Consistency of Relations over Monoids

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The interplay between local consistency and global consistency has been the object of study in several different areas, including probability theory, relational databases, and quantum information. For relational databases, Beeri, Fagin, Maier, and Yannakakis showed that a database schema is acyclic if and only if it has the local-to-global consistency property for relations, which means that every collection of pairwise consistent relations over the schema is globally consistent. More recently, the same result has been shown under bag semantics. In this paper, we carry out a systematic study of local vs. global consistency for relations over positive commutative monoids, which is a common generalization of ordinary relations and bags. Let $\mathbb{K}$ be an arbitrary positive commutative monoid. We begin by showing that acyclicity of the schema is a necessary condition for the local-to-global consistency property for $\mathbb{K}$-relations to hold. Unlike the case of ordinary relations and bags, however, we show that acyclicity is not always sufficient. After this, we characterize the positive commutative monoids for which acyclicity is both necessary and sufficient for the local-to-global consistency property to hold; this characterization involves a combinatorial property of monoids, which we call the *transportation property*. We then identify several different classes of monoids that possess the transportation property. As our final contribution, we introduce a modified notion of local consistency of $\mathbb{K}$-relations, which we call *pairwise consistency up to the free cover*. We prove that, for all positive commutative monoids $\mathbb{K}$, even those without the transportation property, acyclicity is both necessary and sufficient for every family of $\mathbb{K}$-relations that is pairwise consistent up to the free cover to be globally consistent.

CCS Concepts: • **Information systems → Relational database model**; **Data provenance**; • **Theory of computation → Data provenance**; • **Mathematics of computing → Hypergraphs**.

Additional Key Words and Phrases: local consistency; global consistency; acyclic hypergraphs; positive commutative monoids; positive semirings

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1 INTRODUCTION

The interplay between local consistency and global consistency has been investigated in several different settings. In each such setting, the concepts “local", “global", and “consistent" are defined rigorously and a study is carried out as to when objects that are locally consistent are also globally consistent. In probability theory, Vorob’ev [14] studied when, for a collection of probability distributions on overlapping sets of variables, there is a global probability distribution whose

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marginals coincide with the probability distributions in that collection. In quantum mechanics, Bell's theorem [5] is about *contextuality* phenomena, where empirical local measurements may be locally consistent but there is no global explanation for these measurements in terms of hidden local variables. In relational databases, there has been an extensive study of the universal relation problem [1, 9, 13]: given relations $R_1, \ldots, R_m$, is there a relation W such that, for each relation R_i, the projection of W on the attributes of R_i is equal to R_i? If the answer is positive, the relations $R_1, \ldots, R_m$ are said to be *globally consistent* and W is a *universal relation* for them. If the relations $R_1, \ldots, R_m$ are globally consistent, they are *pairwise consistent* (i.e., every two of them are globally consistent), but the converse need not hold.

Beeri, Fagin, Maier, and Yannakakis [4] showed that a relational schema is *acyclic* if and only if the *local-to-global consistency property for relations* over that schema holds, which means that every collection of pairwise consistent relations over the schema is globally consistent. Thus, for acyclic schemas, pairwise consistency and global consistency coincide. Note that set semantics is used in this result, i.e., the result is about ordinary relations. More recently, in [2] it was shown that an analogous result holds also under bag semantics: a relational schema is acyclic if and only if the local-to-global consistency property for bags holds, where in the definitions of pairwise consistency and global consistency for bags, the projection operation adds the multiplicities of all tuples in the relation that are projected to the same tuple. It should be pointed out, however, that there are significant differences between set semantics and bag semantics as regards consistency properties. In particular, under set semantics, the relational join of two consistent relations is the largest witness of their consistency, while, under bag semantics, the join of two consistent bags need not even be a witness of their consistency [2].

During the past two decades and starting with the paper [8], there has been a growing study of $\mathbb{K}$-relations, where tuples in $\mathbb{K}$-relations are annotated with values from the universe of a fixed semiring $\mathbb{K}$. Clearly, ordinary relations are $\mathbb{B}$-relations, where $\mathbb{B}$ is the Boolean semiring, while bags are $\mathbb{N}$-relations, where $\mathbb{N}$ is the semiring of non-negative integers. Originally, $\mathbb{K}$-relations were studied in the context of provenance in databases [8]; later on, the study was expanded to other fundamental problems in databases, including the query containment problem [7, 10]. Note that in the study of both provenance and query containment, the definitions of the basic concepts involve both the addition operation and the multiplication operation of the semiring $\mathbb{K}$.

Aiming to obtain a common generalization of the results in [4] and in [2], we carry out a systematic investigation of local consistency vs. global consistency for relations whose tuples are annotated with values from the universe of some suitable algebraic structure. At first sight, semirings appear to be the most general algebraic structures for this purpose. Upon closer reflection, however, one realizes that the definition of a projection of $\mathbb{K}$-relation involves only the addition operation of the semiring (and not the multiplication operation), hence so do the definitions of the notions of local and global consistency for $\mathbb{K}$-relations. For this reason, we embark on a study of the interplay between local vs. global consistency for $\mathbb{K}$-relations, where $\mathbb{K} = (K, +, 0)$ is a commutative monoid. In addition, we require the monoid $\mathbb{K}$ to be *positive*, which means that the sum of non-zero elements from K is non-zero. This condition is needed in key technical results, but it also ensures that the support of the projection of a $\mathbb{K}$-relation is equal to the support of that relation.

Let $\mathbb{K}$ be an arbitrary positive commutative monoid. Our first result asserts that if a hypergraph H is not acyclic, then there is a collection of pairwise consistent $\mathbb{K}$-relations over H that are not globally consistent; in other words, acyclicity is a necessary condition for the local-to-global consistency property for $\mathbb{K}$-relations to hold. The construction of such $\mathbb{K}$-relations is similar to the one used for bags in [2], which, in turn, was inspired from an earlier construction of hard-to-prove tautologies in propositional logic by Tseitin [12].

Unlike the Boolean monoid $\mathbb{B}$ (case of ordinary relations) and the monoid $\mathbb{N}$ of non-negative integers (case of bags), however, we show that there are positive commutative monoids $\mathbb{K}$ for which acyclicity is not a sufficent condition for the local-to-global consistency property for $\mathbb{K}$-relations to hold. We then go on to characterize the positive commutative monoids for which acyclicity is both necessary and sufficient for the local-to-global consistency property to hold. In fact, we obtain two different characterizations, a semantic one, which we call the *inner consistency property*, and a combinatorial one, which we call the *transportation property*. The inner consistency property asserts that if two $\mathbb{K}$-relations have the same projection on the set of their common attributes, then they are consistent (note that the converse is always true). The transportation property asserts that every balanced instance of the transportation problem with values from $\mathbb{K}$ has a solution in $\mathbb{K}$; these concepts and the terminology are as in the well-studied transportation problem in linear programming.

We then identify several different classes of monoids that possess the transportation property. Special cases include the Boolean monoid $\mathbb{B}$, the monoid $\mathbb{N}$ of non-negative integers, the monoid $\mathbb{R}^{\geq 0}$ of the non-negative real numbers with addition, the monoids obtained by restricting tropical semirings to their additive structure, various monoids of provenance polynomials, and the free commutative monoid on a set of indeterminates. Furthermore, for each such class of monoids, we give either an explicit construction or a procedure for computing a witness to the consistency of two consistent $\mathbb{K}$-relations.

