On the Optimal Cost and Asymptotic Stability in Two-Player Zero-Sum Set-Valued Hybrid Games

Santiago J. Leudo, Francesco Ferrante, and Ricardo G. Sanfelice

***Abstract*— In this paper, we formulate a two-player zero-sum game under dynamic constraints formulated in terms of a hybrid inclusion. The game consists of a min-max problem involving a cost functional associated to the actions and corresponding (potentially nonunique) solutions to the system. We present sufficient conditions given in terms of Hamilton-Jacobi-Isaacs-like equations to establish a bound on the worst-case cost under the optimal strategy and to exactly evaluate it. Under additional conditions, we show that the proposed optimal state-feedback laws render a set of interest pre-asymptotically stable for the resulting hybrid closed-loop system. The results are illustrated in a numerical example.**

I. INTRODUCTION

Optimal control analysis tools are powerful for the study of multi-agent systems operating in contested scenarios in which each of the agents (or players) are dynamic and select their control actions so as to optimize a cost functional. When the constraints are given in terms of differential equations, such problems are referred to as differential games [1]. The presence of dynamic constraints involving both continuous and discrete dynamics imposes challenges to computing optimal feedback laws and to assesing the cost of solutions. Such a combination of continuous and discrete constraints can be efficiently captured by hybrid system models, giving rise to *hybrid dynamic constraints*. Approaches based on Hamilton-Jacobi-Bellman equations, e.g., [2], [3], [4], are limited to continuous-time and discrete-time dynamics, and fall short when employed to compute and evaluate the optimal cost in scenarios with hybrid constraints, which we refer to as *hybrid games*.

In this paper, following [5], [6], a hybrid dynamical system is denoted $\mathcal{H}$ and is given by the hybrid inclusion

$$\mathcal{H} : \begin{cases} \dot{x} & \in & F(x, u_{C1}, u_{C2}) & (x, u_{C1}, u_{C2}) \in C \\ x^+ & \in & G(x, u_{D1}, u_{D2}) & (x, u_{D1}, u_{D2}) \in D \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $(u_{C1}, u_{D1}) \in \mathbb{R}^{m_{C1}} \times \mathbb{R}^{m_{D1}}$ is the input chosen by player P_1, $(u_{C2}, u_{D2}) \in \mathbb{R}^{m_{C2}} \times \mathbb{R}^{m_{D2}}$ is the input[1] chosen by player P_2, and (C, F, D, G) is the data of $\mathcal{H}$. Continuous evolution of $\mathcal{H}$ is allowed when the state and the input are in the *flow set* C, and is governed by the *flow map* $F : \mathbb{R}^n \times \mathbb{R}^{m_C} \rightrightarrows \mathbb{R}^n$. Discrete evolution (or a jump) of $\mathcal{H}$ is allowed when the state and the input are in the *jump set* D, and the new value of the state after a jump is captured by the *jump map* $G : \mathbb{R}^n \times \mathbb{R}^{m_D} \rightrightarrows \mathbb{R}^n$.

For such type of systems and for the case of one player, a cost functional has been proposed in [7] and optimality is certified via Lyapunov-like conditions. The results in [8] provide cost evaluation tools for the case in which the data is given in terms of set-valued maps. The work in [9] provides sufficient conditions to guarantee the existence of optimal solutions. A receding-horizon algorithm for optimal control of hybrid systems with single-valued flow and jumps that extends the model predictive paradigm to such setting is presented in [10]. Informally, a zero-sum two-player game with hybrid conditions is given as

$$\min_{(u_{C1}, u_{D1})} \max_{(u_{C2}, u_{D2})} \mathcal{J}(\xi, u_{C1}, u_{C2}, u_{D1}, u_{D2}). \quad (2)$$

where $\mathcal{J}$ is a cost functional associated to the solutions to $\mathcal{H}$ from the initial condition ξ. The evolution of this hybrid game is determined by the selection of the inputs which is made by the players. The outcome of this selection can be determined by computing $\mathcal{J}$.

In our previous work, cost evaluation results were established to guarantee optimality and asymptotic stability of a set of interest in a discrete-time setting [11]. A finite-horizon hybrid game, where the dynamical constraints are expressed in terms of hybrid systems, is studied in [12]. As an extension of our previous work in [13], we propose a framework for the study of two-player zero-sum hybrid games with set-valued flow and jump maps. Compared to [13], we relax the assumption on uniqueness of solutions to the hybrid system defining the constraints of the game. Although, it might not be possible to construct a saddle-point equilibrium as the solution to the game when the dynamics admit nonunique solutions (due to the game being ill-defined by the nouniqueness of costs associated to a given input), a weak saddle-point equilibrium and an upper value function are provided herein. Specifically, we optimize the worst-case (due to nonuniqueness of solutions) value of the cost functional $\mathcal{J}$ as in (2), which is conveniently defined to penalize the evolution of the state and the input during flow, at jumps and at their final value.

The main contributions of this paper are a formulation of two-player zero-sum games with hybrid dynamic set-valued constraints. In Section III, Theorem 3.7 provides sufficient conditions to characterize the solution of the min-max problem (2), bounds as well as an expression for the

S. Jimenez Leudo and R. G. Sanfelice are with Electrical and Computer Engineering Department, University of California, Santa Cruz, CA 95064. F. Ferrante is with Department of Engineering, University of Perugia. Email: {sjimen28, ricardo}@ucsc.edu francesco.ferrante@unipg.it

Research partially supported by NSF Grants no. CNS-2039054 and CNS-2111688, by AFOSR Grants nos. FA9550-19-1-0169, FA9550-20-1-0238, FA9550-23-1-0145, and FA9550-23-1-0313, by AFRL Grant nos. FA8651-22-1-0017 and FA8651-23-1-0004, by ARO Grant no. W911NF-20-1-0253, and by DoD Grant no. W911NF-23-1-0158.

