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Prescribed-time stability in switching systems with resets: A hybrid dynamical systems approach☆

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ABSTRACT

We consider the problem of achieving prescribed-time stability (PT-S) in a class of hybrid dynamical systems that incorporate switching nonlinear dynamics, exogenous inputs, and resets. By "prescribed-time stability", we refer to the property of having the main state of the system converge to a particular compact set of interest before a given time defined *a priori* by the user. We focus on hybrid systems that achieve this property via time-varying gains. For continuous-time systems, this approach has received significant attention in recent years, with various applications in control, optimization, and estimation problems. However, its extensions beyond continuous-time systems have been limited. This gap motivates this paper, which introduces a novel class of switching conditions for switching systems with resets that incorporate time-varying gains, ensuring the PT-S property even in the presence of unstable modes. The analysis leverages tools from hybrid dynamical system's theory, and a contraction–dilation property that is established for the hybrid time domains of the solutions of the system. We present the model and main results in a general framework, and subsequently apply them to two different problems: (a) PT control of dynamic plants with uncertainty and intermittent feedback; and (b) PT decision-making in non-cooperative switching games using algorithms that incorporate momentum, resets, and dynamic gains. Numerical results are presented to illustrate all our results.

1. Introduction

Recent advances in nonlinear control analysis and design [1–4] have reinvigorated the concept of Prescribed-Time Stability (PT-S), leading to successful applications across various domains, including nonlinear regulation [1,2], adaptive control [3], systems with delays [5], partial differential equations [6], and stochastic systems [7]. In contrast to asymptotic or exponential stability, the PT-S property guarantees that the system's trajectories will converge to the desired compact set within a predetermined time, regardless of the initial conditions. As such, achieving this property requires either time-varying or non-Lipschitz vector fields in the dynamics of the system. Non-Lipschitz autonomous systems that achieve convergence to the point (or set) of interest before a fixed time have been studied in [8–10]. The state of the art of this property, usually called "fixed-time" (FxT) stability, was recently reviewed in [11], with some recent applications in certain classes of hybrid systems under homogeneity conditions [12,13], continuous-time systems in canonical forms with switching gains [14], and non-switching impulsive systems [15]. In contrast to this line of research, this paper we study systems that achieve convergence to the target before a prescribed time using the "time-varying gain approach" introduced for ODEs in [1], usually refereed to as "prescribe-time control". This method has a long history in optimal control and tactical missile guidance systems [16], and it has recently gained renewed attention due to breakthroughs in the design and analysis of nonlinear and adaptive controllers in continuous-time systems with finite-time convergence properties. For a recent survey, see [11] and recent works on adaptive systems [1–4,17], PDEs [5,6,18,19], and systems with delays [20,21]. Since this control approach uses "blow-up" gains over bounded time domains, the solutions of these systems are also defined only over finite-time intervals. For comprehensive discussions on practical applications, strategies to extend the solution domains, and the advantages and limitations of PT control, we refer the reader to recent works [1,2,11,17,22].

While the study of Prescribed-Time stability properties in continuous-time systems modeled as ordinary differential equations (ODEs) has seen significant progress, PT-S tools for hybrid dynamical systems (HDS) have remained mostly unexplored. For example, switching systems with time-varying gains were studied in [23] using a common

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Lyapunov function. Similarly, stable controllers that deactivate, or "clip", the high gains before the prescribed time is reached were also discussed in [24]. However, such results consider only one vector field during the convergence phase, and the switching rules can lead to HDS that are not well-posed in the sense of [25]. To the best of our knowledge, general results on PT-S for switching and HDS, similar to those existing for asymptotic or exponential stabilization [26], are still absent in the literature. Since switching and hybrid controllers have been shown to provide powerful solutions to complex control [27,28], optimization [29,30], and learning problems [31], there is a clear need for the development of PT-S tools that enable the analysis and design of new algorithms able to simultaneously leverage the advantages of both PT-S and hybrid control.

In this paper, we address this problem by showing that the PT-S property can be naturally incorporated into a class of HDS that model nonlinear switching systems with resets, allowing the switching signals to incorporate the dynamic effects of time-varying gains, while preserving the structure of the hybrid arcs associated to the solutions of the system. Specifically, the main contributions of this paper are as follows:

(a) First, we introduce a class of switching signals that preserve the PT-S property in systems switching between a finite number of PT-S vector fields with exogenous inputs and state resets. To derive these conditions, we reformulate the overall switching system as a HDS with dynamic gains that induce appropriate time dilation and contraction in the hybrid time domains of its solutions. By leveraging Lyapunov-based constructions for a suitably normalized HDS evolving on a hybrid dilated time-scale, we show that the original system is PT-Stable, provided the switching signal satisfies a novel "blow-up" average dwell-time (BU-ADT) condition. This condition allows (but does not impose) a non-linear increase in the number of jumps and switches as the total flow time in the system approaches the prescribed convergence time. To study the effect of exogenous inputs and/or disturbances in the system, we establish results via ISS-like bounds "with the convergence property", paralleling those in the literature on PT-S for ODEs [1, Def. 2]. However, unlike the existing results for ODEs, our convergence bounds, presented in Theorem 1, are written in "hybrid time" and highlight the potentially (asymptotically) stabilizing effect of the resets, as well as the order of the dynamics generating the "blow-up" gains. To our knowledge, this is the first result connecting the existing tools on Prescribed-Time Stability for ODEs [1] with the setting of HDS [25].

(b) Next, we incorporate unstable modes into the switching systems, and we characterize a novel "blow-up" average-activation-time (BU-AAT) condition on the amount of time that the system can spend on the unstable modes while preserving the PT-S property. In our model, the unstable modes are also allowed to have "blow-up" time-varying gains with finite-escape times, as well as exogenous inputs and/or disturbances. To study this setting, we construct a HDS with time-ratio monitors, similar in spirit to those considered in [26,31,32], but incorporating the blow-up gains into their dynamics, enabling faster switching between the stable and unstable modes as the total amount of flow time in the system approaches the prescribed time. A Lyapunov-based construction on a dilated-time scale, and a contraction argument on the hybrid time domains, are used to establish in Theorem 2 a PT-ISS-like result for switched systems with stable and unstable modes.

(c) To illustrate the applicability of our model and results, we synthesize two different PT-Stable algorithms for the solution of different control and decision-making problems with prescribed-time convergence requirements. First, in Proposition 3 we consider the problem of PT regulation of input-affine systems under intermittent feedback, and we propose a new feedback law that extends the results of [1] to plants modeled as switching systems. Finally, we consider the problem of prescribed-time Nash equilibrium seeking in games with switching payoffs via hybrid algorithms with resets. We show in Proposition 4 that such algorithms fit into our model and can be studied using the analytical tools presented in the paper.

The rest of this paper is organized as follows: Section 2 introduces some preliminaries on dynamical systems. Sections 3 and 4 present the main analytical results and the proofs. Section 5 presents three different applications, and Section 6 ends with the conclusions.

2. Preliminaries

2.1. Notation

Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $z \in \mathbb{R}^n$, we use $|z|_{\mathcal{A}} := \inf_{s\in\mathcal{A}} \|z-s\|_2$, where $\|\cdot\|_2$ represents the standard Euclidean norm. A set-valued mapping $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ is outer semicontinuous (OSC) at z if for each sequence $\{z_i, s_i\} \to (z,s) \in \mathbb{R}^p \times \mathbb{R}^n$ satisfying $s_i \in M(z_i)$ for all $i \in \mathbb{Z}_{\geq 0}$, we have $s \in M(z)$. A mapping M is locally bounded (LB) at z if there exists an open neighborhood $N_z \subset \mathbb{R}^p$ of z such that $M(N_z)$ is bounded. The mapping M is OSC and LB relative to a set $K \subset \mathbb{R}^p$ if the mapping from $\mathbb{R}^p$ to $\mathbb{R}^n$ defined by $M(z)$ for $z \in K$, and $\emptyset$ for $z \notin K$, is OSC and LB at each $z \in K$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and satisfies $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r,s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r,s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. A function $\tilde{\beta} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KLL}$ if for every $s \in \mathbb{R}_{\geq 0}$, $\tilde{\beta}(\cdot, s, \cdot)$ and $\tilde{\beta}(\cdot,\cdot,s)$ belong to class $\mathcal{KL}$ [33]. Throughout the paper, for two (or more) vectors $u, v \in \mathbb{R}^n$, we write $(u,v) = [u^\top, v^\top]^\top$ to denote their concatenation. We use $\text{diag}\left(\{B_j\}_{j=1}^J\right)$ to denote the block-diagonal matrix obtained from the set of matrices $\{B_j\}_{j=1}^J$. Given a set $\mathcal{O} \subset \mathbb{R}^n$, we use $\mathbb{I}_{\mathcal{O}}(\cdot)$ to denote the indicator function that satisfies $\mathbb{I}_{\mathcal{O}}(x) = 1$ if $x \in \mathcal{O}$, and $\mathbb{I}_{\mathcal{O}}(x) = 0$ if $x \notin \mathcal{O}$.

2.2. Switching systems

In this paper, we consider switching systems with inputs, with the general form $\dot{x} = \tilde{f}_{\sigma(t)}(x,u,t)$, where $x_0 \in \mathbb{R}^n$ is the initial condition, $x \in \mathbb{R}^n$ is the main state, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is an exogenous input assumed to be continuous and bounded, and $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{Q}$ is a right-continuous, piecewise constant, signal that maps the current time t to a finite set of modes $\mathcal{Q} = \{1,2,\ldots,\overline{q}\}$, where $\overline{q} \in \mathbb{Z}_{\geq 1}$. For each $q \in \mathcal{Q}$, $\tilde{f}_q : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is assumed to be continuous with respect to all arguments. Following the notation of [26], we use $\mathcal{S}$ to denote the set of all right-continuous, piecewise constant, signals from $\mathbb{R}_{\geq 0}$ to $\mathcal{Q}$, with a locally finite number of discontinuities. Such functions are referred to as switching signals. For each signal $\sigma \in \mathcal{S}$, we also define the collection of switching instants $\mathcal{W}(\sigma) := \{t \geq 0 : \sigma(t) \neq \sigma(t^-)\}$. In this way, the switching system of interest evolves according to

$$\dot{x} = \tilde{f}_{\sigma(t)}(x,u,t), \quad \forall\, t \notin \mathcal{W}(\sigma), \tag{1}$$

where the solutions x to (1) are understood in the Caratheodory sense over any interval $[t_a, t_b)$ where σ is constant. During switching times $t \in \mathcal{W}(\sigma)$, we allow "jumps" in the state x via mode-dependent *reset maps* of the form

$$x(t) = R_{\sigma(t^-)}(x(t^-)), \quad \forall\, t \in \mathcal{W}(\sigma), \tag{2}$$

where the function $R_q : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuous for each $q \in \mathcal{Q}$. Throughout the paper, we will refer to switching systems of the form (1)–(2) as R-Switching systems.

Remark 1. By taking R_q equal to the identity map, system (1)–(2) recovers a standard switching system [28]. However, other choices of reset maps open the door to study PT-S results in reset control systems [34] (such as impulsive systems by taking $\mathcal{Q} = \{1\}$) as well as more general switched reset controllers (when $|\mathcal{Q}| > 1$), see [26]. It is also possible to consider discontinuous functions $\tilde{f}_q, R_q$ by working with their corresponding Krasovskii regularizations [25, Def. 4.13]. However, for the sake of clarity, we focus on R-switching systems with continuous maps $\tilde{f}_q$ and R_q. □

2.3. Hybrid dynamical systems with inputs

Since R-Switching systems incorporate continuous-time and discrete-time dynamics, for the purpose of analysis they are usually modeled as hybrid dynamical systems (HDS) [25,26]. Such systems can be modeled as

$$(z,u) \in \tilde{C} := C \times \mathbb{R}^m, \qquad \dot{z} \in F(z,u), \tag{3a}$$

$$(z,u) \in \tilde{D} := D \times \mathbb{R}^m, \qquad z^+ \in G(z), \tag{3b}$$

where $z \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is an input, $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the *flow map*, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the *jump map*, $\tilde{C} \subset \mathbb{R}^n \times \mathbb{R}^m$ is the *flow set*, and $\tilde{D} \subset \mathbb{R}^n \times \mathbb{R}^m$ is the *jump set*. We use $(\tilde{C}, F, \tilde{D}, G)$ to denote the *data* of the HDS. HDS of the form (3) are a generalization of continuous-time systems ($\tilde{D} = \emptyset$) and discrete-time systems ($\tilde{C} = \emptyset$). Time-varying systems can also be represented as (3) via an auxiliary state $\tau \in \mathbb{R}$ with dynamics $\dot{\tau} \geq 0$ and $\tau^+ = \tau$. Solutions to system (3) are parameterized by a continuous-time index $t \in \mathbb{R}_{\geq 0}$, which increases continuously during flows, and a discrete-time index $j \in \mathbb{Z}_{\geq 0}$, which increases by one during jumps. Therefore, solutions to (3) are defined on *hybrid time domains* (HTDs) [25, Ch. 2]. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact* HTD if $E = \cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \ldots \leq t_J$. The set E is a HTD if for all $(T,J) \in E$, $E \cap ([0,T] \times \{0,\ldots,J\})$ is a compact HTD. Given a HTD E, we use

$$\sup_t E := \sup \left\{ t \in \mathbb{R}_{\geq 0} : \exists\, j \in \mathbb{Z}_{\geq 0}, \text{ such that } (t,j) \in E \right\}$$

$$\sup_j E := \sup \left\{ j \in \mathbb{Z}_{\geq 0} : \exists\, t \in \mathbb{R}_{\geq 0}, \text{ such that } (t,j) \in E \right\}.$$

Also, we let $\sup E := (\sup_t E, \sup_j E)$, and $\text{length}(E) := \sup_t E + \sup_j E$. The following definition is borrowed from [33].

