}' \in \mathcal{HC}_{\hat{\lambda}}$ (resp. $\mathcal{M} \otimes_U^L \mathcal{M}', \mathcal{M}, \mathcal{M}' \in \mathcal{D}(\mathcal{HC}_{\hat{\lambda}})$) defines a monoidal structure on $\mathcal{HC}_{\hat{\lambda}}$ (resp. $\mathcal{D}(\mathcal{HC}_{\hat{\lambda}})$).

Recall that $\lambda \in \mathfrak{t}$ is called regular if $\dot{W}_{\lambda} = 0$, that is, λ does not lie on any coroot hyperplane shifted by $-\rho$, and it is called dominant if the value of λ at any positive coroot is not a negative integer.

Proposition 6.5 ([BFO, Proposition 3.1]). *Let $\chi \in \check{T}$ and $\lambda \in \mathfrak{t}$ be a dominant regular lifting of χ . The functor*

$$R\Gamma^{\hat{\lambda}, -\widehat{\lambda-2\rho}} : (\mathcal{D}(M_{\chi, \chi^{-1}}), *) \simeq (\mathcal{D}(\mathcal{HC}_{\hat{\lambda}}), \otimes_U^L)$$

is an equivalence of monoidal categories. Here $R\Gamma^{\hat{\lambda}, -\widehat{\lambda-2\rho}}$ is the functor of taking the maximal summand of $R\Gamma(\mathcal{F})$, $\mathcal{F} \in \mathcal{D}(M_{\chi, \chi^{-1}})$ where $\mathfrak{t} \times \mathfrak{t}$ acts with the generalized eigenvalue $(\lambda, -\lambda - 2\rho)$.

Let $\chi \in \tilde{T}$ and $\lambda \in \mathfrak{t}$ be as in Proposition 6.5 and consider the equivalence of monoidal categories

$$(6.5) \quad \mathbf{M} : (\mathcal{D}(H_{\chi, \chi}), *) \simeq (\mathcal{D}(M_{\chi, \chi^{-1}}), *) \xrightarrow{R\Gamma^{\hat{\lambda}, -\hat{\lambda}-2\rho}} (\mathcal{D}(\mathcal{HC}_{\hat{\lambda}}), \otimes_U^L).$$

We denote by $\mathcal{HC}$ the category of finitely generated Harish-Chandra bimodules over U (no restriction on the action of the center Z). We denote by $\mathcal{HC}_{\hat{\chi}}$ the category of finitely generated Harish-Chandra bimodules over the product $\prod_{\mu \in \lambda + \Lambda / W'_\chi} U_{\hat{\mu}}$ (here W'_χ acts on $\mu \in \lambda + \Lambda$ via the dot action). We denote by $Z(\mathcal{HC}_{\hat{\lambda}}, \otimes_U)$ (resp. $Z(\mathcal{HC}, \otimes_U)$, $Z(\mathcal{HC}_{\hat{\chi}}, \otimes_U)$, $Z(H_{\xi, \xi}^t, *^t)$) the Drinfeld center of the monoidal category $(\mathcal{HC}_{\hat{\lambda}}, \otimes_U)$ (resp. $(\mathcal{HC}, \otimes_U)$, $(\mathcal{HC}_{\hat{\chi}}, \otimes_U)$, $(H_{\chi, \chi}^t, *^t)$). Recall an element in $Z(\mathcal{HC}_{\hat{\lambda}}, \otimes_U)$ consists of an element $\mathcal{M} \in \mathcal{HC}_{\hat{\lambda}}$ together with family of compatible isomorphisms $b_{\mathcal{F}} : \mathcal{M} \otimes_U \mathcal{F} \simeq \mathcal{F} \otimes_U \mathcal{M}$ for $\mathcal{F} \in \mathcal{HC}_{\hat{\lambda}}$. Recall the notion of translation functor $\theta_{\hat{\lambda}}^{\mu} : \mathcal{HC}_{\hat{\lambda}} \rightarrow \mathcal{HC}_{\hat{\mu}}$ where $\mu \in \lambda + \Lambda$ (see, e.g., [BG]).

Theorem 6.6.

(1) [BFO, Lemma 3.7] *There is a lifting $\theta_{\hat{\lambda}}^{\mu} : Z(\mathcal{HC}_{\hat{\lambda}}, \otimes_U) \rightarrow Z(\mathcal{HC}_{\hat{\mu}}, \otimes_U)$ such that the functor $\mathbf{F} : Z(\mathcal{HC}_{\hat{\lambda}}, \otimes_U) \rightarrow Z(\mathcal{HC}_{\hat{\chi}}, \otimes_U)$, $L \rightarrow \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} \theta_{\hat{\lambda}}^{\mu}(L)$ define an equivalence of braided monoidal categories.*

(2) [BFO, Theorem 3.6, Lemma 3.8] *For any $\mathcal{M} \in \mathcal{D}_G(G)^{\vee}$, the global section $\Gamma(\mathcal{M})$ is naturally a Harish-Chandra bimodule, with a canonical central structure and the resulting functor $\Gamma : \mathcal{D}_G(G)^{\vee} \rightarrow Z(\mathcal{HC}, \otimes_U)$ is an equivalence of abelian categories. Moreover, the equivalence above restricts to an equivalence $\mathcal{CS}(G)_{\theta} \simeq Z(\mathcal{HC}_{\hat{\chi}}, \otimes_U)$ and the composed equivalence*

$$\mathcal{CS}(G)_{\theta} \simeq Z(\mathcal{HC}_{\hat{\chi}}, \otimes_U) \xrightarrow{\mathbf{F}^{-1}} Z(\mathcal{HC}_{\hat{\lambda}}, \otimes_U)$$

is isomorphic to $R\Gamma^{\hat{\lambda}, -\hat{\lambda}-2\rho} \circ \pi_Y^{\circ} \circ \mathbf{HC}$. Here $\pi_Y : Y \rightarrow Y/T$ is the projection map.

6.5. Harish-Chandra bimodules $\mathcal{Z}_{\lambda}$. In this subsection we attach to each dominant regular lift $\lambda \in \mathfrak{t}$ of $\chi \in \tilde{T}$ an element $\mathcal{Z}_{\lambda} \in Z(\mathcal{HC}_{\hat{\lambda}}, \otimes_U)$ in the Drinfeld center of Harish-Chandra bimodules, and identify it with the local system $\mathcal{E}_{\chi}$ introduced in Section 5.1 under the equivalence in (6.5). We will use the notation $R_T = \text{Sym}(\pi_1(T(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{C})$ in Section 5.1 and the identification $R_T \simeq \text{Sym}(\mathfrak{t}) \simeq \mathcal{O}(\mathfrak{t})$ for the rest of the section.

Recall the identification $W_{a, \lambda} \simeq W_{\chi}$ in (4.4). We have a natural inclusion $\dot{W}_{\lambda} \subset W_{a, \lambda}$ sending $\dot{w}$ to $(w, w^{-1}v - v) = (w, \rho - w^{-1}\rho)$ and it is straightforward to check that the identification above maps $\dot{W}_{\lambda} \subset W_{a, \lambda}$ isomorphically onto the subgroup $W_{\lambda+\rho} \subset W_{\chi}$. Since the map $m_{-\lambda} : \mathfrak{t} \rightarrow \mathfrak{t}$ of translation by $-\lambda$ intertwines the $W_{a, \lambda}$ -action with the W_{χ} -action, we have an isomorphism $m_{-\lambda}^* : R_0^{W_{\lambda+\rho}} \simeq R_{\hat{\lambda}}^{\dot{W}_{\lambda}}$, where $R_0^{W_{\lambda+\rho}}$ (resp. $R_{\hat{\lambda}}^{\dot{W}_{\lambda}}$) is the $W_{\lambda+\rho}$ -invariant of the completion R_0 of $R_T \simeq \mathcal{O}(\mathfrak{t})$ at the origin 0 (resp. $\dot{W}_{\lambda}$ -invariant of the completion $R_{\hat{\lambda}}$ of R_T at λ). Consider the following quotient

$$R_{\chi}^{W_{\lambda+\rho}} = R_0^{W_{\lambda+\rho}} / \langle R_{0,+}^{W_{\chi}} \rangle,$$

where $\langle R_{0,+}^{W_{\chi}} \rangle$ is the ideal generated by the augmentation ideal $R_{0,+}^{W_{\chi}}$ of $R_0^{W_{\chi}}$. We introduce the following Harish-Chandra bimodule

$$(6.6) \quad \mathcal{Z}_{\lambda} = U_{\hat{\lambda}} \otimes_{R_0^{W_{\lambda+\rho}}} R_{\chi}^{W_{\lambda+\rho}} \in \mathcal{HC}_{\hat{\lambda}}.$$

Here $R_0^{W_{\lambda+\rho}}$ acts on $U_{\hat{\lambda}}$ via the the map

$$(6.7) \quad b_{\lambda} : R_0^{W_{\lambda+\rho}} \xrightarrow{m_{\lambda}^*} R_{\hat{\lambda}}^{W_{\lambda}} \rightarrow Z(\tilde{U}_{\hat{\lambda}}^{W_{\lambda}}) \rightarrow Z(U_{\hat{\lambda}}),$$

where the last map is induced from the isomorphism $\tilde{U}_{\hat{\lambda}}^{W_{\lambda}} \simeq U_{\hat{\lambda}}$. Equivalently, consider the $R_{\hat{\lambda}}^{W_{\lambda}}$ -module

$$(6.8) \quad R_{\chi}^{W_{\lambda}} = R_{\hat{\lambda}}^{W_{\lambda}} / \langle R_{\hat{\lambda},+}^{W_{a,\lambda}} \rangle$$

here $\langle R_{\hat{\lambda},+}^{W_{a,\lambda}} \rangle$ is the ideal generated by the augmentation ideal $R_{\hat{\lambda},+}^{W_{a,\lambda}}$ associated to the evaluation morphism $\epsilon_{\lambda} : R_{\hat{\lambda}}^{W_{a,\lambda}} \rightarrow \mathbb{C}$ at λ . Then we have

$$(6.9) \quad Z_{\lambda} \simeq U \otimes_Z R_{\chi}^{W_{\lambda}},$$

where Z acts on $R_{\chi}^{W_{\lambda}}$ via the Harish-Chandra isomorphism $hc : Z \simeq \mathcal{O}(\mathfrak{t})^{\dot{W}} \simeq R_T^{\dot{W}}$.

Proposition 6.7. *Let $\lambda \in \check{\mathfrak{t}}$ be a dominant regular lift of $\chi \in \check{T}$.*

- (1) *To every $\mathcal{M} \in \mathcal{HC}_{\hat{\lambda}}$ there is a canonical isomorphism*

$$b_{\mathcal{M}} : Z_{\lambda} \otimes_U \mathcal{M} \simeq \mathcal{M} \otimes_U Z_{\lambda}$$

such that the data $(Z_{\lambda}, b_{\mathcal{M}})_{\mathcal{M} \in \mathcal{HC}_{\hat{\lambda}}}$ defines an element in the Drinfeld center $Z(\mathcal{HC}_{\hat{\lambda}}, \otimes_U)$.