[1]Here, $m_C = m_{C1} + m_{C2}$ and $m_D = m_{D1} + m_{D2}$.

exact value of the worst-case cost over the set of adversarial strategies, and the characterization of the feedback law that attains it. Connections between optimality and stability for the studied type of games are established in Section IV, while a numerical example is presented in Section V.

Notation. Given two vectors, x and y, we denote $(x, y) = [x^\top y^\top]^\top$. The symbol $\mathbb{N}$ denotes the set of natural numbers including zero. The symbol $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. Given a vector x and a nonempty set $\mathcal{A}$, the distance of x to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x - y|$.

II. Preliminaries

A. Hybrid Systems with Inputs

Since solutions to the dynamical system $\mathcal{H}$ as in (1) can exhibit both continuous and discrete behavior, we use ordinary time t to determine the amount of flow, and a counter $j \in \mathbb{N}$ that counts the number of jumps. Based on this parametrization of time, the concepts of a hybrid time domain, in which solutions are fully described, defined as in [5], is employed here.

A hybrid signal is a function defined on a hybrid time domain. Given a hybrid signal ϕ and $j \in \mathbb{N}$, we define $I^j_\phi := \{t : (t, j) \in \operatorname{dom} \phi\}$.

Definition 2.1: (Hybrid arc) *A hybrid signal $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is called a hybrid arc if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval I^j_ϕ. A hybrid arc ϕ is compact if $\operatorname{dom} \phi$ is compact.*

Let $\mathcal{X}$ be the set of hybrid arcs $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ and $\mathcal{U} = \mathcal{U}_C \times \mathcal{U}_D$ the set of hybrid inputs $u = (u_C, u_D) : \operatorname{dom} u \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, where $u_C = (u_{C1}, u_{C2})$, $m_{C1} + m_{C2} = m_C$, $u_D = (u_{D1}, u_{D2})$, and $m_{D1} + m_{D2} = m_D$. A solution to the hybrid system $\mathcal{H}$ with input is defined as follows.

Definition 2.2: (Solution to $\mathcal{H}$) *A hybrid signal (ϕ, u) defines a solution pair to (1) if $\phi \in \mathcal{X}$, $u = (u_C, u_D) \in \mathcal{U}$, $\operatorname{dom} \phi = \operatorname{dom} u$, and*

- $(\phi(0, 0), u_C(0, 0)) \in \overline{C}$ *or* $(\phi(0, 0), u_D(0, 0)) \in D$,
- *For each $j \in \mathbb{N}$ such that I^j_ϕ has a nonempty interior* $\operatorname{int} I^j_\phi$, *we have, for all $t \in \operatorname{int} I^j_\phi$,*
$$(\phi(t, j), u_C(t, j)) \in C$$
and, for almost all $t \in I^j_\phi$,
$$\frac{d}{dt}\phi(t, j) \in F(\phi(t, j), u_C(t, j))$$
- *For all $(t, j) \in \operatorname{dom} \phi$ such that $(t, j + 1) \in \operatorname{dom} \phi$,*
$$\begin{aligned}(\phi(t, j), u_D(t, j)) &\in D\\ \phi(t, j + 1) &\in G(\phi(t, j), u_D(t, j))\end{aligned}$$

A solution pair (ϕ, u) is a compact solution pair if ϕ is a compact hybrid arc.

A solution pair (ϕ, u) to $\mathcal{H}$ from $\xi \in \mathbb{R}^n$ is nontrivial if $\operatorname{dom}(\phi, u)$ contains at least two points. It is complete if $\operatorname{dom}(\phi, u)$ is unbounded. It is maximal if its domain cannot be extended. We denote by $\hat{\mathcal{S}}_{\mathcal{H}}(M)$ the set of solution pairs (ϕ, u) to $\mathcal{H}$ as in (1) such that $\phi(0, 0) \in M$. The set $\mathcal{S}_{\mathcal{H}}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}}(M)$ denotes all maximal solution pairs from M and the set $\mathcal{U}_{\mathcal{H}}(M)$ denotes all input actions that yield maximal solutions from M. For a given $u \in \mathcal{U}$, we denote the set of maximal state trajectories to $\mathcal{H}$ from ξ for u by $\mathcal{R}(\xi, u) = \{\phi : (\phi, u) \in \mathcal{S}_{\mathcal{H}}(\xi)\}$. We say u renders maximal a trajectory ϕ to $\mathcal{H}$ from ξ if $\phi \in \mathcal{R}(\xi, u)$.

We define the projections of C and D onto $\mathbb{R}^n$, respectively, as

$$\Pi(C) = \{\xi \in \mathbb{R}^n : \exists u_C \in \mathbb{R}^{m_C} \text{ s.t. } (\xi, u_C) \in C\}$$
$$\Pi(D) = \{\xi \in \mathbb{R}^n : \exists u_D \in \mathbb{R}^{m_D} \text{ s.t. } (\xi, u_D) \in D\}$$

We define $\sup_t \operatorname{dom} \phi := \sup\{t \in \mathbb{R}_{\geq 0} : \exists j \in \mathbb{N} \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$ and $\sup_j \operatorname{dom} \phi := \sup\{j \in \mathbb{N} : \exists t \in \mathbb{R}_{\geq 0} \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$. See [5], [6] for more details.

