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Derived invariants from topological Hochschild homology

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Abstract

We consider derived invariants of varieties in positive characteristic arising from topological Hochschild homology. Using theory developed by Ekedahl and Illusie–Raynaud in their study of the slope spectral sequence, we examine the behavior under derived equivalences of various p-adic quantities related to Hodge–Witt and crystalline cohomology groups, including slope numbers, domino numbers, and Hodge–Witt numbers. As a consequence, we obtain restrictions on the Hodge numbers of derived equivalent varieties, partially extending results of Popa–Schnell to positive characteristic.

1. Introduction

In this paper we study derived invariants of varieties in positive characteristic.

- Let X and Y be smooth proper k-schemes for some field k. We say that X and Y are *Fourier–Mukai equivalent*, or *FM-equivalent*, if there is a complex $P \in \mathcal{D}^b(X \times_k Y)$ such that the induced functor $\Phi_P \colon \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is an equivalence, where $\mathcal{D}^b(-)$ denotes the dg category of bounded complexes of coherent sheaves. This is equivalent to asking for $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ to be equivalent as k-linear dg categories (see [Toë07]). When X and Y are smooth and projective, they are FM-equivalent if and only if there is a k-linear triangulated equivalence $\mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$, by Orlov's theorem (see [Huy06, Theorem 5.14]).
- Let h be a numerical or categorical invariant of smooth proper k-schemes. We say that h is a *derived invariant* if whenever there is a Fourier–Mukai equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$, we have $h(X) = h(Y)$.

The Hochschild homology groups $\mathrm{HH}_*(X/k)$ of a smooth proper variety X, which are finite-dimensional vector spaces over k, are derived invariants. In characteristic zero and in characteristic p for $p \geqslant d = \dim X$, the Hochschild–Kostant–Rosenberg isomorphism [HKR62] relates the Hochschild homology groups to Hodge cohomology groups $\mathrm{H}^*(X, \Omega^*_X)$. We briefly review this story in Section 2. This relationship has been extensively studied and plays a key role in our understanding of derived categories of varieties, especially over the complex numbers.

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Now suppose that k is a perfect field of characteristic $p > 0$. With a smooth proper variety X over k we may associate via topological methods certain p-adic analogs of Hochschild homology: the topological Hochschild homology groups $\mathrm{THH}_*(X)$, as well as the related groups $\mathrm{TR}_*(X)$ and $\mathrm{TP}_*(X)$. We recall this theory in Section 3. These are modules over $W = W(k)$ (the ring of Witt vectors of k) which are equipped with certain extra semilinear structures, and whose construction moreover depends only on $\mathcal{D}^b(X)$. Furthermore, by a result of Hesselholt [Hes96], the $\mathrm{TR}_*(X)$ may be computed in terms of the Hodge–Witt cohomology groups $\mathrm{H}^*(W\Omega^*_X)$; compare with the classical relationship between Hochschild homology and Hodge cohomology. Our goal in this paper is to study these objects as derived invariants of the variety X.

Our key technical results are obtained in Section 4, where we analyze the spectral sequence connecting Hodge–Witt cohomology groups to $\mathrm{TR}_*(X)$ and study the extent to which the extra structures of F, V, d are preserved. We also recall the theory of coherent R-modules introduced by Illusie and Raynaud [IR83] in their study of the slope spectral sequence and explain how our noncommutative structures fit into this framework.

For the remainder of the paper, we study consequences of this theory. In Section 5, we observe the following fact, which extends results of Bragg–Lieblich [BL18]. Given a smooth proper d-dimensional k-scheme, following Artin–Mazur [AM77], we let Φ^d_X be the functor on augmented Artin local k-algebras defined by $\Phi^d_X(A) = \ker\left(\mathrm{H}^d(X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, A, \mathbb{G}_m) \to \mathrm{H}^d(X, \mathbb{G}_m)\right)$.

Proposition 1.1. *Let X and Y be Calabi–Yau d-folds over a perfect field k of positive characteristic. If X and Y are FM-equivalent, then $\Phi^d_X \cong \Phi^d_Y$. In particular, the heights of X and Y are equal.*

When $d = 2$, one calls Φ^2_X the *formal Brauer group* of X. Proposition 1.1 implies in particular that the height of a K3 surface is a derived invariant, which was already known.

We then study derived invariants of surfaces in more detail. We find another proof that the Artin invariant is a derived invariant of supersingular K3 surfaces and recover a result of Tirabassi on Enriques surfaces.

In Sections 5.2–5.4, we introduce various numerical p-adic invariants. Specializing to the case of varieties of dimension $d \leqslant 3$, we prove the following (for definitions, see the body of the paper).

Theorem 1.2. *Let k be a perfect field of positive characteristic p, let $W = W(k)$ be the ring of p-typical Witt vectors over k, and let $K = W\left[p^{-1}\right]$ be the fraction field of W. The following are derived invariants of smooth proper varieties of dimension $d \leqslant 3$ over k:*

(i) *the slopes of Frobenius with multiplicity acting on the rational Hodge–Witt cohomology groups $\mathrm{H}^j\left(W\Omega^i_X\right) \otimes_W K$ and rational crystalline cohomology groups $\mathrm{H}^i(X/K)$,*
(ii) *the domino numbers $T^{i,j}$,*
(iii) *the Hodge–Witt numbers $h^{i,j}_W$,*
(iv) *the Zeta function $\zeta(X)$ (if X is projective and defined over a finite field), and*
(v) *the Betti numbers $b_i = \dim_K \mathrm{H}^i(X/K)$.*

Parts (iv) and (v) were previously proved by Honigs in [Hon18], as well as by Achter, Casalaina–Martin, and Vial in [ACV19]. The methods of these papers also suffice to prove part (i). Parts (ii) and (iii) are new, and their proof crucially relies on the topological derived invariants discussed in this paper.

In Section 5.5, we consider the question of whether the Hodge numbers $h^{i,j} = \dim_k \mathrm{H}^j\left(X, \Omega^i_X\right)$ are derived invariants. For context, we note that a conjecture of Orlov [Orl05] states that the rational Chow motive $\mathfrak{h}_X$ of a variety X is a derived invariant. In characteristic zero, the Hodge numbers are determined by the rational Chow motive, and so Orlov's conjecture implies that the Hodge numbers are derived invariant. This consequence has been verified by Popa–Schnell [PS11] for varieties of dimension $d \leqslant 3$. However, as discussed in Section 5.5, their proof breaks in several ways in positive characteristic. In characteristic p, the Hodge numbers are related (in a somewhat subtle way) to Hodge–Witt cohomology groups. Using this relationship and Theorem 1.2, we prove the following.

Theorem 1.3. *Suppose that X and Y are FM-equivalent smooth proper varieties of dimension d over a field k of positive characteristic.*

(i) *If $d \leqslant 2$, then $h^{i,j}(X) = h^{i,j}(Y)$ for all i, j.*
(ii) *If $d \leqslant 3$, then $\chi(\Omega^i_X) = \chi(\Omega^i_Y)$ for all i.*

We remark that our proof of this result uses topological Hochschild homology constructions in a key way. Even in the case of surfaces, we do not know a direct proof using only Hochschild homology.

Under a mild additional assumption, we are able to strengthen this result for $d = 3$. We say that a smooth proper variety X over a perfect field k of positive characteristic is *Mazur–Ogus* if the Hodge–de Rham spectral sequence for X degenerates at E_1 and the crystalline cohomology groups of X are torsion-free. The class of Mazur–Ogus schemes includes smooth complete intersections, abelian varieties, and K3 surfaces.

Theorem 1.4. *Suppose that X and Y are FM-equivalent smooth proper varieties of dimension $d = 3$ over a perfect field k of positive characteristic $p \geqslant 3$. If X is Mazur–Ogus, then so is Y, and $h^{i,j}(X) = h^{i,j}(Y)$ for all i, j.*

However, recent work [AB21] of Addington and the second-named author gives examples of derived equivalent smooth projective 3-folds in characteristic 3 with different Hodge numbers. The appendix of Petrov to their paper shows that $h^{0,3}$ is not a derived invariant in any characteristic.

Finally, in Section 6, we compute TR and TP for twisted K3 surfaces and discuss how to recover the fine structure of the Mukai lattice from TP.

Conventions. We will use many spectral sequences in this paper. They all converge for smooth and proper schemes or dg categories, so we will say nothing more about convergence in our discussion.

If X is a smooth proper scheme over k, we will write $\mathrm{H}^*(X/W)$ for the crystalline cohomology groups of X relative to $W = W(k)$ and $\mathrm{H}^*(W\Omega^*_X) = \mathrm{H}^*(X, W\Omega^*_X)$ for the Hodge–Witt cohomology groups of X, as defined in [Ill79].

2. Hochschild homology

Let k be a commutative ring. For any k-linear dg category $\mathcal{C}$, the Hochschild homology of $\mathcal{C}$ over k is an object $\mathrm{HH}(\mathcal{C}/k) \in \mathcal{D}(k)$ which is equipped with an action of the circle S^1. The homology groups $\mathrm{H}_i(\mathrm{HH}(\mathcal{C}/k)) = \mathrm{HH}_i(\mathcal{C}/k)$ are the Hochschild homology groups of $\mathcal{C}$ over k. For a scheme X/k, we let $\mathrm{HH}(X/k) = \mathrm{HH}(\mathcal{P}\mathrm{erf}(X)/k)$, where $\mathcal{P}\mathrm{erf}(X)$ is the k-linear dg category of perfect complexes on X. This is a *noncommutative invariant* of k-schemes, meaning in particular that if $\mathcal{P}\mathrm{erf}(X) \simeq \mathcal{P}\mathrm{erf}(Y)$, then $\mathrm{HH}(X/k) \simeq \mathrm{HH}(Y/k)$ as complexes with S^1-action. For some details on Hochschild homology and the constructions below from a classical perspective, see [Lod98]; for details on Hochschild homology from a modern perspective, see [BMS19].

While the Hochschild homology of a smooth proper k-scheme X is a derived invariant, one often computes it via the following spectral sequence, which is not.

Definition 2.1. The *Hochschild–Kostant–Rosenberg (HKR) spectral sequence*

$$\mathrm{E}_2^{s,t} = \mathrm{H}^t\big(X, \Omega^s_X\big) \Rightarrow \mathrm{HH}_{s-t}(X/k) \tag{1}$$

is the descent, or local-to-global, spectral sequence for Hochschild homology.[1] Here, $\Omega^*_X = \Omega^*_{X/k}$ are the sheaves of de Rham forms relative to k.

The HKR spectral sequence is known to degenerate for smooth schemes in characteristic zero, or more generally when $\dim(X)!$ is invertible in k by [Yek02]. It also degenerates in characteristic p when $\dim(X) \leqslant p$, by [AV20]. In general, when $\dim(X) > p$, the HKR spectral sequence does not degenerate; for examples, see [ABM21]. If (1) degenerates, then there exist noncanonical isomorphisms

$$\mathrm{HH}_i(X/k) \cong \bigoplus_j \mathrm{H}^{j-i}\big(X, \Omega^j_X\big)$$

of k-vector spaces for each i. The above discussion implies the following well-known result.

Theorem 2.2. *Let X and Y be FM-equivalent smooth proper schemes of dimension d over a field k of characteristic p. If $p = 0$ or $p \geqslant d$, there exist isomorphisms*

$$\bigoplus_j \mathrm{H}^{j-i}\big(X, \Omega^j_X\big) \cong \bigoplus_j \mathrm{H}^{j-i}\big(Y, \Omega^j_Y\big)\,.$$

In particular,

$$\sum_j h^{j,j-i}(X) = \sum_j h^{j,j-i}(Y)$$

for all i.

From Hochschild homology, one constructs several other noncommutative invariants, namely the cyclic homology $\mathrm{HC}(\mathcal{C}/k) = \mathrm{HH}(\mathcal{C}/k)_{hS^1}$ obtained using the S^1-homotopy orbits, the negative cyclic homology $\mathrm{HC}^-(\mathcal{C}/k) = \mathrm{HH}(\mathcal{C}/k)^{hS^1}$ obtained using the S^1-homotopy fixed points, and the periodic cyclic homology $\mathrm{HP}(\mathcal{C}/k) = \mathrm{HH}(\mathcal{C}/k)^{tS^1}$ obtained using the S^1-Tate construction. See [Lod98] for background. For a k-scheme X, each of these theories is computed by two spectral sequences: a noncommutative spectral sequence and a de Rham spectral sequence. We review the theory for $\mathrm{HP}(X/k)$; the other cases are similar.

Definition 2.3. By the definition of the Tate construction (see for example [NS18]), there is a *Tate spectral sequence*

$$\mathrm{E}_2^{s,t} = \widehat{\mathrm{H}}^s\big(\mathbb{CP}^\infty, \mathrm{HH}_t(X/k)\big) \Rightarrow \mathrm{HP}_{t-s}(X/k) \tag{2}$$

computing $\mathrm{HP}(X/k)$, with differentials d_r of bidegree $(r, r-1)$, where $\widehat{\mathrm{H}}^*\big(\mathbb{CP}^\infty, -\big)$ is a 2-periodic version of the cohomology of $\mathbb{CP}^\infty$. When computing $\mathrm{HP}_*(X/k)$ via a mixed complex as in [Lod98], this is the spectral sequence arising from the filtration by columns. This is often called the noncommutative Hodge–de Rham spectral sequence.

[1] With this indexing, the differentials d_r have bidegree $(r-1, r)$; this convention has the advantage that the E_2-page of (1) agrees with the E_1-page of the Hodge–de Rham spectral sequence (4).

Definition 2.4. Let X be a smooth and proper k-scheme. There is a *de Rham–HP spectral sequence*

$$\mathrm{E}_2^{s,t} = \mathrm{H}_{\mathrm{dR}}^{s-t}(X/k) \Rightarrow \mathrm{HP}_{-s-t}(X/k)\,, \tag{3}$$

with differentials d_r of bidegree $(r, 1-r)$, where $\mathrm{R}\Gamma_{\mathrm{dR}}(X/k)$ is the de Rham cohomology of X/k and $\mathrm{H}_{\mathrm{dR}}^{s-t}(X/k) = \mathrm{H}^{s-t}(\mathrm{R}\Gamma_{\mathrm{dR}}(X/k))$. This spectral sequence was constructed in [BMS19] in the p-adically complete situation and in [Ant19] in general. In characteristic zero, it can easily be extracted from [TV11].

Finally, for a smooth proper k-scheme, we have the *Hodge–de Rham spectral sequence*

$$\mathrm{E}_1^{s,t} = \mathrm{H}^t\big(X, \Omega^s_{X/k}\big) \Rightarrow \mathrm{H}_{\mathrm{dR}}^{s+t}(X/k)\,, \tag{4}$$

which has differentials d_r of bidegree $(r, 1-r)$. We summarize our situation in Figure 1.

$$\begin{tikzcd}
& \mathrm{HH}_*(X/k) \arrow[dr, Rightarrow, "\text{Tate}"] & \\
\mathrm{H}^*(X,\Omega^*_X) \arrow[ur, Rightarrow, "\text{HKR}"] \arrow[dr, Rightarrow, "\text{Hodge–de Rham}"'] & & \mathrm{HP}_*(X/k) \\
& \mathrm{H}^*_{\mathrm{dR}}(X/k) \arrow[ur, Rightarrow, "\text{de Rham–HP}"'] &
\end{tikzcd}$$

Figure 1. Four spectral sequences associated with a smooth proper scheme X over k

The Tate spectral sequence (2) itself, meaning the collection of pages and differentials, is a derived invariant. However, there is no reason for the de Rham–HP spectral sequence (3) to be a derived invariant, although the objects it computes are derived invariants.

