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A vanishing conjecture: the GL_n case

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Abstract

In this article we propose a vanishing conjecture for a certain class of ℓ-adic complexes on a reductive group G which can be regarded as a generalization of the acyclicity of the Artin–Schreier sheaf. We show that the vanishing conjecture contains, as a special case, a conjecture of Braverman and Kazhdan on the acyclicity of ρ-Bessel sheaves (Braverman and Kazhdan in Geom Funct Anal I:237–278, 2002). Along the way, we introduce a certain class of Weyl group equivariant ℓ-adic complexes on a maximal torus called *central complexes* and relate the category of central complexes to the Whittaker category on G. We prove the vanishing conjecture in the case when $G = \mathrm{GL}_n$.

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Contents

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1 Introduction

The Artin–Schreier sheaf $\mathcal{L}_\psi$ on the additive group $\mathbb{G}_a$ over an algebraic closure of a finite field has the following basic and important cohomology vanishing property

$$\mathrm{H}_c^*(\mathbb{G}_a, \mathcal{L}_\psi) = 0.$$

If we identify $\mathbb{G}_a$ as the unipotent radical U of the standard Borel B in SL_2, then the acyclicity of the Artin–Schreier sheaf above can be restated as follows. Let $\mathrm{tr} : \mathrm{SL}_2 \to \mathbb{G}_a$ be the trace map and consider the pull back $\Phi = \mathrm{tr}^*\mathcal{L}_\psi$. For any $x \in \mathrm{SL}_2 \backslash B$, we have the following cohomology vanishing property

$$\mathrm{H}_c^*(Ux, i^*\Phi) = 0. \tag{1.1}$$

Here $i : Ux \to \mathrm{SL}_2$ is the natural inclusion map. Indeed, it follows from the fact that the trace map tr restricts to an isomorphism $Ux \simeq \mathbb{G}_a$ for $x \in \mathrm{SL}_2 \backslash B$.

In this article we propose a vanishing conjecture for a certain class of ℓ-adic complexes on a reductive group G generalizing the cohomology vanishing in (1.1), and hence can be regarded as a generalization of the acyclicity of the Artin–Schreier sheaf.

We show that the vanishing conjecture implies a conjecture of Braverman and Kazhdan on the acyclicity of ρ-Bessel sheaves [4] (Theorem 1.4 and Corollary 1.5) and we prove the vanishing conjecture in the case $G = \mathrm{GL}_n$ (Theorem 1.6). Along the way, we introduce a certain class of Weyl group equivariant ℓ-adic complexes on a maximal torus called *central complexes* and relate the category of central complexes to the Whittaker category on G (or rather, the de Rham counterpart of the Whittaker category).

The proof of the vanishing conjecture for GL_n generalizes the one in [8] for the proof of Braverman-Kazhdan conjecture for GL_n using mirabolic subgroups. A new ingredient here is a generalization of Deligne's result of symmetric group actions on Kloosterman sheaves to the setting of central complexes (Proposition 7.6).

We now describe the paper in more details.

1.1 Central complexes

Let k be an algebraic closure of a finite field k_0 with q-element of characteristic $p > 0$. We fix a prime number ℓ different from p. Let G be a connected reductive group over

k. Let T be a maximal tour of G and B be a Borel subgroup containing T with unipotent radical U. Denote by $\mathrm{W} = T\backslash N_G(T)$ the Weyl group, where $N_G(T)$ is the normalizer of T in G. Let $\mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ be the set consisting of characters of the tame étale fundamental group $\pi_1(T)^t$ (see Sect. 4.1). For any $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ we denote by $\mathcal{L}_\chi$ the corresponding tame local system on T (a.k.a the Kummer local system associated to χ). The Weyl group W acts naturally on $\mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ and for any $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ we denote by W'_χ the stabilizer of χ in W and $\mathrm{W}_\chi \subset \mathrm{W}'_\chi$, the subgroup of W'_χ generated by those reflections s_α such that the pull-back $(\check{\alpha})^*\mathcal{L}_\chi$ is isomorphic to the trivial local system, where $\check{\alpha} : \mathbb{G}_m \to T$ is the coroot associated to α. The group W_χ is a normal subgroup of W'_χ and, in general, we have $\mathrm{W}_\chi \subsetneq \mathrm{W}'_\chi$ (see Example 4.1).[1]

Denote by $\mathscr{D}_{\mathrm{W}}(T)$ the W-equivariant derived category of constructible ℓ-adic complexes on T. For any $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$ and $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$, the W-equivariant structure on $\mathcal{F}$ together with the natural W'_χ-equivariant structure on $\mathcal{L}_\chi$ give rise to an action of W'_χ on the étale cohomology groups $\mathrm{H}^*_c(T, \mathcal{F} \otimes \mathcal{L}_\chi)$ (resp. $\mathrm{H}^*(T, \mathcal{F} \otimes \mathcal{L}_\chi)$). In particular, we get an action of the subgroup $\mathrm{W}_\chi \subset \mathrm{W}'_\chi$ on the cohomology groups above. Denote by $\mathrm{sign}_{\mathrm{W}} : \mathrm{W} \to \{\pm 1\}$ the sign character of W.

We have the following key definition.

Definition 1.1 A W-equivariant complex $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$ is called *central* (resp. $*$-*central*) if for any tame character $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$, the group W_χ acts on

$$\mathrm{H}^*_c(T, \mathcal{F} \otimes \mathcal{L}_\chi) \quad (\text{resp. } \mathrm{H}^*(T, \mathcal{F} \otimes \mathcal{L}_\chi))$$

via the sign character $\mathrm{sign}_{\mathrm{W}}$.

Remark 1.1 The Verdier duality maps central objects to $*$-central objects and vice versa.

Example 1.2 Consider the case $G = \mathrm{SL}_2$. Let $\mathrm{tr}_T : T \simeq \mathbb{G}_m \to \mathbb{G}_a, t \to t + t^{-1}$ and consider $\mathcal{F} = \mathrm{tr}_T^*\mathcal{L}_\psi[1]$ with the canonical W-equivariant structure. We claim that $\mathcal{F}$ is central. For this we observe that $\mathrm{W}_\chi \neq e$ if and only if χ is trivial. Thus $\mathcal{F}$ is central if and only if W acts on $\mathrm{H}^*_c(T, \mathcal{F})$ by the sign character, equivalently, the non-trivial involution $\sigma \in \mathrm{W}$ acts by -1 on $\mathrm{H}^*_c(T, \mathcal{F})$. On the other hand, we have $\mathrm{H}^*_c(T, \mathcal{F})^\sigma \simeq \mathrm{H}^*_c(\mathbb{G}_a, \mathcal{L}_\psi \otimes (\mathrm{tr}_{T,!}\mathbb{Q}_\ell)^\sigma) \simeq \mathrm{H}^*_c(\mathbb{G}_a, \mathcal{L}_\psi) = 0$ and it implies $\mathcal{F}$ is central.

Example 1.3 Using Mellin transforms in [11], one can associate to each W-orbit θ in $\mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ a tame central local system on T (see Sect. 4.5).

1.2 Statement of the vanishing conjecture

We have the induction functor

$$\mathrm{Ind}_{T\subset B}^G : \mathscr{D}(T) \to \mathscr{D}(G)$$

[1] The group W_χ plays an important role in the study of representations of finite reductive groups and character sheaves (see, e.g., [14]).

between the derived categories of ℓ-adic sheaves on T and G. For $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$, the W-equivariant structure on $\mathcal{F}$ defines a W-action on $\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})$ and we denote by

$$\Phi_{\mathcal{F}} := \mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})^{\mathrm{W}}$$

the W-invariant factor in $\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})$ (see Sect. 3.2). We propose the the following conjecture on acyclicity of $\Phi_{\mathcal{F}}$ over certain affine subspaces of G, called the vanishing conjecture:

Conjecture 1.2 *Assume $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$ is central (resp. $*$-central). For any $x \in G\backslash B$, we have the following cohomology vanishing*

$$\mathrm{H}_c^*(Ux, i^*\Phi_{\mathcal{F}}) = 0 \qquad (\textit{resp.}\ \mathrm{H}^*(Ux, i^!\Phi_{\mathcal{F}}) = 0) \tag{1.2}$$

where $i : Ux \to G$ is the natural inclusion map. Equivalently, the complex $\pi_!(\Phi_{\mathcal{F}})$ (resp. $\pi_*(\Phi_{\mathcal{F}})$) is supported on the closed subset $T = U\backslash B \subset U\backslash G$. Here $\pi : G \to U\backslash G$ is the quotient map.

Remark 1.4 Note that the Verdier duality $\mathbb{D}$ interchanges central objects with $*$-central objects and $\mathrm{H}_c^*(Ux, i^*\Phi_{\mathcal{F}})$ is dual to $\mathrm{H}^*(Ux, i^!\mathbb{D}\Phi_{\mathcal{F}})$. Thus the conjecture above for central objects implies the one for $*$-central objects and vice versa.

Remark 1.5 The acyclicity of Artin–Schreier sheaf is an essential ingredient in the proof that the ℓ-adic Fourier transform is a t-exact equivalence of categories. This property of ℓ-adic Fourier transform had found several applications in number theory and representation theory. We expect the vanishing conjecture would also have applications in number theory and representation theory (see Sects. 1.3 and 1.4 below for applications in the Braverman-Kazhdan conjecture).

Remark 1.6 Denote by $\mathscr{D}_G(G)$ the G-conjugation equivariant derived category on G and by $\mathscr{D}_U(U\backslash G)$, the Hecke category of U-equivariant derived category on $U\backslash G$. Let $\pi : G \to U\backslash G$ be the quotient map. The push-forward $\pi_!$ induces a functor

$$\pi_! : \mathscr{D}_G(G) \to \mathscr{D}_U(U\backslash G),$$

and it is known that for any $\mathcal{M} \in \mathscr{D}_G(G)$, the image of $\pi_!(\mathcal{M}) \in \mathscr{D}_U(U\backslash G)$ carries a canonical central structure, that is, we have a canonical isomorphism $\pi_!(\mathcal{M}) *_! \mathcal{G} \simeq \mathcal{G} *_! \pi_!(\mathcal{M})$ for any $\mathcal{G} \in \mathscr{D}_U(U\backslash G)$ (here $*_!$ is the convolution product on the Hecke category with respect to shriek push-forward). It follows from Conjecture 1.2 that, for any central complex $\mathcal{F}$, we have $\pi_!(\Phi_{\mathcal{F}}) \simeq \mathcal{F}$ (as plain objects in $\mathscr{D}(T)$). In particular, it implies that any central complex $\mathcal{F}$ carries a canonical central structure. This explains the origin of the name "central complexes".

Example 1.7 Let $\mathcal{F} = \mathrm{tr}_T^*\mathcal{L}_\psi[1]$ be as in Example 1.2. We claim that $\Phi_{\mathcal{F}} \simeq \mathrm{tr}^*\mathcal{L}_\psi[\dim \mathrm{SL}_2]$. Indeed, both complexes are isomorphic to the IC-extensions of their restrictions to the regular semi-simple locus $\mathrm{SL}_2^{\mathrm{rs}}$, and using the fact that Grothendieck-Springer simultaneous resolution (7.1) is a Cartesian over $\mathrm{SL}_2^{\mathrm{rs}}$, it is easy to show that $\Phi_{\mathcal{F}}|_{\mathrm{SL}_2^{\mathrm{rs}}} \simeq \mathrm{tr}^*\mathcal{L}_\psi[\dim \mathrm{SL}_2]|_{\mathrm{SL}_2^{\mathrm{rs}}}$. Thus the vanishing conjecture becomes (1.1), which is exactly the acyclicity of Artin–Schreier sheaf.

1.3 Braverman–Kazhdan conjecture

We recall a construction, due to Braverman and Kazhdan, of ρ-Bessel sheaf $\Phi_{G,\rho}$ attached to a r-dimensional complex representation ρ of the complex dual group $\check{G}$.