B. Closed-loop Hybrid Systems

Given a hybrid system $\mathcal{H}$ and a function $\kappa := (\kappa_C, \kappa_D)$ with $\kappa : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, the autonomous hybrid system resulting from assigning $u = \kappa(x)$, namely, the hybrid closed-loop system, is given by

$$\mathcal{H}_\kappa : \begin{cases} \dot{x} \in F(x, \kappa_C(x)) & x \in C_\kappa \\ x^+ \in G(x, \kappa_D(x)) & x \in D_\kappa \end{cases} \tag{3}$$

where $C_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_C(x)) \in C\}$ and $D_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_D(x)) \in D\}$.

III. Two-player Zero-sum Hybrid Games

A. Game Formulation

Following the formulation in [14], for each $i \in \{1, 2\}$, consider the i-th player P_i with dynamics described by $\mathcal{H}_i$ as in (1) with data (C_i, F_i, D_i, G_i), state $x_i \in \mathbb{R}^{n_i}$, and input $u_i = (u_{Ci}, u_{Di}) \in \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$, where $C_i \subset \mathbb{R}^n \times \mathbb{R}^{m_C}$, $F_i : \mathbb{R}^n \times \mathbb{R}^{m_C} \rightrightarrows \mathbb{R}^{n_i}$, $D_i \subset \mathbb{R}^n \times \mathbb{R}^{m_D}$ and $G_i : \mathbb{R}^n \times \mathbb{R}^{m_D} \rightrightarrows \mathbb{R}^{n_i}$, with $\star_1 + \star_2 = \star$ for $\star \in \{n, m_C, m_D\}$. We denote by $\mathcal{U}_i = \mathcal{U}_{Ci} \times \mathcal{U}_{Di}$ the set of hybrid inputs for $\mathcal{H}_i$; see Definition 2.3.

Definition 3.1: (Elements of a two-player zero-sum hybrid game) *A two-player zero-sum hybrid game is composed by*

1) *The state $x = (x_1, x_2) \in \mathbb{R}^n$, where, for each $i \in \{1, 2\}$, $x_i \in \mathbb{R}^{n_i}$ is the state of player P_i.*
2) *The set of joint input actions $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ with elements $u = (u_1, u_2)$, where, for each $i \in \{1, 2\}$, $u_i = (u_{Ci}, u_{Di})$ is a hybrid input. For each $i \in \{1, 2\}$, P_i selects u_i independently from P_{3-i}, who selects u_{3-i}, namely, the joint input action u has components u_i that are independently chosen by each player.*

3) *The dynamics of the game, described as in* (1) *and denoted by* $\mathcal{H}$, *with data* (C, F, D, G) *given by*

$$\begin{aligned} C &:= C_1 \cap C_2 \\ F(x, u_C) &:= (F_1(x, u_C), F_2(x, u_C)) \\ D &:= D_1 \cup D_2 \\ G(x, u_D) &:= \{\hat{G}_i(x, u_D) : (x, u_D) \in D_i, i \in \{1, 2\}\} \end{aligned}$$

where $\hat{G}_1(x, u_D) = (G_1(x, u_D), I_{n_2})$, $\hat{G}_2(x, u_D) = (I_{n_1}, G_2(x, u_D))$, $u_C = (u_{C1}, u_{C2})$, *and* $u_D = (u_{D1}, u_{D2})$.

4) *For each* $i \in \{1, 2\}$, *a strategy space* $\mathcal{K}_i$ *of* P_i *defined as a collection of mappings* $\kappa_i : \mathbb{R}^n \to \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$. *The strategy space of the game, namely* $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$, *is the collection of mappings with elements* $\kappa = (\kappa_1, \kappa_2)$, *where* $\kappa_i \in \mathcal{K}_i$ *for each* $i \in \{1, 2\}$. *Each* $\kappa_i \in \mathcal{K}_i$ *is said to be a permissible pure*[2] *strategy for* P_i.

5) *A scalar-valued functional*[3] $(\xi, u) \mapsto \mathcal{J}_i(\xi, u)$ *defined for each* $i \in \{1, 2\}$, *and called the cost associated to* P_i. *For each* $u \in \mathcal{U}$, *we refer to* $\mathcal{J} := \mathcal{J}_1 = -\mathcal{J}_2$ *as the worst-case cost of solutions to* $\mathcal{H}$ *from the initial condition* ξ *for the hybrid input* u.

In this type of game, for each $i \in \{1, 2\}$, the player P_i aims to minimize the cost $\mathcal{J}_i$, which, thanks to the definition of $\mathcal{J}$, allows to define a min-max problem in terms of $\mathcal{J}$ alone.

B. Equilibrium Solution Concept

Given the formulation of a zero-sum hybrid game in Definition 3.1, its solution is defined as follows.

