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Optimal Mitigation of SIR Epidemics Under Model Uncertainty

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***Abstract*—We study the impact of model parameter uncertainty on optimally mitigating the spread of epidemics. We capture the epidemic spreading process using a susceptible-infected-removed (SIR) epidemic model and consider testing for isolation as the control strategy. We use a testing strategy to remove a portion of the infected population. Our goal is to maintain the infected population below a certain level, while minimizing the total number of tests. Distinct from existing works on leveraging control strategies in epidemic spreading, we propose a testing strategy by overestimating the seriousness of the epidemic and study the feasibility of the system under the impact of model parameter uncertainty. Compared to the optimal testing strategy, we establish that the proposed strategy under model parameter uncertainty will flatten the curve effectively but require more tests and a longer time period.**

I. Introduction

Resource allocation for epidemic mitigation is of great importance for both resource and risk management during a pandemic. In response to the ongoing COVID-19 pandemic, researchers have studied the use of optimal control formulations [1]–[4]. In addition to optimal control strategies, researchers leveraged model predictive control frameworks [5]–[8], and other strategies [9], [10] to generate optimal/sub-optimal policies for epidemic mitigation. Other works considering epidemic control and resource allocation include [11]–[19]. The aforementioned research was established upon the prior knowledge of the epidemic model parameters. Nevertheless, works regarding real-time epidemic modeling and prediction [20]–[22] have shown that it is difficult to predict the behavior of epidemic spreading processes. Hence, obtaining accurate epidemic spreading parameters is challenging when formulating real-time epidemic modeling and control problems. In this work, we tackle optimal epidemic control problems under the impact of parameter uncertainties. We aim to modify the optimal epidemic mitigation strategy in [23] by leveraging a range of known model parameters generated by epidemic parameter learning processes instead of accurate model parameters. We consider a testing-for-isolation strategy [23], which removes the infected population from the infected group through uniform random sampling, i.e., the control input variable. Similar to vaccination strategies that remove the susceptible population from the mixed group [24], the testing-for-isolation strategy is another widely adopted method for epidemic mitigation [3], [23].

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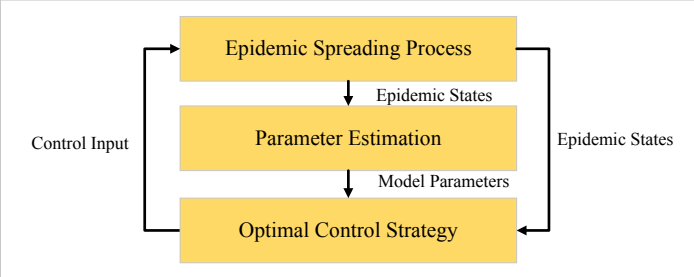

Figure 1: Control Framework

Our main contribution is to propose a testing strategy for epidemic mitigation under the impact of model uncertainties introduced by real-time epidemic modeling parameter estimating, and state estimation. Specifically, we bridge the gap between parameter estimation for epidemic spreading processes and theoretical analysis of optimal control strategies for epidemic mitigation. Assuming the range of the model parameters and states are obtained by any given method, we adapt testing-for-isolation strategies [3], [23] to study the additional control cost of the parameter and state uncertainties on the proposed optimal testing policy [23]. We propose a testing strategy by overestimating the seriousness of the epidemic to adapt the optimal testing policy under the ranges of the obtained parameters and states to guarantee the system feasibility. Further, by comparing the testing cost of the proposed testing strategy with the optimal testing policy, we conclude that the proposed testing strategy under the parameter learning and state estimation processes can flatten the curve effectively, but will cost more tests and time.

The paper is organized as follows. In Section II, we introduce the optimal epidemic mitigation problem and the goal of this work. In Section III, we propose a testing strategy to study the feasibility of the control problem under the parameter and state uncertainties. We characterize the control cost via comparison with the optimal testing strategy generated under accurate models and states. In Section IV, we illustrate the proposed control strategy through simulations. Section V presents the conclusions and future work.

II. Problem Formulation

In this section, we introduce the epidemic spreading model and formulate the optimal epidemic resource allocation for mitigation problem. Our goal is to propose a potential way for policy-makers to implement a feedback testing strategy to mitigate an epidemic. As illustrated by the arrows from the top and middle blocks to the bottom block in Fig. 1, we leverage the model parameters and epidemic states with uncertainties to study the control policy.

A. Testing for Mitigation

In this subsection, we present the model for the epidemic control problem. We consider the following closed-loop Susceptible-Infected-Recovered/Removed (SIR) model:

$$\frac{dS(t)}{dt} = -\beta S(t)I(t), \tag{1a}$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - (\gamma + u(t))I(t), \tag{1b}$$

$$\frac{dR(t)}{dt} = (\gamma + u(t))I(t). \tag{1c}$$

The parameters β and γ represent the transmission rate and removal rate, respectively, and the control input $u(t)$ captures testing strategies that isolate/remove $u(t) \times 100\%$ of the detected infected population from the infected group, represented by $u(t)I(t)$. In this work, we assume the removal rate captures any processes that separate the detected infected group from the whole population, which include the recovery process, hospitalization, deaths, etc. We define mitigation as maintaining the infection level under a certain threshold through control strategies. Note that when $u(t) = 0$, the system in (1) becomes the classic SIR model [25].