- (2) *Let $\mathcal{E}_{\chi} \in H_{\chi}$ be the local system in Section 5.1, viewed as an object in $H_{\chi,\chi}$ via the natural embedding $H_{\chi} \subset H_{\chi,\chi}$, and consider its image $\mathbf{M}(\mathcal{E}_{\chi}) \in \mathcal{D}(\mathcal{HC}_{\hat{\lambda}})$. We have $\mathbf{M}(\mathcal{E}_{\chi}) \simeq Z_{\lambda}$.*

Proof. Proof of (1). We have $W_{\lambda+\rho} = \dot{W}_{\lambda} = e$ since λ is regular. To every $\mathcal{M} \in \mathcal{HC}_{\hat{\lambda}}$, the map b_{λ} in (6.7) gives rise to an action of $R_0 \otimes_{\mathbb{C}} R_0 = R_0^{W_{\lambda+\rho}} \otimes_{\mathbb{C}} R_0^{W_{\lambda+\rho}}$ on $\mathcal{M}$ and it follows from [MS, Theorem 4.1] that this action factors through $R_0 \otimes_{\mathbb{C}} R_0 \rightarrow R_0 \otimes_{R_0^{W_{\chi}}} R_0$. Indeed, it is shown in *loc. cit.* that for every finite dimensional representation V of $\mathfrak{g}$, the action of $R_0 \otimes_{\mathbb{C}} R_0$ on the bimodule $\text{pr}_{\hat{\lambda}}(V \otimes_{\mathbb{C}} U / U I_{\lambda}^n) \in \mathcal{HC}_{\hat{\lambda}}$, $n \in \mathbb{Z}_{\geq 0}$ factors through $R_0 \otimes_{R_0^{W_{\chi}}} R_0$. Here $\text{pr}_{\hat{\lambda}}(-)$ is the projection of the summand on which the action of I_{λ} is locally nilpotent.⁸ Since every object in $\mathcal{HC}_{\hat{\lambda}}$ is isomorphic to a quotient of $\text{pr}_{\hat{\lambda}}(V \otimes_{\mathbb{C}} U / U I_{\lambda}^n)$ for some V and n , the claim follows. Therefore, for every $\mathcal{M} \in \mathcal{HC}_{\hat{\lambda}}$, we have a canonical isomorphism

$$b_{\mathcal{M}} : Z_{\lambda} \otimes_U \mathcal{M} \simeq R_0 / \langle R_{0,+}^{W_{\chi}} \rangle \otimes_{R_0} \mathcal{M} \simeq \mathcal{M} \otimes_{R_0} R_0 / \langle R_{0,+}^{W_{\chi}} \rangle \simeq \mathcal{M} \otimes_U Z_{\lambda}.$$

It follows from the construction that those isomorphisms satisfy the required compatibility conditions and the data $(Z_{\lambda}, b_{\mathcal{M}})$ defines an element in $Z(\mathcal{HC}_{\hat{\lambda}}, \otimes_U)$.

Proof of (2). Let $\tilde{\mathcal{E}}_{\chi} \in M_{\chi}$ be the image of $\mathcal{E}_{\chi}$ under the equivalence $(i^0)^{-1} : H_{\chi} \simeq M_{\chi}$ in Lemma 6.2. Then by definition we have $\mathbf{M}(\mathcal{E}_{\chi}) \simeq R\Gamma^{\hat{\lambda}, -\lambda-2\rho}(\pi_Y^{\circ} \tilde{\mathcal{E}}_{\chi})$, where $\pi_Y : Y \rightarrow Y/T$. Consider the map

$$a : T \times (G/U \times G/U)/T \rightarrow (G/U \times G/U)/T, (t, gU, g'U) \rightarrow (gtU, g'U).$$

⁸In fact, it was shown in [MS, Theorem 4.1] that, for any $\lambda \in \check{\mathfrak{t}}(\mathbb{C})$ and any finite dimensional representation V , the action of $R_0 \otimes_{\mathbb{C}} R_0$ on $\text{pr}_{\hat{\lambda}}(V \otimes_{\mathbb{C}} U / U I_{\lambda}^n)$ factors through $R_0 \otimes_{R_0^{W_{\chi}}} R_0$. But the proof in *loc. cit.* actually shows that, if we fix $\lambda \in \check{\mathfrak{t}}(\mathbb{C})$, the action factors through $R_0 \otimes_{R_0^{W_{\chi}}} R_0$.

Then it follows from the definition of $(i^0)^{-1}$ that $\tilde{\mathcal{E}}_\chi = a_*(\mathcal{E}_\chi \boxtimes \Delta_* \mathcal{O}_{G/B})$, here $\Delta : G/B \rightarrow (G/U \times G/U)/T$ is the embedding $gB \rightarrow (gU, gU) \bmod T$. It is shown in [BFO, Proposition 3.1] that $R\Gamma(\Delta_* \mathcal{O}_{G/B} \otimes p_2^* \omega_{G/B}) \simeq \tilde{U}$ (here p_2 is the right projection map $(G/U \times G/U)/T \rightarrow G/B$), hence by Lemma 6.3 we get

$$\begin{aligned} R\Gamma^{\hat{\lambda}, -\widehat{\lambda-2\rho}}(\pi_Y^\circ \tilde{\mathcal{E}}_\chi) &\simeq R\Gamma^{\hat{\lambda}}(\tilde{\mathcal{E}}_\chi \otimes p_2^* \omega_{G/B}) = R\Gamma^{\hat{\lambda}}(a_*(\mathcal{E}_\chi \boxtimes (\Delta_* \mathcal{O}_{G/B} \otimes p_2^* \omega_{G/B}))) \\ &\simeq R\Gamma(\Delta_* \mathcal{O}_{G/B} \otimes p_2^* \omega_{G/B}) \otimes_{R_T}^L R\Gamma^{\hat{\lambda}}(\mathcal{E}_\chi) \simeq \tilde{U} \otimes_{R_T} \Gamma^{\hat{\lambda}}(\mathcal{E}_\chi). \end{aligned}$$

On the other hand, Proposition 5.1 implies that, as a $\mathcal{O}(\mathfrak{t})$ -module, we have $\Gamma(\mathcal{E}_\chi) \simeq \Gamma(\mathcal{R}_\chi) \simeq \bigoplus_{\mu \in \mathfrak{t}, \exp(\mu)=\chi} \Gamma(m_\mu^*(\mathcal{O}_{D_\lambda})) \stackrel{(5.2)}{\simeq} \bigoplus_{\mu \in \mathfrak{t}, \exp(\mu)=\chi} m_\mu^* R_\chi$. Thus $\Gamma^{\hat{\lambda}}(\mathcal{E}_\chi) \simeq m_{-\lambda}^* R_\chi \simeq m_{-\lambda}^* R_\chi^{W_{\lambda+\rho}} \stackrel{(6.8)}{\simeq} R_\chi^{W_\lambda}$ (since λ is regular we have $W_{\lambda+\rho} = e$) and we conclude that

$$\mathbf{M}(\mathcal{E}_\chi) \simeq R\Gamma^{\hat{\lambda}, -\widehat{\lambda-2\rho}}(\pi_Y^\circ \tilde{\mathcal{E}}_\chi) \simeq \tilde{U} \otimes_{R_T} \Gamma^{\hat{\lambda}}(\mathcal{E}_\chi) \simeq \tilde{U} \otimes_{R_T} R_\chi^{W_\lambda} \simeq \mathcal{Z}_\lambda.$$

Part (2) follows. $\square$

We finish this section with a lemma to be used in the next section.

Lemma 6.8. *Let $\lambda \in \mathfrak{t}$ be regular dominant weight and let μ be a dominant weight in $\lambda + \Lambda$. We have $\theta_\lambda^\mu(\mathcal{Z}_\lambda) \simeq \mathcal{Z}_\mu$.*

Proof. Note that $\theta_\lambda^\mu : \mathcal{HC}_{\hat{\lambda}} \rightarrow \mathcal{HC}_{\hat{\mu}}$ is monoidal, so we have $\theta_\lambda^\mu(U_{\hat{\lambda}}) \simeq U_{\hat{\mu}}$. Note also that, by [BG, Proposition 1.4], for any $r \in R_0^{W_\chi}$ and $\mathcal{M} \in \mathcal{HC}_{\hat{\lambda}}$, we have $\theta_\lambda^\mu(b_\lambda(r) \cdot m) = b_\mu(r) \cdot \theta_\lambda^\mu(m)$, where $m \in \mathcal{M}$ and $b_\nu : R_0^{W_{\mu+\rho}} \rightarrow Z(U_{\hat{\nu}})$, $\nu \in \mathfrak{t}$ is the map in (6.7). Since $\mathcal{Z}_\lambda$ (resp. $\mathcal{Z}_\mu$) is isomorphic to the coinvariant algebra of $U_{\hat{\lambda}}$ (resp. $U_{\hat{\mu}}$) with respect to the action of $R_0^{W_\chi}$ via the map b_λ (resp. b_μ), it follows that $\theta_\lambda^\mu(\mathcal{Z}_\lambda) \simeq \mathcal{Z}_\mu$. $\square$

6.6. Proof of Theorem 6.1 in the de Rham setting. Let $\mathcal{Z}_\lambda \in Z(\mathcal{HC}_{\hat{\lambda}}, \otimes_U)$ and let $\tilde{\mathcal{E}}_\chi \in M_\chi$ be the image of $\mathcal{E}_\chi$ under the equivalence $(i^0)^{-1} : H_\chi \simeq M_\chi$. Define $\tilde{\mathcal{E}}_\theta \simeq \bigoplus_{\chi \in \theta} \tilde{\mathcal{E}}_\chi \in \mathcal{D}(G \backslash Y)^\vee$. By the discussion above there exists a character D -module $\mathcal{M}'_\theta \in \mathcal{CS}(G)_\theta$ such that

$$R\Gamma^{\hat{\lambda}, -\widehat{\lambda-2\rho}} \circ \pi_Y^\circ \circ \mathrm{HC}(\mathcal{M}'_\theta) = \mathcal{Z}_\lambda.$$

Hence by Proposition 6.7, we have

$$R\Gamma^{\hat{\lambda}, -\widehat{\lambda-2\rho}} \circ \pi_Y^\circ \circ \mathrm{HC}(\mathcal{M}'_\theta) \simeq R\Gamma^{\hat{\lambda}, -\widehat{\lambda-2\rho}}(\pi^\circ \tilde{\mathcal{E}}_\chi)$$

for any regular dominant $\lambda \in \mathfrak{t}$ mapping to χ . Since $\pi_Y^\circ \circ \mathrm{HC} : \mathcal{D}(CS_\theta) \rightarrow \bigoplus_{\chi \in \theta} \mathcal{D}(M_{\chi, \chi^{-1}})$ and $R\Gamma^{\hat{\lambda}, -\widehat{\lambda-2\rho}} : \mathcal{D}(M_{\chi, \chi^{-1}}) \simeq \mathcal{HC}_{\hat{\lambda}}$ is an equivalence of category for any regular dominant λ , it follows that

$$(6.10) \quad \mathrm{HC}(\mathcal{M}'_\theta) \simeq \tilde{\mathcal{E}}_\theta.$$

Applying the equivalence $i^\circ : \mathcal{D}(M_\chi) \simeq \mathcal{D}(H_\chi)$ and using Lemma 6.2 (2), we see that

$$(6.11) \quad \mathrm{Av}_*^U(\mathcal{M}'_\theta) \simeq i^\circ \circ \mathrm{HC}(\mathcal{M}'_\theta) \simeq i^\circ \tilde{\mathcal{E}}_\theta \simeq \mathcal{E}_\theta.$$

Note that (6.10) and (6.11) together with Lemma 6.2 (3) imply that $\mathcal{M}'_\theta$ is a direct summand of

$$\mathrm{CH} \circ \mathrm{HC}(\mathcal{M}'_\theta) \simeq \mathrm{CH}(\tilde{\mathcal{E}}_\theta) \simeq \mathrm{Ind}_{T \subset B}^G(\mathcal{E}_\theta).$$

Since $\text{Ind}_{T \subset B}^G(\mathcal{E}_\theta)$ is isomorphic to the intermediate extension of its restriction to G^{rs} , as a direct factor of $\text{Ind}_{T \subset B}^G(\mathcal{E}_\theta)$, we also have $\mathcal{M}'_\theta \simeq j_{!*} j^* \mathcal{M}'_\theta$. Therefore, by Proposition 6.4 (3), there is a canonical W -equivariant structure $\phi' = \{\phi'_w\}$ on $\mathcal{E}_\theta \simeq \text{Res}_{T \subset B}^G(\mathcal{M}'_\theta)$ such that $\mathcal{M}'_\theta \simeq \text{Ind}_{T \subset B}^G(\mathcal{E}_\theta)^W$. We claim that the above W -equivariant structure on $\mathcal{E}_\theta$ is isomorphic to the one $\phi = \{\phi_w\}$ constructed in Section 5.1. Thus we have $\mathcal{M}'_\theta \simeq \mathcal{M}_\theta$. The theorem follows.