Definition 3.2: (Saddle-point equilibrium) *Consider a two-player zero-sum game, with dynamics* $\mathcal{H}$ *as in (1) with* $\mathcal{J}_1 = \mathcal{J}, \mathcal{J}_2 = -\mathcal{J}$, *for a given cost functional* $\mathcal{J} : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$. *We say that a strategy* $\kappa = (\kappa_1, \kappa_2) \in \mathcal{K}$ *is a saddle-point equilibrium if for each* $\xi \in \Pi(\overline{C} \cup D)$, *every*[4] *solution pair* $(\phi^*, u^*) = (\phi^*, (u_1^*, u_2^*)) \in \mathcal{S}_{\mathcal{H}}(\xi)$ *attaining the worst-case cost and input components defined, for each* $i \in \{1, 2\}$, *as* $\operatorname{dom} \phi^* \ni (t, j) \mapsto u_i^*(t, j) = \kappa_i(\phi^*(t, j))$, *satisfies*

$$\mathcal{J}(\xi, (u_1^*, u_2)) \leq \mathcal{J}(\xi, u^*) \leq \mathcal{J}(\xi, (u_1, u_2^*)) \tag{4}$$

for all hybrid inputs u_1 *and* u_2 *such that* $\mathcal{R}(\xi, (u_1, u_2^*))$ *and* $\mathcal{R}(\xi, (u_1^*, u_2))$ *are nonempty, respectively.*

Definition 3.2 is a generalization of the classical pure strategy Nash equilibrium [14, (6.3)] to the case where the players exhibit dynamics expressed in terms of hybrid inclusions and opposite optimization goals.

[2]This is in contrast to when $\mathcal{K}_i$ is defined as a probability distribution, in which case $\kappa_i \in \mathcal{K}_i$ is referred to as a mixed strategy.

[3]Given that we do not insist on having unique solutions to $\mathcal{H}$, the cost $\mathcal{J}$ measures the worst-case cost among that of each solution to $\mathcal{H}$ from ξ for a given hybrid input u. Thus, its second argument is given by hybrid inputs rather than solution pairs.

[4]Notice that a given strategy κ can lead to multiple input actions due to $C \cap D$ being nonempty.

C. Problem Statement

We formulate an optimization problem to solve the two-player zero-sum hybrid game and provide sufficient conditions to characterize its solution. Following Definition 3.1, consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (1). Given $\xi \in C \cup D$, a joint input action $u = (u_C, u_D) \in \mathcal{U}$, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solutions to $\mathcal{H}$ from the initial condition ξ and for the hybrid input u, as

$$\mathcal{J}(\xi, u) := \sup_{\phi \in \mathcal{R}(\xi, u)} \widetilde{\mathcal{J}}(\phi, u) \tag{5}$$

where[5]

$$\begin{aligned} \widetilde{\mathcal{J}}(\phi, u) := &\sum_{j=0}^{\sup_j \operatorname{dom} \phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t, j), u_C(t, j)) dt \\ &+ \sum_{j=0}^{\sup_j \operatorname{dom} \phi - 1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) \\ &+ \limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} q(\phi(t, j)), \end{aligned} \tag{6}$$

$\{t_j\}_{j=0}^{\sup_j \operatorname{dom} \phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of (ϕ, u) and $\mathcal{R}(\xi, u)$ is the set of maximal state trajectories to $\mathcal{H}$ from the initial condition ξ and for the hybrid input u, as defined in Section II-A. The cost $\mathcal{J}$ is defined as the worst-case cost over all solutions from ξ.

A solution to the two-player zero-sum hybrid game can be obtained by solving the following problem.

Problem ($\diamond$)*:* Given $\xi \in \mathbb{R}^n$, solve

$$\underset{u=(u_1,u_2) \in \mathcal{U}_{\mathcal{H}}(\xi)}{\underset{u_1}{\text{minimize}} \ \underset{u_2}{\text{maximize}}} \ \mathcal{J}(\xi, u) \tag{7}$$

where $\mathcal{U}_{\mathcal{H}}$ is the set of joint input actions yielding maximal solutions to $\mathcal{H}$, as defined in Section II.A.

Definition 3.3: (Value function) *Given* $\xi \in \Pi(\overline{C} \cup D)$, *the value function at* ξ, *when it exists, is given by*

$$\begin{aligned} \mathcal{J}^*(\xi) := &\min_{u_1} \max_{u_2}{}_{u=(u_1,u_2) \in \mathcal{U}_{\mathcal{H}}(\xi)} \mathcal{J}(\xi, u) \\ = &\max_{u_2} \min_{u_1}{}_{u=(u_1,u_2) \in \mathcal{U}_{\mathcal{H}}(\xi)} \mathcal{J}(\xi, u) \end{aligned} \tag{8}$$

D. Weak Saddle-point Equilibrium Solution

In general, the cost evaluation tools employed in approaches based on dynamic programming fall short to characterize strategies to attain a saddle-point equilibrium solution for a two-player zero-sum game with dynamics given

[5]Notice that $\mathcal{J}$ depends on the initial condition ξ and input u, while $\widetilde{\mathcal{J}}$ depends on the solution pair (ϕ, u) with $\phi(0, 0) = \xi$.

by hybrid inclusions. The classical conditions involved in dynamic programming do not guarantee the existence of a lower bound for the cost of solutions to $\mathcal{H}$ from a given initial condition and for an input action. Nevertheless, conditions can still be established to characterize the worst-case cost (due to the set-valued dynamics) associated to it. Thus, in this section, we provide sufficient conditions to solve Problem ($\diamond$) via finding a control strategy that minimizes the worst-case cost under the maximizing adversarial action. This leads to a solution of a min-max problem with potentially nonunique solutions, due to F or G being possibly set valued, or $C \cap D$ being nonempty. In addition, the provided sufficient conditions allow to evaluate the value function without computing solutions. First, we provide pointwise conditions that allow to upper bound the cost for a initial condition and input action.

