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All-loop group-theory constraints for four-point amplitudes of SU(N), SO(N), and Sp(N) gauge theories

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Abstract: In the decomposition of gauge-theory amplitudes into kinematic and color factors, the color factors (at a given loop order L) span a proper subspace of the extended trace space (which consists of single and multiple traces of generators of the gauge group, graded by powers of N). Using an iterative process, we systematically construct the L-loop color space of four-point amplitudes of fields in the adjoint representation of SU(N), SO(N), or Sp(N). We define the null space as the orthogonal complement of the color space. For SU(N), we confirm the existence of four independent null vectors (for $L \geq 2$) and for SO(N) and Sp(N), we establish the existence of seventeen independent null vectors (for $L \geq 5$). Each null vector corresponds to a group-theory constraint on the color-ordered amplitudes of the gauge theory.

Keywords: 1/N Expansion, Duality in Gauge Field Theories, Gauge Symmetry, Scattering Amplitudes

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1 Introduction

Gauge-theory scattering amplitudes at tree and loop level may be represented in a gauge-invariant way in terms of color-ordered (or partial) amplitudes [1, 2]. The color-ordered amplitudes for a particular process are not independent but satisfy a number of constraints. Some of these constraints are a consequence of color-kinematic duality [3, 4], a property

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possessed by the amplitudes of a wide class of gauge theories, whose most notable consequence is the gauge-gravity correspondence (see ref. [5] for a comprehensive review). Color-kinematic duality implies the existence of the Bern-Carrasco-Johansson relations of tree-level amplitudes [3] which were proven in refs. [6–9].

There are, however, other constraints on color-ordered amplitudes that are more basic because they follow directly from group theory, such as the Kleiss-Kuijf relations among tree-level n-point amplitudes [10, 11], the Bern-Kosower relations among one-loop SU(N) n-point amplitudes [11, 12], and a two-loop relation that holds for four-point SU(N) color-ordered amplitudes [13]. These group-theory relations for four-point SU(N) amplitudes were generalized to all loop orders by one of the current authors using an iterative procedure [14]. This iterative technique was subsequently used by Edison and one of the current authors to derive all-loop-order relations for five-point SU(N) amplitudes [15] and for six-point SU(N) amplitudes [16]. These results have been used in refs. [17–29]. Other work on loop-level relations among n-point amplitudes includes refs. [30–32].

The primary focus of this paper is to derive all-loop-order group-theory constraints for four-point amplitudes of fields in the adjoint representation of the classical groups SO(N) and Sp(N), while also confirming the results of ref. [14] for SU(N). While SU(N) is obviously most phenomenologically relevant in the standard model context, SO(N) and Sp(N) could become relevant for theories beyond the standard model, e.g. grand unified theories. In previous work, Huang [33, 34] generalized the iterative procedure of ref. [14] to obtain group-theory constraints for four- and five-point amplitudes of SO(N) and Sp(N) up to four loops, but did not uncover any patterns that could generalize to an arbitrary number of loops.

In this paper, we develop a refined version of the iterative approach that allows us obtain the all-loop structure of the space of color factors for all of the classical groups: SU(N), SO(N), and Sp(N). Not surprisingly, for SU(N) we rederive the *four* group-theory constraints for L-loop amplitudes (for $L \geq 2$) obtained in ref. [14]. For SO(N) and Sp(N), we uncover a substantially more intricate structure that implies the existence of *seventeen* group-theory constraints for L-loop amplitudes (for $L \geq 5$). (See tables 1 and 2 for the number of constraints for all values of L.)

Obtaining group-theory constraints for color-ordered amplitudes boils down to a problem in linear algebra. One begins with the amplitude (at some loop order L) expressed in a basis of color factors [11, 35]

$$\mathcal{A}^{(L)} = \sum_i a_i^{(L)} C_i^{(L)} \tag{1.1}$$

where $a_i^{(L)}$ carries the momentum and polarization dependence of the amplitude, and the color factors $C_i^{(L)}$ are obtained by sewing together group-theory factors from all the vertices of the contributing Feynman diagrams. In a theory that contains only fields in the adjoint representation of the gauge group, such as pure or supersymmetric Yang-Mills theory, each cubic vertex contributes a factor of the structure constants $\tilde{f}^{abc}$ of the gauge group G, whereas each quartic vertex contributes a sum of products of $\tilde{f}^{abc}$, each of which are equivalent (from a purely color perspective) to a pair of cubic vertices sewn along one leg. Hence a complete set of color factors $\{C_i^{(L)}\}$ may be constructed from L-loop diagrams with cubic vertices only. The color factors constructed from the set of *all* cubic diagrams are generally not independent but are related by Jacobi relations. We denote the number of independent color factors (i.e., the dimension of the space of color factors) as $n_{\rm color}$. An independent basis of color factors for tree-level and one-loop n-point amplitudes was described in refs. [11, 35]. One of our goals is obtain an independent basis of color factors for four-point amplitudes at any loop order for SU(N), SO(N), and Sp(N).

One may alternatively decompose the amplitude in a trace basis [1, 2]

$$\mathcal{A}^{(L)} = \sum_{\lambda} A_{\lambda}^{(L)} t_{\lambda}^{(L)} \tag{1.2}$$

whose coefficients are gauge-invariant color-ordered amplitudes $A_{\lambda}^{(L)}$ and the basis $\{t_{\lambda}^{(L)}\}$ consists of single and (at loop level) multiple traces of gauge group generators T^a in the defining representation of the gauge group G. The explicit form and dimensionality $n_{\rm trace}$ of this (extended) trace basis depends on the gauge group. For $G = SU(\mathrm{N})$, one has $n_{\rm trace} = 3L + 3$ while for $G = \mathrm{SO}(N)$ or $\mathrm{Sp}(N)$, one has $n_{\rm trace} = 6L + 3$. The dimension of the trace basis is always larger than that of the independent color basis ($n_{\rm trace} > n_{\rm color}$) so there is redundancy among the color-ordered amplitudes, expressed below as group-theory relations (1.8).

The color (1.1) and trace (1.2) decompositions are related by writing the structure constants as

$$\tilde{f}^{abc} = \mathrm{Tr}(T^a, [T^b, T^c]) \tag{1.3}$$

and then using group-dependent identities satisfied by the generators (see section 2) to express each color factor $C_i^{(L)}$ as a linear combination of trace factors

$$C_i^{(L)} = \sum_{\lambda} M_{i\lambda}^{(L)} t_{\lambda}^{(L)}\,. \tag{1.4}$$

Since $n_{\rm trace} > n_{\rm color}$, the linear combinations given by eq. (1.4) span a proper subspace (which we will refer to as the color space) of the extended trace space. Consequently, the transformation matrix $M_{i\lambda}^{(L)}$ possesses a set of independent null eigenvectors

$$\sum_{\lambda} M_{i\lambda}^{(L)} r_{\lambda m}^{(L)} = 0\,, \qquad m = 1, \cdots, n_{\rm null} \tag{1.5}$$

whose number $n_{\rm null}$ is the difference between the dimensions of the trace space and the color space. The null vectors, defined by $r_m^{(L)} = \sum_{\lambda} r_{\lambda m}^{(L)} t_{\lambda}^{(L)}$, are orthogonal to the color factors $C_i^{(L)}$ with respect to the inner product

$$(t_{\lambda}^{(L)}, t_{\lambda'}^{(L)}) = \delta_{\lambda\lambda'} \tag{1.6}$$

and hence span the orthogonal complement of the color space; we refer to this as the *null space*.

One combines eq. (1.4) with eqs. (1.1) and (1.2) to express the color-ordered amplitudes as

$$A_{\lambda}^{(L)} = \sum_{i} a_i^{(L)} M_{i\lambda}^{(L)}\,. \tag{1.7}$$

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number of loops	0	1	2	3	4	5	6	$L \geq 2$
n_{color}	2	3	5	8	11	14	17	$3L-1$
n_{trace}	3	6	9	12	15	18	21	$3L+3$
n_{null}	1	3	4	4	4	4	4	4

Table 1. Dimensions of color, trace, and null spaces for SU(N) amplitudes.

number of loops	0	1	2	3	4	5	6	$L \geq 5$
n_{color}	2	3	5	8	11	16	22	$6L-14$
n_{trace}	3	9	15	21	27	33	39	$6L+3$
n_{null}	1	6	10	13	16	17	17	17

Table 2. Dimensions of color, trace, and null spaces for SO(N) and Sp(N) amplitudes.

Applying eq. (1.5) to eq. (1.7) implies the set of constraints

$$\sum_{\lambda} A_{\lambda}^{(L)} r_{\lambda m}^{(L)} = 0\,, \qquad m = 1, \cdots, n_{\text{null}} \tag{1.8}$$

which we refer to as group-theory relations. Hence, specifying the null space is equivalent to specifying the complete set of group-theory relations satisfied by the color-ordered amplitudes.

The iterative approach taken in this paper involves attaching a rung across any pair of external legs of an arbitrary L-loop color factor. We make the assumption that doing this to all diagrams spanning the space of L-loop color factors generates the space of $(L+1)$-loop color factors, an assumption borne out in practice. Starting with the tree-level color space, we explicitly construct a set of color factors spanning the color space at each loop order for SU(N), SO(N), and Sp(N). In tables 1 and 2, we list the dimensions of these color spaces, together with the dimensions of trace and null spaces, where $n_{\text{null}} = n_{\text{trace}} - n_{\text{color}}$. The dimensions in table 1 confirm the results of ref. [14] for four-point SU(N) amplitudes at all loop orders. The dimensions in table 2 are in agreement with ref. [33] for four-point SO(N) and Sp(N) amplitudes for $0 \leq L \leq 4$. Ref. [33] did not go beyond four loops.

Since the complete set of color factors at a given loop order is invariant under permutation of the external legs, the color space forms a representation of S_4, which can be decomposed into irreducible representations of one and two dimensions, denoted in this paper by u and x respectively. The trace and null spaces also decompose into u- and x-type irreducible representations. For SU(N), there are generically (for $L \geq 2$) four null vectors, two of u-type and two of x-type, for which we determine the explicit forms. For SO(N) and Sp(N), there are generically (for $L \geq 5$) seventeen null vectors, seven of u-type and ten of x-type. We determine (for arbitrary L) the explicit forms of the ten x-type null vectors in this paper, leaving the seven u-type null vectors to future work.

This paper is structured as follows. In section 2 we review the color and trace spaces for L-loop four-point amplitudes through two loops, decomposing them into irreducible representations of S_4. In section 3 we review and refine the iterative procedure for generating the $(L+1)$-loop color space from the L-loop color space. In section 4 we employ this refined iterative procedure to generate the L-loop color space for SU(N). In section 5 after defining

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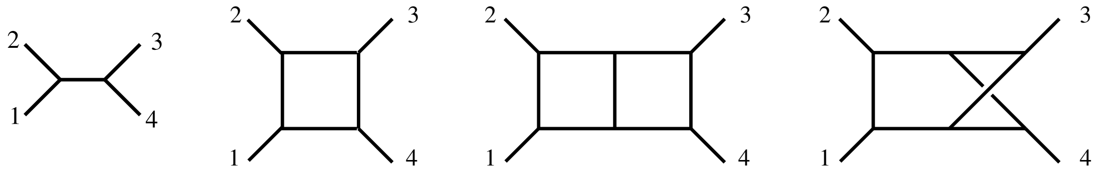

Figure 1. Tree-level, one-loop, and two-loop four-point color factors.

an inner product on the trace space, we determine the L-loop null space for SU(N), the orthogonal complement of the L-loop color space with respect to this inner product. In section 6 we generate the L-loop color space for SO(N), and in section 7 we obtain the complete set of x-type null vectors for SO(N). Section 8 briefly explains how the results from Sp(N) are related to those of SO(N). Section 9 concludes the paper, and some technical details are relegated to two appendices.

2 Trace and color spaces

In this section, we describe in some detail the trace and color spaces associated with color factors for SU(N), SO(N), and Sp(N) four-point amplitudes through two loops. This will set the stage for the subsequent discussion of all-loop color factors in the remainder of the paper. First, we describe the decomposition of L-loop color factors into the trace basis for each group. The span of these color factors gives the L-loop color space. We then break these color spaces into irreducible representations of S_4, the permutation group of the external legs of the amplitude, which allows for the most efficient representation of these spaces.

2.1 Trace basis decomposition of low-loop color factors

Color factors for amplitudes of fields in the adjoint representation are constructed by contracting structure constants $\tilde{f}^{abc}$ of the associated group. For example, for the four-point diagrams shown in figure 1, the s-channel tree-level color factor is given by

$$C^{(0)}_{1234} = \tilde{f}^{a_1a_2e}\tilde{f}^{a_3a_4e}\,, \tag{2.1}$$

the one-loop box color factor is

$$C^{(1)}_{1234} = \tilde{f}^{ea_1b}\tilde{f}^{ba_2c}\tilde{f}^{ca_3d}\tilde{f}^{da_4e}\,, \tag{2.2}$$

and the two-loop planar and nonplanar color factors are

$$C^{(2P)}_{1234} = \tilde{f}^{ea_1b}\tilde{f}^{ba_2c}\tilde{f}^{cgd}\tilde{f}^{dfe}\tilde{f}^{ga_3h}\tilde{f}^{ha_4f}\,, \tag{2.3}$$

$$C^{(2NP)}_{1234} = \tilde{f}^{ea_1b}\tilde{f}^{ba_2c}\tilde{f}^{cgd}\tilde{f}^{hfe}\tilde{f}^{ga_3h}\tilde{f}^{da_4f}\,. \tag{2.4}$$

The decomposition of these color factors into the trace basis is accomplished by rewriting the structure constants in terms of generators T^a in the defining representation using

$$\tilde{f}^{abc} = \mathrm{Tr}(T^a,[T^b,T^c])\,, \qquad [T^a,T^b] = \tilde{f}^{abc}T^c\,, \qquad \mathrm{Tr}(T^aT^b) = \delta^{ab}\,. \tag{2.5}$$

By repeatedly using trace identities specific to each gauge group (as described below), one can reduce all color factors to a linear combination of traces and products of traces of generators. In particular, four-point color factors may be written in terms of a six-dimensional basis $T_{[\lambda]}$ of single and double traces of generators

$$C = \sum_{\lambda=1}^{6} C_{[\lambda]}\, T_{[\lambda]}\,. \tag{2.6}$$

For $\mathrm{SU}(N)$, we will use the following basis[1]

$$\begin{aligned}
T_{[1]} &= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}), & T_{[4]} &= 2\,\mathrm{Tr}(T^{a_1}T^{a_3})\,\mathrm{Tr}(T^{a_2}T^{a_4}),\\
T_{[2]} &= \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}) + \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}), & T_{[5]} &= 2\,\mathrm{Tr}(T^{a_1}T^{a_4})\,\mathrm{Tr}(T^{a_2}T^{a_3}),\\
T_{[3]} &= \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3}) + \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}), & T_{[6]} &= 2\,\mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4})\,.
\end{aligned} \tag{2.7}$$

For $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$, the trace of a product B of generators is equal (up to a possible sign) to the trace of the generators written in reverse order B^R

$$\mathrm{Tr}(B^R) = (-1)^{n_B}\,\mathrm{Tr}(B) \qquad\qquad \text{for } \mathrm{SO}(N) \text{ and } \mathrm{Sp}(N) \tag{2.8}$$

where n_B denotes the number of factors in B, using eqs. (A.9) and (A.14) in appendix A. This implies that for $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$, the trace basis (2.7) simplifies to[2]

$$\begin{aligned}
T_{[1]} &= 2\,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,, & T_{[4]} &= 2\,\mathrm{Tr}(T^{a_1}T^{a_3})\,\mathrm{Tr}(T^{a_2}T^{a_4})\,,\\
T_{[2]} &= 2\,\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})\,, & T_{[5]} &= 2\,\mathrm{Tr}(T^{a_1}T^{a_4})\,\mathrm{Tr}(T^{a_2}T^{a_3})\,,\\
T_{[3]} &= 2\,\mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})\,, & T_{[6]} &= 2\,\mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4})\,.
\end{aligned} \tag{2.9}$$

Note that $\mathrm{Tr}(T^a) = 0$ for all the groups considered, so that there are no other terms in the four-point trace basis.