Definition 1. A hybrid signal is a function defined on a HTD. A hybrid signal $u : \text{dom}(u) \to \mathbb{R}^m$ is called a hybrid input if $u(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A hybrid signal $z : \text{dom}(z) \to \mathbb{R}^n$ is called a hybrid arc if $z(\cdot, j)$ is locally absolutely continuous for each j such that the interval $I_j := \{t : (t,j) \in \text{dom}(z)\}$ has nonempty interior. A hybrid arc $z : \text{dom}(z) \to \mathbb{R}^n$ and a hybrid input $u : \text{dom}(u) \to \mathbb{R}^m$ form a solution pair (z,u) to (3) if $\text{dom}(z) = \text{dom}(u)$, $(z(0,0), u(0,0)) \in \tilde{C} \cup \tilde{D}$, and:

1. For all $j \in \mathbb{Z}_{\geq 0}$ such that I_j has nonempty interior, and for almost all $t \in I_j$, $(z(t,j), u(t,j)) \in \tilde{C}$ and $\dot{z}(t,j) \in F(z(t,j), u(t,j))$.
2. For all $(t,j) \in \text{dom}(z)$ such that $(t,j+1) \in \text{dom}(z)$, $(z(t,j), u(t,j)) \in \tilde{D}$ and $z(t,j+1) \in G(z(t,j))$. □

Remark 2. By Definition 1, solutions to (3) are required to satisfy $\text{dom}(z) = \text{dom}(u)$. To establish this correspondence, we obtain the input u in (3) from u in (1) using (with some abuse of notation) $u(t,j) = u(t)$ during flows for each fixed j, and by keeping u constant during the jumps (3b). □

A hybrid solution pair (z,u) is said to be maximal if it cannot be further extended. A hybrid solution pair (z,u) is said to be complete if length $\text{dom}(z) = \infty$. This does not necessarily imply that $\sup_t \text{dom}(z) = \infty$, or that $\sup_j \text{dom}(z) = \infty$, although at least one of these two conditions should hold when z is complete. To simplify notation, in this paper we use $|u|_{(t,j)} = \sup_{\substack{(0,0)\leq(\tilde{t},\tilde{j})\leq(t,j) \\ (t,j)\in\text{dom}(z)}} |u(\tilde{t},\tilde{j})|$, and we use $|u|_\infty$ to denote $|u|_{(t,j)}$ when $t+j \to \infty$.

3. PT-ISS in hybrid dynamical systems

Motivated by the PT-S property studied for ODEs [1–5], and before specializing our results to R-switching systems of the form (1)–(2), in this section we introduce PT-S properties for general HDS of the form (3). In particular, we consider systems with state $z = (\psi, \mu_k) \in \mathbb{R}^n \times \mathbb{R}_{\geq 1}$, set C given by:

$$C := \Psi_C \times \mathbb{R}_{\geq 1}, \tag{4a}$$

and flow-map defined as:

$$\dot{z} = \begin{pmatrix} \dot{\psi} \\ \dot{\mu}_k \end{pmatrix} \in F(z,u) := \begin{pmatrix} \mu_k \cdot F_\Psi(\psi, \mu_k, u) \\ \dfrac{k}{T}\mu_k^{1+\frac{1}{k}} \end{pmatrix}, \tag{4b}$$

where $T > 0$ and $k \geq 1$ are tunable parameters, and $F_\Psi : \mathbb{R}^n \times \mathbb{R}_{\geq 1} \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is a set-valued mapping that we will specify below. The set D is given by

$$D = \Psi_D \times \mathbb{R}_{\geq 1}, \tag{4c}$$

and the jump map is given by:

$$z^+ = \begin{pmatrix} \psi^+ \\ \mu_k^+ \end{pmatrix} \in G(z) := \begin{pmatrix} G_\Psi(\psi) \\ \mu_k \end{pmatrix}, \tag{4d}$$

where $G_\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is also to be specified. We denote the HDS (3) with data given by (4) as $\mathcal{H}$. It is assumed that this system satisfies the following standard hybrid basic conditions [25, Assumption 6.5]. These conditions are standard in the hybrid dynamical systems literature [33], and they will be satisfied by construction later when we specialize the results of this section to R-Switching systems with unstable and stable modes.

Assumption 1. The sets $\Psi_C, \Psi_D \subset \mathbb{R}^n$ are closed. The set-valued maps F_Ψ and G_Ψ are OSC and LB with respect to Ψ_C, and Ψ_D, respectively; and F_Ψ is convex for all $(\psi, \mu_k, u) \in \Psi_C \times \mathbb{R}_{\geq 1} \times \mathbb{R}^m$. □

Since in (4b) the dynamics of μ_k are independent of ψ, system (4) has a cascade structure. However, for system (4) the dynamics of ψ will mostly determine the structure of the HTDs of the solutions z, e.g., purely continuous, purely discrete, eventually continuous, etc. To study PT-S properties, in this paper we consider signals μ_k generated by (4b), exhibiting finite escape times that are "controlled" by the parameters (T,k) and by $\mu_k(0)$. This property can be established for the dynamics of μ_k in (4b) by direct integration, and it is formalized in Lemma 1. The proof is presented in the Appendix.

Lemma 1. *Let $k \geq 1$, and consider the "blow-up" (BU)-ODE $\dot{\mu}_k = \frac{k}{T}\mu_k^{1+\frac{1}{k}}$ with $\mu_k(0) = \mu_0 \in \mathbb{R}_{\geq 1}$. Then, its unique solution satisfies:*

$$\mu_k(t) = \frac{T^k}{\left(\Upsilon_{T,k} - t\right)^k} \geq 1, \qquad \forall\, t \in [0, \Upsilon_{T,k}), \tag{5}$$

where $\Upsilon_{T,k} := T\mu_0^{-\frac{1}{k}}$. □

For each $k \geq 1$, $\mu_k(\cdot)$ is continuous in its domain, strictly increasing, and satisfies $\lim_{t \to \Upsilon_{T,k}} \mu_k(t) = \infty$. Hence, the next lemma follows directly by the definition of solutions to HDS.

Lemma 2 (*Bounded Flow-Time*)**.** *Let z be a maximal solution to $\mathcal{H}$. Then, the HTD of z satisfies $\sup_t(\text{dom}(z)) \leq \Upsilon_{T,k}$.* □

Lemma 2 states that the total amount of flow-time of *every* solution of $\mathcal{H}$ will be upper bounded by $\Upsilon_{T,k}$. We will refer to this quantity as the *prescribed time* (PT), and we emphasize its dependency on the initial value μ_0 and the constants (T,k). In the literature on PT-S in continuous-time, μ_0 is usually equal to one. However, we will consider any $\mu_0 \in \mathbb{R}_{\geq 1}$.

A useful property of the BU-ODE studied in Lemma 1, is that, when normalized by μ_k, the resulting ODE has solutions that are complete and lower bounded by 1. The following Lemma is also proved in Appendix.

Lemma 3. *Let $k \geq 1$, and consider the normalized-by-μ_k BU-ODE $\frac{d\hat{\mu}_k}{ds} = \frac{k}{T}\hat{\mu}_k^{\frac{1}{k}}$ with $\hat{\mu}_k(0) = \mu_0 \in \mathbb{R}_{\geq 1}$, evolving in the s-time scale. Then, its unique solution satisfies: (a) For $k = 1$: $\hat{\mu}_k(s) = \mu_0 e^{\frac{s}{T}} \geq 1$ for all $s \geq 0$; (b) For $k > 1$: $\hat{\mu}_k(s) = \left(\frac{(k-1)}{T}s + \mu_0^{\frac{k-1}{k}}\right)^{\frac{k}{k-1}} \geq 1$, for all $s \geq 0$.* □

3.1. Time-scaling of hybrid time domains

The signals μ_k generated by the dynamics (4b) will be used to define a suitable dilation and contraction on the HTD of the solutions to $\mathcal{H}$. To do this, for each $(T,k)\in\mathbb{R}_{>0}\times\mathbb{R}_{\geq 1}$, and $1\leq a\leq b$, let the function $\omega_k:\mathbb{R}_{\geq 1}\times\mathbb{R}_{\geq 1}\to\mathbb{R}_{\geq 0}$ be defined as

$$\omega_k(b,a):=\frac{T}{k}\left(\frac{b^{\rho(k)}-a^{\rho(k)}}{\rho(k)}\right),\quad \forall\, k>1, \tag{6}$$

and $\omega_1(b,a):=\lim_{k\to 1^+}\omega_k(b,a)$, where $\rho(k):=\frac{k-1}{k}$. The following proposition states some important properties of $\omega_k(\cdot,\cdot)$ when evaluated along μ_k. The proof is presented in the Appendix.

Proposition 1. *Let $(T,k)\in\mathbb{R}_{>0}\times\mathbb{R}_{\geq 1}$, μ_k be given by (5), and let $\mathcal{T}_k:[0,\Upsilon_{T,k})\to\mathbb{R}_{\geq 0}$ be the function*

$$\mathcal{T}_k(t):=\omega_k(\mu_k(t),\mu_k(0)),\quad \forall\, t\in[0,\Upsilon_{T,k}). \tag{7}$$

Then, $\mathcal{T}_k(\cdot)$ satisfies the following properties:

(P1) $\lim_{t\to\Upsilon_{T,k}}\mathcal{T}_k(t)=\infty$.

(P2) For any pair $t_2,t_1\in[0,\Upsilon_{T,k})$ such that $t_2\geq t_1$:

$$\mathcal{T}_k(t_2)-\mathcal{T}_k(t_1)=\omega_k(\mu_k(t_2),\mu_k(t_1)).$$

(P3) For all $t\in[0,\Upsilon_{T,k})$, we have

$$\frac{d\mathcal{T}_k(t)}{dt}=\mu_k(t),\quad \mathcal{T}_k(0)=0. \tag{8}$$

(P4) For all $t\in[0,\Upsilon_{T,k})$, $\mathcal{T}_k$ has a well-defined inverse $\mathcal{T}_k^{-1}:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$, which is given by

$$\mathcal{T}_k^{-1}(s)=\Upsilon_{T,k}\left(1-\left(1+\frac{(k-1)s}{\Upsilon_{T,k}\mu_0}\right)^{\frac{1}{1-k}}\right),\quad k>1, \tag{9}$$

and by $\mathcal{T}_1^{-1}(s)=\lim_{k\to 1^+}\mathcal{T}_k^{-1}(s)$.

(P5) For all $s\in\mathbb{R}_{\geq 0}$, $\mathcal{T}_k^{-1}$ satisfies

$$\frac{d}{ds}\mathcal{T}_k^{-1}(s)=\frac{1}{\mu_k\left(\mathcal{T}_k^{-1}(s)\right)},\quad \mathcal{T}_k^{-1}(0)=0. \tag{10}$$

(P6) $\lim_{T\to\infty}\mathcal{T}_k(t)=\mu_0^{\frac{k-1}{k^2}}\,t$ for $k>1$, and $\lim_{T\to\infty}\mathcal{T}_1(t)=\mu_0 t$ for all $t\geq 0$. □

Remark 3. To contextualize Proposition 1, consider the special case $k=1$, which is commonly used in the literature on PT-control of ODEs [1,2]. In this case, Proposition 1 yields the following "standard" mappings:

$$\mathcal{T}_1^{-1}(s)=\Upsilon_{T,1}\left(1-e^{-\frac{1}{T}s}\right),\quad \forall\, s\in\mathbb{R}_{\geq 0}, \tag{11a}$$

$$\mathcal{T}_1(t)=T\left(\ln\left(\frac{\mu_1(t)}{\mu_1(0)}\right)\right),\quad \forall\, t\in[0,\Upsilon_{T,1}). \tag{11b}$$

Indeed, note that (9) can be written as: $\mathcal{T}_k^{-1}(s)=\Upsilon_{T,1}\left(1-\left(1+\frac{s}{n(k)T}\right)^{-n(k)}\right)$, with $n(k)=\frac{1}{k-1}$. Using $e^{\frac{s}{T}}=\lim_{n\to\infty}\left(1+\frac{s}{nT}\right)^n$ and the fact that $n\to\infty$ when $k\to 1^+$, we obtain (11a). Similarly, using $\lim_{\rho\to 0}\frac{\mu_1^{\rho}-1}{\rho}=\ln(\mu_1)$, and the fact that $\rho(k)\to 0$ if and only if $k\to 1$, (11b) follows directly from (6) and the definition of ω_1 by applying the product law for limits. □

The properties established in Proposition 1 are used to derive the following result, which provides a suitable dilation/contraction of the HTDs of $\mathcal{H}$ with data defined by (4) when analyzed in a different hybrid time scale (s,j) induced by the transformation $s=\mathcal{T}_k(t)$, see Fig. 1. Note that, since μ_k does not change during the jumps (4d), when evaluating (7) along (hybrid) solutions of μ_k generated by (4c) we can omit the dependence of $\mathcal{T}_k$ on j.

Proposition 2 (*Dilation and Contraction of HTD*). *Let $(T,\mu_0,k)\in\mathbb{R}_{>0}\times\mathbb{R}_{\geq 1}\times\mathbb{R}_{\geq 1}$, and $\mathcal{T}_k$ be given by (7). Consider the following HDS, denoted by $\widehat{\mathcal{H}}$, evolving on the (s,j)-hybrid time scale, with state $\hat{z}=(\hat{\psi},\hat{\mu}_k)$ and input $\hat{u}$:*

$$(\hat{z},\hat{u})\in\tilde{C}=C\times\mathbb{R}^m,\quad \dot{\hat{z}}_s\in\frac{1}{\hat{\mu}_k}F(\hat{z},\hat{u}). \tag{12a}$$

$$(\hat{z},\hat{u})\in\tilde{D}=D\times\mathbb{R}^m,\quad \hat{z}^+\in G(\hat{z}). \tag{12b}$$

where $(\tilde{C},F,\tilde{D},G)$ in (12) are the same as in (4), and where $\dot{\hat{z}}_s:=\frac{d}{ds}\hat{z}$. Then, the following holds:

(a) If $(\hat{z},\hat{u})$ is a maximal solution pair of $\widehat{\mathcal{H}}$ from the initial condition z_0, then the pair of hybrid signals defined as $(z(t,j),u(t,j)):=(\hat{z}(s,j),\hat{u}(s,j))$, for all $(s,j)\in\mathrm{dom}(\hat{z})$, is also a maximal solution pair of $\mathcal{H}$ from the initial condition z_0 via the time dilation $s=\mathcal{T}_k(t)$.

(b) If (z,u) is a maximal solution pair of $\mathcal{H}$ from the initial condition z_0, then the pair of hybrid signals defined as $(\hat{z}(s,j),\hat{u}(s,j)):=(z(t,j),u(t,j))$ for all $(t,j)\in\mathrm{dom}(z)$, is also a maximal solution pair of $\widehat{\mathcal{H}}$ from the initial condition z_0 via the time contraction $t=\mathcal{T}_k^{-1}(s)$.