Proposition 3.4: (Upper bound for a given input) *Given a hybrid system $\mathcal{H}$ as in (1) with data (C, F, D, G), stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\Pi(C)$ such that*

$$L_C(x, u_C) + \sup_{f \in F(x,u_C)} \langle \nabla V(x), f \rangle \leq 0 \qquad \forall (x, u_C) \in C, \tag{9}$$

$$L_D(x, u_D) + \sup_{g \in G(x,u_D)} V(g) - V(x) \leq 0 \qquad \forall (x, u_D) \in D. \tag{10}$$

Let (ϕ, u) be a solution to $\mathcal{H}$ from $\xi \in \Pi(\overline{C} \cup D)$. Then,

$$\widetilde{\mathcal{J}}(\phi, u) \leq V(\xi) \tag{11}$$

where $\widetilde{\mathcal{J}}$ is defined in (6).

In the following result we study a special hybrid system, whose solutions are a subset of the solutions to $\mathcal{H}$ as in (1) and attain the worst-case cost due to nonuniqueness of solutions to $\mathcal{H}$. Following [15], we provide conditions to exactly evaluate such a cost and show how it is an upper bound for the cost of any other solution to $\mathcal{H}$.

Proposition 3.5: (Maximal System) *Consider a hybrid system $\mathcal{H}$ as in (1) with data (C, F, D, G), where F and G are compact for each $(x, u_C) \in C$ and each $(x, u_D) \in D$, respectively, stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, and suppose that there exists a continuous function $V : \operatorname{dom} V \to \mathbb{R}$, $\operatorname{dom} V \supset \Pi(\overline{C}) \cup \Pi(D) \cup G(D)$, that is continuously differentiable on a neighborhood of $\Pi(C)$. Given $\xi \in \Pi(\overline{C} \cup D)$ and a solution*[6] *(ϕ^*, u) to*

$$\mathcal{H}_{\max} : \begin{cases} \dot{x} \in \operatorname*{argmax}_{f \in F(x,u_C)} \langle \nabla V(x), f \rangle & (x, u_C) \in C \\ x^+ \in \operatorname*{argmax}_{g \in G(x,u_D)} V(g) & (x, u_D) \in D \end{cases} \tag{12}$$

from ξ with $u = (u_C, u_D)$, if

$$0 = L_C(x, u_C) + \sup_{f \in F(x,u_C)} \langle \nabla V, f \rangle \qquad \forall (x, u_C) \in C, \tag{13}$$

$$0 = L_D(x, u_D) + \sup_{g \in G(x,u_D)} V(g) \qquad \forall (x, u_D) \in D, \tag{14}$$

and

$$\limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi^* \\ (t,j) \in \operatorname{dom} \phi^*}} V(\phi^*(t, j)) = \limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi^* \\ (t,j) \in \operatorname{dom} \phi^*}} q(\phi^*(t, j)), \tag{15}$$

then

$$\mathcal{J}(\xi, u) = \tilde{\mathcal{J}}(\phi^*, u) \tag{16}$$

and

$$V(\xi) = \mathcal{J}(\xi, u). \tag{17}$$

A solution to (12) attains the worst-case cost among the potential nonunique solutions to (1). Furthermore, the worst-case cost associated to the input that satisfies (13) and (14) can be evaluated without computing solutions as it is equal to $V(\xi)$.

Corollary 3.6: (Change of Signs) *If the conditions in Proposition 3.5 hold with inequality, namely, if in* (13) *and* (14) *"=" is replaced with "$\leq$" (or "$\geq$"), then* (17) *holds with "$\leq$" (or "$\geq$", respectively).*

Based on Proposition 3.4, which provides an upper bound on the cost $\widetilde{\mathcal{J}}$, and the exact cost evaluation in Proposition 3.5, we introduce sufficient conditions in terms of Hamilton-Jacobi-Isaacs-like equations to characterize the saddle-point equilibrium strategy and evaluate the value function without computing solutions.

Theorem 3.7: (Sufficient conditions to solve Problem ($\diamond$)) *Given a hybrid system $\mathcal{H}$ as in (1) with data (C, F, D, G), stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, suppose the following hold:*

1) *There exists a continuous function $V : \operatorname{dom} V \to \mathbb{R}$, $\operatorname{dom} V \supset \Pi(\overline{C}) \cup \Pi(D) \cup G(D)$, that is continuously differentiable on a neighborhood of $\Pi(C)$ and a feedback law $\kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2})) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ such that $F(x, \kappa_C(x))$ and $G(x, \kappa_D(x))$ are compact for every x such that $(x, \kappa_C(x)) \in C$ and $(x, \kappa_D(x)) \in D$, respectively, and such that the functions $\mathcal{L}_C(x, u_C) := L_C(x, u_C) + \sup_{f \in F(x,u_C)} \langle \nabla V(x), f \rangle$, and $\mathcal{L}_D(x, u_D) := L_D(x, u_D) + \sup_{g \in G(x,u_D)} V(g)$ satisfy*

[6] Solutions to the "maximal system" in (12) exist under compactness of the set-valued maps, regularity of V, and a proper selection of the initial condition.