6.7. Proof of the claim. The proof is a bit long but the idea behind it is simple. Since we are in the de Rham setting, to identify the two W -equivariant structures ϕ and ϕ' on $\mathcal{E}_\theta$, it suffices to identify the corresponding W -actions on the space of global sections $\Gamma(\mathcal{E}_\theta)$. To this end, we will give explicit descriptions of various D -modules and morphisms introduced in the previous section at the level of global sections.

6.7.1. Let $x \in N(T)$ and $w \in W$ its image in the Weyl group. Consider the maps

$$\Gamma(\phi_w) : \Gamma(\mathcal{E}_\theta) \rightarrow \Gamma(w^* \mathcal{E}_\theta) \simeq \Gamma(\mathcal{E}) \quad \Gamma(\phi'_w) : \Gamma(\mathcal{E}_\theta) \rightarrow \Gamma(w^* \mathcal{E}_\theta) \simeq \Gamma(\mathcal{E})$$

$$\Gamma(\phi_x) : \Gamma(\mathcal{M}'_\theta) \rightarrow \Gamma(\text{Ad}_x^* \mathcal{M}'_\theta) \simeq \Gamma(\mathcal{M}'_\theta)$$

coming from the W -equivariant structures $\phi_w, \phi'_w : \mathcal{E}_\theta \simeq w^* \mathcal{E}_\theta$ and the G -equivariant structure $\phi_x : \mathcal{M}'_\theta \simeq \text{Ad}_x^* \mathcal{M}'_\theta$ on $\mathcal{E}_\theta$ and $\mathcal{M}'_\theta$ respectively.⁹ Our goal is to show $\Gamma(\phi_w) = \Gamma(\phi'_w)$.

Consider the following commutative diagram

$$(6.12) \quad \begin{array}{ccccc} G & \xrightarrow{\pi} & G/U & \xleftarrow{\iota} & T \\ \downarrow \text{Ad}_x & & \downarrow \text{Ad}_x & & \downarrow \text{Ad}_x = w \\ G & \xrightarrow{\pi_w} & G/U_w & \xleftarrow{\iota_w} & T \end{array}$$

where $U_w = xUx^{-1}$ with Lie algebra $\mathfrak{n}_w$ and the horizontal maps are the natural quotient maps or embeddings. The diagram (6.12) gives rise to natural isomorphisms

$$\Gamma(\iota_* \phi_w) : \Gamma(\iota_* \mathcal{E}_\theta) \simeq \Gamma(\iota_* w^* \mathcal{E}) \simeq \Gamma(\text{Ad}_x^* (\iota_w)_* \mathcal{E}_\theta) \simeq \Gamma((\iota_w)_* \mathcal{E}_\theta)$$

$$\Gamma(\iota_* \phi'_w) : \Gamma(\iota_* \mathcal{E}_\theta) \simeq \Gamma(\iota_* w^* \mathcal{E}) \simeq \Gamma(\text{Ad}_x^* (\iota_w)_* \mathcal{E}_\theta) \simeq \Gamma((\iota_w)_* \mathcal{E}_\theta)$$

$$\Gamma(\text{Av}_*^U(\phi_x)) : \Gamma(\text{Av}_*^U \mathcal{M}'_\theta) \simeq \Gamma(\text{Av}_*^U \text{Ad}_x^* \mathcal{M}'_\theta) \simeq \Gamma(\text{Ad}_x^* \text{Av}_*^{U_w} \mathcal{M}'_\theta) \simeq \Gamma(\text{Av}_*^{U_w} \mathcal{M}'_\theta).$$

By Lemma 6.3 and Example 6.1, we have

$$(6.13) \quad \Gamma(\text{Av}_*^U(\mathcal{M}'_\theta)) \simeq (U/U\mathfrak{n}) \otimes_U \Gamma(\mathcal{M}'_\theta) \quad \Gamma(\iota_* \mathcal{E}_\theta) = (U/U\mathfrak{n}) \otimes_{R_T} \Gamma(\mathcal{E}_\theta)$$

$$(6.14) \quad \Gamma(\text{Av}_*^{U_w}(\mathcal{M}'_\theta)) \simeq (U/U\mathfrak{n}_w) \otimes_U \Gamma(\mathcal{M}'_\theta) \quad \Gamma((\iota_w)_* \mathcal{E}_\theta) = (U/U\mathfrak{n}_w) \otimes_{R_T} \Gamma(\mathcal{E}_\theta)$$

and the maps $\Gamma(\iota_* \phi_w)$, $\Gamma(\iota_* \phi'_w)$, and $\Gamma(\text{Av}_*^U(\phi_x))$ above are given by

$$(6.15) \quad \begin{aligned} \Gamma(\iota_* \phi_w) &= \text{Ad}_x(-) \otimes \Gamma(\phi_w) : \Gamma(\iota_* \mathcal{E}_\theta) \\ &= U/U\mathfrak{n} \otimes_{R_T} \Gamma(\mathcal{E}_\theta) \longrightarrow \Gamma((\iota_w)_* \mathcal{E}_\theta) \\ &= U/U\mathfrak{n}_w \otimes_{R_T} \Gamma(\mathcal{E}_\theta) \end{aligned}$$

⁹Recall that there are canonical identifications of vector spaces $\Gamma(\text{Ad}_x^* \mathcal{M}'_\theta) \simeq \Gamma(\mathcal{M}'_\theta)$, $\Gamma(w^* \mathcal{E}_\theta) \simeq \Gamma(\mathcal{E}_\theta)$

$$\begin{aligned}
(6.16) \quad \Gamma(\iota_* \phi'_w) &= \text{Ad}_x(-) \otimes \Gamma(\phi'_w) : \Gamma(\iota_* \mathcal{E}_\theta) \\
&= \mathbf{U} / \mathbf{U} \mathbf{n} \otimes_{R_T} \Gamma(\mathcal{E}_\theta) \longrightarrow \Gamma((\iota_w)_* \mathcal{E}_\theta) \\
&= \mathbf{U} / \mathbf{U} \mathbf{n}_w \otimes_{R_T} \Gamma(\mathcal{E}_\theta)
\end{aligned}$$

$$\begin{aligned}
(6.17) \quad \Gamma(\text{Av}_*^U(\phi_x)) &= \text{Ad}_x(-) \otimes \Gamma(\phi_x) : \Gamma(\text{Av}_*^U \mathcal{M}'_\theta) \\
&= \mathbf{U} / \mathbf{U} \mathbf{n} \otimes_{\mathbf{U}} \Gamma(\mathcal{M}'_\theta) \rightarrow \Gamma(\text{Av}_*^{U_w} \mathcal{M}'_\theta) \\
&= \mathbf{U} / \mathbf{U} \mathbf{n}_w \otimes_{\mathbf{U}} \Gamma(\mathcal{M}'_\theta).
\end{aligned}$$

Thus to prove the claim, that is $\Gamma(\phi_w) = \Gamma(\phi'_w)$, it suffices to show (6.16) = (6.15). On the other hand, it follows from the construction of the $\mathbf{W}$ -equivariant structure ϕ in Proposition 6.4 that the map (6.16) is, by definition, given by the composition

$$\Gamma(\iota_* \mathcal{E}_\theta) \stackrel{(6.11)}{\simeq} \Gamma(\text{Av}_*^U \mathcal{M}'_\theta) \stackrel{(6.17)}{\simeq} \Gamma(\text{Av}_*^{U_w} \mathcal{M}'_\theta) \stackrel{(6.11)}{\simeq} \Gamma((\iota_w)_* \mathcal{E}_\theta).$$

Here the last arrow is induced by the isomorphism $\text{Av}_*^{U_x}(\mathcal{M}'_\theta) \simeq (\iota_w)_* \mathcal{E}_\theta$ in (6.11) in the case when the unipotent radical is U_w . Thus we need to identify the map above with (6.15), or equivalently, to prove the commutativity of the following diagram

$$\begin{array}{ccc}
(6.18) \quad \Gamma(\text{Av}_*^U \mathcal{M}'_\theta) & \xrightarrow{(6.17)} & \Gamma(\text{Av}_*^{U_w} \mathcal{M}'_\theta) \\
(6.11) \downarrow & & \downarrow (6.11) \\
\Gamma(\iota_* \mathcal{E}_\theta) & \xrightarrow{(6.15)} & \Gamma((\iota_w)_* \mathcal{E}_\theta).
\end{array}$$

6.7.2. We shall describe the map (6.11) in diagram (6.18) (see line (6.22)). We first describe $\Gamma(\mathcal{M}'_\theta)$. By Theorem 6.6, we have $\Gamma(\mathcal{M}'_\theta) \simeq \bigoplus_{\mu \in \lambda + \Lambda / \mathbf{W}'_\chi} \theta_\lambda^\mu(\mathcal{Z}_\lambda)$. Since $\theta_\lambda^\mu(\mathcal{Z}_\lambda) \simeq \mathcal{Z}_\mu$ for any dominant $\mu \in \lambda + \Lambda$ (Lemma 6.8) and every $\mathbf{W}'_\chi$ -orbit in $\lambda + \Lambda$ (under the dot action) contains a unique dominant weight, we have

$$(6.19) \quad \Gamma(\mathcal{M}'_\theta) \simeq \bigoplus_{\mu \in \lambda + \Lambda / \mathbf{W}'_\chi} \theta_\lambda^\mu(\mathcal{Z}_\lambda) \simeq \bigoplus_{\mu \in \lambda + \Lambda / \mathbf{W}'_\chi} \mathcal{Z}_\mu \stackrel{(6.9)}{\simeq} \bigoplus_{\mu \in \lambda + \Lambda / \mathbf{W}'_\chi} \mathbf{U} \otimes_Z R_\chi^{\dot{\mathbf{W}}\mu}.$$

We claim that there is a natural isomorphism of $Z \simeq R_T^{\dot{\mathbf{W}}} \simeq R_T^{\mathbf{W}_{a,-\rho}}$ -modules (note that $\dot{\mathbf{W}} = \mathbf{W}_{a,-\rho}$ as a subgroup of $\mathbf{W}_a$)

$$(6.20) \quad \bigoplus_{\mu \in \lambda + \Lambda / \mathbf{W}'_\chi} R_\chi^{\dot{\mathbf{W}}\mu} \simeq \Gamma(\mathcal{E}_\theta)^{\mathbf{W}_{a,-\rho}}.$$

Indeed, the map $\check{\mathfrak{t}} // \dot{\mathbf{W}}_\mu \rightarrow \check{\mathfrak{t}} // \dot{\mathbf{W}}$ is étale at the image of $\nu \in \dot{\mathbf{W}}_\mu \subset \check{\mathfrak{t}}$ along the projection $\check{\mathfrak{t}} \rightarrow \check{\mathfrak{t}} // \dot{\mathbf{W}}_\mu$, thus we have

$$\check{\mathfrak{t}} \times_{\check{\mathfrak{t}} // \dot{\mathbf{W}}} (\check{\mathfrak{t}} // \dot{\mathbf{W}}_\mu)_{\hat{\nu}} = \bigsqcup_{\nu \in \dot{\mathbf{W}}_\mu} \check{\mathfrak{t}} \times_{\check{\mathfrak{t}} // \dot{\mathbf{W}}_\nu} (\check{\mathfrak{t}} // \dot{\mathbf{W}}_\nu)_{\hat{\nu}},$$

(where $(\check{\mathfrak{t}} // \dot{\mathbf{W}}_\nu)_{\hat{\nu}}$ is the completion at image of ν in $\check{\mathfrak{t}} // \dot{\mathbf{W}}_\nu$) and it follows that

$$R_T \otimes_Z R_\chi^{\dot{\mathbf{W}}\mu} \simeq R_T \otimes_{R_T^{\dot{\mathbf{W}}}} R_\chi^{\dot{\mathbf{W}}\mu} \simeq \bigoplus_{\nu \in \dot{\mathbf{W}}_\mu, \chi_\nu = \exp(\nu)} R_T \otimes_{R_T^{\dot{\mathbf{W}}_\nu}} R_{\chi_\nu}^{\dot{\mathbf{W}}_\nu}.$$