□

Proof. We prove each item separately:

(a) Let $(\hat{z},\hat{u})$ be a maximal solution pair of $\widehat{\mathcal{H}}$ from z_0. Then, for each $j\in\mathbb{Z}_{\geq 0}$ such that the interior of $\hat{I}_j:=\{s\geq 0:(s,j)\in\mathrm{dom}(\hat{z})\}$ is nonempty, $\hat{z}$ satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{z}(s,j)\in\frac{1}{\hat{\mu}_k(s,j)}F(\hat{z}(s,j),\hat{u}(s,j)), \tag{13}$$

for almost all $s\in I_j$. Using the chain rule, z satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t,j)=\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}(\mathcal{T}_k(t),j)=\frac{\mathrm{d}}{\mathrm{d}s}\hat{z}(s,j)\cdot\dot{\mathcal{T}}_k(t),$$

and since $\dot{\mathcal{T}}_k(t)=\mu_k(t,j)$ for all $t\in[0,\Upsilon_{T,k})$ due to (8), and given that μ_k does not change during the jumps (4d), by using (13) we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t,j)=\mu_k(t,j)\frac{\mathrm{d}}{\mathrm{d}s}\hat{z}(s,j)\in\frac{\mu_k(t,j)}{\hat{\mu}_k(s,j)}F(\hat{z}(s,j),\hat{u}(s,j)).$$

By construction, $\mu_k(t,j)=\hat{\mu}_k(s,j)$, $u(t,j)=\hat{u}(s,j)$ and $z(t,j)=\hat{z}(s,j)$ via the time dilation $s=\mathcal{T}_k(t)$. Therefore, substituting in the above inclusion we obtain that $\dot{z}(t,j)$ satisfies (4b) for almost all $t\in I_j:=\{t\geq 0:(t,j)\in\mathrm{dom}(z)\}$. Moreover, note that $\mathcal{T}_k(\underline{t}_j)=\underline{s}_j$ and $\mathcal{T}_k(\bar{t}_j)=\bar{s}_j$ where $\underline{t}_j:=\min I_j$, $\bar{t}_j=\sup I_j$, $\underline{s}_j:=\min\hat{I}_j$, $\bar{s}_j=\sup\hat{I}_j$. Similarly, for every $(s,j)\in\mathrm{dom}(\hat{z})$ such that $(s,j+1)\in\mathrm{dom}(\hat{z})$, we have that $\hat{z}(s,j+1)\in G(\hat{z}(s,j))$, and therefore $z(t,j+1)\in G(z(t,j))$. Thus (z,u) is a maximal solution to $\mathcal{H}$.

(b) Let (z,u) be a maximal solution pair of $\mathcal{H}$ from z_0. Using again the chain rule, and the definition of $\hat{z}$, we obtain that for each j for which the interior of $I_j:=\{t\geq 0:(t,j)\in\mathrm{dom}(z)\}$ is nonempty, the signal $\hat{z}$ satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{z}(s,j)=\frac{\mathrm{d}z}{\mathrm{d}\mathcal{T}_k^{-1}}\frac{\mathrm{d}\mathcal{T}_k^{-1}}{\mathrm{d}s}=\frac{\dot{z}(t,j)}{\mu_k(t,j)}\in\frac{F(z(t,j),u(t,j))}{\mu_k(t,j)},$$

where we used (10) and (4b). Note that by construction $\hat{z}(s,j)=z(t,j)$, $\hat{\mu}_k(s,j)=\mu_k(t,j)$, and $\hat{u}(s,j)=u(t,j)$ via the time contraction $t=\mathcal{T}_k^{-1}(s)$. Then, by substituting in the above expression we obtain that $\hat{z}$ satisfies $\dot{\hat{z}}_s\in\frac{1}{\hat{\mu}_k}F(\hat{z},\hat{u}_k)$ for almost all $s\in\hat{I}_j=\{s\geq 0:(s,j)\in\mathrm{dom}(\hat{z})\}$. Moreover, note that $\mathcal{T}_k^{-1}(\underline{s}_j)=\underline{t}_j$ and $\mathcal{T}_k^{-1}(\bar{s}_j)=\bar{t}_j$ where $\underline{t}_j:=\min I_j$, $\bar{t}_j=\sup I_j$, $\underline{s}_j:=\min\hat{I}_j$, $\bar{s}_j=\sup\hat{I}_j$. Since for every $(t,j)\in\mathrm{dom}(z)$ such that $(t,j+1)\in\mathrm{dom}(z)$, we have that $z(t,j+1)\in G(z(t,j))$, and therefore $\hat{z}(s,j+1)\in G(\hat{z}(s,j))$, it follows that $(\hat{z},\hat{u})$ is a maximal solution pair to $\widehat{\mathcal{H}}$. ■

Remark 4. Proposition 2 establishes a relationship between the solutions of the HDS $\mathcal{H}$ in the (t,j) time scale, and the solutions of $\widehat{\mathcal{H}}$ in the (s,j) time scale via the family of k-parameterized dilations $s=\mathcal{T}_k(t)$ and contractions $\mathcal{T}_k^{-1}(s)$. In particular, the function $\mathcal{T}_k:[0,\Upsilon_{T,k})\to\mathbb{R}_{\geq 0}$

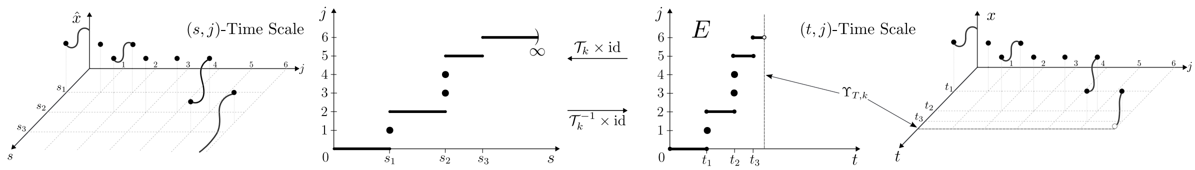

Fig. 1. Dilation and contraction of hybrid time domains and hybrid arcs. The structure of the hybrid time domain E in the (t,j)-time scale is preserved under the diffeomorphism $\mathcal{T}_k \times \mathrm{id}$ in the (s,j)-time scale.

will define a *diffeomorphism* that preserves the structure of the HTD of the hybrid arcs of $\widehat{\mathcal{H}}$. This observation is central to our analysis, as it enables us to conduct the stability analysis of the original HDS $\mathcal{H}$ by first studying the qualitative behavior of the solutions of system $\widehat{\mathcal{H}}$. In particular, note that $\widehat{\mathcal{H}}$ has a flow map that is normalized by $\hat{\mu}_k$, which removes the finite escape times in $\hat{\mu}_k$ (c.f., Lemma 3). This normalized HDS can be viewed as a "target" system that can be first designed and studied using the extensive set of tools available in the literature on HDS [25,27]. □

Remark 5. Using (5) with $k > 1$, $\mathcal{T}_k$ can be written as

$$\mathcal{T}_k(t) = \frac{T\mu_0^{\frac{k-1}{k}}}{k-1}\left(\frac{T^{k-1}}{\left(T - t\mu_0^{\frac{1}{k}}\right)^{k-1}} - 1\right), \ \forall\, t \in [0, \Upsilon_{T,k}), \tag{14}$$

which recovers the common dilation used for ODEs when $\mu_0 = 1$, see [1]. Other types of transformations are presented in [4] for the study of finite-time control of ODEs. Proposition 2 provides an extension of these results to hybrid systems. □

Remark 6. Analyses of HDS based on the time scaling of the flow map are not new, and they have been extensively explored in the context of singular perturbations [35,36] and averaging theory [31,37]. However, in contrast to (14), the time scaling in those scenarios is usually *linear*. □

3.2. PT-S via flows in HDS

Since solutions to system $\mathcal{H}$, whose data is described by (4), can only flow for a total amount of time upper bounded by $\Upsilon_{T,k}$, in this paper we are interested in regulating the state z to a general closed set $\mathcal{A}$, as $t \to \Upsilon_{T,k}$ (or before $\Upsilon_{T,k}$), where

$$\mathcal{A} = \mathcal{A}_\psi \times \mathbb{R}_{\geq 1}, \tag{15}$$

and where $\mathcal{A}_\psi$ is an application-dependent compact set. For systems with inputs, the following definition aims to capture this property, which makes use of the transformation $\mathcal{T}_k$ defined in (7), and which extends [1, Def. 1] from ODEs to HDS.

Definition 2. Let $\mathcal{A}$ be given by (15), where $\mathcal{A}_\psi \subset \mathbb{R}^n$ is compact. The set $\mathcal{A}$ is said to be *Prescribed-Time Input-to-State Stable via Flows* (PT-ISS$_\mathrm{F}$) for the HDS $\mathcal{H}$ if there exists $\beta \in \mathcal{KLL}$ and $\gamma \in \mathcal{K}$ such that for every $z(0,0) \in C \cup D$, all solutions z satisfy:

$$|z(t,j)|_{\mathcal{A}} \leq \beta\big(|z(0,0)|_{\mathcal{A}}, \mathcal{T}_k(t), j\big) + \gamma\left(|u|_{(t,j)}\right), \tag{16}$$

for all $(t,j) \in \mathrm{dom}(z)$. If (16) holds with $u \equiv 0$, the set $\mathcal{A}$ is said to be *Prescribed-Time Stable via Flows* (PT-S$_\mathrm{F}$). □

In some cases, it might be possible to completely suppress the residual effect of the input u in the bound (16) via PT feedback. This property, termed *PT-ISS with Convergence* in [1, Def. 1], can also be obtained in hybrid systems:

Definition 3. Let $\mathcal{A}$ be given by (15), where $\mathcal{A}_\psi \subset \mathbb{R}^n$ is compact. The set $\mathcal{A}$ is said to be *Prescribed-Time Input-to-State Stable with Convergence via Flows* (PT-ISS-C$_\mathrm{F}$) for the HDS $\mathcal{H}$ if there exists $\beta \in \mathcal{KLL}$, $\gamma \in \mathcal{K}$, and $\beta_c \in \mathcal{KL}$ such that for every $z(0,0) \in C \cup D$, all solutions z satisfy:

$$|z(t,j)|_{\mathcal{A}} \leq \beta_c\left(\beta\left(|z(0,0)|_{\mathcal{A}}, \mathcal{T}_k(t), j\right) + \gamma\left(|u|_{(t,j)}\right), \mathcal{T}_k(t)\right), \tag{17}$$

for all $(t,j) \in \mathrm{dom}(z)$. □

Remark 7 (*On the Use of $\mathcal{KLL}$ Functions*)**.** The use of $\mathcal{KLL}$ functions in Definitions 2 and 3 enable us to differentiate convergence behaviors in the continuous-time domain from those in the discrete-time domain. This type of comparison function is common in the analysis of HDS with inputs [33]. Additionally, since by construction $|z(t,j)|_{\mathcal{A}} = |\psi(t,j)|_{\mathcal{A}_\psi}$ for all $(t,j) \in \mathrm{dom}(z)$ (because $|\mu_k(t,j)|_{\mathbb{R}_{\geq 1}} = 0$), we can equivalently express the bounds (16)–(17) with z replaced by ψ, and $\mathcal{A}$ replaced by $\mathcal{A}_\psi$. □

Remark 8 (*On the Lack of Uniformity with Respect to μ_0*)**.** Definitions 2 and 3 extend Prescribed-Time Stability (PT-S) notions, studied in the literature of ODEs, [1, Def. 1] to hybrid systems. The $\mathcal{KLL}$ function β and the $\mathcal{KL}$ function β_c in the bounds (16) and (17) are independent of the initial conditions on $z = (\psi, \mu)$. However, as defined in (7), the diffeomorphism $\mathcal{T}_k$ clearly depends on the initial value of μ_k via (7), which parameterizes the prescribed time $\Upsilon_{T,k}$. Yet, the bounds (16) and (17) are uniform across the initial conditions of ψ, which is the main state of interest in the system. □

The following example, which follows as a particular case of the main results in the next section, illustrates the previous discussions:

Example 1. Consider the HDS $\mathcal{H}$ with $k = 1$, $T = 1$, $\psi = (x,\tau)$, $F_\psi = \{-x+u\} \times \{1\}$, $G_\psi = \{\frac{1}{2}x\} \times \{0\}$, $\Psi_C = \mathbb{R}^n \times [0,1]$, $\Psi_D = \mathbb{R}^n \times \{1\}$, and u is continuous and bounded. Then, every solution $z = (x,\tau,\mu_1)$ satisfies the following bound (see proof of Theorem 1):

$$|\psi(t,j)|_{\mathcal{A}_\psi} \leq k_1 e^{-k_2\mathcal{T}_1(t)}\left(e^{-k_3(\mathcal{T}_1(t)+j)}|\psi(0,0)|_{\mathcal{A}_\psi} + k_4|u|_{(t,j)}\right),$$

where $k_i > 0$ and $\mathcal{A}_\psi = \{0\} \times [0,1]$, for all $(t,j) \in \mathrm{dom}(z)$. Moreover, using (11b), the above bound can be written as:

$$|\psi(t,j)|_{\mathcal{A}_\psi} \leq \frac{\mu_1(0,0)^{\alpha_1}}{\mu_1(t,j)^{\alpha_2}}\left(\frac{e^{-qj}}{\mu_1(t,j)^{\alpha_3}}|\psi(0,0)|_{\mathcal{A}_\psi} + \alpha_4 \cdot |u|_{(t,j)}\right),$$

where $\alpha_i > 0$, $\mu_1(0,0) = \mu_0 \geq 1$, and for all $(t,j) \in \mathrm{dom}(z)$. It follows that $\lim_{(t,j)\in\mathrm{dom}(z), t\to\Upsilon_{1,1}} \psi(t,j) = 0$. □

It is important to note that, unlike ODEs, for HDS the existence of bounds of the form (16)–(17) does not necessarily guarantee that the internal state ψ will converge to $\mathcal{A}_\psi$ as $t \to \Upsilon_{T,k}$, for any $\Upsilon_{T,k} > 0$, even if $u \equiv 0$ and z is complete. The following scalar example illustrates this scenario.

Example 2. Consider the HDS $\mathcal{H}$ with $k = 1$, main state $\psi \in \mathbb{R}$, $u \equiv 0$, functions $F_\psi = \{-\psi\}$, $G_\psi = \frac{1}{2}\psi$, and sets $\Psi_C = (-\infty,-1] \cup [1,\infty)$, and $\Psi_D = [-1,1]$. For this system, we can study stability of ψ with respect to the set $\mathcal{A}_\psi = \{0\}$. For any initial condition to $\mathcal{H}$, $z(0,0) = (\psi_0,\mu_0)$, satisfying $|\psi_0| > 1$ and $\mu_0 = 1$, the unique maximal solution to the HDS satisfies $\psi(t,0) = \psi_0\left(\frac{T-t}{T}\right)^T$, for all $(t,j) \in [0,t'] \times \{0\}$, where $t' =$

$T(1-|\psi_0|^{-\frac{1}{T}})$, and $\psi(t,j) = \left(\frac{1}{2}\right)^j \psi(t',0)$, for all $(t,j) \in \bigcup_{j\in\mathbb{Z}_{\geq 1}} \{t'\}\times\{j\}$. It follows that $\psi(t,j) \to \mathcal{A}_\psi$ only as $j \to \infty$. Yet, every maximal solution z of the HDS satisfies (16) with $u = 0$. This follows by a direct application of item (a) of Proposition 2, the result of [38, Thm. 1], and item (b) of Proposition 2, in that order. □

The previous example shows that bounds of the form (16) or (17) only guarantee PT-S-like behaviors via the *flows* of the HDS. Therefore, to emulate the existing PT-S bounds obtained for ODEs [1,2], the "target" HDS $\widehat{\mathcal{H}}$ in (12) must generate maximal solutions with hybrid time domains E satisfying $\sup_t E = \infty$, such as those in Example 1. In general, this is not possible whenever $C = \emptyset$, or whenever $\widehat{\mathcal{H}}$ has eventually discrete, Zeno, or purely discrete solutions. However, as shown in the next section, for R-Switching systems, discrete solutions can be ruled out by designing appropriate switching signals generated by hybrid automatons that additionally exploit the "blow-up" nature of the functions μ_k.