After this extended investigation of classes of positive commutative monoids with the transportation property, we revisit the broader question of characterizing the local-to-global consistency property for collections of $\mathbb{K}$-relations on acyclic schemas for *arbitrary* positive commutative monoids $\mathbb{K}$. By the “no-go examples" in the first part of the paper, we know that any such characterization that applies to all positive commutative monoids must either require more than just pairwise consistency or settle for less than global consistency.

In [3], the second scenario was explored. Specifically, by relaxing the notion of consistency to what was called there *consistency up to normalization*, it was shown that the local-to-global consistency property up to normalization holds precisely for the acyclic schemas. While this result is a common generalization of the theorems by Vorob'ev [14] and by Beeri et al. [4] (because for ordinary relations and for probability distributions the relaxed concept of consistency up to normalization agrees with the standard one), it fails to generalize the local-to-global consistency property for bags from [2]. Furthermore, the definition of this relaxed notion of consistency required $\mathbb{K}$ to come equipped with a multiplication operation making it into a positive semiring, hence the result in [3] does not apply to arbitrary positive commutative monoids.

Here, we explore the first scenario by introducing a stronger notion of consistency, which we call *consistency up to the free cover* (the term reflects the role that the free commutative monoid plays in the definition of this notion). First, we prove that the local-to-global consistency property with consistency strengthened to consistency up to the free cover holds precisely for the acyclic schemas. Second and perhaps unexpectedly, by exploiting the universal property of the free commutative monoid, we establish that the notion of global consistency up to the free cover is *absolute*, in the sense that global consistency holds up to the free cover if and only if it holds in the standard sense. As a consequence, we have that for every positive commutative monoid $\mathbb{K}$, a schema H is acyclic precisely when every collection of $\mathbb{K}$-relations over H that is pairwise consistent up to the free cover is indeed globally consistent. Vice versa, every collection of $\mathbb{K}$-relations that is globally consistent is pairwise consistent up to the free cover. We view these results as an answer to the question of characterizing the global consistency of relations for acyclic schemas in the broader setting of relations over arbitrary positive commutative monoids.

2 PRELIMINARIES

A *commutative monoid* is a structure $\mathbb{K} = (K, +, 0)$, where $+$ is a binary operation on the universe K of $\mathbb{K}$ that is associative, commutative, and has 0 as its neutral element, i.e., $p + 0 = p = 0 + p$ holds for all $p \in K$. A *positive commutative monoid* is a commutative monoid $\mathbb{K} = (K, +, 0)$ such that for all elements $p, q \in K$ with $p + q = 0$, we have that $p = 0$ and $q = 0$. We will only consider *non-trivial* monoids, i.e., those whose universes have at least two elements.

As an example, the structure $\mathbb{B} = (\{0, 1\}, \vee, 0)$ with disjunction $\vee$ as its operation and 0 (false) as its neutral element is a positive commutative monoid. Other examples of positive commutative monoids include the structures $\mathbb{N} = (Z^{\geq 0}, +, 0,)$, $\mathbb{Q}^{\geq 0} = (Q^{\geq 0}, +, 0)$, $\mathbb{R}^{\geq 0} = (R^{\geq 0}, +, 0)$, where $Z^{\geq 0}$ is the set of non-negative integers, $Q^{\geq 0}$ is the set of non-negative rational numbers, $R^{\geq 0}$ is the set of non-negative real numbers, and $+$ is the standard addition operation. In contrast, the structure $\mathbb{Z} = (Z, +, 0)$, where Z is the set of integers, is a commutative monoid, but not a positive one. Two examples of positive commutative monoids of different flavor are the structures $\mathbb{T} = (R \cup \{\infty\}, \min, \infty)$ and $\mathbb{V} = ([0, 1], \max, 0)$, where R is the set of real numbers, and min and max are the standard minimum and maximum operations. Finally, if A is a set and $\mathcal{P}(A)$ is its powerset, then the structure $\mathbb{P}(A) = (\mathcal{P}(A), \cup, \emptyset)$ is a positive commutative monoid, where $\cup$ is the union operation on sets.

An *attribute* A is a symbol with an associated set $\mathrm{Dom}(A)$, called its *domain*. If X is a finite set of attributes, then we write $\mathrm{Tup}(X)$ for the set of X*-tuples*, i.e., $\mathrm{Tup}(X)$ is the set of functions that take each attribute $A \in X$ to an element of its domain $\mathrm{Dom}(A)$. Note that $\mathrm{Tup}(\emptyset)$ is non-empty as it contains the *empty tuple*, i.e., the unique function with empty domain. If $Y \subseteq X$ is a subset of attributes and t is an X-tuple, then the *projection of* t *on* Y, denoted by $t[Y]$, is the unique Y-tuple that agrees with t on Y. In particular, $t[\emptyset]$ is the empty tuple.

Let $\mathbb{K} = (K, +, 0)$ be a positive commutative monoid and let X be a finite set of attributes. A $\mathbb{K}$-*relation over* X is a function $R : \mathrm{Tup}(X) \to K$ that assigns a value $R(t)$ in K to every X-tuple t in $\mathrm{Tup}(X)$. We will often write $R(X)$ to indicate that R is a $\mathbb{K}$-relation over X, and we will refer to X as the set of attributes of R. These notions make sense even if X is the empty set of attributes, in which case a $\mathbb{K}$-relation over X is simply a single value from K that is assigned to the empty tuple. Clearly, the $\mathbb{B}$-relations are just the ordinary relations, while the $\mathbb{N}$-relations are the *bags* or *multisets*, i.e., each tuple has a non-negative integer associated with it that denotes the *multiplicity* of the tuple.

The *support* of a $\mathbb{K}$-relation $R(X)$, denoted by $\mathrm{Supp}(R)$, is the set of X-tuples t that are assigned non-zero value, i.e.,

$$\mathrm{Supp}(R) := \{t \in \mathrm{Tup}(X) : R(t) \neq 0\}. \tag{1}$$

When this does not lead to confusion, we write R' to denote $\mathrm{Supp}(R)$. Note that R' is an ordinary relation over X. A $\mathbb{K}$-relation is *finitely supported* if its support is a finite set. In this paper, all $\mathbb{K}$-relations considered will be finitely supported, and we omit the term; thus, from now on, a $\mathbb{K}$-relation is a finitely supported $\mathbb{K}$-relation. When R' is empty, we say that R is the empty $\mathbb{K}$-relation over X.

If $Y \subseteq X$, then the *marginal* $R[Y]$ *of* R *on* Y is the $\mathbb{K}$-relation over Y such that for every Y-tuple t, we have that

$$R[Y](t) := \sum_{\substack{r \in R': \\ r[Y]=t}} R(r). \tag{2}$$

The value $R[Y](t)$ is the *marginal of* R *over* t. In what follows and for notational simplicity, we will often write $R(t)$ for the marginal of R over t, instead of $R[Y](t)$. It will be clear from the context (e.g., from the arity of the tuple t) if $R(t)$ is indeed the marginal of R over t (in which case t must be a Y-tuple) or $R(t)$ is the actual value of R on t as a mapping from $\mathrm{Tup}(X)$ to K (in which case t must be an X-tuple). If R is an ordinary relation or a bag (i.e., R is a $\mathbb{B}$-relation or an $\mathbb{N}$-relation), then the marginal $R[Y]$ is the projection of R on Y under set semantics or under bag semantics, respectively.