B. Optimal Testing Problem

In this subsection, we introduce the optimal control framework. Consider the system formulated in (1). The goal for the epidemic mitigation problem is to optimally allocate the testing resources during the pandemic such that the daily infected population is maintained at/below the desired infection threshold. In this work, we consider mitigating the epidemic by minimizing the total number of tests during the epidemic through the following cost function

$$J(u(t)) = \int_0^{+\infty} u(t)dt. \tag{2}$$

In order to obtain the testing-for-isolation strategy that minimizes the total number of tests needed during the epidemic spreading process while ensuring that the fraction of infected individuals remains below a desired threshold, we formulate the following optimization problem,

$$\min_{u(t), 0\le t\le +\infty} \quad J(u(t)) \tag{3a}$$

$$\text{s.t. } \dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), u(t)), \tag{3b}$$

$$0 \le I(t) \le \bar{I}, \underline{u} \le u(t) \le \bar{u}, \forall t \in [0, +\infty), \tag{3c}$$

where $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), u(t))$ denotes the closed-loop dynamics in (1). The state constraint $\bar{I}$ describes the *infection threshold* for the fraction of the infected undetected population. In addition, the control input constraints $\underline{u}$ and $\bar{u}$ define the lower and upper bounds on the testing rates, respectively.

C. Goals

In this work, we assume the ranges of the model parameters and states in (1) are given via potential existing real-time modeling and estimation techniques at any given time $t \ge 0$. We use $S(t), I(t), R(t)\ \forall t \ge 0$ to denote the true susceptible, infected, and removal states, respectively, while $\hat{S}(t)$, $\hat{I}(t)$, $\hat{R}(t)\ \forall t \ge 0$ represent the corresponding estimated states. Distinct from the true model parameters β and γ, we use $\hat{\beta}(t)$ and $\hat{\gamma}(t)\ \forall t \ge 0$ to represent the estimated parameters at any given time $t \ge 0$. In addition, we assume $\hat{\beta}(t), \beta \in [\hat{\beta}_{\min}(t), \hat{\beta}_{\max}(t)]$; $\hat{\gamma}(t), \gamma \in [\hat{\gamma}_{\min}(t), \hat{\gamma}_{\max}(t)]$; $\hat{S}(t), S(t) \in [\hat{S}_{\min}(t), \hat{S}_{\max}(t)]$; and $\hat{I}(t), I(t) \in [\hat{I}_{\min}(t), \hat{I}_{\max}(t)]\ \forall t \ge 0$. Moreover, we use $S^*(t)$, $I^*(t)$, $R^*(t)$ to represent the true states under the optimal control strategy $u^*(t)\ \forall t \ge 0$ for the problem defined in (3).

We focus on the theoretical analysis of the optimal control for the epidemic mitigation problem defined in (3), under the impact of the parameter and state uncertainties. We study optimal control strategies of (3) in order to propose a testing strategy by leveraging the estimated model parameters and states. We explore the *additional control cost* by comparing the total number of tests generated from the proposed control strategy with the tests under the optimal testing strategy. We aim to show the effectiveness of the proposed testing strategy through overestimating the seriousness of the epidemic under the existence of parameter and state uncertainties.

III. Testing for Epidemic Mitigation

We explore the feasibility and additional cost of the optimal control framework proposed in Fig. 1 in this section.

A. Feasibility and the Optimal Testing Strategy

We first study the optimal control framework in (3) under accurate model parameters and states. Let $t = 0$ denote the very beginning of an epidemic, and t_p denote the time when the infection state reaches the peak value during the epidemic spreading process, i.e., $I(t_p) \ge I(t)\ \forall t \ge 0$. The following lemma characterizes the peak value $I(t_p)$ in (1).

Lemma 1. *Starting from* $\boldsymbol{x}(t_a) = [S(t_a) \quad I(t_a) \quad R(t_a)]^\top$ *and* $u(t_a) = \underline{u}$ *at time* $t_a < t_p$*, if the system in* (1) *under the fixed control input* $u(t) = \underline{u}$ *reaches a peak infection value* $I(t_p)$*, we have* $I(t_p) = \rho(\ln\rho - 1 - \ln S(t_a)) + S(t_a) + I(t_a)$*, where* $\rho = \frac{\gamma+\underline{u}}{\beta}$.

The proof of Lemma 1 is included in [26]. Lemma 1 calculates the peak infection value $I(t_p)$ from any initial condition $\boldsymbol{x}(t_a)$ and $u(t_a)$ before t_p, under the fixed control input $u(t) = \underline{u}\ \forall t \ge 0$. Note that if $\underline{u} = \bar{u} = 0$, Lemma 1 characterizes the peak infection for the classic SIR model.