Let $\rho : \check{G} \to \mathrm{GL}(V_\rho)$ be such a representation. The restriction of ρ to the maximal torus $\check{T}$ is diagonalizable and there exists a collection of weights

$$\underline{\lambda} = \{\lambda_1, ..., \lambda_r\} \subset \mathbb{X}^\bullet(\check{T}) := \mathrm{Hom}(\check{T}, \mathbb{C}^\times)$$

such that there is an eigenspace decomposition

$$V_\rho = \bigoplus_{i=1}^r V_{\lambda_i}$$

of V_ρ, where $\check{T}$ acts on V_{λ_i} via the character λ_i. One can regard $\underline{\lambda}$ as collection of co-characters of T using the the canonical isomorphism $\mathbb{X}^\bullet(\check{T}) \simeq \mathbb{X}_\bullet(T)$, and define

$$\Phi_{T,\rho} = \mathrm{pr}_{\underline{\lambda},!}\mathrm{tr}^*\mathcal{L}_\psi[r] \qquad \Phi^*_{T,\rho} = \mathrm{pr}_{\underline{\lambda},*}\mathrm{tr}^*\mathcal{L}_\psi[r]$$

where

$$\mathrm{pr}_{\underline{\lambda}} := \prod_{i=1}^r \lambda_i : \mathbb{G}_m^r \longrightarrow T, \quad \mathrm{tr} : \mathbb{G}_m^r \longrightarrow \mathbb{G}_a, (x_1, ..., x_r) \to \sum_{i=1}^r x_i.$$

It is shown in [5], using Deligne's result about symmetric group actions on hypergeometric sheaves (see Proposition 5.2), that both $\Phi_{T,\rho}$ and $\Phi^*_{T,\rho}$ carry natural W-equivariant structures and the resulting objects in $\mathscr{D}_{\mathrm{W}}(T)$, denote again by $\Phi_{T,\rho}$ and $\Phi^*_{T,\rho}$, are called ρ-Bessel sheaves on T.[2] The ρ-Bessel sheaves on G attached to ρ, denoted by $\Phi_{G,\rho}$ and $\Phi^*_{G,\rho}$, are defined as

$$\Phi_{G,\rho} = \mathrm{Ind}^G_{T\subset B}(\Phi_{T,\rho})^{\mathrm{W}} \qquad \Phi^*_{G,\rho} = \mathrm{Ind}^G_{T\subset B}(\Phi^*_{T,\rho})^{\mathrm{W}}.$$

In [4, Conjecture 9.12], Braverman-Kazhdan proposed the following conjecture on acyclicity of ρ-Bessel sheaves over certain affine subspaces of G:

Conjecture 1.3 *Let $\Phi_{G,\rho}$ (resp. $\Phi^*_{G,\rho}$) be the ρ-Bessel sheaf attached to a representation $\rho : \check{G} \to \mathrm{GL}(V_\rho)$ of the dual group. Then for any $x \in G\backslash B$, we have the following cohomology vanishing*

$$\mathrm{H}^*_c(Ux, i^*\Phi_{G,\rho}) = 0 \quad (\textit{resp.}\ \ \mathrm{H}^*(Ux, i^!\Phi^*_{G,\rho}) = 0) \tag{1.3}$$

[2] In [4,5], the authors called $\Phi_{T,\rho}$ γ-sheaves on T. However, based on the fact that the classical γ-function is the Mellin transform of the Bessel function, we follow [17] and use the term ρ-Bessel sheaves instead of γ-sheaves.

where $i : Ux \to G$ is the natural inclusion map. Equivalently, the complex $\pi_!(\Phi_{G,\rho})$ (resp. $\pi_*(\Phi^*_{G,\rho})$) is supported on the closed subset $T = U\backslash B \subset U\backslash G$. Here $\pi : G \to U\backslash G$ is the quotient map.

Remark 1.8 In *loc.cit.* Conjecture 1.3 was stated for those representations ρ with σ-positive weights (see Sect. 5.1 for the definition of σ-positive weights). It is shown in [5,8] that, under this positivity assumption, the ρ-Bessel sheaves $\Phi_{G,\rho}$ and $\Phi^*_{G,\rho}$ (resp. $\Phi_{T,\rho}$ and $\Phi^*_{T,\rho}$) are in fact perverse sheaves and we have $\Phi_{G,\rho} \simeq \Phi^*_{G,\rho}$ (resp. $\Phi_{T,\rho} \simeq \Phi^*_{T,\rho}$) (see Proposition 5.1). This is a generalization of Deligne's theorem on Kloosterman sheaves [9]. We will see below that the vanishing conjecture (Conjecture 1.2) implies the Braverman-Kazhdan conjecture for ρ-Bessel sheaves attached to arbitrary representation ρ of the dual group $\check{G}$.

Remark 1.9 For a motivated introduction to the Braverman-Kazhdan conjecture and its role in the Langlands program see [4,5,17]

1.4 Main results

The following is the first main result of the paper. Let $\Phi_{T,\rho}$ (resp. $\Phi^*_{T,\rho}$) be the ρ-Bessel sheaf on T attached to a representation $\rho : \check{G} \to \mathrm{GL}(V_\rho)$ of the dual group.

Theorem 1.4 *For any tame character χ of T, the stabilizer subgroup W'_χ acts on*

$$\mathrm{H}^*_c(T, \Phi_{T,\rho} \otimes \mathcal{L}_\chi) \quad (\textit{resp.} \quad \mathrm{H}^*(T, \Phi^*_{T,\rho} \otimes \mathcal{L}_\chi))$$

via the sign character $\mathrm{sign}_{\mathrm{W}}$. In particular, the ρ-Bessel sheaf $\Phi_{T,\rho}$ (resp. $\Phi^*_{T,\rho}$) is central (resp. $*$-central)

Corollary 1.5 *Conjecture 1.2 implies Conjecture 1.3*

Remark 1.10 Theorem 1.4 and Corollary 1.5 put the Braverman–Kazhdan conjecture in a wider context: it is a special case of a more general vanishing conjecture whose formulation does not involve representations of the dual group.

Here is the second main result of the paper.

Theorem 1.6 *Conjecture 1.2 is true for $G = \mathrm{GL}_n$.*

Now Corollary 1.5 immediately implies:

Corollary 1.7 *Conjecture 1.3 is true for $G = \mathrm{GL}_n$.*

Conjecture 1.3 for GL_n was proved by Cheng and Ngô [8, Theorem 2.4] under some assumption on ρ. The proof of Conjecture 1.2 for GL_n, and hence Conjecture 1.3 for GL_n and arbitrary ρ, follows the one in [8] using mirabolic subgroups. However, in order to deal with the more general case of central complexes, one needs a new ingredient: a generalization of Deligne's result of symmetric group action on hypergeometric sheaves to the setting of central complexes (see Proposition 7.6).

The discussions in the previous sections have obvious counterpart in the de Rham setting, where Artin–Schreier sheaf $\mathcal{L}_\psi$ is replaced by the exponential D-module e^x and ℓ-adic sheaves are replaced by holonomic D-modules. The proof of the main results that will be given in this paper is entirely geometric and hence can be also applied to the de Rham setting.

1.5 Whittaker categories

The de Rham counterpart of the category of $*$-central perverse sheaves on T, to be called the category of $*$-central D-modules on T, also appears in the recent works of Ginzburg and Lonergan [10,13] on Whittaker D-modules, nil Hecke algebras, and quantum Toda lattices. To be specific, in *loc.cit.* the authors proved that the category of Whittaker D-modules on G, denoted by $\mathrm{Whit}(G)$, is equivalent to a certain full subcategory of the category of W-equivariant D-modules on T. It turns out that, as proven in [7, Theorem 1.8], the latter full subcategory, and hence the Whittaker category $\mathrm{Whit}(G)$, is equivalent to the category of $*$-central D-modules on T. It would be interesting to establish similar results in the ℓ-adic setting and use it to give a description of ℓ-adic counterpart of the Whittaker category in terms of W-equivariant sheaves on the maximal torus T.[3]

The equivalence between $\mathrm{Whit}(G)$ and the category of $*$-central D-modules on T gives rise to a functor $\mathrm{Whit}(G) \to \mathrm{D\text{-}mod}(G\backslash G)$, where $\mathrm{D\text{-}mod}(G\backslash G)$ is the category of G-conjugation equivariant D-modules on G. In [3], Ben-Zvi and Gunningham gave another construction of a functor from $\mathrm{Whit}(G)$ to $\mathrm{D\text{-}mod}(G\backslash G)$ and they conjectured that the essential image of their functor satisfies the same acyclicity in (1.2), see [3, Conjecture 2.9 and 2.14]. Our results might be useful for studying their conjecture.

1.6 Recent developments

In the recent work [7], we established the vanishing conjecture for general reductive group G and hence, by Corollary 1.5, the Braverman-Kazhdan conjecture for almost all characteristics. The proof given in *loc. cit.* is different form the one given here: it first established the vanishing conjecture in the de Rham setting using the theory of Harish-Chandra bimodules and character D-modules and then deduced the positive characteristic case using character sheaves in mixed-characteristic and a spreading out argument. The proof makes use of Harish-Chandra bimodules and hence is algebraic. It will be interesting to have a geometric proof of the vanishing conjecture for general reductive groups, like the proof given here for GL_n, which treats the case of various ground fields and sheaf theories uniformly.

In the recent work [15], G. Laumon and E. Letellier established the function theoretical version of Braverman-Kazhdan conjecture for all reductive groups G.

[3] An question raised by V.Drinfeld according to [10, Section 1.5].

1.7 Organization

We briefly summarize here the main goals of each section. In Sect. 2 we collect standard notation in algebraic geometry and ℓ-adic sheaves. In Sect. 3 we study induction and restriction functors for ℓ-adic sheaves on reductive groups. In Sect. 4 we give a characterization of central complexes using the ℓ-adic Mellin transform and use it to construct examples of tame central (resp. $*$-central) local systems on T. In Sect. 5 we prove Theorem 1.4. In Sect. 7 we prove Theorem 1.6.

2 Notations

In this article k will be an algebraic closure of a finite field k_0 with q-element of characteristic $p > 0$. We fix a prime number ℓ different from p.

For an algebraic stack $\mathcal{X}$ over k, we denote by $\mathscr{D}(\mathcal{X}) = D^b_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$ the bounded derived category of constructible ℓ-adic complexes on $\mathcal{X}$. For a representable morphism $f : \mathcal{X} \to \mathcal{Y}$, the six functors $f^*, f_*, f_!, g^!, \otimes, \underline{\mathrm{Hom}}$ are understood in the derived sense. For a k-scheme X, sometimes we will write $R\Gamma(X, \overline{\mathbb{Q}}_\ell) = f_*\overline{\mathbb{Q}}_\ell$ (resp. $R\Gamma_c(X, \overline{\mathbb{Q}}_\ell) = f_!\overline{\mathbb{Q}}_\ell$), where $f : X \to \operatorname{Spec} k$ is the structure map.

For an algebraic group H over k acting on a k-scheme X, we denote by $H\backslash X$, the corresponding quotient stack. Consider the case when H is a finite group. Then the pull-back along the quotient map $X \to H\backslash X$ induces an equivalence between $\mathscr{D}(H\backslash X)$ and the (naive) H-equivariant derived category of ℓ-adic complexes on X, denoted by $\mathscr{D}_H(X)$, whose objects consist of pair $(\mathcal{F}, \phi)$, where $\mathcal{F} \in \mathscr{D}(X)$ and $\phi : a^*\mathcal{F} \simeq \mathrm{pr}^*\mathcal{F}$ is an isomorphism satisfying the usual compatibility conditions (here a and pr are the action and projection map from $H \times X$ to X respectively).[4] We will call an object $(\mathcal{F}, \phi)$ in $\mathscr{D}_H(X)$ a H-equivariant complex and ϕ a H-equivariant structure on $\mathcal{F}$. For simplicity, we will write $\mathcal{F} = (\mathcal{F}, \phi)$ for an object in $\mathscr{D}_H(X)$.

The category $\mathscr{D}(\mathcal{X})$ has a natural perverse t-structure and we denote by $\mathscr{D}(\mathcal{X})^\heartsuit$ the corresponding heart and ${}^p\tau_{\leq n}, {}^p\tau_{\geq n}$ the perverse truncation functors. For any $\mathcal{F} \in \mathscr{D}(\mathcal{X})$, the n-the perverse cohomology sheaf is defined as ${}^p\mathscr{H}^n(\mathcal{F}) = {}^p\tau_{\geq n}{}^p\tau_{\leq n}(\mathcal{F})[n]$.

We will denote by $\mathbb{G}_a$ the additive group over k and $\mathbb{G}_m$ the multiplicative group over k. We will fix a non-trivial character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ and denote by $\mathcal{L}_\psi$ the corresponding Artin–Schreier sheaf on $\mathbb{G}_a$.

3 Induction and restriction functors

Let G be a connected reductive group over k. Let T be a maximal torus of G, B a Borel subgroup containing T with unipotent radical U. We denote by $\mathrm{W} = T\backslash N_G(T)$ the Weyl group of G, where $N_G(T)$ is the normalizer of T in G. We denote by $\mathrm{sign} = \mathrm{pr}^* \mathrm{sign}_\mathrm{W} \in \mathscr{D}_\mathrm{W}(T)$ the pull back of the sign representation sign_W of W

[4] This holds in a more general situation when the neutral component of H is unipotent.