LEMMA 2.1. *Let $\mathbb{K}$ be a positive commutative monoid and let $R(X)$ be a $\mathbb{K}$-relation. The following statements hold:*

(1) *For all $Y \subseteq X$, we have $R'[Y] = R[Y]'$.*
(2) *For all $Z \subseteq Y \subseteq X$, we have $R[Y][Z] = R[Z]$.*

If X and Y are sets of attributes, then we write XY as shorthand for the union $X \cup Y$. Accordingly, if x is an X-tuple and y is a Y-tuple with the property that $x[X \cap Y] = y[X \cap Y]$, then we write xy to denote the XY-tuple that agrees with x on X and on y on Y. We say that x *joins with* y, and that y *joins with* x, to *produce* the tuple xy.

A *schema* is a sequence $X_1, \ldots, X_m$ of sets of attributes. A *collection of* $\mathbb{K}$*-relations over the schema* $X_1, \ldots, X_m$ is a sequence $R_1(X_1), \ldots, R_m(X_m)$ of $\mathbb{K}$-relations, where $R_i(X_i)$ is a $\mathbb{K}$-relation over X_i, for $i = 1, \ldots, m$.

3 CONSISTENCY OVER MONOIDS

The following definitions directly generalize the standard notions of consistency for collections of ordinary relations to collections of $\mathbb{K}$-relations, where $\mathbb{K}$ is an arbitrary positive commutative monoid.

Definition 3.1. Let $\mathbb{K}$ be a positive commutative monoid, let $R_1(X_1), \ldots, R_m(X_m)$ be a collection of $\mathbb{K}$-relations over the schema $X_1, \ldots, X_m$, and let k be a positive integer. We say that the collection $R_1, \ldots, R_m$ is *k-wise consistent* if for all $q \in [k]$ and $i_1, \ldots, i_q \in [m]$, there exists a $\mathbb{K}$-relation $W(X_{i_1} \cdots X_{i_q})$ such that $W[X_i] = R_i$ holds for all $i \in [q]$. If $k = 2$, we say that the collection $R_1, \ldots, R_m$ is *pairwise consistent.* If $k = m$, we say that the collection $R_1, \ldots, R_m$ is *globally consistent.* In all such cases we say that $W(X_{i_1} \cdots X_{i_q})$ *witnesses* the consistency of $R_{i_1}, \ldots, R_{i_q}$.

From Definition 3.1, it follows that if a collection of $\mathbb{K}$-relations is $(k+1)$-wise consistent, then it is also k-wise consistent. In particular, if a collection of $\mathbb{K}$-relations is globally consistent, then it is also pairwise consistent. Our goal in this paper is to investigate when the converse is true. In other words, we focus on the following question: under what conditions on the positive commutative monoid $\mathbb{K}$ and on the schema $X_1, \ldots, X_m$ is it the case that every collection of $\mathbb{K}$-relations of schema $X_1, \ldots, X_m$ that is pairwise consistent is also globally consistent? Our investigation begins by identifying a very broad necessary condition.

3.1 Acyclicity is Always Necessary

A *hypergraph* is a pair $H = (V, E)$, where V is a set of *vertices* and E is a set of *hyperedges*, each of which is a non-empty subset of V. Every collection $X_1, \ldots, X_m$ of sets of attributes can be identified with a hypergraph $H = (V, E)$, where $V = X_1 \cup \cdots \cup X_m$ and $E = \{X_1, \ldots, X_m\}$. Conversely, every hypergraph $H = (V, E)$ gives rise to a collection $X_1, \ldots, X_m$ of sets of attributes, where $X_1, \ldots, X_m$ are the hyperedges of H. Thus, we can move seamlessly between collections of sets of attributes and hypergraphs.

Acyclic Hypergraphs. The notion of an *acyclic* hypergraph generalizes the notion of an acyclic graph. Since we will not work directly with the definition of an acyclic hypergraph, we refer the reader to [4] for the precise definition. Instead, we focus on other notions that are equivalent to hypergraph acyclicity and will be of interest to us in the sequel.

The *primal* graph of a hypergraph $H = (V, E)$ is the undirected graph that has V as its set of vertices and has an edge between any two distinct vertices that appear together in at least one hyperedge of H. A hypergraph H is *conformal* if the set of vertices of every clique (i.e., complete subgraph) of the primal graph of H is contained in some hyperedge of H. A hypergraph H is *chordal* if its primal graph is chordal, that is, if every cycle of length at least four of the primal graph of H has a chord. To illustrate these concepts, let $V_n = \{A_1, \ldots, A_n\}$ be a set of n vertices and consider the hypergraphs

$$P_n = (V_n, \{A_1, A_2\}, \ldots, \{A_{n-1}, A_n\}) \quad (3)$$
$$C_n = (V_n, \{A_1, A_2\}, \ldots, \{A_{n-1}, A_n\}, \{A_n, A_1\}) \quad (4)$$
$$H_n = (V_n, \{V_n \setminus \{A_i\} : 1 \leq i \leq n\}) \quad (5)$$

If $n \geq 2$, then the hypergraph P_n is both conformal and chordal. The hypergraph $C_3 = H_3$ is chordal, but not conformal. For every $n \geq 4$, the hypergraph C_n is conformal, but not chordal, while the hypergraph H_n is chordal, but not conformal.

We say that a hypergraph H has the *running intersection property* if there is a listing $X_1, \ldots, X_m$ of all hyperedges of H such that for every $i \in [m]$ with $i \geq 2$, there exists a $j \in \{1, \ldots, i-1\}$ such that $X_i \cap (X_1 \cup \cdots \cup X_{i-1}) \subseteq X_j$.

Local-to-Global Consistency Property. We say that a hypergraph H has the *local-to-global consistency property for relations* if every collection of relations over H that is pairwise consistent is also globally consistent. This generalizes naturally to $\mathbb{K}$-relations.

Definition 3.2. Let $\mathbb{K}$ be a positive commutative monoid, and let $X_1, \ldots, X_m$ be a listing of all the hyperedges of a hypergraph H. We say that H has the *local-to-global consistency property for* $\mathbb{K}$*-relations* if every collection $R_1(X_1), \ldots, R_m(X_m)$ of $\mathbb{K}$-relations that is pairwise consistent is also globally consistent.

The main theorem in [4] is about $\mathbb{B}$-relations.

Theorem 3.3 (Theorem 3.4 in [4]). *Let H be a hypergraph. The following statements are equivalent:*

(a) *H is an acyclic hypergraph.*
(b) *H is a conformal and chordal hypergraph.*
(c) *H has the running intersection property.*
(d) *H has the local-to-global consistency property for relations.*

Our first result states that the implication (d) $\Longrightarrow$ (a) holds for $\mathbb{K}$-relations, where $\mathbb{K}$ is an arbitrary positive commutative monoid.

Theorem 3.4. *Let $\mathbb{K}$ be a positive commutative monoid and let H be a hypergraph. If H has the local-to-global consistency property for $\mathbb{K}$-relations, then H is an acyclic hypergraph.*

To prove Theorem 3.4, one needs to find a more general construction than the one devised in [4] since the construction given there uses some special properties of ordinary (set-theoretic) relations. Our construction generalizes the one for bags in [2].

3.2 Acyclicity is Not Always Sufficient

In this section, we show that the acyclicity of a schema is not a sufficient condition for the local-to-global consistency property to hold for arbitrary positive commutative monoids.