Corollary 1. *Assume the closed-loop system in* (1) *starts from* $\boldsymbol{x}(t_a) = [S(t_a) \quad I(t_a) \quad R(t_a)]^\top$ *and* $u(t_a) = \underline{u}$ *at time* t_a*. If* $\exists\ t_p$ *s.t.* $I(t_p) \ge I(t)\ \forall t \ge t_a$*, the peak infection value* $I(t_p)$ *will increase as* β *increases; decrease as* γ *increases; and decrease as* $\underline{u}$ *increases.*

The proof of Corollary 1 is included in [26]. Corollary 1 implies that, under the same initial conditions, the peak infection value $I(t_p)$ will decrease with higher β and/or lower γ. Further, Corollary 1 states that increasing the lower bound on the testing rate $\underline{u}$ will lower the peak infection value. Hence, if $\underline{u}$ in (3) is sufficiently high, such that $I(t_p) \le \bar{I}$ when $u(t) = \underline{u}\ \forall t \ge 0$, the optimal control strategy will be $u(t) = \underline{u}\ \forall t \ge 0$.

Corollary 2 (Optimal Testing Strategy 1)**.** *The optimal testing strategy for the problem in* (3) *is* $u^*(t) = \underline{u}\ \forall t \ge 0$*, if* $I^*(t_p) = \rho(\ln\rho - 1 - \ln S^*(0)) + S^*(0) + I^*(0) \le \bar{I}$.

Corollary 2 is a direct result from Lemma 1 and Corollary 1, thus the proof is omitted. For the optimal control

problem in (3), if there is no risk for the infection state to exceed the infection threshold $\bar{I}$, maintaining the testing at $\underline{u}$ is the best way to reduce the cost. For the control framework in (3), we consider the case when $I(t_p) > \bar{I}$ under $u(t) = \underline{u}$, $\forall t \geq 0$, and develop the following theorem to study the feasibility of the framework in (3).

Theorem 1. *Starting from $t = t_a \geq 0$, if $\exists t_b \geq t_a$ s.t. $I(t_b) = \bar{I}$ for the first time, then the control framework in* (3) *is feasible if and only if $\exists u(t_b) \in (\underline{u}, \bar{u}]$ s.t. $u(t_b) = \beta S(t_b) - \gamma$.*

The proof of Theorem 1 is included in [26]. In this work, we study the case that satisfies Theorem 1: the upper bound on the testing rate $\bar{u}$ is sufficiently large such that we can always find a $u(t_b) \in (\underline{u}, \bar{u}]$, to satisfy $u(t_b) = \beta S(t_b) - \gamma$. Under such condition, the optimal testing strategy is given by the following proposition, where a^*, $a \in \{S, I, R, t_b, t_h\}$ represents the state or the time step of the system in (1) under the optimal control strategy $u^*(t)$. Note that t_b^* is the time step when $I^*(t)$ $t \geq 0$ reaches $\bar{I}$ under the optimal testing strategy $u^*(t)$ for the first time. In addition, t_h^* is time step when the epidemic reaches herd immunity under the optimal testing strategy $u^*(t)$ for the first time, i.e., $\frac{dI(t_h^*)}{dt} = (\beta S(t_h^*) - (\gamma + \underline{u}))I(t_h^*) = 0$. Furthermore, we have $\frac{dI(t_h^*)}{dt} \leq 0$, $\forall u(t) \in [\underline{u}, \bar{u}]$, $\forall t \geq t_h^*$.

Proposition 1 (Optimal Testing Strategy 2)**.** *[3, Theorem 1] The optimal testing strategy for the problem in* (3) *can be cast into three stages:*

1) *At the early stage of the epidemic, when $I^*(t) < \bar{I}$, $\forall t \in [0, t_b^*)$, $u^*(t) = \underline{u}$;*
2) *During the outbreak, starting from $I^*(t_b^*) = \bar{I}$, $\forall t \in [t_b^*, t_h^*)$, $u^*(t) = \beta S^*(t) - \gamma$;*
3) *When the epidemic reaches herd immunity at t_h^*, i.e., $\beta S^*(t_h^*) = \gamma + \underline{u}$, $\forall t \geq t_h^*$, $u(t) = \underline{u}$.*

The proof of Proposition 1 is the same as the proof of [3, Theorem 1], although the lower bound on the testing rate is $\underline{u} = 0$ in [3, Theorem 1]. Proposition 1 separates the testing strategy into three stages via considering the first time when the infection state reaches $\bar{I}$, i.e., t_b^*, and the herd immunity time step t_h^* as the switching time steps. In the following subsection, we aim to explore testing strategies under the guidance of the optimal testing strategy in Proposition 1, with parameter and state uncertainties.

B. Testing Strategy under Uncertainties

In this subsection, we propose a testing strategy for the problem in (3) with parameter and state uncertainties captured by the ranges given in Section II. Recall that we define $\hat{X}(t)$, $X \in \{S, I, R\}$, $\forall t \geq 0$ as the estimated states. We use $\hat{t}_b$ to denote the time step when the overestimated state $\hat{I}_{\max}(t)$ reaches the infection threshold $\bar{I}$ for the first time. In addition, we use $\hat{t}_h$ to represent the time step when $\hat{\beta}_{\max}(\hat{t}_h)\hat{S}_{\max}(\hat{t}_h) = \hat{\gamma}_{\min}(\hat{t}_h) + \underline{u}$ for the first time, i.e., the computed herd immunity time step by overestimating the epidemic states and spreading parameters. We use $\hat{u}(t)$ $\forall t \geq 0$ to represent the generated testing strategy by leveraging the overestimated epidemic spreading process and the corresponding computed time steps $\hat{t}_h$ and $\hat{t}_b$.