(regarding as an object in $\mathscr{D}_{\mathrm{W}}(\mathrm{pt})$) along the projection $\mathrm{pr}: T \to \mathrm{pt}$. Throughout the paper, we assume p does not divide the order of the Weyl group W.
3.2 Recall the Grothendieck-Springer simultaneous resolution of the Steinberg map $c: G \to \mathrm{W}\backslash\backslash T$:

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\tilde{q}} & T \\ \downarrow{\scriptstyle \tilde{c}} & & \downarrow{\scriptstyle q} \\ G & \xrightarrow{c} & \mathrm{W}\backslash\backslash T \end{array} \tag{3.1}$$

where $\widetilde{G}$ is the closed subvariety of $G \times G/B$ consisting of pairs (g, xB) such that $x^{-1}gx \in B$. The map $\tilde{c}$ is given by $(g, xB) \to g$, and the map $\tilde{q}$ is given by $(g, xB) \to x^{-1}gx \mod U \in B/U = T$. The induction functor $\mathrm{Ind}_{T\subset B}^{G}: \mathscr{D}(T) \to \mathscr{D}(G)$ is given by

$$\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F}) = \tilde{c}_!\tilde{q}^*(\mathcal{F})[\dim G - \dim T].$$

We have the following equivalent construction of $\mathrm{Ind}_{T\subset B}^{G}$. Consider the fiber product $S = G \times_{\mathrm{W}\backslash\backslash T} T$. The diagram (7.1) induces a map

$$h: \widetilde{G} \to S = G \times_{\mathrm{W}\backslash\backslash T} T \tag{3.2}$$

which is proper and small, and an isomorphism over $S^{rs} = G^{rs} \times_{\mathrm{W}\backslash\backslash T} T^{rs}$. It follows that

$$h_!\overline{\mathbb{Q}}_\ell[\dim G] \simeq j_{!*}\overline{\mathbb{Q}}_\ell[\dim G] := \mathrm{IC}(S, \overline{\mathbb{Q}}_\ell) \tag{3.3}$$

where $j: S^{rs} \to S$ is the open embedding and $j_{!*}$ is the intermediate extension functor. We have

$$\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F}) \simeq (p_G)_!(p_T^*(\mathcal{F}) \otimes \mathrm{IC}(S, \overline{\mathbb{Q}}_\ell))[-\dim T] \tag{3.4}$$

where $p_T: S \to T$ and $p_G: S \to G$ are the natural projection map.

The functor $\mathrm{Ind}_{T\subset B}^{G}$ admits a left adjoint $\mathrm{Res}_{T\subset B}^{G}: \mathscr{D}(G) \to \mathscr{D}(T)$, called the restriction functor, which is given by

$$\mathrm{Res}_{T\subset B}^{G}(\mathcal{F}) = (q_B)_!i_B^*(\mathcal{F})$$

where $i_B: B \to G$ is the natural inclusion and $q_B: B \to T = U\backslash B$ is the quotient map. More generally, one could define $\mathrm{Res}_{L\subset P}^{G}: \mathscr{D}(G) \to \mathscr{D}(L)$, for any pair (L, P) where L is a Levi subgroup of a parabolic subgroup P of G.

We have the following exactness properties of induction and restriction functors:

Proposition 3.1 [6, Theorem 5.4] *(1) The functor* $\mathrm{Ind}_{T\subset B}^{G}$ *maps perverse sheaves on* T *to perverse sheaves on* G. *(2) The functor* $\mathrm{Res}_{L\subset P}^{G}$ *maps* G*-conjugation equivariant perverse sheaves on* G *to* L*-conjugation equivariant perverse sheaves on* L.

Remark 3.1 The proposition above generalizes well-known results of Lusztig's on exactness of induction and restriction functors for character sheaves [14].

3.1 W-action

Proposition 3.2 *(1) Let $\mathcal{F} \in \mathscr{D}(T)$. For every $w \in \mathrm{W}$ one has a canonical isomorphism*

$$\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F}) \simeq \mathrm{Ind}_{T\subset B}^{G}(w^*\mathcal{F}).$$

(2) Let $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$. There is a natural W-action on $\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F}) \in \mathscr{D}(G)$ and, for any irreducible representation $\xi : \mathrm{W} \to \mathrm{GL}(V_\xi)$ of W, there is canonical direct summand

$$\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})^{\mathrm{W},\xi} \in \mathscr{D}(G) \tag{3.5}$$

of $\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})$ such that we have a W-equivariant decomposition in $\mathscr{D}(G)$

$$\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F}) \simeq \bigoplus_{(\xi, V_\xi)\in \mathrm{Irr}\,\mathrm{W}} V_\xi \otimes \mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})^{\mathrm{W},\xi}. \tag{3.6}$$

(3) Let $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)^{\heartsuit}$ be a W-equivariant perverse sheaf. Then the direct summand $\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})^{\mathrm{W},\xi}$ in (3.6) is also a perverse sheaf.

Proof We have a W-action on $S = G \times_{\mathrm{W}\backslash\backslash T} T$ given by $(g, t) \to (g, wt)$, $w \in \mathrm{W}$, and the projection maps p_G and p_T form S to G and T are W-equivariant (for p_G, the W-action on G is the trivial action). Consider the following commutative diagram

$$\begin{array}{ccccc} G & \xleftarrow{p_G} & S & \xrightarrow{p_T} & T \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ G & \xleftarrow{p'_G} & \mathrm{W}\backslash S & \xrightarrow{p'_T} & \mathrm{W}\backslash T \end{array} \tag{3.7}$$

where π', p'_G, p'_T are the natural quotient maps. Note that, since S^{rs} is W-invariant, the IC-complex $\mathrm{IC}(S, \overline{\mathbb{Q}}_\ell) = j_{*!}\overline{\mathbb{Q}}_\ell[\dim G]$ (here $j : S^{rs} \to S$) descends to the IC-complex $\mathrm{IC}(\mathrm{W}\backslash S, \overline{\mathbb{Q}}_\ell)$ on $\mathrm{W}\backslash S$, in particular, $\mathrm{IC}(S, \overline{\mathbb{Q}}_\ell)$ is W-equivariant. It follows from (3.4) that

$$\begin{aligned} \mathrm{Ind}_{T\subset B}^{G}(w^*\mathcal{F}) &\simeq (p_G)_!(w^* p_T^*(\mathcal{F}) \otimes \mathrm{IC}(S, \overline{\mathbb{Q}}_\ell))[-\dim T] \simeq \\ &\simeq (p_G)_!(w^* p_T^*(\mathcal{F}) \otimes w^*\mathrm{IC}(S, \overline{\mathbb{Q}}_\ell))[-\dim T] \\ &\simeq (p_G)_! w^*\big(p_T^*(\mathcal{F}) \otimes \mathrm{IC}(S, \overline{\mathbb{Q}}_\ell)\big)[-\dim T] \simeq \\ &\simeq (p_G)_!\big(p_T^*(\mathcal{F}) \otimes \mathrm{IC}(S, \overline{\mathbb{Q}}_\ell)\big)[-\dim T] \simeq \mathrm{Ind}_{T\subset B}^{G}(\mathcal{F}). \end{aligned} \tag{3.8}$$

Part (1) follows.

Let $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$. Since $w^*\mathcal{F} \simeq \mathcal{F}$ for any $w \in \mathrm{W}$, we have a canonical isomorphism

$$a_w : \mathrm{Ind}_{T\subset B}^G(\mathcal{F}) \simeq \mathrm{Ind}_{T\subset B}^G(w^*\mathcal{F}) \overset{(3.8)}{\simeq} \mathrm{Ind}_{T\subset B}^G(\mathcal{F}),$$

and the assignment $w \to a_w$, $w \in \mathrm{W}$ defines a W-action on $\mathrm{Ind}_{T\subset B}^G(\mathcal{F})$. To show (3.6), we observe that

$$\mathrm{IC}(S, \overline{\mathbb{Q}}_\ell) \simeq (\pi')^*\mathrm{IC}(\mathrm{W}\backslash S, \overline{\mathbb{Q}}_\ell),$$

and it implies

$$(\pi')_!\mathrm{IC}(S, \overline{\mathbb{Q}}_\ell) \simeq (\pi')_!(\pi')^*\mathrm{IC}(\mathrm{W}\backslash S, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{(\xi, V_\xi)\in \mathrm{Irr} \mathrm{W}} V_\xi \otimes \big(\mathrm{IC}(\mathrm{W}\backslash S, \overline{\mathbb{Q}}_\ell) \otimes V_{\xi,S}\big) \tag{3.9}$$

where $V_{\xi,S} \in \mathscr{D}(\mathrm{W}\backslash S)$ is the pull back of $(\xi, V_\xi) \in \mathrm{Rep}\,\mathrm{W} \simeq \mathscr{D}(\mathrm{W}\backslash \mathrm{pt})^\heartsuit$ along the projection $\mathrm{W}\backslash S \to \mathrm{W}\backslash \mathrm{pt}$.

Let $\mathcal{F}' \in \mathscr{D}(\mathrm{W}\backslash T)$ be such that $\pi^*\mathcal{F}' \simeq \mathcal{F}$. It follows that

$$\begin{aligned}
\mathrm{Ind}_{T\subset B}^G(\mathcal{F}) &\simeq (p_G)_!\big(p_T^*(\mathcal{F}) \otimes \mathrm{IC}(S, \overline{\mathbb{Q}}_\ell)\big)[-\dim T] \simeq \\
&\simeq (p'_G)_!(\pi')_!\big((\pi')^*(p'_T)^*\mathcal{F}' \otimes \mathrm{IC}(S, \overline{\mathbb{Q}}_\ell)\big)[-\dim T] \\
&\simeq (p'_G)_!\big((p'_T)^*\mathcal{F}' \otimes (\pi'_!)\mathrm{IC}(S, \overline{\mathbb{Q}}_\ell)\big)[-\dim T] \overset{(3.9)}{\simeq} \\
&\simeq \bigoplus_{(\xi, V_\xi)\in \mathrm{Irr}\,\mathrm{W}} V_\xi \otimes \mathrm{Ind}_{T\subset B}^G(\mathcal{F})^{\mathrm{W},\xi}
\end{aligned}$$

where

$$\mathrm{Ind}_{T\subset B}^G(\mathcal{F})^{\mathrm{W},\xi} := (p'_G)_!\big((p'_T)^*\mathcal{F}' \otimes \mathrm{IC}(\mathrm{W}\backslash S, \overline{\mathbb{Q}}_\ell) \otimes V_{\xi,S}\big)[-\dim T]. \tag{3.10}$$

Part (2) follows. Part (3) follows from Proposition 3.1. □

Remark 3.2 Part (1) of the proposition generalizes [4, Theorem 2.5(3)].

Definition 3.3 For any W-equivariant complex $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$, we will write

$$\mathrm{Ind}_{T\subset B}^G(\mathcal{F})^{\mathrm{W}} := \mathrm{Ind}_{T\subset B}^G(\mathcal{F})^{\mathrm{W},\mathrm{triv}} = (p'_G)_!\big((p'_T)^*\mathcal{F}' \otimes \mathrm{IC}(\mathrm{W}\backslash S, \overline{\mathbb{Q}}_\ell)\big)[-\dim T], \tag{3.11}$$

(here p'_G, p'_T are the morphisms in (3.7)) for the summand in (3.5) corresponding to the trivial representation (triv, $V_{\mathrm{triv}} = \overline{\mathbb{Q}}_\ell$).

In the case when $\mathcal{F}$ is a W-equivariant perverse local system on T, we have the following description of (3.11): Let $q^{rs} : T^{rs} \to \mathrm{W}\backslash\backslash T^{rs}$ and $c^{rs} : G \to \mathrm{W}\backslash\backslash T^{rs}$ be the restriction of the maps in (7.1) to the regular semi-simple locus. As q^{rs} is an étale covering, the restriction of $\mathcal{F}$ to T^{rs} descends to a perverse local system $\mathcal{F}'$ on $\mathrm{W}\backslash\backslash T^{rs}$ and we have

$$\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})^{\mathrm{W}}\simeq j_{!*}(c^{rs})^{*}\mathcal{F}'[\dim G-\dim T].$$

Let P be a standard parabolic subgroup containing B and let L be the unique Levi subgroup of P containing T with Borel subgroup $B_L=B\cap L$. Let W_L be the Weyl group of L, which is naturally a subgroup of W.