Let $\mathbb{N}_2 = (\{0, 1, 2\}, \oplus, 0)$ be the positive commutative monoid with the set $\{0, 1, 2\}$ as its universe and addition rounded to 2 as its operation, i.e., $1 \oplus 1 = 2 \oplus 1 = 2 \oplus 2 = 2$, and $0 \oplus x = x \oplus 0 = x$ for all $x \in \{0, 1, 2\}$. Let P_3 be the *path-of-length-3* hypergraph whose vertices form the set $\{A, B, C, D\}$ and whose edges form the set $\{\{A, B\}, \{B, C\}, \{C, D\}\}$. Clearly, P_3 is an acyclic hypergraph.

Proposition 3.5. *The path-of-length-3 hypergraph P_3 does not have the local-to-global consistency property for $\mathbb{N}_2$-relations.*

The proof of Proposition 3.5 will actually be subsumed by the main result of the next section.

4 THE TRANSPORTATION PROPERTY

As seen in the previous section, there exist positive commutative monoids $\mathbb{K}$ for which acyclicity of a hypergraph is not a sufficient condition for it to have the local-to-global consistency property for $\mathbb{K}$-relations. In this section we pursue sufficient conditions.

4.1 The Inner Consistency Property

Let $\mathbb{K}$ be a positive commutative monoid. It is not difficult to see that if $R(X)$ and $S(Y)$ are consistent $\mathbb{K}$-relations, then $R[X \cap Y] = S[X \cap Y]$, i.e., $R(X)$ and $S(Y)$ have the same marginals on the set of their common attributes. Motivated by this, we introduce the following two notions.

Definition 4.1. Let $\mathbb{K}$ be a positive commutative monoid. Two $\mathbb{K}$-relations $R(X)$ and $S(Y)$ are *inner consistent* if $R[X \cap Y] = S[X \cap Y]$ holds. The *inner consistency property holds for $\mathbb{K}$-relations* if whenever two $\mathbb{K}$-relations $R(X)$ and $S(Y)$ are inner consistent, then $R(X)$ and $S(Y)$ are also consistent.

The main result of this section asserts that the inner consistency property holds for $\mathbb{K}$-relations if and only if every acyclic hypergraph has the local-to-global consistency property for $\mathbb{K}$-relations. Rather unexpectedly, it turns out that this last property is equivalent to just having it for the path-of-length three hypergraph P_3. To prove this, we will introduce a combinatorial property of monoids.

Definition 4.2. Let $\mathbb{K} = (K, +, 0)$ be a positive commutative monoid. The *transportation problem for $\mathbb{K}$* is the following decision problem: given two positive integers m and n, a column m-vector $b = (b_1, \ldots, b_m) \in K^m$ with entries in K, and a row n-vector $c = (c_1, \ldots, c_n) \in K^n$ with entries in K, does there exist an $m \times n$ matrix $D = (d_{ij} : i \in [m], j \in [n]) \in K^{m \times n}$ with entries in K such that $d_{i1} + \cdots + d_{im} = b_i$ for all $i \in [m]$ and $d_{1j} + \cdots + d_{mj} = c_j$ for all $j \in [n]$? In words, this means that the rows of D sum to b and the columns of D sum to c.

We view an instance $b = (b_1, \ldots, b_m)$ and $c = (c_1, \ldots, c_n)$ of the transportation problem as a system of linear equations having mn variables and $m + n$ equations. We represent the first m equations horizontally and the next n equations vertically, in accordance with the convention that b is a column vector and c is a row vector:

$$\begin{array}{ccccccccc} x_{11} & + & x_{12} & + & \cdots & + & x_{1n} & = & b_1 \\ + & & + & & & & + & & \\ x_{21} & + & x_{22} & + & \cdots & + & x_{2n} & = & b_2 \\ + & & + & & & & + & & \\ \vdots & & \vdots & & \ddots & & \vdots & & \\ + & & + & & & & + & & \\ x_{m1} & + & x_{m2} & + & \cdots & + & x_{mn} & = & b_m \\ \shortparallel & & \shortparallel & & & & \shortparallel & & \\ c_1 & & c_2 & & & & c_n & & \end{array} \tag{6}$$

The term “transportation problem" comes from linear programming, where this problem has the following interpretation. Suppose a product is manufactured in m different factories, where factory i produces b_i units of the product, $i \in [m]$. The units produced have to be transported to n

different markets, where the demand of the product at market j is c_j units, $j \in [n]$. The question is whether there is a way to ship every unit produced at each factory, so that the demand at each market is met; thus, the variable x_{ij} represents the number of units produced in factory i that are shipped to market j, where $i \in [m]$ and $j \in [n]$.

A necessary condition for an instance of the transportation problem to have a solution is that this instance is *balanced*, i.e., $b_1 + \cdots + b_n = c_1 + \cdots + c_m$. In words, the total supply must be equal to the total demand. This motivates the following notion.

Definition 4.3. Let $\mathbb{K} = (K, +, 0)$ be a positive commutative monoid. We say that $\mathbb{K}$ has the *transportation property* if for every two positive integers m and n, every column m-vector $b = (b_1, \dots, b_m) \in K^m$ with entries in K and every row n-vector $c = (c_1, \dots, c_n) \in K^n$ with entries in K such that $b_1 + \cdots + b_m = c_1 + \cdots + c_n$ holds, we have that there exists an $m \times n$ matrix $D = (d_{ij} : i \in [m], j \in [n]) \in K^{m \times n}$ with entries in K whose rows sum to b and whose columns sum to c, i.e., $d_{i1} + \cdots + d_{im} = b_i$ for all $i \in [m]$ and $d_{1j} + \cdots + d_{mj} = c_j$ for all $j \in [n]$.

The Boolean monoid $\mathbb{B}$, the bag monoid $\mathbb{N}$, and the the non-negative reals $\mathbb{R}^{\geq 0}$ will turn out to have the transportation property.

4.2 Transportation Property and Acyclicity

We are now ready to state and prove the main result of this section.

Theorem 4.4. *Let $\mathbb{K}$ be a positive commutative monoid. The following statements are equivalent:*

(1) *$\mathbb{K}$ has the transportation property.*
(2) *The inner consistency property holds for $\mathbb{K}$-relations.*
(3) *Every acyclic hypergraph has the local-to-global consistency property for $\mathbb{K}$-relations.*
(4) *The hypergraph P_3 has the local-to-global consistency property for $\mathbb{K}$-relations.*

Proof. (1) $\Longrightarrow$ (2). Suppose that $\mathbb{K}$ has the transportation property. Let $R(X)$ and $S(Y)$ be two inner consistent $\mathbb{K}$-relations and let $Z = X \cap Y$. For each Z-tuple w in the support of $R[Z] = S[Z]$, let $u_1, \dots, u_{m_w}$ be an enumeration of the X-tuples that are in the support R' of R and extend w, and let $v_1, \dots, v_{n_w}$ be an enumeration of the Y-tuples that are in the support S' of S and extend w. Let $b_w = (b_{w,1}, \dots, b_{w,m_w})$ be the column vector defined by $b_{w,j} := R(u_j)$ for $j \in [m_w]$, and let $c_w = (c_{w,1}, \dots, c_{w,n_w})$ be the row vector defined by $c_{w,i} := S(v_i)$ for $i \in [n_w]$. Since R and S are inner consistent, we have that $R(w) = S(w)$, hence

$$b_{w,1} + \cdots + b_{w,m_w} = c_{w,1} + \cdots + c_{w,n_w}. \tag{7}$$

By the transportation property of $\mathbb{K}$, there exists an $m_w \times n_w$ matrix $M_w = (d_w(i,j) : i \in [m_w], j \in [n_w])$ that has b_w as column sum and c_w as row sum. Let $T(XY)$ be the $\mathbb{K}$-relation defined for every XY-tuple t by $T(t) := d_w(i,j)$ where $w = t[Z]$ and i and j are such that $t[X] = u_i$ and $t[Y] = v_j$ in the enumerations of the tuples in R' and S' that are used in defining b_w and c_w. For any other XY-tuple t, set $T(t) := 0$. It follows from the definitions that T is a $\mathbb{K}$-relation that witnesses the consistency of R and S.