$$0 = \mathcal{L}_C(x, \kappa_C(x)) \qquad \forall x : (x, \kappa_C(x)) \in C, \quad (18)$$

$$0 \le \mathcal{L}_C(x, (u_{C1}, \kappa_{C2}(x))) \qquad \forall (x, u_{C1}) : (x, (u_{C1}, \kappa_{C2}(x))) \in C, \quad (19)$$

$$0 \ge \mathcal{L}_C(x, (\kappa_{C1}(x), u_{C2})) \qquad \forall (x, u_{C2}) : (x, (\kappa_{C1}(x), u_{C2})) \in C, \quad (20)$$

$$V(x) = \mathcal{L}_D(x, \kappa_D(x)) \qquad \forall x : (x, \kappa_D(x)) \in D, \quad (21)$$

$$V(x) \le \mathcal{L}_D(x, (u_{D1}, \kappa_{D2}(x))) \qquad \forall (x, u_{D1}) : (x, (u_{D1}, \kappa_{D2}(x)))) \in D, \quad (22)$$

$$V(x) \ge \mathcal{L}_D(x, (\kappa_{D1}(x), u_{D2})) \qquad \forall (x, u_{D2}) : (x, (\kappa_{D1}(x), u_{D2})) \in D, \quad (23)$$

2) *For each* $\xi \in \Pi(\overline{C} \cup D)$, *each* $\phi \in \mathcal{S}_{\mathcal{H}_\kappa}(\xi)$ *satisfies*

$$\limsup_{\substack{(t,j)\to\sup\operatorname{dom}\phi \\ (t,j)\in\operatorname{dom}\phi}} V(\phi(t,j)) = \limsup_{\substack{(t,j)\to\sup\operatorname{dom}\phi \\ (t,j)\in\operatorname{dom}\phi}} q(\phi(t,j)), \quad (24)$$

Then

$$\mathcal{J}^*(\xi) = V(\xi) \qquad \forall \xi \in \Pi(\overline{C} \cup D). \qquad (25)$$

Remark 3.8: (Weak optimality of the saddle-point equilibrium) *When both players play the saddle-point equilibrium strategy, due to nonuniqueness of solutions, there is no reason to expect that the worst cost is attained, implying that such a strategy is not necessarily optimal in the min-max sense. Nevertheless, by playing the saddle-point equilibrium, the worst-case cost is minimized under the adversarial action that aims to maximize it. See the example in Section V.*

IV. Asymptotic Stability for Hybrid Games

We present a result that connects optimality and asymptotic stability for two-player zero-sum hybrid games. First, we introduce a class of positive definite functions.

Definition 4.1: (Positive definite functions) *A function* $\rho : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\ge 0}$ *is said to be positive definite with respect to the set* $\mathcal{A} \subset \mathbb{R}^n$, *in composition with* $\kappa : \mathbb{R}^n \to \mathbb{R}^m$, *also written as* $\rho \in \mathcal{PD}_\kappa(\mathcal{A})$, *if* $\rho(x, \kappa(x)) > 0$ *for all* $x \in \mathbb{R}^n \setminus \mathcal{A}$ *and* $\rho(\mathcal{A}, \kappa(\mathcal{A})) = \{0\}$.

Theorem 4.2: (Saddle-point equilibrium under the existence of a Lyapunov function) *Consider a two-player zero-sum hybrid game with closed-loop dynamics* $\mathcal{H}_\kappa$ *as in (3) with data* (C, F, D, G), *and* $\kappa := (\kappa_C, \kappa_D) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ *such that* $C_\kappa = \Pi(C)$ *and* $D_\kappa = \Pi(D)$. *Given a closed set* $\mathcal{A} \subset \mathbb{R}^n$, *continuous functions* $L_C : C \to \mathbb{R}_{\ge 0}$ *and* $L_D : D \to \mathbb{R}_{\ge 0}$ *defining the stage costs for flows and jumps, respectively, and* $q : \mathbb{R}^n \to \mathbb{R}$ *defining the terminal cost, suppose there exists a function* $V : \mathbb{R}^n \to \mathbb{R}$ *that is continuously differentiable on an open set containing* $\overline{C_\kappa}$, *satisfying (18)-(23), and such that, for each* $\xi \in \overline{C_\kappa} \cup D_\kappa$, *each* $\phi \in \mathcal{S}_{\mathcal{H}_\kappa}(\xi)$ *satisfies* (24). *Furthermore, suppose that there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that*

$$\alpha_1(|x|_\mathcal{A}) \le V(x) \le \alpha_2(|x|_\mathcal{A}) \qquad \forall x \in \overline{C_\kappa} \cup D_\kappa \cup G(D_\kappa) \quad (26)$$

and one of the following conditions[7] *holds:*

1) $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ *and* $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$;
2) $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$ *and there exists a continuous function* $\eta \in \mathcal{PD}$ *such that* $L_C(x, \kappa_C(x)) \ge \eta(|x|_\mathcal{A})$ *for all* $x \in C_\kappa$;
3) $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ *and there exists a continuous function* $\eta \in \mathcal{PD}$ *such that* $L_D(x, \kappa_D(x)) \ge \eta(|x|_\mathcal{A})$ *for all* $x \in D_\kappa$.