Definition 1 (Testing Strategy under Uncertainties)**.** *The testing strategy for the problem in* (3) *follows the rules:*

1) *At the early stage of the epidemic, when the overestimated infection state is smaller than the infection threshold $\bar{I}$, the testing strategy is given by $\hat{u}(t) = \underline{u}$, $\forall t \in [0, \hat{t}_b)$;*
2) *From the time step $\hat{t}_b$ to the computed herd immunity time step $\hat{t}_h$, the testing strategy is given by $\hat{u}(t) = \hat{\beta}_{\max}(t)\hat{S}_{\max}(t) - \hat{\gamma}_{\min}(t)$, $\forall t \in [\hat{t}_b, \hat{t}_h)$;*
3) *Starting from the computed herd immunity time step $\hat{t}_h$, the testing strategy is given by $\hat{u}(t) = \underline{u}$, $\forall t \geq \hat{t}_h$.*

Definition 1 modifies the optimal testing strategy in Proposition 1 by proposing a testing policy under the given ranges of estimated parameters and states. Definition 1 implies that without accurate model parameters and states, if we know the range of the parameters and states, the testing strategy will always assume the worst case scenario at any given time step to generate the testing policy, i.e., to overestimate the seriousness of the epidemic.

We discuss the feasibility of the system in (1) under the proposed testing strategy in Definition 1 by first studying the situation where $\hat{\beta}_{\max}(t) = \hat{\beta}_{\max} \geq \beta$, $\hat{\gamma}_{\min}(t) = \hat{\gamma}_{\min} \leq \gamma$, $\forall t \geq 0$, and $\hat{S}(t) \in [S(t), \hat{S}_{\max}(t)]$, $\hat{I}(t) \in [\hat{I}(t), \hat{I}_{\max}(t)]$ $\forall t \geq 0$. This case assumes the estimated ranges of the parameters are time-invariant. Recall that $S^*(t)$, $I^*(t)$, $R^*(t)$ denote the system's trajectories under the optimal testing strategy $u^*(t)$, $\forall t \geq 0$. Similar to the definitions of t_b^* and t_h^*, we define $\hat{t}_b$ and $\hat{t}_h$ as the time steps when $\hat{I}(\hat{t}_b) = \bar{I}$ for the first time and $\underline{u} = \hat{\beta}_{\max}\hat{S}(\hat{t}_h) - \hat{\gamma}_{\min}$ for the first time, respectively. We plot both trajectories of the system under the optimal testing strategy $u^*(t)$ and the strategy $\hat{u}(t)$ from Definition 1 in Fig. 2, in order to better explain t_b^*, t_h^*, $\hat{t}_b$, and $\hat{t}_h$. Fig. 2 compares the behavior of the epidemic under the testing strategy in Definition 1 when overestimating the spreading parameters, with the behavior of the epidemic under the optimal testing strategy in Proposition 1 when the true spreading parameters are known. Consider an epidemic spreading process with $\beta = 0.016$ and $\gamma = 0.033$. The infection threshold is set as $\bar{I} = 0.01$. The lower bound on the testing rate is $\bar{u} = 0.03$. We use $S^*(t)$, $I^*(t)$, and $R^*(t)$ $t \geq 0$ to represent the states generated by $u^*(t)$ following the Optimal Testing Strategy 1 in Proposition 1. We use $S(t)$, $I(t)$, and $R(t)$ $t \geq 0$ to denote the true states generated by $\hat{u}(t)$, when implementing the testing strategy given in Definition 1 and leveraging the overestimated spreading parameters $\hat{\beta}(t) = 1.05\beta$ and $\hat{\gamma}(t) = 0.95\gamma$ $\forall t \geq 0$, and noisy states $\hat{S}(t)$ and $\hat{I}(t)$ $\forall t \geq 0$. From Definition 1, we will leverage Fig. 2 to illustrate the following result.