(2) $\Longrightarrow$ (3). Assume that the hypergraph H is acyclic and therefore it has the running intersection property. Hence, there is a listing $X_1, \dots, X_m$ of its hyperedges such that for every $i \in [m]$ with $i \geq 2$, there is a $j \in [i-1]$ such that $X_i \cap (X_1 \cup \cdots \cup X_{i-1}) \subseteq X_j$. Let $R_1(X_1), \dots, R_m(X_m)$ be a collection of $\mathbb{K}$-relations that is pairwise consistent. By induction on $i = 1, \dots, m$, we show that there is a $\mathbb{K}$-relation T_i over $X_1 \cup \cdots \cup X_i$ that witnesses the global consistency of the $\mathbb{K}$-relations $R_1, \dots, R_i$. For $i = 1$ the claim is obvious by taking $T_1 = R_1$. Assume then that $i \geq 2$ and that the claim is true for all smaller indices. Let $X := X_1 \cup \cdots \cup X_{i-1}$. By the running intersection property, let $j \in [i-1]$ be such that $X_i \cap X \subseteq X_j$. By induction hypothesis, there is a $\mathbb{K}$-relation $T_{i-1}(X)$ that witnesses

the global consistency of $R_1, \ldots, R_{i-1}$. First, we show that T_{i-1} and R_i are consistent. Since, by assumption, the inner consistency property for $\mathbb{K}$-relations holds, it suffices to show that T_{i-1} and R_i are inner consistent, i.e., that $T_{i-1}[X \cap X_i] = R_i[X \cap X_i]$. Let $Z = X \cap X_i$, so $Z \subseteq X_j$ by the choice of j, and indeed $Z = X_j \cap X_i$. Since $j \leq i-1$, we have $R_j = T_{i-1}[X_j]$. Since $Z \subseteq X_j$, we have $R_j[Z] = T_{i-1}[X_j][Z] = T_{i-1}[Z]$. By assumption, also R_j and R_i are consistent, and if W is any $\mathbb{K}$-relation that witnesses their consistency and $Z = X_j \cap X_i$, then $R_j[Z] = W[X_j][Z] = W[Z] = W[X_i][Z] = R_i[Z]$. By transitivity we get $T_{i-1}[Z] = R_i[Z]$, as was to be proved to show that T_{i-1} and R_i are consistent. Now, let T_i be a $\mathbb{K}$-relation that witnesses the consistency of T_{i-1} and R_i. We show that T_i witnesses the global consistency of $R_1, \ldots, R_i$. Since T_{i-1} and R_i are consistent and T_i is a witness, we have $T_{i-1} = T_i[X]$ and $R_i = T_i[X_i]$. Now fix $k \leq i-1$ and note that $R_k = T_{i-1}[X_k] = T_i[X][X_k] = T_i[X_k]$, where the first equality follows from the fact that T_{i-1} witnesses the consistency of $R_1, \ldots, R_{i-1}$ and $k \leq i-1$, and the other two equalities follow from $T_{i-1} = T_i[X]$ and the fact that $X_k \subseteq X$. Thus, T_i witnesses the consistency of $R_1, \ldots, R_i$, which was to be shown.

(3) $\Longrightarrow$ (4). This statement is obvious.

(4) $\Longrightarrow$ (1). Assume that the path-of-length-3 hypergraph P_3 has the local-to-global consistency property for $\mathbb{K}$-relations. Let $(b_1, \ldots, b_m)$ and $(c_1, \ldots, c_n)$ be the two vectors of a balanced instance of the transportation problem for $\mathbb{K}$. Consider the associated system of equations as in (6). Let $a = b_1 + \cdots + b_m = c_1 + \cdots + c_n$. If $a = 0$, then $b_1 = \cdots = b_m = c_1 = \cdots = c_n = 0$ by the positivity of $\mathbb{K}$, and then setting $x_{ij} = 0$ for all i and j we get a solution to (6). Assume then that $a \neq 0$. Based on this instance, we first build three $\mathbb{K}$-relations $R(AB), S(BC), T(CD)$, then we show that they are pairwise consistent, and finally we show how to use any witness of their global consistency to build a solution to the given balanced instance of the transportation problem. The three $\mathbb{K}$-relations are given by the following tables, where the third column is the annotation value from $\mathbb{K}$ for the tuple on its left:

A	B	:	R
u_1	0	:	b_1
$\vdots$	$\vdots$		$\vdots$
u_m	0	:	b_m
v_1	1	:	c_1
$\vdots$	$\vdots$		$\vdots$
v_n	1	:	c_m

B	C	:	S
0	0	:	a
1	1	:	a

C	D	:	T
1	u_1	:	b_1
$\vdots$	$\vdots$		$\vdots$
1	u_m	:	b_m
0	v_1	:	c_1
$\vdots$	$\vdots$		$\vdots$
0	v_n	:	c_n

As witnesses to the pairwise consistency of these three $\mathbb{K}$-relations, consider the $\mathbb{K}$-relations defined by $U(u_i, 0, 0) = V(1, 1, u_i) = W(u_i, 0, 1, v_i) = b_i$ and $U(v_i, 1, 1) = V(0, 0, v_i) = W(v_i, 1, 0, u_i) = c_i$. By construction, we have $U[AB] = R$ and $U[BC] = S$, also $V[BC] = S$ and $V[CD] = T$, and $W[AB] = R$ and $W[CD] = T$. By the assumption that the hypergraph P_3 has the local-to-global consistency property for $\mathbb{K}$-relations, there is a $\mathbb{K}$-relation $Y(ABCD)$ that witnesses the global consistency of R, S, T. Since $Y[BC] = S$, for every tuple (a, b, c, d) in the support Y' of Y, we have $b = c = 0$ or $b = c = 1$. Similarly, since $Y[AB] = R$, we have that if $b = 0$ then $a = u_i$ for some $i \in [m]$, and since $Y[CD] = T$, we have that if $c = 0$ then $d = v_j$ for some $j \in [n]$. Now, set $d_{ij} := Y(u_i, 0, 0, v_j)$ for every $i \in [m]$ and $j \in [n]$. For every $i \in [m]$ we have $\sum_j d_{ij} = \sum_j Y(u_i, 0, 0, v_j) = \sum_{i,c,d} Y(u_i, 0, c, d) = R(u_i, 0) = b_i$, where the first equality follows from the choice of d_{ij}, the second follows from the above-mentioned properties of the tuples (a, b, c, d) in the support Y' of Y, the third follows from $Y[AB] = R$, and the last follows from the choice of R. Similarly, for every $j \in [n]$ we have $\sum_i d_{ij} = \sum_i Y(u_i, 0, 0, v_j) = \sum_{a,b,j} Y(a, b, 0, v_j) = T(0, v_j) = c_j$, with very similar justifications for each step. This proves that $D = (d_{ij} : i \in [m], j \in [n])$ is a solution to the balanced instance of the transportation property of $\mathbb{K}$ given by the vectors $(b_1, \dots, b_m)$ and $(c_1, \dots, c_n)$, which completes the proof. □

Theorems 3.4 and 4.4 yield the following result.