Remark 2.5. Playing these spectral sequences off of each other can be profitable. For example, if k is a field and if X/k is smooth and proper and if the HKR spectral sequence (1) degenerates (for example, if $\dim(X) \leqslant p$), then the degeneration of the Tate spectral sequence (2) implies degeneration of the de Rham–HP spectral sequence (3) *and* the Hodge–de Rham spectral sequence (4). Similarly, if the HKR and Hodge–de Rham spectral sequences degenerate, then the Tate spectral sequence degenerates if and only if the de Rham–HP spectral sequence degenerates.

If k is a perfect field, the Tate spectral sequence (2) computing HP degenerates when $\mathcal{C}$ is smooth and proper over k, $\mathrm{HH}_i(\mathcal{C}/k) = 0$ for $i \notin [-p, p]$, and $\mathcal{C}$ lifts to $W_2(k)$, by work of Kaledin [Kal08, Kal17] (see also Mathew's paper [Mat20]). Using this fact, we prove a theorem which implies Hodge–de Rham degeneration in many cases.

Theorem 2.6. *Let k be a perfect field of positive characteristic p. Let X and Y be smooth proper schemes such that $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ and $\dim(X) = \dim(Y) \leqslant p$. If X lifts to $W_2(k)$, then the Hodge–de Rham spectral sequence degenerates for Y.*

Proof. The HKR spectral sequence (1) degenerates for both X and Y by [AV20]. Since X lifts to $W_2(k)$, the Tate spectral sequence (2) degenerates for X. This tells us the total dimension of $\mathrm{HP}(X/k) \simeq \mathrm{HP}(Y/k)$ and hence implies that the Tate spectral sequence (2) degenerates for Y as well. The existence of the convergent de Rham–HP spectral sequence (3) now implies that the Hodge–de Rham spectral sequence (4) degenerates for Y, as one can see by counting dimensions. □

The theorem is some evidence for a positive answer to the following question.

Question 2.7 (Lieblich). Let X and Y be FM-equivalent smooth proper varieties over a perfect field k of positive characteristic p. If X lifts to characteristic zero (or lifts to $W_2(k)$, etc.), does Y also lift?

3. Topological Hochschild homology

In this section, we introduce the main tools of this paper, which are topological analogs of the invariants of the previous section. Here, topological means that one works relative to the sphere spectrum $\mathbb{S}$, which is the initial commutative ring (spectrum) in homotopy theory. With any stable ∞-category or dg category $\mathcal{C}$, one can associate a spectrum $\mathrm{THH}(\mathcal{C}) = \mathrm{HH}(\mathcal{C}/\mathbb{S})$ with S^1-action. This is again a noncommutative invariant, and there are various analogs of the spectral sequences of the previous section. We will especially be interested in the *topological periodic cyclic homology*

$$\mathrm{TP}(\mathcal{C}) = \mathrm{THH}(\mathcal{C})^{tS^1}.$$

Topological Hochschild homology $\mathrm{THH}(\mathcal{C})$ is equipped with an even richer structure than simply an S^1-action: it is a cyclotomic spectrum, a notion introduced by Bökstedt–Hsiang–Madsen [BHM93] to study algebraic K-theory and recently recast by Nikolaus and Scholze in [NS18]. We use the following definition, which is a slight alteration of the main definition of [NS18].

Definition 3.1. A *p-typical cyclotomic spectrum* is a spectrum X with an S^1-action together with an S^1-equivariant map $X \to X^{tC_p}$, called the cyclotomic Frobenius, where X^{tC_p} is equipped with the S^1-action coming from the isomorphism $S^1 \cong S^1/C_p$. We let $\mathbf{CycSp}_p$ denote the stable ∞-category of p-typical cyclotomic spectra. If k is a commutative ring, then $\mathrm{THH}(k)$ is a commutative algebra object of $\mathbf{CycSp}_p$ and we let $\mathbf{CycSp}_{\mathrm{THH}(k)}$ denote the stable ∞-category of $\mathrm{THH}(k)$-modules in p-typical cyclotomic spectra.

The exact nature of a cyclotomic spectrum will not concern us much, except in the extraction of homotopy objects with respect to a natural t-structure on cyclotomic spectra studied in [AN21].

Definition 3.2. A *p-typical Cartier module* is an abelian group M equipped with endomorphisms F and V such that $FV = p$ on M. A *Dieudonné module* over a perfect field k is a W-module (recall that $W = W(k)$) M equipped with endomorphisms F and V satisfying $FV = VF = p$ and which are compatible with the Witt vector Frobenius σ in the sense that

$$F(am) = \sigma(a)F(m) \quad \text{and} \quad V(\sigma(a)m) = aV(m)$$

for $a \in W$ and $m \in M$. Note that we do not require that M be finitely generated or torsion-free.

A Cartier or Dieudonné module M is *derived V-complete* if the natural map $M \to \mathrm{R}\lim_n M//V^n$ is an equivalence, where $M//V^n$ is the cofiber of $V^n \colon M \to M$ in the derived category of abelian groups. Let $\widehat{\mathrm{Cart}}_p$ denote the abelian category of derived V-complete Cartier modules, and let $\widehat{\mathrm{Dieu}}_k$ denote the abelian category of derived V-complete Dieudonné modules over k.

The full subcategory of $\mathbf{CycSp}_p$ of p-typical cyclotomic spectra X such that $\pi_i X = 0$ for $i < 0$ defines the connective part of a t-structure on $\mathbf{CycSp}_p$ and similarly for $\mathbf{CycSp}_{\mathrm{THH}(k)}$. The main theorem of [AN21] identifies the heart.

Theorem 3.3 ([AN21]). *Let k be a perfect field of positive characteristic p. There are equivalences of abelian categories $\mathbf{CycSp}_p^{\heartsuit} \simeq \widehat{\mathrm{Cart}}_p$ and $\mathbf{CycSp}_{\mathrm{THH}(k)}^{\heartsuit} \simeq \widehat{\mathrm{Dieu}}_k$.*

With a p-typical cyclotomic spectrum X, we can associate a new spectrum $\mathrm{TR}(X)$ with S^1-action and with natural endomorphisms F and V making the homotopy groups of $\mathrm{TR}(X)$ into Cartier modules. The construction of $\mathrm{TR}(X)$ was introduced by Hesselholt in [Hes96]. It is proven in [AN21] that $\pi_i\mathrm{TR}(X) \cong \pi_i^{\mathrm{cyc}}(X)$ as Cartier modules under the equivalence of Theorem 3.3, where $\pi_i^{\mathrm{cyc}}(X)$ denotes the ith homotopy object of X with respect to the t-structure on cyclotomic spectra. In the case of a scheme X, we let $\mathrm{TR}(X) = \mathrm{TR}(\mathcal{P}\mathrm{erf}(X))$. If k is a perfect field of positive characteristic p, then for any cyclotomic spectrum $X \in \mathbf{CycSp}_{\mathrm{THH}(k)}$, the homotopy groups $\mathrm{TR}_i(X) = \pi_i\mathrm{TR}(X)$ are equipped with differentials

$$\mathrm{TR}_i(X) \xrightarrow{d} \mathrm{TR}_{i+1}(X)$$

coming from the S^1-action making $\mathrm{TR}_*(X)$ into an R-module, where R is the Cartier–Dieudonné–Raynaud ring; see Section 4.

Example 3.4. When k is a perfect field of positive characteristic p, the object $\mathrm{THH}(k)$ is in the heart $\mathbf{CycSp}^{\heartsuit}_{\mathrm{THH}(k)}$ and corresponds to the ring of Witt vectors $W = W(k)$ with its Witt vector Frobenius and Vershiebung maps. In this language, the result is due to [AN21, Theorem 2], but the underlying computation that $\pi_*\mathrm{TR}(k) \cong W$ is due to Hesselholt–Madsen [HM97, Theorem 5.5]. The fact should be compared to the fundamental computation of Bökstedt which says that $\pi_*\mathrm{THH}(k) \cong k[b]$, where $|b| = 2$ (see [Bök85] for the case $k = \mathbb{F}_p$ and [HM97, Corollary 5.5] for the general case).

Let A be a smooth commutative k-algebra, where k is a perfect field of positive characteristic p. The homotopy groups of $\mathrm{THH}(A)$ are computed in [Hes96], where it is shown that $\pi_*\mathrm{THH}(A) \cong \Omega^*_{A/k} \otimes_k k[b]$, where $|b| = 2$ and $\Omega^i_{A/k}$ lives in homological degree i. This is analogous to the HKR isomorphism for Hochschild homology but more complicated thanks to Bökstedt's class b.

However, when working with TR instead, Hesselholt proved in [Hes96] an exact de Rham–Witt analog of the HKR isomorphism: there is a graded isomorphism $\mathrm{TR}_*(A) \cong W\Omega^*_A$ compatible with the F, V, and d operations, where $W\Omega^\bullet_A$ is the de Rham–Witt complex of A as studied in [Ill79], and we write $W\Omega^*_A$ for the graded abelian group underlying the complex $W\Omega^\bullet_A$.

Now, we can define the topological or crystalline analogs of the four spectral sequences from Section 2.

Definition 3.5. Let X be a smooth proper scheme over a perfect field k of positive characteristic p, and let $\mathcal{C}$ be a smooth proper dg category over k.

(i) Using Hesselholt's local calculation, the *descent spectral sequence* for TR is

$$\mathrm{E}_2^{s,t} = \mathrm{H}^t\big(X, W\Omega^s_X\big) \Rightarrow \mathrm{TR}_{s-t}(X)\,. \tag{5}$$

With this indexing, the differentials d_r have bidegree $(r-1, r)$. This is the topological analog of (1).

(ii) There is a *Tate spectral sequence*

$$\mathrm{E}_2^{s,t} = \widehat{\mathrm{H}}^s\big(\mathbb{CP}^\infty, \mathrm{TR}_t(\mathcal{C})\big) \Rightarrow \mathrm{TP}_{t-s}(\mathcal{C}) \tag{6}$$

computing $\mathrm{TP}(\mathcal{C})$, with differentials d_r of bidegree $(r, r-1)$. This is the TP analog of (2). It was constructed in [AN21, Corollary 10]. See Figure 3. By analogy with (2), this spectral sequence could be called the "noncommutative slope spectral sequence."

(iii) There is a *crystalline–*TP *spectral sequence*

$$\mathrm{E}_2^{s,t} = \mathrm{H}^{s-t}_{\mathrm{crys}}(X/W) \Rightarrow \mathrm{TP}_{-s-t}(X)\,, \tag{7}$$

with differentials d_r of bidegree $(r, 1-r)$. This is the topological analog of (3) and is due to [BMS19].

(iv) The Hodge–de Rham spectral sequence (4) is replaced by the *slope spectral sequence*

$$\mathrm{E}_1^{s,t} = \mathrm{H}^t\big(X, W\Omega^s_X\big) \Rightarrow \mathrm{H}^{s+t}_{\mathrm{crys}}(X/W) \tag{8}$$

of [Ill79]; it has differentials d_r of bidegree $(r, 1-r)$.

Figure 2 gives the topological analog of Figure 1.

$$\begin{tikzcd}
& \mathrm{TR}_*(X) \arrow[dr, Rightarrow, "\text{Tate}"] & \\
\mathrm{H}^*(X, W\Omega^*_X) \arrow[ur, Rightarrow, "\text{descent}"] \arrow[dr, Rightarrow, "\text{slope}"'] & & \mathrm{TP}_*(X) \\
& \mathrm{H}^*(X/W) \arrow[ur, Rightarrow, "\text{crystalline–TP}"'] &
\end{tikzcd}$$

FIGURE 2. Four spectral sequences associated with a smooth proper scheme X over a perfect field k of positive characteristic

Remark 3.6. As in the case of Hochschild homology, the Tate spectral sequence is a noncommutative invariant, but *a priori* the other three spectral sequences are not.

Remark 3.7. The W-modules appearing in Figure 2 are equipped with various extra structures, and the interactions of these structures with the four spectral sequences will be critical in our analysis. Specifically, we have the following:

(i) The groups $\mathrm{H}^*(W\Omega^*_X)$ and $\mathrm{TR}_*(X)$ are Dieudonné modules, and the descent spectral sequence (5) takes place in the abelian category of derived V-complete Dieudonné modules. In particular, the differentials on each page commute with F and V.

(ii) The groups $\mathrm{H}^*(X/K) = \mathrm{H}^*(X/W) \otimes_W K$ and $\mathrm{TP}_*(X) \otimes_W K$ are F-isocrystals (see Definition 5.9), and up to certain Tate twists, the crystalline–TP spectral sequence (7) is compatible with these F-isocrystal structures.

(iii) The differentials in the slope (8) and Tate (6) spectral sequences do *not* commute with F and V. Rather, they satisfy Relations 4.2.

We will need the following proposition.

PROPOSITION 3.8. *If X is smooth and proper over a perfect field of positive characteristic p, then the four spectral sequences of Definition* 3.5 *degenerate rationally.*

Proof. For the slope spectral sequence, this is due to Illusie–Raynaud [IR83]. For the crystalline–TP spectral sequence, it is proved by Elmanto in [Elm18] using an argument of Scholze. The other two cases follow by counting dimensions. □

Remark 3.9. Suppose that $\mathcal{C}$ is a smooth proper dg category over a perfect field k, and suppose that $\mathcal{C}$ lifts to a smooth proper dg category $\widetilde{\mathcal{C}}$ over W. Then, $\mathrm{TP}(\mathcal{C})$ is equivalent to $\mathrm{HP}\big(\widetilde{\mathcal{C}}/W\big)$ by an unpublished argument of Scholze. In this way, one might view TP as a noncommutative version of crystalline cohomology. See also work of Petrov and Vologodsky [PV19].

$$\begin{tikzcd}
 & \vdots & & \vdots & & \vdots & \\
\cdots & \mathrm{TR}_{t+1}(\mathcal{C}) & 0 & \mathrm{TR}_{t+1}(\mathcal{C}) & 0 & \mathrm{TR}_{t+1}(\mathcal{C}) & \cdots \\
\cdots & \mathrm{TR}_{t}(\mathcal{C}) \arrow[urr] & 0 & \mathrm{TR}_{t}(\mathcal{C}) \arrow[urr] & 0 & \mathrm{TR}_{t}(\mathcal{C}) & \cdots \\
\cdots & \mathrm{TR}_{t-1}(\mathcal{C}) \arrow[urr] & 0 & \mathrm{TR}_{t-1}(\mathcal{C}) \arrow[urr] & 0 & \mathrm{TR}_{t-1}(\mathcal{C}) & \cdots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

Figure 3. A portion of the E_2-page of the Tate spectral sequence (6) computing TP. The Tate spectral sequence is 2-periodic in the columns, and for $\mathcal{C}$ smooth and proper over a perfect field of characteristic p it is bounded in the rows by [AN21, Corollary 5].

4. Compatibility of the descent and slope spectral sequences

For the remainder of the paper, we use the following notation.

Notation 4.1. Unless otherwise stated, k denotes a perfect field of characteristic $p > 0$. We write $W = W(k)$ for the ring of Witt vectors of k and $K = W\left[p^{-1}\right]$ for the field of fractions of W. We let σ denote the Frobenius on W.

Let X be a smooth proper scheme over k. The Hodge–Witt cohomology groups $\mathrm{H}^*(W\Omega_X^*)$ and the homotopy groups $\mathrm{TR}_*(X)$ are modules over W and are equipped with σ-linear operators F, V, which satisfy the relations $FV = VF = p$. That is, they are Dieudonné modules in the sense of Definition 3.2. The differentials $\mathrm{H}^j\left(W\Omega_X^i\right) \to \mathrm{H}^j\left(W\Omega_X^{i+1}\right)$ and $\mathrm{TR}_i(X) \to \mathrm{TR}_{i+1}(X)$ do not commute with F and V but instead satisfy Relations 4.2. Following Illusie–Raynaud [IR83], we formalize these properties using the *Cartier–Dieudonné–Raynaud algebra* relative to k, which is the graded ring

$$R = R^0 \oplus R^1 \tag{9}$$

generated by W and by operations F and V in degree 0 and by d in degree 1, subject to Relations 4.2.