Using $R_T \otimes_{R_T^{\dot{W}_\nu}} R_{\chi_\nu}^{\dot{W}_\nu} \simeq m_{-\nu}^*(R_T / \langle R_{T,+}^{\dot{W}_{\chi_\nu}} \rangle) \simeq \Gamma^{\dot{\nu}}(\mathcal{E}_\theta)$, we obtain

$$\begin{aligned} \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} R_T \otimes_{R_T^{\dot{W}}} R_\chi^{\dot{W}_\mu} &\simeq \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} \bigoplus_{\nu \in \dot{W}\mu, \chi_\nu = \exp(\nu)} R_T \otimes_{R_T^{\dot{W}_\nu}} R_{\chi_\nu}^{\dot{W}_\nu} \\ &\simeq \bigoplus_{\nu \in \dot{W}(\lambda + \Lambda)} \Gamma^{\dot{\nu}}(\mathcal{E}_\theta) \simeq \Gamma(\mathcal{E}_\theta). \end{aligned}$$

Taking $W_{a,-\rho}$ -invariants on both sides of the above isomorphism, we obtain the desired isomorphism (6.20). Combining (6.19) and (6.20), we obtain

$$(6.21) \quad \Gamma(\mathcal{M}'_\theta) \simeq U \otimes_Z \Gamma(\mathcal{E}_\theta)^{\dot{W}} \simeq U \otimes_{R_T^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}}.$$

Now using the isomorphism $\Gamma(\mathcal{E}_\theta) \simeq R_T \otimes_{R_T^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}}$ (see Theorem 4.2), we obtain the following description of (6.11):

$$\begin{aligned} (6.22) \quad \Gamma(\text{Av}_*^U(\mathcal{M}'_\theta)) &\stackrel{(6.13)}{\simeq} (U / U\mathfrak{n}) \otimes_U \Gamma(\mathcal{M}'_\theta) \stackrel{(6.21)}{\simeq} (U / U\mathfrak{n}) \otimes_{R_T^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}} \simeq \\ &\simeq (U / U\mathfrak{n}) \otimes_{R_T} R_T \otimes_{R_T^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}} \simeq (U / U\mathfrak{n}) \otimes_{R_T} \Gamma(\mathcal{E}_\theta) \stackrel{(6.13)}{\simeq} \Gamma(\iota_* \mathcal{E}_\theta). \end{aligned}$$

Remark 6.2. The discussion above depends on the choice of the Borel subgroup B . If we use the Borel subgroup $B_w = xBx^{-1}$ (where x and w be as in Section 6.7.1), then the isomorphisms (6.21) and (6.22) become

$$\Gamma(\mathcal{M}'_\theta) \simeq U \otimes_{R_T^{W_{a,-w(\rho)}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-w(\rho)}}$$

and

$$\begin{aligned} \Gamma(\text{Av}_*^{U_w}(\mathcal{M}'_\theta)) &\simeq (U / U\mathfrak{n}_w) \otimes_{R_T^{W_{a,-w(\rho)}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-w(\rho)}} \\ &\simeq (U / U\mathfrak{n}_w) \otimes_{R_T} \Gamma(\mathcal{E}_\theta) \simeq \Gamma((\iota_w)_* \mathcal{E}_\theta), \end{aligned}$$

where $w(\rho)$ is equal to the half sum of positive roots associated to B_w . Moreover, the composed isomorphism

$$U \otimes_{R_T^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}} \simeq \Gamma(\mathcal{M}'_\theta) \simeq U \otimes_{R_T^{W_{a,-w(\rho)}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-w(\rho)}}$$

is given by $\text{Id} \otimes \Gamma(\phi_w)$ (note that the isomorphism $\Gamma(\phi_w) : \Gamma(\mathcal{E}_\theta) \rightarrow \Gamma(\mathcal{E}_\theta)$ maps $\Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}}$ isomorphically onto $\Gamma(\mathcal{E}_\theta)^{W_{a,-w(\rho)}}$).

6.7.3. We shall describe the map (6.17) in diagram (6.18) (see line (6.23)) and use it to verify the desired commutativity of diagram (6.18). Theorem 6.6 and Remark 6.2 imply that, under the isomorphism

$$\Gamma(\mathcal{M}'_\theta) \simeq U \otimes_{R_T^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}} \quad (\text{resp. } \Gamma(\mathcal{M}'_\theta) \simeq U \otimes_{R_T^{W_{a,-w(\rho)}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-w(\rho)}})$$

the map $\Gamma(\phi_x) : \Gamma(\mathcal{M}'_\theta) \rightarrow \Gamma(\mathcal{M}'_\theta)$ is given by

$$\Gamma(\phi_x) = \text{Ad}_x(-) \otimes \Gamma(\phi_w) : U \otimes_{R_T^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}} \longrightarrow U \otimes_{R_T^{W_{a,-w(\rho)}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-w(\rho)}}.$$

It follows that, under the isomorphism

$$\begin{aligned} \Gamma(\text{Av}_*^U \mathcal{M}'_\theta) &\simeq (U / U\mathfrak{n}) \otimes_{R_T^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}} \\ (\text{resp. } \Gamma(\text{Av}_*^{U_w} \mathcal{M}'_\theta) &\simeq (U / U\mathfrak{n}_w) \otimes_{R_T^{W_{a,-w(\rho)}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-w(\rho)}}) \end{aligned}$$

the map $\Gamma(\mathrm{Av}_*^U(\phi_x)) : \Gamma(\mathrm{Av}_*^U \mathcal{M}'_\theta) \rightarrow \Gamma(\mathrm{Av}_*^{U_w} \mathcal{M}'_\theta)$ in (6.17) is given by

$$(6.23) \quad \mathrm{Ad}_x(-) \otimes \Gamma(\phi_w) : (\mathrm{U} / \mathrm{U} \mathfrak{n}) \otimes_{R_T^{\mathrm{W}_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{\mathrm{W}_{a,-\rho}} \rightarrow (\mathrm{U} / \mathrm{U} \mathfrak{n}_w) \otimes_{R_T^{\mathrm{W}_{a,-w(\rho)}}} \Gamma(\mathcal{E}_\theta)^{\mathrm{W}_{a,-w(\rho)}}.$$

Now the description of (6.11) in (6.22) implies that (6.17) fits into the following commutative diagram

$$(6.24) \quad \begin{array}{ccc} \Gamma(\mathrm{Av}_*^U \mathcal{M}'_\theta) & \xrightarrow{(6.17)=(6.23)} & \Gamma(\mathrm{Av}_*^{U_w} \mathcal{M}'_\theta) \\ (6.11)=(6.22) \downarrow & & \downarrow (6.11)=(6.22) \\ \Gamma(\iota_* \mathcal{E}_\theta) \simeq (\mathrm{U} / \mathrm{U} \mathfrak{n}) \otimes_{R_T} \Gamma(\mathcal{E}) & \longrightarrow & \Gamma((\iota_w)_* \mathcal{E}_\theta) \simeq (\mathrm{U} / \mathrm{U} \mathfrak{n}_w) \otimes_{R_T} \Gamma(\mathcal{E}), \end{array}$$

where the bottom arrow is given by $\mathrm{Ad}_x(-) \otimes \Gamma(\phi_w)$. Since $\mathrm{Ad}_x(-) \otimes \Gamma(\phi_w) = (6.15)$, diagram (6.24) is equal to diagram (6.18) and the commutativity of diagram (6.18) follows. The proof is complete.

6.8. Mixed characteristic lifting of $\mathcal{E}_\theta$ and $\mathcal{M}_\theta$. Let $G_{\mathbb{Z}}$ be a split reductive group over $\mathbb{Z}$. Let $T_{\mathbb{Z}}$ be a maximal torus of $G_{\mathbb{Z}}$ and let $B_{\mathbb{Z}}$ be a Borel subgroup containing $T_{\mathbb{Z}}$ with unipotent radical $U_{\mathbb{Z}}$. For any ring R (resp. any scheme S), we denote by G_R , B_R , etc. (resp. G_S , B_S , etc.) the base change of $G_{\mathbb{Z}}$, $B_{\mathbb{Z}}$, etc. along $\mathrm{Spec}(R) \rightarrow \mathrm{Spec}(\mathbb{Z})$ (resp. $S \rightarrow \mathrm{Spec}(\mathbb{Z})$).

Let A be a strictly henselian discrete valuation ring between $\mathbb{Z}[1/\ell]$ and $\mathbb{C}$ with residue field k . Let $\chi \in \mathcal{C}(T_k)(\overline{\mathbb{Q}}_\ell)$ and $\theta = W\chi$ be the W -orbit of χ . Let $\mathcal{E}_\theta$ be the W -equivariant ℓ -adic local system on T_k in Section 5.1.

Lemma 6.9. *There exists a W -equivariant ℓ -adic local system $\mathcal{E}_{\theta,A}$ on T_A which, after base change $A \rightarrow k$, becomes $\mathcal{E}_\theta$.*

Proof. Let ρ_θ be the ℓ -adic representation of $W \ltimes \pi_1^t(T_k)$ associated to $\mathcal{E}_\theta$. The specialization isomorphism $\mathrm{sp} : \pi_1^t(T_k) \rightarrow \pi_1^t(T_A)$ (see [SGA-1]) induces an isomorphism $W \ltimes \pi_1^t(T_k) \simeq W \ltimes \pi_1^t(T_A)$, thus we can identify ρ_θ as a ℓ -adic representation of $W \ltimes \pi_1^t(T_A)$ and we denote by $\mathcal{E}_{\theta,A}$ the corresponding local sysetm on T_A . It is obvious that $\mathcal{E}_{\theta,A}$ satisfies the desired property. $\square$

Let $\mathcal{M}_\theta$ be the character sheaf associated to θ . Our next goal is to construct a mixed characteristic version of $\mathcal{M}_\theta$. For this we observe that for any flat group scheme $\mathcal{G}$ of finite type over $\mathbb{Z}$ and a closed subgroup scheme $\mathcal{H} \subset \mathcal{G}$ flat over $\mathbb{Z}$, the universal geometric quotient $\mathcal{G}/\mathcal{H}$ exists [A]. Note also that the Chevalley isomorphism holds $G_R //_{\mathrm{ad}} G_R \simeq T_R // W_R$ for any ring R and the formation commutes with arbitrary base change $R \rightarrow R'$ [Le]. It follows that the quotient map $\pi : G \rightarrow G/U$ and the Grothendieck-Springer simultaneous resolution in (3.1) make sense over any ring R . Moreover, the formation commutes with arbitrary base change $R \rightarrow R'$. We denote them by $\pi_R : G_R \rightarrow G_R/U_R$ and

$$(6.25) \quad \begin{array}{ccc} \tilde{G}_R & \xrightarrow{\tilde{q}_R} & T_R \\ \downarrow \tilde{c}_R & & \downarrow q_R \\ G_R & \xrightarrow{c_R} & T_R // W_R. \end{array}$$

Denote by $h_R : \tilde{G}_R \rightarrow Z_R := G_R \times_{T_R // W} T_R$ the induced map. We define

$$(6.26) \quad \mathrm{IC}(Z_R) := (h_Z)_! \overline{\mathbb{Q}}_\ell[\dim G_{\mathbb{C}}] \in \mathcal{D}(Z_R).$$

When $R = k$ is an algebraically closed field of characteristic not equal to ℓ , $\mathrm{IC}(Z_k)$ is the IC-complex of Z_k and there is a canonical W -equivariant structure $(\mathrm{IC}(Z_k), \phi_k) \in \mathcal{D}_W(Z_k)$ (see Section 3.5).