Proposition 3.4 *Let $\mathcal{F}\in\mathscr{D}_{\mathrm{W}}(T)^{\heartsuit}$. (1) We have a canonical isomorphism*

$$\mathrm{Res}_{L\subset P}^{G}\circ\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F}_T)\simeq\mathrm{Ind}_{\mathrm{W}_L}^{\mathrm{W}}\mathrm{Ind}_{T\subset B_L}^{L}(\mathcal{F}_T)$$

which is compatible with the nature W*-actions on both sides. (2) There is a canonical isomorphism*

$$\mathrm{Res}_{L\subset P}^{G}\circ\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})^{\mathrm{W}}\simeq\mathrm{Ind}_{T\subset B_L}^{L}(\mathcal{F})^{\mathrm{W}_L}.$$

Proof Part (2) follows from part (1) by taking W-invariants on both sides. Part (1) is proved in [5, Theorem 2.7]. [5] □

4 Characterization of central complexes

4.1 The scheme of tame characters

Let $\pi_1(T)$ be the étale fundamental group of T and let $\pi_1(T)^t$ be its tame quotient. A continuous character $\chi:\pi_1(T)\to\overline{\mathbb{Q}}_\ell^{\times}$ is called *tame* if it factors though $\pi_1(T)^t$. For any continuous character $\chi:\pi_1(T)\to\overline{\mathbb{Q}}_\ell^{\times}$, we denote by $\mathcal{L}_\chi$ the corresponding rank one local system on T. A rank one local system $\mathcal{L}$ on T is called *tame* if $\mathcal{L}\simeq\mathcal{L}_\chi$ for a tame character χ.

In [11], a $\overline{\mathbb{Q}}_\ell$-scheme $\mathcal{C}(T)$ is defined, whose $\overline{\mathbb{Q}}_\ell$-points are in bijection with tame characters of $\pi_1(T)$. There is decomposition

$$\mathcal{C}(T)=\bigsqcup_{\chi_f\in\mathcal{C}(T)_f}\{\chi_f\}\times\mathcal{C}(T)_\ell \tag{4.1}$$

into connected components, where $\mathcal{C}(T)_f\subset\mathcal{C}(T)$ is the subset consisting of tame characters of order prime to ℓ and $\mathcal{C}(T)_\ell$ is the connected component of $\mathcal{C}(T)$ containing the trivial character. It is shown in *loc. cit.* that $\mathcal{C}(T)$ is Noetherian and regular and there is an isomorphism

$$\mathcal{C}(T)_\ell\simeq\mathrm{Spec}(\overline{\mathbb{Q}}_\ell\otimes_{\mathbb{Z}_\ell}\mathbb{Z}_\ell[[x_1,...,x_r]]).$$

[5] In *loc. cit.* the authors assumed $\mathcal{F}$ is irreducible with support a W-stable sub-torus in T. This is because the proof makes use of the isomorphism $\mathrm{Ind}_{T\subset B}^{G}(\mathcal{F})\simeq\mathrm{Ind}_{T\subset B}^{G}(w^*\mathcal{F})$ which was constructed only for those $\mathcal{F}$ satisfying the assumption above (see [4, Theorem 2.5(3)]). Now, with Proposition 3.2, the same argument works for arbitrary W-equivariant perverse sheaves on T.

In addition, the $\overline{\mathbb{Q}}_\ell$-points of $\mathcal{C}(T)_\ell$ are in bijection with pro-ℓ characters of $\pi_1(T)$ (i.e. characters of the pro-ℓ quotient $\pi_1(T)_\ell$ of $\pi_1(T)^t$).

4.2 The group W_χ

The Weyl group W acts naturally on $\mathcal{C}(T)$. For any $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ we denote by W'_χ the stabilizer of χ in W. Equivalently, W_χ is the group consisting of $w \in \mathrm{W}$ such that $w^*\mathcal{L}_\chi \simeq \mathcal{L}_\chi$. Let W_χ to be the subgroup of W'_χ generated by reflections s_α satisfying the following property: Let $\check{\alpha} : \mathbb{G}_m \to T$ be the coroot corresponding to α. The pullback $(\check{\alpha})^*\mathcal{L}_\chi$ is isomorphic to the trivial local system on $\mathbb{G}_m$.

Example 4.1 Let $G = \mathrm{SL}_2$ and let $\chi : \pi_1(T)^t \to \{\pm 1\}$ be the character corresponding to the tame covering $T \to T, x \to x^2$. Then we have $\mathrm{W}'_\chi = \mathrm{W}$ but $\mathrm{W}_\chi = e$ is trivial.

4.3 Central complexes

Let $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$. For any tame character $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ the stabilizer W'_χ, hence also its subgroup W_χ, acts naturally on the étale cohomology groups $\mathrm{H}^*_c(T, \mathcal{F}_T \otimes \mathcal{L}_\chi)$ (resp. $\mathrm{H}^*(T, \mathcal{F}_T \otimes \mathcal{L}_\chi)$).

Definition 4.1 A W-equivariant complex $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$ is called *central* (resp. $*$-*central*) if the following holds: for any tame character $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$, the group W_χ acts on

$$\mathrm{H}^*_c(T, \mathcal{F} \otimes \mathcal{L}_\chi) \qquad (\text{resp.}\ \ \mathrm{H}^*(T, \mathcal{F} \otimes \mathcal{L}_\chi))$$

through the sign character $\mathrm{sign}_{\mathrm{W}} : \mathrm{W}_\chi \to \{\pm 1\}$.

4.4 Mellin transforms

In [11], the authors constructed the *Mellin transforms*

$$\mathcal{M}_! : \mathscr{D}(T) \to D^b_{coh}(\mathcal{C}(T))$$
$$\mathcal{M}_* : \mathscr{D}(T) \to D^b_{coh}(\mathcal{C}(T))$$

with the following properties:

(1) Let $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ and $i_\chi : \{\chi\} \to \mathcal{C}(T)$ be the natural inclusion. We have

$$R\Gamma_c(T, \mathcal{F} \otimes \mathcal{L}_\chi) \simeq i^*_\chi \mathcal{M}_!(\mathcal{F})$$
$$R\Gamma(T, \mathcal{F} \otimes \mathcal{L}_\chi) \simeq i^*_\chi \mathcal{M}_*(\mathcal{F}). \tag{4.2}$$

(2) We have natural isomorphism $\mathbb{D}(\mathcal{M}_!(\mathcal{F})) \simeq \mathrm{inv}^*\, \mathcal{M}_*(\mathbb{D}(\mathcal{F}))$.
(3) The functor $\mathcal{M}_*$ is t-exact with respect to the perverse t-structure on $\mathscr{D}(T)$ and the natural t-structure on $D^b_{coh}(\mathcal{C}(T))$. Moreover, for any $\mathcal{F} \in \mathscr{D}(T)$, $\mathcal{F}$ is perverse if and only if $\mathcal{M}_*(\mathcal{F})$ is a coherent complex in degree zero.

(4) We have

$$\mathcal{M}_!(\mathcal{F} *_! \mathcal{F}') \simeq \mathcal{M}_!(\mathcal{F}) \otimes \mathcal{M}_!(\mathcal{F}')$$
$$\mathcal{M}_*(\mathcal{F} * \mathcal{F}') \simeq \mathcal{M}_*(\mathcal{F}) \otimes \mathcal{M}_*(\mathcal{F}')$$

where $\mathcal{F} *_! \mathcal{F}' = m_!(\mathcal{F} \boxtimes \mathcal{F}')$, $\mathcal{F} * \mathcal{F}' = m_*(\mathcal{F} \boxtimes \mathcal{F}')$, and $m : T \times T \to T$ is the multiplication map.

(5) For any $\chi \in \mathcal{C}(T)_f$ we have

$$\mathcal{M}_*(\mathcal{F})|_{\{\chi\}\times\mathcal{C}(T)_\ell} \simeq \mathcal{M}_*(\mathcal{F} \otimes \mathcal{L}_\chi)|_{\mathcal{C}(T)_\ell}.$$

(6) The Mellin transforms restricts to an equivalence

$$\mathcal{M}_!, \mathcal{M}_* : \mathscr{D}(T)_{\text{mon}} \simeq D^b_{coh}(\mathcal{C}(T))_f \tag{4.3}$$

between the full subcategory $\mathscr{D}(T)_{\text{mon}}$ of monodromic ℓ-adic complexes on T and the full subcategory $D^b_{coh}(\mathcal{C}(T))_f$ of coherent complexes on $\mathcal{C}(T)$ with finite support.

The Weyl group W acts naturally on $\mathcal{C}(T)$ and it follows from the construction of Mellin transforms that for $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$, the W-equivariant structure on $\mathcal{F}$ gives rise to a W-equivariant structure on $\mathcal{M}_!(\mathcal{F})$ (resp. $\mathcal{M}_*(\mathcal{F})$), such that for any $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$, the isomorphism (4.2) above is compatible with the natural W'_χ-actions.

We have the following characterization of central complexes:

Proposition 4.2 *Let $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$ be a* W*-equivariant complex. The following are equivalent:*

(1) $\mathcal{F}$ is central.
(2) For any $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ the action of W_χ on $i_\chi^*\mathcal{M}_!(\mathcal{F} \otimes \mathrm{sign})$ is trivial.
Assume further that $\mathrm{W}_\chi = \mathrm{W}'_\chi$ for all $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$, then above statements are equivalent to
(3) The restriction of the Mellin transfrom $\mathcal{M}_!(\mathcal{F} \otimes \mathrm{sign}) \in D^b_{coh}(\mathcal{C}(T))$ to each connected component $\mathcal{C}(T)_{\ell,\chi_f} := \{\chi_f\} \times \mathcal{C}(T)_\ell$ of $\mathcal{C}(T)$ (see (4.1)) descends to the quotient $\mathrm{W}_{\chi_f}\backslash\backslash\mathcal{C}(T)_{\ell,\chi_f}$.[6]

The same is true for $*$-central complexes if we replace $\mathcal{M}_!$ by $\mathcal{M}_*$.

Proof (1) $\Leftrightarrow$ (2) follows from the property (1) of Mellin transform above. Assume $\mathrm{W}_\chi = \mathrm{W}'_\chi$ for all χ. Then for any $\chi \in \mathcal{C}(T)_{\ell,\chi_f}$ we have $\mathrm{W}_\chi = \mathrm{W}'_\chi \subset \mathrm{W}'_{\chi_f} = \mathrm{W}_{\chi_f}$ and it follows that the stabilizer of χ in W_{χ_f} is equal to W_χ and (2) $\Leftrightarrow$ (3) follows from the descent criterion for coherent complexes in [16, Theorem 1.3]. □

6 Note that $\mathcal{C}(T)_{\ell,\chi_f}$ is stable under the W_{χ_f}-action.

4.5 Examples of tame central local systems

Consider the quotient map $\pi_\chi : \mathcal{C}(T)_\ell \to \mathrm{W}_\chi\backslash\backslash\mathcal{C}(T)_\ell$. Let $0 \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ be the trivial character and let $D_\chi = \pi_\chi^{-1}(\pi_\chi(0))$ be the scheme theoretic pre-image of $\pi_\chi(0)$ for the map π_χ. We define $\mathcal{R}_\chi^{uni} = \mathcal{O}_{D_\chi}$, which is a W_χ-equivariant coherent sheaf on $\mathcal{C}(T)_\ell$. Note that, as W_χ is normal subgroup of W'_χ, the W'_χ-action on $\mathcal{C}(T)_\ell$ descends to a W'_χ-action on the quotient $\mathrm{W}_\chi\backslash\backslash\mathcal{C}(T)_\ell$ fixing $\pi_\chi(0)$ and it follows that W'_χ stabilizes D_χ and $\mathcal{R}_\chi^{uni}$ has a canonical W'_χ-equivariant structure.

We will regard $\mathcal{R}_\chi^{uni}$ as a coherent sheaf on $\mathcal{C}(T)$ supported at the component $\mathcal{C}(T)_\ell = \{0\} \times \mathcal{C}(T)_\ell$ and define $\mathcal{R}_\chi = m_\chi^*(\mathcal{R}_\chi^{uni})$ where $m_\chi : \mathcal{C}(T) \to \mathcal{C}(T)$ be the morphism of translation by χ. Since m_χ intertwines the W'_χ-action on $\mathcal{C}(T)$, $\mathcal{R}_\chi$ is W'_χ-equivariant, moreover, there is a natural isomorphism

$$w^*\mathcal{R}_\chi \simeq \mathcal{R}_{w^{-1}\chi} \tag{4.4}$$

for any $w \in \mathrm{W}$.[7] Thus for any W-orbit θ in $\mathcal{C}(T)$, we have the following W-equivariant coherent sheaf with finite support

$$\mathcal{R}_\theta = \bigoplus_{\chi\in\theta} \mathcal{R}_\chi$$

where the W-equivariant structure is given by the isomorphisms in (4.4).