Relations 4.2. The following relations hold in the Raynaud ring:

- $FV = VF = p$;
- $Fa = \sigma(a)F$, $V\sigma(a) = aV$, and $da = ad$ for all $a \in W$;
- $d^2 = 0$ and $FdV = d$.

Note that $VF = p = FV$ and $FdV = d$ imply the standard relations $Vd = pdV$ and $dF = pFd$. In particular, $R^0 = W_\sigma[F, V]$ is the usual Raynaud algebra. Note also that a (left) module over R^0 is the same thing as a Dieudonné module in the sense of Definition 3.2. We will use the terms interchangeably. A graded left R-module is a complex

$$M^\bullet = \left[\cdots \to M^{i-1} \xrightarrow{d} M^i \xrightarrow{d} M^{i+1} \to \cdots\right]$$

of W-modules whose terms are R^0-modules and whose differential d satisfies $FdV = d$. We will refer to such an object simply as an *R-module*.

Remark 4.3. A similar structure is studied by Bhatt–Lurie–Mathew in [BLM21]. They introduce the notion of a Dieudonné complex. A saturated Dieudonné complex in the sense of [BLM21, Definition 2.2.1] is naturally an R-module for the Raynaud algebra relative to $\mathbb{F}_p$ by [BLM21, Proposition 2.2.4].

If M is an R-module, we denote by $M[n]$ the graded module with degree shifted by n so that $M[n]^i = M^{i-n}$. Note that this notation is not consistent with that of [Eke86, Eke84], although we will use the sign conventions from op. cit.

For each j, the differentials in the de Rham–Witt complex make the complex

$$\mathrm{H}^j\big(W\Omega^\bullet_X\big) \overset{\text{def}}{=\!=} \big[0 \to \mathrm{H}^j(W\mathscr{O}_X) \xrightarrow{d} \mathrm{H}^j\big(W\Omega^1_X\big) \xrightarrow{d} \mathrm{H}^j\big(W\Omega^2_X\big) \xrightarrow{d} \cdots\big] \tag{10}$$

into an R-module, where $\mathrm{H}^j\big(W\Omega^i_X\big)$ is placed in degree i (see [Ill79]).[2] Similarly, by [AN21, Section 6.2], the S^1-action on $\mathrm{TR}(X)$ gives rise to a complex

$$\mathrm{TR}_\bullet(X) \overset{\text{def}}{=\!=} \big[\cdots \xrightarrow{d} \mathrm{TR}_{t-1}(X) \xrightarrow{d} \mathrm{TR}_t(X) \xrightarrow{d} \mathrm{TR}_{t+1}(X) \xrightarrow{d} \cdots\big], \tag{11}$$

which is an R-module, where $\mathrm{TR}_t(X)$ is placed in degree t.

The E_1-page of the slope spectral sequence (8) is the same as the E_2-page of the descent spectral sequence (5) and is depicted in Figure 4.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & \mathrm{H}^{t+3}\big(W\Omega^{s-1}_X\big) \arrow[r] & \mathrm{H}^{t+3}\big(W\Omega^{s}_X\big) \arrow[r] & \mathrm{H}^{t+3}\big(W\Omega^{s+1}_X\big) \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^{t+2}\big(W\Omega^{s-1}_X\big) \arrow[r] & \mathrm{H}^{t+2}\big(W\Omega^{s}_X\big) \arrow[r] & \mathrm{H}^{t+2}\big(W\Omega^{s+1}_X\big) \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^{t+1}\big(W\Omega^{s-1}_X\big) \arrow[r] \arrow[uur] & \mathrm{H}^{t+1}\big(W\Omega^{s}_X\big) \arrow[r] \arrow[uur] & \mathrm{H}^{t+1}\big(W\Omega^{s+1}_X\big) \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^{t}\big(W\Omega^{s-1}_X\big) \arrow[r] \arrow[uur] & \mathrm{H}^{t}\big(W\Omega^{s}_X\big) \arrow[r] \arrow[uur] & \mathrm{H}^{t}\big(W\Omega^{s+1}_X\big) \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Figure 4. The E_1-page of the slope spectral sequence (8) (with horizontal differentials) and the E_2-page of the descent spectral sequence for TR (5) (with diagonal differentials)

Let $\mathrm{F}^\star\mathrm{TR}(X) = \mathrm{R}\Gamma(X, \tau_{\geqslant\star}\mathrm{TR}(\mathcal{O}_X))$ be the filtration giving rise to the descent spectral sequence. The differentials in the slope and descent spectral sequences are compatible in the following sense.

Lemma 4.4. *Let X be a smooth proper variety over a perfect field k of positive characteristic. The de Rham–Witt differential $\mathrm{H}^t\big(W\Omega^s_X\big) \to \mathrm{H}^t\big(W\Omega^{s+1}_X\big)$ is compatible with the descent spectral sequence* (5). *Specifically,*

[2] In [Eke84], this complex is denoted by $\mathrm{R}^j\Gamma\big(X, W\Omega^\bullet_X\big)$ (see Section 0, p. 190). We will avoid this notation due to its potential for confusion with the jth hypercohomology of $W\Omega^\bullet_X$.

(i) *there is a self-map* $\mathrm{E}_*^{s,t} \to \mathrm{E}_*^{s+1,t}$ *of degree* $(1,0)$ *of the descent spectral sequence which on the* E_2*-page is the de Rham–Witt differential;*

(ii) *for each* $r \geqslant 2$*, the resulting complexes*

$$\mathrm{E}_r^{\bullet,t} = \left[\cdots \to \mathrm{E}_r^{s-1,t} \to \mathrm{E}_r^{s,t} \to \mathrm{E}_r^{s+1,t} \to \cdots\right]$$

are naturally R*-modules;*

(iii) *for each* $t \in \mathbb{Z}$*, the differential* d_r *in the descent spectral sequence is a map of* R*-modules* $\mathrm{E}_r^{\bullet,t} \to \mathrm{E}_r^{\bullet,t+r}[r-1]$*;*

(iv) *the filtration on* $\mathrm{TR}_*(X)$ *coming from the spectral sequence may be interpreted as a filtration by* R*-modules*

$$0 = F_\bullet^{\dim(X)+1} \subseteq F_\bullet^{\dim(X)} \subseteq \cdots \subseteq F_\bullet^1 \subseteq F_\bullet^0 = \mathrm{TR}_\bullet(X)\,,$$

where

$$F_i^t = \mathrm{im}\big(\pi_i \mathrm{F}^{i+t}\mathrm{TR}(X) \to \mathrm{TR}_i(X)\big)$$

with isomorphisms

$$F_\bullet^t / F_\bullet^{t+1} \cong \mathrm{E}_\infty^{\bullet,t}[t]\,.$$

Note that for each i, the filtration on $\mathrm{TR}_i(X)$ induced by the F_i^t is a shift of the usual filtration coming from the descent spectral sequence.

Proof. Consider the sheaf $\mathrm{TR}(\mathcal{O}_X)$ of spectra with S^1-action on the Zariski site of X and the filtration $\mathrm{F}^\star\mathrm{TR}(\mathcal{O}_X) = \tau_{\geqslant\star}\mathrm{TR}(\mathcal{O}_X)$ arising from the Postnikov tower in sheaves of spectra with S^1-action. The graded piece $\mathrm{gr}^t\mathrm{TR}(\mathcal{O}_X)$ is equivalent to $W\Omega_X^t[t]$. Taking global sections, we obtain a filtration $\mathrm{F}^\star\mathrm{TR}(X) = \mathrm{F}^\star\mathrm{R}\Gamma(X, \mathrm{TR}(\mathcal{O}_X))$ with graded pieces $\mathrm{gr}^t\mathrm{TR}(X) = \mathrm{R}\Gamma(X, W\Omega_X^t)[t]$. The descent spectral sequence is by definition the spectral sequence of this filtration (with a commonly used reindexing so that it begins with the E_2-page). Now, consider the S^1-action on $\mathrm{TR}(\mathcal{O}_X)$. This induces a map $\mathrm{TR}(\mathcal{O}_X) \to \mathrm{TR}(\mathcal{O}_X)[-1]$ which automatically respects the filtration since it is just the canonical Postnikov filtration. In particular, this means that $\mathrm{TR}(X) \to \mathrm{TR}(X)[-1]$ induces a filtered map

$$\mathrm{F}^\star\mathrm{TR}(X) \to \mathrm{F}^{\star+1}\mathrm{TR}(X)[-1]\,.$$

It follows that there is a map of spectral sequences associated with the two filtrations. But the spectral sequence coming from $\mathrm{F}^{\star+1}\mathrm{TR}(X)[-1]$ is just a regrading of spectral sequence coming from $\mathrm{F}^\star\mathrm{TR}(X)$. In particular, we can view the differential as a self-map of the spectral sequence (5) of degree $(1,0)$. On the graded pieces of the filtrations, we get maps

$$\mathrm{R}\Gamma\big(X, W\Omega_X^t\big)[t] \to \mathrm{R}\Gamma\big(X, W\Omega_X^{t+1}\big)[t]\,.$$

Hesselholt checks in [Hes96, Theorem C] that this map is induced by the de Rham–Witt differential $W\Omega_X^t \to W\Omega_X^{t+1}$. In particular, on the E_2-page, we see the de Rham–Witt differential. This proves part (i).

We have already remarked that the spectral sequence (5) takes places in the abelian category of Dieudonné modules; in other words, the differentials in the descent spectral sequence commute with F and V. But, by part (i), the differentials in the descent spectral sequence also commute with the de Rham–Witt differential d in the sense that for all t and $r \geqslant 2$, the diagram

$$\begin{tikzcd} \mathrm{E}_r^{s,t} \arrow[r] \arrow[d] & \mathrm{E}_r^{s+1,t} \arrow[d] \\ \mathrm{E}_r^{s+r-1,t+r} \arrow[r] & \mathrm{E}_r^{s+r,t+r} \end{tikzcd}$$

commutes. Now, parts (ii) and (iii) follow by induction, where the base case is the R-module (10).

Finally, to prove part (iv), it is enough to note that the filtered map $d\colon \mathrm{F}^\star\mathrm{TR}(X) \to \mathrm{F}^{\star+1}\mathrm{TR}(X)[-1]$ implies that the inclusion $F^t_\bullet \subseteq \mathrm{TR}_\bullet(X)$ is compatible with the differential. Since it is also compatible with F and V and since $\mathrm{TR}_\bullet(X)$ is an R-module, it follows that $F^t_\bullet$ is an R-submodule. This completes the proof. □

To investigate the fine structure of the Hodge–Witt cohomology groups, Illusie and Raynaud [IR83] introduced a certain subcategory of the category of R-modules, which we briefly review. Forgetting the F and V operations, an R-module gives rise to a complex of W-modules with cohomology groups $\mathrm{H}^*(M)$. This complex comes equipped with a canonical decreasing $(V + dV)$-filtration given by $\mathrm{Fil}^n M^i = V^n M^i + dV^n M^{i-1} \subseteq M^i$. We say that M is *complete* if each M^i is complete and separated for the $(V + dV)$-topology. We say that a complete R-module M is *profinite* if $M^i/\mathrm{Fil}^n M^i$ has finite length as a $W(k)$-module for each i and n. An R-module M is *coherent* if it is bounded (that is, $M^i = 0$ for $|i|$ sufficiently large), profinite, and $\mathrm{H}^i(M)$ is a finitely generated W-module for all i (see [IR83, Théorème I.3.8 et Définition I.3.9]).

Illusie and Raynaud showed in [IR83, Théorème II.2.2] that the Hodge–Witt complex $\mathrm{H}^j\big(W\Omega^\bullet_X\big)$ is coherent for each j.[3] In the derived setting, we consider the R-module $\mathrm{TR}_\bullet(X)$ from (11).

Proposition 4.5. *If X is a smooth proper k-scheme, then $\mathrm{TR}_\bullet(X)$ is a coherent R-module.*

Proof. As recorded in [Eke84, Section 0, p. 191], the category of coherent R-modules is closed under kernels, cokernels, and extensions in the category of all R-modules. As remarked above, the Hodge–Witt complex $\mathrm{H}^j\big(W\Omega^\bullet_X\big)$ is coherent for each j. It follows from the compatibility of Lemma 4.4 that the R-modules $\mathrm{E}_r^{\bullet,t}$ appearing as the rows of the pages in the descent spectral sequence for X are coherent for each $r \geqslant 2$. The descent spectral sequence degenerates at some finite stage for degree reasons, so the $\mathrm{E}_\infty^{\bullet,t}$ are also coherent. But $\mathrm{TR}_\bullet(X)$ admits a finite filtration by R-submodules whose successive quotients are isomorphic to the $\mathrm{E}_\infty^{\bullet,t}$ by Lemma 4.4. The category of coherent R-modules is closed under extensions, so it follows inductively that each piece of the filtration is coherent. In particular, $\mathrm{TR}_\bullet(X)$ is coherent, as desired. □

Using results of Illusie–Raynaud and Ekedahl, we obtain the following consequences for the structure of the R-module $\mathrm{TR}_\bullet(X)$.

Proposition 4.6. *If X is a smooth proper k-scheme of dimension d over a perfect field of positive characteristic, then for each $i \geqslant d-2$,*

(i) *$\mathrm{TR}_i(X)$ is finitely generated over W, and*

(ii) *the differential $d\colon \mathrm{TR}_{i-1}(X) \to \mathrm{TR}_i(X)$ vanishes.*

[3] Note that this implies that while some $\mathrm{H}^j\big(W\Omega^i_X\big)$ might be non-finitely generated as a W-module, the terms appearing in the E_2-page of the slope spectral sequence (8) are all finitely generated W-modules.

Proof. As explained in [Ill83, Section 3.1], the domino numbers $T^{i,j}(X)$ of X are zero if $j \leqslant 1$ or $i \geqslant d-1$, and therefore $\mathrm{H}^j\big(W\Omega^i_X\big)$ is finitely generated over W if $j \leqslant 1$ or $i = d$ (we review the definition of the domino numbers in Section 5.3). The descent spectral sequence then gives claim (i). Claim (ii) then follows from Proposition 4.5 and Lemma 4.7. $\square$

Lemma 4.7. *Let $M^\bullet$ be a coherent R-module. For each i, the following are equivalent:*

(i) *The module M^i is finitely generated over W.*

(ii) *The differentials $M^{i-1} \to M^i$ and $M^i \to M^{i+1}$ vanish.*

Proof. That part (ii) implies part (i) follows from the definition of coherence. For the converse, see [IR83, Corollaires II.3.8 et II.3.9] for the case of the de Rham–Witt complex and [IR83, Section II.3.1(f)] for a general coherent R-module. $\square$

5. Derived invariants of varieties

Let X be a smooth proper variety over k of dimension d. By construction, the Dieudonné modules $\mathrm{TR}_*(X)$ together with their differentials are derived invariants of X. Our goal in the following sections is to relate their structure to that of the crystalline and Hodge–Witt cohomology groups of X. Our main tool is the descent spectral sequence (5), which relates the $\mathrm{TR}_*(X)$ to the Hodge–Witt cohomology groups $\mathrm{H}^*(W\Omega^*_X)$ of X.

We begin with the observation that the descent spectral sequence induces natural isomorphisms

$$\mathrm{TR}_{-d}(X) \cong \mathrm{H}^d(W\mathscr{O}_X) \quad \text{and} \quad \mathrm{TR}_d(X) \cong \mathrm{H}^0\big(W\Omega^d_X\big)\,.$$

It follows that the Dieudonné modules $\mathrm{H}^d(W\mathscr{O}_X)$ and $\mathrm{H}^0\big(W\Omega^d_X\big)$ are derived invariants. We record a few easy consequences of this. Recall that if X is a d-fold such that $h^{0,d-1} = \mathrm{H}^{d-1}(X, \mathscr{O}_X) = 0$, then Artin–Mazur's functor $\Phi^d(X, \mathbb{G}_m)$ is a smooth formal group of dimension $h^{0,d}$ (see [AM77, Corollary 4.2 and further]). Furthermore, by [AM77, Corollary 4.3], the Dieudonné module of $\Phi^d(X, \mathbb{G}_m)$ is $\mathrm{H}^d(W\mathscr{O}_X)$.