4. PT-ISS in R-switching systems

In this section, we apply Proposition 2 to study a class of R-switching systems (1)–(2) characterized by the following dynamics:

$$\dot{x} = \mu_k(t)\cdot f_{\sigma(t)}(x,\mu_k(t),u,\tau), \qquad t \notin \mathcal{W}(\sigma), \tag{18a}$$

$$x(t) = R_{\sigma(t^-)}\left(x(t^-)\right), \qquad t \in \mathcal{W}(\sigma). \tag{18b}$$

For generality, in (18a) we allow f_σ to depend on μ_k and also on a signal τ that is generated by the following hybrid dynamics

$$\dot{\tau} \in \left[0, \frac{\mu_k(t)}{\tau_d}\right], \qquad t \notin \mathcal{W}(\sigma), \tag{19a}$$

$$\tau^+ = \tau - 1, \qquad t \in \mathcal{W}(\sigma), \tag{19b}$$

where μ_k is given by (5) and $\tau_d > 0$. To contextualize this model, some remarks are in order.

Remark 9. When $\mu_k \equiv 1$, $R_\sigma = \mathrm{id}(\cdot)$, and f_σ does not depend on τ and u, Eq. (18) coincides with the conventional nonlinear switching systems examined in [39,40]. On the other hand, when f_σ depends on u, (18a) captures nonlinear switching systems with inputs, similar to those studied [26,41]. □

Remark 10. When $\mu_k \equiv 1$ and f_σ depends on τ, system (18) describes a class of τ-parameterized nonlinear switching systems. In this class, τ is not necessarily constant throughout time, and the function $t \mapsto \tau(t)$ may not be differentiable or even continuous. Such models emerge in, for example, a class of time-triggered reset systems [42,43] suitable for optimization and learning problems; see also Section 5.2 for a specific application. □

Remark 11. In many applications, the system of interest might not match the exact form of (18). This is often the case in PT-regulation and feedback control of affine dynamical systems with non-zero drift, where multiplying the entire vector fields by the gain μ_k is not feasible. However, as shown later in Section 5, appropriate feedback design or variable transformation can reformulate these systems into the form (18). □

To have a well-posed system, we make the following regularity assumption on system (18a):

Assumption 2. For each $q \in \mathcal{Q}$, $f_q : \mathbb{R}^n \times \mathbb{R}_{\geq 1} \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is locally Lipschitz, $R_q : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is continuous and bounded. □

We consider R-switching systems (18) with a mix of stable and unstable modes. We denote the set of stable modes as $\mathcal{Q}_s$ and the set of unstable modes as $\mathcal{Q}_u$, such that $\mathcal{Q}_s \cup \mathcal{Q}_u = \mathcal{Q}$ and $\mathcal{Q}_u \cap \mathcal{Q}_s = \emptyset$. To leverage this partition and derive prescribed-time stability results, we proceed to introduce specific stability assumptions for our "target" HDS $\widehat{\mathcal{H}}$ defined in (12). Central to these assumptions is the role of a function $\Delta(\hat{\mu}_k)$ that characterizes the effect of the time-varying gain $\hat{\mu}_k$ on the input u in (18). In our subsequent analysis, we focus on three specific cases: $\Delta(\hat{\mu}_k) = 0$, $\Delta(\hat{\mu}_k) = 1$, and $\Delta(\hat{\mu}_k) = \hat{\mu}_k^{-\ell}$ with $\ell > 0$.

Assumption 3. There exist $\tau_d \in \mathbb{R}_{>0}$, $N_0 \in \mathbb{R}_{\geq 1}$, smooth functions $V_{\hat{q}} : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where $\hat{q} \in \mathcal{Q}$, and constants $c_{\hat{q},i} > 0$, $i \in \{1,2,3,4,5\}$, $p > 0$, such that:

(a) For all $(\hat{x},\hat{\tau},\hat{q}) \in \mathbb{R}^n \times [0,N_0] \times \mathcal{Q}$:

$$c_{\hat{q},1}|\hat{x}|^p \leq V_{\hat{q}}(\hat{x},\hat{\tau}) \leq c_{\hat{q},2}|\hat{x}|^p. \tag{20a}$$

(b) For all $(\hat{x},\hat{\tau},\hat{q},\hat{\mu}_k,\eta) \in \mathbb{R}^n \times [0,N_0] \times \mathcal{Q}_s \times \mathbb{R}_{\geq 1} \times [0,\tau_d^{-1}]$ and for all $u \in \mathbb{R}^m$, we have:

$$\left\langle \nabla V_{\hat{q}}(\hat{x},\hat{\tau}), \begin{pmatrix} f_{\hat{q}}(\hat{x},\hat{\mu}_k,u,\hat{\tau}) \\ \eta \end{pmatrix} \right\rangle \leq -c_{\hat{q},3}V_{\hat{q}}(\hat{x},\hat{\tau}) + c_{\hat{q},4}\Delta(\hat{\mu}_k)|u|^p. \tag{20b}$$

(c) For all $(\hat{x},\hat{\tau},\hat{q},\hat{\mu}_k,\eta) \in \mathbb{R}^n \times [0,N_0] \times \mathcal{Q}_u \times \mathbb{R}_{\geq 1} \times [0,\tau_d^{-1}]$ and for all $u \in \mathbb{R}^m$, we have:

$$\left\langle \nabla V_{\hat{q}}(\hat{x},\hat{\tau}), \begin{pmatrix} f_{\hat{q}}(\hat{x},\hat{\mu}_k,u,\hat{\tau}) \\ \eta \end{pmatrix} \right\rangle \leq c_{\hat{q},5}V_{\hat{q}}(\hat{x},\hat{\tau}) + c_{\hat{q},4}\Delta(\hat{\mu}_k)|u|^p. \tag{20c}$$

(d) For all $(\hat{x},\hat{\tau}) \in \mathbb{R}^n \times [1,N_0]$ and $\hat{o},\hat{q} \in \mathcal{Q}$ such that $\hat{q} \neq \hat{o}$:

$$V_{\hat{q}}(R_{\hat{o}}(\hat{x}),\hat{\tau}-1) \leq \chi V_{\hat{o}}(\hat{x},\hat{\tau}), \tag{20d}$$

where $\chi > 0$. □

Remark 12. Inequalities (20a)–(20b) are common in the context of exponential stability in continuous-time and hybrid systems. For the case when the vector field f_q in (18a) does not depend on τ, the function $V_{\hat{q}}$ can also be taken to be independent of $\hat{\tau}$. This is the most common situation in switching systems and systems with resets. An example where f_q does depend on τ will be studied in Section 5.2. □

Remark 13. Inequality (20b) in item (b) gives a standard decrease condition on the Lyapunov functions $V_{\hat{q}}$, for each stable mode $\hat{q} \in \mathcal{Q}_s$, and up to a neighborhood of the origin, whose size is parameterized by $\Delta(\hat{\mu}_k)|u|^p$. When $\Delta(\hat{\mu}_k) = 0$, and by [38, Thm. 1], conditions (20a)–(20b) imply that each mode $\hat{q} \in \mathcal{Q}_s$ renders the origin exponentially stable in the dilated time scale $s = \mathcal{T}_k(t)$ (see Proposition 2). When $\Delta(\hat{\mu}_k) = 1$, and by [33, Prop. 1], conditions (20a)–(20b) imply that each mode $\hat{q} \in \mathcal{Q}_s$ renders the origin ISS with exponential decay in the dilated time scale. The case $\Delta(\hat{\mu}_k) = \hat{\mu}_k^{-\ell}$, with $\ell > 0$, will emerge in the context of PT-regulation where convergence bounds of the form (17) are sought-after. An example in this direction is presented in Section 5. □

Remark 14. Inequality (20c) in item (c) rules out finite escape times for the unstable modes $\hat{q} \in \mathcal{Q}_u$. Similar assumptions are considered in the context of asymptotic/exponential stability in switching systems [31,41]. When $\mathcal{Q}_u = \emptyset$ (i.e., there are no unstable modes), item (c) holds vacuously. □

Remark 15. Inequality (20d) in item (d) considers the effect of the resets on the Lyapunov functions related to each of the modes. Usually (e.g., in standard switching systems) $R_{\hat{q}} = \mathrm{id}(\cdot)$ and $V_{\hat{q}}$ is independent of $\hat{\tau}$, and in this case, inequality (20d) holds trivially with $\chi = 1$. When $V_{\hat{q}}$ is independent of $\hat{\tau}$ but $R_{\hat{q}} \neq \mathrm{id}(\cdot)$, item (d) recovers the main assumptions of [26]. □

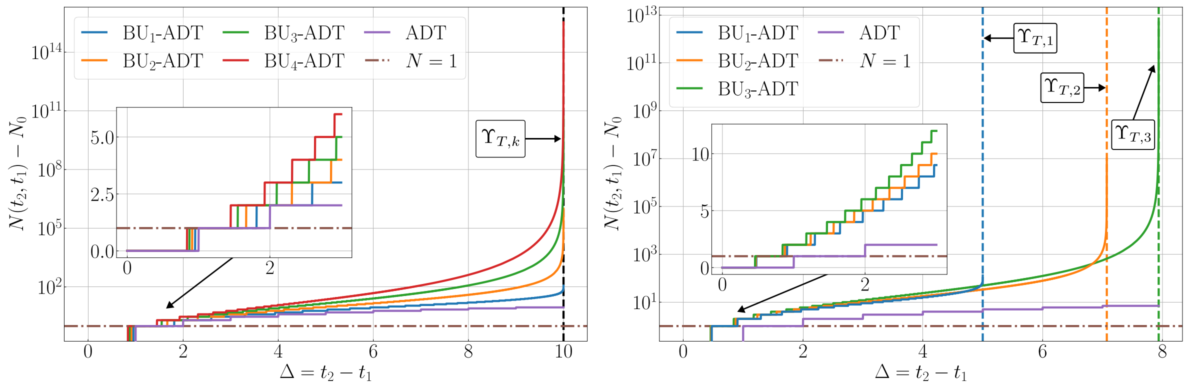

Fig. 2. BU_k-ADT condition (22) for $k \in \{1,2,3,4\}$. Left: When $\mu_0 = 1$, $T = 10$, and $t_1 = 0$, there exists a single common terminal time $T = \Upsilon_{T,k}$ for all k. Right: When $\mu_0 = 2$, $T = 10$, and $t_1 = 0$, the dependence of $\Upsilon_{T,k}$ on μ_0 (see Lemma 1) leads to the emergence of three distinct terminal times.

4.1. Blow-up average dwell-time conditions

To achieve asymptotic stability in systems that switch between a finite number of stable modes, it is common to assume that for all times $t_2 \geq t_1 \geq 0$, the switching signal σ satisfies an average dwell-time (ADT) condition of the form:

$$N(t_2,t_1) \leq \frac{1}{\tau_d}(t_2 - t_1) + N_0, \tag{21}$$

where $N(t_2,t_1)$ is the number of switches of σ in the interval $(t_1, t_2]$, $\tau_d > 0$ is called the dwell-time, and $N_0 \geq 1$ is the chatter bound, see [39,40], [25, Ch. 2.4]. However, unlike asymptotic convergence results, PT-S properties are defined only over the finite interval $[0, \Upsilon_{T,k})$. Therefore, we consider switching signals defined on similar intervals, which are additionally allowed to have a switching frequency that becomes unbounded as $t \to \Upsilon_{T,k}$.

Definition 4. Let μ_k be given by (5). A switching signal $\sigma : [0, \Upsilon_{T,k}) \to \mathcal{Q}$ is said to satisfy the *blow-up average dwell-time condition of order* k (BU_k-ADT) if there exist $N_0 \geq 1$ and $\tau_d > 0$ such that for all $t_2, t_1 \in \mathrm{dom}(\sigma)$:

$$N(t_2,t_1) \leq \frac{1}{\tau_d}\omega_k\left(\mu_k(t_2), \mu_k(t_1)\right) + N_0, \tag{22}$$

where $\omega_k(\cdot,\cdot)$ is given by (6). We use $\Sigma_{\mathrm{BU}_k-\mathrm{ADT}}(\tau_d, N_0, T, \mu_0)$ to denote the family of such signals. □

Fig. 2 illustrates the BU_k-ADT condition by comparing various bounds derived from (22) (plotted on a logarithmic scale) as functions of $\Delta = t_2 - t_1$, with $t_1 = 0$, and for different values of $k \in \mathbb{Z}_{\geq 1}$, with $\mu_0 = 1$ (left plot) and $\mu_0 = 2$ (right plot). The standard ADT bound (21) is also shown in color purple. Unlike the ADT bound, the BU_k-ADT bound grows to infinity as $\Delta \to \Upsilon_{T,k}$, allowing an increasing number of switches as $t \to \Upsilon_{T,k}$. However, in any compact sub-interval of $[0, \Upsilon_{T,k})$ the allowable number of switches is bounded. The following lemma shows that switching signals satisfying the ADT condition (21) also satisfy the BU_k-ADT condition (22) when their domain is appropriately restricted. The implication follows directly because the right-hand side of (21) can be upper-bounded by the right-hand side of (22). The proof is presented in the Appendix.

Lemma 4. *Let $T > 0$, $\mu_0 \geq 1$, and σ be a switching signal satisfying the ADT condition (21) with $\tau_d > 0$ and $N_0 \geq 1$. Then, $\sigma(t)$ satisfies the BU_k-ADT condition (22) for all $k \in \mathbb{Z}_{\geq 1}$ and all $0 \leq t_1 \leq t_2 < \Upsilon_{T,k}$, with the same τ_d, N_0.* □

Next, we present a lemma that provides an equivalent formulation of the BU_k-ADT condition, as well as its limiting behavior when the prescribed-time $\Upsilon_{T,k}$ goes to infinity. The proof is presented in the Appendix.