Lemma 2. *When $\hat{\beta}(t) = \hat{\beta}_{\max} \geq \beta$, $\hat{\gamma}(t) = \hat{\gamma}_{\min} \leq \gamma$, $\hat{S}(t) \in [S(t), \hat{S}_{\max}(t)]$, $\hat{I}(t) \in [\hat{I}(t), \hat{I}_{\max}(t)]$, $\forall t \geq 0$, the system in* (1) *under the control strategy $\hat{u}(t)$ generated by leveraging $\hat{\beta}_{\max}$, $\hat{\gamma}_{\min}$, $\hat{S}(t)$, $\hat{I}(t)$ $\forall t \geq 0$, from Definition 1 is feasible. The control strategy satisfies $\hat{u}(t) \geq u^*(t)$, $\forall t \geq 0$,*

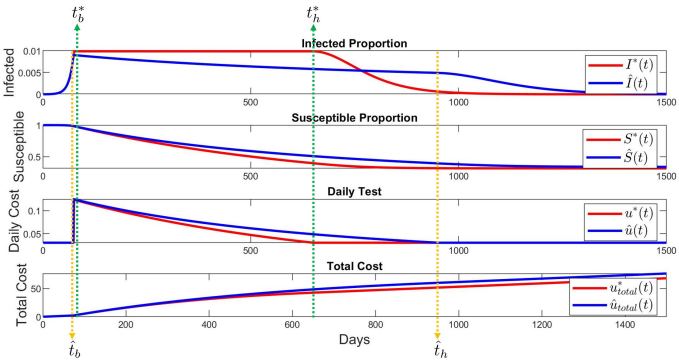

Figure 2: Comparison of Lemma 2 with the Optimal Testing Strategy. Larger version of the figure is included in [26].

Proof. We compare $u^*(t)$ and $\hat{u}(t)$ by considering $t \in [0, \hat{t}_b) \cup [\hat{t}_b, t_b^*] \cup [t_b^*, t_h^*] \cup (t_h^*, \hat{t}_h] \cup (\hat{t}_h, +\infty)$, where the chronological order will be demonstrated within the context. First, we show the system in (1) under the testing policy $\hat{u}(t)\ \forall t \geq 0$ is feasible. We analyze the testing strategy by considering three main testing stages. Recall that the control framework first switches its testing policy when $\hat{I}(\hat{t}_b) = \bar{I}$ ($\hat{t}_b$ is the first time when $\hat{I}(t)$ reaches $\bar{I}$, as shown in the top plot of Fig. 2). Since $\hat{I}(t) \geq I(t)$, $\forall t \geq 0$, we have $I^*(\hat{t}_b) = I(\hat{t}_b) \leq \hat{I}(\hat{t}_b) = \bar{I}$. Hence, compared to using the optimal testing policy $u^*(t)\ \forall t \in [0, t_b^*]$, the system, by leveraging larger estimated infection states, will start to raise the testing rate away from the lower bound earlier, i.e., at $\hat{t}_b$. Hence, we have $\hat{t}_b \leq t_b^*$, as illustrated in Fig. 2. In addition, at the early stage of the epidemic, when $\hat{I}(t) < \bar{I}$, $\forall t \in [0, \hat{t}_b)$, we have $\hat{u}(t) = u^*(t) = \underline{u}$, $\forall t \in [0, \hat{t}_b)$. Then we consider the time step when $\hat{I}(\hat{t}_b) = \bar{I}$. From Definition 1, we have $(\beta S(t) - (\gamma + \hat{u}(t)) \leq (\hat{\beta}_{\max}\hat{S}(t) - (\hat{\gamma}_{\min} + \hat{u}(t)) = 0$. Thus $\frac{dI(t)}{dt} \leq 0, \forall t \in [\hat{t}_b, \hat{t}_h]$, where $\hat{t}_h$ is the computed herd immunity time step under the condition that $\hat{S}(\hat{t}_h)\hat{\beta}_{\max} - \hat{\gamma}_{\min} = \underline{u}$ (shown in Fig. 2). Hence, the infection state $I(t)$ is non-increasing under $\hat{u}(t)$, and $I(t) \leq \bar{I}$, $\forall t \in [\hat{t}_b, \hat{t}_h]$. Lastly, after reaching the computed herd immunity time step $\hat{t}_h$, from Definition 1, we have $\hat{u}(t) = \underline{u}$ and $(\beta S(t) - (\gamma + \underline{u})) \leq (\hat{\beta}_{\max}\hat{S}(t) - (\hat{\gamma}_{\min} + \underline{u})) \leq 0$, $\forall t \geq \hat{t}_h$. Therefore, $I(t)$ $\forall t \geq \hat{t}_h$ will monotonically decrease, and thus cannot reach $\bar{I}$ again. The trajectories of the optimal states under $u^*(t)$ and the true states under the testing strategy $\hat{u}(t)\ \forall t \geq 0$ are shown in Fig. 2. In summary, starting from $t = 0$, $I(t)$ cannot exceed $\bar{I}$ under the given control policy $\hat{u}(t)\ \forall t \geq 0$, which completes the proof of feasibility.