We now describe the process of decomposing the color factors shown in figure 1 into the trace basis (2.7) for $\mathrm{SU}(N)$ and (2.9) for $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$. For all groups G, the tree-level color factor (2.1) reduces to

$$C^{(0)}_{1234} = \mathrm{Tr}(T^{a_1},[T^{a_2},T^e])\tilde{f}^{a_3a_4e} = \mathrm{Tr}(T^{a_1},[T^{a_2},[T^{a_3},T^{a_4}]]) = T_{[1]} - T_{[2]} \tag{2.10}$$

where we have used eq. (2.5). Similarly, the one-loop color factor (2.2) becomes

$$C^{(1)}_{1234} = \mathrm{Tr}(T^e,[T^{a_1},T^b])\tilde{f}^{ba_2c}\tilde{f}^{ca_3d}\tilde{f}^{da_4e} = \mathrm{Tr}(T^e,[T^{a_1},[T^{a_2},[T^{a_3},[T^{a_4},T^e]]]]) \tag{2.11}$$

where we are left with a contraction over T^e. The remainder of the calculation depends on the group G. For $\mathrm{SU}(N)$, one uses the identities (A.5) valid for generators in the defining representation

$$\begin{aligned}
\mathrm{Tr}(AT^a)\,\mathrm{Tr}(BT^a) &= \mathrm{Tr}(AB) - \frac{1}{N}\,\mathrm{Tr}(A)\,\mathrm{Tr}(B)\,,\\
\mathrm{Tr}(AT^aBT^a) &= \mathrm{Tr}(A)\,\mathrm{Tr}(B) - \frac{1}{N}\,\mathrm{Tr}(AB)
\end{aligned} \tag{2.12}$$

[1]We are following the convention of ref. [16] rather than that of ref. [14]. By including factors of two in the double-trace terms, this basis generalizes more naturally to the trace basis for higher-point amplitudes [16].

[2]We retain the factors of 2 for consistency with eq. (2.7), but they may easily be removed.

where A and B are arbitrary products of generators. Then eq. (2.11) yields

$$C^{(1)}_{1234} = N T_{[1]} + T_{[4]} + T_{[5]} + T_{[6]} \qquad \text{for } \mathrm{SU}(N)\,. \tag{2.13}$$

For $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$, one uses instead the identities (A.10) and (A.15)

$$\begin{aligned} \mathrm{Tr}(AT^a)\,\mathrm{Tr}(BT^a) &= \frac{1}{2}\Big[\,\mathrm{Tr}(AB) - (-1)^{n_B}\,\mathrm{Tr}(AB^R)\Big]\,, \\ \mathrm{Tr}(AT^aBT^a) &= \frac{1}{2}\Big[\,\mathrm{Tr}(A)\,\mathrm{Tr}(B) \mp (-1)^{n_B}\,\mathrm{Tr}(AB^R)\Big] \end{aligned} \tag{2.14}$$

in which case eq. (2.11) reduces to

$$C^{(1)}_{1234} = \frac{1}{2}(N \mp 4) T_{[1]} \mp \Big(T_{[2]} + T_{[3]}\Big) + \frac{1}{2}\Big(T_{[4]} + T_{[5]} + T_{[6]}\Big) \quad \text{for } \mathrm{SO}(N) \text{ and } \mathrm{Sp}(N)\,. \tag{2.15}$$

The two-loop color factors (2.3) and (2.4) may be similarly reduced to the six-dimensional trace basis in this way.

It is convenient to represent a color factor in the trace basis (2.6) as a six-dimensional row vector

$$C = (C_{[1]},\ C_{[2]},\ C_{[3]};\ C_{[4]},\ C_{[5]},\ C_{[6]})\,. \tag{2.16}$$

Thus for $\mathrm{SU}(N)$, the tree-level, one-loop, and two-loop color factors are represented as

$$\begin{aligned} C^{(0)}_{1234} &= (1,\ -1,\ 0;\ 0,\ 0,\ 0)\,, \\ C^{(1)}_{1234} &= (N,\ 0,\ 0;\ 1,\ 1,\ 1)\,, \\ C^{(2P)}_{1234} &= (N^2+2,\ 2,\ -4;\ 0,\ 0,\ 3N)\,, \\ C^{(2NP)}_{1234} &= (2,\ 2,\ -4;\ -N,\ -N,\ 2N)\,. \end{aligned} \tag{2.17}$$

For $\mathrm{SO}(N)$, the tree-level, one-loop, and two-loop color factors are represented as

$$\begin{aligned} C^{(0)}_{1234} &= (1,\ -1,\ 0;\ 0,\ 0,\ 0)\,, \\ C^{(1)}_{1234} &= \frac{1}{2}(N-4,\ -2,\ -2;\ 1,\ 1,\ 1)\,, \\ C^{(2P)}_{1234} &= \frac{1}{4}(N^2-7N+16,\ -3N+12,\ -12;\ 2,\ 2,\ 3N-10)\,, \\ C^{(2NP)}_{1234} &= \frac{1}{4}(-N+8,\ -N+8,\ 2N-16;\ -N+4,\ -N+4,\ 2N-8)\,. \end{aligned} \tag{2.18}$$

For $\mathrm{Sp}(N)$, the tree-level, one-loop, and two-loop color factors are represented as

$$\begin{aligned} C^{(0)}_{1234} &= (1,\ -1,\ 0;\ 0,\ 0,\ 0)\,, \\ C^{(1)}_{1234} &= \frac{1}{2}(N+4,\ 2,\ 2;\ 1,\ 1,\ 1)\,, \\ C^{(2P)}_{1234} &= \frac{1}{4}(N^2+7N+16,\ 3N+12,\ -12;\ -2,\ -2,\ 3N+10)\,, \\ C^{(2NP)}_{1234} &= \frac{1}{4}(N+8,\ N+8,\ -2N-16;\ -N-4,\ -N-4,\ 2N+8)\,. \end{aligned} \tag{2.19}$$

We will use these results to decompose the color spaces into irreducible representations of S_4.

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2.2 Trace space and color space

As one can see from the low-loop examples (2.17)–(2.19) in the previous subsection, in an L-loop color factor of the form (2.16), the first three terms $C_{[1]}$, $C_{[2]}$, and $C_{[3]}$ are polynomials in N of maximal degree L and the second three terms $C_{[4]}$, $C_{[5]}$, and $C_{[6]}$ are polynomials in N of maximal degree $L-1$. Furthermore, for SU(N), $C_{[1]}$, $C_{[2]}$, and $C_{[3]}$ are polynomials of even/odd degree depending on whether L is even/odd, and vice versa for $C_{[4]}$, $C_{[5]}$, and $C_{[6]}$.

Color factors can be regarded as belonging to a vector space $V^{(L)}$, which we call the L-loop trace space, consisting of all such polynomials. Of course, an L^{th} degree polynomial may be regarded as an element of an $(L+1)$-dimensional vector space, whose components are given by the coefficients of the polynomial.[3] Thus the dimension of the L-loop trace space is

$$\dim V^{(L)} = \begin{cases} 3L+3 & \text{for SU}(N), \\ 6L+3 & \text{for SO}(N) \text{ and Sp}(N)\,. \end{cases} \tag{2.20}$$

In ref. [14], we defined an explicit basis for the trace space of SU(N), called the extended trace basis $t_\lambda^{(L)}$, whose elements were of the form $N^n T_{[\lambda]}$. Similarly, an extended trace basis for SO(N) and Sp(N) was defined in ref. [33]. In this paper, it is more convenient to express color factors in polynomial form.

The set of all L-loop color factors, formed from all possible cubic diagrams, spans a proper subspace of $V^{(L)}$. We call this subspace the L-loop color space. In the color space we must include all permutations of external legs of the color factors. For example, the tree-level color space includes not only the s-channel diagram shown in figure 1, but the t- and u-channel diagrams obtained by permutations of the external legs. Given the trace decomposition (2.16) of a particular cubic diagram, the trace decompositions of the same color factor with permutations of the external legs are given by

$$\begin{aligned}
C_{1234} &= (C_{[1]},\ C_{[2]},\ C_{[3]};\ C_{[4]},\ C_{[5]},\ C_{[6]})\,,\\
C_{1243} &= (C_{[2]},\ C_{[1]},\ C_{[3]};\ C_{[5]},\ C_{[4]},\ C_{[6]})\,,\\
C_{1342} &= (C_{[3]},\ C_{[1]},\ C_{[2]};\ C_{[6]},\ C_{[4]},\ C_{[5]})\,,\\
C_{1324} &= (C_{[3]},\ C_{[2]},\ C_{[1]};\ C_{[6]},\ C_{[5]},\ C_{[4]})\,,\\
C_{1423} &= (C_{[2]},\ C_{[3]},\ C_{[1]};\ C_{[5]},\ C_{[6]},\ C_{[4]})\,,\\
C_{1432} &= (C_{[1]},\ C_{[3]},\ C_{[2]};\ C_{[4]},\ C_{[6]},\ C_{[5]})
\end{aligned} \tag{2.21}$$

as may easily be seen by examining eqs. (2.7) and (2.9).

2.3 Irreducible subspaces

Because the set of all possible L-loop cubic diagrams necessarily includes all permutations of the external legs, the L-loop color space forms a (reducible) representation of S_4, the group of permutations of the external legs. This representation can be reduced to a set of irreducible one- and two-dimensional representations in the form of Kronecker products

$$[P,Q]\otimes u, \qquad [P,Q]\otimes x^i, \quad (i=1,2) \tag{2.22}$$

[3] If the polynomial is even or odd, the vector space has dimension $\lceil \frac{L+1}{2} \rceil$.

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where

$$u = (1, 1, 1), \qquad x^1 = (1, -1, 0), \qquad x^2 = (0, 1, -1), \tag{2.23}$$

and P and Q are polynomials in N of maximal degree L and $L-1$ respectively. That is,

$$\begin{aligned} [P, Q] \otimes u &\equiv (P, P, P; Q, Q, Q)\,, \\ [P, Q] \otimes x^1 &\equiv (P, -P, 0; Q, -Q, 0)\,, \\ [P, Q] \otimes x^2 &\equiv (0, P, -P; 0, Q, -Q)\,. \end{aligned} \tag{2.24}$$

The decomposition of the single- and double-trace bases into irreducible representations of S_n was described in detail in ref. [16].

An arbitrary color factor (2.16) may be decomposed into irreducible representations of S_4 as follows:

$$\begin{aligned} C = & \frac{1}{3}\Big(\, [C_{[1]} + C_{[2]} + C_{[3]}, C_{[4]} + C_{[5]} + C_{[6]}] \otimes u \\ & + [2C_{[1]} - C_{[2]} - C_{[3]}, 2C_{[4]} - C_{[5]} - C_{[6]}] \otimes x^1 \\ & + [C_{[1]} + C_{[2]} - 2C_{[3]}, C_{[4]} + C_{[5]} - 2C_{[6]}] \otimes x^2 \Big) \end{aligned} \tag{2.25}$$

as is easily verified using eq. (2.23). For example, the tree-level color factor (2.1) and its permutations are given by

$$\begin{aligned} C^{(0)}_{1234} &= (1, -1, 0; 0, 0, 0) = [1, 0] \otimes x^1\,, \\ C^{(0)}_{1342} &= (0, 1, -1; 0, 0, 0) = [1, 0] \otimes x^2\,, \\ C^{(0)}_{1423} &= (-1, 0, 1; 0, 0, 0) = [1, 0] \otimes (-x^1 - x^2)\,. \end{aligned} \tag{2.26}$$

These three color factors thus span a 2-dimensional representation $[1, 0] \otimes x^i$ of S_4. (They are not independent due to the Jacobi identity $C^{(0)}_{1234} + C^{(0)}_{1342} + C^{(0)}_{1423} = 0$.) The one-loop $\mathrm{SU}(N)$ color factor $C^{(1)}_{1234} = (N, 0, 0; 1, 1, 1)$ decomposes into

$$C^{(1)}_{1234} = \frac{1}{3}[N, 3] \otimes u + \frac{2}{3}[N, 0] \otimes x^1 + \frac{1}{3}[N, 0] \otimes x^2\,. \tag{2.27}$$

This color factor and its permutations

$$\begin{aligned} C^{(1)}_{1342} &= (0, N, 0; 1, 1, 1)\,, \\ C^{(1)}_{1423} &= (0, 0, N; 1, 1, 1) \end{aligned} \tag{2.28}$$

span a 3-dimensional representation of S_4 which reduces to a 1-dimensional representation $[N, 3] \otimes u$ and a 2-dimensional representation $[N, 0] \otimes x^i$. The two-loop planar and nonplanar $\mathrm{SU}(N)$ color factors (2.17) decompose into

$$\begin{aligned} C^{(2P)}_{1234} &= \frac{1}{3}[N^2, 3N] \otimes u + \frac{1}{3}[2N^2 + 6, -3N] \otimes x^1 + \frac{1}{3}[N^2 + 12, -6N] \otimes x^2\,, \\ C^{(2NP)}_{1234} &= [2, -N] \otimes x^1 + [4, -2N] \otimes x^2 \end{aligned} \tag{2.29}$$

so that the two-loop color space consists of the 1-dimensional representation $[N^2, 3N] \otimes u$ and two 2-dimensional representations $[N^2, 0] \otimes x^i$ and $[2, -N] \otimes x^i$.

Summarizing our results, we see that the low-loop SU(N) color spaces are spanned by

$$\begin{aligned}
\text{Tree-level:}\quad & [1, 0] \otimes x^i\,, \\
\text{One-loop:}\quad & [N, 3] \otimes u\,, \\
& [N, 0] \otimes x^i\,, \\
\text{Two-loop:}\quad & [N^2, 3N] \otimes u\,, \\
& [N^2, 0] \otimes x^i\,, \\
& [2, -N] \otimes x^i\,.
\end{aligned} \tag{2.30}$$

The same procedure employed for the SO(N) color factor spaces yields

$$\begin{aligned}
\text{Tree-level:}\quad & [1, 0] \otimes x^i\,, \\
\text{One-loop:}\quad & [N-8, 3] \otimes u\,, \\
& [N-2, 0] \otimes x^i\,, \\
\text{Two-loop:}\quad & [(N-2)(N-8), 3(N-2)] \otimes u\,, \\
& [(N-2)^2, 0] \otimes x^i\,, \\
& [N-8, N-4] \otimes x^i
\end{aligned} \tag{2.31}$$

and for the Sp(N) color spaces

$$\begin{aligned}
\text{Tree-level:}\quad & [1, 0] \otimes x^i\,, \\
\text{One-loop:}\quad & [N+8, 3] \otimes u\,, \\
& [N+2, 0] \otimes x^i\,, \\
\text{Two-loop:}\quad & [(N+2)(N+8), 3(N+2)] \otimes u\,, \\
& [(N+2)^2, 0] \otimes x^i\,, \\
& [N+8, N+4] \otimes x^i\,.
\end{aligned} \tag{2.32}$$

These results will be useful in generating the color space for an arbitrary loop amplitude.

We observe that the dimensions of the color spaces are 2, 3, and 5 for $L = 0$, 1, and 2, respectively for all three groups, as reflected in tables 1 and 2. We will see below that this equality between the groups breaks down for $L \geq 5$.

3 Iterative procedure

In this section, we review the iterative procedure introduced by one of the current authors in ref. [14] to generate a complete set of color factors at all loop orders. We then present a refined version of the iterative approach that takes into account the decomposition of color spaces into irreducible representations of S_4.

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The iterative approach involves attaching a rung between any two external legs of an L-loop color factor to generate an $(L+1)$-loop color factor. By considering all possible attachments of rungs, one generates the space of color factors at $(L+1)$ loops. The orthogonal complement of the color space in the trace space defines the null space, i.e., the space of null eigenvectors of the transformation matrix (1.5). Each null eigenvector then corresponds to a group-theory constraint on the color-ordered amplitudes.