Theorem 5.1. *If X and Y are derived equivalent Calabi–Yau d-folds, then $\Phi^d(X, \mathbb{G}_m) \cong \Phi^d(Y, \mathbb{G}_m)$. In particular, the heights of the formal groups of X and Y are the same.*

In the case of K3 surfaces over an algebraically closed field, this is well known, being an easy consequence of the derived invariance of the rational Mukai crystal $\widetilde{\mathrm{H}}(X/K)$ introduced in [LO15, Section 2.2].

Remark 5.2. In fact, the formal group $\Phi^d(X, \mathbb{G}_m)$ is also a twisted derived invariant. Indeed, if $\alpha \in \mathrm{Br}(X)$, then $\mathrm{TR}_{-d}(X, \alpha) \cong \mathrm{H}^d(W\mathcal{O}_X)$ as well.

Example 5.3. Theorem 5.1 is especially interesting in light of Yobuko's result [Yob19] which says that any finite-height Calabi–Yau variety lifts to $W_2(k)$. We see that if X and Y are derived equivalent Calabi–Yau varieties and if X has finite height, then so does Y and they both lift to $W_2(k)$. This gives some cases in which Question 2.7 has a positive answer.

The relationship between the remaining Hodge–Witt cohomology groups and the $\mathrm{TR}_*(X)$ is subtle in general. There are several difficulties here: First, there may be nontrivial differentials in the descent spectral sequence, which need to be understood to relate the Hodge–Witt cohomology groups to the E_∞-page. Second, the E_∞-page only determines the graded pieces of the induced filtration on $\mathrm{TR}_\bullet(X)$. Unlike in the case of Hochschild homology, this is not enough to determine $\mathrm{TR}_\bullet(X)$ up to isomorphism, as there are nontrivial extensions in the category of R-modules. Finally, as in the case of Hochschild homology, to compute the E_∞-page from $\mathrm{TR}_\bullet(X)$ one needs the additional data of the filtration on $\mathrm{TR}_\bullet(X)$, which is not a derived invariant.

In the remainder of this section, we will attempt to overcome these obstacles in various situations. We begin in Section 5.1 by analyzing the case of surfaces in detail. Here, the situation is sufficiently restricted to allows us to (almost) prove that the entire first page of the slope spectral sequence is a derived invariant.

We then discuss the groups $\mathrm{TR}_*(X)$ up to isogeny, that is, after inverting p. This is the classical theory of *slopes*. A major simplification occurs here, in that all of the relevant spectral sequences degenerate after inverting p, by Proposition 3.8. The resulting analysis follows the well-established patterns involved in extracting derived invariants from Hochschild homology (see also Remark 5.20).

We then study the torsion in the Hodge–Witt cohomology, which contains information lost upon inverting p. We focus on the dominoes associated with differentials on the first page of the slope spectral sequence. These encode in an elegant way the infinitely generated p-torsion in the Hodge–Witt cohomology groups, which is present in many interesting (and geometrically well-behaved) examples, such as certain K3 surfaces and abelian varieties. Here we introduce new nonclassical derived invariants, defined using the differentials in the Tate spectral sequence (6). We then discuss Hodge–Witt numbers, which combine the information from the slopes of isocrystals and the domino numbers.

We finally apply these results to study the derived invariance of Hodge numbers in positive characteristic. We note that in order to obtain information on the Hodge numbers from Hodge–Witt cohomology, it is necessary to remember the infinite p-torsion. The information from the slopes of the isocrystals, while perhaps more elementary, does not suffice.

We are able to obtain the strongest results for surfaces and threefolds. For future use, we will record a visualization of the information contained in the slope and descent spectral sequences in these cases. We record the following lemma.

Lemma 5.4. *Let X be a smooth proper k-scheme of dimension d.*

(i) *If $d \leqslant 2$, then the descent spectral sequence for X degenerates at* E_2.

(ii) *If $d = 3$, then the only possibly nonzero differentials on the* E_2*-page of the descent spectral sequence for X are as pictured in Figure 6.*

Proof. Note that, in general, the descent spectral sequence degenerates at E_{d+1} for degree reasons. This gives the result for $d = 1$. If $d = 2$, the only possible differentials in the E_2-page are $\mathrm{H}^0(W\mathcal{O}_X) \to \mathrm{H}^2(W\Omega^1_X)$ and $\mathrm{H}^0(W\Omega^1_X) \to \mathrm{H}^2(W\Omega^2_X)$. By Proposition 3.8, the differentials vanish after inverting p. Since $\mathrm{H}^2(W\Omega^2_X)$ is torsion-free, the latter differential is zero. The former is zero by functoriality and the fact that $\mathrm{H}^0(W\mathcal{O}_X) \cong \mathrm{TR}_0(k)$. By the same reasoning, when $d = 3$, we conclude that the differentials $\mathrm{H}^0(W\mathscr{O}_X) \to \mathrm{H}^2(W\Omega^1_X)$ and $\mathrm{H}^1(W\Omega^2_X) \to \mathrm{H}^3(W\Omega^3_X)$ vanish. $\square$

Now suppose that X is a smooth proper surface over k. The first page of the slope spectral sequence for X is given in Figure 5.

The horizontal arrow is the only possibly nonzero differential on the first page of the slope spectral sequence. All differentials on all pages of the descent spectral sequence vanish (see Lemma 5.4). The dotted box (the small 1×1 box on the upper left) indicates the source of the

$$\begin{tikzcd}
\mathrm{H}^2(W\mathscr{O}_X) \arrow[r] & \mathrm{H}^2(W\Omega^1_X) & \mathrm{H}^2(W\Omega^2_X) \\
\mathrm{H}^1(W\mathscr{O}_X) & \mathrm{H}^1(W\Omega^1_X) & \mathrm{H}^1(W\Omega^2_X) \\
\mathrm{H}^0(W\mathscr{O}_X) & \mathrm{H}^0(W\Omega^1_X) & \mathrm{H}^0(W\Omega^2_X)
\end{tikzcd}$$

Figure 5. The E_1-page of the slope spectral sequence and the E_2-page of the descent spectral sequence for TR of a smooth proper surface X over k

only possibly nonzero domino, the dashed box (the 1×2 box) indicates the possible locations of nilpotent torsion, and the solid box (the stair-step shaped box) indicates the possible locations of semisimple torsion (see Remark 5.25).

If X is a smooth proper threefold over k, then the first page of the slope spectral sequence is given in Figure 6.

$$\begin{tikzcd}
\mathrm{H}^3(W\mathscr{O}_X) \arrow[r] & \mathrm{H}^3(W\Omega^1_X) \arrow[r] & \mathrm{H}^3(W\Omega^2_X) & \mathrm{H}^3(W\Omega^3_X) \\
\mathrm{H}^2(W\mathscr{O}_X) \arrow[r] & \mathrm{H}^2(W\Omega^1_X) \arrow[r] & \mathrm{H}^2(W\Omega^2_X) & \mathrm{H}^2(W\Omega^3_X) \\
\mathrm{H}^1(W\mathscr{O}_X) \arrow[uur] & \mathrm{H}^1(W\Omega^1_X) \arrow[uur] & \mathrm{H}^1(W\Omega^2_X) & \mathrm{H}^1(W\Omega^3_X) \\
\mathrm{H}^0(W\mathscr{O}_X) & \mathrm{H}^0(W\Omega^1_X) \arrow[uur] & \mathrm{H}^0(W\Omega^2_X) \arrow[uur] & \mathrm{H}^0(W\Omega^3_X)
\end{tikzcd}$$

Figure 6. The E_1-page of the slope spectral sequence and the E_2-page of the descent spectral sequence for TR of a smooth proper threefold X over k

The horizontal arrows are all possibly nonzero differentials on the first page of the slope spectral sequence. The diagonal arrows are all possibly nonzero differentials on the second page of the descent spectral sequence (see Lemma 5.4). The descent spectral sequence degenerates at E_3. The dotted box (the 2×2 box) indicates the sources of the possibly nonzero dominoes, the dashed box (the 2×3 box) indicates the possible locations of nilpotent torsion, and the solid box (the stair-step shaped box) indicates the possible locations of semisimple torsion (see Remark 5.25).

5.1 Derived invariants of surfaces

Suppose that X is a surface. By Proposition 4.6, the R-module $\mathrm{TR}_\bullet(X)$ has only one possibly nonzero differential and so looks like

$$\mathrm{TR}_{-2}(X) \xrightarrow{d} \mathrm{TR}_{-1}(X) \xrightarrow{0} \mathrm{TR}_0(X) \xrightarrow{0} \mathrm{TR}_1(X) \xrightarrow{0} \mathrm{TR}_2(X)\,.$$

Furthermore, $\mathrm{TR}_i(X)$ is finitely generated for $i \geqslant 0$. By Lemma 5.4, the descent spectral sequence degenerates at E_2, so by Lemma 4.4, we have a filtration

$$0 = F^3_\bullet \subset F^2_\bullet \subset F^1_\bullet \subset F^0_\bullet = \mathrm{TR}_\bullet(X)$$

by coherent sub-R-modules such that

$$F^i_\bullet / F^{i+1}_\bullet \cong \mathrm{H}^i\big(W\Omega^\bullet_X\big)[i]\,.$$

This filtration yields short exact sequences

$$0 \to F^2_\bullet \to \mathrm{TR}_\bullet(X) \to \frac{\mathrm{TR}_\bullet}{F^2_\bullet} \to 0 \quad \text{and} \quad 0 \to \frac{F^1_\bullet}{F^2_\bullet} \to \frac{\mathrm{TR}_\bullet(X)}{F^2_\bullet} \to \frac{\mathrm{TR}_\bullet(X)}{F^1_\bullet(X)} \to 0\,,$$

and hence we have commuting diagrams

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
\mathrm{H}^2(W\mathscr{O}_X) \arrow[r, "d"] \arrow[d, "\wr"] & \mathrm{H}^2\big(W\Omega^1_X\big) \arrow[r, "0"] \arrow[d] & \mathrm{H}^2\big(W\Omega^2_X\big) \arrow[d] & & \\
\mathrm{TR}_{-2}(X) \arrow[r, "d"] & \mathrm{TR}_{-1}(X) \arrow[r, "0"] \arrow[d] & \mathrm{TR}_0(X) \arrow[r, "0"] \arrow[d] & \mathrm{TR}_1(X) \arrow[r, "0"] \arrow[d, "\wr"] & \mathrm{TR}_2(X) \arrow[d, "\wr"] \\
 & \mathrm{H}^1(W\mathscr{O}_X) \arrow[r, "0"] \arrow[d] & \big(F^0_\bullet/F^1_\bullet\big)_0 \arrow[r, "0"] \arrow[d] & \mathrm{TR}_1(X) \arrow[r, "0"] & \mathrm{TR}_2(X) \\
 & 0 & 0 & &
\end{tikzcd} \tag{12}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\mathrm{H}^1(W\mathscr{O}_X) \arrow[r, "0"] \arrow[d, "\wr"] & \mathrm{H}^1\big(W\Omega^1_X\big) \arrow[r, "0"] \arrow[d] & \mathrm{H}^1\big(W\Omega^2_X\big) \arrow[d] & \\
\mathrm{H}^1(W\mathscr{O}_X) \arrow[r, "0"] & \big(F^0_\bullet/F^1_\bullet\big)_0 \arrow[r, "0"] \arrow[d] & \mathrm{TR}_1(X) \arrow[r, "0"] \arrow[d] & \mathrm{TR}_2(X) \arrow[d, "\wr"] \\
 & \mathrm{H}^0(W\mathscr{O}_X) \arrow[r, "0"] \arrow[d] & \mathrm{H}^0\big(W\Omega^1_X\big) \arrow[r, "0"] \arrow[d] & \mathrm{H}^0\big(W\Omega^2_X\big)\,, \\
 & 0 & 0 &
\end{tikzcd} \tag{13}$$

where the columns are exact sequences of R-modules and the rows are R-modules.

Theorem 5.5. *Let X and Y be smooth proper surfaces over k. If X and Y are FM-equivalent, then for*

$$(i,j) \in \{(0,0),(1,0),(2,0),(1,1),(2,1),(0,2),(2,2)\}\,,$$

there exists an isomorphism

$$\mathrm{H}^j\big(W\Omega^i_X\big) \cong \mathrm{H}^j\big(W\Omega^i_Y\big)$$

of Dieudonné modules. Furthermore, there exist isomorphisms

$$\begin{aligned} \mathrm{H}^1(W\mathscr{O}_X)\otimes K &\cong \mathrm{H}^1(W\mathscr{O}_Y)\otimes K\,,\\ \mathrm{H}^2\big(W\Omega^1_X\big)\otimes K &\cong \mathrm{H}^2\big(W\Omega^1_Y\big)\otimes K\,,\\ \mathrm{H}^2\big(W\Omega^1_X\big)[p^\infty] &\cong \mathrm{H}^2\big(W\Omega^1_Y\big)[p^\infty] \end{aligned}$$

of R^0-modules and a commutative diagram

$$\begin{tikzcd} \mathrm{H}^2(W\mathscr{O}_X) \arrow[r, "d"] \arrow[d, "\wr"] & \mathrm{H}^2\big(W\Omega^1_X\big)[p^\infty] \arrow[d, "\wr"] \\ \mathrm{H}^2(W\mathscr{O}_Y) \arrow[r, "d"] & \mathrm{H}^2\big(W\Omega^1_Y\big)[p^\infty]\,. \end{tikzcd}$$

Proof. By diagram (13), we have an isomorphism $\mathrm{TR}_2(X)\xrightarrow{\sim}\mathrm{H}^0\big(W\Omega^2_X\big)$, and by diagram (12), we have an isomorphism $\mathrm{H}^2\big(W\mathscr{O}_X\big)\xrightarrow{\sim}\mathrm{TR}_{-2}(X)$. This produces the desired isomorphisms for $(i,j)\in\{(2,0),(0,2)\}$. By Corollary 5.14, the isogeny class of $\mathrm{H}^0\big(W\Omega^1_X\big)$ is a derived invariant. But $\mathrm{H}^0\big(W\Omega^1_X\big)$ is torsion-free of slope zero, so in fact $\mathrm{H}^0\big(W\Omega^1_X\big)\cong\mathrm{H}^0\big(W\Omega^1_Y\big)$. We consider the short exact sequence

$$0\to\mathrm{H}^1\big(W\Omega^2_X\big)\to\mathrm{TR}_1(X)\to\mathrm{H}^0\big(W\Omega^1_X\big)\to 0\,,$$

whose terms are finitely generated R^0-modules, and similarly for Y. As $\mathrm{H}^0\big(W\Omega^1_X\big)$ is torsion-free of slope zero, this sequence splits, and we conclude that $\mathrm{H}^1\big(W\Omega^2_X\big)\cong\mathrm{H}^1\big(W\Omega^2_Y\big)$. We conclude the result for $(i,j)\in\{(0,0),(1,1),(2,2)\}$ from the derived invariance of $\mathrm{TR}_0(X)$. Next, consider the short exact sequence

$$0\to\mathrm{H}^2\big(W\Omega^1_X\big)\to\mathrm{TR}_{-1}(X)\to\mathrm{H}^1(W\mathscr{O}_X)\to 0$$

coming from diagram (12). As $\mathrm{H}^1(W\mathscr{O}_X)$ is torsion-free, the map $\mathrm{H}^2\big(W\Omega^1_X\big)\to\mathrm{TR}_{-1}(X)$ induces an isomorphism on torsion. □

Remark 5.6. Theorem 5.5 almost shows that the entire first page of the slope spectral sequence is a derived invariant.

As a corollary, we recover the following result.