Now we compare $\hat{u}(t)$ and $u^*(t)$. Recall at the early stage of the epidemic, when $\hat{I}(t) < \bar{I}$, $\forall t \in [0, \hat{t}_b)$, $\hat{u}(t) = u^*(t) = \underline{u}$. Starting from $\hat{t}_b$, we have $\hat{u}(t) = \hat{S}(t)\hat{\beta}_{\max} - \hat{\gamma}_{\min} \geq \underline{u} = u^*(t)\ \forall t \in [\hat{t}_b, t_b^*]$. Note that $\hat{u}(t)$ is not the optimal control strategy (but a strategy that ensures the system is feasible) for the problem defined in (3). Moreover, [3, Lemma 5] states that, among all the feasible frameworks, the system in (1) reaches the herd immunity time step t_h^* the fastest, under the optimal testing strategy $u^*(t)$. Hence, we have $\hat{t}_h \geq t_h \geq t_h^*$. Recall that $\hat{S}(t)$ and $S(t)\ \forall t \geq 0$ are the estimated susceptible state and the corresponding true state under the control policy from Lemma 2, respectively. In addition, t_h and $\hat{t}_h$ are the time steps when $S(t_h)\beta - \gamma = \underline{u}$ and $\hat{S}(\hat{t}_h)\beta_{\max} - \gamma_{\min} = \underline{u}$ under the control policy $\hat{u}(t)$, respectively. The inequality $\hat{t}_h \geq t_h$ implies that when $S(t_h)\beta - \gamma = \underline{u}$, the estimated parameters and states still satisfy $\hat{S}(t_h)\hat{\beta}_{\max} - \hat{\gamma}_{\min} \geq \underline{u}$. Thus, it will take longer for the system to reach the estimated herd immunity time step $\hat{t}_h$. Further, the system in (1) under the optimal control policy $u^*(t)$ will reach the herd immunity time step t_h^* faster (or equal to) the system in (1) under $\hat{u}(t)$ (i.e., the estimated herd immunity time $\hat{t}_h$). From Proposition 1 and Definition 1, $u^*(t) = \underline{u}$, $\forall t \geq t_h^*$, and $\hat{u}(t) = \hat{S}(t)\hat{\beta}_{\max} - \hat{\gamma}_{\min} \geq \underline{u}$, $\forall t \in [t_h^*, \hat{t}_h]$. In addition, we have $\hat{u}(t) = \underline{u}$, $\forall t \geq \hat{t}_h$, which leads to $\underline{u} = u^*(t) \leq \hat{u}(t)$ $\forall t \geq t_h^*$, eventually. Lastly, we analyze both testing policies when $t \in [t_b^*, t_h^*]$. Following the discussion from the feasibility and the fact that the optimal control strategy $u^*(t)$ maintains $I^*(t) = \bar{I}\ \forall t \in [t_b^*, t_h^*]$, we have $I(t) \leq \bar{I} = I^*(t)$, $\forall t \in [t_b^*, t_h^*]$. Hence, from the integration of (1a) (dividing $S(t)$ on both sides): $log(S(t)) = log(S(t_b^*)) - \int_{t_b^*}^{t}(\beta I(\tau))d\tau$, if $I(t) \leq \bar{I} = I^*(t)$, $\forall t \in [t_b^*, t_h^*]$, then $\hat{S}(t) \geq S(t) \geq S^*(t)$, $\forall t \in [t_b^*, t_h^*]$ (note that $S(t_b^*) \geq S^*(t_b^*)$). From the fact that $\hat{S}(t) \geq S^*(t)$, $\forall t \in [t_b^*, t_h^*]$, and $\hat{\beta}_{\max} \geq \beta$, $\hat{\gamma}_{\min} \leq \gamma$, we have $\hat{u}(t) \geq u^*(t)$, $\forall t \in [t_b^*, t_h^*]$. □

Lemma 2 explores the case where the estimated upper and lower bounds on the parameters β and γ are time-invariant, and the states are overestimated. Lemma 2 implies that $\hat{u}(t) = u^*(t) = \underline{u}$, $\forall t \in [0, \hat{t}_b) \cup [\hat{t}_h, +\infty)$. In addition, compared to $u^*(t)$, the proposed testing policy $\hat{u}(t)$ from Lemma 2 starts to raise the testing rate from $\underline{u}$ earlier, and switches back to $\underline{u}$ later. Thus, to compare the cost between $\hat{u}(t)$ and the optimal control policy $u^*(t)$, we have the following lemma.

Lemma 3. *The overall cost by leveraging $\hat{\beta}(t) = \hat{\beta}_{\max} \geq \beta$, $\hat{\gamma}(t) = \hat{\gamma}_{\min} \leq \gamma$, $\hat{S}(t) \in [S(t), \hat{S}_{\max}(t)]$, $\hat{I}(t) \in [\hat{I}(t), \hat{I}_{\max}(t)]\ \forall t \geq 0$, is higher than the optimal cost by $\int_{\hat{t}_b}^{\hat{t}_h}(\beta(S(t) - S^*(t)))dt - log(I(\hat{t}_h)) + log(I^*(\hat{t}_h))$.*

The proof of Lemma 3 is included in [26]. The difference between $\hat{u}(t)$ and $u^*(t)$ is captured by the difference between the susceptible states $S(t)$ and $S^*(t)$, and the infection states when the systems reach the computed herd immunity time step $\hat{t}_h$. Lemma 2 and 3 study one approach to guarantee the system's feasibility when knowing the ranges of the parameters and states. The next result analyzes the testing strategy given in Definition 1 with possibly time-varying estimates by leveraging the analysis from Lemma 2 and 3.