Then

$$\mathcal{J}^*(\xi) = V(\xi) \qquad \forall \xi \in \overline{C_\kappa} \cup D_\kappa \qquad (27)$$

Moreover, the feedback law κ *is the saddle-point equilibrium (see Definition 3.2) and it renders* $\mathcal{A}$ *uniformly globally pre-asymptotically stable for* $\mathcal{H}_\kappa$, *as in [5, Definition 3.6].*

V. Example: Scalar Set-Valued Hybrid Game

The following example characterizes both the saddle-point equilibrium and the value function in a two-player zero-sum game with a scalar state associated to player P_1. Thus, $n_1 = 1, n_2 = 0$, and the role of player P_2 reduces to select the action u_{C2}. Specifically, consider a hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}$, input $u_C := (u_{C1}, u_{C2}) \in \mathbb{R}^2$, and dynamics

$$\begin{aligned} \dot{x} &\in F(x, u_C) := [\underline{a}, \overline{a}]x + Bu_C & x &\in [0, \overline{\sigma}] \cup [\mu, \delta] \\ x^+ &\in G(x) := [\underline{\sigma}, \overline{\sigma}] & x &= \mu \end{aligned} \qquad (28)$$

where[8] $\underline{a} < \overline{a} < 0, B = [b_1\ b_2]$ and $\delta \ge \mu > \overline{\sigma} > \underline{\sigma} > 0$. Consider the cost functions $L_C(x, u_C) := x^2 Q_C + u_C^\top R_C u_C$, $L_D(x) := P(x^2 - \overline{\sigma}^2)$, and terminal cost $q(x) := Px^2$, defining $\mathcal{J}$ as in (6), with $R_C := \left[\begin{smallmatrix} R_{C1} & 0 \\ 0 & R_{C2} \end{smallmatrix}\right]$, Q_C, R_{C1}, $-R_{C2}$, $P \in \mathbb{R}_{>0}$, such that

$$Q_C + 2P\overline{a} - P^2(b_1^2 R_{C1}^{-1} + b_2^2 R_{C2}^{-1}) = 0. \qquad (29)$$

Here, u_{C1} is designed by player P_1, which aims to minimize a cost functional $\mathcal{J}$, while player P_2 seeks to maximize it by means of u_{C2}. This is formulated as a two-player zero-sum hybrid game, for which we solve Problem ($\diamond$) in Section III-C. The function $V(x) := Px^2$ satisfies the sufficient condition for (18)-(20) in Theorem 3.7 given as

$$\min_{\substack{u_{C1} \\ u_C = (u_{C1}, u_{C2}) \in \mathbb{R}^2}} \max_{u_{C2}} \left\{ L_C(x, u_C) + \sup_{f \in F(x, u_C)} \langle \nabla V, f \rangle \right\} = 0 \qquad (30)$$

which holds for all $x \in [0, \overline{\sigma}] \cup [\mu, \delta]$. In fact, the min-max in (30) is attained by $\kappa_C(x) = (-R_{C1}^{-1} b_1 P x, -R_{C2}^{-1} b_2 P x)$. In particular, thanks to (29), we have $-L_C(x, \kappa_C(x)) =$

[7]The subindexes in the sets of positive definite functions $\mathcal{PD}_*$ denote the feedback law that they are composed with, as in Definition 4.1.

[8]Given that $\mu > \delta$, flow from μ is not possible.

$\sup_{f\in F(x,\kappa_C(x))} \langle \nabla V(x), f\rangle$. Then, $V(x) = Px^2$ is a solution to (18)-(20). In addition, the function V satisfies the sufficient condition for (21)-(23) in Theorem 3.7 given as

$$L_D(x) + \sup_{g\in G(x)} V(g(x)) = Px^2 \tag{31}$$

at $x = \mu$, which makes $V(x) = Px^2$ a solution to (21)-(23) with saddle-point equilibrium κ_C. Given that V is continuously differentiable on $\mathbb{R}$, and that (18)-(23) hold thanks to (30) and (31), from Theorem 3.7 we have that the value function is $\mathcal{J}^*(\xi) := P\xi^2$ for any $\xi \in [0, \overline{\sigma}] \cup [\mu, \delta]$.

To study in detail the nonunique solutions yielded by the feedback law κ_C, notice that, given that $\underline{a} < \overline{a} < 0$, solutions from $x = \delta$ flow and then jump at $x = \mu$ to any value $\sigma_s \in [\underline{\sigma}, \overline{\sigma}]$. Consider a solution ϕ_h with domain $\operatorname{dom}\phi_h = ([0,t^h]\times\{0\}) \cup ([t^h,\infty)\times\{1\})$, and given by $\phi_h(t,0) = \delta\exp((a_s - R_{C1}^{-1}b_1P - R_{C2}^{-1}b_2P)t)$, $\phi_h(t,1) = \sigma_s\exp((a_s - R_{C1}^{-1}b_1P - R_{C2}^{-1}b_2P)(t-t^h))$ with $a_s \in [\underline{a}, \overline{a}]$. In simple words, ϕ_h flows from δ to μ in t^h units of time, then it jumps to σ_s, and flows converging (exponentially fast) to zero. Notice that κ_C as defined above also yields a solution ϕ_κ with domain $\operatorname{dom}\phi_\kappa = ([0,t^\kappa]\times\{0\}) \cup ([t^\kappa,\infty)\times\{1\})$, and given by $\phi_\kappa(t,0) = \delta\exp((\overline{a} - R_{C1}^{-1}b_1P - R_{C2}^{-1}b_2P)t)$, $\phi_\kappa(t,1) = \overline{\sigma}\exp((\overline{a} - R_{C1}^{-1}b_1P - R_{C2}^{-1}b_2P)(t-t^\kappa))$ attaining the worst-case cost. Figure 1 illustrates the similar behavior of the solutions ϕ_h and ϕ_κ, yielded by κ_C, with the cost of the latter equating $P\delta^2$.