Corollary 5.7 ([Bra21, Corollary 3.4.3]). *Suppose that X and Y are FM-equivalent K3 surfaces over k. If X is supersingular, then so is Y, and $\sigma_0(X)=\sigma_0(Y)$.*

Proof. The image of the differential

$$d\colon \mathrm{H}^2(W\mathscr{O}_X)\to\mathrm{H}^2\big(W\Omega^1_X\big)$$

is p-torsion. Moreover, the Artin invariant of X is equal to the dimension of the k-vector space $\ker d$. This follows for instance from the descriptions given in [Ill79, Section II.7.2]. The result follows from Theorem 5.5. □

We also find another proof of the following result of Tirabassi [Tir18] on derived equivalences of Enriques surfaces. Recall from [BM76] that an Enriques surface X in characteristic 2 is either classical, singular, or supersingular, depending on whether the Picard scheme $\mathrm{Pic}_{X/k}$ is $\mathbb{Z}/2$, μ_2, or α_2, respectively. We call this the type of the Enriques surface.

Corollary 5.8 (Tirabassi). *If X is an Enriques surface over an algebraically closed field of characteristic 2, then the type of X is a derived invariant.*

Proof. The E_1-pages of the slope spectral sequences of Enriques surfaces in characteristic 2 are recorded in [Ill79, Proposition II.7.3.6]. In particular, we see by Theorem 5.5 that in this case the first page of the slope spectral sequence is a derived invariant, and this is more than enough to recover the type of X. □

5.2 Slopes and isogeny invariants

In this section, we investigate derived invariants arising from topological Hochschild homology after inverting p. Recall that k is a perfect field of characteristic p, $W = W(k)$, and $K = W[1/p]$. We let σ denote the Frobenius automorphism of k, W, or K.

Definition 5.9. An *F-isocrystal* is a finite-dimensional K-vector space V equipped with a σ-linear map $\Phi\colon V \to V$. A morphism of F-isocrystals is a map of vector spaces commuting with the respective semilinear maps.

By fundamental results of Dieudonné and Manin, it is known that when k is algebraically closed, the category of F-isocrystals is abelian semisimple, and its simple objects are in bijection with rational numbers $\lambda \in \mathbb{Q}$ (see for example [Dem72]). The *slopes* of an F-isocrystal (M, Φ) are the collection (with multiplicities) of the rational numbers appearing in the decomposition of $M \otimes_K K^{\mathrm{un}}$ into simple objects, where $K^{\mathrm{un}} = W\big(\overline{k}\big)[1/p]$. Given a subset $S \subset \mathbb{Q}$, we write M_S for the sub-isocrystal of M whose slopes are those in the subset S. If (M, Φ) is an isocrystal and j is an integer, we let $M(j)$ denote the isocrystal $\big(M, p^{-j}\Phi\big)$. We have $M(j)_\lambda = M_{j+\lambda}$.

For a smooth proper k-scheme X, we let $\mathrm{R}\Gamma(X/W) = \mathrm{R}\Gamma_{\mathrm{crys}}(X/W)$ denote the crystalline cohomology of X over W, and we let $\mathrm{R}\Gamma(X/K) = \mathrm{R}\Gamma(X/W) \otimes_W K$. Each rational crystalline cohomology group $\mathrm{H}^i(X/K)$ of X comes with an endomorphism Φ induced by the absolute Frobenius of X, and the pair $\big(\mathrm{H}^i(X/K), \Phi\big)$ is an F-isocrystal. Given a rational number λ, we define the *slope number* of the ith crystalline cohomology of X by

$$h^i_{\mathrm{crys},\lambda}(X) = \dim_K \mathrm{H}^i(X/K)_{[\lambda]}\,. \tag{14}$$

There are two facts which allow us to get some control over the slope numbers. The first is that Poincaré duality implies the existence of a perfect pairing

$$\mathrm{H}^i(X/K) \otimes_K \mathrm{H}^{2d-i}(X/K) \to \mathrm{H}^{2d}(X/K) \cong K(-d)$$

of isocrystals, where $K(-d)$ is the 1-dimensional isocrystal of slope d. It follows that for each $\lambda \in \mathbb{Q}$, we have a perfect pairing

$$\mathrm{H}^i(X/K)_{[\lambda]} \otimes_K \mathrm{H}^{2d-i}(X/K)_{[d-\lambda]} \to K(-d)\,.$$

This implies

$$h^i_{\mathrm{crys},\lambda} = h^{2d-i}_{\mathrm{crys},d-\lambda}\,. \tag{15}$$

Suppose that X is projective. The hard Lefschetz theorem in crystalline cohomology (see [KM74] for the finite field case and [Mor19, Corollary A.10] for the general case) implies that if $u = c_1(L) \in \mathrm{H}^2(X/K)$ is the rational crystalline Chern class of an ample line bundle, then cupping with powers of u gives isomorphisms

$$u^i\colon\; \mathrm{H}^{d-i}(X/K) \cong \mathrm{H}^{d+i}(X/K)\,.$$

Since u generates a 1-dimensional subspace of $\mathrm{H}^2(X/K)$ which is closed under Frobenius and has pure slope 1, it follows that u^i induces isomorphisms

$$u^i\colon\; \mathrm{H}^{d-i}(X/K)_{[\lambda]} \cong \mathrm{H}^{d+i}(X/K)_{[i+\lambda]}(i)\,. \tag{16}$$

This implies

$$h^{d-i}_{\mathrm{crys},\lambda} = h^{d+i}_{\mathrm{crys},i+\lambda}$$

or, equivalently,

$$h^{i}_{\mathrm{crys},\lambda} = h^{i}_{\mathrm{crys},i-\lambda}\,. \tag{17}$$

In fact, by [Suh12, Corollary 2.2.4], the relation (17) still holds under only the assumption that X is smooth and proper.

The Frobenius endomorphism on rational crystalline cohomology comes from an endomorphism of complexes $\mathrm{R}\Gamma(X/W)$. In particular, there is a Frobenius-fixed W-lattice $\mathrm{H}^i(X/W)/\operatorname{tors}$ inside $\mathrm{H}^i(X/K)$. This implies that the slopes λ appearing in crystalline cohomology are all nonnegative. Equation (15) implies that the slopes are additionally bounded above by d, and finally (17) implies that the slopes of $\mathrm{H}^i(X/K)$ are bounded above by i.

The Hodge–Witt cohomology groups $\mathrm{H}^j\big(W\Omega^i_X\big)$ also come with a σ-linear operator, denoted by F. Again, writing $\mathrm{H}^j\big(W\Omega^i_X\big)_K = \mathrm{H}^j\big(W\Omega^i_X\big)\otimes_W K$, the pair $\big(\mathrm{H}^j\big(W\Omega^i_X\big)_K, F\big)$ is an F-isocrystal, and given a rational number $\lambda \geqslant 0$, we write

$$h^{i,j}_{\mathrm{dRW},\lambda}(X) = \dim_K \big(\mathrm{H}^j\big(X, W\Omega^i_X\big)_K, F\big)_{[\lambda]}\,. \tag{18}$$

By [Ill79, Corollaire II.3.5], we have a canonical isomorphism

$$\mathrm{H}^{j-i}\big(W\Omega^i_X\big)_K(-i) \cong \mathrm{H}^j(X/K)_{[i,i+1)} \tag{19}$$

of F-isocrystals, and hence a canonical decomposition

$$\mathrm{H}^j(X/K) \cong \bigoplus_i \mathrm{H}^{j-i}\big(W\Omega^i_X\big)_K(-i)\,. \tag{20}$$

In particular, $h^{i,j}_{\mathrm{dRW},\lambda}$ is nonzero only if $\lambda \in [0,1)$, in which case

$$h^{i,j}_{\mathrm{dRW},\lambda} = h^{i+j}_{\mathrm{crys},i+\lambda}\,. \tag{21}$$

It follows from (17) that

$$h^{i,j}_{\mathrm{dRW},\lambda} = h^{i+j}_{\mathrm{crys},i+\lambda} = h^{d-(d-i-j)}_{\mathrm{crys},i+\lambda} = h^{2d-i-j}_{\mathrm{crys},d-j+\lambda} = h^{d-j,d-i}_{\mathrm{dRW},\lambda}\,, \tag{22}$$

which we will use below. This last equality is a crystalline analog of Hodge symmetry, which we do not have access to in de Rham cohomology.

We now discuss these invariants under FM-equivalence.

Theorem 5.10. *If X and Y are FM-equivalent smooth proper k-schemes, then there are canonical isomorphisms*

$$\bigoplus_j \mathrm{H}^{2j}(X/K)(j) \cong \bigoplus_j \mathrm{H}^{2j}(Y/K)(j) \quad \textit{and} \quad \bigoplus_j \mathrm{H}^{2j+1}(X/K)(j) \cong \bigoplus_j \mathrm{H}^{2j+1}(Y/K)(j)\,.$$

Proof. By Proposition 3.8, the crystalline–TP spectral sequence (7) degenerates rationally. Moreover, by an argument of Scholze, the induced filtration on each $\mathrm{TP}_n \otimes_W K$ is canonically split by Adams operations; see [Elm18]. Introducing the appropriate Tate twists, we obtain canonical isomorphisms of isocrystals

$$\mathrm{TP}_0(X)_K \cong \bigoplus_j \mathrm{H}^{2j}(X/K)(j) \quad \text{and} \quad \mathrm{TP}_1(X)_K \cong \bigoplus_j \mathrm{H}^{2j+1}(X/K)(j)\,. \tag{23}$$

We deduce the result from the derived invariance of TP. □

We record the following result, which is entirely analogous to Theorem 2.2 (note, however, that we need no restrictions on p).

Theorem 5.11. *If X and Y are FM-equivalent smooth proper k-schemes, then for each integer i, there is an isomorphism*

$$\bigoplus_j \mathrm{H}^{j-i}\big(W\Omega^j_X\big)_K \cong \bigoplus_j \mathrm{H}^{j-i}\big(W\Omega^j_Y\big)_K$$

of F-isocrystals. In particular, for any i and any λ, we have the equality

$$\sum_j h^{j,j-i}_{\mathrm{dRW},\lambda}(X) = \sum_j h^{j,j-i}_{\mathrm{dRW},\lambda}(Y)\,.$$

Proof. Combining (23) with the decomposition (20) and reindexing, we obtain for each integer l canonical isomorphisms of isocrystals

$$(\mathrm{TP}_0(X)_K)_{[l,l+1)}(l) \cong \bigoplus_j \mathrm{H}^{j-2l}\big(W\Omega^j_X\big)_K \quad \text{and} \quad (\mathrm{TP}_1(X)_K)_{[l,l+1)}(l) \cong \bigoplus_j \mathrm{H}^{j-2l+1}\big(W\Omega^j_X\big)_K\,,$$

and similarly for Y. Using the derived invariance of TP, the former gives the desired isomorphism for even values of i, and the latter gives the desired isomorphism for odd values of i. □

We recall the following result of Popa–Schnell (extended to positive characteristic in [Hon18, Theorem A.1] by Achter, Casalaina-Martin, Honigs, and Vial).

Theorem 5.12. *If X and Y are FM-equivalent smooth proper varieties over an arbitrary field k, then $\big(\mathrm{Pic}^0_X\big)_{\mathrm{red}}$ is isogenous to $\big(\mathrm{Pic}^0_Y\big)_{\mathrm{red}}$.*

Remark 5.13. The theorem is stated in [Hon18] only for smooth projective varieties. But, the only place projectivity is used in the proof is to guarantee the existence of a FM-equivalence, which we assume to exist.

Equivalently, the isogeny class of the Albanese variety is a derived invariant. This has the following immediate consequence.

Corollary 5.14. *If X and Y are FM-equivalent smooth proper varieties over our perfect field k of positive characteristic, then there exists an isomorphism $\mathrm{H}^1(X/K) \cong \mathrm{H}^1(Y/K)$ of F-isocrystals, and we have the equality $h^1_{\mathrm{crys},\lambda}(X) = h^1_{\mathrm{crys},\lambda}(Y)$ of slope numbers for all $\lambda \in [0,1]$.*

Remark 5.15. In any characteristic, the tangent space to Pic^0_X at the origin is naturally identified with $\mathrm{H}^1(X,\mathscr{O}_X)$. If the characteristic of k is zero, then Pic^0_X is automatically reduced, so Theorem 5.12 implies that the Hodge number $h^{0,1}$ is a derived invariant in characteristic zero. Similarly, in characteristic zero, the Hodge number $h^{1,0}$ is determined by the dimension of the Albanese of X and hence is also a derived invariant. In positive characteristic, the isogeny class of the Albanese does not, in general, determine the Hodge numbers $h^{0,1}$ and $h^{1,0}$, and therefore Theorem 5.12 does not imply the invariance of $h^{0,1}$ or $h^{1,0}$ under derived equivalence.

In dimension $d \leqslant 3$, combining Theorem 5.11 and Corollary 5.14 with the constraints (15) and (17) give us complete control of the slopes of the isocrystals $\mathrm{H}^i(X/K)$ and $\mathrm{H}^j\big(W\Omega^i_X\big)_K$.

Theorem 5.16. *If X and Y are FM-equivalent smooth projective k-schemes of dimension $d \leqslant 3$, then for each i and j, there exist isomorphisms*

$$\mathrm{H}^j\big(W\Omega^i_X\big)_K \cong \mathrm{H}^j\big(W\Omega^i_Y\big)_K \quad \textit{and} \quad \mathrm{H}^i(X/K) \cong \mathrm{H}^i(Y/K)$$

of F-isocrystals. In particular, we have

$$h^{i,j}_{\mathrm{dRW},\lambda}(X) = h^{i,j}_{\mathrm{dRW},\lambda}(Y) \quad \text{and} \quad h^{i}_{\mathrm{crys},\lambda}(X) = h^{i}_{\mathrm{crys},\lambda}(Y)$$

for all i, j, and λ.

Proof. Via (19), the isocrystals coming from crystalline cohomology determine the Hodge–Witt isocrystals. Thus, it will suffice to prove the derived invariance of the rational crystalline cohomology. We have $\mathrm{H}^0(X/K) \cong K$ and $\mathrm{H}^6(X/K) \cong K(-3)$, and similarly for Y. By Corollary 5.14, we have $\mathrm{H}^1(X/K) \cong \mathrm{H}^1(Y/K)$. By (16), we have $\mathrm{H}^1(X/K) \cong \mathrm{H}^5(X/K)(2)$, and similarly for Y. We conclude that $\mathrm{H}^i(X/K) \cong \mathrm{H}^i(Y/K)$ for $i \in \{0,1,5,6\}$.

To obtain the desired isomorphism for the remaining values of i, we consider the decompositions (23). On the one hand, we have that

$$\mathrm{TP}_1(X)_K \cong \mathrm{H}^1(X/K) \oplus \mathrm{H}^3(X/K)(1) \oplus \mathrm{H}^5(X/K)(2)$$

is a derived invariant. It follows from the Krull–Schmidt theorem [Ati56, Theorem 1] that $\mathrm{H}^3(X/K)(1) \cong \mathrm{H}^3(Y/K)(1)$. We also have that

$$\mathrm{TP}_0(X)_K \cong \mathrm{H}^0(X/K) \oplus \mathrm{H}^2(X/K)(1) \oplus \mathrm{H}^4(X/K)(2) \oplus \mathrm{H}^6(X/K)(3)$$

is a derived invariant. By (16), we have $\mathrm{H}^2(X/K)(1) \cong \mathrm{H}^4(X/K)(2)$, and similarly for Y. Again by the Krull–Schmidt theorem, we deduce that $\mathrm{H}^2(X/K)(1) \cong \mathrm{H}^2(Y/K)(1) \cong \mathrm{H}^4(X/K)(2) \cong \mathrm{H}^4(Y/K)(2)$. This completes the proof. □

Remark 5.17. If X and Y are only assumed smooth and proper, then the conclusions of Theorem 5.16 regarding the equality of slope numbers still hold. To see this, run the same argument as Theorem 5.16 with slope numbers instead of isocrystals, and replace (16) with the weaker slope number relation (17).