Finally, we define the following W-equivariant perverse local systems on T:

$$\mathcal{E}_\theta^! := \mathcal{M}_!^{-1}(\mathcal{R}_\theta) \otimes \mathrm{sign}$$
$$\mathcal{E}_\theta := \mathcal{M}_*^{-1}(\mathcal{R}_\theta) \otimes \mathrm{sign}\,.$$

Here $\mathcal{M}_!^{-1}, \mathcal{M}_*^{-1}$ are the inverse of the Mellin transforms in (4.3).[8]

Lemma 4.3 *$\mathcal{E}_\theta^!$ is central and $\mathcal{E}_\theta$ is $*$-central.*

Proof Since $\mathcal{R}_\theta$ is supported on $\theta^{-1} = \{\chi^{-1}|\chi \in \theta\}$, by Proposition 4.2, it suffices to show that the action of $\mathrm{W}_{\chi^{-1}}$ (note that $\mathrm{W}_{\chi^{-1}} = \mathrm{W}_\chi$) on the fiber

$$i_{\chi^{-1}}^*\mathcal{M}_!(\mathcal{E}_\theta^! \otimes \mathrm{sign}) \simeq i_{\chi^{-1}}^*\mathcal{M}_!(\mathcal{M}_!^{-1}(\mathcal{S}_\theta)) \simeq i_{\chi^{-1}}^*\mathcal{R}_\theta \simeq i_{\chi^{-1}}^*\mathcal{R}_\chi$$

is trivial. Let $\mathcal{C}(T)_{\ell,\chi^{-1}} \subset \mathcal{C}(T)$ be the component containing χ^{-1}. Then the translation map m_χ restricts to a map $m_\chi : \mathcal{C}(T)_{\ell,\chi^{-1}} \to \mathcal{C}(T)_\ell$, moreover, we have the

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[7] Indeed, we have $w^*\mathcal{R}_\chi \simeq w^*m_\chi^*\mathcal{R}_\chi^{uni} \simeq m_{w^{-1}\chi}^* w^*\mathcal{R}_\chi^{uni} \simeq m_{w^{-1}\chi}^*\mathcal{R}_{w^{-1}\chi}^{uni} = \mathcal{R}_{w^{-1}\chi}$, where we use the observation that $w^{-1}\mathrm{W}_\chi w = \mathrm{W}_{w^{-1}\chi}$ and hence $w^*\mathcal{R}_\chi^{uni} \simeq \mathcal{R}_{w^{-1}\chi}^{uni}$.

[8] Note that $\mathcal{R}_\theta$ has finite support and hence the inverse of the Mellin transform is well-defined by (4.3).

following commutative diagram

$$\begin{array}{ccc} \mathcal{C}(T)_{\ell,\chi^{-1}} & \xrightarrow{m_\chi} & \mathcal{C}(T)_\ell \\ \downarrow{\scriptstyle \pi_{\ell,\chi^{-1}}} & & \downarrow{\scriptstyle \pi_\chi} \\ \mathrm{W}_{\chi^{-1}}\backslash\backslash\mathcal{C}(T)_{\ell,\chi^{-1}} & \xrightarrow{\bar{m}_\chi} & \mathrm{W}_\chi\backslash\backslash\mathcal{C}(T)_\ell \end{array}$$

where $\bar{m}_\chi$ is the descent of m_χ. It follows that the restriction of $\mathcal{R}_\chi = m_\chi^*\pi_\chi^*\mathcal{O}_{\pi_\chi(0)} \simeq \pi_{\ell,\chi^{-1}}^*\bar{m}_\chi^*\mathcal{O}_{\pi_\chi(0)}$ to $\mathcal{C}(T)_{\ell,\chi^{-1}}$ descends to $\mathrm{W}_{\chi^{-1}}\backslash\backslash\mathcal{C}(T)_{\ell,\chi^{-1}}$ and it implies the action of $\mathrm{W}_{\chi^{-1}}$ on $i_{\chi^{-1}}^*\mathcal{R}_\chi$ is trivial. □

5 ρ-Bessel sheaves

5.1 Hypergeometric sheaves

Let $\underline{\lambda} = \{\lambda_1, ..., \lambda_r\} \subset \mathbb{X}_\bullet(T)$ be a collection of possible repeated cocharacters. Consider the following maps

$$\mathbb{G}_a \xleftarrow{\mathrm{tr}} \mathbb{G}_m^r \xrightarrow{\mathrm{pr}_{\underline{\lambda}}} T$$

where $\mathrm{tr}(x_1, ..., x_r) = \sum_{i=1}^r x_i$ and $\mathrm{pr}_{\underline{\lambda}}(x_1, ..., x_r) = \prod_{i=1}^r \lambda_i(x_i)$. Consider the following complexes:

$$\Phi_{\underline{\lambda}} := \mathrm{pr}_{\underline{\lambda},!}\mathrm{tr}^*\mathcal{L}_\psi[r], \quad \Phi_{\underline{\lambda}}^* := \mathrm{pr}_{\underline{\lambda},*}\mathrm{tr}^*\mathcal{L}_\psi[r] \tag{5.1}$$

in $D(T)$. Note that we have a natural forget supports map

$$\Phi_{\underline{\lambda}} \to \Phi_{\underline{\lambda}}^*. \tag{5.2}$$

Following Katz [12], we will call the complexes in (5.1) hypergeometric sheaves.

Let $\sigma : T \to \mathbb{G}_m$ be a character. A cocharacter λ is called σ-positive if $\langle\sigma, \lambda\rangle$ is positive.

Proposition 5.1 *Let $\underline{\lambda} = \{\lambda_1, ..., \lambda_r\} \subset \mathbb{X}_\bullet(T)$ be a collection of cocharacters.*

(1) Assume that each λ_i is nontrivial. Then $\Phi_{\underline{\lambda}}$ and $\Phi_{\underline{\lambda}}^*$ are perverse sheaves.
(2) Assume that each λ_i is σ-positive. Then the map $\Phi_{\underline{\lambda}} \to \Phi_{\underline{\lambda}}^*$ in (5.2) is an isomorphism and $\Phi_{\underline{\lambda}} \simeq \Phi_{\underline{\lambda}}^*$ is a perverse local system over the image of $p_{\underline{\lambda}}$, which is a subtorus of T.

Proof This is [5, Theorem 4.2], and [8, Appendix B] □

Example 5.1 For $T = \mathbb{G}_m$ and $\lambda_i = \mathrm{id} : \mathbb{G}_m \to \mathbb{G}_m$ for all i. Then each λ_i is σ-positive for $\sigma = \mathrm{id}$, and the corresponding perverse local system $\Phi_{\underline{\lambda}}$ on $\mathbb{G}_m$ is the Kloosterman sheaf considered by Deligne in [9].

Let $S_{\underline{\lambda}}$ be the subgroup of the symmetric group S_r consisting of permutations τ such that for all $i \in \{1, ..., r\}$, we have $\lambda_{\tau(i)} = \lambda_i$. We have the following result due to Deligne [9, Proposition 7.20]:

Proposition 5.2 *The group* $S_{\underline{\lambda}}$ *acts on* $\Phi_{\underline{\lambda}}$ *(resp.* $\Phi^*_{\underline{\lambda}}$*) via the sign character* $\mathrm{sign}_r : S_r \to \{\pm 1\}$.

5.2 ρ-Bessel sheaves on T

We recall the construction of Braverman-Kazhdan's ρ-Bessel sheaves on T attached to a representation of the dual group $\check{G}$. Let $\rho : \check{G} \to \mathrm{GL}(V_\rho)$ be a r-dimensional complex representation of $\check{G}$. The restriction of ρ to $\check{T}$ is diagonalizable and there exists a collection of weights

$$\underline{\lambda} = \{\lambda_1, ..., \lambda_r\} \subset \mathbb{X}^\bullet(\check{T}) := \mathrm{Hom}(\check{T}, \mathbb{C}^\times)$$

such that there is an eigenspace decomposition

$$V_\rho = \bigoplus_{i=1}^{r} V_{\lambda_i}$$

of V_ρ, where $\check{T}$ acts on V_{λ_i} via the character λ_i.

We can regard $\underline{\lambda}$ as collection of cocharacters of T using the the canonical isomorphism $\mathbb{X}^\bullet(\check{T}) \simeq \mathbb{X}_\bullet(T)$ and we denote by

$$\Phi_{T,\rho} := \Phi_{\underline{\lambda}}, \quad \Phi^*_{T,\rho} := \Phi^*_{\underline{\lambda}}$$

the hypergeometric sheaves associated to $\underline{\lambda}$ in (5.1). We will call them ρ-Bessel sheaves.

Following [5] (see also [8]), we shall construct a W-equivariant structure on ρ-Bessel sheaves. Let $\{\lambda_{i_1}, ..., \lambda_{i_k}\}$ be the distinct cocharacters appearing in $\underline{\lambda}$ and m_l be the multiplicity of $\lambda_{i_l} \in \underline{\lambda}$. Let $A_m = \{\lambda_j | \lambda_j = \lambda_{i_m}\}$. Then we have $\{\lambda_1, ..., \lambda_r\} = A_1 \sqcup ... \sqcup A_k$. The symmetric group on r-letters S_r acts naturally on $\{\lambda_1, ..., \lambda_r\}$ and we define $S_{\underline{\lambda}} = \{\sigma \in S_r | \sigma(A_i) = A_i \text{ for all } i\}$. There is a canonical isomorphism

$$S_{\underline{\lambda}} \simeq S_{m_1} \times \cdots \times S_{m_k}.$$

Define $S'_{\underline{\lambda}} = \{\eta \in S_r | \text{such that, for all } i,\ \eta(A_i) = A_{\tau(i)} \text{ for a } \tau \in S_k\}$. We have a natural map $\pi_k : S'_{\underline{\lambda}} \to S_k$ sending η to τ. The kernel of π_k is isomorphic to $S_{\underline{\lambda}}$ and its image, denote by $S_{k,\underline{\lambda}}$, consists of $\tau \in S_k$ such that $m_i = m_{\tau(i)}$. In other words, there is a short exact sequence

$$0 \to S_{\underline{\lambda}} \to S'_{\underline{\lambda}} \overset{\pi_k}{\to} S_{k,\underline{\lambda}} \to 0.$$

Notice that the Weyl group W acts on $\{\lambda_{i_1}, ..., \lambda_{i_k}\}$ and the induced map $W \to S_k$ has image $S_{k,\underline{\lambda}}$. So we have a map $\rho : W \to S_{k,\underline{\lambda}}$. Pulling back the short exact sequence above along ρ, we get an extension W' of W by $S_{\underline{\lambda}}$

$$0 \to S_{\underline{\lambda}} \to W' \to W \to 0$$

where an element in $w' \in W'$ consists of pair $(w, \eta) \in W \times S'_{\underline{\lambda}}$ such that $\rho(w) = \pi_k(\eta) \in S_{k,\underline{\lambda}}$.

The group W' acts on $\mathbb{G}_m^r$ (resp. T) via the composition of the action of S_r (reps. W) with the natural projection $W' \to S'_{\underline{\lambda}} \subset S_r$ (resp. $W' \to W$) and the map $\mathrm{pr}_{\underline{\lambda}} : \mathbb{G}_m^r \to T$ and $\mathrm{tr} : \mathbb{G}_m^r \to \mathbb{G}_a$ is W'-equivariant where W' acts trivially on $\mathbb{G}_a$.

Since $\Phi_{T,\rho} = \mathrm{pr}_{\underline{\lambda},!}\mathrm{tr}^*\mathcal{L}_\psi[r]$ (resp. $\Phi^*_{T,\rho} = \mathrm{pr}_{\underline{\lambda},*}\mathrm{tr}^*\mathcal{L}_\psi[r]$), the discussion above implies for each $w' = (w, \eta) \in W'$ there is an isomorphism

$$\begin{aligned} i'_{w'} &: \Phi_{T,\rho} \simeq w^*\Phi_{T,\rho} \\ &(\text{resp. } i'_{w'} : \Phi^*_{T,\rho} \simeq w^*\Phi^*_{T,\rho}). \end{aligned} \tag{5.3}$$

We define

$$\begin{aligned} i_{w'} &= \mathrm{sign}_W(w)\,\mathrm{sign}_r(\eta) i'_{w'} : \Phi_{T,\rho} \simeq w^*\Phi_{T,\rho} \\ &(\text{resp.} \quad i_{w'} = \mathrm{sign}_W(w)\,\mathrm{sign}_r(\eta) i'_{w'} : \Phi^*_{T,\rho} \simeq w^*\Phi^*_{T,\rho}) \end{aligned} \tag{5.4}$$

where sign_r is the sign character of S_r. It follows from Proposition 5.2 that the isomorphism $i_{w'}$ depends only on w. Denote the resting isomorphism by i_w, then the data $(\Phi_{T,\rho}, \{i_w\}_{w\in W})$ (resp. $(\Phi^*_{T,\rho}, \{i_w\}_{w\in W})$) defines an object in $\mathscr{D}_W(T)$ which, by abuse of notation, we still denote by $\Phi_{T,\rho}$ (resp. $\Phi^*_{T,\rho}$).