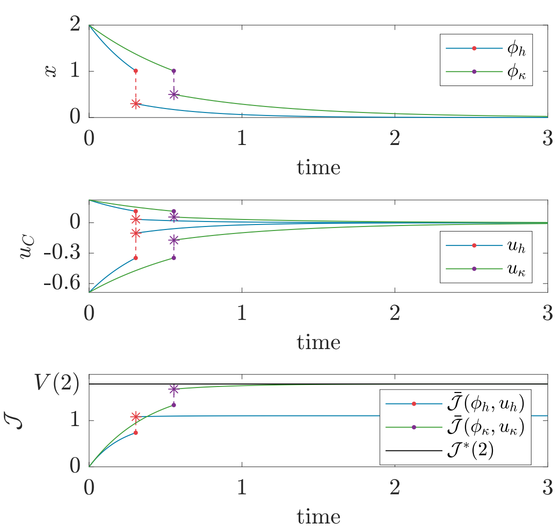

Fig. 1. Nonnunique solutions due to set-valued dynamics for $\underline{a} = -2, \overline{a} = -1, b_1 = b_2 = 1, \delta = \xi = 2, \mu = 1, \underline{\sigma} = 0.3, \overline{\sigma} = 0.5, Q_C = 1, R_{C1} = 1.304, R_{C2} = -4$, and $P = 0.4481$. Worst-case cost solution (green and pruple). Arbitrary solution (blue and red).

This shows that the weak saddle-point equilibrium κ_C is not necessarily optimal in the min-max sense. Nevertheless, by playing κ_C, player P_1 minimizes the worst-case cost under the maximizing adversarial action.

VI. Conclusion and Future Work

In this paper, we formulate a two-player zero-sum game under dynamic constraints given in terms of hybrid dynamical systems, as in [5]. Scenarios in which the control action is selected by a player P_1 to accomplish an objective and countereffect the damage of an adversarial player P_2 are studied. By encoding the objectives of the players in the optimization of a cost functional, sufficient conditions are provided to bound and exactly evaluate it. The main result characterizes the strategy of P_1 that minimizes the worst-case cost under the maximizing adversarial action. Additional conditions are proposed to allow the saddle-point strategy to render a set of interest asymptotically stable by letting the value function take the role of a Lyapunov function.

Future work includes studying conditions to guarantee the existence of a solution to Problem $(\diamond)$ based on smoothness and regularity of the data of the system, as in [9].

References

[1] R. Isaacs, *Differential games: a mathematical theory with applications to warfare and pursuit, control and optimization*. Courier Corporation, 1999.

[2] D. S. Bernstein, "Nonquadratic cost and nonlinear feedback control," *International Journal of Robust and Nonlinear Control*, vol. 3, no. 3, pp. 211–229, 1993.

[3] A. L'Afflitto, "Differential games, continuous Lyapunov functions, and stabilisation of non-linear dynamical systems," *IET Control Theory & Applications*, vol. 11, no. 15, pp. 2486–2496, 2017.

[4] W. M. Haddad and A. L'Afflitto, "Finite-time stabilization and optimal feedback control," *IEEE Transactions on Automatic Control*, vol. 61, no. 4, pp. 1069–1074, 2016.

[5] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. New Jersey: Princeton University Press, 2012.

[6] R. G. Sanfelice, *Hybrid Feedback Control*. New Jersey: Princeton University Press, 2021.

[7] F. Ferrante and R. G. Sanfelice, "Certifying optimality in hybrid control systems via Lyapunov-like conditions," *Proceedings of the 11th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, vol. 52, no. 16, pp. 245–250, 2019.

[8] ——, "On the optimality of Lyapunov-based feedback laws for constrained difference inclusions," in *Proceedings of the American Control Conference*. IEEE, 2018, pp. 3435–3440.

[9] R. Goebel, "Existence of optimal controls on hybrid time domains," *Nonlinear Analysis: Hybrid Systems*, vol. 31, pp. 153–165, 2019.

[10] B. Altin and R. G. Sanfelice, "Asymptotically stabilizing model predictive control for hybrid dynamical systems," in *Proceedings of the 2019 American Control Conference (ACC)*, July 2019, pp. 3630–3635.

[11] S. J. Leudo, F. Ferrante, and R. G. Sanfelice, "Upper bounds and cost evaluation in dynamic two-player zero-sum games," in *Proceedings of the 2020 59th IEEE Conference on Decision and Control (CDC)*, December 2020, pp. 424–429.

[12] S. Jimenez Leudo and R. G. Sanfelice, "Sufficient conditions for optimality in finite-horizon two-player zero-sum hybrid games," in *Proceedings of the 2022 61th IEEE Conference on Decision and Control (CDC)*, December 2022, pp. 3268–3273.

[13] ——, "Sufficient conditions for optimality and asymptotic stability in two-player zero-sum hybrid games," in *Proceedings of the 25th ACM International Conference on Hybrid Systems: Computation and Control*, ser. HSCC '22. New York, NY, USA: Association for Computing Machinery, 2022. [Online]. Available: https://doi.org/10.1145/3501710.3519514

[14] T. Basar and G. J. Olsder, *Dynamic noncooperative game theory*. Siam, 1999, vol. 23.

[15] F. Ferrante and R. G. Sanfelice, "Cost evaluation for hybrid inclusions: A Lyapunov approach," in *Proceedings of the 2018 IEEE Conference on Decision and Control*, December 2018, pp. 855–860.