We obtain another proof of the following result of Honigs [Hon18].

Theorem 5.18. *Suppose that X and Y are smooth projective schemes over a finite field $\mathbb{F}_q$ of dimension $d \leqslant 3$. If X and Y are FM-equivalent, then $\zeta(X) = \zeta(Y)$.*

Proof. We will verify that $\#X(\mathbb{F}_{q^n}) = \#Y(\mathbb{F}_{q^n})$ for all $n \geqslant 1$. We use the Lefschetz trace formula

$$\#X(\mathbb{F}_{q^n}) = \sum_i (-1)^i \mathrm{tr}\big(\varphi^n | \mathrm{H}^i(X/K)\big)$$

(see, for example, [Kat81, Section I, p. 169]). Here, $\varphi = \Phi^m$ denotes the mth power of the Frobenius operator Φ on $\mathrm{H}^i(X/K)$, where m is the integer such that $q = p^m$ (note that φ is linear). The traces of powers of φ on the $\mathrm{H}^i(X/K)$ are determined by the isocrystals $\big(\mathrm{H}^i(X/K), \Phi\big)$, which by Theorem 5.16 are derived invariants in dimensions $d \leqslant 3$. □

Recall that the Betti numbers of X are defined to be $b_n(X) = \dim_K \mathrm{H}^n(X/K)$ for X smooth and proper over k. In general, we have only an inequality $b_n(X) \leqslant \dim_k \mathrm{H}^n_{\mathrm{dR}}(X/k)$, with equality if and only if $\mathrm{H}^n(X/W)$ and $\mathrm{H}^{n+1}(X/W)$ are torsion-free.

Corollary 5.19. *If X and Y are FM-equivalent smooth proper k-schemes of dimension $d \leqslant 3$, then $b_n(X) = b_n(Y)$ for each n.*

Proof. This follows from the equality of slope numbers $h^i_{\mathrm{crys},\lambda}(X) = h^i_{\mathrm{crys},\lambda}(Y)$ in Theorem 5.16 (see Remark 5.17 for the proper case). □

Remark 5.20. Given a Fourier–Mukai equivalence $\Phi_P\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$, the crystalline Mukai vector $v(P)$ of P gives rise to a correspondence $\mathrm{H}^*(X/K) \to \mathrm{H}^*(Y/K)$. The usual formalism shows that this correspondence is an isomorphism of K-vector spaces. The results of Section 5.2 can be proven by an analysis of the Künneth components of $v(P)$. In particular, via this approach one can access the information contained in the Hodge–Witt and crystalline cohomology groups up to isogeny without using TP and TR. However, we do not know how to control the torsion in the Hodge–Witt cohomology groups without using the topological constructions. Remembering this information is crucial in order to access the Hodge numbers.

5.3 Domino numbers

To control the infinitely generated p-torsion in the Hodge–Witt cohomology groups, Illusie and Raynaud introduce in [IR83] certain structures called dominoes and domino numbers. We review their definition and prove that the domino numbers are derived invariants in low dimensions. If M is an R-module, we set

$$V^{-\infty}Z^iM = \{x \in M^i \mid dV^n(x) = 0 \text{ for all } n \geqslant 0\} \quad \text{and}$$
$$F^{\infty}B^iM = \{x \in M^i \mid x \in F^n d(M^{i-1}) \text{ for some } n \geqslant 0\}.$$

Definition 5.21. A coherent R-module M is a *domino* if there exists an integer i such that $M^n = 0$ for $n \notin \{i, i+1\}$, $V^{-\infty}Z^i = 0$, and $F^\infty B^{i+1} = M^{i+1}$.

For a further explication of this definition, we refer the reader to [IR83, Définition 2.16] and the surrounding material, as well as [Ill83, Section 2.5]. Given any R-module M, each differential $M^i \to M^{i+1}$ of M factors as

$$\begin{tikzcd} M^i \arrow[rr, "d"] \arrow[dr] & & M^{i+1} \\ & M^i/V^{-\infty}Z^i \arrow[r] & F^\infty B^{i+1} \arrow[u] \end{tikzcd} \tag{24}$$

and if M is coherent, then the R-module

$$\mathrm{Dom}^i(M) = \left[M^i/V^{-\infty}Z^i \to F^\infty B^{i+1}\right]$$

is a domino. We sometimes refer to this as the domino associated with the differential $M^i \to M^{i+1}$ of M.

Definition 5.22. If D is a domino supported in degrees i and $i+1$, then

$$T(D) = \dim_k \left(D^i/VD^i\right) \tag{25}$$

is finite. We refer to it as the *dimension* of the domino D. If M is a coherent R-module, we let $T^i(M) = T\big(\mathrm{Dom}^i(M)\big)$.

Illusie and Raynaud define in [IR83, Section 1.2.D] certain simple 1-dimensional dominoes U_σ depending on an integer σ. By [IR83, Proposition 1.2.15], every domino is a finite iterated extension of the U_σ. We will use the notation

$$\mathrm{Dom}^{i,j}(X) \overset{\text{def}}{=\!=} \mathrm{Dom}^i\left(\mathrm{H}^j\big(W\Omega^\bullet_X\big)\right), \tag{26}$$

$$T^{i,j}(X) \overset{\text{def}}{=\!=} T^i\big(\mathrm{H}^j\big(W\Omega^\bullet_X\big)\big). \tag{27}$$

Example 5.23. If X is a K3 surface over a perfect field, then the differential

$$d\colon \mathrm{H}^2(W\mathscr{O}_X) \to \mathrm{H}^2\big(W\Omega^1_X\big) \tag{28}$$

is nonzero if and only if X is supersingular, in which case it is a domino of dimension 1, isomorphic to U_{σ_0}, where σ_0 is the Artin invariant of X (see [Ill79, Section II.7.2]).

In [Eke84, Theorem IV.3.5], Ekedahl showed that $\mathrm{Dom}^{i,j}(X)$ and $\mathrm{Dom}^{d-i-2,d-j+2}(X)$ are naturally dual (in a certain sense); as a consequence, we have the equality

$$T^{i,j} = T^{d-i-2,d-j+2} \tag{29}$$

of domino numbers [Eke84, Corollary IV.3.5.1].

Remark 5.24. The domino associated with an R-module supported in two degrees whose differential is zero is zero. Thus, we have $T^{i,j} = 0$ if $i \geqslant d$ or $j > d$. By (29), this implies the vanishing of various other domino numbers which are not obviously zero. For instance, if X is a surface, then we see that the only possible nontrivial domino number of X is $T^{0,2}$, the dimension of the domino associated with the differential (28) in the slope spectral sequence.

Remark 5.25. Ekedahl studies in [Eke84] a certain canonical filtration of a coherent R-module M, one piece of which is composed of the dominoes defined above (see also [Ill83]). From this filtration, Ekedahl shows how to partition the torsion of M according to its behavior under V and F: semisimple torsion, nilpotent torsion, and dominoes. For surfaces and threefolds, the possible degrees in which each of these types of torsion may appear are indicated in Figures 5 and 6. We will study only the domino torsion in this document, although the semisimple and nilpotent torsion are undoubtedly interesting as well.

Let X be a smooth and proper k-scheme. We now consider the complex $\mathrm{TR}_\bullet(X)$, which by Proposition 4.5 is a coherent R-module. We define

$$\mathrm{Dom}_i^{\mathrm{cyc}}(X) \overset{\mathrm{def}}{=\!=} \mathrm{Dom}^i(\mathrm{TR}_\bullet(X))\,, \tag{30}$$

$$T_i^{\mathrm{cyc}}(X) \overset{\mathrm{def}}{=\!=} T^i(\mathrm{TR}_\bullet(X))\,. \tag{31}$$

By construction, the T_i^{cyc} are derived invariants of X, and we refer to them as the *derived domino numbers* of X. We will use the spectral sequence (5) to relate them to the usual domino numbers $T^{i,j}$ of X. We note the following lemma.

Lemma 5.26. *If $0 \to L \to M \to N \to 0$ is an exact sequence of coherent R-modules, then for each i, we have $T^i(M) = T^i(L) + T^i(N)$.*

Proof. See [MR15, Lemma 2.5]. □

Proposition 5.27. *If X is a smooth proper k-scheme of dimension d, then $T^{0,d}(X) = T_{-d}^{\mathrm{cyc}}(X)$. In particular, $T^{0,d}(X)$ is a derived invariant.*

Proof. We interpret the rows of the pages of the descent spectral sequence for X as R-modules by Lemma 4.4. We have an exact sequence

$$\mathrm{H}^{d-2}\big(W\Omega_X^\bullet\big)[-1] \xrightarrow{d_2} \mathrm{H}^d\big(W\Omega_X^\bullet\big) \to \mathrm{E}_3^{\bullet,d} \to 0\,.$$

The image of d_2 is a coherent R-submodule $\mathrm{im}(d_2) \subseteq \mathrm{H}^d\big(W\Omega_X^\bullet\big)$, and $\mathrm{Dom}^0(\mathrm{im}(d_2)) = 0$ for degree reasons. Hence, $T^{0,d}(X) = T^0\big(\mathrm{H}^d\big(W\Omega_X^\bullet\big)\big) = T^0\big(\mathrm{E}_3^{\bullet,d}\big)$. The first two terms of $\mathrm{E}_3^{\bullet,d}$ do not see any further differentials, and so

$$T^{0,d}(X) = T^0\big(\mathrm{E}_3^{\bullet,d}\big) = T^0\big(\mathrm{E}_\infty^{\bullet,d}\big)\,. \tag{32}$$

Consider the filtration $F^i_\bullet$ of $\mathrm{TR}_\bullet(X)$ from Lemma 4.4. It follows inductively from Lemma 5.26 and the isomorphisms of Lemma 4.4(iv) that $T^{-d}\big(\mathrm{TR}_\bullet(X)/F^i_\bullet\big) = 0$ for all i. Combining this with (32), we obtain

$$T^{0,d}(X) = T^0\big(\mathrm{E}^{\bullet,d}_\infty\big) = T^{-d}\big(F^d_\bullet\big) = T^{-d}(\mathrm{TR}_\bullet(X)) = T^{\mathrm{cyc}}_{-d}(X)\,,$$

as desired. □

Definition 5.28. Let X be a smooth proper k-scheme. We say that the descent spectral sequence for X is *degenerate at the level of dominoes* if for each i and j, we have $T^{i,j}(X) = T^i\big(\mathrm{E}^{\bullet,j}_r\big)$ for all $r \geqslant 2$, where $\mathrm{E}^{\bullet,j}_r$ denotes the coherent R-module from Lemma 4.4 arising in the descent spectral sequence.

In low dimensions, this condition is automatic.

Lemma 5.29. *Let X be a smooth proper k-scheme. If X has dimension $d \leqslant 3$, then the descent spectral sequence for X is degenerate at the level of dominoes.*

Proof. If $d \leqslant 2$, the descent spectral sequence is degenerate. Suppose $d = 3$. The only possibly nonzero dominoes of X are depicted in Figure 6. The result follows immediately from Lemma 5.26. □

Proposition 5.30. *Let X be a smooth proper k-scheme. If the descent spectral sequence for X is degenerate at the level of dominoes, then for each i, we have*

$$T^{\mathrm{cyc}}_i(X) = \sum_{j\geqslant 0} T^{i+j,j}(X)\,.$$

Proof. We have

$$T^i\big(F^j_\bullet/F^{j+1}_\bullet\big) = T^i\big(\mathrm{E}^{\bullet,j}_\infty[j]\big) = T^{i+j}\big(\mathrm{E}^{\bullet,j}_\infty\big) = T^{i+j,j}(X)\,.$$

The result follows from Lemma 5.26. □

Theorem 5.31. *If X and Y are FM-equivalent smooth proper k-schemes of dimension $d \leqslant 3$, then for all i, j, we have $T^{i,j}(X) = T^{i,j}(Y)$.*

Proof. If X and Y are surfaces, the result follows from Proposition 5.27. The only possibly nonzero domino numbers of a threefold are $T^{0,2}$, $T^{0,3}$, $T^{1,2}$, and $T^{1,3}$. By Proposition 5.27, the number $T^{0,3} = T^{\mathrm{cyc}}_{-3}$ is a derived invariant. By duality (29), the number $T^{1,2} = T^{0,3}$ is also a derived invariant. By Lemma 5.29, the descent spectral sequence of a threefold is degenerate at the level of dominoes, so by Proposition 5.30, we have that $T^{\mathrm{cyc}}_{-2} = T^{0,2} + T^{1,3}$ is a derived invariant. But, by duality again, $T^{0,2} = T^{1,3}$, and hence both terms are themselves derived invariants. □

5.4 Hodge–Witt numbers

We recall certain p-adic invariants introduced by Ekedahl in [Eke86, Section IV]. We also refer the reader to Crew's article [Cre85] and Illusie's article [Ill83]. Let X be a smooth and proper k-scheme. We define the *Hodge–Newton numbers* of X by

$$m^{i,j} = \sum_{\lambda\in[i,i+1)} (i+1-\lambda)h^{i+j}_{\mathrm{crys},\lambda} + \sum_{\lambda\in[i-1,i)} (\lambda-i+1)h^{i+j}_{\mathrm{crys},\lambda}\,. \qquad (33)$$

One can show that the $m^{i,j}$ are in fact nonnegative integers, and by [Eke86, Lemma VI.3.1], they satisfy the relations

$$m^{i,j} = m^{j,i}\,, \tag{34}$$

$$m^{i,j} = m^{d-i,d-j}\,. \tag{35}$$

The *Hodge–Witt numbers* of X are defined as

$$h_W^{i,j} = m^{i,j} + T^{i,j} - 2T^{i-1,j+1} + T^{i-2,j+2}\,. \tag{36}$$

By [Eke86, Propositions VI.3.2 and VI.3.3], these satisfy

$$h_W^{i,j} = h_W^{d-i,d-j}\,, \tag{37}$$

and if X has dimension $d \leqslant 3$, one has

$$h_W^{i,j} = h_W^{j,i}\,. \tag{38}$$

By [Ill83, Equations (6.3.10.2), (6.3.5), and (6.3.2)], the Hodge–Witt numbers are related to the Hodge numbers $h^{i,j} = h^j(X, \Omega_X^i)$ by the inequalities

$$h_W^{i,j} \leqslant h^{i,j} \tag{39}$$

and by *Crew's formula*, which states that

$$\sum_j (-1)^j h_W^{i,j} = \sum_j (-1)^j h^{i,j} = \chi\big(\Omega_X^i\big)\,. \tag{40}$$

Finally, we have

$$b_n = \sum_{i+j=n} h_W^{i,j} \tag{41}$$

for each n.

As an immediate consequence of Theorems 5.16 and 5.31, we have the following result.

Theorem 5.32. *If X and Y are FM-equivalent smooth proper k-schemes of dimension $d \leqslant 3$, then for all i, j, we have $h_W^{i,j}(X) = h_W^{i,j}(Y)$.*

5.5 Hodge numbers

Let us turn to the question of whether the Hodge numbers $h^{i,j}$ are derived invariants of smooth proper k-schemes. The answer to this is known to be yes up to dimension 3 in characteristic zero by [PS11]. For curves (in any characteristic), it is an easy consequence of the Hochschild–Kostant–Rosenberg (HKR) theorem. For surfaces in characteristic zero, it follows from the HKR theorem together with Hodge symmetry and Serre duality; for threefolds in characteristic zero, it follows with the additional input of the theorem of Popa and Schnell (Theorem 5.12).

These arguments fail in several places in positive characteristic. First, the HKR isomorphism is only known to hold in general if $d \leqslant p$. Second, Hodge symmetry fails in general already for surfaces. Finally, in positive characteristic, the isogeny class of the Albanese does not determine the Hodge numbers $h^{0,1}$ or $h^{1,0}$ (see Remark 5.15).