Corollary 4.5. *Let $\mathbb{K}$ be a positive commutative monoid that has the transportation property. For every hypergraph H, the following statements are equivalent:*

(1) *H is an acyclic hypergraph.*
(2) *H has the local-to-global consistency property for $\mathbb{K}$-relations.*

Since the transportation property holds for $\mathbb{B}$ and since the $\mathbb{B}$-relations are the ordinary relations, Corollary 4.5 contains the Beeri-Fagin-Maier-Yannakakis Theorem from [4] as a special case. In the next section, we identify several different classes of positive commutative monoids that have the transportation property; therefore, Corollary 4.5 applies to all such monoids.

5 TRANSPORTATION MONOIDS

We now turn to the question of identifying broad classes of positive commutative monoids that do have the transportation property.

5.1 Expansions to Semirings and Standard Joins

A *semiring* is a structure $\mathbb{K} = (K, +, \times, 0, 1)$ such that:

- $(K, +, 0)$ and $(K, \times, 1)$ are commutative monoids;
- $\times$ distributes over $+$, i.e., $p \times (q + r) = p \times q + p \times r$.
- 0 annihilates, i.e., $0 \times p = p \times 0 = 0$.

An *additively positive semiring* is a semiring $\mathbb{K} = (K, +, \times, 0, 1)$ whose additive reduct $(K, +, 0)$ is a positive monoid. If $R(X)$ and $S(Y)$ are two $\mathbb{K}$-relations, then the *standard $\mathbb{K}$-join* of R and S, denoted by $R \bowtie_{\mathbb{K},S} S$, is the $\mathbb{K}$-relation $W(XY)$ defined for every XY-tuple t by the equation $W(t) = R(t[X]) \times S(t[Y])$. We say that *the inner consistency property holds for $\mathbb{K}$-relations via the standard $\mathbb{K}$-join* if the inner consistency property holds for $\mathbb{K}$-relations and, moreover, the standard $\mathbb{K}$-join witnesses the consistency of two consistent $\mathbb{K}$-relations. Clearly, if $\mathbb{K}$ is the Boolean semiring $\mathbb{B}$, then the standard $\mathbb{K}$-join coincides with the relational join. Unfortunately, if $\mathbb{K}$ is an arbitrary positive semiring, then the standard $\mathbb{K}$-join need not always be a witness to the consistency of two consistent $\mathbb{K}$-relations. This fails even for bags as pointed out in [2].

Our aim is to characterize the additively positive semirings $\mathbb{K}$ for which the inner consistency property holds for $\mathbb{K}$-relations via the standard $\mathbb{K}$-join. We need two definitions. We say that $\mathbb{K}$ is *additively absorptive* if for all $p, q \in K$ it holds that $p + p \times q = p$. We say that $\mathbb{K}$ is *multiplicatively idempotent* if for all $p \in K$ it holds that $p \times p = p$. It turns out that, if $\mathbb{K}$ is additively absorptive, then $\mathbb{K}$ is additively positive. Indeed, suppose that p and q are two elements of K such that $p + q = 0$. Then $p = p + (p + q) = (p + p) + q = p + q = 0$, where the first and last equalities follow from the assumption that $p + q = 0$, and the second and third equalities follow from associativity and additive idempotence, respectively. Similarly, we get $q = (p + q) + q = p + (q + q) = p + q = 0$, hence $p = q = 0$.

Proposition 5.1. *Let $\mathbb{K}$ be a semiring. The following statements are equivalent.*

(1) *$\mathbb{K}$ is additively absorptive and multiplicatively idempotent.*
(2) *$\mathbb{K}$ is additively positive and the inner consistency property holds for $\mathbb{K}$-relations via the standard $\mathbb{K}$-join.*

Examples of additively positive semirings to which Proposition 5.1 applies include the Boolean semiring $\mathbb{B}$, all bounded distributive lattices such as the max/min semirings $(A, \max, \min, a, b)$ where $(A, \leq)$ is a totally ordered set with minimum a and maximum b, and the Boolean positive expressions up to equivalence.

5.2 Expansions to Semifields and Vorob'ev Joins

A *semifield* is a structure $\mathbb{K} = (K, +, \times, 0, 1)$ such that:

- $\mathbb{K} = (K, +, \times, 0, 1)$ is a semiring.
- For every element $p \neq 0$ in K, there exists an element q in K such that $p \times q = 1 = q \times p$; since, as can be seen, this q is unique, we write $1/p$ for q, and r/p for $r \times (1/p)$.

An *additively positive semifield* is a semifield $\mathbb{K} = (K, +, \times, 0, 1)$ in which the underlying additive monoid $(K, +, 0)$ is positive. For two inner consistent $\mathbb{K}$-relations $R(X)$ and $S(Y)$, the *Vorob'ev $\mathbb{K}$-join* of R and S, denoted by $R \bowtie_{\mathbb{K},V} S$, is the $\mathbb{K}$-relation $W(XY)$ defined for every XY-tuple t as follows: $W(t) = R(t[X]) \times S(t[Y])/R(t[X \cap Y]) = R(t[X]) \times S(t[Y])/S(t[X \cap Y])$ if $R(t[X \cap Y]) = S(t[X \cap Y]) \neq 0$, and $W(t) = 0$ otherwise. The Vorob'ev $\mathbb{K}$-join of two $\mathbb{K}$-relations is well-defined because $R(X)$ and $S(Y)$ were assumed to be inner consistent $\mathbb{K}$-relations. We say that *the inner consistency property holds for $\mathbb{K}$-relations via the Vorob'ev $\mathbb{K}$-join* if the inner consistency property holds for $\mathbb{K}$-relations and, moreover, the Vorob'ev $\mathbb{K}$-join witnesses the consistency of two consistent $\mathbb{K}$-relations.

Proposition 5.2. *If $\mathbb{K}$ is an additively positive semifield, then the inner consistency property holds for $\mathbb{K}$-relations via the Vorob'ev $\mathbb{K}$-join.*

Two well known examples of positive semifields are the semiring $\mathbb{R}^{\geq 0} = (R^{\geq 0}, +, \times, 0, 1)$ of non-negative real numbers and its rational substructure $\mathbb{Q}^{\geq 0} = (Q^{\geq 0}, +, \times, 0, 1)$. Others examples include the tropical semiring $\mathbb{T}_{\min} = ((-\infty, +\infty], \min, +, +\infty, 0)$, and its smooth variant, which is called the log semiring.

5.3 Northwest Corner Method

Let $\mathbb{K} = (K, +, 0)$ be a positive commutative monoid. Consider the binary relation $\sqsubseteq$ on K defined, for all $b, c \in K$, by $b \sqsubseteq c$ if and only if there exists some $a \in K$ such that $b + a = c$. The binary relation $\sqsubseteq$ is reflexive and transitive, and is hence a pre-order, called the *canonical pre-order* of $\mathbb{K}$. We say that $\mathbb{K}$ is *totally canonically pre-ordered* if $b \sqsubseteq c$ or $c \sqsubseteq b$ for all $b, c \in K$, and *weakly cancellative* if $a + b = a + c$ implies $b = c$ or $b = 0$ or $c = 0$ for all $a, b, c \in K$.