Theorem 2. *The testing strategy $\hat{u}(t)$ from Definition 1 by leveraging $\hat{\beta}_{\max}(t)$, $\hat{\gamma}_{\min}(t)$, $\hat{S}_{\max}(t)$ and $\hat{I}_{\max}(t)$, $\forall t \geq 0$ satisfies $\hat{u}(t) \geq u^*(t)\ \forall t \geq 0$. Further, the optimality gap is bounded by $\int_{\hat{t}_b}^{\hat{t}_h}(\beta(S(t) - S^*(t)))dt - log(I(\hat{t}_h)) + log(I^*(\hat{t}_h))$, where $S(t)$ and $I(t)\ \forall t \geq 0$ are the true states generated by using $\hat{u}(t)$.*

The proof of Theorem 2 is included in [26]. Theorem 2 studies the testing strategy proposed in Definition 1. Under the condition that the ranges of the learned parameters and estimated states are known, i.e., $\hat{\beta}(t), \beta \in [\hat{\beta}_{\min}(t), \hat{\beta}_{\max}(t)]$;

$\hat{\gamma}(t), \gamma \in [\hat{\gamma}_{\min}(t), \hat{\gamma}_{\max}(t)]$; $\hat{S}(t), S(t) \in [\hat{S}_{\min}(t), \hat{S}_{\max}(t)]$; $\hat{I}(t), I(t) \in [\hat{I}_{\min}(t), \hat{I}_{\max}(t)]$ $\forall t \geq 0$, Definition 1 casts the testing by overestimating the seriousness of the epidemic at any given time step. Theorem 2 ensures the system in (1) is feasible via leveraging Definition 1. Further, Theorem 2 provides a bound on the testing cost under uncertainties captured by the ranges of the learned parameters and estimated states. In addition, Theorem 2 shows that, by leveraging Definition 1, the susceptible state dominates the trajectory of the optimal susceptible state $\forall t \in [0, t_h^*]$.

Corollary 3. *For any time t up to the herd immunity time step t_h^*, $t \in [0, t_h^*]$, the cumulative number of people infected for the optimal testing strategy, $I^*(t)+R^*(t)$, will be greater than or equal to the cumulative number of people infected from the proposed testing strategy in Definition 1, $I(t)+R(t)$.*

From Lemma 2, 3, Theorem 2, and Corollary 3, we reach the following conclusions on the testing strategy given below.

Remark 1. *When learning and estimation strategies offer $\hat{\beta}(t) \in [\hat{\beta}_{\min}(t), \hat{\beta}_{\max}(t)]$, $\hat{\gamma}(t) \in [\hat{\gamma}_{\min}(t), \hat{\gamma}_{\max}(t)]$, $\hat{S}(t) \in [\hat{S}_{\min}(t), \hat{S}_{\max}(t)]$, $\hat{I}(t) \in [\hat{I}_{\min}(t), \hat{I}_{\max}(t)]$ $\forall t \geq 0$, compared to the optimal control strategy given in Proposition 1, the testing strategy from Definition 1 under uncertainties captured by the parameter learning and state estimation process will:*

1) *Overestimate the seriousness of the epidemic at any given time step;*
2) *React earlier to the outbreak and switch back to the lower bound on the testing rate later;*
3) *Cost more or the same in terms of testing at each time t $\forall t \geq 0$;*
4) *Generate fewer or equal total uninfected individuals in the population at any given time step up to t_h^*.*

IV. Simulation

We now illustrate the proposed testing strategy from Definition 1 via simulations. Consider an epidemic spreading process in (1) with $\beta = 0.16$ and $\gamma = 0.033$. The goal is to minimize the total number of tests during the epidemic given by (3) while maintaining the infection level under (or equal to) 1% of the population, i.e. $\bar{I} = 0.01$. We update the parameters, states, and testing policies daily, under the condition that the daily upper and lower bounds on the testing rates are $\bar{u} = 15\%$ and $\underline{u} = 3\%$, respectively. The initial conditions are $I(0) = 0.00001$, $R(0) = 0$, $S(0) = 1 - I(0)$. The observed data sets are corrupted with noise, and the signal-to-noise ratio is $55dB$. From Fig. 1, the observed data will impact both model parameter estimation and the computation of the control input. We compare the results via testing policies given by Proposition 1 and Definition 1.

Besides the optimal testing strategy that leverages the true parameters and states, we consider two types of testing strategies. The first testing strategy (Strategy 1) is to leverage Proposition 1 by considering the noisy data and estimated parameters as the states and model parameters for policy-making, respectively. The second testing strategy (Strategy 2) is to leverage Definition 1, where the ranges of the parameters and states are given daily. Fig. 3 shows the comparison between the epidemic dynamics under three testing strategies, while the parameter estimation process via generalized linear regression [27] is shown in Fig. 4. From Fig. 4, we find that the transmission rate $\hat{\beta}(t)$ is highly underestimated during the spreading process, which may lead to the underestimation of the seriousness of the epidemic. Hence, we can compare the robustness of Strategy 1 and Strategy 2 against model uncertainties. Note that we use $I^*(t)$, $S^*(t)$ to represent the system trajectories under the optimal daily testing rate $u^*(t)$ and the cumulative cost $u^*_{total}(t)$. Similarly, we use $I_1(t)$ and $S_1(t)$ and $I_2(t)$, $S_2(t)$ to denote the true system trajectories under Strategy 1: $\hat{u}_1(t)$ and Strategy 2: $\hat{u}_2(t)$, respectively. Note that the corresponding noisy states ($\hat{I}_1(t)$, $\hat{S}_1(t)$, $\hat{I}_2(t)$, and $\hat{S}_2(t)$) which we leverage for parameter estimation and control policy generation are not shown in these plots. We compare the trajectories in Fig. 3. The