Lemma 5. *The following holds:*

(a) If $k = 1$, then (22) is equivalent to

$$N(t_2,t_1) \leq \frac{T}{\tau_d}\ln\left(\frac{\Upsilon_{T,1} - t_1}{\Upsilon_{T,1} - t_2}\right) + N_0. \tag{23}$$

(b) If $k \in \mathbb{Z}_{>1}$, then (22) is equivalent to

$$N(t_2,t_1) \leq \frac{\gamma_k(t_1,t_2)}{\tau_d}\left((t_2 - t_1) + \sum_{\ell=2}^{k-1}\tilde{c}_{\ell,k}\left(t_2^{\ell} - t_1^{\ell}\right)\right) + N_0,$$

where $\tilde{c}_{\ell,k} := (-1)^{\ell+1}\frac{b_{k,l}}{k-1}\Upsilon_{T,k}^{1-\ell}$, $b_{k,l} = \frac{(k-1)!}{\ell!(k-\ell-1)!}$ and

$$\gamma_k(t_1,t_2) := \mu_0\left(\frac{\Upsilon_{T,k}^2}{\left(\Upsilon_{T,k} - t_2\right)\left(\Upsilon_{T,k} - t_1\right)}\right)^{k-1}.$$

(c) For all $k \in \mathbb{Z}_{\geq 1}$ and all $t_2 \geq t_1 \geq 0$ the bound (22) satisfies

$$\lim_{T\to\infty}\frac{1}{\tau_d}\omega_k\left(\mu_k(t_2), \mu_k(t_1)\right) + N_0 = \frac{\mu_0}{\tau_d}(t_2 - t_1) + N_0,$$

thus recovering the ADT condition (21) when $\mu_0 = 1$. □

4.2. PT-ISS in R-switching systems with stable modes

When all the modes f_q are stable, i.e., $\mathcal{Q}_u = \emptyset$ and $\mathcal{Q} = \mathcal{Q}_s$, we can study PT-S properties of (18) by considering switching signals that satisfy the BU_k-ADT bound. In this case, the R-Switching system (18) can be analyzed by considering the HDS $\mathcal{H}$ with data (4), state $\psi = (x, \tau, q) \in \mathbb{R}^{n+2}$, and

$$F_\Psi(\psi, \mu_k, u) := \{f_q(x, \mu_k, u, \tau)\} \times \left[0, \frac{1}{\tau_d}\right] \times \{0\}, \tag{24a}$$

$$G_\Psi(\psi, u) := \{R_q(x)\} \times \{\tau - 1\} \times \mathcal{Q}_s \setminus \{q\}, \tag{24b}$$

$$\Psi_C = \mathbb{R}^n \times [0, N_0] \times \mathcal{Q}_s, \quad \Psi_D = \mathbb{R}^n \times [1, N_0] \times \mathcal{Q}_s. \tag{24c}$$

As established in the next lemma, there is a close connection between the HTDs of the solutions of system $\mathcal{H}$ with data (24), and the signals σ that satisfy the BU_k-ADT condition.

Lemma 6. *Let $(F_\Psi, G_\Psi, \Psi_C, \Psi_D)$ be given by (24a)–(24c), and consider the HDS $\mathcal{H}$ under Assumptions 2 and 3. Then, Assumption 1 holds, and:*

(a) For every maximal solution z and for any pair $(t_1, j_1), (t_2, j_2) \in \mathrm{dom}(z)$, with $t_2 > t_1$, inequality (22) holds with $N(t_2, t_1) = j_2 - j_1$.
(b) For every HTD satisfying property (a), there exists a solution z of the HDS $\mathcal{H}$ having the said HTD. □

Proof. The HDS $\mathcal{H}$ given by (4) has state $z = (\psi, \mu_k) \in \mathbb{R}^{n+3}$ with $\psi = (x, \tau, q) \in \mathbb{R}^{n+2}$, and dynamics

$$z \in C := \mathbb{R}^n \times [0, N_0] \times \mathcal{Q}_s \times \mathbb{R}_{\geq 1} \tag{25a}$$

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{\tau} \\ \dot{q} \\ \dot{\mu}_k \end{pmatrix} \in F(z,u) := \begin{pmatrix} \mu_k \cdot f_q(x,\mu_k,u,\tau) \\ \left[0, \frac{\mu_k}{\tau_d}\right] \\ 0 \\ \frac{k}{T}\mu_k^{1+\frac{1}{k}} \end{pmatrix}, \tag{25b}$$

$$z \in D := \mathbb{R}^n \times [1, N_0] \times \mathcal{Q}_s \times \mathbb{R}_{\geq 1}, \tag{25c}$$

$$z^+ = \begin{pmatrix} x^+ \\ \tau^+ \\ q^+ \\ \mu_k^+ \end{pmatrix} \in G(z) := \begin{pmatrix} R_q(x) \\ \tau - 1 \\ \mathcal{Q}_s \backslash \{q\} \\ \mu_k \end{pmatrix}. \tag{25d}$$

Since the function μ_k generated by (25) is precisely (5), any solution $z : \text{dom}(z) \to \mathbb{R}^{n+3}$ to (25) will necessarily satisfy $\text{length}_t(\text{dom}(z)) \leq \Upsilon_{T,k}$. By Proposition 2, the corresponding HDS (12) in the (s,j)-time scale is given by:

$$\hat{z} \in C,\ \dot{\hat{z}}_s = \begin{pmatrix} \dot{\hat{x}}_s \\ \dot{\hat{\tau}}_s \\ \dot{\hat{q}}_s \\ \dot{\hat{\mu}}_{k_s} \end{pmatrix} \in \hat{F}_T(\hat{z},\hat{u}) := \begin{pmatrix} f_{\hat{q}}(\hat{x},\hat{\mu}_k,\hat{u},\hat{\tau}) \\ \left[0, \frac{1}{\tau_d}\right] \\ 0 \\ \frac{k}{T}\hat{\mu}_k^{\frac{1}{k}} \end{pmatrix}, \tag{26a}$$

$$\hat{z} \in D, \qquad \hat{z}^+ \in G(\hat{z}), \tag{26b}$$

where C, D, and G were defined in (25). Since Assumption 3 ensures that the state $\hat{x}$ does not exhibit finite escape times, by noting that the dynamics of $(\hat{\tau},\hat{q})$ are decoupled from $\hat{\mu}_k$, and since $\hat{\mu}_k$ remains constant during jumps, we can directly obtain $\hat{\mu}_k(s,j)$ for any $(s,j) \in \text{dom}(\hat{z})$ using Lemma 3: $\hat{\mu}_k(s,j) = \left(\frac{(k-1)}{T}s + \hat{\mu}(\underline{s}_j,j)^{\frac{k-1}{k}}\right)^{\frac{k}{k-1}}$, for $k > 1$, and $\hat{\mu}_k(s,j) = \hat{\mu}_k(\underline{s}_j,j)e^{\frac{s}{T}}$, for $k = 1$, where $\underline{s}_j := \min\{s \geq 0 : (s,j) \in \text{dom}(\hat{z})\}$. By [25, Ex. 2.15] it follows that every solution $\hat{z}$ of (26) has a HTD that satisfies the ADT bound in the (s,j)-time scale:

$$j_2 - j_1 \leq \frac{1}{\tau_d}(s_2 - s_1) + N_0, \tag{27}$$

for all $(s_1,j_1),(s_2,j_2) \in \text{dom}(\hat{z})$, with $s_2 > s_1 \geq 0$. Additionally, by [25, Ex. 2.15], for every hybrid time domain satisfying (27), there exists a solution to the HDS (26) having said hybrid time domain. Thus, it remains to show that (27) is equivalent to (22) in the original (t,j)-time scale. Using the time scaling function $\mathcal{T}_k$ given by (7), for any solution z of (25) and all $(t_1,j_1),(t_2,j_2) \in \text{dom}(z)$ with $0 \leq t_1 < t_2$, we have that $(s_1,j_1),(s_2,j_2) \in \text{dom}(\hat{z})$, where $s_1 = \mathcal{T}_k(t_1)$, $s_2 = \mathcal{T}_k(t_2)$, and $0 \leq s_1 < s_2$. Substituting in (27):

$$j_2 - j_1 \leq \frac{1}{\tau_d}(\mathcal{T}_k(t_2) - \mathcal{T}_k(t_1)) + N_0.$$

The result follows now by using (P2) in Proposition 1. ■

One of the main consequences of the equivalence established in Lemma 6 is that analyzing the stability properties of the R-switching system (18) under the family of switching signals $\Sigma_{\text{BU-ADT}}(\tau_d,N_0,T,\mu_0)$ is equivalent to examining the stability properties of the HDS $\mathcal{H}$ with $(F_\Psi,G_\Psi,\Psi_C,\Psi_D)$ defined by (24a)–(24c). In this case, we can study the stability properties of this HDS with respect to the set $\mathcal{A}$ given by (15), where $\mathcal{A}_\psi$ is the following compact set

$$\mathcal{A}_\psi = \{0\} \times [0,N_0] \times \mathcal{Q}_s. \tag{28}$$

The following Theorem is the first main result of this paper.

Theorem 1. *Let $N_0 \geq 1$, $\mathcal{Q}_s \neq \emptyset$, $\mathcal{Q}_u = \emptyset$, and consider the HDS $\mathcal{H}$ with $(F_\Psi,G_\Psi,\Psi_C,\Psi_D)$ given by (24a)–(24c). Suppose that Assumptions 2–3 hold, and*

$$\tau_d > \frac{\ln(r)}{\min_{q\in\mathcal{Q}} c_{q,3}}, \tag{29}$$

where $r := \max\{1,\chi\}$, and $\chi > 0$ is given in Assumption 3. For each $(T,k) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 1}$, the following holds:

(a) If $\Delta(\mu_k) = 0$, then the set $\mathcal{A}$ is PT-S_F for $\mathcal{H}$.
(b) If $\Delta(\mu_k) = 1$, then the set $\mathcal{A}$ is PT-ISS$_F$ for $\mathcal{H}$.
(c) If $\Delta(\mu_k) = \mu_k^{-\ell}$, then for any $\ell > 0$ the set $\mathcal{A}$ is PT-ISS-C$_F$ for $\mathcal{H}$.

□

Proof. The proof has three main steps.

Step 1: Stability of the "target" HDS $\hat{\mathcal{H}}$ in the (s,j)-Hybrid Time Scale: The overall HDS is given by (25), which in the (s,j)-time scale is given by (26).

To study the stability properties of system (26), we consider the Lyapunov function $W(\hat{z}) := V_{\hat{q}}(\hat{x},\hat{\tau})e^{\ln(r)\hat{\tau}}$. By Assumption 3, this function satisfies $\underline{c}|\hat{z}|^p_{\mathcal{A}} \leq W(\hat{z}) \leq \overline{c}|\hat{z}|^p_{\mathcal{A}}$, $\forall\ \hat{z} \in C \cup D$, with $\underline{c} := \min_{p\in\mathcal{Q}} c_{1,p}$, $\overline{c} := e^{\ln(r)N_0}\overline{c}_2$, and $\overline{c}_2 := \max_{p\in\mathcal{Q}} c_{2,p}$. When $\hat{z} \in C$, for all $\eta \in [0,1/\tau_d]$, we have:

$$\begin{aligned}\langle \nabla W(\hat{z}), \hat{F}_T(\hat{z},\hat{u})\rangle &= \left\langle \nabla V_{\hat{q}}(\hat{x},\hat{\tau}), \begin{pmatrix} f_{\hat{q}}(\hat{x},\hat{u},\hat{\tau}) \\ \eta \end{pmatrix}\right\rangle e^{\ln(r)\hat{\tau}} \\ &\quad + \langle \ln(r)V_{\hat{q}}(\hat{x},\hat{\tau})e^{\ln(r)\hat{\tau}}, \hat{\tau}_s\rangle \\ &\leq -\underline{c}_3\left(1 - \frac{\ln(r)}{\underline{c}_3\tau_d}\right)W(\hat{z}) + \overline{c}_4 e^{\ln(r)N_0}\Delta(\hat{\mu}_k)|\hat{u}|^p,\end{aligned}$$

where $\underline{c}_3 := \min_{p\in\mathcal{Q}} c_{p,3}$, $\overline{c}_4 := \max_{p\in\mathcal{Q}} c_{p,4}$, and where we used item (b) in Assumption 3. On the other hand, when $\hat{z} \in D$ we can use Assumption 3-(d) to obtain

$$\begin{aligned}W(\hat{z}^+) &= V_{\hat{q}^+}(\hat{x}^+,\hat{\tau}^+)e^{\ln(r)\hat{\tau}^+} = V_{\hat{q}^+}\left(R_{\hat{q}}(\hat{x}),\hat{\tau}-1\right)e^{\ln(r)(\hat{\tau}-1)} \\ &\leq \chi V_{\hat{q}}(\hat{x},\hat{\tau})e^{\ln(r)(\hat{\tau}-1)} = \frac{\chi}{r}W(\hat{z}).\end{aligned}$$

Thus, using the definition of r, during jumps we obtain $W(\hat{z}^+) - W(\hat{z}) \leq -\left(1-\frac{\chi}{\max\{1,\chi\}}\right)W(\hat{z}) \leq 0$. Using Lemma 10 in the Appendix, we conclude that every solution $\hat{z}$ of system (26) satisfies:

$$|\hat{z}(s,j)|_{\mathcal{A}} \leq \kappa_1 e^{-\kappa_2(s+j)}|\hat{z}(0,0)|_{\mathcal{A}} + \kappa_3 \cdot \sup_{0\leq\zeta\leq s}|\hat{\Delta}(\zeta)|, \tag{30}$$

for all $(s,j) \in \text{dom}(\hat{z})$, where $\kappa_1 = \left(\overline{c}/\underline{c}\right)^{1/p} e^{\frac{\lambda}{2p}\frac{\tau_d}{1+\tau_d}N_0}$ $\kappa_2 = \lambda\tau_d/(2p(1+\tau_d))$, $\kappa_3 = \left(2\overline{c}_4 r^{N_0}/[\lambda\underline{c}]\right)^{1/p}$, $\lambda = \underline{c}_3 - \ln(r)/\tau_d$, and $\hat{\Delta}(s) := \Delta(\hat{\mu}_k(s))\hat{u}(s)$. Moreover, when $\Delta(\hat{\mu}_k) = \hat{\mu}_k^{-\ell}$, via Lemma 11 in the Appendix, there exists $\beta_k \in \mathcal{KL}$ such that every solution $\hat{z}$ of system (26) satisfies:

$$|\hat{z}(s,j)|_{\mathcal{A}} \leq \beta_k\left(\overline{\kappa}_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\overline{\kappa}_2(s+j)} + \overline{\kappa}_3|\hat{u}|_{(s,j)}, s\right), \tag{31}$$

for all $(s,j) \in \text{dom}(\hat{z})$, with $\overline{\kappa}_1 := \kappa_1$, $\kappa_2 := \frac{\kappa_2}{2}$, $\overline{\kappa}_3 := 2\kappa_3$.