We will show that if a positive commutative monoid $\mathbb{K}$ is weakly cancellative and totally canonically pre-ordered, then $\mathbb{K}$ has the inner consistency property for $\mathbb{K}$-relations. This will be achieved by using the *northwest corner method* of linear programming for finding solutions for the transportation problem, and hence witnesses of consistency. We say that the inner consistency property holds for $\mathbb{K}$-relations *via the northwest (NW) corner method.*

Intuitively, the NW corner method starts by assigning a value to the variable in the northwest corner of the system of equations, eliminating at least one equation, and iterating this process by considering next the variable in the northwest corner of the resulting system. Unlike the case of linear programming, here we cannot subtract values; instead, we have to use the assumption that the monoid is weakly cancellative and totally canonically pre-ordered.

Proposition 5.3. *If $\mathbb{K}$ is a weakly cancellative and totally canonically pre-ordered positive commutative monoid, then $\mathbb{K}$ has the inner consistency property for $\mathbb{K}$-relations via the NW corner method.*

The prime example to which Proposition 5.3 applies is the bag monoid $\mathbb{N}$. Other examples include the additive reduct of the log semiring, and its non-negative version that has domain $[0, +\infty]$.

5.4 Powers, Polynomials, and Free Monoids

Let I be a finite or infinite index set and let $\mathbb{K} = (K, +, 0)$ be a monoid. The *power monoid* $\mathbb{K}^I$ has as elements the maps $f : I \to K$ with addition $f + g$ defined componentwise by the equation $(f + g)(i) = f(i) + g(i)$. The *finite support power* $\mathbb{K}^I_{\text{fin}}$ is the substructure whose elements are the maps of finite support. It is easy to check that if $\mathbb{K}$ is positive, then $\mathbb{K}^I$ and $\mathbb{K}^I_{\text{fin}}$ are positive as well.

Proposition 5.4. *Let I be a non-empty index set and let $\mathbb{K}$ be a positive commutative monoid. The following are equivalent:*

(1) $\mathbb{K}$ *has the transportation property,*
(2) $\mathbb{K}^I$ *has the transportation property,*
(3) $\mathbb{K}^I_{\text{fin}}$ *has the transportation property.*

Viewing a polynomial as an indexed collection of coefficients for its monomials, examples of the form $\mathbb{K}^I_{\text{fin}}$ include the monoids of polynomials with coefficients in $\mathbb{K}$ and variables in a set of indeterminates X, denoted by $\mathbb{K}[X]$. Their restriction to monomials of total degree d are the *degree-d forms*, and it is denoted by $\mathbb{K}[X]_d$. If $\mathbb{K}$ is a semiring, then $\mathbb{K}[X]$ can also be viewed as a semiring. The special case $\mathbb{N}[X]$ is called *the most informative* of the *provenance semirings* (see [8]). In fact, the additive reducts of *all* provenance semirings considered in [8] have the form $\mathbb{K}^I_{\text{fin}}$ for some positive monoid $\mathbb{K}$ and hence have the transportation property. The monoid $\mathbb{N}[X]_1$ of linear forms with non-negative integer coefficients is isomorphic to the free commutative monoid generated by X (see Chapter 10 in [6]), denoted here by $\mathbb{F}(X)$.

Proposition 5.5. *For every set X of indeterminates, the free commutative monoid generated by X is isomorphic to $\mathbb{N}[X]_1$, and it is a positive commutative monoid that has the transportation property.*

The free commutative monoids $\mathbb{F}(X)$ will reappear in Section 6.

5.5 Some Important Non-Examples

As we have seen, many important positive commutative monoids have the transportation property. Unfortunately there are positive commutative monoids of different character that fail to have the transportation property. Here we present a few such examples.

The first non-example was already defined in Section 3.2, i.e., the monoid $\mathbb{N}_2$ of natural numbers with addition truncated to 2. Consider the $\mathbb{N}_2$-relations $R(AC)$ and $S(BC)$ defined by $R(a_1, c) = R(a_2, c) = S(b_1, c) = 1$ and $S(b_2, c) = 2$, and no other tuples in their supports. It is easy to see that this is a counterexample to the inner consistency property for $\mathbb{N}_2$-relations. Another non-example with a similar counterexample is $\mathbb{R}_1 = (\{0\} \cup [1, +\infty], +, 0)$, the non-negative reals with a gap. Note that $\mathbb{N}_2$ is finite, while $\mathbb{R}_1$ is infinite.

Our third non-example involves a natural positive commutative monoid for which the failure of the transportation property is conceptually significant as it corresponds to the deep fact of quantum mechanics that there exist pairs of binary observables that cannot be jointly measured. This is a manifestation of the *Heisenberg uncertainty principle* for positive-operator-valued measures [11].

Let $n \geq 1$ be a positive integer and let $\mathbb{PSD}_n$ be the set of positive semidefinite matrices in $\mathbb{R}^{n\times n}$ with standard matrix addition. The properties of PSD matrices can be used to show that this is a positive monoid. For $n = 2$, there is an instance of the transportation problem involving the (real) Pauli matrices I, X, Z that has no solution over $\mathbb{PSD}_2$, concretely the instance given by $b = ((I + X)/2, (I - X)/2)$ and $c = ((I + Z)/2, (I - Z)/2)$. This can be extended to all $n \geq 2$ by padding the matrices with zeros. The bottom case $\mathbb{PSD}_1$ has the transportation property as it is isomorphic to $\mathbb{R}^{\geq 0}$.

6 LOCAL CONSISTENCY UP TO A COVER

Here, we investigate whether there is a suitably modified notion of local consistency of $\mathbb{K}$-relations that has the effect of capturing the global consistency of $\mathbb{K}$-relations for precisely the acyclic hypergraphs, but that applies to *every* positive commutative monoid. We achieve this by strengthening the notion of locality.

6.1 Consistency up to a Cover

Let $\mathbb{K}$ be a positive commutative monoid. A *cover of* $\mathbb{K}$ is a positive commutative monoid $\mathbb{K}^*$ such that there is a surjective homomorphism h from $\mathbb{K}^*$ onto $\mathbb{K}$. The *identity cover* is the cover where $\mathbb{K}^*$ is $\mathbb{K}$ itself and h is the identity map. A cover of $\mathbb{K}$ is given by the pair $(\mathbb{K}^*, h)$ of both objects; we use the notation $h : \mathbb{K}^* \xrightarrow{s} \mathbb{K}$ to say that the pair $(\mathbb{K}^*, h)$ is a cover of $\mathbb{K}$. For the definitions of the next paragraph, fix such a cover.

For a $\mathbb{K}$-relation $R(Y)$, an *h-lift of R* is a $\mathbb{K}^*$-relation $R^*(Y)$ such that $h(R^*(t)) = R(t)$ holds for every Y-tuple t, i.e., $h \circ R^* = R$ holds. In most of the cases that follow, the cover will be clear from the context, and we simply say that R^* is a lift of R, without any reference to h. Note that, since the homomorphism h is surjective onto $\mathbb{K}$, every $\mathbb{K}$-relation R has at least one h-lift R^*. Consider the special case where $h : \mathbb{K}^* \xrightarrow{s} \mathbb{K}$ is a *retraction*, meaning that $K \subseteq K^*$ and h is the identity on K, where K and K^* are the universes of $\mathbb{K}$ and $\mathbb{K}^*$, respectively; in this case, the *direct h-lift of R* is the $\mathbb{K}^*$-relation R^* defined by $R^*(t) = R(t)$, for every Y-tuple t.