Example 5.2 Consider the case $G = \mathrm{GL}_r$ and ρ is the standard representation of $\check{G} = \mathrm{GL}_r(\mathbb{C})$. We have $\Phi_{T,\rho} = \mathrm{tr}^*\mathcal{L}_\psi[r] \in \mathscr{D}_W(T)$, where $\mathrm{tr} : \mathbb{G}_m^r \to \mathbb{G}_a$ is the trace map.

6 Proof of Theorem 1.4

Recall the maps $\mathbb{G}_a \xleftarrow{\mathrm{tr}} \mathbb{G}_m^r \xrightarrow{\mathrm{pr}_{\underline{\lambda}}} T$. Let $\chi \in \mathcal{C}(\mathbb{G}_m^r)(\overline{\mathbb{Q}}_\ell)$ be a tame character. The permutation action of S_r on $\mathbb{G}_m^r$ induces an action of S_r on $\mathcal{C}(\mathbb{G}_m^r)$ and let $S_{r,\chi}$ be the stabilizer of χ in S_r. The pullback $\mathrm{tr}^*\mathcal{L}_\psi$ is naturally S_r-equivariant and we have a natural $S_{r,\chi}$-action on the cohomology $\mathrm{H}^*(\mathbb{G}_m^r, \mathrm{tr}^*\mathcal{L}_\psi \otimes \mathcal{L}_\chi)$ (resp. $\mathrm{H}_c^*(\mathbb{G}_m^r, \mathrm{tr}^*\mathcal{L}_\psi \otimes \mathcal{L}_\chi)$).

Lemma 6.1 *The* $S_{r,\chi}$*-action on* $\mathrm{H}_c^*(\mathbb{G}_m^r, \mathrm{tr}^*\mathcal{L}_\psi \otimes \mathcal{L}_\chi)$ *is given by* $\mathrm{sign}_r : S_{r,\chi} \to \{\pm 1\}$.

Proof Let $\sigma = (i, j) \in S_{r,\chi}$ be a simple reflection. It suffices to show that σ acts on $\mathrm{H}^*(\mathbb{G}_m^r, \mathrm{tr}^*\mathcal{L}_\psi \otimes \mathcal{L}_\chi)$ by -1, or equivalently, the σ-invariant $\mathrm{H}_c^*(\mathbb{G}_m^r, \mathrm{tr}^*\mathcal{L}_\psi \otimes \mathcal{L}_\chi)^\sigma$ is zero. Write $\mathcal{L}_\chi = \mathcal{L}_1 \boxtimes \cdots \boxtimes \mathcal{L}_r$, where each $\mathcal{L}_k$ is a tame local system on $\mathbb{G}_m$.

Note that $\sigma^*\mathcal{L}_\chi \simeq \mathcal{L}_\chi$ implies $\mathcal{L}_i \simeq \mathcal{L}_j := \mathcal{L}$. Consider the quotient map

$$q : \mathbb{G}_m^r \to \sigma\backslash\backslash\mathbb{G}_m^r \simeq \mathbb{A}^1 \times \mathbb{G}_m \times \prod_{k\in\{1,\ldots,r\},k\neq i,j} \mathbb{G}_m,$$

given by

$$q(x_1, \ldots, x_r) = (x_i + x_j, x_i x_j, \prod_{k\in\{1,\ldots,r\},k\neq i,j} x_k).$$

Using the fact that $\mathcal{L} \boxtimes \mathcal{L} \simeq m^*\mathcal{L}$, where $m : \mathbb{G}_m^2 \to \mathbb{G}_m$ is the multiplication map, we see that

$$\mathrm{tr}^*\mathcal{L}_\psi \otimes \mathcal{L}_\chi \simeq q^*\big(\mathcal{L}_\psi \boxtimes \mathcal{L} \boxtimes (\mathrm{tr}^*\mathcal{L}_\psi \otimes \prod_{k\neq i,j} \mathcal{L}_k)\big).$$

The permutation σ acts naturally on $q_*(\mathrm{tr}^*\mathcal{L}_\psi \otimes \mathcal{L}_\chi)$ and it follows from the isomorphism above that

$$\begin{aligned}&(q_*(\mathrm{tr}^*\mathcal{L}_\psi \otimes \mathcal{L}_\chi))^\sigma \\ &\quad\simeq (q_*q^*\big(\mathcal{L}_\psi \boxtimes \mathcal{L} \boxtimes (\mathrm{tr}^*\mathcal{L}_\psi \otimes \prod_{k\neq i,j} \mathcal{L}_k)\big))^\sigma \simeq \mathcal{L}_\psi \boxtimes \mathcal{L} \boxtimes (\mathrm{tr}^*\mathcal{L}_\psi \otimes \prod_{k\neq i,j} \mathcal{L}_k).\end{aligned}$$

This implies

$$\mathrm{H}_c^*(\mathbb{G}_m^r, \mathrm{tr}^*\mathcal{L}_\psi \otimes \mathcal{L}_\chi)^\sigma \simeq \mathrm{H}_c^*(\sigma\backslash\backslash\mathbb{G}_m^r, \mathcal{L}_\psi \boxtimes \mathcal{L} \boxtimes (\mathrm{tr}^*\mathcal{L}_\psi \otimes \prod_{k\neq i,j} \mathcal{L}_k)) = 0$$

where the last equality follows from the cohomology vanishing $\mathrm{H}_c^*(\mathbb{A}^1, \mathcal{L}_\psi) = 0$. The lemma follows. □

To proceed, let $\chi \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ and let $\chi' = \mathrm{pr}_{\underline{\lambda}}^*\chi \in \mathcal{C}(\mathbb{G}_m^r)(\overline{\mathbb{Q}}_\ell)$ be the pull back of χ. We have $\mathrm{pr}_{\underline{\lambda}}^*\mathcal{L}_\chi \simeq \mathcal{L}_{\chi'}$ and

$$\mathrm{H}_c^*(T, \Phi_{T,\rho}\otimes\mathcal{L}_\chi) \simeq \mathrm{H}_c^*(T, \mathrm{pr}_{\underline{\lambda},!}\mathrm{tr}^*\mathcal{L}_\psi[r]\otimes\mathcal{L}_\chi) \simeq \mathrm{H}_c^*(\mathbb{G}_m^r, \mathrm{tr}^*\mathcal{L}_\psi[r]\otimes\mathcal{L}_{\chi'}) \quad (6.1)$$

where the second isomorphism comes from the projection formula. Let $w \in \mathrm{W}'_\chi$ and choose a lift $w' = (w, \eta) \in \mathrm{W}'$ of w. Note that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{G}_m^r & \xrightarrow{\mathrm{pr}_{\underline{\lambda}}} & T \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle w} \\ \mathbb{G}_m^r & \xrightarrow{\mathrm{pr}_{\underline{\lambda}}} & T \end{array} .$$

It follows that $\eta^*\mathcal{L}_{\chi'} \simeq \eta^*\mathrm{pr}^*_{\underline{\lambda}}\mathcal{L}_\chi \simeq \mathrm{pr}^*_{\underline{\lambda}} w^*\mathcal{L}_\chi \simeq \mathrm{pr}^*_{\underline{\lambda}}\mathcal{L}_\chi \simeq \mathcal{L}_{\chi'}$, that is, $\eta \in \mathrm{S}_{r,\chi'}$. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^*_c(T, \Phi_{T,\rho}\otimes\mathcal{L}_\chi) & \longrightarrow & \mathrm{H}^*_c(\mathbb{G}^r_m, \mathrm{tr}^*\mathcal{L}_\psi[r]\otimes\mathcal{L}_{\chi'}) \\ \downarrow{\scriptstyle i'_{w'}} & & \downarrow{\scriptstyle \eta} \\ \mathrm{H}^*_c(T, \Phi_{T,\rho}\otimes\mathcal{L}_\chi) & \longrightarrow & \mathrm{H}^*_c(\mathbb{G}^r_m, \mathrm{tr}^*\mathcal{L}_\psi[r]\otimes\mathcal{L}_{\chi'}) \end{array}$$

where the horizontal arrows are the isomorphism (6.1), the left vertical arrow is the isomorphism induced by the isomorphism the $i'_{w'}$ in (5.3), and the right vertical arrow is the action of η on $\mathrm{H}^*_c(\mathbb{G}^r_m, \mathrm{tr}^*\mathcal{L}_\psi[r]\otimes\mathcal{L}_{\chi'})$ coming from the $\mathrm{S}_{r,\chi'}$-equivariant structure on $\mathrm{tr}^*\mathcal{L}_\psi[r]\otimes\mathcal{L}_{\chi'}$. Therefore, by Lemma 6.1, we see that $i'_{w'} = \mathrm{sign}_r(\eta)$ and it follow from the definition of W-equivariant structure of ρ-Bessel sheaf in (5.4) that the action of w on the cohomology group $\mathrm{H}^*_c(T, \Phi_{T,\rho}\otimes\mathcal{L}_\chi)$ is given by

$$i_{w'} = \mathrm{sign}_{\mathrm{W}}(w)\,\mathrm{sign}_r(\eta) i'_{w'} = \mathrm{sign}_{\mathrm{W}}(w)\,\mathrm{sign}_r(\eta)\,\mathrm{sign}_r(\eta) = \mathrm{sign}_{\mathrm{W}}(w).$$

The proof of Theorem 1.4 is complete.

7 Proof of Theorem 1.6

In this section we shall prove the vanishing conjecture for $G = \mathrm{GL}_n$.

7.1 The GL_2-example

Let us first consider the simple but important case $G = \mathrm{GL}_2$. Let $\mathcal{F} \in \mathcal{D}_{\mathrm{W}}(T)$ be a W-equivariant complex on $T = \mathbb{G}^2_m$ and let $\Phi_{\mathcal{F}} = \mathrm{Ind}^G_{T\subset B}(\mathcal{F})^{\mathrm{W}}$. Note that, for $x \in G\backslash B$, the coset $Ux \subset G^{\mathrm{reg}}$ consists of regular elements. Note also that the Grothendieck-Springer simultaneous resolution is Cartesian over G^{reg}:

$$\begin{array}{ccc} \widetilde{G}^{\mathrm{reg}} \simeq G^{\mathrm{reg}} \times_{\mathrm{W}\backslash\backslash T} T & \xrightarrow{\tilde{q}} & T \\ \downarrow{\scriptstyle \tilde{c}} & & \downarrow{\scriptstyle q} \\ G^{\mathrm{reg}} & \xrightarrow{c} & \mathrm{W}\backslash\backslash T \end{array} \tag{7.1}$$

All together, we obtain

$$\mathrm{H}^*_c(Ux, \Phi_{\mathcal{F}}) \simeq \mathrm{H}^*_c(Ux, \tilde{c}_!\tilde{q}^*\mathcal{F})^{\mathrm{W}} \simeq \mathrm{H}^*_c(Ux, c^*q_!\mathcal{F})^{\mathrm{W}}$$

Under the identification $\mathrm{W}\backslash\backslash T \simeq \mathbb{G}_a \times \mathbb{G}_m$, the maps c (resp. q) is given by $c(g) = (\mathrm{tr}_G(g), \det(g))$ (resp. $q(t) = (\mathrm{tr}_T(t), \det(t))$), and a direct calculation shows that c

restricts to an isomorphism

$$c : Ux \simeq \mathbb{G}_a \times \det(x) \subset \mathbb{G}_a \times \mathbb{G}_m \simeq \mathrm{W}\backslash\backslash T. \tag{7.2}$$

Thus we have

$$\mathrm{H}_c^*(Ux, \Phi_{\mathcal{F}}) \simeq \mathrm{H}_c^*(Ux, c^* q_! \mathcal{F})^{\mathrm{W}} \simeq (m_! \mathcal{F})^{\mathrm{W}}|_{\det(x)}, \tag{7.3}$$

where $m = \mathrm{pr}_{\mathbb{G}_m} \circ q : T = \mathbb{G}_m^2 \to \mathbb{G}_m, (x, y) \to xy$. As $\det|_{G\backslash B} : G\backslash B \to \mathbb{G}_m$ is surjective, it follows from (7.3) that $\mathrm{H}_c^*(Ux, \Phi_{\mathcal{F}}) = 0$ for all $x \in G\backslash B$ if and only if W acts on $m_!\mathcal{F}$ via the sign character. We claim that the later property of $m_!\mathcal{F}$ is equivalent to $\mathcal{F}$ being central. Thus we conclude that $\mathrm{H}_c^*(Ux, \Phi_{\mathcal{F}}) = 0$ for all $x \in G\backslash B$ if and only if $\mathcal{F}$ is central. This completes the proof of the vanishing conjecture for GL_2.