In the case of surfaces, we are able to overcome these difficulties with the additional input of our results on Hodge–Witt numbers from Section 5.4, which in turn rely on the results on domino numbers of Section 5.3. Despite its elementary statement, we do not know a direct proof of Theorem 5.33 avoiding topological Hochschild homology machinery.

Theorem 5.33. *Suppose that X and Y are smooth proper surfaces over an arbitrary field k. If X and Y are FM-equivalent, then $h^{i,j}(X) = h^{i,j}(Y)$ for all i, j.*

Proof. As described in Theorem 2.2, the HKR isomorphism implies that $h^{2,0}$, $h^{1,1}$, and $h^{0,2}$ are derived invariants, as are the sums $h^{0,1} + h^{1,2}$ and $h^{1,0} + h^{2,1}$. If k has characteristic zero, Serre duality and Hodge symmetry give the result. Suppose that k has positive characteristic. We can assume that k is perfect since passage to the perfection does not change the Hodge numbers. Theorem 5.32 and (40) give that $\chi\big(\Omega^i_X\big)$ is a derived invariant for each i. It follows that $h^{0,1}$ and $h^{1,0}$ are derived invariants. By the HKR isomorphism, $h^{1,2}$ and $h^{2,1}$ are derived invariants. □

Note that the proof of Theorem 5.33 does not use the characteristic p Popa–Schnell result of [Hon18].

Remark 5.34. For the positive characteristic case of Theorem 5.33, we may alternatively argue as follows. As observed in the beginning of Section 5.1, the cohomology group $\mathrm{H}^2(W\mathscr{O}_X)$, together with its R^0-module structure (that is, with its action of F and V), is a derived invariant. The length of the V-torsion of $\mathrm{H}^2(W\mathscr{O}_X)$ is equal to the dimension of the tangent space at the origin of Pic^0_X minus the dimension of the tangent space at the origin of $\big(\mathrm{Pic}^0_X\big)_{\mathrm{red}}$ (see for instance [Ill79, Remarque II.6.4]). By Theorem 5.12, the latter is a derived invariant as well. It follows that the dimension of the tangent space of Pic^0_X at the origin, which is equal to the Hodge number $h^{0,1}$, is a derived invariant. We then conclude by Serre duality and the HKR theorem, as before.

We next consider threefolds in positive characteristic. To ensure that the HKR spectral sequence degenerates, we might restrict our attention to characteristic $p \geqslant 3$.[4] Even with this restriction, the failure of Hodge symmetry and of Popa–Schnell to determine $h^{0,1}$ means that we do not have enough control to prove derived invariance of Hodge numbers of threefolds in general (see, however, Theorem 5.39). We record the following consequence of Theorem 5.32.

Theorem 5.35. *Suppose that X and Y are smooth proper schemes of dimension 3 over an arbitrary field k. If X and Y are FM-equivalent, then $\chi\big(\Omega^i_X\big) = \chi\big(\Omega^i_Y\big)$ for each $0 \leqslant i \leqslant 3$.*

Proof. If k has characteristic zero, the result follows as in [PS11] (and, in fact, all Hodge numbers are derived invariants). Suppose that k has positive characteristic. We may then assume moreover that k is perfect, and the result follows immediately from Theorem 5.32 and (40). □

Remark 5.36. Suppose that k has characteristic $p \geqslant 3$, so that the spectral sequence (1) associated with a threefold X degenerates. The HKR isomorphism then gives that certain sums of Hodge numbers of X are derived invariants, as described in Theorem 2.2. Combining this with Serre duality and the obvious Hodge number $h^{0,0} = 1$, we obtain 13 linearly independent relations which are preserved by derived equivalences on the total 16 Hodge numbers.

It is not hard to check that the relations in the conclusion of Theorem 5.35 are *not* in the span of these relations. Precisely, the result of Theorem 5.35 gives exactly one new linear relation on Hodge numbers that is preserved under derived equivalence.

5.6 Mazur–Ogus and Hodge–Witt varieties

In this section, we prove the derived invariance of certain conditions on de Rham and Hodge–Witt cohomology. We keep using Notation 4.1, so that k is a perfect field of positive characteristic p.

Following Joshi [Jos20, Section 2.31], we make the following definition.

[4] The examples of [ABM21] of varieties with nondegenerate HKR spectral sequence are $2p$-dimensional. We do not know an example of a threefold in characteristic 2 with nondegenerate HKR spectral sequence.

Definition 5.37. Let X be a smooth proper k-scheme. We say that X is *Mazur–Ogus* if

(i) the Hodge–de Rham spectral sequence for X degenerates at E_1, and
(ii) the crystalline cohomology groups of X are torsion-free.

We view this as a rather mild set of assumptions, which still allow for a lot of interesting behaviors in the Hodge–Witt cohomology of X. For instance, K3 surfaces, abelian varieties, and complete intersections in projective space are Mazur–Ogus.

Lemma 5.38. *If X is a smooth proper k-scheme, then the following conditions are equivalent:*

(i) *The scheme X is Mazur–Ogus.*
(ii) *We have $b_n(X) = \sum_{i+j=n} h^{i,j}(X)$ for all n.*
(iii) *We have $h^{i,j}(X) = h^{i,j}_W(X)$ for all i, j.*

Proof. In general, we have inequalities

$$b_n(X) \leqslant \dim \mathrm{H}^n_{\mathrm{dR}}(X/k) \leqslant \sum_{i+j=n} h^{i,j}(X)\,.$$

The first of these is an equality if and only if $\mathrm{H}^n(X/W)$ and $\mathrm{H}^{n+1}(X/W)$ are torsion-free, and the second is an equality if and only if the Hodge–de Rham spectral sequence in degree n degenerates at E_1. This shows (i) $\iff$ (ii). Using (39) and (41), we deduce (ii) $\iff$ (iii). □

Theorem 5.39. *Suppose $p \geqslant 3$. Let X be a smooth proper k-scheme of dimension $d \leqslant 3$. If X is Mazur–Ogus and Y is a smooth proper k-scheme such that $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$, then Y is Mazur–Ogus and we have $h^{i,j}(X) = h^{i,j}(Y)$ for all i, j.*

Proof. Using Lemma 5.38, Theorem 5.32, and (39), we have

$$h^{i,j}(X) = h^{i,j}_W(X) = h^{i,j}_W(Y) \leqslant h^{i,j}(Y)$$

for each i, j. Using the assumption $p \geqslant 3$, by Theorem 2.2, we have

$$\sum_j h^{j,j-i}(X) = \sum_j h^{j,j-i}(Y)$$

for each i. We conclude that $h^{i,j}(X) = h^{i,j}(Y)$ for all i, j and hence $h^{i,j}_W(Y) = h^{i,j}(Y)$ for all i and j. By Lemma 5.38, we conclude that Y is Mazur–Ogus. □

Definition 5.40. Following [IR83, Section IV.4], we say that a smooth proper k-scheme is *Hodge–Witt* if $\mathrm{H}^j\big(X, W\Omega^i_X\big)$ is finitely generated as a W-module for all i, j. We say that X is *derived Hodge–Witt* if $\mathrm{TR}_i(X)$ is finitely generated as a W-module for all i.

Hodge–Witt is implied by ordinarity but is weaker than it. For example, a K3 surface is Hodge–Witt if and only if it is nonsupersingular, whereas it is ordinary if and only if the associated formal group has height 1 (see the [Ill79, Section II.7.2]).

Proposition 5.41. *Let X be a smooth proper k-scheme. The following are equivalent:*

(i) *The scheme X is Hodge–Witt.*
(ii) *The slope spectral sequence for X degenerates at E_1.*
(iii) *We have $T^{i,j}(X) = 0$ for all i, j.*

Proof. By [IR83, conditions (IV.4.6.2)], the scheme X is Hodge–Witt if and only if the slope spectral sequence for X degenerates at E_1, and so we have (i) $\Longleftrightarrow$ (ii). We have (i) $\Longleftrightarrow$ (iii) by, for instance, [Ill83, Section 3.1.4]. □

Theorem 5.42. *Let X be a smooth proper k-scheme. If X is of dimension $d \leqslant 3$, then X is Hodge–Witt if and only if it is derived Hodge–Witt.*

Proof. By [Ill83, Section 3.1.4], we have that for $j \in \{0, 1\}$, the Hodge–Witt cohomology groups $\mathrm{H}^j\big(W\Omega^i_X\big)$ are finitely generated for each i. The differentials in the descent spectral sequence have vertical degree 2 (with our conventions), and hence X is Hodge–Witt if and only if the terms $\mathrm{E}^{i,j}_\infty$ appearing on the E_∞-page of the descent spectral sequence are finitely generated for all i, j.

Furthermore, each of the $\mathrm{TR}_i(X)$ admits a filtration whose successive quotients are given by terms appearing on the E_∞-page of the descent spectral sequence for X. In particular, we see that all of the $\mathrm{TR}_i(X)$ are finitely generated W-modules if and only if $\mathrm{E}^{i,j}_\infty$ is finitely generated for all i, j. □

Corollary 5.43. *Let X and Y be smooth proper threefolds over k. If $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ and if X is Hodge–Witt, then so is Y.*

In particular, the equivalent conditions recorded in Proposition 5.41 are all derived invariants in dimension $d \leqslant 3$.

Example 5.44. Joshi shows in [Jos07, Corollary 6.2] that if X is an F-split threefold, then it is Hodge–Witt. Thus, Corollary 5.43 applies to F-split threefolds. This has recently been extended to quasi-F-split threefolds by Nakkajima [Nak19, Corollary 1.8] and in particular then applies to all finite-height Calabi–Yau threefolds by [Yob19].

Remark 5.45. Using the equivalent conditions of Proposition 5.41, one can give a less direct proof of Theorem 5.42 using Proposition 5.30.

6. Twisted K3 surfaces

In this section, we will study some examples with interesting behavior in the slope and descent spectral sequences. Specifically, we will completely compute TR and TP for twisted K3 surfaces. This will give a different perspective on the twisted K3 crystals defined and studied in [BL18]. For the remainder of this section, we let k be an algebraically closed field of positive characteristic p.

While we have only discussed derived invariants of varieties so far, much of our discussion carries over unchanged to twisted varieties. For instance, TR and TP are defined for an abstract dg category. Given $\alpha \in \mathrm{Br}(X)$, a Brauer class on a smooth proper variety X, we let $\mathrm{TR}_*(X, \alpha)$ and $\mathrm{TP}_*(X, \alpha)$ denote their application to the natural enhancement of the bounded derived category of α-twisted coherent sheaves on X. There is a descent spectral sequence

$$\mathrm{E}^{s,t}_2 = \mathrm{H}^t\big(X, W\Omega^s_X\big) \Rightarrow \mathrm{TR}_{s-t}(X, \alpha)\,, \tag{42}$$

which computes the $\mathrm{TR}_*(X, \alpha)$ in terms of the Hodge–Witt cohomology groups of the underlying variety X. We remark that the Hodge–Witt cohomology groups of a K3 surface are determined very explicitly in [Ill79, Section II.7.2]. We will see that although the objects appearing on the E_2-page of (42) are the same as those on the E_2-page of the descent spectral sequence for X, the differentials may be different. We let d^α_r denote the differentials in the α-twisted descent spectral sequence.

Let X be a K3 surface over k. We begin by computing the topological periodic cyclic homology groups $\mathrm{TP}_*(X)$ of X in terms of crystalline cohomology. Since the crystalline cohomology groups of X are all torsion-free, the crystalline–TP spectral sequence (7) degenerates; the corresponding filtration splits canonically and gives an isomorphism of W-modules

$$\mathrm{TP}_{2i}(X) \cong \mathrm{H}^0(X/W) \oplus \mathrm{H}^2(X/W) \oplus \mathrm{H}^4(X/W)$$

for each i. In general, the K-vector spaces obtained by tensoring TP_i with $\mathbb{Q}$ admit a natural Frobenius operator coming from the cyclotomic Frobenius and are thus endowed with a structure of F-isocrystal. However, after inverting p, this isomorphism does not carry the Frobenius on the left-hand side to the natural Frobenius operator Φ on crystalline cohomology. Rather, consider the Mukai crystal as introduced in [LO15]:

$$\widetilde{\mathrm{H}}(X/W) = \mathrm{H}^0(X/W)(-1) \oplus \mathrm{H}^2(X/W) \oplus \mathrm{H}^4(X/W)(1)\,.$$

The above can then be upgraded to isomorphisms $\mathrm{TP}_{2i}(X)_{\mathbb{Q}} \cong \widetilde{\mathrm{H}}(X/K)(i+1)$ of F-isocrystals for each i, where $\widetilde{\mathrm{H}}(X/K) = \widetilde{\mathrm{H}}(X/W) \otimes_W K$. In fact, for any (possibly twisted) surface, the Frobenius is defined integrally[5] on $\mathrm{TP}_i(X,\alpha)$ for $i \leqslant -2$. The filtration on TP can even be split integrally, and so we obtain an isomorphism

$$\mathrm{TP}_{2i}(X) \cong \widetilde{\mathrm{H}}(X/W)(i+1) \tag{43}$$

of F-crystals for each $i \leqslant -1$.

6.1 Finite height

Let (X,α) be a twisted K3 surface over k, and suppose that X has finite height. By the computation of the Hodge–Witt cohomology groups of X in [Ill79, Section II.7.2], we see that the slope spectral sequence for X and the descent spectral sequences for X and (X,α) are all degenerate for degree reasons. Hence, both $\mathrm{TR}_0(X)$ and $\mathrm{TR}_0(X,\alpha)$ admit a filtration by R^0-submodules with graded pieces $\mathrm{H}^0(W\mathscr{O}_X)$, $\mathrm{H}^1\big(W\Omega^1_X\big)$, and $\mathrm{H}^2\big(W\Omega^2_X\big)$. As these groups are all torsion-free, this filtration certainly splits at the level of W-modules. In fact, by computing the appropriate Ext groups, one can show that it even splits at the level of R^0-modules. We conclude that there exist (noncanonical) isomorphisms

$$\mathrm{TR}_i(X) \cong \mathrm{TR}_i(X,\alpha) \cong \begin{cases} \mathrm{H}^2(W\mathcal{O}_X) & \text{if } i=-2\,,\\ \mathrm{H}^0(W\mathscr{O}_X) \oplus \mathrm{H}^1\big(W\Omega^1_X\big) \oplus \mathrm{H}^2\big(W\Omega^2_X\big) & \text{if } i=0\,,\\ \mathrm{H}^0\big(W\Omega^2_X\big) & \text{if } i=2 \end{cases} \tag{44}$$

of R^0-modules, and $\mathrm{TR}_i(X) = \mathrm{TR}_i(X,\alpha) = 0$ otherwise.

We next compute TP. Because TR is concentrated in even degrees, the Tate spectral sequences for X and (X,α) degenerate and TP is also concentrated in even degrees. We have a filtration on $\mathrm{TP}_{2i}(X,\alpha)$ with graded pieces $\mathrm{H}^0\big(W\Omega^2_X\big)$, $\mathrm{TR}_0(X,\alpha)$, and $\mathrm{H}^2(W\mathcal{O}_X)$. Moreover, for $i \leqslant -1$, keeping track of the appropriate Tate twists, this yields a filtration by F-crystals,[6] whose graded pieces are $\mathrm{H}^0\big(W\Omega^2_X\big)(i-1)$, $\mathrm{TR}_0(X,\alpha)(i)$, and $\mathrm{H}^2(W\mathcal{O}_X)(i+1)$.

In particular, this determines the $\mathrm{TP}_{2i}(X,\alpha)$ as F-isocrystals. To determine their structure as F-crystals, one needs some additional input. This can be done by comparison with the B-field constructions of [BL18], from which one can compute the Hodge polygon of $\mathrm{TP}_{2i}(X,\alpha)$.

[5] To see this, one must use the Nygaard filtration.

[6] By [Ill79, Section II.7.2(a)], the Hodge–Witt cohomology groups of X are finitely generated and torsion-free.