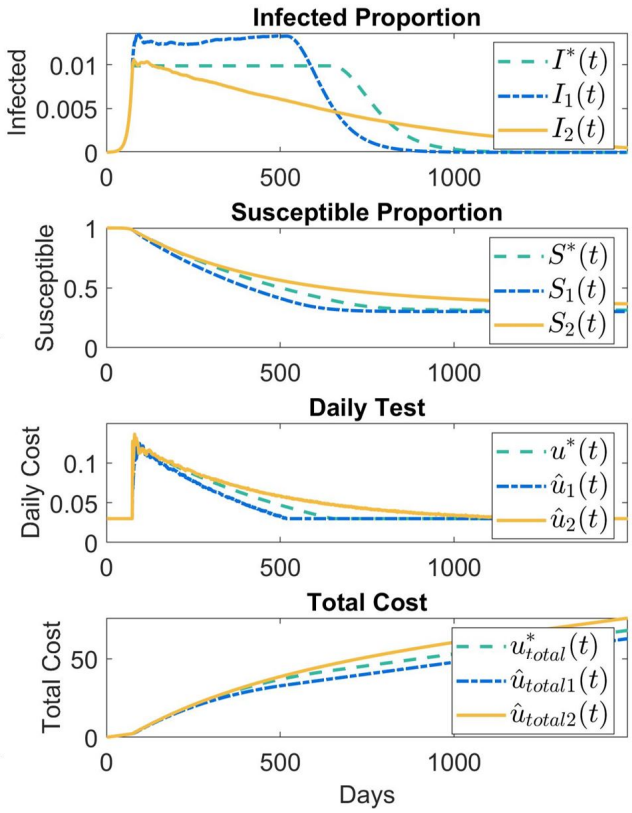

Figure 3: Comparison Between Testing Strategies

simulation illustrates that the control system is nearly feasible by leveraging Strategy 2, as demonstrated by $I_2(t)$ in Fig. 3. However, when leveraging the learned parameters directly (Strategy 1), the system becomes infeasible. As shown in Fig. 3, the infection state $I_1(t)$ is still increasing after reaching $\bar{I}$. The cause of this phenomenon is that when the system starts to change the testing policy from $\hat{u}_1(t) = \underline{u}$ to $\hat{u}_1(t) = \hat{S}_1(t)\hat{\beta}(t) - \hat{\gamma}(t)$ at the time step when $\hat{I}_1(t) \geq \bar{I}$, the highly underestimated transmission rate $\hat{\beta}(t)$, shown in Fig. 4, leads to the underestimation of the seriousness of the epidemic, and the testing rate $\hat{u}_1(t)$. Further, as illustrated in Fig. 3, $\hat{u}_1(t) \leq u^*(t)$ during the

Figure 4: Parameter Estimation

epidemic outbreak, which will generate insufficient testing resources to maintain the infection level under the infection threshold $\bar{I}$. Recall from Lemma 3, the optimal control policy is the pointwise smallest testing strategy we can leverage to ensure the system is feasible. Hence, $\hat{u}_1(t) \leq u^*(t)$ during the outbreak will lead to the system becoming infeasible. Regarding the second statement of Remark 1, the simulation shows that it takes longer for the system under Strategy 2 to reach the herd immunity, compared to the system under the optimal testing strategy. The daily testing generated through Strategy 2 is higher than the optimal daily testing, captured by $\hat{u}_1(t) \geq u^*(t)\ \forall t \geq 0$. By comparing the simulated susceptible states, we see Strategy 2 generates fewer or equal total uninfected population at any given time step, i.e., $S_1(t) \geq S^*(t)\ \forall t \geq 0$, which implies that Strategy 2 will cause fewer people to be infected over the course of the outbreak, that is, $I^*(t) + R^*(t) \geq I_1(t) + R(t)$ for all $t \geq 0$.

V. Conclusion

In this work, we study the impact of uncertainties introduced by parameter learning and state estimation in real-time optimal epidemic mitigation. We show the effectiveness of the proposed testing strategy when overestimating the seriousness of the epidemic under the condition that the ranges of the parameters and states are known. Compared to the optimal testing strategy, the proposed strategy can flatten the curve effectively with more cost in terms of testing and time. However, we have shown analytically that the proposed strategy generates fewer or equal cumulative infected individuals at any given time step up to the optimal herd immunity point and it appears, via simulations, to be true for all time. Future work will propose strategies to learn the parameters and embed the parameter learning techniques into the proposed testing and isolation framework.

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