Lemma 6.10. *There exists a positive integer N , depending on the group $G_{\mathbb{Z}}$, satisfying the following. Let A be a strictly henselian discrete valuation ring between $\mathbb{Z}[1/N\ell]$ and $\mathbb{C}$ with residue field k . One can endow the ℓ -adic complex $\mathrm{IC}(Z_A)$ in (6.26) with a W -equivariant structure $(\mathrm{IC}(Z_A), \phi_A) \in \mathcal{D}_W(Z_A)$ which, under the base change $A \rightarrow k$, becomes $(\mathrm{IC}(Z_k), \phi_k) \in \mathcal{D}_W(Z_k)$.*

Proof. According to [BBD, Section 6.1] (or the discussion in [Dr, Section 4]), one can choose a (large enough) positive integer N , a stratification $\mathcal{T}$ of $Z_{\mathbb{Z}[1/N\ell]}$, and for each $T \in \mathcal{T}$ a (finite) collection $\mathcal{L}(T)$ of ℓ -adic local systems on T , satisfying the following: (1) We have $w^*\mathrm{IC}(Z_{\mathbb{Z}[1/N\ell]}) \in \mathcal{D}_{\mathcal{T}, \mathcal{L}}(Z_{\mathbb{Z}[1/N\ell]})$ for $w \in W$. (2) Let A, k be as above. Let $i : Z_k \rightarrow Z_A$ be the imbedding. The functor $i^* : \mathcal{D}_{\mathcal{T}, \mathcal{L}}(Z_A) \rightarrow \mathcal{D}_{\mathcal{T}, \mathcal{L}}(Z_k)$ is an equivalence. Here $\mathcal{D}_{\mathcal{T}, \mathcal{L}}(Z_A)$ (resp. $\mathcal{D}_{\mathcal{T}, \mathcal{L}}(Z_k)$) is the full subcategory of $\mathcal{D}(Z_A)$ (resp. $\mathcal{D}(Z_k)$) generated by the $*$ -restriction of $\mathcal{L}(T), T \in \mathcal{T}$ to Z_A (resp. Z_k).

Note that, by (1) above, we have $w^*\mathrm{IC}(Z_A) \in \mathcal{D}_{\mathcal{T}, \mathcal{L}}(Z_A)$ and $w^*\mathrm{IC}(Z_k) \in \mathcal{D}_{\mathcal{T}, \mathcal{L}}(Z_k)$. Let $\phi_{k,w} : \mathrm{IC}(Z_k) \simeq w^*\mathrm{IC}(Z_k)$ be the isomorphism coming from the W -equivariant structure ϕ_k on $\mathrm{IC}(Z_k)$. Since $i^*\mathrm{IC}(Z_A) \simeq \mathrm{IC}(Z_k)$, it follows from (2) above that there exists an isomorphism $\phi_{A,w} : \mathrm{IC}(Z_A) \simeq w^*\mathrm{IC}(Z_A)$ which, under the base change $A \rightarrow k$, becomes $\phi_{k,w}$. Now the collection $\{\phi_{A,w} | w \in W\}$ defines a W -equivariant structure ϕ_A on $\mathrm{IC}(Z_A)$ satisfying the required property. $\square$

Proposition 6.11. *There exists a positive integer N , depending on the group $G_{\mathbb{Z}}$, satisfying the following: Let A be a strictly henselian discrete valuation ring between $\mathbb{Z}[1/N\ell]$ and $\mathbb{C}$ with residue field k . There is a ℓ -adic complex $\mathcal{M}_{\theta,A}$ on G_A which, under the base change $A \rightarrow k$, becomes $\mathcal{M}_{\theta}$.*

Proof. Let N, A, k be as in Lemma 6.10. Let $\mathcal{E}_{\theta,A} \in \mathcal{D}_W(T_A)$ be the lift of $\mathcal{E}_{\theta}$ in Lemma 6.9. Define

$$\begin{aligned} \mathrm{Ind}_{T_A \subset B_A}^{G_A}(\mathcal{E}_{\theta,A}) &:= (\tilde{c}_A)_!(\tilde{q}_A)^*(\mathcal{E}_{\theta,A})[\dim G_{\mathbb{C}} - \dim T_{\mathbb{C}}] \simeq \\ &\simeq p_{G_A,!}(p_{T_A}^*(\mathcal{E}_{\theta,A}) \otimes \mathrm{IC}(Z_A))[\dim G_{\mathbb{C}} - \dim T_{\mathbb{C}}]. \end{aligned}$$

Here $p_{T,A}$ and $p_{G,A}$ are the natural projections from Z_A to T_A and G_A respectively. The W -equivariant structures on $\mathcal{E}_{\theta,A}$ and $\mathrm{IC}(Z_A)$ give rise to a W -action on $\mathrm{Ind}_{T_A \subset B_A}^{G_A}(\mathcal{E}_{\theta,A})$ and we define $\mathcal{M}_{\theta,A} = \mathrm{Ind}_{T_A \subset B_A}^{G_A}(\mathcal{E}_{\theta,A})^W$ to be the W -invariant factor. Since the base change of $\mathcal{E}_{\theta,A}$ along $A \rightarrow k$ is isomorphic to $\mathcal{E}_{\theta}$, it follows that the base change of $\mathcal{M}_{\theta,A}$ along $A \rightarrow k$ is isomorphic to $\mathcal{M}_{\theta} = \mathrm{Ind}_{T_k \subset B_k}^{G_k}(\mathcal{E}_{\theta})^W$. $\square$

6.9. ULA property of the pushforward functors. We first review the notion of universal local acyclicity (ULA) following [De2, Z1].

Let S be a Noetherian scheme. Let $f : X \rightarrow S$ be a morphism of finite type and let $\mathcal{F} \in \mathcal{D}(X)$. Let s be a geometric point of S and let $S_{(s)}$ be the strict henselisation at s . We recall the following definition in [De2]:

Definition 6.12. A ℓ -adic complex $\mathcal{F} \in \mathcal{D}(X)$ is called locally acyclic with respect to $f : X \rightarrow S$ if for every geometric point $x \in X$ and every geometric point $t \in S_{(f(x))}$, the natural map $H^*(X_{(x)}, \mathcal{F}) \rightarrow H^*((X_{(x)})_t, \mathcal{F})$ is an isomorphism, where $(X_{(x)})_t = (X_{(x)}) \times_{S_{(f(x))}} t$. It is called universally locally acyclic (ULA) if it is locally acyclic after arbitrary base change $S' \rightarrow S$.

One can reformulate local acyclicity as follows. Let t be a geometric point of $S_{(s)}$. Denote by $j_t : X \times_S t \rightarrow X$ and $i_s : X \times_S s \rightarrow X$ the natural maps. We write

$$(6.27) \quad \Psi_{t \rightarrow s}(\mathcal{F}) := i_s^*(j_t)_* j_t^*(\mathcal{F}).$$

It is shown in [Z1, Lemma A 2.2] that $\mathcal{F}$ is locally acyclic with respect to f if and only if the natural map

$$(6.28) \quad i_s^* \mathcal{F} \rightarrow \Psi_{t \rightarrow s}(\mathcal{F})$$

is an isomorphism.

Remark 6.3. In the case when $S = \operatorname{Spec}(A)$ is the spectrum of a strictly henselian discrete valuation ring A with s the closed point S and t a geometric point over the generic point of S , $\Psi_{t \rightarrow s}(\mathcal{F})$ in (6.27) is by definition the nearby cycles $\Psi(\mathcal{F})$ of $\mathcal{F}$ and (6.28) is an isomorphism if and only if the vanishing cycles $\Phi(\mathcal{F})$ vanishes.

We have the following ULA property of the pushforward functors.

Proposition 6.13. *Let $f : X_{\mathbb{Z}[1/N\ell]} \rightarrow Y_{\mathbb{Z}[1/N\ell]}$ be a morphism of smooth schemes over $\mathbb{Z}[1/N\ell]$, separated and of finite type. There exists a subring R of $\mathbb{C}$, which is finitely generated and smooth over $\mathbb{Z}[1/N\ell]$, such that the following holds. Let A be a strictly henselian discrete valuation ring between R and $\mathbb{C}$ with residue field k and let $f_A : X_A \rightarrow Y_A$ be the base change of f along $\mathbb{Z}[1/N\ell] \rightarrow A$. For any ℓ -adic local system $\mathcal{L}_A$ on X_A the pushforward $(f_A)_* \mathcal{L}_A$ is ULA with respect to $Y_A \rightarrow \operatorname{Spec}(A)$, of formation compatible with arbitrary change of base on $\operatorname{Spec} A$.*

Proof. We first claim that there exists a subring R of $\mathbb{C}$, which is finitely generated and smooth over $\mathbb{Z}[1/N\ell]$, such that the base change $f_R : X_R \rightarrow Y_R$ of f along $\mathbb{Z}[1/N\ell] \rightarrow R$ admits a normal crossing compactification $\bar{f}_R : \bar{X}_R \rightarrow Y_R$, i.e., $\bar{X}_R$ is smooth R -scheme containing X_R as a dense open subscheme, such that $\bar{X}_R \setminus X_R$ is a union of smooth over R divisors which have normal crossings relative to $\operatorname{Spec}(R)$, and $\bar{f}_R$ is a proper morphism extending f_R . Indeed, Nagata compactification theorem and resolution of singularities imply the existence of such a normal crossing compactification $\bar{f}_{\mathbb{C}} : \bar{X}_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ for the base change $f_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ of f along the embedding $\mathbb{Z}[1/N\ell] \rightarrow \mathbb{C}$. Then we could apply the standard “spreading out” techniques to produce the desired ring $\mathbb{Z}[1/N\ell] \subset R \subset \mathbb{C}$ and the normal crossing compactification for f_R (see, e.g., [EGA-IV, Section 7 and Section 8] or [BBD, Section 6.1.7]).

Let A and $\mathcal{L}_A$ be as in the proposition. The base change $\bar{f}_A : \bar{X}_A \rightarrow Y_A$ of $\bar{f}_R$ along $R \rightarrow A$ gives rise to a normal crossing compactification of f_A . Denote by $j_A : X_A \rightarrow \bar{X}_A$ the open embedding. Since A has generic characteristic zero, any ℓ -adic local system $\mathcal{L}_A$ on X_A is tamely ramified along the normal crossing divisors $\bar{X}_A \setminus X_A$ at infinity [SGA-1, XIII 2.3 (a)]. Thus the results in [De3, Lemma 2.1.10 and Lemma 2.1.11] imply that the pushforward $(j_A)_* \mathcal{L}_A$ is ULA with respect to $\bar{X}_A \rightarrow A$ of formation compatible with arbitrary change of base on $\operatorname{Spec}(A)$. Indeed, the results in *loc. cit.* show that the formation $(j_A)_* \mathcal{L}_A$ commutes with arbitrary base change on $\operatorname{Spec}(A)$ and the vanishing cycle $\Phi((j_A)_! \mathcal{L}_A)$ is zero. Since the vanishing cycles functor $\Phi(-)$ commutes with Verdier duality $\mathbb{D}$, it implies $\Phi((j_A)_* \mathcal{L}_A) = \mathbb{D}\Phi((j_A)_!(\mathbb{D}\mathcal{L}_A)) = 0$ and hence $(j_A)_* \mathcal{L}_A$ is ULA by Remark 6.3.¹⁰

¹⁰Note that $\mathbb{D}\mathcal{L}_A$ is again a ℓ -adic local system (up to cohomological shift) tamely ramified along the normal crossing divisors at infinity.

Since the map $\bar{f}_A : \bar{X}_A \rightarrow Y_A$ is proper, the functor $(\bar{f}_A)_*$ maps ULA complexes to ULA complexes [Z1, Theorem A.2.5], and hence $(f_A)_*(\mathcal{L}_A) \simeq (\bar{f}_A)_*((j_A)_*\mathcal{L}_A)$ is ULA with respect to $Y_A \rightarrow A$, of formation compatible with arbitrary change of base on $\text{Spec} A$. The proof is complete. $\square$

Let A be a strictly henselian local ring and let $S = \text{Spec}(A)$. Let s be the closed point of S and let t be a geometric point of S . Let $f : X \rightarrow S$ be a scheme over S and let $Y \subset X$ be an open subscheme over S .