Given an L-loop color factor $C^{a_1a_2a_3a_4}$, attaching a rung between external legs 1 and 2 yields an $(L+1)$-loop color factor given by

$$C^{a_1a_2a_3a_4} \longrightarrow \tilde{f}^{a_1b_1c}\tilde{f}^{cb_2a_2}C^{b_1b_2a_3a_4} \tag{3.1}$$

with similar expressions for the color factors obtained by attachments of rungs between other legs. To determine the effect of attaching rungs to an arbitrary color factor, we define an iterative matrix $G(e_{12}, e_{13}, e_{14})$ by attaching rungs between different pairs of legs of the trace basis $T_{[\lambda]}^{a_1a_2a_3a_4}$ and decomposing the result in the trace basis

$$\begin{aligned} &e_{12}\tilde{f}^{a_1b_1c}\tilde{f}^{cb_2a_2}T_{[\lambda]}^{b_1b_2a_3a_4} + e_{13}\tilde{f}^{a_1b_1c}\tilde{f}^{cb_3a_3}T_{[\lambda]}^{b_1a_2b_3a_4} + e_{14}\tilde{f}^{a_1b_1c}\tilde{f}^{cb_4a_4}T_{[\lambda]}^{b_1a_2a_3b_4} \\ &= \sum_\kappa G_{\lambda\kappa}(e_{12}, e_{13}, e_{14})T_{[\kappa]}^{a_1a_2a_3a_4}\,. \end{aligned} \tag{3.2}$$

Thus the coefficient of e_{12} gives the result of attaching a rung between legs 1 and 2, etc. (We need not consider the effect of attaching rungs between legs 2 and 3, etc., as they are redundant.) The 6×6 matrix $G_{\lambda\kappa}$ can be written in block diagonal form, with the N dependence made explicit:

$$G(e_{12}, e_{13}, e_{14}) = \begin{pmatrix} NA+E & B \\ C & ND+F \end{pmatrix} \tag{3.3}$$

where A through F are 3×3 matrices that depend on e_{1j}. For SU(N), one finds[4]

$$\begin{aligned} A &= \begin{pmatrix} e_{12}+e_{14} & 0 & 0 \\ 0 & e_{12}+e_{13} & 0 \\ 0 & 0 & e_{13}+e_{14} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & e_{14}-e_{13} & e_{12}-e_{13} \\ e_{13}-e_{14} & 0 & e_{12}-e_{14} \\ e_{13}-e_{12} & e_{14}-e_{12} & 0 \end{pmatrix}, \\ C &= 2\begin{pmatrix} 0 & e_{12}-e_{14} & e_{14}-e_{12} \\ e_{12}-e_{13} & 0 & e_{13}-e_{12} \\ e_{14}-e_{13} & e_{13}-e_{14} & 0 \end{pmatrix}, \qquad D = 2\begin{pmatrix} e_{13} & 0 & 0 \\ 0 & e_{14} & 0 \\ 0 & 0 & e_{12} \end{pmatrix}, \end{aligned} \tag{3.4}$$

with E and F vanishing. For SO(N) (upper sign) and Sp(N) (lower sign), one finds

$$\begin{aligned} A &= \frac{1}{2}\begin{pmatrix} e_{12}+e_{14} & 0 & 0 \\ 0 & e_{12}+e_{13} & 0 \\ 0 & 0 & e_{13}+e_{14} \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 0 & e_{14}-e_{13} & e_{12}-e_{13} \\ e_{13}-e_{14} & 0 & e_{12}-e_{14} \\ e_{13}-e_{12} & e_{14}-e_{12} & 0 \end{pmatrix}, \\ C &= 2\begin{pmatrix} 0 & e_{12}-e_{14} & e_{14}-e_{12} \\ e_{12}-e_{13} & 0 & e_{13}-e_{12} \\ e_{14}-e_{13} & e_{13}-e_{14} & 0 \end{pmatrix}, \quad D = \begin{pmatrix} e_{13} & 0 & 0 \\ 0 & e_{14} & 0 \\ 0 & 0 & e_{12} \end{pmatrix}, \end{aligned}$$

[4]These matrices differ slightly from those in ref. [14] because of the factors of two multiplying the double-trace basis elements. See footnote 1.

$$
\begin{aligned}
E &= \pm\frac{1}{2}\begin{pmatrix} 2e_{13}-3e_{12}-3e_{14} & e_{13}-e_{12} & e_{13}-e_{14} \\ e_{14}-e_{12} & 2e_{14}-3e_{12}-3e_{13} & e_{14}-e_{13} \\ e_{12}-e_{14} & e_{12}-e_{13} & 2e_{12}-3e_{13}-3e_{14} \end{pmatrix}, \\
F &= \mp 2\begin{pmatrix} e_{13} & 0 & 0 \\ 0 & e_{14} & 0 \\ 0 & 0 & e_{12} \end{pmatrix}.
\end{aligned} \tag{3.5}
$$

The effect of attaching rungs to an arbitrary color factor $C = \sum_\lambda C_{[\lambda]} T_{[\lambda]}$ results in

$$
C_{[\lambda]} \longrightarrow \sum_\kappa C_{[\kappa]}\; G_{\kappa\lambda}(e_{12}, e_{13}, e_{14}) \tag{3.6}
$$

that is, one multiplies the row vector C by the matrix G. To give some simple examples, attaching a rung between legs 1 and 4 of the tree-level s-channel diagram (2.1) yields the one-loop box diagram (2.2) so that

$$
C^{(0)}_{1234} G(0,0,1) = C^{(1)}_{1234} \tag{3.7}
$$

while attaching a rung between legs 1 and 2 of the one-loop box diagram yields the two-loop planar diagram (2.3) so that

$$
C^{(1)}_{1234} G(1,0,0) = C^{(2P)}_{1234}\,. \tag{3.8}
$$

These may be confirmed using eqs. (2.30)–(2.32) and (3.3)–(3.5).

3.1 Iterative matrices for irreducible representations of S_4

We explained in section 2.3 how color factors may be written in terms of irreducible representations of S_4:

$$
[P,Q]\otimes u, \qquad [P,Q]\otimes x^i \qquad (i=1,2)\,. \tag{3.9}
$$

In general $G(e_{12}, e_{13}, e_{14})$ will act on these color factors to produce linear combinations of u and x^i types, but we may define G matrices for certain choices of the parameters e_{12}, e_{13}, and e_{14} that produce pure u and x^i types. One may then write the action of G in terms of four 2×2 matrices g_1, g_{ux}, g_{xu}, and g_{xx} which act on the two-dimensional row vector $[P,Q]$. This gives a refined approach to generate the color space for any L in terms of u- and x^i-type irreducible representations.

First we choose $e_{12} = e_{13} = e_{14} = \frac{1}{2}e$ which makes $G(e_{12}, e_{13}, e_{14})$ proportional to the unit matrix, mapping u-type color factors to u-type, and x^i-type to x^i-type:

$$
\begin{aligned}
([P,Q]\otimes u)\, G\left(\frac{1}{2}e, \frac{1}{2}e, \frac{1}{2}e\right) &= [P,Q] g_1 \otimes u\,, \\
\left([P,Q]\otimes x^i\right) G\left(\frac{1}{2}e, \frac{1}{2}e, \frac{1}{2}e\right) &= [P,Q] g_1 \otimes x^i\,.
\end{aligned} \tag{3.10}
$$

One may verify that

$$
g_1 = e\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} \quad \text{for SU}(N), \qquad g_1 = \frac{1}{2}e\begin{pmatrix} N\mp 2 & 0 \\ 0 & N\mp 2 \end{pmatrix} \quad \text{for} \begin{cases} \text{SO}(N) \\ \text{Sp}(N) \end{cases}. \tag{3.11}
$$

Another choice of e_{1j} takes u-type color factors to x^i-type color factors:

$$\begin{aligned}([P,Q]\otimes u)\,G(0,-e,e) &= [P,Q]g_{ux}\otimes x^1\,,\\ ([P,Q]\otimes u)\,G(e,0,-e) &= [P,Q]g_{ux}\otimes x^2\,.\end{aligned} \tag{3.12}$$

In this case, one finds

$$g_{ux} = e\begin{pmatrix} N & -3 \\ 6 & -2N\end{pmatrix} \quad \text{for SU}(N), \qquad g_{ux} = \frac{1}{2}e\begin{pmatrix} N\mp 5 & -3 \\ 12 & -2N\pm 4\end{pmatrix} \quad \text{for} \begin{cases}\text{SO}(N)\\ \text{Sp}(N)\end{cases}. \tag{3.13}$$

Yet another choice of e_{1j} takes x^i-type color factors to x^i-type:

$$\begin{aligned}\left([P,Q]\otimes x^1\right)G(e,0,0) &= [P,Q]g_{xx}\otimes x^1\,,\\ \left([P,Q]\otimes x^2\right)G(0,e,0) &= [P,Q]g_{xx}\otimes x^2\,.\end{aligned} \tag{3.14}$$

One then obtains

$$g_{xx} = e\begin{pmatrix} N & 0 \\ -2 & 0\end{pmatrix} \quad \text{for SU}(N), \qquad g_{xx} = \frac{1}{2}e\begin{pmatrix} N\mp 2 & 0 \\ -4 & 0\end{pmatrix} \quad \text{for} \begin{cases}\text{SO}(N)\\ \text{Sp}(N)\end{cases}. \tag{3.15}$$

Finally one must act on the two x^i-type color factors with different choices of e_{1j} to obtain a u-type color factor

$$\left([P,Q]\otimes x^1\right)G(0,-e,e) + \left([P,Q]\otimes x^2\right)G(e,-e,0) = [P,Q]g_{xu}\otimes u\,. \tag{3.16}$$

One then obtains

$$g_{xu} = e\begin{pmatrix} N & 3 \\ 0 & -2N\end{pmatrix} \quad \text{for SU}(N), \qquad g_{xu} = \frac{1}{2}e\begin{pmatrix} N\mp 8 & 3 \\ 0 & -2N\pm 4\end{pmatrix} \quad \text{for} \begin{cases}\text{SO}(N)\\ \text{Sp}(N)\end{cases}. \tag{3.17}$$

The iterative matrices g_1, g_{ux}, g_{xu}, and g_{xx} will be used to generate the L-loop color spaces for SU(N) in section 4 and for SO(N) in section 6. In section 8, we will show that the L-loop color spaces for Sp(N) are obtained from those for SO(N) by some simple sign changes.

4 L-loop SU(N) color space

The goal of this section is to explicitly construct the space of L-loop color factors for SU(N). As already discussed in section 2.3, an L-loop color factor may be expressed in terms of one- and two-dimensional irreducible representations of S_4 as

$$[P,Q]\otimes u, \qquad [P,Q]\otimes x^i \qquad (i=1,2) \tag{4.1}$$

where P and Q are polynomials in N of maximal degree L and $L-1$ respectively. For SU(N) color factors, the polynomials P are of even/odd degree depending on whether L is even/odd, and vice versa for Q. Thus, L-loop SU(N) color factors inhabit a vector space $V^{(L)}$ of dimension $3L+3$ (the trace space). The polynomials P and Q corresponding to color factors, however, are not completely arbitrary but satisfy certain constraints. Consequently, the set of all L-loop color factors spans a proper subspace (the color space) of $V^{(L)}$.

In this section, we iteratively construct an explicit basis for the L-loop SU(N) color space, beginning with the single tree-level irreducible representation $[1,0]\otimes x^i$ and acting repeatedly with the iterative matrices for SU(N) obtained in section 3

$$g_1 = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}, \qquad g_{xx} = \begin{pmatrix} N & 0 \\ -2 & 0 \end{pmatrix}, \qquad g_{xu} = \begin{pmatrix} N & 3 \\ 0 & -2N \end{pmatrix}, \qquad g_{ux} = \begin{pmatrix} N & -3 \\ 6 & -2N \end{pmatrix} \tag{4.2}$$

where we have chosen to set $e=1$. These 2×2 matrices act on the $[P,Q]$ part of the color factor, while the subscripts indicate their action on the x or u part of the color factor. Specifically:

1. g_{xx} takes an x-type color factor to an x-type color factor,
2. g_{xu} takes an x-type color factor to a u-type color factor,
3. g_{ux} takes a u-type color factor to an x-type color factor,
4. g_1 takes x to x and u to u.

We will show that these matrices generate a basis consisting of polynomials multiplied by one of four specific (linearly independent) types:

$$\begin{aligned} x_a &\equiv [1,0]\otimes x^i\,, \\ x_b &\equiv [2,-N]\otimes x^i\,, \\ u_a &\equiv [N,3]\otimes u\,, \\ u_b &\equiv [N,N^2+3]\otimes u\,. \end{aligned} \tag{4.3}$$

Hints of these types have already appeared in eq. (2.30). Our first step is to ascertain how each of the operators (4.2) act on the types of color factors (4.3). First, the operator g_1 just rescales each type by N

$$\begin{aligned} x_a g_1 &= N x_a\,, \\ x_b g_1 &= N x_b\,, \\ u_a g_1 &= N u_a\,, \\ u_b g_1 &= N u_b\,. \end{aligned} \tag{4.4}$$

Second, the operator g_{xx} acts on the x-type color factors as

$$\begin{aligned} x_a g_{xx} &= N x_a\,, \\ x_b g_{xx} &= 4N x_a\,. \end{aligned} \tag{4.5}$$

Since the action of g_{xx} on x_a is identical to the action of g_1 (and therefore redundant), we will restrict our attention to its action on x_b, defining $g_{ba} = \frac{1}{4} g_{xx}$ with

$$x_b g_{ba} = N x_a\,. \tag{4.6}$$

Third, the operator g_{xu} takes an x_a-type color factor to a u_a-type color factor, and an x_b-type color factor to a u_b-type color factor:

$$\begin{aligned} x_a g_{xu} &= u_a \,, \\ x_b g_{xu} &= 2u_b \,. \end{aligned} \tag{4.7}$$

Finally, the operator g_{ux} acts on u-type color factors to give linear combinations of x_a and x_b types:

$$\begin{aligned} u_a g_{ux} &= N^2 x_a + 9x_b \,, \\ u_b g_{ux} &= 3N^2 x_a + (2N^2 + 9)x_b \,. \end{aligned} \tag{4.8}$$

With these in hand, we now generate the SU(N) color space through three-loop order. We begin with the single tree-level irreducible representation

$$\text{Tree level:} \qquad x_a \,. \tag{4.9}$$

Acting on x_a with g_1 using eq. (4.4) and with g_{xu} using eq. (4.7), we obtain the three-dimensional space spanned by two irreducible representations

$$\text{One loop:} \qquad N x_a, \quad u_a \,. \tag{4.10}$$

We then act on each of these one-loop color factors with g_1 to obtain $N^2 x_a$ and $N u_a$. The action of g_{xu} on $N x_a$ is redundant, but we can act on the u_a-type color factor with g_{ux} to obtain $[N^2 + 18, -9N] \otimes x^i$, which is a linear combination of x_a and x_b types, as shown in eq. (4.8). Since we already have $N^2 x_a$ in the color space, we subtract it and divide by 9 to obtain x_b. Thus the two-loop color space is five-dimensional, spanned by three irreducible representations

$$\text{Two loops:} \qquad N^2 x_a \,, \quad N u_a \,, \quad x_b \,. \tag{4.11}$$

It is reassuring that eqs. (4.10) and (4.11) agree with the results we obtained earlier in eq. (2.30). The three-loop color factors are then obtained by acting on each of the two-loop color factors with g_1. The action of g_{ux} on $N u_a$ is redundant. We can also act on x_b with g_{ba} using eq. (4.6) and with g_{xu} using eq. (4.7). The three-loop color space is thus eight-dimensional, spanned by five irreducible representations

$$\text{Three loops:} \qquad N^3 x_a, \quad N x_a, \quad N^2 u_a, \quad N x_b, \quad u_b \,. \tag{4.12}$$

We now make some general observations that allow us to determine the complete span of color factors at arbitrary loop order L.