Step 2: PT-ISS$_F$ of the HDS in the (t,j) - Time Scale: We now use the properties of the solutions $\hat{z}$ of system (26) to establish properties for the solutions z of system (25). First, we use Proposition 2 and let $s = \mathcal{T}_k(t)$, which yields $e^{-\kappa_2(s+j)} = e^{-\kappa_2(\mathcal{T}_k(t)+j)}$, and $|\hat{z}(\mathcal{T}_k(t),j)|_{\mathcal{A}} = \left|z\left(\mathcal{T}_k^{-1}(\mathcal{T}_k(t)),j\right)\right|_{\mathcal{A}} = |z(t,j)|_{\mathcal{A}}$. Then, by substituting in (30) and noting that $\mathcal{T}_k(0) = \mathcal{T}_k^{-1}(0) = 0$, it follows that when $\Delta = 0$ or $\Delta = 1$, every solution $z = (\psi,\mu_k)$ of the HDS (25) with $\mu_k(0,0) = \mu_0 \geq 1$ satisfies the bound:

$$|z(t,j)|_{\mathcal{A}} \leq \kappa_1 e^{-\kappa_2(\mathcal{T}_k(t)+j)}|z(0,0)|_{\mathcal{A}} + \kappa_3\Delta|u|_{(t,j)}, \tag{32}$$

for all $(t,j) \in \text{dom}(\psi)$, which implies that $\mathcal{A}$ is PT-ISS$_F$. Similarly, when $\Delta(\hat{\mu}_k) = \hat{\mu}_k^{-\ell}$, (31) leads to:

$$|z(t,j)|_{\mathcal{A}} \leq \beta_k\left(\overline{\kappa}_1|z(0,0)|_{\mathcal{A}}e^{-\overline{\kappa}_2(\mathcal{T}_k(t)+j)} + \overline{\kappa}_3|u|_{(t,j)}, \mathcal{T}_k(t)\right), \tag{33}$$

for all $(t,j) \in \text{dom}(z)$. Inequality (33) implies that $\mathcal{A}$ is PT-ISS-C$_F$.

Step 3: Length of solutions in the (t,j) - Time Scale: Finally, we show that $\sup_t(\text{dom}(z)) = \Upsilon_{T,k}$ for all solutions z of (25). First, note that by the definition of $\mathcal{T}_k$ and Proposition 2, we have $\sup_t(\text{dom}(z)) = \Upsilon_{T,k}$ if and only if $\sup_s(\text{dom}(\hat{z})) = \infty$. Furthermore, based on the bound (27), we obtain hat $j \leq \frac{1}{\tau_d}s + N_0$ for any $(s,j) \in \text{dom}(\hat{z})$. Since every complete solution $\hat{z}$ of (26) satisfies $\text{length}(\text{dom}(\hat{z})) = \infty$, and noting that length(dom($\hat{z}$)) = $\sup_s(\text{dom}(\hat{z})) + \sup_j(\text{dom}(\hat{z}))$, we can infer that if $j \to \infty$, then $s \to \infty$. Consequently, every complete solution of (26) must satisfy $\sup_s(\text{dom}(\hat{z})) = \infty$, which in turn implies that $\sup_t(\text{dom}(z)) = \Upsilon_{T,k}$ for such solutions. ■

The following Corollary covers the case $k = 1$, which is the most common in the literature of PT-S [1,24].

Corollary 1. *Suppose that all the assumptions of Theorem 1 hold, and that $k = 1$. Then, for every solution $z = (x, \tau, q, \mu_k)$ to $\mathcal{H}$, and all $(t,j) \in \text{dom}(z)$, the state x satisfies the following properties:*

1. *If (20b) holds with $\Delta(\mu_1) = 0$ or $\Delta(\mu_1) = 1$, then*
$$|x(t,j)| \leq \kappa_1 \left(\frac{\mu_0}{\mu_1(t,j)}\right)^{\kappa_2 T} e^{-\kappa_2 j}|x(0,0)| + \kappa_3 \Delta |u|_{(t,j)}, \tag{34}$$
where $\kappa_i > 0$ for $i \in \{1,2,3\}$.
2. *If (20b) holds with $\Delta(\mu_1) = \mu_1^{-\ell}$, then:*
$$|x(t,j)| \leq \frac{\alpha_1 \mu_0^{\alpha_2}}{\mu_1(t,j)^{\alpha_3}} \left(\frac{e^{-\alpha_4 j}}{\mu_1(t,j)^{\alpha_5}}|x(0,0)| + \alpha_6 |u|_{(t,j)}\right), \tag{35}$$
where $\alpha_i > 0$ for $i \in \{1,2,\ldots,6\}$. □

Proof. Using (11b) and the bounds obtained in Step 2 of the proof of Theorem 1, it follows that $e^{-\kappa_2(\mathcal{T}_k(t)+j)} = e^{-\alpha \ln\left(\frac{\mu_1(t)}{\mu_0}\right)} e^{-\kappa_2 j} = \left(\frac{\mu_0}{\mu_1(t)}\right)^{\alpha} e^{-\frac{\alpha}{T}j}$, where $\alpha = \kappa_2 T$. Since, by definition, $|z|_{\mathcal{A}} = |x|$ for every solution, inequality (32) becomes (34). Similarly, inequality (33) becomes (35) with $\alpha_1 := \overline{\kappa}_1$, $\alpha_2 := (\overline{\kappa}_3 + \overline{\kappa}_2)T$, $\alpha_3 := \overline{\kappa}_3 T$, $\alpha_4 := \overline{\kappa}_2$, $\alpha_5 := \overline{\kappa}_2 T$, and $\alpha_6 := \overline{\kappa}_4$. ■

4.3. PT-ISS in R-switching systems with unstable modes

We now consider the scenario where some of the modes f_q in (18) are unstable, i.e., $\mathcal{Q}_u \neq \emptyset$ and $\mathcal{Q} = \mathcal{Q}_s \cup \mathcal{Q}_u$. To study this case, we introduce a *blow-up average activation-time* (BU_k-AAT) condition on the amount of time that the unstable modes can remain active in any sub-interval of $[0, \Upsilon_{T,k})$.

Definition 5. A switching signal $\sigma : [0, \Upsilon_{T,k}) \to \mathcal{Q}$ is said to satisfy the *blow-up average activation-time condition of order* k (BU_k-AAT) if there exist $T_0 > 0$ and $\tau_a > 1$ such that for each pair of times $t_2, t_1 \in \text{dom}(\sigma)$:
$$\int_{t_1}^{t_2} \mu_k(t) \cdot \mathbb{I}_{\mathcal{Q}_u}(\sigma(t))dt \leq \frac{1}{\tau_a}\omega_k\left(\mu_k(t_2), \mu_k(t_1)\right) + T_0, \tag{36}$$
where μ_k is given by (5). We denote the family of such signals as $\Sigma_{\text{BU}_k-\text{AAT}}(\mathcal{Q}_u, \tau_a, T_0, T, \mu_0)$. □

Remark 16. For asymptotic and exponential stability results in switching systems with both stable and unstable modes [26,31,41], it is common to restrict the family of admissible switching signals to those that satisfy the ADT condition (27) *and* the following average activation-time (AAT) condition:
$$\int_{t_1}^{t_2} \mathbb{I}_{\mathcal{Q}_u}(\sigma(t))dt \leq \frac{1}{\tau_a}(t_2 - t_1) + T_0, \tag{37}$$
where $\tau_a > 1$, and $T_0 > 0$. This bound can be recovered from (36) by taking the limit as $T \to \infty$ in both sides of (36) and using $\mu_0 = 1$. Also, note that for $k = 1$, the BU_1-AAT condition reduces to:
$$\int_{t_1}^{t_2} \frac{\mathbb{I}_{\mathcal{Q}_u}(\sigma(t))}{\Upsilon_{T,1} - t}dt \leq \frac{1}{\tau_a}\ln\left(\frac{T - t_1\mu_0}{T - t_2\mu_0}\right) + T_0.$$
Similar bounds can be obtained for $k \in \mathbb{Z}_{\geq 2}$ using (5). □

Fig. 3 compares the BU_k-AAT bounds and the traditional AAT bound (37). The left plot shows the left-hand side of (36) for different values of k, under a particular switching signal σ that switches between one stable mode and one unstable mode. The classic AAT bound is shown in purple color. The right plot shows (36) for $k = 1$ and different values of τ_a.

To study the PT-S properties of the R-Switching system (18) when $\mathcal{Q}$ contains unstable modes, we now consider the HDS $\mathcal{H}$ with state $\psi = (x, \tau, \rho, q) \in \mathbb{R}^{n+3}$, set-valued mappings:
$$F_\Psi := \{f_q(x, \mu_k, u, \tau)\} \times \left[0, \frac{1}{\tau_d}\right] \times \left(\left[0, \frac{1}{\tau_a}\right] - \mathbb{I}_{\mathcal{Q}_u}(q)\right) \times \{0\}, \tag{38a}$$
$$G_\Psi := \{R_q(x)\} \times \{\tau - 1\} \times \{\rho\} \times \mathcal{Q}\backslash\{q\}, \tag{38b}$$
and sets:
$$\Psi_C = \mathbb{R}^n \times [0, N_0] \times [0, T_0] \times \mathcal{Q}, \tag{38c}$$
$$\Psi_D = \mathbb{R}^n \times [1, N_0] \times [0, T_0] \times \mathcal{Q}. \tag{38d}$$
There is a close connection between the hybrid time domains of the solutions generated by the HDS $\mathcal{H}$ with data (38), and the switching signals that simultaneously satisfy (22) and (36).

Lemma 7. *Let $(F_\Psi, G_\Psi, \Psi_C, \Psi_D)$ be given by (38a)–(38c), and consider the HDS $\mathcal{H}$ given by (4), under Assumption 2–3. Then, Assumption 1 holds, and:*

- *(a) For every maximal solution z to $\mathcal{H}$ and for any pair $(t_1, j_1), (t_2, j_2) \in \text{dom}(z)$, with $t_2 > t_1$, inequality (22) holds with $N(t_2, t_1) = j_2 - j_1$, and inequality (36) holds with $\sigma(t) = q(t, \underline{j}(t))$, where $\underline{j}(t) := \min\{j \in \mathbb{Z}_{\geq 0} : (t,j) \in \text{dom}(z)\}$.*
- *(b) For every HTD satisfying property (a), there exists a solution z of $\mathcal{H}$ having the said HTD.* □

Proof. The overall HDS has state $z = (\psi, \mu_k) \in \mathbb{R}^{n+4}$ with $\psi = (x, \tau, \rho, q,)$, and the following dynamics:
$$z \in C := \mathbb{R}^n \times [0, N_0] \times [0, T_0] \times \mathcal{Q} \times \mathbb{R}_{\geq 1}, \tag{39a}$$
$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{\tau} \\ \dot{\rho} \\ \dot{q} \\ \dot{\mu}_k \end{pmatrix} \in F(z,u) := \begin{pmatrix} \mu_k \cdot f_q(x, \mu_k, u, \tau) \\ \left[0, \frac{\mu_k}{\tau_d}\right] \\ \left[0, \frac{\mu_k}{\tau_a}\right] - \mu_k \mathbb{I}_{\mathcal{Q}_u}(q) \\ 0 \\ \frac{k}{T}\mu_k^{1+\frac{1}{k}} \end{pmatrix}, \tag{39b}$$
$$z \in D := \mathbb{R}^n \times [1, N_0] \times [0, T_0] \times \mathcal{Q} \times \mathbb{R}_{\geq 1}, \tag{39c}$$
$$z^+ = \begin{pmatrix} x^+ \\ \tau^+ \\ \rho^+ \\ q^+ \\ \mu_k^+ \end{pmatrix} \in G(z,u) := \begin{pmatrix} R_q(x) \\ \tau - 1 \\ \rho \\ \mathcal{Q}\setminus\{q\} \\ \mu_k \end{pmatrix}. \tag{39d}$$
This system has a finite escape time at $t = \Upsilon_{T,k}$, induced by μ_k. Note that, by construction, the states (τ, ρ, q) are confined to the compact sets $[0, N_0]$, $[0, T_0]$, and $\mathcal{Q}$ respectively. Using the time variable $s = \mathcal{T}_k(t)$ defined in (7), and Proposition 2, we obtain the following HDS in the (s,j)-time scale:
$$\hat{z} \in C,\ \dot{\hat{z}}_s = \left(\dot{\hat{x}}_s \dot{\hat{\tau}}_s \dot{\hat{\rho}}_s \dot{\hat{q}}_s \dot{\hat{\mu}}_{k_s}\right) \in \hat{F}(\hat{z}, \hat{u}) := \begin{pmatrix} f_q(\hat{x}, \hat{\mu}_k, \hat{u}, \hat{\tau}) \\ \left[0, \frac{1}{\tau_d}\right] \\ \left[0, \frac{1}{\tau_a}\right] - \mathbb{I}_{\mathcal{Q}_u}(\hat{q}) \\ 0 \\ \frac{k}{T}\hat{\mu}_k^{\frac{1}{k}} \end{pmatrix}, \tag{40a}$$

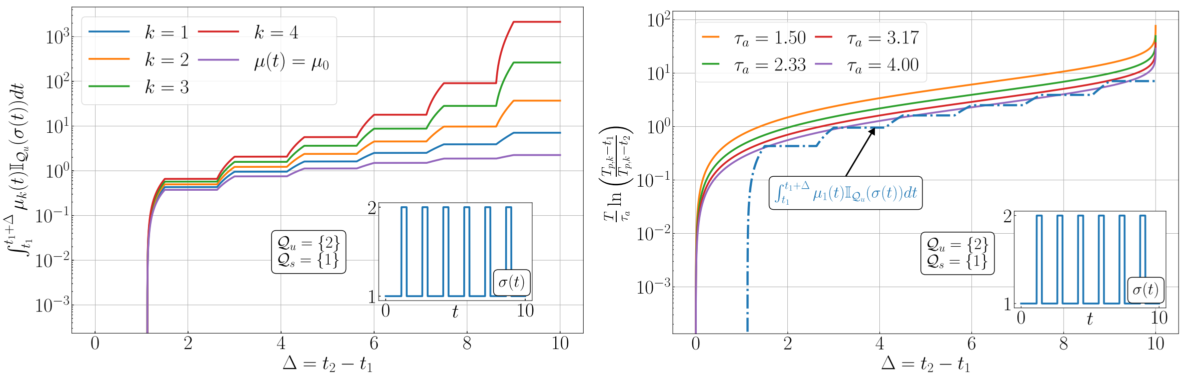

Fig. 3. Functions appearing in the BU_k-AAT condition (36) using the switching signal $\sigma(\cdot)$ (see inset), $T=10$, and $\mu_0=1$.