Lemma 6.14. *Let $\mathcal{F}$ be a ℓ -adic complex on X ULA with respect to $X \rightarrow S$. Let $\mathcal{F}_s$ and $\mathcal{F}_t$ be the restriction of $\mathcal{F}$ to the fiber X_s and X_t respectively. Then $\mathcal{F}_t|_{Y_t} \simeq 0$ implies $\mathcal{F}_s|_{Y_s} \simeq 0$.*

Proof. Indeed, since $Y \rightarrow X$ is smooth and the functor $\Psi_{t \rightarrow s}$ in (6.27) commutes with smooth pull back, the isomorphism $\mathcal{F}_s \simeq \Psi_{t \rightarrow s}(\mathcal{F})$ (coming from the ULA property) implies $\mathcal{F}_s|_{Y_s} \simeq \Psi_{t \rightarrow s}(\mathcal{F})|_{Y_s} \simeq \Psi_{t \rightarrow s}(\mathcal{F}|_Y) \simeq i_s^*(j_t)_*(\mathcal{F}_t|_{Y_t}) \simeq 0$. $\square$

6.10. Proof of Theorem 6.1 in the ℓ -adic setting. We shall show that there exists a positive integer N , depending only on $G_{\mathbb{Z}}$, such that for any algebraic closure k of a finite field of characteristic not dividing $N\ell$ and a W -orbit $\theta = W\chi$ of a tame character $\chi \in \mathcal{C}(T_k)(\bar{\mathbb{Q}}_{\ell})$, the averaging $\text{Av}_*^{U_k}(\mathcal{M}_{\theta})$ is supported on $T_k = B_k/U_k \subset G_k/U_k$. Equivalently, the restriction of $\text{Av}_*^{U_k}(\mathcal{M}_{\theta})$ to the open complement $Y_k = (G_k/U_k) \setminus (B_k/U_k)$ is zero.

To this end, let N be as in Proposition 6.11 and let $R \subset \mathbb{C}$ be the finitely generated smooth $\mathbb{Z}[1/N\ell]$ -algebra as in Proposition 6.13 in the case when $f : X_{\mathbb{Z}[1/N\ell]} \rightarrow Y_{\mathbb{Z}[1/N\ell]}$ is equal to

$$f : \tilde{G}_{\mathbb{Z}[1/N\ell]} \xrightarrow{\tilde{c}_{\mathbb{Z}[1/N\ell]}} G_{\mathbb{Z}[1/N\ell]} \xrightarrow{\pi_{\mathbb{Z}[1/N\ell]}} G_{\mathbb{Z}[1/N\ell]}/U_{\mathbb{Z}[1/N\ell]},$$

where $\tilde{c}_{\mathbb{Z}[1/N\ell]}$ is the map in (6.25). By replacing N with a larger positive integer, we can assume the map $\text{Spec}(R) \rightarrow \text{Spec}(\mathbb{Z}[1/N\ell])$ is a smooth surjection. Then for any geometric k -point of $\text{Spec}(\mathbb{Z}[1/N\ell])$, there exists a strictly henselian discrete valuation ring $R \subset A \subset \mathbb{C}$ with residue field k (see, e.g., [BBD, Section 6.1.8]). Let $\mathcal{M}_{\theta,A}, \mathcal{E}_{\theta,A}$ be as in Proposition 6.11. We will write $\mathcal{E}_{\theta,A'}, \mathcal{M}_{\theta,A'}$ for the base change of $\mathcal{E}_{\theta,A}, \mathcal{M}_{\theta,A}$ along $A \rightarrow A'$. Note that, by Lemma 6.9 and Proposition 6.11, we have $\mathcal{E}_{\theta,k} \simeq \mathcal{E}_{\theta}$ and $\mathcal{M}_{\theta,k} \simeq \mathcal{M}_{\theta}$.

Lemma 6.15. *$(\pi_A)_*\mathcal{M}_{\theta,A}$ is ULA with respect to $G_A/U_A \rightarrow \text{Spec}(A)$, of formation compatible with arbitrary base change on $\text{Spec}(A)$.*

Proof. Let $\mathcal{L}_A = (\tilde{q}_A)^*\mathcal{E}_{\theta,A}$ be the ℓ -adic local system on $\tilde{G}_A$ given by the pull-back of $\mathcal{E}_{\theta,A}$ along the map $\tilde{q}_A : \tilde{G}_A \rightarrow T_A$ in (6.25). By Proposition 6.13, $(f_A)_*\mathcal{L}_A$ is ULA with respect to $G_A/U_A \rightarrow \text{Spec}(A)$, of formation compatible with arbitrary base change on $\text{Spec}(A)$. Since $\mathcal{M}_{\theta,A}$ is the W -invariant direct factor of

$$\text{Ind}_{T_A \subset B_A}^{G_A}(\mathcal{E}_{\theta,A}) = (\tilde{c}_A)_*\mathcal{L}_A[\dim G_{\mathbb{C}} - \dim T_{\mathbb{C}}]$$

and

$$(f_A)_*\mathcal{L}_A = (\pi_A)_*(\tilde{c}_A)_*\mathcal{L}_A \simeq (\pi_A)_*\text{Ind}_{T_A \subset B_A}^{G_A}(\mathcal{E}_{\theta,A})[\dim T_{\mathbb{C}} - \dim G_{\mathbb{C}}].$$

It follows that $(\pi_A)_*\mathcal{M}_{\theta,A}$ is a direct factor of $(\pi_A)_*\mathcal{L}_A$ and hence is also ULA with respect to $G_A/U_A \rightarrow \text{Spec}(A)$, of formation compatible with arbitrary base change on $\text{Spec}(A)$. $\square$

Lemma 6.16. $(\pi_{\mathbb{C}})_*\mathcal{M}_{\theta,\mathbb{C}}$ is supported on $T_{\mathbb{C}} = B_{\mathbb{C}}/U_{\mathbb{C}} \subset G_{\mathbb{C}}/U_{\mathbb{C}}$.

Proof. Let $D_c^b(G_{\mathbb{C}}, \overline{\mathbb{Q}}_{\ell})$ be the bounded derived category of constructible ℓ -adic complexes on $G_{\mathbb{C}}$ and let $D_c^b(G_{\mathbb{C}}(\mathbb{C}), \mathbb{C})$ be the usual bounded derived category of $\mathbb{C}$ -constructible complexes on the complex Lie group $G_{\mathbb{C}}(\mathbb{C})$. We fix an isomorphism $\overline{\mathbb{Q}}_{\ell} \simeq \mathbb{C}$. Then according to [BBD, Section 6.1], there is a comparison functor

$$\epsilon^* : D_c^b(G_{\mathbb{C}}, \overline{\mathbb{Q}}_{\ell}) \rightarrow D_c^b(G_{\mathbb{C}}(\mathbb{C}), \mathbb{C})$$

which is fully-faithful and commutes with six functor formalism. Let $\mathcal{E}_{\theta,\mathbb{C}}$ be the base change of $\mathcal{E}_{\theta,A}$ along $A \rightarrow \mathbb{C}$. We claim that there exists a character $\chi_{\mathbb{C}}$ of the topological fundamental group $\pi_1(T(\mathbb{C}))$ such that the local system $\epsilon^*\mathcal{E}_{\theta,\mathbb{C}}$ corresponds to the de Rham local system $\mathcal{E}_{\theta_{\mathbb{C}}}$ in Section 5.1 under the Riemann-Hilbert correspondence. Here $\theta_{\mathbb{C}} = W_{\chi_{\mathbb{C}}}$ is the W -orbit of $\chi_{\mathbb{C}}$. Assume the claim for the moment, then the perverse sheaf

$$\epsilon^*\mathcal{M}_{\theta,\mathbb{C}} \simeq \epsilon^*((\mathrm{Ind}_{T_{\mathbb{C}} \subset B_{\mathbb{C}}}^{G_{\mathbb{C}}}(\mathcal{E}_{\theta,\mathbb{C}}))^W) \simeq (\mathrm{Ind}_{T_{\mathbb{C}} \subset B_{\mathbb{C}}}^{G_{\mathbb{C}}}(\epsilon^*\mathcal{E}_{\theta,\mathbb{C}}))^W$$

corresponds to the character D -module $\mathrm{Ind}_{T_{\mathbb{C}} \subset B_{\mathbb{C}}}^{G_{\mathbb{C}}}(\mathcal{E}_{\theta_{\mathbb{C}}})^W = \mathcal{M}_{\theta_{\mathbb{C}}}$ under the Riemann-Hilbert correspondence and the lemma follows from Theorem 6.1 in the de Rham setting.

To prove the claim, we observe that the constructions of $\mathcal{E}_{\theta}$ and $\mathcal{E}_{\theta,A}$ in Section 5.1 and Lemma 6.9 imply that $\mathcal{E}_{\theta,\mathbb{C}}$ corresponds to a ℓ -adic representation $\mathrm{Ind}_{W'_{\chi_{\mathbb{C}},\ell}}^W(R_{\chi_{\mathbb{C}},\ell} \otimes \chi_{\mathbb{C},\ell})$ where $\chi_{\mathbb{C},\ell}$ is a ℓ -adic character of $\pi_1(T_{\mathbb{C}})_{\ell}$ with $W'_{\chi_{\mathbb{C}},\ell} = W'_{\chi}$, and $R_{\chi_{\mathbb{C}},\ell}$ is the representation of $W'_{\chi_{\mathbb{C}},\ell} \times \pi_1(T_{\mathbb{C}})$ in $\overline{\mathbb{Q}}_{\ell}[[\pi_1(T_{\mathbb{C}})_{\ell}]]/\langle \overline{\mathbb{Q}}_{\ell}[[\pi_1(T_{\mathbb{C}})_{\ell}]]_+^{W_{\chi_{\mathbb{C}},\ell}} \rangle$ given by the $\overline{\mathbb{Q}}_{\ell}[[\pi_1(T_{\mathbb{C}})_{\ell}]]$ -module structure. Here $\pi_1(T_{\mathbb{C}})$ is the étale fundamental group of $T_{\mathbb{C}}$ and $\overline{\mathbb{Q}}_{\ell}[[\pi_1(T_{\mathbb{C}})_{\ell}]]$ is the completed group algebra of the pro- ℓ quotient $\pi_1(T_{\mathbb{C}})_{\ell}$ of $\pi_1(T_{\mathbb{C}})$. Note that the restriction of the functor ϵ^* to the subcategory of ℓ -adic local systems on $T_{\mathbb{C}}$ is induced by the natural embedding

$$(6.29) \quad \pi_1(T(\mathbb{C})) \rightarrow \pi_1(T(\mathbb{C}))_{\ell} \simeq \pi_1(T_{\mathbb{C}})_{\ell}.$$

Note also that (6.29) induces an isomorphism

$$R_{\chi_{\mathbb{C}}} = R_T / \langle R_{T,+}^{W_{\chi_{\mathbb{C}}}} \rangle \simeq \overline{\mathbb{Q}}_{\ell}[[\pi_1(T_{\mathbb{C}})_{\ell}]] / \langle \overline{\mathbb{Q}}_{\ell}[[\pi_1(T_{\mathbb{C}})_{\ell}]]_+^{W_{\chi_{\mathbb{C}},\ell}} \rangle \simeq R_{\chi_{\mathbb{C}},\ell}$$

compatible with the $W'_{\chi_{\mathbb{C}}} = W'_{\chi_{\mathbb{C}},\ell}$ -action. Let $\chi_{\mathbb{C}}$ be the pullback of the character $\chi_{\mathbb{C},\ell}$ along (6.29). Then the pullback of the representation $\mathrm{Ind}_{W'_{\chi_{\mathbb{C}},\ell}}^W(R_{\chi_{\mathbb{C}},\ell} \otimes \chi_{\mathbb{C},\ell})$ along (6.29) is isomorphic to $\mathrm{Ind}_{W'_{\chi_{\mathbb{C}}}}^W(R_{\chi_{\mathbb{C}}} \otimes \chi_{\mathbb{C}})$. Since $\mathcal{E}_{\theta_{\mathbb{C}}}$ is the de Rham local system corresponding to $\mathrm{Ind}_{W'_{\chi_{\mathbb{C}}}}^W(R_{\chi_{\mathbb{C}}} \otimes \chi_{\mathbb{C}})$ under the Riemann-Hilbert correspondence, the desired claim follows. $\square$

Applying Lemma 6.14 to the case $\mathcal{F} = (\pi_A)_*\mathcal{M}_{\theta,A}$, $X = G_A/U_A$, $Y = Y_S = (G_A/U_A) \setminus (B_A/U_A)$, and using Lemma 6.15 and Lemma 6.16, we conclude that $\mathrm{Av}_*^{U_k}(\mathcal{M}_{\theta}) \simeq (\pi_k)_*\mathcal{M}_{\theta,k}$ is supported on T_k . To finish the proof of Theorem 6.1 it remains to show that $\iota^{\circ} \mathrm{Av}_*^{U_k}(\mathcal{M}_{\theta}) \simeq \mathrm{Res}_{T \subset B}^G(\mathcal{M}_{\theta})$ is isomorphic to $\mathcal{E}_{\theta}$ but it follows from Proposition 3.2.