(Observation 1). *All L-loop u-type color factors are generated by the action of g_{xu} on the complete set of x-type color factors at $(L-1)$ loops using eq. (4.7).* The only possible exception would be through g_1 acting on an $(L-1)$-loop u-type color factor. But since (by hypothesis) the latter can be obtained through g_{xu} acting on an $(L-2)$-loop x-type color factor, and since g_1 commutes with g_{xu}, the same color factor can obtained by the action of g_{xu} on an $(L-1)$-loop x-type color factor.

(Observation 2). *All L-loop x-type color factors are obtained from x-type color factors at $(L-1)$ and $(L-2)$ loops.* To see this, observe that all L-loop x-type color factors are obtained from $(L-1)$-loop color factors through the action of g_1, g_{ba}, and g_{ux}. However, from observation (1), any color factor obtained using g_{ux} on an $(L-1)$-loop u-type color factor can be obtained directly from an $(L-2)$-loop x-type color factor using $g_{xu}g_{ux}$. This action typically produces a linear combination of x_a- and x_b-type factors, so it will be useful to replace $g_{xu}g_{ux}$ with two-step operators (i.e., ones that map $(L-2)$-loop x-type color factors to L-loop x-type color factors) that produce color factors of pure type:

$$g_{ab}^{(2)} = \frac{1}{9}\left(g_{xu}g_{ux} - g_1^2\right) ,$$

$$g_{bb}^{(2)} = \frac{1}{18}\left(g_{xu}g_{ux} - 4g_1^2 - 6g_{ba}g_1\right) . \tag{4.13}$$

These act on x_a- and x_b-type color factors respectively at $(L-2)$ loops to yield x_b-type color factors at L loops

$$x_a g_{ab}^{(2)} = x_b ,$$

$$x_b g_{bb}^{(2)} = x_b \tag{4.14}$$

which are easily verified using eqs. (4.4)–(4.8). Thus we have shown that all L-loop x-type color factors may be obtained from x-type color factors at $(L-1)$ and $(L-2)$ loops through the action of the four operators g_1, g_{ba}, $g_{ab}^{(2)}$, and $g_{bb}^{(2)}$.

(Observation 3). *All L-loop x_a-type color factors can be obtained from g_{ba} acting on an $(L-1)$-loop x_b-type color factor using eq. (4.6) with one exception, namely $N^L x_a$, which results from g_1 acting repeatedly on the tree-level color factor x_a.* From observation (2), all x-type color factors are obtained from $(L-1)$-loop x-type color factors using g_1, g_{ba}, $g_{ab}^{(2)}$, and $g_{bb}^{(2)}$, but the last two always land on x_b-type color factors. Because g_1 commutes with g_{ba}, the action of g_1 on an $(L-1)$-loop x_a-type color factor that is obtained from g_{ba} acting on an $(L-2)$-loop x_b-type color factor can also be obtained by g_{ba} acting on an $(L-1)$-loop x_b-type color factor. The remaining possibility is an x_a-type color factor obtained through g_1 acting L times on the tree-level color factor x_a.

(Observation 4). *All L-loop x_b-type color factors for $L > 2$ can be obtained from g_1 and $g_{bb}^{(2)}$ acting on x_b-type color factors at $L-1$ and $L-2$ loops.* From observation (2), we know that all L-loop x_b-type color factors may be obtained from g_1 and $g_{bb}^{(2)}$ acting on x_b-type color factors at $L-1$ and $L-2$ loops respectively, and $g_{ab}^{(2)}$ acting on x_a-type color factors at $L-2$ loops. From observation (3), the latter may be replaced (with one possible exception discussed below) by $g_{ba}g_{ab}^{(2)}$ acting on an x_b-type color factor at $L-3$ loops. However, observe using eqs. (4.6) and (4.14) that

$$x_b g_{ba} g_{ab}^{(2)} = N x_a g_{ab}^{(2)} = N x_b = x_b g_1 g_{bb}^{(2)} \tag{4.15}$$

thus the same color factor is obtained using g_1 and $g_{bb}^{(2)}$ alone. The possible exception mentioned above is $g_{ab}^{(2)}$ acting on $N^{L-2}x_a$. However since

$$N^{L-2} x_a g_{ab}^{(2)} = x_a g_1^{L-2} g_{ab}^{(2)} = x_a g_{ab}^{(2)} g_1^{L-2} = x_b g_1^{L-2} \tag{4.16}$$

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this is equivalent to g_1 acting repeatedly on the two-loop x_b-type color factor. Thus, we have shown that all x_b-type color factors for $L > 2$ can be generated by the action of two (commuting) operators

$$
\begin{aligned}
x_b g_1 &= N x_b \,, \\
x_b g_{bb}^{(2)} &= x_b
\end{aligned}
\tag{4.17}
$$

on lower-loop x_b-type color factors.

From the observations above, we now determine the complete set of L-loop color factors. The most general L-loop x_b-type color factor is obtained by acting with an arbitrary combination of g_1 and $g_{bb}^{(2)}$ on the two-loop color factor x_b to give

$$
x_b g_1^{n_1} g_{bb}^{(2)n_2} = N^{n_1} x_b \qquad \text{where} \qquad n_1 + 2n_2 = L - 2 \tag{4.18}
$$

where n_1 and n_2 are non-negative integers. Thus, the space of L-loop x_b-type color factors is spanned by $\lfloor L/2 \rfloor$ irreducible representations

$$
N^n x_b \,, \qquad 0 \le n \le L-2, \qquad n = L \mod 2\,. \tag{4.19}
$$

Using observation (3), the space of L-loop x_a-type color factors (for $L \ge 1$) is spanned by $\lfloor (L+1)/2 \rfloor$ irreducible representations

$$
N^n x_a \,, \qquad 1 \le n \le L, \qquad n = L \mod 2\,. \tag{4.20}
$$

Taking eqs. (4.19) and (4.20) together, the number of L-loop x-type irreducible representations (for $L \ge 1$) is given by L.

Using observation (1), the space of L-loop u_a-type color factors (for $L \ge 2$) is spanned by $\lfloor L/2 \rfloor$ irreducible representations

$$
N^n u_a \,, \qquad 1 \le n \le L-1, \qquad n = L-1 \mod 2 \tag{4.21}
$$

and the space of L-loop u_b-type color factors is spanned by $\lfloor (L-1)/2 \rfloor$ irreducible representations

$$
N^n u_b \,, \qquad 0 \le n \le L-3, \qquad n = L-1 \mod 2\,. \tag{4.22}
$$

Taking eqs. (4.21) and (4.22) together, the number of L-loop u-type irreducible representations (for $L \ge 2$) is given by $L-1$.

Table 3 summarizes the counting of irreducible representations spanning the L-loop color space for each value of L. The total dimension of the L-loop color space given in the last row is the sum of these basis elements, taking into account that x-type elements are two-dimensional representations (of S_4) while u-type elements are one-dimensional.

5 L-loop SU(N) null space

In the previous section, we generated a complete set of color factors spanning the L-loop color space for SU(N), which (for $L \ge 2$) is a $(3L-1)$-dimensional subspace of the $3L+3$ dimensional trace space. In this section, we will determine the vectors that span the L-loop null space, which is the four-dimensional orthogonal complement to the L-loop color space. This will consist of two u-type null vectors and one x-type irreducible representation. To do this, we first need to define an inner project on the trace space.

# of loops L	0	1	2	3	4	5	6	$L \geq 2$
# of x_a-type irreps	1	1	1	2	2	3	3	$\lfloor (L+1)/2 \rfloor$
# of x_b-type irreps	0	0	1	1	2	2	3	$\lfloor L/2 \rfloor$
# of u_a-type irreps	0	1	1	1	2	2	3	$\lfloor L/2 \rfloor$
# of u_b-type irreps	0	0	0	1	1	2	2	$\lfloor (L-1)/2 \rfloor$
total # of color factors	2	3	5	8	11	14	17	$3L-1$

Table 3. Number of irreducible representations spanning the L-loop color space for SU(N).

5.1 Inner product

To define an inner project, we need to represent color factors in a slightly different way. Up to this point, we have represented a color factor as a six-dimensional vector

$$C = (C_{[1]},\ C_{[2]},\ C_{[3]};\ C_{[4]},\ C_{[5]},\ C_{[6]}) \tag{5.1}$$

whose coefficients are polynomials in N. We now express each of these polynomials as a vector. An L^{th} degree polynomial may be written as an infinite-dimensional row vector

$$P(N) = \sum_{\ell=0}^{L} P_\ell N^\ell \quad \to \quad \mathbf{P} = (P_0, P_1, P_2, \cdots, P_L, 0, \cdots) \tag{5.2}$$

with all but the first $L+1$ entries of $\mathbf{P}$ vanishing, that is

$$P \text{ is an } L^{\text{th}} \text{ degree polynomial} \quad \Longrightarrow \quad \mathbf{P} = \mathbf{P}\Pi_L \quad \text{where} \quad \Pi_L = \begin{pmatrix} \mathbf{1}_{(L+1)\times(L+1)} & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.3}$$

We observe that for SU(N), where the polynomials are of even or odd degree, every other entry of $\mathbf{P}$ vanishes. We may compactly express eq. (5.2) as

$$P(N) = \mathbf{P}\mathbf{N}^T, \qquad \text{where} \qquad \mathbf{N} = (1, N, N^2, \cdots)\,. \tag{5.4}$$

Given two polynomials $P = \mathbf{P}\mathbf{N}^T$ and $P' = \mathbf{P}'\mathbf{N}^T$, we may define a natural inner product $\langle P'|P\rangle$ by

$$\langle P'|P\rangle = \mathbf{P}'\mathbf{P}^T\,. \tag{5.5}$$

Extending this definition to color factors (5.1) we have

$$\langle C'|C\rangle = \sum_{\lambda=1}^{6} \mathbf{C}'_{[\lambda]}\mathbf{C}_{[\lambda]}{}^T \tag{5.6}$$

where $C_{[\lambda]} = \mathbf{C}_{[\lambda]}\mathbf{N}^T$. If the color factor has the form $C = [P, Q] \otimes v$ where $v = u$, x^1, or x^2, then the inner product become

$$\langle C'|C\rangle = \left(\mathbf{P}'\mathbf{P}^T + \mathbf{Q}'\mathbf{Q}^T\right)\gamma_{v'v} \quad \text{where} \quad \gamma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{5.7}$$

The main point here is that u-type color factors are orthogonal to x-type color factors. Since we are only using the inner product to determine orthogonality, we will ignore the $\gamma_{v'v}$ piece and redefine

$$\langle C'|C\rangle = \left(\mathbf{P}'\mathbf{P}^T + \mathbf{Q}'\mathbf{Q}^T\right)\delta_{v'v} \quad \text{where} \quad v = u \text{ or } x\,. \tag{5.8}$$

Next we observe that the color factors are of the form $C = c[p,q]\otimes v$, where p and q are (at most) degree-two polynomials, $p = p_0 + p_1 N + p_2 N^2$ and $q = q_0 + q_1 N + q_2 N^2$, and c is a common factor of P and Q. Then the associated row vectors satisfy

$$P = cp \implies \mathbf{P} = \mathbf{c}\mathcal{P} \quad \text{where} \quad \mathcal{P} = \begin{pmatrix} p_0 & p_1 & p_2 & 0 & \cdots \\ 0 & p_0 & p_1 & p_2 & \cdots \\ 0 & 0 & p_0 & p_1 & \cdots \\ 0 & 0 & 0 & p_0 & \cdots \end{pmatrix},$$
$$Q = cq \implies \mathbf{Q} = \mathbf{c}\mathcal{Q} \quad \text{where} \quad \mathcal{Q} = \begin{pmatrix} q_0 & q_1 & q_2 & 0 & \cdots \\ 0 & q_0 & q_1 & q_2 & \cdots \\ 0 & 0 & q_0 & q_1 & \cdots \\ 0 & 0 & 0 & q_0 & \cdots \end{pmatrix}. \tag{5.9}$$

The inner product (5.8) between color factors $C = c[p,q]\otimes v$ and $C' = c'[p',q']\otimes v$ becomes

$$\langle C'|C\rangle = \mathbf{c}' M \mathbf{c}^T \delta_{v'v} \quad \text{where} \quad M = \mathcal{P}'\mathcal{P}^T + \mathcal{Q}'\mathcal{Q}^T\,. \tag{5.10}$$

We are interested in finding a set of null vectors R which are orthogonal to the color factors. If R has the form

$$R = r[\tilde{p},\tilde{q}]\otimes v \tag{5.11}$$

then its inner product with a color factor $C = c[p,q]\otimes v'$ is

$$\langle C|R\rangle = \mathbf{c} M \mathbf{r}^T \delta_{v'v} \quad \text{where} \quad M = \mathcal{P}\tilde{\mathcal{P}}^T + \mathcal{Q}\tilde{\mathcal{Q}}^T \tag{5.12}$$

where $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ are defined analogously to eq. (5.9). An astute choice of $\tilde{p}$ and $\tilde{q}$ can ensure orthogonality of R and C. In particular, for degree one polynomials $p = p_0 + p_1 N$ and $q = q_0 + q_1 N$, we define $\tilde{p} = q_1 + q_0 N$ and $\tilde{q} = -p_1 - p_0 N$ (possibly up to an overall sign for both). For degree two polynomials $p = p_0 + p_1 N + p_2 N^2$ and $q = q_0 + q_1 N + q_2 N^2$, we define $\tilde{p} = q_2 + q_1 N + q_0 N^2$ and $\tilde{q} = -p_2 - p_1 N - p_0 N^2$ (again possibly up to an overall sign). Under these conditions, one may easily verify that the matrix M in eq. (5.12) automatically vanishes, so that $\langle C|R\rangle = 0$. This will be useful in defining the null space.

5.2 SU(N) null vectors

In section 4, we determined a complete set of color factors that span the L-loop color space for SU(N), namely,

$$\begin{aligned}
C_{xa}^{(L)} &= c_{xa}^{(L)} x_a, & c_{xa}^{(L)} &\in \{N^n \mid 1 \le n \le L, & n &= L \mod 2\},\\
C_{xb}^{(L)} &= c_{xb}^{(L)} x_b, & c_{xb}^{(L)} &\in \{N^n \mid 0 \le n \le L-2, & n &= L \mod 2\},\\
C_{ua}^{(L)} &= c_{ua}^{(L)} u_a, & c_{ua}^{(L)} &\in \{N^n \mid 1 \le n \le L-1, & n &= L-1 \mod 2\},\\
C_{ub}^{(L)} &= c_{ub}^{(L)} u_b, & c_{ub}^{(L)} &\in \{N^n \mid 0 \le n \le L-3, & n &= L-1 \mod 2\}
\end{aligned} \tag{5.13}$$

where we recall that

$$
\begin{aligned}
x_a &= [1,0] \otimes x^i\,, \\
x_b &= [2,-N] \otimes x^i\,, \\
u_a &= [N,3] \otimes u\,, \\
u_b &= [N,N^2+3] \otimes u\,.
\end{aligned}
\tag{5.14}
$$

In this subsection, we will obtain a complete set of L-loop null vectors $R^{(L)}$, defined to be orthogonal to the set (5.13) with respect to the inner project defined in the previous subsection. We will show that the SU(N) null vectors can be of four possible types, namely,

$$
\begin{aligned}
x_\alpha &= [0,1] \otimes x^i\,, \\
x_\beta &= [1,2N] \otimes x^i\,, \\
u_\alpha &= [3N,-1] \otimes x^i\,, \\
u_\beta &= [3N^2+1,-N] \otimes x^i\,.
\end{aligned}
\tag{5.15}
$$

These are chosen, using the prescription from the previous subsection, so that x_α-type null vectors are automatically orthogonal to x_a-type color factors, x_β-type null vectors orthogonal to x_b-type color factors, etc. Also x-type null vectors are automatically orthogonal to u-type color factors, and vice versa. We use the remaining orthogonality condition to fully determine the form of the null vectors.