Definition 6.1. Let $\mathbb{K}$ be a positive commutative monoid, let $h : \mathbb{K}^* \xrightarrow{s} \mathbb{K}$ be a cover of $\mathbb{K}$, let $R_1(X_1), \ldots, R_m(X_m)$ be a collection of $\mathbb{K}$-relations over the schema $X_1, \ldots, X_m$, and let k be a positive integer. We say that the collection $R_1, \ldots, R_m$ is *k-wise consistent up to the cover* $h : \mathbb{K}^* \xrightarrow{s} \mathbb{K}$ if there exists a collection $R_1^*, \ldots, R_m^*$ of h-lifts of $R_1, \ldots, R_m$ that is k-wise consistent (as a collection of $\mathbb{K}^*$-relations). If $k = 2$, then we say that the collection $R_1, \ldots, R_m$ is *pairwise consistent up to the cover.* If $k = m$, then we say that the collection $R_1, \ldots, R_m$ is *globally consistent up to the cover.* When $k = m = 2$, we just say that R_1 and R_2 are *consistent up to the cover.*

It is worth pointing out that, in the definition of consistency up to a cover, not only the choice of the cover $h : \mathbb{K}^* \xrightarrow{s} \mathbb{K}$ potentially matters, but also the choice of h-lifts $R_1^*, \ldots, R_m^*$ matters.

The following simple but important observation about global consistency up to covers is key to our development.

Proposition 6.2 (Absoluteness). *Let $\mathbb{K}$ be a positive commutative monoid and let $R_1, \ldots, R_m$ be a collection of $\mathbb{K}$-relations. The following statements are equivalent.*

(1) $R_1, \ldots, R_m$ *is globally consistent,*
(2) $R_1, \ldots, R_m$ *is globally consistent up to every cover of* $\mathbb{K}$*,*
(3) $R_1, \ldots, R_m$ *is globally consistent up to some cover of* $\mathbb{K}$*.*

In view of Proposition 6.2, we say that the notion of global consistency up to covers is *absolute* as if it holds for some cover, then it holds for all covers. Unlike the global notion, the local notion is *not* but at least 2/3 of Proposition 6.2 descend to local consistency.

Proposition 6.3. *Let k be a positive integer, let $\mathbb{K}$ be a positive commutative monoid, and let $R_1, \ldots, R_m$ be a collection of $\mathbb{K}$-relations. The following statements are equivalent.*

(1) $R_1, \ldots, R_m$ *is k-wise consistent,*
(2) $R_1, \ldots, R_m$ *is k-wise consistent up to some cover of* $\mathbb{K}$*.*

Let us point out that the monoid $\mathbb{N}_2$ of Section 5.5 and the counterexample to the inner consistency property given there can be used to show that local consistency is not absolute in the sense that a third clause with all covers cannot be added to Proposition 6.3.

6.2 Local-to-Global Consistency up to Covers

The local-to-global consistency property up to a cover is defined to generalize Definition 3.2 as follows:

Definition 6.4. Let $\mathbb{K}$ be a positive commutative monoid, let $h : \mathbb{K}^* \xrightarrow{s} \mathbb{K}$ be a cover of $\mathbb{K}$, and let $X_1, \ldots, X_m$ be a listing of all the hyperedges of a hypergraph H. We say that H has the *local-to-global consistency property for* $\mathbb{K}$*-relations up to the cover* $h : \mathbb{K}^* \xrightarrow{s} \mathbb{K}$ if every collection $R_1(X_1), \ldots, R(X_m)$ of $\mathbb{K}$-relations that is pairwise consistent up to the cover is globally consistent.

Recall that, by Proposition 5.5, the free commutative monoid $\mathbb{F}(X)$ for a finite or infinite set of indeterminates X has the transportation property. In the statement of the following theorem, for $\mathbb{K} = (K, +, 0)$, let $\mathbb{F}(K^+)$ denote the free commutative monoid generated by the set K^+ of non-zero elements in K seen as indeterminates. The *free cover of* $\mathbb{K}$ refers to the cover $h : \mathbb{F}(K^+) \xrightarrow{s} \mathbb{K}$ provided by the homomorphism h from $\mathbb{F}(K^+)$ to $\mathbb{K}$ given by the universal property of $\mathbb{F}(K^+)$; concretely, viewing $\mathbb{F}(K^+)$ as isomorphic to $\mathbb{N}[K^+]_1$, the homomorphism h is the natural *evaluation map* that takes the linear form $f \in \mathbb{N}[K^+]_1$ to its evaluation in $\mathbb{K}$ on the indeterminates interpreted by the corresponding element in K. Clearly, h is surjective onto K, so $h : \mathbb{F}(K^+) \xrightarrow{s} \mathbb{K}$ is indeed a cover.

Theorem 6.5. *Let* $\mathbb{K}$ *be a positive commutative monoid and let* H *be a hypergraph. The following statements are equivalent:*

(1) *H is an acyclic hypergraph,*
(2) *H has the local-to-global consistency property up to the free cover of* $\mathbb{K}$*,*
(3) *H has the local-to-global consistency property up to some cover of* $\mathbb{K}$*.*

Proof. Let $Y_1, \ldots, Y_m$ be a listing of the hyperedges of H.

(1) $\Longrightarrow$ (2). Let $R_1(Y_1), \ldots, R_m(Y_m)$ be a collection of $\mathbb{K}$-relations and assume that it is pairwise consistent up to the free cover $h : \mathbb{F}(K^+) \xrightarrow{s} \mathbb{K}$. Let $R_1^*, \ldots, R_m^*$ be a collection of $\mathbb{F}(K^+)$-relations that are h-lifts of $R_1, \ldots, R_m$, respectively, and assume that the collection $R_1^*, \ldots, R_m^*$ is pairwise consistent. By Proposition 5.5 and Corollary 4.5, the hypergraph H has the local-to-global consistency property for $\mathbb{F}(K^+)$-relations, so the collection $R_1^*, \ldots, R_m^*$ is globally consistent. But, then, the collection $R_1, \ldots, R_m$ itself is globally consistent by Proposition 6.2.

(2) $\Longrightarrow$ (3). This is obvious because the free cover is a cover.

(3) $\Longrightarrow$ (1). A close inspection of the proof of Theorem 3.4 shows that the construction gives a collection of $\mathbb{K}$-relations that is pairwise consistent up to every cover of $\mathbb{K}$. □

7 CONCLUDING REMARKS

We conclude by describing a few open problems and directions for research motivated by the work reported here.

As seen earlier, there are finite positive commutative monoids that have the transportation property (e.g., $\mathbb{B}$) and others that do not (e.g., $\mathbb{N}_2$). How difficult is it to decide whether or not a given finite positive commutative monoid $\mathbb{K}$ has the transportation property? Is this problem decidable or undecidable? The same question can be asked when the given monoid is *finitely presentable*. Note that the transportation property is defined using an infinite set of first-order axioms in the language of monoids. Thus, a related question is whether or not the transportation property is finitely axiomatizable.

We exhibited several classes of monoids that have the transportation property. In each case, we gave an explicit construction or a procedure for finding a witness to the consistency of two consistent $\mathbb{K}$-relations. In some cases (e.g., when the monoid has an expansion to a semifield), there is a suitable join operation that yields a *canonical* such witness. However, in some other cases (e.g., when the northwest corner method is used), no *canonical* such witness seems to exist. Is there a way to compare the different witnesses to consistency and classify them according to some desirable property, such as maximizing some carefully chosen objective function?

Finally, the work presented here expands the study of relations with annotations over semirings to relations with annotations over monoids. As explained in the Introduction, consistency notions only require the use of an addition operation (and not a multiplication operation). What other fundamental problems in databases can be studied in this broader framework of relations with annotations over monoids?

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