To prove the claim we observe that W acts on $m_!\mathcal{F}$ via the sign character if and only if $(m_!\mathcal{F})^{\mathrm{W}} = 0$. By [11, Proposition 3.4.5], $(m_!\mathcal{F})^{\mathrm{W}} = 0$ is equivalent to

$$\mathrm{H}_c^*(\mathbb{G}_m, (m_!\mathcal{F})^{\mathrm{W}} \otimes \mathcal{L}') \simeq \mathrm{H}_c^*(\mathbb{G}_m, m_!\mathcal{F} \otimes \mathcal{L}')^{\mathrm{W}} \simeq \mathrm{H}_c^*(T, \mathcal{F} \otimes m^*\mathcal{L}')^{\mathrm{W}} = 0 \tag{7.4}$$

for all tame local system $\mathcal{L}'$ on $\mathbb{G}_m$. Note that we have $\mathrm{W}_\chi \neq e$ if and only if $\mathcal{L}_\chi \simeq \mathcal{L}' \boxtimes \mathcal{L}' \simeq m^*\mathcal{L}'$ for some tame local system $\mathcal{L}'$ on $\mathbb{G}_m$, thus (7.4) is equivalent to the condition that W_χ acts on $\mathrm{H}_c^*(T, \mathcal{F} \otimes \mathcal{L}_\chi)$ via the sign character for any tame character χ, that is, $\mathcal{F}$ is central. The claim follows.

We shall generalize the proof above for GL_2 to GL_n. The argument involves mirabolic subgroups of GL_n as an essential ingredient.

7.2 Mirabolic subgroups

We recall some geometric facts about Mirabolic subgroups, established in [8], that will be used in the proof of Theorem 1.6.[9]

Let $V = \mathbb{A}^n$ be the standard n-dimensional vector space over k with the standard basis $e_1, ..., e_n$. For any $1 \leq m \leq n$ We define F_m (resp. E_m) to be the subspace generated by $e_1, ..., e_m$ (resp. $e_{m+1}, ..., e_n$).

Let Q be the mirabolic subgroup of $G = \mathrm{GL}(V)$ consisting of $g \in G$ fixing the line generated by $v := e_1$. Let U_Q be the unipotent radical of Q. Consider the Q-conjugation equivariant stratification of G

$$G = \bigsqcup_{m=1}^{n} X_m, \tag{7.5}$$

where X_m the subset of G consisting of $g \in G$ such that the span of the vectors $v, gv, g^2v, ...$ is of dimension m.

9 As mentioned in *loc. cit.* the geometry of the conjugation action of Mirabolic has been described by Bernstein in [1].

Lemma 7.1 *Consider the Chevalley quotient map $c : G \to W\backslash\backslash T \simeq \mathbb{A}^{n-1} \times \mathbb{G}_m$, $c(x) = (a_1, ..., a_n)$ where $t^n + a_1 t^n + \cdots + a_n$ is the characteristic polynomial of $x \in G$ Let $x \in X_n$. Then the map $u \to c(ux) - c(x)$ induces a linear isomorphism*

$$U_Q \simeq \mathbb{A}^{n-1} \times \{0\} \subset \mathbb{A}^{n-1} \times \mathbb{G}_m$$

between U_Q and the subspace of $\mathbb{A}^{n-1} \times \mathbb{G}_m$ defined by $a_n = 0$. In particular, the map c restricts to an isomorphism $U_Q x \simeq \mathbb{A}^{n-1} \times \{a_n\} \subset \mathbb{A}^{n-1} \times \mathbb{G}_m$.

Proof This is [8, Proposition 4.1]. □

Remark 7.1 The lemma above is a generalization of the isomorphism in (7.2) to the setting of mirabolic subgroups.

Lemma 7.2 *Let $x \in X_m$. There exists an element $q \in Q$ such that qxq^{-1} is of the form*

$$\begin{bmatrix} x_{F_m} & y \\ 0 & x_{E_m} \end{bmatrix} \tag{7.6}$$

where $x_{E_m} \in \mathrm{GL}(E_m)$, and $x_{F_m} \in \mathrm{GL}(F_m)$ has the form of a companion matrix

$$x_{F_m} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & -a_m \\ 1 & 0 & \dots & 0 & 0 & -a_{m-1} \\ 0 & 1 & \dots & 0 & 0 & -a_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_2 \\ 0 & 0 & \dots & 0 & 1 & -a_1 \end{bmatrix} \tag{7.7}$$

Proof Straightforward exercise in linear algebra. □

Let P_m be the parabolic group of G consisting of g such that $gF_m = F_m$. Let L_m be the standard Levi subgroup of P_m and U_m be the unipotent radical of P_m consisting of matrices of the form

$$u_m = \begin{bmatrix} \mathrm{Id}_{F_m} & v_m \\ 0 & \mathrm{Id}_{E_m} \end{bmatrix}$$

where $u_m \in \mathrm{Hom}(E_m, F_m)$. Let U_1 and U_{m-1} be the subgroup of U_m consisting of u_m as above with $v_m \in \mathrm{Hom}(E_m, F_1)$ and $v_m \in \mathrm{Hom}(E_m, F_{m-1})$ respectively.

Lemma 7.3 *Let $x_{F_m} \in \mathrm{GL}(F_m)$ be a linear map such that, for any $1 \leq j \leq m$, $x_{F_m}(F_j) \subset F_{j+1}$ and the induced map $F_j/F_{j-1} \to F_{j+1}/F_j$ is an isomorphism. Let $x_{E_m} \in \mathrm{GL}(E_m)$ be an arbitrary linear map. Then the action of $U_1 \times U_{m-1}$ on the space of matrices of the form*

$$x = \begin{bmatrix} x_{F_m} & y \\ 0 & x_{E_m} \end{bmatrix} \tag{7.8}$$

given by $(u_1, u_{m-1})x = u_{m-1}u_1 x u_{m-1}^{-1}$ is simply transitive.[10]

[10] Note that $u_{m-1}u_1 x u_{m-1}^{-1} = u_1 u_{m-1} x u_{m-1}^{-1}$.

Proof This is [8, Lemma 3.2].[11] □

Let $Q_{L_m} = L_m \cap Q$ be a mirabolic subgroup of L_m and let $U_{Q_{L_m}}$ be the unipotent radical of Q_{L_m} consisting of matrices

$$\begin{bmatrix} 1 & v & 0 \\ 0 & \mathrm{Id}_{m-1} & 0 \\ 0 & 0 & \mathrm{Id}_{n-m} \end{bmatrix}$$

where $v = (v_1, ..., v_{m-1}) \in \mathbb{A}^{m-1}$ is a row vector.

Lemma 7.4 *Let $x \in X_m$ be as in (7.6) with x_{F_m} being as in (7.7). The morphism*

$$U_Q \times U_{m-1} \to U_m U_{Q_{L_m}} x$$

given by $(u_Q, u_{m-1}) \to u_{m-1} u_Q x u_{m-1}^{-1}$ is an isomorphism.

Proof This is proved in [8, p17]. □

7.3 Proof of Theorem 1.6

Lemma 7.5 *Assume $G = \mathrm{GL}_n$. If Conjecture 1.2 is true for perverse central sheaves (resp. perverse $*$-central sheaves), then is it true for arbitrary central complexes (resp. arbitrary $*$-central complexes).*[12]

Proof By Remark 1.4, it is enough to prove the statement for $*$-central complexes. By induction on the (finite) number of non vanishing perverse cohomology sheaves, we can assume the conjecture is true for $*$-central complexes in ${}^p\mathscr{D}_{\mathrm{W}}^{[a,b]}(T)$, $|a-b| \leq l$. Let $\mathcal{F} \in {}^p\mathscr{D}_{\mathrm{W}}^{[a,b+1]}(T)$ be a $*$-central complex. We need to show that $\pi_*\Phi_{\mathcal{F}}$ is supported on $T = U\backslash B \subset U\backslash G$, where $\pi : G \to U\backslash G$ is the quotient map. Consider the following distinguished triangle

$${}^p\tau_{\leq b}\mathcal{F} \to \mathcal{F} \to {}^p\mathscr{H}^{b+1}(\mathcal{F})[-b-1] \to .$$

We claim that both ${}^p\tau_{\leq b}\mathcal{F}$, ${}^p\mathscr{H}^{b+1}(\mathcal{F}) \in \mathscr{D}_{\mathrm{W}}(T)$ are $*$-central. Applying the functor $\mathrm{Ind}_{T\subset B}^{G}(-)^{\mathrm{W}}$ to the above distinguished triangle, we obtain

$$\Phi_{{}^p\tau_{\leq b}\mathcal{F}} \to \Phi_{\mathcal{F}} \to \Phi_{{}^p\mathscr{H}^{b+1}(\mathcal{F})}[-b-1] \to .$$

By induction, both $\pi_*(\Phi_{{}^p\tau_{\leq b}\mathcal{F}})$ and $\pi_*(\Phi_{{}^p\mathscr{H}^{b+1}(\mathcal{F})})$ are supported on $T = U\backslash B \subset U\backslash G$ and it implies $\pi_*\Phi_{\mathcal{F}}$ is also supported on T.

---

[11] There is minor mistake in the computation of $u_1 u_m x u_m^{-1}$ in *loc. cit.*: is should be $u_1 u_m x u_m^{-1} = \begin{bmatrix} x_F & y + v_1 x_E + v_{m-1} x_E - x_F v_{m-1} \\ 0 & x_E \end{bmatrix}$. The same proof goes through after this minor correction.

[12] In fact, the Lemma remains true without the assumption $G = \mathrm{GL}_n$, see [7, Lemma 7.5].

It remains to prove the claim. Let $\chi_f \in \mathcal{C}(T)_f$ and let $\mathcal{C}(T)_{\ell,\chi_f} = \{\chi_f\} \times \mathcal{C}(T)_\ell$ be the corresponding component. Let $q : \mathcal{C}(T)_{\ell,\chi_f} \to \mathrm{W}_{\chi_f}\backslash\backslash\mathcal{C}(T)_{\ell,\chi_f}$ be the quotient map. Note that for $G = \mathrm{GL}_n$ we have $\mathrm{W}_\chi = \mathrm{W}'_\chi$ for all χ and hence by Proposition 4.2, we have

$$\mathcal{M}_*(\mathcal{F} \otimes \mathrm{sign})|_{\mathcal{C}(T)_{\ell,\chi_f}} = q^*\mathcal{G}$$

for a $\mathcal{G} \in D^b_{\mathrm{coh}}(\mathrm{W}_{\chi_f}\backslash\backslash\mathcal{C}(T)_{\ell,\chi_f})$. Since both $\mathcal{M}_*$ and q^* are exact, we have

$$\mathcal{M}_*({}^p\tau_{\leq b}(\mathcal{F} \otimes \mathrm{sign}))|_{\mathcal{C}(T)_{\ell,\chi_f}} = q^*(\tau_{\leq b}\mathcal{G}),\ \ \mathcal{M}_*({}^p\mathscr{H}^{b+1}(\mathcal{F} \otimes \mathrm{sign}))|_{\mathcal{C}(T)_{\ell,\chi_f}} \\ = q^*\mathscr{H}^{b+1}(\mathcal{G}),$$

and by Proposition 4.2 again it implies ${}^p\tau_{\leq b}\mathcal{F}$ and ${}^p\mathscr{H}^{b+1}(\mathcal{F})$ are $*$-central. The proof is complete. □

Consider $G = (\prod_{i\neq j} \mathrm{GL}_{n_i}) \times \mathrm{GL}_{n_j}$ with maximal torus $T = (\prod_{i\neq j} T_i) \times T_j$, and Weyl group $\mathrm{W} = (\prod_{i\neq j} \mathrm{W}_i) \times \mathrm{W}_j$. Here W_k denotes the symmetric group of degree n_k. Consider the following map

$$\eta = \mathrm{id} \times \det : T = (\prod_{i\neq j} T_i) \times T_j \longrightarrow T' := (\prod_{i\neq j} T_i) \times \mathbb{G}_m.$$

The symmetric group W_j of degree n_j acts on T via the permutation action on the factor T_j and η is W_j-invariant. Thus for any $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$ the push-forward $\eta_!(\mathcal{F})$ carries a natural W_j-action. We have the following generalization of Deligne's result about $\mathrm{S}_{\underline{\lambda}}$-action on hypergeometric sheaves $\Phi_{\underline{\lambda}}$ (see Proposition 5.2) to central complexes.