(1) x_α-type null vectors. Consider an L-loop null vector of the form

$$
R^{(L)}_{x\alpha} = r^{(L)}_{x\alpha} x_\alpha \tag{5.16}
$$

where $r^{(L)}_{x\alpha}$ is a polynomial in N of maximal degree $L-1$ and is odd/even for L even/odd. As we just remarked, orthogonality to $C^{(L)}_{xa}$, $C^{(L)}_{ua}$, and $C^{(L)}_{ub}$ is automatic from the definition of x_α. To impose the final orthogonality condition, we compute

$$
\langle C^{(L)}_{xb} | R^{(L)}_{x\alpha} \rangle = \mathbf{c}^{(L)}_{xb} M \mathbf{r}^{(L)T}_{x\alpha}, \qquad M = \begin{pmatrix} 0 & -1 & 0 & \cdots \\ 0 & 0 & -1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.17}
$$

where M is defined by eq. (5.12), using the $\mathcal{P}$ and $\mathcal{Q}$ matrices appropriate to x_b and x_α. Requiring $\langle C^{(L)}_{xb} | R^{(L)}_{x\alpha} \rangle = 0$ for any $c^{(L)}_{xb}$ belonging to the set defined in eq. (5.13), we find that $r^{(L)}_{x\alpha}$ must vanish if L is even, whereas for L odd, the only null vector is $R^{(L)}_{x\alpha} = x_\alpha$.

(2) x_β-type null vectors. Next consider an L-loop null vector of the form

$$
R^{(L)}_{x\beta} = r^{(L)}_{x\beta} x_\beta \tag{5.18}
$$

where $r^{(L)}_{x\beta}$ is a polynomial in N of maximal degree $L-2$ and is even/odd for L even/odd. Orthogonality to $C^{(L)}_{xb}$, $C^{(L)}_{ua}$, and $C^{(L)}_{ub}$ is automatic from the definition of x_β. To impose the final orthogonality condition, we compute

$$\langle C_{xa}^{(L)} | R_{x\beta}^{(L)} \rangle = \mathbf{c}_{xa}^{(L)} M \mathbf{r}_{x\beta}^{(L)T}, \qquad M = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.19}$$

where M is defined by eq. (5.12), using the $\mathcal{P}$ and $\mathcal{Q}$ matrices appropriate to x_a and x_β. Requiring $\langle C_{xa}^{(L)} | R_{x\beta}^{(L)} \rangle = 0$ for any $c_{xa}^{(L)}$ belonging to the set defined in eq. (5.13), we see that $r_{x\beta}^{(L)}$ must vanish if L is odd, whereas for L even, the only null vector is $R_{x\beta}^{(L)} = x_\beta$.

(3) $\boldsymbol{u_\alpha}$-type null vectors. Next consider an L-loop null vector of the form

$$R_{u\alpha}^{(L)} = r_{u\alpha}^{(L)} u_\alpha \tag{5.20}$$

where $r_{u\alpha}^{(L)}$ is a polynomial in N of maximal degree $L-1$ and is odd/even for L even/odd. Orthogonality to $C_{xa}^{(L)}$, $C_{xb}^{(L)}$, and $C_{ua}^{(L)}$ is automatic from the definition of u_α. To impose the final orthogonality condition, we compute

$$\langle C_{ub}^{(L)} | R_{u\alpha}^{(L)} \rangle = \mathbf{c}_{ub}^{(L)} M \mathbf{r}_{u\alpha}^{(L)T}, \qquad M = \begin{pmatrix} 0 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.21}$$

where M is defined by eq. (5.12), using the $\mathcal{P}$ and $\mathcal{Q}$ matrices appropriate to u_b and u_α. Requiring $\langle C_{ub} | R_{u\alpha}^{(L)} \rangle = 0$ for any $c_{ub}^{(L)}$ belonging to the set defined in eq. (5.13) yields the null vector $R_{u\alpha}^{(L)} = N u_\alpha$ for even L, and $R_{u\alpha}^{(L)} = u_\alpha$ for odd L.

(4) $\boldsymbol{u_\beta}$-type null vectors. Finally consider an L-loop null vector of the form

$$R_{u\beta}^{(L)} = r_{u\beta}^{(L)} u_\beta \tag{5.22}$$

where $r_{u\beta}$ is a polynomial in N of maximal degree $L-2$ and is even/odd for L even/odd. Orthogonality to $C_{xa}^{(L)}$, $C_{xb}^{(L)}$, and $C_{ub}^{(L)}$ is automatic from the definition of u_β. To impose the final orthogonality condition, we compute

$$\langle C_{ua}^{(L)} | R_{u\beta}^{(L)} \rangle = \mathbf{c}_{ua}^{(L)} M \mathbf{r}_{u\beta}^{(L)T}, \qquad M = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix}. \tag{5.23}$$

where M is defined by eq. (5.12), using the $\mathcal{P}$ and $\mathcal{Q}$ matrices appropriate to u_a and u_β. Requiring $\langle C_{ua}^{(L)} | R_{u\beta}^{(L)} \rangle = 0$ for any $c_{ua}^{(L)}$ belonging to the set defined in eq. (5.13), yields the null vector $R_{u\beta}^{(L)} = u_\beta$ for even L, and $R_{u\beta}^{(L)} = N u_\beta$ for odd L.

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Even-loop null space. To summarize the results of this section, the L-loop null space for even L (with $L \geq 2$) is spanned by

$$\text{Even loop } (L \geq 2): \qquad x_\beta, \qquad N u_\alpha, \qquad u_\beta\,. \tag{5.24}$$

We may replace u_β with $u_\beta - N u_\alpha = [1,0] \otimes u$, and write the null vectors explicitly as

$$\text{Even loop } (L \geq 2): \qquad [1,2N] \otimes x^i, \qquad [3N^2, -N] \otimes u, \qquad [1,0] \otimes u\,. \tag{5.25}$$

At tree level, the only null vector is $[1,0] \otimes u$.

Odd-loop null space. The L-loop null space for odd L (with $L \geq 3$) is spanned by

$$\text{Odd loop } (L \geq 3): \qquad x_\alpha, \qquad u_\alpha, \qquad N u_\beta\,. \tag{5.26}$$

Writing the null vectors explicitly, we have

$$\text{Odd loop } (L \geq 3): \qquad [0,1] \otimes x^i, \qquad [3N, -1] \otimes u, \qquad [3N^3 + N, -N^2] \otimes u\,. \tag{5.27}$$

At one loop, the null vectors are $[0,1] \otimes x^i$ and $[3N, -1] \otimes u$.

Eqs. (5.25) and (5.27) agree precisely with the results obtained in refs. [14, 16], taking into account footnote 1. Thus, as stated in the introduction, for SU(N) there are precisely four L-loop null vectors for all $L \geq 2$.

6 L-loop SO(N) color space

The goal of this section is to explicitly construct the space of L-loop color factors for SO(N). The procedure is analogous to that employed in section 4. An L-loop color factor may be expressed as

$$[P,Q] \otimes u, \qquad\qquad [P,Q] \otimes x^i \qquad (i = 1,2) \tag{6.1}$$

where P and Q are polynomials in N of maximal degree L and $L-1$ respectively. Thus, L-loop SO(N) color factors inhabit a vector space $V^{(L)}$ of dimension $6L+3$ (the trace space). The polynomials P and Q corresponding to color factors, however, are not completely arbitrary but satisfy certain constraints. Consequently, the space of all L-loop color factors spans a proper subspace (the color space) of $V^{(L)}$.

As before, we iteratively construct an explicit basis for the L-loop SO(N) color space, beginning with the single tree-level irreducible representation $[1,0] \otimes x^i$ and acting repeatedly with the iterative matrices for SO(N) obtained in section 3

$$g_1 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad g_{xx} = \begin{pmatrix} K & 0 \\ -4 & 0 \end{pmatrix}, \quad g_{xu} = \begin{pmatrix} K-6 & 3 \\ 0 & -2K \end{pmatrix}, \quad g_{ux} = \begin{pmatrix} K-3 & -3 \\ 12 & -2K \end{pmatrix} \tag{6.2}$$

where we have chosen to set $e = 2$. Moreover, we find it convenient to express these matrices in terms of the SO(N) quadratic Casimir $K = N - 2$ rather than in terms of N. As before, these 2×2 matrices act on the $[P,Q]$ part of the color factor, while the subscripts indicate their action on the x or u part of the color factor, so that, for example, g_{xu} takes an x-type color factor to a u-type color factor, etc. We will show that these matrices generate a basis consisting of polynomials multiplied by one of four specific (linearly independent) types:

$$\begin{aligned} x_a &\equiv [1,0]\otimes x^i\,, \\ x_b &\equiv [K+3,1]\otimes x^i\,, \\ u_a &\equiv [K-6,3]\otimes u\,, \\ u_b &\equiv [(K+3)(K-6),K+9]\otimes u\,. \end{aligned} \tag{6.3}$$

Our first step is to ascertain how each of the operators (6.2) act on the types of color factors (6.3). First, the operator g_1 just rescales each type by K

$$\begin{aligned} x_a g_1 &= K x_a\,, \\ x_b g_1 &= K x_b\,, \\ u_a g_1 &= K u_a\,, \\ u_b g_1 &= K u_b\,. \end{aligned} \tag{6.4}$$

Second, the operator g_{xx} acts on the x-type color factors as

$$\begin{aligned} x_a g_{xx} &= K x_a\,, \\ x_b g_{xx} &= (K+4)(K-1)x_a\,. \end{aligned} \tag{6.5}$$

Since the action of g_{xx} on x_a is identical to the action of g_1 (and therefore redundant), we will restrict our attention to its action on x_b, defining $g_{ba} = g_{xx}$ with

$$x_b g_{ba} = (K+4)(K-1)x_a\,. \tag{6.6}$$

Third, the operator g_{xu} takes an x_a-type color factor to a u_a-type color factor, and an x_b-type color factor to a u_b-type color factor:

$$\begin{aligned} x_a g_{xu} &= u_a\,, \\ x_b g_{xu} &= u_b\,. \end{aligned} \tag{6.7}$$

Finally, the operator g_{ux} acts on u-type color factors to give linear combinations of x_a and x_b types:

$$\begin{aligned} u_a g_{ux} &= 10K^2\, x_a - 9(K-2)x_b\,, \\ u_b g_{ux} &= 6K(K-1)(K+4)x_a - (5K^2+9K-54)x_b\,. \end{aligned} \tag{6.8}$$

Our next step is to generate the SO(N) color space through three-loop order. We begin with the single tree-level irreducible representation

$$\text{Tree level:} \qquad x_a\,. \tag{6.9}$$

Acting on x_a with g_1 using eq. (6.4) and with g_{xu} using eq. (6.7), we obtain the three-dimensional space spanned by two irreducible representations

$$\text{One loop:} \qquad Kx_a, \quad u_a\,. \tag{6.10}$$

We then act on each of these one-loop color factors with g_1 to obtain K^2x_a and Ku_a. The action of g_{xu} on Kx_a is redundant, but we can act on the u_a-type color factor with g_{ux} to obtain $[K^2-9K+54,-9K+18]\otimes x^i$, which is a linear combination of x_a and x_b types, as shown in eq. (6.8). Since we already have K^2x_a in the color space, we subtract $10K^2x_a$ and divide by -9 to obtain $(K-2)x_b$. Thus the two-loop color space is five-dimensional, spanned by three irreducible representations

$$\text{Two loops:} \qquad K^2x_a\,, \quad Ku_a\,, \quad (K-2)x_b\,. \tag{6.11}$$

Observe that all these results are consistent with the results obtained earlier in eq. (2.31), noting that $[N-8,N-4]\otimes x^i$ is a linear combination of K^2x_a and $(K-2)x_b$. The three-loop color factors are then obtained by acting on each of the two-loop color factors with g_1. The action of g_{ux} on Ku_a is redundant. We can also act on $(K-2)x_b$ with g_{ba} using eq. (6.6) and with g_{xu} using eq. (6.7). The three-loop color space is thus eight-dimensional, spanned by five irreducible representations

$$\text{Three loops:} \quad K^3x_a\,, \quad (K+4)(K-1)(K-2)x_a\,, \quad K^2u_a\,, \quad K(K-2)x_b\,, \quad (K-2)u_b\,. \tag{6.12}$$

We now make some general observations that allow us to determine the complete span of color factors at arbitrary loop order L. We omit the arguments for these when they are identical to those given for SU(N) in section 4.

(Observation 1). *All L-loop u-type color factors are generated by the action of g_{xu} on the complete set of x-type color factors at $(L-1)$ loops using eq. (6.7).*

(Observation 2). *All L-loop x-type color factors are obtained from x-type color factors at $(L-1)$ and $(L-2)$ loops.* The two-step operators that produce color factors of pure type are

$$\begin{aligned} g^{(2)}_{ab} &= \frac{1}{9}\left(-g_{xu}g_{ux}+10g_1^2\right)\,, \\ g^{(2)}_{bb} &= \frac{1}{9}\left(-g_{xu}g_{ux}-5g_1^2+6g_{ba}g_1\right) \end{aligned} \tag{6.13}$$

which act on x_a- and x_b-type color factors respectively at $(L-2)$ loops to yield x_b-type color factors at L loops

$$\begin{aligned} x_a g^{(2)}_{ab} &= (K-2)x_b\,, \\ x_b g^{(2)}_{bb} &= (K-6)x_b \end{aligned} \tag{6.14}$$

easily verified using eqs. (6.4)–(6.8). Thus all L-loop x-type color factors may be obtained from x-type color factors at $(L-1)$ and $(L-2)$ loops through the action of the four operators g_1, g_{ba}, $g^{(2)}_{ab}$, and $g^{(2)}_{bb}$.

(Observation 3). *All L-loop x_a-type color factors can be obtained from g_{ba} acting on an $(L-1)$-loop x_b-type color factor using eq. (6.6) with one exception, namely K^Lx_a, which results from g_1 acting repeatedly on the tree-level color factor x_a.*

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(Observation 4). *All L-loop x_b-type color factors for $L > 2$ can be obtained from g_1, $g_{bb}^{(2)}$, and $g_{bb}^{(3)}$ acting on x_b-type color factors at $L-1$, $L-2$, and $L-3$ loops.* From observation (2), we know that all L-loop x_b-type color factors may be obtained from g_1 and $g_{bb}^{(2)}$ acting on x_b-type color factors at $L-1$ and $L-2$ loops respectively, and $g_{ab}^{(2)}$ acting on x_a-type color factors at $L-2$ loops. From observation (3), the latter may be replaced (with one possible exception) by $g_{ba}g_{ab}^{(2)}$ acting on an x_b-type color factor at $L-3$ loops. The one possible exception is $g_{ab}^{(2)}$ acting on $K^{L-2}x_a$. However since

$$K^{L-2}x_a g_{ab}^{(2)} = x_a g_1^{L-2} g_{ab}^{(2)} = x_a g_{ab}^{(2)} g_1^{L-2} = (K-2)x_b g_1^{L-2} \tag{6.15}$$

this is equivalent to g_1 acting repeatedly on the two-loop x_b-type color factor. It is convenient to replace $g_{ba}g_{ab}^{(2)}$ with a three-step operator

$$g_{bb}^{(3)} = \frac{1}{4}\left[-g_{ba}g_{ab}^{(2)} + g_1^3 + g_1 g_{bb}^{(2)}\right] = \frac{1}{36}\left[g_{ba}g_{xu}g_{ux} - g_1 g_{xu}g_{ux} - 4g_{ab}g_1^2 + 4g_1^2\right] \tag{6.16}$$

which maps an $(L-3)$-loop x_b-type color factor to an L-loop x_b-type color factor

$$x_b g_{bb}^{(3)} = (K-2)x_b\,. \tag{6.17}$$

To summarize, all x_b-type colors for $L > 2$ can be generated by the action of three (commuting) operators

$$\begin{aligned} x_b g_1 &= K x_b\,, \\ x_b g_{bb}^{(2)} &= (K-6)x_b\,, \\ x_b g_{bb}^{(3)} &= (K-2)x_b \end{aligned} \tag{6.18}$$

acting on lower-loop x_b-type color factors.