$$\hat{z}\in D, \qquad \hat{z}^+\in G(\hat{z}), \tag{40b}$$

where the subscript s in (40a) indicates that the time derivative is taken with respect to s. Since (40) incorporates an ADT automaton $\hat{\tau}$ and a time-ratio monitor $\hat{\rho}$, by [31, Lemma 7] every solution $\hat{z}$ of (40) has a hybrid time domain such that for any pair $(s_1,j_1),(s_2,j_2)\in\mathrm{dom}(\hat{z})$ the bound (27) is satisfied, as well as the following bound:

$$\mathbb{T}(s_1,s_2):=\int_{s_1}^{s_2}\mathbb{I}_{\mathcal{Q}_u}(\hat{q}(s,\underline{\hat{j}}(s)))ds\le\frac{1}{\tau_a}(s_2-s_1)+T_0, \tag{41}$$

where $\underline{\hat{j}}(s) := \min\{j\in\mathbb{Z}_{\ge0} : (s,j)\in\mathrm{dom}(\hat{q})\}$. Moreover, by [31, Lemma 7] every hybrid arc satisfying (41) can be generated by the HDS (40). Using $s=\mathcal{T}_k(t)$, the left-hand side of (41) can be expressed in the t-variable as:

$$\begin{aligned}\mathbb{T}(\mathcal{T}_k(t_2),\mathcal{T}_k(t_1))&=\int_{t_1}^{t_2}\frac{\partial\mathcal{T}_k(t)}{\partial t}\cdot\mathbb{I}_{\mathcal{Q}_u}\left(\hat{q}\left(\mathcal{T}_k(t),\underline{\hat{j}}\left(\mathcal{T}_k(t)\right)\right)\right)dt\\&=\int_{t_1}^{t_2}\mu_k(t)\cdot\mathbb{I}_{\mathcal{Q}_u}\left(q(t,\underline{j}(t))\right)dt,\end{aligned} \tag{42}$$

where we used Proposition 1-(P3), together with the equality

$$\hat{q}(\mathcal{T}_k(t),\underline{\hat{j}}\left(\mathcal{T}_k(t)\right))=q\left(\mathcal{T}_k^{-1}(\mathcal{T}_k(t)),\ \underline{j}\left(\mathcal{T}_k^{-1}\left(\mathcal{T}_k(t)\right)\right)\right)=q(t,\underline{j}(t)).$$

Using (41)–(42), together with Proposition 1-(P2), the AAT condition in the (t,j)-time scale becomes

$$\int_{t_1}^{t_2}\mu_k(t)\cdot\mathbb{I}_{\mathcal{Q}_u}(q(t,\underline{j}(t)))dt\le\frac{1}{\tau_a}\omega_k\left(\mu_k(t_2),\mu_k(t_1)\right)+T_0,$$

which is precisely (36). The fact that inequality (22) holds follows by Lemma 6. ■

Similar to Lemma 6, the result of Lemma 7 enables the study of the stability properties of the R-Switching system (18), under switching signals σ satisfying (22) *and* (36), by studying the stability properties of the HDS (39). In this case, we consider the set $\mathcal{A}$ given by (15), where $\mathcal{A}_\Psi$ is now given by

$$\mathcal{A}_\Psi=\{0\}\times[0,N_0]\times[0,T_0]\times\mathcal{Q}. \tag{43}$$

The next theorem is the second main result of this paper.

Theorem 2. *Let $N_0\ge1$, $T_0>0$, $\mathcal{Q}_u\ne\emptyset$, $\mathcal{Q}_s\ne\emptyset$, and consider the HDS $\mathcal{H}$ given by (4) with $(F_\Psi,G_\Psi,\Psi_C,\Psi_D)$ given by (38a)–(38c). Suppose that Assumptions 2–3 hold, and that*

$$1>\frac{1}{\underline{c}_3\tau_d}\ln(r)+\frac{1}{\tau_a}\left(1+\frac{\overline{c}_5}{\underline{c}_3}\right), \tag{44}$$

where $r=\max\{1,\chi\}$, $\chi>0$ is given in Assumption 3, $\underline{c}_3=\min_{p\in\mathcal{Q}}c_{q,3}$, and $\overline{c}_5=\max_{p\in\mathcal{Q}}c_{q,5}$. For each $(T,k)\in\mathbb{R}_{>0}\times\mathbb{R}_{\ge1}$ the following holds:

(a) If $\Delta(\mu_k)\triangleq0$, then the set $\mathcal{A}$ is PT-S_F.
(b) If $\Delta(\mu_k)\triangleq1$, then the set $\mathcal{A}$ is PT-ISS$_F$.
(c) If $\Delta(\mu_k)\triangleq\mu_k^{-\ell}$, $\ell>0$, then the set $\mathcal{A}$ is PT-ISS-C$_F$. □

Proof. The proof follows the same three steps as in the proof of Theorem 1. We start by using the time dilation $\mathcal{T}_k^{-1}$ and Proposition 2. Hence, we consider the HDS (40) in the (s,j)-time scale, with state $\hat{z}=(\hat{x},\hat{\tau},\hat{\rho},\hat{q},\hat{\mu}_k)$. To study the stability properties of this system, let $\hat{\xi}:=\ln(r)\hat{\tau}+(\underline{c}_3+\overline{c}_5)\hat{\rho}$, and consider the Lyapunov function $W_2(\hat{z})=V_{\hat{q}}(\hat{x},\hat{\tau})e^{\hat{\xi}}$, which, by Assumption 3-(a), satisfies the inequalities $\underline{\varphi}|\hat{z}|^2_{\mathcal{A}}\le W_2(\hat{z})\le\overline{\varphi}|\hat{z}|^2_{\mathcal{A}}$, with $\underline{\varphi}:=\min_{p\in\mathcal{Q}}c_{p,1}$ and $\overline{\varphi}:=\max_{p\in\mathcal{Q}}c_{p,2}e^{\ln(r)N_0+(\underline{c}_3+\overline{c}_5)T_0}$. When $\hat{z}\in C$, the time derivative of $\hat{\xi}$ with respect to s satisfies:

$$\dot{\hat{\xi}}_s=\ln(r)\dot{\hat{\tau}}_s+(\underline{c}_3+\overline{c}_5)\dot{\hat{\rho}}_s\in[0,\delta]-(\underline{c}_3+\overline{c}_5)\mathbb{I}_{\mathcal{Q}_u}(\hat{q}),$$

where $\delta:=\frac{1}{\tau_d}\ln(r)+\frac{1}{\tau_a}(\underline{c}_3+\overline{c}_5)$. Using the above expression together with Assumption 3, we evaluate the change of W_2 during the flows of stable and unstable modes. In particular, when $\hat{z}\in C$ and $\hat{q}\in\mathcal{Q}_s$, we have

$$\begin{aligned}\langle\nabla W_2(\hat{z}),\dot{\hat{z}}_s\rangle&=e^{\hat{\xi}}\left\langle\nabla V_{\hat{q}}(\hat{x},\hat{\tau}),\ \dot{\hat{x}}_s\right\rangle+e^{\hat{\xi}}V_{\hat{q}}(\hat{x},\hat{\tau})\dot{\hat{\xi}}_s\\&\le-(\underline{c}_3-\delta)W_2(\hat{z})+\frac{\overline{c}_4}{\underline{c}_2}\overline{\varphi}\hat{\Delta}(s)|\hat{u}|^p,\end{aligned} \tag{45}$$

where $\hat{\Delta}(s):=\Delta(\hat{\mu}_k(s))\hat{u}(s)$, $\overline{c}_2:=\max_{p\in\mathcal{Q}}c_{2,p}$ and $\overline{c}_4=\max_{p\in\mathcal{Q}}c_{4,p}$, and where $\underline{c}_3-\delta>0$ since (44) is satisfied by assumption. On the other hand, when $\hat{z}\in C$ and $\hat{q}\in\mathcal{Q}_u$:

$$\begin{aligned}\langle\nabla W_2(\hat{z}),\dot{\hat{z}}_s\rangle&\le\left(\overline{c}_5V_{\hat{q}}(\hat{x},\hat{\tau})+\overline{c}_4\hat{\Delta}(s)|\hat{u}|\right)e^{\hat{\xi}}+V_{\hat{q}}(\hat{x},\hat{\tau})e^{\hat{\xi}}\dot{\hat{\xi}}_s\\&\le\left(\delta-\underline{c}_3\right)W_2(\hat{z})+\overline{c}_4\hat{\Delta}(s)|\hat{u}|e^{\hat{\xi}}\\&\le-\left(\underline{c}_3-\delta\right)W_2(\hat{z})+\frac{\overline{c}_4}{\underline{c}_2}\overline{\varphi}\hat{\Delta}(s)|\hat{u}|^p,\end{aligned}$$

which is the same bound as (45).

During jumps, it follows that $\hat{\xi}^+=\ln(r)\hat{\tau}^++(\underline{c}_3+\overline{c}_4)\hat{\rho}^+=\hat{\xi}-\ln(r)$ for all $\hat{z}\in D$. Then, using Assumption 3, the Lyapunov function satisfies:

$$\begin{aligned}W_2(\hat{z}^+)&=V_{\hat{q}^+}(\hat{x}^+,\hat{\tau}^+)e^{\hat{\xi}^+}=V_{\hat{q}^+}\left(R_{\hat{q}}(\hat{x}),\hat{\tau}-1\right)e^{\hat{\xi}-\ln(r)}\\&\le\chi V_{\hat{q}}(\hat{x},\hat{\tau})e^{\hat{\xi}-\ln(r)}=\frac{\chi}{\max\{1,\chi\}}W_2(\hat{z})\le W_2(\hat{x}).\end{aligned}$$

It follows that $W_2(\hat{z}^+)-W_2(\hat{z})\le0$ for all $\hat{z}\in D$. Using Lemma 10 in the Appendix, we conclude that every solution $\hat{z}$ satisfies the bound

$$|\hat{z}(s,j)|_{\mathcal{A}}\le\kappa_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\kappa_2(s+j)}+\kappa_3\hat{\Delta}(s)|\hat{u}|_{(s,j)},$$

for all $(s,j)\in\mathrm{dom}(\hat{z})$, where $\kappa_1=\left(\overline{\varphi}/\underline{\varphi}\right)^{1/p}e^{\frac{\lambda}{2p}\frac{\tau_d}{1+\tau_d}N_0}$, $\kappa_2=\lambda\tau_d/(2p(1+\tau_d))$, $\kappa_3=\left(2\overline{c}_4\overline{\varphi}/[\underline{c}_2\lambda\underline{\varphi}]\right)^{1/p}$, $\lambda=\underline{c}_3-\delta$, and $\hat{\Delta}(s):=\Delta(\hat{\mu}_k(s))\hat{u}(s)$. From here, the bounds (16)–(17) are obtained following the exact same arguments used in Steps 2 and 3 of the proof of Theorem 1. ■

Remark 17 (*Switching with Non-PT Unstable Modes*). It is reasonable to consider a situation where the unstable modes in (18a) do not have time-varying gains, i.e., $\mu_k \equiv 1$ when $q \in \mathcal{Q}_u$. In particular, consider a system switching between the following two families of systems:

$$\dot{x} = \mu_k f_q(x), \quad q \in \mathcal{Q}_s, \quad \text{and} \quad \dot{x} = f_p(x), \quad p \in \mathcal{Q}_u,$$

where the modes in $\mathcal{Q}_s$ satisfy (20b), and the modes in $\mathcal{Q}_u$ satisfy (20c) with $u \equiv 0$. Following the same approach of Theorem 2, and operating in the s-time scale for the flows, we now obtain the following two type of modes:

$$\dot{\hat{x}}_s = f_q(\hat{x}), \quad q \in \mathcal{Q}_s, \quad \text{and} \quad \dot{\hat{x}}_s = \frac{1}{\hat{\mu}_k} f_p(x), \quad p \in \mathcal{Q}_u.$$

For this system, the same Lyapunov-based analysis can be applied as in the proof of Theorem 2 to obtain the bound (45) for all $q \in \mathcal{Q}_s$. On the other hand, for $q \in \mathcal{Q}_u$, we now obtain $\langle \nabla W_2(\hat{z}), \dot{\hat{z}}_s \rangle \leq -\left(\underline{c}_3 - \delta\right) W_2(\hat{z}) - \overline{c}_5 \left(1 - \frac{1}{\hat{\mu}_k}\right) W_2(\hat{z})$. Note that $1 - \frac{1}{\hat{\mu}_k} \geq 0$ since $\hat{\mu}_k \geq 1$ by Lemma 3. This implies that $\langle \nabla W_2(\hat{z}), \dot{\hat{z}}_s \rangle \leq -\left(\underline{c}_3 - \delta\right) W_2(\hat{z})$. From here, the proofs follow the same steps as in the proof of Theorem 2. □

Remark 18. While all our results assumed that the resets (18b) were stabilizing, or at least, not destabilizing, it is possible to extend Theorems 1–2 to cases where the resets are destabilizing, provided the flows of the HDS are "sufficiently" frequent compared to the jumps. In this case, stability can be established by a simple modification of the Lyapunov functions used to study the target systems $\widehat{\mathcal{H}}$ as in [25, Prop. 3.29]. □

We conclude this section by noting that, with some additional effort, the stability results of Theorems 1–2 could be extended to systems for which Lyapunov functions with monomial bounds do not exist. While this represents an interesting research direction, such characterizations are beyond the scope of this paper and could be more appropriately studied in the future within the context of integral-ISS, as described in [26]. For our applications of interest, discussed in the next section, as well as others not detailed here due to space constraints (e.g., concurrent learning [44], extremum seeking [17], feedback-optimization), Assumption 3 is typically satisfied.

5. Applications to PT-control and PT-decision making

This section presents two applications that illustrate our main results. Throughout this section, the state q and the blow-up gain μ_k are assumed to follow the hybrid dynamics $\mathcal{H}$ defined in (4), with data given by (24) or (38). Since practical implementations of PT-Stable algorithms typically involve early terminations to avoid numerical instabilities, as well as techniques such as clipping and saturation [2, 11,17], for all our numerical simulations we employ a fourth-order Runge–Kutta method with fixed time step $\delta t = 10^{-6}$ and we saturate the blow-up gain μ_k at 1×10^3.

5.1. PT-regulation with intermittent feedback

Consider a switched input-affine system with intermittent feedback, of the form:

$$\dot{x} = d_q(x) + \mathbb{I}_{\mathcal{Q}_s}(q)\, b_q(x) u_q(x, \mu_k), \tag{46}$$

where $x \in \mathbb{R}^n$, $q \in \mathcal{Q} = \mathcal{Q}_s \cup \mathcal{Q}_u$ is a logic state and $\mathcal{Q}_u \neq \emptyset$. The blow-up gain μ_k is as defined in (5), $d_q(x) \in \mathbb{R}^n$ and $b_q(x) \in \mathbb{R}^{n\times n}$ denote mode-dependent drift and input vector fields, respectively, $u_q : \mathbb{R}^n \times \mathbb{R}_{\geq 1} \to \mathbb{R}^n$ is the control input, and $\mathbb{I}_{\mathcal{Q}_s}(q)$, is an indicator function representing the intermittent nature of the feedback. Such input-affine switching systems model diverse phenomena, ranging from gene regulatory networks in biology [45] to hybrid locomotion in robotics [46]. Incorporating intermittent feedback enhances the practical relevance of these models by addressing challenges such as limited sensor availability, and adversarial operating environments. The implementation of prescribed time controllers proves crucial in scenarios demanding strict time constraints thereby extending the applicability of these models to time-sensitive applications.