7. PROOF OF THE VANISHING CONJECTURE

In this section we prove the vanishing conjecture (Conjecture 1.2) for almost all characteristic:

Theorem 7.1. *There exists a positive integer N depending only on the type of the group G such that the following holds. Assume $k = \mathbb{C}$ or $\text{char } k = p$ is not dividing $N\ell$. Let $\mathcal{F} \in \mathcal{D}_W(T)$ be a central complex (resp. $*$ -central complex) on T and let $\Phi_{\mathcal{F}} = \text{Ind}_{T \subset B}^G(\mathcal{F})^W \in \mathcal{D}(G)$. For any $x \in G \setminus B$, we have the following cohomology vanishing*

$$H_c^*(xU, i^* \Phi_{\mathcal{F}}) = 0 \quad (\text{resp. } H^*(xU, i^! \Phi_{\mathcal{F}}) = 0),$$

where $i : xU \rightarrow G$ is the natural inclusion map. Equivalently, $\text{Av}_!^U(\Phi_{\mathcal{F}})$ (resp. $\text{Av}_*^U(\Phi_{\mathcal{F}})$) is supported on the closed subset $T = B/U \subset G/U$.

7.1. Reduction to perverse sheaves.

Lemma 7.2. *If Conjecture 1.2 holds for central perverse sheaves (resp. $*$ -central perverse sheaves), then it holds for arbitrary central complexes (resp. $*$ -central complex).*

Proof. Since the Verdier duality interchanges central complexes with $*$ -central complexes, it is enough to verify the lemma for $*$ -central complexes. Let $\mathcal{F}$ be a $*$ -central complex. We claim that both ${}^p\tau_{\leq b}(\mathcal{F})$ and ${}^p\mathcal{H}^b(\mathcal{F})$, $b \in \mathbb{Z}$, are $*$ -central. Assuming the claim for the moment, then an induction argument on the (finite) number of non vanishing perverse cohomology sheaves of $\mathcal{F}$ in [C, Lemma 7.5] implies the lemma.

We shall prove the claim. We first deal with the ℓ -adic setting. Let $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_{\ell})$ and let I_{χ} be the maximal ideal corresponding to χ . Write $\mathcal{C}(T)_{\hat{\chi}}$ for the completion of $\mathcal{C}(T)$ at χ . Let $q_{\hat{\chi}} : \mathcal{C}(T)_{\hat{\chi}} \rightarrow \mathcal{C}(T)_{\hat{\chi}} // W_{\chi}$ be the quotient map. Since χ is the unique closed point of $\mathcal{C}(T)_{\hat{\chi}}$ and the action of W_{χ} on the fiber $i_{\chi}^*(\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)|_{\mathcal{C}(T)_{\hat{\chi}}}) \simeq i_{\chi}^*\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)$ is trivial, by [Ne, Theorem 1.3], there exists $\mathcal{G} \in D_{coh}^b(\mathcal{C}(T)_{\hat{\chi}} // W_{\chi})$ such that

$$\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)|_{\mathcal{C}(T)_{\hat{\chi}}} \simeq q_{\hat{\chi}}^* \mathcal{G} \in D_{coh}^b(\mathcal{C}(T)_{\hat{\chi}} // W_{\chi}).$$

Since the functors $\mathfrak{M}$, $q_{\hat{\chi}}^*$ and taking completions $(-)|_{\mathcal{C}(T)_{\hat{\chi}}}$ are t-exact, we have

$$\mathfrak{M}({}^p\tau_{\leq b}(\mathcal{F}) \otimes \text{sign}_W)|_{\mathcal{C}(T)_{\hat{\chi}}} \simeq \tau_{\leq b}(\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)|_{\mathcal{C}(T)_{\hat{\chi}}}) \simeq \tau_{\leq b}(q_{\hat{\chi}}^* \mathcal{G}) \simeq q_{\hat{\chi}}^*(\tau_{\leq b}(\mathcal{G})).$$

Similarly, we have

$$\mathfrak{M}({}^p\mathcal{H}^b(\mathcal{F}) \otimes \text{sign}_W)|_{\mathcal{C}(T)_{\hat{\chi}}} \simeq q_{\hat{\chi}}^* \mathcal{H}^b(\mathcal{G}).$$

It follows that W_{χ} acts trivially on the fibers $i_{\chi}^*\mathfrak{M}({}^p\tau_{\leq b}(\mathcal{F}) \otimes \text{sign}_W)$ and $i_{\chi}^*\mathfrak{M}({}^p\mathcal{H}^b(\mathcal{F}) \otimes \text{sign}_W)$ and, by Lemma 4.1, we conclude that ${}^p\tau_{\leq b}(\mathcal{F})$ and ${}^p\mathcal{H}^b(\mathcal{F})$ are $*$ -central.

The de Rham setting. Let $\lambda \in \check{\mathfrak{t}}(\mathbb{C})$. By Theorem 4.4, the Mellin transform $\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)$ descends to $\check{\mathfrak{t}} // W_{a,\lambda}$, that is, we have

$$\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W) \simeq \pi_{\lambda}^* \mathcal{G}$$

for some $\mathcal{G} \in D_{coh}^b(\check{\mathfrak{t}} // W_{a,\lambda})$. Since $\mathfrak{M}$ and π_{λ}^* are exact functors, we have

$$\mathfrak{M}(\tau_{\leq b} \mathcal{F} \otimes \text{sign}_W) \simeq \tau_{\leq b}(\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)) \simeq \tau_{\leq b}(\pi_{\lambda}^* \mathcal{G}) \simeq \pi_{\lambda}^*(\tau_{\leq b} \mathcal{G})$$

$$\mathfrak{M}(\mathcal{H}^b(\mathcal{F}) \otimes \text{sign}_W) \simeq \mathcal{H}^b(\mathfrak{M}(\mathcal{F} \otimes \text{sign}_W)) \simeq \mathcal{H}^b(\pi_{\lambda}^* \mathcal{G}) \simeq \pi_{\lambda}^*(\mathcal{H}^b \mathcal{G})$$

and hence by Theorem 4.4 again we conclude that both $\tau_{\leq b}\mathcal{F}$ and $\mathcal{H}^b(\mathcal{F})$ are $*$ -central.

□

7.2. Reduction to reductive groups with connected center.

Lemma 7.3. *Let $f_G : G \rightarrow G_1$ be a central isogeny. If Conjecture 1.2 holds for G_1 then it holds for G .*

Proof. We retain the notation from Section 4.4. Let $B_1 = f(B)$ and $U_1 = f(U)$. Since f maps the root subgroups of G isomorphically to the root subgroups of G_1 and the kernel of f is in the center of G , $B_1 = T_1 U_1$ is a Borel subgroup of G_1 with unipotent radical U_1 . In addition, we have the following Cartesian diagrams

$$\begin{array}{ccccc} G & \xleftarrow{\tilde{c}} & \tilde{G} & \xrightarrow{\tilde{q}} & T \\ \downarrow f_G & & \downarrow & & \downarrow f_T \\ G_1 & \xleftarrow{\tilde{c}_1} & \tilde{G}_1 & \xrightarrow{\tilde{q}_1} & T_1 \end{array}$$

where the horizontal arrows are the maps in (3.1). It follows that for any $\mathcal{F} \in \mathcal{D}_W(T)$ we have

$$\mathrm{Ind}_{T_1 \subset B_1}^{G_1}((f_T)_! \mathcal{F})^{W_1} \simeq (f_G)_!(\mathrm{Ind}_{T \subset B}^G(\mathcal{F})^W)$$

(recall that f_G induces an isomorphism $f_W : W \simeq W_1$ (4.12)). From the above isomorphism we obtain

$$\begin{aligned} (7.1) \quad (\pi_1)_!(\mathrm{Ind}_{T_1 \subset B_1}^{G_1}((f_T)_! \mathcal{F})^{W_1}) &\simeq (\pi_1)_!((f_G)_!(\mathrm{Ind}_{T \subset B}^G(\mathcal{F})^W)) \\ &\simeq (f_{G/U})_!(\pi_!(\mathrm{Ind}_{T \subset B}^G(\mathcal{F})^W)), \end{aligned}$$

where $\pi : G \rightarrow G/U$, $\pi_1 : G_1 \rightarrow G_1/U_1$ and $f_{G/U} : G/U \rightarrow G_1/U_1$. Since $f_{G/U}^{-1}(T_1) = T$ and for any $\mathcal{G} \in \mathcal{D}(G/U)$ we have $\mathrm{supp}((f_{G/U})_! \mathcal{G}) = f_{G/U}(\mathrm{supp}(\mathcal{G}))$ (as $f_{G/U}$ is surjective étale), the isomorphism (7.1) implies that if $(\pi_1)_!(\mathrm{Ind}_{T_1 \subset B_1}^{G_1}((f_T)_! \mathcal{F})^{W_1})$ is supported on T_1 then $\pi_!(\mathrm{Ind}_{T \subset B}^G(\mathcal{F})^W)$ is supported on T . Now the lemma follows from Lemma 4.5.

□

Corollary 7.4. *If Conjecture 1.2 holds for connected reductive groups with connected center then it holds for arbitrary connected reductive groups.*

Proof. Let G_{der} and Z_G be the derived subgroup and the center of G respectively and let $G_1 = G/G_{\mathrm{der}} \times G/Z_G$. Note that G_1 has connected center and the natural map $f_G : G \rightarrow G_1$ is a central isogeny. Thus the corollary follows from Lemma 7.3.

□

7.3. Convolution with $\mathcal{M}_\theta$.

Proposition 7.5. *Assume G has connected center. Let $\mathcal{F} \in \mathcal{D}_W(T)^\heartsuit$ be a $*$ -central perverse sheaf and let θ be a W -orbit through a tame character $\chi \in \mathcal{C}(T)(E)$. There is an isomorphism*

$$\Phi_{\mathcal{F}} * \mathcal{M}_\theta \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{M}_\theta.$$

Proof. By Proposition 5.4, we have

$$(7.2) \quad \mathcal{F} * \mathcal{E}_\theta \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_\theta \in \mathcal{D}(T).$$

Since $\mathrm{Av}_*^U(\mathcal{M}_\theta) \simeq \mathcal{E}_\theta$ is supported on $T = B/U \subset G/U$, Proposition 3.3 implies

$$(7.3) \quad \mathrm{Ind}_{T \subset B}^G(\mathcal{F}) * \mathcal{M}_\theta \simeq \mathrm{Ind}_{T \subset B}^G(\mathcal{F} * \mathcal{E}_\theta) \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathrm{Ind}_{T \subset B}^G(\mathcal{E}_\theta).$$

We claim that the isomorphism above is compatible with the natural W -actions. Taking W -invariance on both sides of (7.3), we get

$$\begin{aligned} \Phi_{\mathcal{F}} * \mathcal{M}_\theta &\simeq \mathrm{Ind}_{T \subset B}^G(\mathcal{F})^W * \mathcal{M}_\theta \\ &\stackrel{(7.3)}{\simeq} H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathrm{Ind}_{T \subset B}^G(\mathcal{E}_\theta)^W \\ &\simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{M}_\theta. \end{aligned}$$