From the observations above, we are now able to determine the complete set of L-loop color factors. Beginning with x_b-type color factors, we observe from eq. (6.11) that the first x_b-type color factor occurs at two loops, namely $(K-2)x_b$. The set of all higher-loop x_b-type color factors is obtained by acting on $(K-2)x_b$ with an arbitrary combination of g_1, $g_{bb}^{(2)}$, and $g_{bb}^{(3)}$:

$$(K-2)x_b g_1^{n_1} g_{bb}^{(2)n_2} g_{bb}^{(3)n_3} = K^{n_1}(K-6)^{n_2}(K-2)^{n_3+1}x_b \tag{6.19}$$

where n_1, n_2, and n_3 are arbitrary non-negative integers that satisfy

$$n_1 + 2n_2 + 3n_3 = L-2\,. \tag{6.20}$$

The right hand side of eq. (6.19) may be written more explicitly as

$$[P,Q]\otimes x^i \qquad \text{with} \begin{cases} P = K^{n_1}(K-6)^{n_2}(K-2)^{n_3+1}(K+3) \\ Q = K^{n_1}(K-6)^{n_2}(K-2)^{n_3+1} \end{cases} \tag{6.21}$$

confirming that P is a polynomial of maximal degree L and Q is a polynomial of maximal degree $L-1$. For $L=3$ through $L=7$, the number of solutions of eq. (6.20) is $L-2$, with

n_3 given by either 0 or 1. Specifically, denoting $n_2 = n$ and $n_1 = L - 2 - 2n - 3n_3$, these solutions correspond to x_b-type irreducible representations

$$\begin{aligned} &K^{L-2-2n}(K-6)^n(K-2)x_b, \quad n = 0, \cdots, \left\lfloor \frac{L-2}{2} \right\rfloor, \\ &K^{L-5-2n}(K-6)^n(K-2)^2 x_b, \quad n = 0, \cdots, \left\lfloor \frac{L-5}{2} \right\rfloor. \end{aligned} \tag{6.22}$$

This set of $L-2$ irreducible representations (for $L \geq 3$) is linearly independent, since the exponents of K are all distinct. Starting at $L = 8$, additional solutions of eq. (6.20) arise, with $n_3 \geq 2$, but we claim that the corresponding color factors are not linearly independent of the set (6.22). We verify this claim in appendix B, where we explicitly construct two x_α-type irreducible representations orthogonal to the entire set (6.19). This establishes that at most $L-2$ of the irreducible representations in eq. (6.19) are independent. With $L-2$ as both lower and upper bound, the $L-2$ irreducible representations belonging to the set (6.22) constitute a complete and independent set of L-loop x_b-type color factors for $L \geq 3$. From these, we may construct the rest of the color space using the observations above.

Let us first consider the x_a-type color factors. From eq. (6.9) through eq. (6.12), we observe that there is one x_a-type irreducible representation for $L = 0$ through $L = 2$, and two x_a-type irreducible representations for $L = 3$. For $L \geq 3$, we can use observation (3) to generate a complete set of linearly independent L-loop x_a-type color factors by acting with g_{ba} on the complete set of $(L-1)$-loop x_b-type color factors in eq. (6.22) and adding in the one exception:

$$\begin{aligned} &K^L x_a\,, \\ &K^{L-3-2n}(K-6)^n(K-2)(K-1)(K+4)x_a, \quad n = 0, \cdots, \left\lfloor \frac{L-3}{2} \right\rfloor, \\ &K^{L-6-2n}(K-6)^n(K-2)^2(K-1)(K+4)x_a, \quad n = 0, \cdots, \left\lfloor \frac{L-6}{2} \right\rfloor. \end{aligned} \tag{6.23}$$

This set contains (for $L \geq 4$) $L-2$ linearly independent x_a-type irreducible representations.

Next we turn to u-type color factors. From eq. (6.12), we observe that the first u_b-type color factor occurs at $L = 3$, namely, $(K-2)u_b$. From observation (1) above, we can generate all L-loop u_b-type color factors for $L \geq 3$ by acting with g_{xu} on the complete set of $(L-1)$-loop x_b-type color factors in eq. (6.22) to give

$$\begin{aligned} &K^{L-3-2n}(K-6)^n(K-2)u_b, \quad n = 0, \cdots, \left\lfloor \frac{L-3}{2} \right\rfloor, \\ &K^{L-6-2n}(K-6)^n(K-2)^2 u_b, \quad n = 0, \cdots, \left\lfloor \frac{L-6}{2} \right\rfloor \end{aligned} \tag{6.24}$$

which is a complete set of (for $L \geq 4$) $L-3$ u_b-type color factors at L loops. Finally, we observe that there is one u_a-type color factor at $L = 1$ through $L = 3$, and two u_a-type color factors at $L = 4$. Using observation (3) above, we can generate all L-loop u_a-type

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# of loops L	0	1	2	3	4	5	6	$L \geq 5$
# of x_a-type irreps	1	1	1	2	2	3	4	$L-2$
# of x_b-type irreps	0	0	1	1	2	3	4	$L-2$
# of u_a-type irreps	0	1	1	1	2	2	3	$L-3$
# of u_b-type irreps	0	0	0	1	1	2	3	$L-3$
total # of color factors	2	3	5	8	11	16	22	$6L-14$

Table 4. Number of irreducible representations spanning the L-loop color space for SO(N).

color factors for $L \geq 4$ by acting with g_{xu} on the complete set of $(L-1)$-loop x_a-type color factors in eq. (6.23) to obtain

$$\begin{aligned}
&K^{L-1}u_a\,, \\
&K^{L-4-2n}(K-6)^n(K-2)(K-1)(K+4)u_a, \qquad && n=0,\cdots,\left\lfloor\frac{L-4}{2}\right\rfloor, \\
&K^{L-7-2n}(K-6)^n(K-2)^2(K-1)(K+4)u_a, \qquad && n=0,\cdots,\left\lfloor\frac{L-7}{2}\right\rfloor
\end{aligned} \tag{6.25}$$

which is a complete set of (for $L \geq 5$) $L-3$ u_a-type color factors at L loops.

We summarize the counting of irreducible representations spanning the L-loop color space in table 4. The total dimension of the L-loop color space given in the last row is the sum of these basis elements, taking into account that x-type elements are two-dimensional representations (of S_4) while u-type elements are one-dimensional. Observe that the dimensions of the color spaces begins to differ from those of SU(N) at $L=5$.

7 L-loop SO(N) null space

In the previous section, we generated a basis of color factors spanning the L-loop color space for SO(N). The numbers of independent color factors for various values of L are given in table 5, divided into x-type and u-type.[5] The dimensions of the associated trace spaces, in which these color factors live, are also listed by type. The differences of these two numbers is the number of null vectors, defined as inhabiting the orthogonal complement to the color space, and are also listed in the table. The last three rows of the table, which list the dimensions of the spaces regardless of type, reproduce table 2.

For $L \geq 5$, the color space is a $(6L-14)$-dimensional subspace of the $(6L+3)$-dimensional trace space, so the null space is therefore (as claimed in the introduction) generically 17-dimensional, and consists of 10 x-type null vectors and 7 u-type null vectors.

The purpose of this section is to derive explicit expressions for the 10 x-type null vectors. (The construction of the 7 u-type null vectors is left to future work.) As before, we first need to define an inner project on the trace space.

[5]The number of x-type color factors is twice the number of x-type irreducible representations because those representations are two-dimensional.

number of loops	0	1	2	3	4	5	6	$L \geq 5$
# x-type color factors	2	2	4	6	8	12	16	$4L-8$
dim x-type trace space	2	6	10	14	18	22	26	$4L+2$
# x-type null vectors	0	4	6	8	10	10	10	10
# u-type color factors	0	1	1	2	3	4	6	$2L-6$
dim u-type trace space	1	3	5	7	9	11	13	$2L+1$
# u-type null vectors	1	2	4	5	6	7	7	7
# color factors	2	3	5	8	11	16	22	$6L-14$
dim trace space	3	9	15	21	27	33	39	$6L+3$
# null vectors	1	6	10	13	16	17	17	17

Table 5. Dimensions of trace, color, and null spaces for SO(N) amplitudes.

7.1 Inner product

We choose an inner project for the SO(N) trace space similar to that defined in section 5.1 for the SU(N) trace space, but with a slight difference. The inner product of two polynomials P and P' is given by

$$\langle P'|P\rangle = \mathbf{P}'\mathbf{P}^T \tag{7.1}$$

except that the row vectors $\mathbf{P} = (P_0, P_1, P_2, \cdots)$ consist of the coefficients of the polynomials expressed in terms of $K = N - 2$ rather than of N:

$$P(K) = \mathbf{P}\mathbf{K}^T, \qquad \text{with} \qquad \mathbf{K} = (1, K, K^2, \cdots)\,. \tag{7.2}$$

Other than this, everything is the same as in section 5.1.

7.2 SO(N) null vectors

In section 6, we determined a complete set of color factors that span the L-loop color space for SO(N), namely,

$$\begin{aligned}
C^{(L)}_{xa} &= c^{(L)}_{xa}\, x_a, & c^{(L)}_{xa} &\in \{K^L\} \cup \{(K-1)(K+4)c^{(L-1)}_{xb}\}\,, \\
C^{(L)}_{xb} &= c^{(L)}_{xb}\, x_b, & c^{(L)}_{xb} &\in \{K^{n_1}(K-6)^{n_2}(K-2)^{n_3+1} \mid n_1 + 2n_2 + 3n_3 = L-2\}\,, \\
C^{(L)}_{ua} &= c^{(L)}_{ua}\, u_a, & c^{(L)}_{ua} &\in \{c^{(L-1)}_{xa}\}\,, \\
C^{(L)}_{ub} &= c^{(L)}_{ub}\, u_b, & c^{(L)}_{ub} &\in \{c^{(L-1)}_{xb}\}
\end{aligned} \tag{7.3}$$

where we recall that

$$\begin{aligned}
x_a &= [1, 0] \otimes x^i, \\
x_b &= [K+3, 1] \otimes x^i, \\
u_a &= [K-6, 3] \otimes u, \\
u_b &= [(K+3)(K-6), K+9] \otimes u\,.
\end{aligned} \tag{7.4}$$

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We will obtain a complete set of x-type null vectors $R_x^{(L)}$ living in the L-loop trace space and orthogonal to the set (7.3). We will show that all SO(N) x-type null vectors can be written in terms of three possible types, namely,

$$\begin{aligned} x_\alpha &= [0,1] \otimes x^i \,, \\ x_\beta &= [K, -3K-1] \otimes x^i \,, \\ x_\gamma &= [1,0] \otimes x^i \,. \end{aligned} \tag{7.5}$$

These are chosen, using the prescription given at the end of section 5.1, so that x_α-type null vectors are automatically orthogonal to x_a-type color factors and the x_β-type null vectors are orthogonal to x_b-type color factors. Unlike SU(N), we will also need a third type, x_γ, of null vector which is not automatically orthogonal to either x_a- or x_b-type color factors. All x-type null vectors, however, are automatically orthogonal to the u-type color factors. The ten x-type null vectors (for $L \geq 5$) consist of five x-type irreducible representations: two each of x_α and x_β type, and one of x_γ type.

(1) x_α-type null vectors. Consider an L-loop null vector of the form

$$R_{x\alpha}^{(L)} = r_{x\alpha}^{(L)} x_\alpha \tag{7.6}$$

where $r_{x\alpha}^{(L)}$ is a polynomial in K of maximal degree $L-1$. Orthogonality to $C_{xa}^{(L)}$, $C_{ua}^{(L)}$, and $C_{ub}^{(L)}$ is automatic from the definition of x_α. The final orthogonality condition gives

$$0 = \langle C_{xb}^{(L)} | R_{x\alpha}^{(L)} \rangle = \mathbf{c}_{xb}^{(L)} M_{b\alpha} \mathbf{r}_{x\alpha}^{(L)T}, \qquad M_{b\alpha} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{7.7}$$

where $M_{b\alpha}$ is defined in eq. (5.12) using the $\mathcal{P}$ and $\mathcal{Q}$ matrices appropriate to x_b and x_α.

Let's now impose $\langle C_{xb}^{(L)} | R_{x\alpha}^{(L)} \rangle = 0$ to determine the form of the x_α-type null vectors. At one loop, this condition is automatically satisfied since there are no one-loop x_b-type color factors, so we have

$$R_{x\alpha}^{(1)} = x_\alpha \,. \tag{7.8}$$

At two loops, we use $c_{xb}^{(2)} = K - 2$ to find

$$R_{x\alpha}^{(2)} = \left(K + \frac{1}{2}\right) x_\alpha \,. \tag{7.9}$$

At three loops, we use $c_{xb}^{(3)} = K(K-2)$ to find $R_{x\alpha}^{(3)} = \left(K^2 + \frac{1}{2}K + \lambda\right) x_\alpha$ where λ is arbitrary. Thus the null space contains a pair of independent x_α-type irreducible representations, which we choose to be (for reasons that will immediately become clear)

$$\begin{aligned} R_{x\alpha,1}^{(3)} &= \left(K^2 + \frac{1}{2}K + \frac{1}{4}\right) x_\alpha \,, \\ R_{x\alpha,2}^{(3)} &= \left(K^2 + \frac{1}{2}K + \frac{5}{36}\right) x_\alpha \,. \end{aligned} \tag{7.10}$$

In appendix B, we prove that, for all $L \geq 3$ there are exactly two x_α-type irreducible representations, given by

$$
\begin{aligned}
R^{(L)}_{x\alpha,j} &= r^{(L)}_{x\alpha,j} x_\alpha \,, \qquad j = 1,2 \\
r^{(L)}_{x\alpha,1} &= \frac{\left[1-(2K)^L\right]}{2^{L-1}(1-2K)} \,, \\
r^{(L)}_{x\alpha,2} &= \frac{4\left[1-(3K)^L\right]}{3^L(1-3K)} + \frac{2\left[1-(-6K)^L\right]}{(-6)^L(1+6K)} \,.
\end{aligned} \tag{7.11}
$$

At three loops, these agree with eq. (7.10), and at four loops, they give

$$
\begin{aligned}
r^{(4)}_{x\alpha,1} &= \left(K^3 + \frac{1}{2}K^2 + \frac{1}{4}K + \frac{1}{8}\right) \,, \\
r^{(4)}_{x\alpha,2} &= \left(K^3 + \frac{1}{2}K^2 + \frac{5}{36}K + \frac{11}{216}\right) \,.
\end{aligned} \tag{7.12}
$$

There is an evident pattern whereby $r^{(L)}_{x\alpha,j}$ is given by $K r^{(L-1)}_{x\alpha,j}$ plus a constant easily obtained from eq. (7.11).