Proposition 7.6 *Let $\mathcal{F}$ be a central complex in $\mathscr{D}_{\mathrm{W}}(T)$. Then the above W_j-action on $\eta_!(\mathcal{F})$ is given by the sign character $\mathrm{sign}_{\mathrm{W}} : \mathrm{W}_j \to \{\pm 1\}$.*

Proof It suffices to show that $\eta_!(\mathcal{F})^{\sigma=\mathrm{id}} = 0$ for any involution $\sigma \in \mathrm{W}_j$. By a result of Laumon [11, Proposition 3.4.5], it is enough to show that for any tame local system $\mathcal{L}_\chi$ on T' we have

$$\mathrm{H}^*_c(T', \eta_!(\mathcal{F})^{\sigma=\mathrm{id}} \otimes \mathcal{L}_\chi) = 0.$$

Note that, for any such $\mathcal{L}_\chi$, we have

$$\mathrm{H}^*_c(T', \eta_!(\mathcal{F})^{\sigma=\mathrm{id}} \otimes \mathcal{L}_\chi) \simeq \mathrm{H}^*_c(T', \eta_!(\mathcal{F}) \otimes \mathcal{L}_\chi)^{\sigma=\mathrm{id}} \simeq \mathrm{H}^*_c(T, \mathcal{F} \otimes \eta^*\mathcal{L}_\chi)^{\sigma=\mathrm{id}}.$$

Since $\eta^*\mathcal{L}_\chi$ is a tame local system on T fixed by σ, the central property of $\mathcal{F}$ implies that the action of σ on $\mathrm{H}^*_c(T, \mathcal{F} \otimes \eta^*\mathcal{L}_\chi)$ is not trivial, but through the sign character. Thus we have $\mathrm{H}^*_c(T, \mathcal{F} \otimes \eta^*\mathcal{L}_\chi)^{\sigma=\mathrm{id}} = 0$ and the isomorphism above implies $\mathrm{H}^*_c(T', \eta_!(\mathcal{F})^{\sigma=\mathrm{id}} \otimes \mathcal{L}_\chi) = 0$ for all tame local system $\mathcal{L}_\chi$. The proposition follows. □

The following proposition generalizes [8, Proposition 5.1] to central complexes:

Proposition 7.7 *Let G be a direct product of general linear group and let Q be a mirabolic subgroup of G of the form*

$$Q=(\prod_{i\neq j}\mathrm{GL}_{n_i})\times Q_j\subset G=\prod_i \mathrm{GL}_{n_i}$$

where Q_j is the mirabolic subgroup of GL_{n_j}. Let $\mathcal{F}\in\mathscr{D}_{\mathrm{W}}(T)^{\heartsuit}$ be a W-equivariant central perverse sheaf on T. Then for any $x\in G\backslash Q$, we have

$$\mathrm{H}^*_c(U_Qx,i^*\Phi_{\mathcal{F}})=0$$

where $i:U_Qx\to G$ is the inclusion map.

Proof If $n_j=1$, then $Q=G$ and the proposition holds vacuously. Assume $n_j\geq 2$. Consider the following stratification of G

$$G=\bigsqcup_{m=1}^{n_j}X_m=\bigsqcup_{m=1}^{n_j}\big(\prod_{i\neq j}\mathrm{GL}_{n_i}\times X_{j,m}\big)$$

where $X_{j,m}\subset \mathrm{GL}_{n_j}$ is the subset introduced in (7.5). We have $Q=X_1$, thus the assumption $x\notin Q$ implies $x\in X_m$ for some $2\leq m\leq n_j$.

Consider the case $x\in X_{j,n_j}$. Since X_{j,n_j} is contained in $\mathrm{GL}_{n_j}^{\mathrm{reg}}$, the Grothendieck-Springer simultaneous resolution implies a Cartesian diagram

$$\begin{array}{ccc}\tilde{X}_{n_j} & \xrightarrow{\tilde{c}} & (\prod_{i\neq j}\mathrm{GL}_{n_i})\times T_j\\ \downarrow{\scriptstyle\tilde{q}} & & \downarrow{\scriptstyle q}\\ X_{n_j} & \xrightarrow{c} & (\prod_{i\neq j}\mathrm{GL}_{n_i})\times \mathrm{W}_j\backslash\backslash T_j\end{array}$$

It follows that

$$\Phi_{\mathcal{F}}|_{X_{n_j}}\simeq c^*q_!(\Phi_{L_j,\mathcal{F}})^{\mathrm{W}_j}$$

where $L_j=(\prod_{i\neq j}\mathrm{GL}_{n_i})\times T_j$ and $\Phi_{L_j,\mathcal{F}}=\mathrm{Ind}_T^{L_j}(\mathcal{F})^{\mathrm{W}_{L_j}}$. Consider the map

$$\det{}_{\mathrm{W}_j}=\mathrm{Id}\times\sigma:(\prod_{i\neq j}\mathrm{GL}_{n_i})\times \mathrm{W}_j\backslash\backslash T_j\to(\prod_{i\neq j}\mathrm{GL}_{n_i})\times\mathbb{G}_m \tag{7.9}$$

where $\sigma:\mathrm{W}_j\backslash\backslash T_j\simeq \mathrm{GL}_{n_j}\backslash\backslash\mathrm{GL}_{n_j}\to\mathbb{G}_m$ is given by the determinant function on GL_{n_j}. By Lemma 7.1, the restriction of c to U_Qx induces an isomorphism between

$U_Q x$ and the fiber of (7.9) over the image $\det_{W_j} \circ c(x)$ of x. Thus, to prove the desired vanishing, it is enough to show that

$$\det_{j,!}(\Phi_{L_j,\mathcal{F}})^{W_j} \simeq \det_{W_j,!} q_!(\Phi_{L_j,\mathcal{F}})^{W_j} = 0 \tag{7.10}$$

where

$$\det_j = \det_{W_j} \circ q : L_j = (\prod_{i\neq j} \mathrm{GL}_{n_i}) \times T_j \to (\prod_{i\neq j} \mathrm{GL}_{n_i}) \times \mathbb{G}_m$$

is the determinant map on the jth component. Consider the following Cartesian diagrams

$$\begin{array}{ccccc} L_j = (\prod_{i\neq j} \mathrm{GL}_{n_i}) \times T_j & \longleftarrow & (\prod_{i\neq j} \widetilde{\mathrm{GL}}_{n_i}) \times T_j & \longrightarrow & T = (\prod_{i\neq j} T_i) \times T_j \\ \downarrow{\scriptstyle \det_j = \mathrm{Id}\times\det} & & \downarrow{\scriptstyle \mathrm{id}\times\det} & & \downarrow{\scriptstyle \mathrm{id}\times\det} \\ L'_j = (\prod_{i\neq j} \mathrm{GL}_{n_i}) \times \mathbb{G}_m & \longleftarrow & \prod_{i\neq j} \widetilde{\mathrm{GL}}_{n_i} \times \mathbb{G}_m & \longrightarrow & T' = (\prod_{i\neq j} T_i) \times \mathbb{G}_m \end{array},$$

where the horizontal arrows are the map induced by the Grothendieck-Springer simultaneous resolution (see (7.1)). It follows that

$$\det_{j,!} \Phi_{L_j,\mathcal{F}} \simeq \det_{j,!} \mathrm{Ind}_T^{L_j}(\mathcal{F})^{W_{L_j} = \prod_{i\neq j} W_i} \simeq \mathrm{Ind}_{T'}^{L'_j}(\mathcal{F}')^{\prod_{i\neq j} W_i} \tag{7.11}$$

where

$$\mathcal{F}' = (\mathrm{Id} \times \det)_!(\mathcal{F}).$$

By Proposition 7.6, the action of W_j on $\mathcal{F}'$ is given by the sign character $\mathrm{sign}_W : W_j \to \{\pm 1\}$. Therefore, by (7.11), the action W_j on $\det_{j,!} \Phi_{L_j,\mathcal{F}}$ is by the sign character and we have

$$\det_{j,!}(\Phi_{L_j,\mathcal{F}})^{W_j} = 0$$

as for $n_j \geq 2$ the sign character of W_j is non-trivial. This concludes the case when $x \in X_{n_j}$.

Consider the general case $x \in X_m$ with $2 \leq m \leq n_j$. By Lemma 7.2, we can assume the jth component $x_j \in \mathrm{GL}_{n_j}$ of x is of the form (7.6) with $x_{j,F}$ being a companion form in (7.7). Let $P = \prod_{i\neq j} \mathrm{GL}_{n_i} \times P_m$ denote the parabolic subgroup of G where P_m is the parabolic subgroup of GL_{n_j} introduced in Sect. 7.2. Let $L = (\prod_{i\neq j} \mathrm{GL}_{n_i}) \times L_m$ and $U_P = (\prod_{i\neq j} \mathrm{Id}_{n_i}) \times U_m$ be the standard Levi subgroup of P and its unipotent radical. We have $x \in P$ and let x_L be its image in L. Consider the mirabolic subgroup Q_L of L with unipotent radical $U_{Q_L} = (\prod_{i\neq j} \mathrm{Id}_{n_i}) \times Q_{L_m}$. By applying the result obtained above to the case to L, we obtain

$$\mathrm{H}_c^*(U_{Q_L} x_L, \Phi_{L,\mathcal{F}}|_{U_{Q_L} x_L}) = 0,$$

where $\Phi_{L,\mathcal{F}} = \operatorname{Ind}_T^L(\mathcal{F})^{W_L}$. Note that, by Proposition 3.4, we have $\Phi_{L,\mathcal{F}} \simeq \operatorname{Res}^G_{L\subset P}\Phi_{\mathcal{F}}$, and above cohomology vanishing implies

$$\mathrm{H}_c^*(U_P U_{Q_L} x, \Phi_{\mathcal{F}}|_{U_P U_{Q_L} x}) = 0. \tag{7.12}$$

Using Lemma 7.4, we see that the map

$$U_Q \times \Big(\prod_{i\neq j} \mathrm{Id}_{n_i} \times U_{m-1}\Big) \to U_P U_{Q_L} x$$

given by $(u_q, u_{m-1}) \to u_{m-1}u_q x u_{m-1}^{-1}$ is an isomorphism. Since $\Phi_{\mathcal{F}}$ is G-conjugation equivariant, (7.12) implies

$$\mathrm{H}_c^*(U_Q x, \Phi_{\mathcal{F}}|_{U_Q x}) = 0.$$

The proposition follows. □

We are ready to prove Theorem 1.6. Let $G = \mathrm{GL}_n$ and let Q be the mirabolic subgroup. Let $\mathcal{F} \in \mathscr{D}_{\mathrm{W}}(T)$ be a central complex. We would like to show that $\mathrm{H}_c^*(Ux, \Phi_{\mathcal{F}}|_{Ux}) = 0$ for $x \in G\backslash B$. By Lemma 7.5, we can assume $\mathcal{F}$ is a perverse sheaf. Consider the case $x \notin Q$. Then $ux \notin Q$ for all $u \in U$ and, by Proposition 7.7, we have $\mathrm{H}_c^*(U_Q ux, \Phi_{\mathcal{F}}|_{U_Q ux}) = 0$. The desired vanishing $\mathrm{H}_c^*(Ux, \Phi_{\mathcal{F}}|_{Ux}) = 0$ follows from the Leray spectral sequence associated to the map $Ux \to U_Q\backslash Ux$. Now assume $x \in Q$. Let x_L be the image of x in the standard Levi L of Q. Note that we have $x_L \notin B_L = B \cap L$ and

$$\mathrm{H}_c^*(Ux, \Phi_{\mathcal{F}}|_{Ux}) = \mathrm{H}_c^*(U_{B_L}x_L, \operatorname{Res}^G_L\Phi_{\mathcal{F}}|_{U_{B_L}x_L}) = \mathrm{H}_c^*(U_{B_L}x_L, \Phi_{L,\mathcal{F}}|_{U_{B_L}x_L})$$

where U_{B_L} is the unipotent radical of B_L. Now, using Proposition 7.7, we can conclude by an induction argument.

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