The proposition follows. $\square$

7.3.1. Proof of the claim. Let us write $\mathrm{Res} = \mathrm{Res}_{T \subset B}^G$, $\mathrm{Ind} = \mathrm{Ind}_{T \subset B}^G$, $\mathcal{M} = \mathcal{M}_\theta$, $\mathcal{E} = \mathcal{E}_\theta$, and $V = H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}})$. Our goal is to show that the isomorphism (7.3)

$$(7.4) \quad \phi : \mathrm{Ind}(\mathcal{F}) * \mathcal{M} \simeq V \otimes \mathrm{Ind}(\mathcal{E})$$

is compatible with the W -actions. To this end, consider the following commutative diagram induced from (7.4):

$$\begin{array}{ccc} \mathrm{End}(\mathrm{Ind}(\mathcal{F}) * \mathcal{M}) & \longrightarrow & \mathrm{End}(\mathrm{Res}(\mathrm{Ind}(\mathcal{F}) * \mathcal{M})) \\ (7.4) \downarrow \sim & & \downarrow \sim (7.4) \\ \mathrm{End}(V \otimes \mathrm{Ind}(\mathcal{E})) & \longrightarrow & \mathrm{End}(\mathrm{Res}(V \otimes \mathrm{Ind}(\mathcal{E}))) \end{array}$$

Since the natural map $\mathrm{End}(\mathrm{Ind}(\mathcal{E})) \rightarrow \mathrm{End}(\mathrm{Res} \circ \mathrm{Ind}(\mathcal{E}))$ is injective,¹¹ the horizontal arrows in the diagram above are injective. Thus it suffices to check the compatibility after applying the functor Res , that is, the map

$$(7.5) \quad \mathrm{Res}(\phi) : \mathrm{Res}(\mathrm{Ind}(\mathcal{F}) * \mathcal{M}) \simeq \mathrm{Res}(V \otimes \mathrm{Ind}(\mathcal{E})) \simeq V \otimes \mathrm{Res} \circ \mathrm{Ind}(\mathcal{E})$$

is compatible with the W -actions. Since $\mathrm{Av}_*^U(\mathcal{M}) \simeq \mathrm{Res}(\mathcal{M}) \simeq \mathcal{E}$ is supported on T , we have a canonical isomorphism

$$\mathrm{Res}(\mathrm{Ind}(\mathcal{F}) * \mathcal{M}) \simeq \mathrm{Res} \circ \mathrm{Ind}(\mathcal{F}) * \mathrm{Res}(\mathcal{M}) \simeq \mathrm{Res} \circ \mathrm{Ind}(\mathcal{F}) * \mathcal{E}^{12}$$

and hence (7.5) can be identified with

$$(7.6) \quad \mathrm{Res} \circ \mathrm{Ind}(\mathcal{F}) * \mathcal{E} \simeq V \otimes \mathrm{Res} \circ \mathrm{Ind}(\mathcal{E})$$

¹¹Indeed, since Res is the right adjoint of Ind , it has a one side inverse $\mathrm{End}(\mathrm{Res} \circ \mathrm{Ind}(\mathcal{E})) \rightarrow \mathrm{End}(\mathrm{Ind}(\mathcal{E}))$ sending $f : \mathrm{Res} \circ \mathrm{Ind}(\mathcal{E}) \rightarrow \mathrm{Res} \circ \mathrm{Ind}(\mathcal{E})$ to $\mathrm{Ind}(\mathcal{E}) \xrightarrow{\eta} \mathrm{Ind} \circ \mathrm{Res} \circ \mathrm{Ind}(\mathcal{E}) \xrightarrow{h} \mathrm{Ind}(\mathcal{E})$, where η is induced by the unit morphism $\mathrm{Id} \rightarrow \mathrm{Res} \circ \mathrm{Ind}$ and h is the image of f under the isomorphism $\mathrm{End}(\mathrm{Res} \circ \mathrm{Ind}) \simeq \mathrm{Hom}(\mathrm{Ind} \circ \mathrm{Res}, \mathrm{Ind}, \mathrm{Ind})$.

¹²The first isomorphism follows from the fact that, for any $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{D}(G/\mathrm{ad}G)$ such that $\mathrm{Av}_*^U(\mathcal{M}_2) \simeq \mathrm{Res}(\mathcal{M}_2)$ is supported on T , we have canonical isomorphism $\mathrm{Res}(\mathcal{M}_1 * \mathcal{M}_2) \simeq \mathrm{Res}(\mathcal{M}_1) * \mathrm{Res}(\mathcal{M}_2)$.

and we reduce to show that (7.6) is compatible with the W -actions. By Proposition 3.2 and Proposition 5.4, there is a commutative diagram

$$\begin{array}{ccc} (E[W] \otimes \mathcal{F}) * \mathcal{E} & \longrightarrow & V \otimes (E[W] \otimes \mathcal{E}) \\ (3.6) \downarrow \sim & & (3.6) \downarrow \sim \\ \text{Res} \circ \text{Ind}(\mathcal{F}) * \mathcal{E} & \xrightarrow{(7.6)} & V \otimes (\text{Res} \circ \text{Ind}(\mathcal{E})) \end{array}$$

where the vertical arrows and the upper horizontal arrow are compatible with the natural W -actions. Therefore (7.6) is also compatible with the W -actions. The claim follows.

7.4. Proof of Theorem 7.1. Since the Verdier duality interchanges central complexes with $*$ -central complexes, it suffices to verify Theorem 7.1 for $*$ -central complexes. Let $\mathcal{F} \in \mathcal{D}_W(T)$ be a $*$ -central complex. We need to show that the natural map

$$(7.7) \quad r : \text{Res}_{T \subset B}^G(\Phi_{\mathcal{F}}) \rightarrow \text{Av}_*^U(\Phi_{\mathcal{F}})$$

is an isomorphism. By Lemma 7.2 and Corollary 7.4, we can assume $\mathcal{F}$ is a perverse sheaf and G has connected center. We claim that, for any W -orbit $\theta = W\chi \subset \mathcal{C}(T)(E)$, the convolution of r with $\mathcal{E}_{\theta}$ is an isomorphism

$$(7.8) \quad \text{Res}_{T \subset B}^G(\Phi_{\mathcal{F}}) * \mathcal{E}_{\theta} \xrightarrow{\sim} \text{Av}_*^U(\Phi_{\mathcal{F}}) * \mathcal{E}_{\theta}.$$

For this, it is enough to show that $\text{Av}_*^U(\Phi_{\mathcal{F}}) * \mathcal{E}_{\theta}$ is supported on T and this follows from Theorem 6.1 and Proposition 7.5. Indeed, we have

$$\begin{aligned} \text{Av}_*^U(\Phi_{\mathcal{F}}) * \mathcal{E}_{\theta} &\stackrel{\text{Thm 6.1}}{\simeq} \text{Av}_*^U(\Phi_{\mathcal{F}}) * \text{Av}_*^U(\mathcal{M}_{\theta}) \\ &\simeq \text{Av}_*^U(\Phi_{\mathcal{F}} * \mathcal{M}_{\theta}) \\ &\stackrel{\text{Prop 7.5}}{\simeq} H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \text{Av}_*^U(\mathcal{M}_{\theta}) \\ &\stackrel{\text{Thm 6.1}}{\simeq} H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_{\theta}. \end{aligned}$$

Since $\mathcal{E}_{\theta} \simeq \bigoplus_{\chi \in \theta} \mathcal{E}_{\chi}$, the isomorphism (7.8) implies that the cone of the map in (7.7), denoted by $\text{cone}(r)$, satisfies $\text{cone}(r) * \mathcal{E}_{\chi} = 0$ for all $\chi \in \mathcal{C}(T)(E)$. As $\mathcal{E}_{\chi}$ is a local system on T with generalized monodromy χ , that is, $\mathcal{E}_{\chi} \otimes \mathcal{L}_{\chi}$ is a unipotent local system, Lemma 7.6 and Lemma 7.7 below imply $\text{cone}(r) = 0$. The theorem follows.

7.5. Vanishing lemmas. Let X be a smooth variety with a free T action $a : T \times X \rightarrow X$. For $\mathcal{L} \in \mathcal{D}(T)$ and $\mathcal{F} \in \mathcal{D}(X)$ we define $\mathcal{L} * \mathcal{F} := a_*(\mathcal{L} \boxtimes \mathcal{F}) \in \mathcal{D}(X)$.

Lemma 7.6. *Let $\mathcal{L}$ be a local system on T with generalized monodromy $\chi \in \mathcal{C}(T)(E)$, that is, $\mathcal{L} \otimes \mathcal{L}_{\chi}$ is a unipotent local system. Let $\mathcal{F} \in \mathcal{D}(X)$ and assume $\mathcal{L} * \mathcal{F} = 0$. Then we have $\mathcal{L}_{\chi} * \mathcal{F} = 0$.*

Proof. There is a filtration $0 = \mathcal{L}^{(0)} \subset \mathcal{L}^{(1)} \subset \dots \subset \mathcal{L}^{(k)} = \mathcal{L}$ such that

$$0 \rightarrow \mathcal{L}^{(i-1)} \rightarrow \mathcal{L}^{(i)} \rightarrow \mathcal{L}^{(i)} / \mathcal{L}^{(i-1)} \simeq \mathcal{L}_{\chi} \rightarrow 0.$$

Assume $\mathcal{L}_{\chi} * \mathcal{F} \neq 0$ and let m be the smallest number such that $\mathcal{H}^{\geq m}(\mathcal{L}_{\chi} * \mathcal{F}) = 0$. We claim that $\mathcal{H}^{\geq m}(\mathcal{L}^{(i)} * \mathcal{F}) = 0$ for $i = 1, \dots, k$. The case $i = 1$ is automatic

since $\mathcal{L}^{(1)} = \mathcal{L}_\chi$. For $i \leq k$, consider the distinguished triangle

$$\mathcal{L}^{(i-1)} * \mathcal{F} \rightarrow \mathcal{L}^{(i)} * \mathcal{F} \rightarrow \mathcal{L}_\chi * \mathcal{F} \rightarrow \mathcal{L}^{(i-1)} * \mathcal{F}[1]$$

induced from above short exact sequence. Then for any $n \geq m$ we obtain an exact sequence

$$\mathcal{H}^n(\mathcal{L}^{(i-1)} * \mathcal{F}) \rightarrow \mathcal{H}^n(\mathcal{L}^{(i)} * \mathcal{F}) \rightarrow \mathcal{H}^n(\mathcal{L}_\chi * \mathcal{F}).$$

By induction, the first and third terms are zero, and hence $\mathcal{H}^n(\mathcal{L}^{(i)} * \mathcal{F}) = 0$. The claim follows.

Now since $\mathcal{L} * \mathcal{F} = 0$, the distinguished triangle

$$\mathcal{L}^{(k-1)} * \mathcal{F} \rightarrow \mathcal{L} * \mathcal{F} \rightarrow \mathcal{L}_\chi * \mathcal{F} \rightarrow \mathcal{L}^{(k-1)} * \mathcal{F}[1]$$

implies

$$\mathcal{L}_\chi * \mathcal{F} \simeq \mathcal{L}^{(k-1)} * \mathcal{F}[1].$$

Therefore we have $\mathcal{H}^{m-1}(\mathcal{L}_\chi * \mathcal{F}) \simeq \mathcal{H}^{m-1}(\mathcal{L}^{(k-1)} * \mathcal{F}[1]) = \mathcal{H}^m(\mathcal{L}^{(k-1)} * \mathcal{F}) = 0$ which contradicts to the fact that m is the smallest number such that $\mathcal{H}^{\geq m}(\mathcal{L}_\chi * \mathcal{F}) = 0$. We are done. $\square$

Lemma 7.7. *Let $\mathcal{F} \in \mathcal{D}(X)$. If $\mathcal{L}_\chi * \mathcal{F} = 0$ for all $\chi \in \mathcal{C}(T)(E)$, then $\mathcal{F} = 0$.*

Proof. Since T acts freely on X we have an embedding $o_x : T \rightarrow X, t \rightarrow t \cdot x$. Moreover, by base change formulas, we have

$$H^*(T, \mathcal{L}_{\chi^{-1}} \otimes^! o_x^! \mathcal{F}) \simeq i_x^!(\mathcal{L}_\chi * \mathcal{F}) = 0$$

for all $\chi \in \mathcal{C}(T)(E)$. Here $i_x : x \rightarrow X$ is the natural inclusion map. By a result of Laumon [GL, Proposition 3.4.5], it implies $o_x^! \mathcal{F} = 0$ for all x . The lemma follows. $\square$

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