(2) x_β-type null vectors. Next consider an L-loop null vector of the form

$$
R^{(L)}_{x\beta} = r^{(L)}_{x\beta} x_\beta \tag{7.13}
$$

where $r^{(L)}_{x\beta}$ is a polynomial in K of maximal degree $L-2$. Orthogonality to $C^{(L)}_{xb}$, $C^{(L)}_{ua}$, and $C^{(L)}_{ub}$ is automatic from the definition of x_β. The final orthogonality condition gives

$$
0 = \langle C^{(L)}_{xa} | R^{(L)}_{x\beta} \rangle = \mathbf{c}^{(L)}_{xa} M_{a\beta} \mathbf{r}^{(L)T}_{x\beta} \,, \qquad M_{a\beta} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{7.14}
$$

where $M_{a\beta}$ is defined in eq. (5.12) using the $\mathcal{P}$ and $\mathcal{Q}$ matrices appropriate to x_a and x_β. One of the L-loop x_a-type color factors is $c^{(L)}_{xa} = K^L$, for which eq. (7.14) is automatically satisfied since $r_{x\beta}$ has maximal degree $L-2$. By observation (3) of section 6, all the other L-loop x_a-type color factors are obtained from $(L-1)$-loop x_b-type color factors,

$$
c^{(L)}_{xa} = (K-1)(K+4) c^{(L-1)}_{xb} \tag{7.15}
$$

which can be expressed in matrix form as

$$
\mathbf{c}^{(L)}_{xa} = \mathbf{c}^{(L-1)}_{xb} G_{ba} \,, \qquad G_{ba} = \begin{pmatrix} -4 & 3 & 1 & 0 & \cdots \\ 0 & -4 & 3 & 1 & \cdots \\ 0 & 0 & -4 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \,. \tag{7.16}
$$

Thus eq. (7.14) implies

$$0 = \mathbf{c}_{xb}^{(L-1)} H \mathbf{r}_{x\beta}^{(L)T}, \qquad H = G_{ba} M_{a\beta} = \begin{pmatrix} 3 & 1 & 0 & 0 & \cdots \\ -4 & 3 & 1 & 0 & \cdots \\ 0 & -4 & 3 & 1 & \cdots \\ 0 & 0 & -4 & 3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}. \tag{7.17}$$

Given that $c_{xb}^{(L-1)}$ and $r_{x\beta}^{(L)}$ are both of maximal degree $L-2$, we may truncate the infinite matrix H to the finite matrix $H^{(L)}$, consisting of the first $L-1$ rows and columns of H. Then eq. (7.17) becomes

$$0 = \mathbf{c}_{xb}^{(L-1)} H^{(L)} \mathbf{r}_{x\beta}^{(L)T}, \qquad H^{(L)} = \Pi_{L-2} H \Pi_{L-2}\,. \tag{7.18}$$

We observe[6] that $\det H^{(L)} > 0$, so that the $(L-1)\times(L-1)$ matrix $H^{(L)}$ is invertible. Since generically we found two solutions to $\mathbf{c}_{xb}^{(L-1)} \mathbf{r}_{x\alpha}^{(L-1)T} = 0$, there are therefore two x_β-type irreducible represention, namely

$$\begin{aligned} \mathbf{r}_{x\beta,1}^{(L)T} &= \left(H^{(L)}\right)^{-1} \mathbf{r}_{x\alpha,1}^{(L-1)T}, \\ \mathbf{r}_{x\beta,2}^{(L)T} &= \left(H^{(L)}\right)^{-1} \mathbf{r}_{x\alpha,2}^{(L-1)T} \end{aligned} \tag{7.19}$$

where $r_{x\alpha,j}^{(L-1)}$ are given in eq. (7.11). For $L=2$ and $L=3$, $\mathbf{r}_{x\beta,1}^{(L)}$ and $\mathbf{r}_{x\beta,2}^{(L)}$ coincide, but they are distinct for $L \geq 4$.

(3) x_γ-type null vectors. Having found (for $L \geq 4$) two irreducible representations of type x_α and two of type x_β, there must be one remaining, which we will show to be of the form

$$R_{x\gamma}^{(L)} = r_{x\gamma}^{(L)} x_\gamma \tag{7.20}$$

where $r_{x\gamma}^{(L)}$ is a polynomial in K of maximal degree L (but see below). Orthogonality to $C_{ua}^{(L)}$ and $C_{ub}^{(L)}$ is automatic. Orthogonality to x_a-type color factors, $\langle C_{xa}^{(L)} | R_{x\gamma}^{(L)} \rangle = 0$, implies

$$0 = \mathbf{c}_{xa}^{(L)} M_{a\gamma} \mathbf{r}_{x\gamma}^{(L)T}, \qquad M_{a\gamma} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{7.21}$$

where $M_{a\gamma}$ is defined in eq. (5.12) using the $\mathcal{P}$ and $\mathcal{Q}$ matrices appropriate to x_a and x_γ. One of the L-loop x_a-type color factors is $c_{xa}^{(L)} = K^L$, so eq. (7.21) implies that $r_{x\gamma}^{(L)}$ is actually of maximal degree $L-1$. All the other L-loop x_a-type color factors are obtained from $(L-1)$-loop x_b-type color factors, so that eq. (7.21) becomes

$$0 = \mathbf{c}_{xb}^{(L-1)} G_{ba} \mathbf{r}_{x\gamma}^{(L)T} \tag{7.22}$$

[6] This follows from $\det H^{(L)} = 3 \det H^{(L-1)} + 4 \det H^{(L-2)}$.

# of loops L	0	1	2	3	4	$L \geq 4$
# of x_α-type irreps	0	1	1	2	2	2
# of x_β-type irreps	0	0	1	1	2	2
# of x_γ-type irreps	0	1	1	1	1	1
total # of x-type irreps	0	2	3	4	5	5

Table 6. Number of independent x-type null vectors for SO(N).

where G_{ba} was defined in eq. (7.16). In addition, orthogonality to x_b-type color factors, $\langle C_{xb}^{(L)} | R_{x\gamma}^{(L)} \rangle = 0$, requires

$$0 = \mathbf{c}_{xb}^{(L)} M_{b\gamma} \mathbf{r}_{x\gamma}^{(L)T}, \qquad M_{b\gamma} = \begin{pmatrix} 3 & 1 & 0 & \cdots \\ 0 & 3 & 1 & \cdots \\ 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{7.23}$$

where $M_{b\gamma}$ is defined in eq. (5.12) using the $\mathcal{P}$ and $\mathcal{Q}$ matrices appropriate to x_b and x_γ. At one loop, eqs. (7.22) and (7.23) are empty, as there are no x_b-type color factors below two loops, so there is a single x_γ-type irreducible representation:

$$R_{x\gamma}^{(1)} = x_\gamma \,. \tag{7.24}$$

At two loops one has $c_{xb}^{(1)} = K - 2$, so that eq. (7.23) again yields a single x_γ-type irreducible representation:

$$R_{x\gamma}^{(2)} = \left(K + \frac{1}{6} \right) x_\gamma \,. \tag{7.25}$$

For $L \geq 3$, one must impose both eqs. (7.22) and (7.23). We show in appendix B that there is also a single x_γ-type irreducible representation that satisfies both eqs. (7.22) and (7.23), which has the form

$$r_{x\gamma}^{(L)} = \frac{2\left[1 - (3K)^L\right]}{3^L(1-3K)} - \frac{2\left[1 - (-6K)^L\right]}{(-6)^L(1+6K)} \tag{7.26}$$

consistent with eqs. (7.24) and (7.25). Again, there is a pattern whereby $r_{x\gamma}^{(L)}$ is given by $K r_{x\gamma}^{(L-1)}$ plus a constant easily obtained from eq. (7.26).

7.3 Summary of null vectors

We have explicitly constructed all the x-type null vectors for SO(N). For $L \geq 4$, there are ten such null vectors, consisting of five irreducible representations whose general forms are given in eqs. (7.11), (7.19), and (7.26). For $L < 4$, the number of null vectors is fewer (see table 6).

For the reader's convenience, we explicitly list the x-type null vectors through four loops here:

$$\begin{aligned}
\text{One loop: } R^{(1)}_{x\alpha} &= x_\alpha\,, \\
R^{(1)}_{x\gamma} &= x_\gamma\,, \\
\text{Two loops: } R^{(2)}_{x\alpha} &= \left(K+\frac{1}{2}\right) x_\alpha\,, \\
R^{(2)}_{x\beta} &= x_\beta\,, \\
R^{(2)}_{x\gamma} &= \left(K+\frac{1}{6}\right) x_\gamma\,, \\
\text{Three loops: } R^{(3)}_{x\alpha,1} &= \left(K^2+\frac{1}{2}K+\frac{1}{4}\right) x_\alpha\,, \\
R^{(3)}_{x\alpha,2} &= \left(K^2+\frac{1}{2}K+\frac{5}{36}\right) x_\alpha\,, \\
R^{(3)}_{x\beta} &= \left(K+\frac{1}{10}\right) x_\beta\,, \\
R^{(3)}_{x\gamma} &= \left(K^2+\frac{1}{6}K+\frac{1}{12}\right) x_\gamma\,, \\
\text{Four loops: } R^{(4)}_{x\alpha,1} &= \left(K^3+\frac{1}{2}K^2+\frac{1}{4}K+\frac{1}{8}\right) x_\alpha\,, \\
R^{(4)}_{x\alpha,2} &= \left(K^3+\frac{1}{2}K^2+\frac{5}{36}K+\frac{11}{216}\right) x_\alpha\,, \\
R^{(4)}_{x\beta,1} &= \left(K^2+\frac{9}{46}K+\frac{11}{92}\right) x_\beta\,, \\
R^{(4)}_{x\beta,2} &= \left(K^2+\frac{57}{382}K+\frac{47}{764}\right) x_\beta\,, \\
R^{(4)}_{x\gamma} &= \left(K^2+\frac{1}{6}K^2+\frac{1}{12}K+\frac{5}{216}\right) x_\gamma
\end{aligned} \tag{7.27}$$

where we have rescaled the x_β-type null vectors.

8 L-loop Sp(N) color space

The L-loop color space for four-point amplitudes with gauge group Sp(N) can be dealt with summarily since the results are nearly identical to those for amplitudes with gauge group SO(N), up to certain relative signs. The iterative matrices for Sp(N) obtained in section 3 are given by

$$g_1 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad g_{xx} = \begin{pmatrix} K & 0 \\ -4 & 0 \end{pmatrix}, \quad g_{xu} = \begin{pmatrix} K+6 & 3 \\ 0 & -2K \end{pmatrix}, \quad g_{ux} = \begin{pmatrix} K+3 & -3 \\ 12 & -2K \end{pmatrix} \tag{8.1}$$

where we have chosen to set $e = 2$ and have expressed these matrices in terms of the Sp(N) quadratic Casimir $K = N + 2$. These matrices generate a basis consisting of polynomials multiplied by one of four specific (linearly independent) types:

$$\begin{aligned}
x_a &\equiv [1,0]\otimes x^i,\\
x_b &\equiv [K-3,1]\otimes x^i,\\
u_a &\equiv [K+6,3]\otimes u,\\
u_b &\equiv [(K-3)(K+6),K-9]\otimes u\,.
\end{aligned}\tag{8.2}$$

Carrying out manipulations exactly analogous to those for SO(N) in section 6, we determine a complete set of color factors that span the L-loop color space for Sp(N):

$$\begin{aligned}
C^{(L)}_{xa} &= c^{(L)}_{xa}x_a, & c^{(L)}_{xa} &\in \{K^L\}\cup\{(K+1)(K-4)c^{(L-1)}_{xb}\}\,,\\
C^{(L)}_{xb} &= c^{(L)}_{xb}x_b, & c^{(L)}_{xb} &\in \{K^{n_1}(K+6)^{n_2}(K+2)^{n_3+1}\mid n_1+2n_2+3n_3=L-2\}\,,\\
C^{(L)}_{ua} &= c^{(L)}_{ua}u_a, & c^{(L)}_{ua} &\in \{c^{(L-1)}_{xa}\}\,,\\
C^{(L)}_{ub} &= c^{(L)}_{ub}u_b, & c^{(L)}_{ub} &\in \{c^{(L-1)}_{xb}\}
\end{aligned}\tag{8.3}$$

which is the same as eq. (7.3) up to certain relative signs.

The orthogonal complement of the space of color factors (8.3) is the L-loop null space, which (for $L\geq 5$) is spanned by ten x-type null vectors and seven u-type null vectors. Again carrying out manipulations exactly analogous to those for SO(N) in section 7, we may determine the explicit forms of all the x-type null vectors, which may be obtained from the SO(N) x-type null vectors (7.11), (7.19), and (7.26) by some obvious changes of relative signs.

9 Conclusions

In this paper, we have analyzed the spaces of color factors associated with L-loop four-point amplitudes of fields transforming in the adjoint representation of gauge groups SU(N), SO(N), or Sp(N) by decomposing them into an extended trace basis. The extended trace basis consists of traces (and products of traces) of generators multiplied by various powers of N (or of K, where K is proportional to the quadratic Casimir, viz., N for SU(N), $N-2$ for SO(N), and $N+2$ for Sp(N)). The dimension of the L-loop extended trace space is $3L+3$ for SU(N) and $6L+3$ for SO(N) and Sp(N), and the L-loop color space spans a proper subspace of the L-loop trace space. Using a refined iterative process, we have determined the dimensions of this subspace for all values of L for the groups SU(N), SO(N), or Sp(N), with the results listed in tables 1 and 2. We observe that the dimensions of these color spaces are the same for all these groups up through four loops, but begin to differ for $L\geq 5$.

As can be seen in tables 1 and 2, the codimensions of the color spaces (vis-a-vis the extended trace space) reach a fixed value for sufficiently large L. Thus these spaces are more efficiently characterized by specifying the null space, i.e., the orthogonal complement of the color space in the trace space. Moreover, the null vectors are directly related to group-theory constraints on the color-ordered amplitudes, as described in the introduction. We established the number of null vectors to be four for SU(N) (for $L\geq 2$) and seventeen for SO(N) and Sp(N) (for $L\geq 5$). For SU(N), we confirmed the forms of the four null vectors (or constraints) found previously. For SO(N) (and Sp(N)), we derived explicit expressions for ten of the seventeen null vectors, namely, the x-type null vectors. Obtaining the remaining seven u-type null vectors is left for future work.

Admittedly the usefulness of the null vectors for SO(N)/Sp(N) is limited because they are constructed with respect to an unconventional inner product. One might ask why we bother to construct these null vectors explicitly. The answer is that proving the existence of these null vectors, which we do by constructing them, is crucial to establish the completeness of the basis of $6L-14$ color factors (for $L \geq 5$) for SO(N) constructed in section 6 and listed in table 4. As explained in section 6, our iterative procedure produces the correct number $(L-2)$ of independent x_b-type irreducible representations through seven loops, but (apparently) produces additional ones for $L \geq 8$, corresponding to solutions of eq. (6.20) with $n_3 \geq 2$. To show that these additional irreps are not independent of the others, we demonstrate in appendix B that they are orthogonal to two x_α-type irreps, which we explicitly construct. There may be other ways to demonstrate the completeness of the color factors, but this is how we have done it. As usual, this is subject to the assumption, stated in the introduction, that the L-loop color space can be obtained by attaching rungs between any two external legs of the set of $(L-1)$-loop color factors; it would be nice to have a proof of this assumption.

Another obvious target for future work is the characterization of the color spaces and the null vectors for five-point (and higher) amplitudes of SO(N) and Sp(N). These were previously found for SU(N) in refs. [15, 16].

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A Group theory identities

Let T^a denote generators in the defining representation of SU(N), SO(N), or Sp(N), a set of $N \times N$ traceless hermitian matrices that in the case of SO(N) and Sp(N) satisfy additional conditions (see below). For Sp(N), N is even.

The generators are chosen to be orthonormal

$$\mathrm{Tr}(T^a T^b) = c\delta^{ab} \tag{A.1}$$

where c denotes the index of the defining representation. These matrices obey commutation relations

$$[T^a, T^b] = \tilde{f}^{abc} T^c \tag{A.2}$$

so that eqs. (A.1) and (A.2) imply[7]

$$\tilde{f}^{abc} = (1/c)\,\mathrm{Tr}(T^a, [T^b, T^c]) \tag{A.3}$$

which is manifestly totally antisymmetric. In the main body of the paper, we adopt the convention $c = 1$ for the index of the defining representation which is commonly used in the

[7]For the groups SU(2) and Sp(2), one has $\tilde{f}^{abc} = i\sqrt{2c}\,\epsilon^{abc}$, while for SO(3), $\tilde{f}^{abc} = i\sqrt{c/2}\,\epsilon^{abc}$.

amplitudes community. It is not difficult, however, to adapt our results to other conventions because all of the quantities considered scale homogeneously with c.

We now discuss each classical group separately.

A.1 SU(N)

Generators in the defining representation of SU(N) obey [36]

$$(T^a)_{ij}(T^a)_{kl} = c\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) . \tag{A.4}$$

Thus for arbitrary products of generators A and B, we have

$$\begin{aligned} \mathrm{Tr}(AT^a)\,\mathrm{Tr}(BT^a) &= c\left[\mathrm{Tr}(AB) - \frac{1}{N}\,\mathrm{Tr}(A)\,\mathrm{Tr}(B)\right] , \\ \mathrm{Tr}(AT^aBT^a) &= c\left[\mathrm{Tr}(A)\,\mathrm{Tr}(B) - \frac{1}{N}\,\mathrm{Tr}(AB)\right] . \end{aligned} \tag{A.5}$$

A.2 SO(N)

The generators for SO(N) satisfy

$$(T^a)^T = -T^a \tag{A.6}$$

where T denotes transpose. That is, they are antisymmetric as well as hermitian (and therefore purely imaginary). Equation (A.6) implies that T^a is traceless.