Using Katz's Newton–Hodge decomposition [Kat79, Theorem 1.6.1], one then deduces that the filtration on $\mathrm{TP}_{2i}(X,\alpha)$ in fact splits canonically, and so we have a canonical isomorphism

$$\mathrm{TP}_{2i}(X,\alpha) = \mathrm{H}^2(W\mathscr{O}_X)(i+1) \oplus \mathrm{TR}_0(X,\alpha)(i) \oplus \mathrm{H}^0\big(W\Omega^2_X\big)(i-1)$$

of F-crystals for each $i \leqslant -1$. In particular, by (44), we see that $\mathrm{TP}_{2i}(X) \cong \mathrm{TP}_{2i}(X,\alpha)$ as F-crystals for each $i \leqslant -1$, although not canonically.

6.2 Supersingular

We now consider a twisted K3 surface (X,α) where X is supersingular. We will examine the descent spectral sequence for (X,α). In particular, we will see that it is not degenerate.

We begin with a few general facts. For any smooth k-scheme X, there is a natural map of étale sheaves

$$\mathbb{G}_m \xrightarrow{\mathrm{dlog}} W\Omega^1_X \tag{45}$$

given on sections by $f \mapsto d[f]/[f]$. There is an induced map on cohomology

$$\mathrm{dlog}\colon \mathrm{H}^2(X,\mathbb{G}_m) \to \mathrm{H}^2\big(W\Omega^1_X\big)\,. \tag{46}$$

Lemma 6.1. *Let X be a smooth proper k-scheme and let $\alpha \in \mathrm{H}^2(X,\mathbb{G}_m)$.*

(i) *The differential*

$$d_2^\alpha\colon\ \mathrm{H}^0(W\mathscr{O}_X) \to \mathrm{H}^2\big(W\Omega^1_X\big)$$

appearing in the E_2-page of the α-twisted descent spectral sequence (42) sends the canonical generator $1 \in W = \mathrm{H}^0(W\mathscr{O}_X)$ to $\mathrm{dlog}(\alpha)$.

(ii) *If X is a surface, then all other differentials in the twisted descent spectral sequence are zero.*

Proof. Let $\mathrm{K}^{\text{ét}}(X,\alpha)$ denote the α-twisted étale K-theory of X. There is a descent spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^t_{\text{ét}}(X,\mathrm{K}_s(\mathcal{O}_X)) \Rightarrow \mathrm{K}^{\text{ét}}_{s-t}(X,\alpha)\,.$$

We also have natural isomorphisms $\mathbb{Z} \cong \mathrm{K}_0(\mathcal{O}_X)$ and $\mathbb{G}_m \cong \mathrm{K}_1(\mathcal{O}_X)$. The d_2^α-differential $\mathrm{H}^0_{\text{ét}}(X,\mathbb{Z}) \to \mathrm{H}^2_{\text{ét}}(X,\mathbb{G}_m)$ sends 1 to α by [Ant11, Theorem 8.5]. Now, the map $\mathrm{K}_1(\mathcal{O}_X) \cong \mathbb{G}_m \to \mathrm{TR}_1(\mathcal{O}_X) \cong W\Omega^1$ is given by the dlog map; see [GH99, Lemma 4.2.3].[7] Thus, part (i) follows from the compatibility between the descent spectral sequences for $\mathrm{K}^{\text{ét}}(X,\alpha)$ and $\mathrm{TR}(X,\alpha)$ using the trace map $\mathrm{K}^{\text{ét}}(X,\alpha) \to \mathrm{TR}(X,\alpha)$ and especially the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0_{\text{ét}}(X,\mathbb{Z}) \arrow[r, "d_2^\alpha"] \arrow[d] & \mathrm{H}^2(X,\mathbb{G}_m) \arrow[d, "\mathrm{dlog}"] \\
\mathrm{H}^0(W\mathcal{O}_X) \arrow[r, "d_2^\alpha"] & \mathrm{H}^2\big(W\Omega^1_X\big)\,.
\end{tikzcd}$$

For part (ii), note that all of the differentials are torsion but that $\mathrm{H}^2\big(W\Omega^2_X\big)$ is torsion-free. Thus, $d_2^\alpha\colon \mathrm{H}^0(W\mathcal{O}_X) \to \mathrm{H}^2\big(W\Omega^1_X\big)$ is the only possible nonzero differential for a surface. □

We conclude from the above that the descent spectral sequence for a twisted surface (X,α) is degenerate at E_2 if and only if $\mathrm{dlog}(\alpha) = 0$ and is always degenerate at E_3 for degree reasons.

[7] Note that in [GH99], the map is given by $-\mathrm{dlog}$, but this sign depends on a choice of the HKR isomorphism $\mathrm{TR}_1(\mathcal{O}_X) \cong W\Omega^1_X$, which amounts to a choice of an orientation on the circle.

Let us now return to the situation where X is a supersingular K3 surface. We record the following result.

LEMMA 6.2. *If X is a supersingular K3 surface over an algebraically closed field k of positive characteristic p, then the sequence*

$$0 \to \mathrm{Br}(X) \xrightarrow{\mathrm{dlog}} \mathrm{H}^2\big(W\Omega^1_X\big) \xrightarrow{1-F} \mathrm{H}^2\big(W\Omega^1_X\big) \to 0$$

is exact, where the left arrow is the map on cohomology induced by (45).

Proof. As $\mathrm{H}^1\big(W\Omega^1_X\big)$ is finitely generated, its endomorphism $1-F$ is surjective by [Ill79, Lemme II.5.3]. By flat duality, $\mathrm{H}^4(X,\mathbb{Z}_p(1)) = 0$. By [Ill79, Théorème II.5.5], we therefore obtain a short exact sequence

$$0 \to \mathrm{H}^3(X,\mathbb{Z}_p(1)) \to \mathrm{H}^2\big(W\Omega^1_X\big) \xrightarrow{1-F} \mathrm{H}^2\big(W\Omega^1_X\big) \to 0\,,$$

where as usual we put

$$\mathrm{H}^3(X,\mathbb{Z}_p(1)) \overset{\mathrm{def}}{=\!=} \varprojlim \mathrm{H}^3(X,\mu_{p^n})\,.$$

Artin showed that as a consequence of flat duality, this inverse system is constant, and hence the natural map

$$\mathrm{H}^3(X,\mathbb{Z}_p(1)) \xrightarrow{\sim} \mathrm{H}^3(X,\mu_p)$$

is an isomorphism. Furthermore, the boundary map induced by the Kümmer sequence gives an isomorphism

$$\mathrm{Br}(X) \xrightarrow{\sim} \mathrm{H}^3(X,\mu_p). \tag{47}$$

For these facts, see [Art74, Proof of Theorem 4.3, p. 559]. We thus find an isomorphism $\mathrm{Br}(X) \xrightarrow{\sim} \mathrm{H}^3(X,\mathbb{Z}_p(1))$. Using the definitions of the maps involved, one checks that the resulting map $\mathrm{Br}(X) \to \mathrm{H}^2\big(W\Omega^1_X\big)$ is the map on cohomology induced by (45). □

We remark that the isomorphism (47) implies that $\mathrm{Br}(X)$ is p-torsion; in fact, as recorded in [Art74], there is an abstract isomorphism of groups $\mathrm{Br}(X) \cong k$.

Lemma 6.2 implies in particular that dlog is injective. The Brauer group of a supersingular K3 surface is nontrivial, so combined with Lemma 6.1, we have produced examples of twisted surfaces with nondegenerate descent spectral sequence. We record the E_2- and E_3-pages in the following figure:

$$\begin{tikzcd}
\mathrm{H}^2(W\mathscr{O}_X) \arrow[r, "d"] & \mathrm{H}^2\big(W\Omega^1_X\big) & \mathrm{H}^2\big(W\Omega^2_X\big) \\
0 & \mathrm{H}^1\big(W\Omega^1_X\big) & 0 \\
\mathrm{H}^0(W\mathscr{O}_X) \arrow[uur, "d_2^\alpha"] & 0 & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{H}^2(W\mathscr{O}_X) \arrow[r, "d"] & \dfrac{\mathrm{H}^2\big(W\Omega^1_X\big)}{\mathrm{dlog}(\alpha)} & \mathrm{H}^2\big(W\Omega^2_X\big) \\
0 & \mathrm{H}^1\big(W\Omega^1_X\big) & 0 \\
\mathrm{ord}(\alpha)W & 0 & 0
\end{tikzcd}$$

FIGURE 7. The E_2- and $\mathrm{E}_3 = \mathrm{E}_\infty$-pages of the descent spectral sequence for (X,α)

The horizontal arrows are the maps induced by the differentials appearing on the E_1-page of the slope spectral sequence for X. The term $\mathrm{ord}(\alpha)W$ is the kernel of the nonzero differential, which is generated by $\mathrm{ord}(\alpha) \in W$, where $\mathrm{ord}(\alpha) = \mathrm{ord}(\mathrm{dlog}(\alpha))$ is the order of α.

As in the finite-height case, we therefore have noncanonical isomorphisms

$$\mathrm{TR}_0(X) \cong \mathrm{TR}_0(X,\alpha) \cong \mathrm{H}^0(W\mathscr{O}_X) \oplus \mathrm{H}^1\big(W\Omega^1_X\big) \oplus \mathrm{H}^2\big(W\Omega^2_X\big) \tag{48}$$

of R^0-modules, where we use $\mathrm{ord}(\alpha)$ to give an isomorphism between $W \cong \mathrm{H}^0(W\mathcal{O}_X)$ and $\mathrm{ord}(\alpha)W$. We have a commuting diagram

$$\begin{tikzcd} \mathrm{H}^2(W\mathscr{O}_X) \arrow[r, "d"] \arrow[d, "\wr"] & \dfrac{\mathrm{H}^2\big(W\Omega^1_X\big)}{\mathrm{dlog}\,\alpha} \arrow[d, "\wr"] & \\ \mathrm{TR}_{-2}(X,\alpha) \arrow[r, "d"] & \mathrm{TR}_{-1}(X,\alpha) \arrow[r, "0"] & \mathrm{TR}_0(X,\alpha)\,, \end{tikzcd}$$

where the vertical arrows are induced by the descent spectral sequence and the right lower differential vanishes by Lemma 4.7. Finally, we have $\mathrm{TR}_1(X,\alpha) = \mathrm{TR}_2(X,\alpha) = 0$.

The differential d is surjective, and we let $K(X,\alpha)$ denote its kernel, so that we have a short exact sequence

$$0 \to K(X,\alpha) \to \mathrm{H}^2(W\mathscr{O}_X) \xrightarrow{d} \frac{\mathrm{H}^2\big(W\Omega^1_X\big)}{\mathrm{dlog}(\alpha)} \to 0\,.$$

We know that $K(X) = K(X,0)$ is a k-vector space of dimension $\sigma_0(X)$. Recall from [BL18, Corollary 3.4.23] that the Artin invariant of a twisted supersingular K3 surface is given by $\sigma_0(X,\alpha) = \sigma_0(X) + 1$ if $\alpha \neq 0$ and $\sigma_0(X,\alpha) = \sigma_0(X)$ otherwise. We conclude that $K(X,\alpha)$ is a k-vector space of dimension $\sigma_0(X,\alpha)$. In particular, this shows that the Artin invariant $\sigma_0(X,\alpha)$ is a derived invariant of (X,α), which gives another proof of [Bra21, Corollary 3.4.3].

The topological periodic cyclic homology $\mathrm{TP}(X,\alpha)$ is computed by the Tate spectral sequence (6), whose E_2 and E_3 pages are pictured in Figures 8 and 9.

$$\begin{tikzcd} \cdots & \mathrm{TR}_0(X,\alpha) & 0 & \mathrm{TR}_0(X,\alpha) & 0 & \mathrm{TR}_0(X,\alpha) & \cdots \\ \cdots & \frac{\mathrm{H}^2\left(W\Omega^1_X\right)}{\mathrm{dlog}(\alpha)} \arrow[urr, "0"] & 0 & \frac{\mathrm{H}^2\left(W\Omega^1_X\right)}{\mathrm{dlog}(\alpha)} \arrow[urr, "0"] & 0 & \frac{\mathrm{H}^2\left(W\Omega^1_X\right)}{\mathrm{dlog}(\alpha)} & \cdots \\ \cdots & \mathrm{H}^2(W\mathscr{O}_X) \arrow[urr, "d"] & 0 & \mathrm{H}^2(W\mathscr{O}_X) \arrow[urr, "d"] & 0 & \mathrm{H}^2(W\mathscr{O}_X) & \cdots \end{tikzcd}$$

Figure 8. A portion of the E_2 page of the Tate spectral sequence for (X,α)

$$\begin{array}{ccccccc} \cdots & \mathrm{TR}_0(X,\alpha) & 0 & \mathrm{TR}_0(X,\alpha) & 0 & \mathrm{TR}_0(X,\alpha) & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & K(X,\alpha) & 0 & K(X,\alpha) & 0 & K(X,\alpha) & \cdots \end{array}$$

Figure 9. A portion of the $\mathrm{E}_3 = \mathrm{E}_\infty$ page of the Tate spectral sequence for (X,α)

We therefore find short exact sequences

$$0 \to \mathrm{TR}_0(X,\alpha) \to \mathrm{TP}_i(X,\alpha) \to K(X,\alpha) \to 0 \tag{49}$$

of W-modules for all even i, and $\mathrm{TP}_i(X,\alpha)=\mathrm{TP}_i(X)=0$ for i odd. In particular, keeping track of the respective Frobenius actions as in the previous section, we find for each $i\leqslant -1$ a short exact sequence

$$0\to \mathrm{TR}_0(X,\alpha)(-i)\to \mathrm{TP}_{2i}(X,\alpha)\to K(X,\alpha)\to 0 \tag{50}$$

of W-modules, where the left arrow is a map of F-crystals.

If $\alpha=0$, then using (43) and (48), one checks that the inclusion $\mathrm{TR}_0(X)(-1)\to \mathrm{TP}_{-2}(X)$ above is isomorphic to the inclusion of the Tate module of $\widetilde{\mathrm{H}}(X/W)$. The cokernel of this inclusion is the same as the cokernel of the inclusion of the Tate module of $\mathrm{H}^2(X/W)$. Thus, $K(X)$ is naturally identified with the characteristic subspace associated with X by Ogus [Ogu79].

In general, one can show that for any $i\leqslant -1$, there is an isomorphism

$$\mathrm{TP}_{2i}(X,\alpha)\cong \widetilde{\mathrm{H}}(X/W,B)(i)$$

of F-crystals, where $\widetilde{\mathrm{H}}(X/W,B)$ is the twisted K3 crystal attached to (X,α) in [Bra21]. As in the classical setting, the construction of $\widetilde{\mathrm{H}}(X/W,B)$ depends on a noncanonical choice of a B-field lift of α, although the isomorphism class of the resulting crystal is independent of this choice. Furthermore, under this isomorphism, the inclusion $\mathrm{TR}_0(X,\alpha)(-1)\to \mathrm{TP}_{-2}(X,\alpha)$ is identified with the inclusion of the Tate module of $\widetilde{\mathrm{H}}(X/W,B)$, so that $K(X,\alpha)$ is identified with the characteristic subspace associated with (X,α) defined in [Bra21]. In particular, as the dimension of $K(X,\alpha)$ is determined by the F-crystal structure on $\mathrm{TP}_{2i}(X,\alpha)$, we see that if $\alpha\neq 0$, then $\mathrm{TP}_{2i}(X)$ is not isomorphic to $\mathrm{TP}_{2i}(X,\alpha)$ as an F-crystal for any $i\leqslant -1$, in contrast to the finite-height case. We remark that the derived Torelli theorem of [Bra21] states that $\mathcal{D}^b(X,\alpha)$ is determined by the F-crystal $\widetilde{\mathrm{H}}(X/W,B)$ together with its Mukai pairing.

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