Generators in the defining representation of SO(N) obey [36]

$$(T^a)_{ij}(T^a)_{kl} = \frac{c}{2}\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right) . \tag{A.7}$$

Hence for arbitrary products of generators A and B, we have

$$\begin{aligned} \mathrm{Tr}(AT^a)\,\mathrm{Tr}(BT^a) &= \frac{c}{2}\Big[\mathrm{Tr}(AB) - \mathrm{Tr}(AB^T)\Big] , \\ \mathrm{Tr}(AT^aBT^a) &= \frac{c}{2}\Big[\mathrm{Tr}(A)\,\mathrm{Tr}(B) - \mathrm{Tr}(AB^T)\Big] . \end{aligned} \tag{A.8}$$

Using eq. (A.6), we have

$$B^T = (-1)^{n_B} B^R \tag{A.9}$$

where B^R denotes the product of generators B in reverse order, and n_B denotes the number of factors in B. Thus we can recast eq. (A.8) as [33]

$$\begin{aligned} \mathrm{Tr}(AT^a)\,\mathrm{Tr}(BT^a) &= \frac{c}{2}\Big[\mathrm{Tr}(AB) - (-1)^{n_B}\,\mathrm{Tr}(AB^R)\Big] , \\ \mathrm{Tr}(AT^aBT^a) &= \frac{c}{2}\Big[\mathrm{Tr}(A)\,\mathrm{Tr}(B) - (-1)^{n_B}\,\mathrm{Tr}(AB^R)\Big] . \end{aligned} \tag{A.10}$$

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A.3 Sp(N)

The generators for Sp(N) satisfy

$$(T^a)^T = JT^aJ \tag{A.11}$$

where J is an $N \times N$ matrix satisfying $J^2 = -1$ and $J^T = -J$, where N is even. Equation (A.11) implies that T^a is traceless.

Generators in the defining representation of Sp(N) obey [36]

$$(T^a)_{ij}(T^a)_{kl} = \frac{c}{2}\left(\delta_{il}\delta_{jk} - J_{ik}J_{jl}\right) . \tag{A.12}$$

Hence for arbitrary products of generators A and B, we have

$$\begin{aligned} \mathrm{Tr}(AT^a)\,\mathrm{Tr}(BT^a) &= \frac{c}{2}\Big[\mathrm{Tr}(AB) + \mathrm{Tr}(AJB^TJ)\Big] , \\ \mathrm{Tr}(AT^aBT^a) &= \frac{c}{2}\Big[\mathrm{Tr}(A)\,\mathrm{Tr}(B) - \mathrm{Tr}(AJB^TJ)\Big] . \end{aligned} \tag{A.13}$$

Using eq. (A.11), we have

$$B^T = (-1)^{n_B-1}JB^RJ \tag{A.14}$$

and so we can recast eq. (A.13) as [33]

$$\begin{aligned} \mathrm{Tr}(AT^a)\,\mathrm{Tr}(BT^a) &= \frac{c}{2}\Big[\mathrm{Tr}(AB) - (-1)^{n_B}\,\mathrm{Tr}(AB^R)\Big] , \\ \mathrm{Tr}(AT^aBT^a) &= \frac{c}{2}\Big[\mathrm{Tr}(A)\,\mathrm{Tr}(B) + (-1)^{n_B}\,\mathrm{Tr}(AB^R)\Big] . \end{aligned} \tag{A.15}$$

B Derivation of SO(N) null vectors

In this appendix, we prove the existence of two x_α-type SO(N) irreducible representations of S_4, which establishes the claim made in section 7 that the x_b-type color space is spanned by $L-2$ irreducible representations. We find the explicit form for these null vectors and also for the single x_γ-type irreducible representation.

B.1 x_α-type null vectors

We recall that the null vector $R^{(L)}_{x\alpha} = r^{(L)}_{x\alpha}x_\alpha$ must satisfy the condition (7.7), which is

$$\mathbf{c}^{(L)}_{xb}\mathbf{r}^{(L)T}_{x\alpha} = 0 \tag{B.1}$$

where $C^{(L)}_{xb} = c^{(L)}_{xb}x_b$ is an arbitrary L-loop x_b-type color factor for SO(N). We develop a recursive proof to construct the null vectors. Recall from eq. (6.18) that any L-loop x_b-type color factor may be expressed in terms of lower loop x_b-type color factors

$$c^{(L)}_{xb} = Kc^{(L-1)}_{xb} , \qquad c^{(L)}_{xb} = (K-6)c^{(L-2)}_{xb} , \qquad c^{(L)}_{xb} = (K-2)c^{(L-3)}_{xb} \tag{B.2}$$

via the operators g_1, $g_{bb}^{(2)}$, and $g_{bb}^{(3)}$. It is useful to express these equations in matrix form:

$$\mathbf{c}_{xb}^{(L)} = \mathbf{c}_{xb}^{(L-1)} G_1 \,, \qquad G_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} ,$$

$$\mathbf{c}_{xb}^{(L)} = \mathbf{c}_{xb}^{(L-2)} G_{bb}^{(2)} \,, \qquad G_{bb}^{(2)} = \begin{pmatrix} -6 & 1 & 0 & 0 & \cdots \\ 0 & -6 & 1 & 0 & \cdots \\ 0 & 0 & -6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} ,$$

$$\mathbf{c}_{xb}^{(L)} = \mathbf{c}_{xb}^{(L-3)} G_{bb}^{(3)} \,, \qquad G_{bb}^{(3)} = \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots \\ 0 & -2 & 1 & 0 & \cdots \\ 0 & 0 & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{B.3}$$

We may freely write

$$\mathbf{c}_{xb}^{(L)} = \mathbf{c}_{xb}^{(L)} \Pi_{L-1} \tag{B.4}$$

where Π_L is defined in eq. (5.3), since $c_{xb}^{(L)}$ is an $(L-1)$th degree polynomial. Using eqs. (B.3) and (B.4), we see that eq. (B.1) may be replaced by the following three equations

$$\begin{aligned} \mathbf{c}_{xb}^{(L-1)} \Pi_{L-2} G_1 \mathbf{r}_{x\alpha}^{(L)T} &= 0 \,, \\ \mathbf{c}_{xb}^{(L-2)} \Pi_{L-3} G_{bb}^{(2)} \mathbf{r}_{x\alpha}^{(L)T} &= 0 \,, \\ \mathbf{c}_{xb}^{(L-3)} \Pi_{L-4} G_{bb}^{(3)} \mathbf{r}_{x\alpha}^{(L)T} &= 0 \,. \end{aligned} \tag{B.5}$$

We will now construct two independent solutions of eq. (B.5). First define an infinite vector depending on an arbitrary real number n:

$$\boldsymbol{\lambda}_n = (1, n, n^2, \cdots) \qquad \Longrightarrow \qquad \lambda_n = \boldsymbol{\lambda}_n \mathbf{K}^T = \frac{1}{1 - nK} \,. \tag{B.6}$$

It is easy to check that

$$G_1 \boldsymbol{\lambda}_n^T = n \boldsymbol{\lambda}_n^T \,, \qquad G_{bb}^{(2)} \boldsymbol{\lambda}_n^T = (n-6) \boldsymbol{\lambda}_n^T \,, \qquad G_{bb}^{(3)} \boldsymbol{\lambda}_n^T = (n-2) \boldsymbol{\lambda}_n^T \,. \tag{B.7}$$

Next we consider a truncated version

$$\boldsymbol{\lambda}_n^{(L)} = \frac{1}{n^{L-1}} \boldsymbol{\lambda}_n \Pi_{L-1} = \frac{1}{n^{L-1}} (1, n, n^2, \cdots, n^{L-1}, 0, 0, \cdots) \Longrightarrow \boldsymbol{\lambda}_n^{(L)T} = \frac{1}{n^{L-1}} \Pi_{L-1} \boldsymbol{\lambda}_n^T \,. \tag{B.8}$$

The following relations are easy to check:

$$\Pi_{L-2} G_1 \Pi_{L-1} = \Pi_{L-2} G_1 \,, \quad \Pi_{L-3} G_{bb}^{(2)} \Pi_{L-1} = \Pi_{L-3} G_{bb}^{(2)} \,, \quad \Pi_{L-4} G_{bb}^{(3)} \Pi_{L-1} = \Pi_{L-4} G_{bb}^{(3)} \,. \tag{B.9}$$

Using eqs. (B.7) and (B.9) we observe that

$$\begin{aligned}
\Pi_{L-2} G_1 \boldsymbol{\lambda}_n^{(L)T} &= \boldsymbol{\lambda}_n^{(L-1)T} \,, \\
\Pi_{L-3} G_{bb}^{(2)} \boldsymbol{\lambda}_n^{(L)T} &= \left(\frac{n-6}{n^2}\right) \boldsymbol{\lambda}_n^{(L-2)T} \,, \\
\Pi_{L-4} G_{bb}^{(3)} \boldsymbol{\lambda}_n^{(L)T} &= \left(\frac{n-2}{n^3}\right) \boldsymbol{\lambda}_n^{(L-3)T} \,.
\end{aligned} \tag{B.10}$$

We now recursively prove that $\boldsymbol{\lambda}_n^{(L)}$ satisfies the conditions (B.5) to be the L-loop null vector $\mathbf{r}_{x\alpha}^{(L)}$. Assuming that $\boldsymbol{\lambda}_n^{(L)}$ satisfies the conditions (B.1) through $L-1$ loops, we plug eq. (B.10) into eq. (B.5) to see that they satisfy those conditions at L loops. We also need, however, to ensure consistency with the base case. Recall from eq. (7.9) that at two loops, the null vector is

$$\mathbf{r}_{x\alpha}^{(2)} = \left(\frac{1}{2}, 1, 0, \cdots\right) . \tag{B.11}$$

This is satisfied by $\boldsymbol{\lambda}_n^{(2)}$ only when $n=2$, so it appears we only have one solution, $\mathbf{r}_{x\alpha,1}^{(L)} = \boldsymbol{\lambda}_2^{(L)}$. However, we may also satisfy eq. (B.5) with a linear combination of two different $\boldsymbol{\lambda}_n^{(L)}$, provided that the constants in parentheses in eq. (B.10) are degenerate, which is the case for $n=3$ and $n=-6$. Thus, for any values of A and B, the vector

$$\mathbf{r}_{x\alpha,2}^{(L)} = A\boldsymbol{\lambda}_3^{(L)} + B\boldsymbol{\lambda}_{-6}^{(L)} \tag{B.12}$$

satisfies

$$\begin{aligned}
\Pi_{L-2} G_1 \mathbf{r}_{x\alpha,2}^{(L)T} &= \mathbf{r}_{x\alpha,2}^{(L-1)T} \,, \\
\Pi_{L-3} G_{bb}^{(2)} \mathbf{r}_{x\alpha,2}^{(L)T} &= -\frac{1}{3} \mathbf{r}_{x\alpha,2}^{(L-2)T} \,, \\
\Pi_{L-4} G_{bb}^{(3)} \mathbf{r}_{x\alpha,2}^{(L)T} &= \frac{1}{27} \mathbf{r}_{x\alpha,2}^{(L-3)T} \,.
\end{aligned} \tag{B.13}$$

Consistency with the base case (B.11) requires $A = \frac{4}{3}$ and $B = -\frac{1}{3}$ so that finally we have two solutions of eq. (B.5)

$$\begin{aligned}
\mathbf{r}_{x\alpha,1}^{(L)} &= \boldsymbol{\lambda}_2^{(L)} \,, \\
\mathbf{r}_{x\alpha,2}^{(L)} &= \frac{4}{3}\boldsymbol{\lambda}_3^{(L)} - \frac{1}{3}\boldsymbol{\lambda}_{-6}^{(L)} \,.
\end{aligned} \tag{B.14}$$

From eq. (B.8), we have

$$\lambda_n^{(L)} = \boldsymbol{\lambda}_n^{(L)} \mathbf{K}^T = \frac{1-(nK)^L}{n^{L-1}(1-nK)} \tag{B.15}$$

so we can conveniently express eq. (B.14) as polynomials of degree $L-1$:

$$\begin{aligned}
r_{x\alpha,1}^{(L)} &= \mathbf{r}_{x\alpha,1}^{(L)} \mathbf{K}^T = \frac{\left[1-(2K)^L\right]}{2^{L-1}(1-2K)} \,, \\
r_{x\alpha,2}^{(L)} &= \mathbf{r}_{x\alpha,2}^{(L)} \mathbf{K}^T = \frac{4\left[1-(3K)^L\right]}{3^L(1-3K)} + \frac{2\left[1-(-6K)^L\right]}{(-6)^L(1+6K)} \,.
\end{aligned} \tag{B.16}$$

B.2 x_γ-type null vectors

We will now establish that the single x_γ-type irreducible representation (for $L \geq 1$) has the form

$$\mathbf{r}_{x\gamma}^{(L)} = \frac{2}{3}\boldsymbol{\lambda}_3^{(L)} + \frac{1}{3}\boldsymbol{\lambda}_{-6}^{(L)} = \left(\frac{2}{3^L}\boldsymbol{\lambda}_3 - \frac{2}{(-6)^L}\boldsymbol{\lambda}_{-6}\right)\Pi_{L-1} \tag{B.17}$$

corresponding to a polynomial of degree $L-1$

$$r_{x\gamma}^{(L)} = \mathbf{r}_{x\gamma}^{(L)}\mathbf{K}^T = \frac{2\left[1-(3K)^L\right]}{3^L(1-3K)} - \frac{2\left[1-(-6K)^L\right]}{(-6)^L(1+6K)}\,. \tag{B.18}$$

We may write eq. (B.17) as

$$\mathbf{r}_{x\gamma}^{(L)} = \left(\frac{2}{3^L}\boldsymbol{\lambda}_3 - \frac{2}{(-6)^L}\boldsymbol{\lambda}_{-6}\right)\Pi_L \tag{B.19}$$

since the K^L term vanishes. For the matrices G_{ba} and $M_{b\gamma}$ defined in eqs. (7.16) and (7.23), one may ascertain that

$$\Pi_{L-2}G_{ba}\Pi_L = \Pi_{L-2}G_{ba}\,, \qquad \Pi_{L-1}M_{b\gamma}\Pi_L = \Pi_{L-1}M_{b\gamma} \tag{B.20}$$

and also that

$$G_{ba}\boldsymbol{\lambda}_n^T = (n+4)(n-1)\boldsymbol{\lambda}_n^T\,, \qquad M_{b\gamma}\boldsymbol{\lambda}_n^T = (n+3)\boldsymbol{\lambda}_n^T\,. \tag{B.21}$$

Using eq. (B.19) in eqs. (B.20) and (B.21) we have

$$\begin{aligned}\Pi_{L-2}G_{ba}\mathbf{r}_{x\gamma}^{(L)T} &= \Pi_{L-2}\left(\frac{28}{3^L}\boldsymbol{\lambda}_3^T - \frac{28}{(-6)^L}\boldsymbol{\lambda}_{-6}^T\right) = \frac{7}{3}\mathbf{r}_{x\alpha,2}^{(L-1)T}\,,\\ \Pi_{L-1}M_{b\gamma}\mathbf{r}_{x\gamma}^{(L)T} &= \Pi_{L-1}\left(\frac{12}{3^L}\boldsymbol{\lambda}_3^T + \frac{6}{(-6)^L}\boldsymbol{\lambda}_{-6}^T\right) = 3\mathbf{r}_{x\alpha,2}^{(L)T}\,.\end{aligned} \tag{B.22}$$

The conditions (7.22) and (7.23) for the x_γ-type null vector may be written, using eq. (B.4), as

$$0 = \mathbf{c}_{xb}^{(L-1)}\Pi_{L-2}G_{ba}\mathbf{r}_{x\gamma}^{(L)T}\,, \qquad 0 = \mathbf{c}_{xb}^{(L)}\Pi_{L-1}M_{b\gamma}\mathbf{r}_{x\gamma}^{(L)T}\,. \tag{B.23}$$

Finally, using eqs. (B.22) and (B.1), we see that eq. (B.23) is satisfied by eq. (B.17).

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