We assume that $b_q(\cdot)$ and $d_q(\cdot)$ are unknown locally Lipschitz functions, which satisfy the following properties:

$$|d_q(x)| \leq \overline{d}_q(x), \qquad \forall\, q \in \mathcal{Q},\ x \in \mathbb{R}^n,$$
$$b_q(x) + b_q(x)^\top \geq \epsilon I_n, \quad \forall\, q \in \mathcal{Q}_s,\ x \in \mathbb{R}^n,$$

where $\epsilon > 0$, and $\overline{d}_q(x) > 0$ is a known scalar-valued function assumed to be continuous for all $x \in \mathbb{R}^n$ and all $q \in \mathcal{Q}$. We also assume that $\overline{d}_q(x)$ is ℓ_q-globally Lipschitz for all $q \in \mathcal{Q}_u$. To regulate the state x to the origin in a prescribed time, we consider the following switching feedback-law:

$$u_q(x, \mu_k) = -\mu_k \left(\eta_q + \delta_q \overline{d}_q(x)^2\right) x, \tag{47}$$

with $\delta_q > 0$ and $\eta_q > 0$ and $k \geq 2$. The closed-loop system has the form of the HDS $\mathcal{H}$ with data (38) and continuous-time dynamics of x given by:

$$\dot{x} = \mu_k(t) f_{\sigma(t)}(x, \mu_k), \tag{48}$$

where, for every $q \in \mathcal{Q}$, $f_q : \mathbb{R}^n \times \mathbb{R}_{\geq 1} \to \mathbb{R}^n_{\geq 0}$ is given by

$$f_q(x, \mu_k) := -\mathbb{I}_{\mathcal{Q}_s}(q) \left(\eta_q + \delta_q \psi_q(x)^2\right) b_q(x) x + \frac{1}{\mu_k} d_q(x).$$

The following proposition extends the results of [1, Sec. 3] to the scenario where the system switches between multiple stable and unstable modes:

Proposition 3. *There exists $\tau_d > 0$ and $\tau_a > 0$ such that the set $\mathcal{A}_\psi \times \mathbb{R}_{\geq 1}$ is PT-ISS-C_F for the closed-loop system, where $\mathcal{A}_\psi$ is as given in (43). Additionally, the switching feedback-law u_q is bounded over the continuous-time interval $[0, \Upsilon_{T,k})$ and converges to 0 as $t \to \Upsilon_{T,k}$.* □

Proof. We show that under Assumption 4 a suitable Lyapunov function can be used to show that Assumption 3 is satisfied. Let $V_{\hat{q}}(\hat{x}) = \frac{1}{2\underline{\sigma}_{\hat{q}}} |\hat{x}|^2$ for every $\hat{q} \in \mathcal{Q}_s$. By employing Young's inequality, we obtain

$$\left\langle \nabla V_{\hat{q}}(\hat{x}),\ f_{\hat{q}}(\hat{x}, \hat{\mu}_k) \right\rangle \leq -2\underline{\sigma}_{\hat{q}} \eta_{\hat{q}} V_{\hat{q}}(\hat{x}) + \frac{1}{\hat{\mu}_k^2} \frac{1}{4\underline{\sigma}_{\hat{q}}^2 \delta_{\hat{q}}}, \tag{49}$$

for all $\hat{q} \in \mathcal{Q}_s$. Similarly, for all $\hat{q} \in \mathcal{Q}_u$ let $V_{\hat{q}}(\hat{x}) = \frac{|\hat{x}|^2}{2}$. Using this function, we obtain

$$\left\langle \nabla V_{\hat{q}}(\hat{x}),\ f_{\hat{q}}(\hat{x}, \hat{\mu}_k) \right\rangle \leq V_{\hat{q}}(\hat{x}) + \frac{1}{\hat{\mu}_k^2} \frac{\overline{d}_{\hat{q}}^2}{2}, \tag{50}$$

for all $\hat{q} \in \mathcal{Q}_u$. Using $c_{\hat{q},1} = c_{\hat{q},2} = 1/2\underline{\sigma}_{\hat{q}}$, $c_{\hat{q},3} = 2\underline{\sigma}_{\hat{q}}\eta_{\hat{q}}$, $c_{\hat{q},4} = 1/4\underline{\sigma}_{\hat{q}}^2\delta_{\hat{q}}$, when $\hat{q} \in \mathcal{Q}_s$, and $c_{\hat{q},1} = c_{\hat{q},2} = 1/2$, $c_{\hat{q},5} = 1$, $c_{\hat{q},4} = \overline{d}_{\hat{q}}^2/2$ when $\hat{q} \in \mathcal{Q}_u$, together with the set of smooth functions $\{V_{\hat{q}}\}_{\hat{q}\in\mathcal{Q}}$, Assumption 3 is satisfied. Thus, we can always pick $\tau_a > 1$ and $\tau_d > 0$ large enough to satisfy the stability condition (44). Additionally, Assumption 2 is satisfied by the Lipschitz properties of both $d_q(\cdot)$ and $b_q(\cdot)$. Assumption 1 is met by the same Lipschitz property and the construction of the HDS $\mathcal{H}$ with data (38). It follow that $\mathcal{A}_\psi \times \mathbb{R}_{\geq 1}$ is PT-ISS-C_F for the closed-loop system via Theorem 2-(c).

We now prove the boundedness and convergence to 0 of the switching feedback-law u_q given in (47). By applying (33) from the proof of Theorem 2-(c), for any $(x_0, \mu_0) \in \mathbb{R}^n \times \mathbb{R}_{\geq 1}$ and any solution $z = (x, \tau, \rho, q, \mu_k)$ to the closed-loop system satisfying $x(0,0) = x_0$ and $\mu_k(0,0) = \mu_0$ we obtain:

$$|x(t,j)| \leq \beta_k(\overline{\kappa}_1 e^{-\overline{\kappa}_2(\mathcal{T}_k(t)+j)} |x(0,0)| + \overline{\kappa}_3 \overline{u}, \mathcal{T}_k(t)), \tag{51}$$

for all $(t,j) \in \mathrm{dom}(z)$, where $\overline{\kappa}_1, \overline{\kappa}_2, \overline{\kappa}_3 > 0$, $\overline{u} := \max\left\{ \max_{q\in\mathcal{Q}_s} \frac{1}{4\underline{\sigma}_q^2 \delta_q},\ \max_{q\in\mathcal{Q}_u} \frac{\overline{d}_q^2}{2} \right\}$, and $\beta_k(r,s) = r \cdot \max\{\overline{\kappa}_1 e^{-\overline{\kappa}_2 s}, \xi_k^{-2}(s)\} =: r \cdot \alpha_k(s)$, with

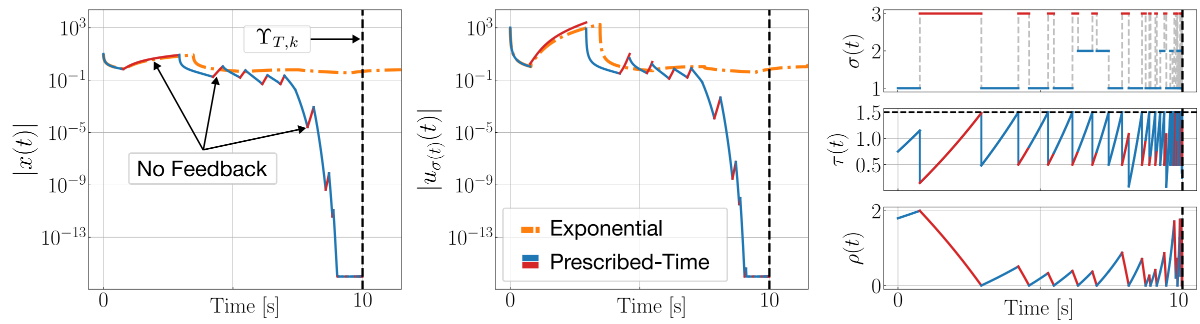

Fig. 4. Comparison between controller with Exponential convergences and PT-Regulation with intermittent feedback. Left: Trajectory of system's state norm plotted in logarithmic scale. Center: Trajectories of the switching feedback law u_q. Right: Trajectories of the switching signal σ (top), the dwell-time state τ (middle), and the monitor state ρ (bottom) for the PT-Regulation mechanism with intermittent feedback.

$\xi_k(s) = \left(\frac{k-1}{T}s+1\right)^{\frac{k}{k-1}}$, is the same $\mathcal{KL}$ function obtained in Lemma 11. Then, from (51) we obtain:

$$|x(t,j)| \le \left(\overline{\kappa}_1 e^{-\overline{\kappa}_2(\mathcal{T}_k(t)+j)}|x(0,0)| + \overline{\kappa}_3\overline{u}\right)\alpha_k(\mathcal{T}(t)),$$

for all $(t,j) \in \text{dom}(z)$. Hence, using Eq. (47) u_q satisfies:

$$\left|u_q(x(t,j),\mu_k(t))\right| \le \tilde{r}_k(t,j)\left|\eta_q+\delta_q \overline{d}_q^{\,2}(x(t,j))\right|\mu_k(t)\alpha_k(\mathcal{T}_k(t)),$$

for all $(t,j) \in \text{dom}(z)$ and all $q \in \mathcal{Q}$, where $\tilde{r}_k(t,j) = \left(\overline{\kappa}_1 e^{-\overline{\kappa}_2(\mathcal{T}_k(t)+j)}|x(0,0)| + \overline{\kappa}_3\overline{u}\right)$. Since $\overline{d}_q(\cdot)$ is assumed to be continuous for all $x \in \mathbb{R}^n$, it is locally bounded. Then, $\tilde{r}_k(t,j)\left|\eta_q + \delta_q \overline{d}_q^{\,2}(x(t,j))\right|$ is bounded as $\tilde{r}(t,j)$ is bounded by definition. Now, note that $\alpha_k(s) = \max\{\overline{\kappa}_1 e^{-\overline{\kappa}_2 s}, \xi_k^{-2}(s)\} = \xi_k^{-2}(s)$ for s sufficiently large since the inverse exponential decays faster than any proper rational function. Additionally, by leveraging the result of Proposition 2 it follows that $\mu_k(t) = \hat{\mu}_k(\mathcal{T}_k(t)) = \left(\frac{k-1}{T}\mathcal{T}_k(t)+\mu_0^{\frac{k-1}{k}}\right)^{\frac{k}{k-1}}$ for $k \ge 2$. Then, as $t \to \Upsilon_{T,k}$ we have that $\mu_k(t)\alpha_k(\mathcal{T}_k(t)) = \left[\left(\frac{k-1}{T}\mathcal{T}_k(t)+\mu_0^{\frac{k-1}{k}}\right)/\left(\frac{k-1}{T}\mathcal{T}_k(t)+1\right)^2\right]^{\frac{k}{k-1}}$ which implies that $\mu_k(t)\alpha_k(\mathcal{T}_k(t)) \to 0$. Using this fact, together with the inequality above and the boundedness of $\tilde{r}_k(t,j)\left|\eta_q + \delta_q \overline{d}_q^{\,2}(x(t,j))\right|$, allows us to conclude that $u_q \to 0$ as $t \to \Upsilon_{T,k}$. ■

To illustrate Proposition 3 with a numerical example, consider $\mathcal{Q}_s = \{1,2\}$, $\mathcal{Q}_u = \{3\}$, and $x \in \mathbb{R}$. Let $d_q(x) = q\tanh(x)$, $b_q(x) = 1$, $\forall q \in \mathcal{Q}$, and consider the control-law $u_q(x,t) = -\mu_2(t)(1+q|x|^2)x$. Then, all the conditions to apply Proposition 3 are satisfied. We numerically verify the PT-ISS-C$_F$ property by using a switching signal $\sigma \in \Sigma_{\text{BU-ADT}}(\tau_d, N_0, T, \mu_0) \cap \Sigma_{\text{BU-AAT}}(\mathcal{Q}_u, \tau_a, T_0, T, \mu_0)$ with $\tau_a = 2$, $\tau_d = 1$, $T = 10$, $T_0 = 2$, and $N_0 = 1.5$. Fig. 4 displays the trajectories of the norm of the state x plotted in logarithmic scale, the switching feedback-law u_q, the switching signal σ, and the associated average dwell-time and average activation time states τ and ρ. As shown in the figure, the state x and the switching feedback-law u_q rapidly approach zero as $t \to \Upsilon_{T,1}$ and converge faster than using a switching feedback with static gains (for exponential convergence). The overshoot occur when the system is in one of the modes without feedback.

5.2. PT-decision-making in switching games

Consider a non-cooperative game with $n \in \mathbb{Z}_{\ge 2}$ players [10], where the cost functions defining the game are allowed to switch in time. Specifically, for each $i \in \mathcal{V} = \{1,2,\dots,n\}$, the ith player has an associated mode-dependent and continuously differentiable cost function $\phi_q^i : \mathbb{R}^n \to \mathbb{R}$, where $q \in \mathcal{Q}$. We refer to the qth game as the game with the set of cost functions $\left\{\phi_q^i\right\}_{i\in\mathcal{V}}$. The action of the ith player is denoted by $x_1^i \in \mathbb{R}$, and the action profile of the game is given by the vector $x_1 := \left(x_1^1, x_1^2, \dots, x_1^n\right) \in \mathbb{R}^n$. The goal of the players is to converge to the unique common Nash equilibrium (NE) of the games [10,47], defined as the vector $\tilde{x} \in \mathbb{R}^n$ that satisfies:

$$\phi_q^i\left(\tilde{x}^i,\ \tilde{x}^{-i}\right) = \inf_{x_1^i\in\mathbb{R}} \phi_q^i\left(x_1^i,\ \tilde{x}^{-i}\right),\quad \forall i \in \mathcal{V},$$

for all $q \in \mathcal{Q}$, where $x_1^{-i} \in \mathbb{R}^{n-1}$ denotes the vector that contains all actions except those of player i. To study this problem, let $\mathcal{G}_q : \mathbb{R}^n \to \mathbb{R}^n$ denote the pseudo-gradient of the qth game, which is given by:

$$\mathcal{G}_q(x_1) := \left(\frac{\partial \phi_q^1}{\partial x_1^1}, \frac{\partial \phi_q^2}{\partial x_1^2}, \dots, \frac{\partial \phi_q^n}{\partial x_1^n}\right).$$

For all $q \in \mathcal{Q}$, we assume that there exists $\kappa_q > 0$ and $\ell_q > 0$ such that $\mathcal{G}_q$ is a κ_q-strongly monotone and ℓ_q-globally Lipschitz mapping. These properties are common in NE seeking problems and they guarantee the existence and uniqueness of the NE $\tilde{x}$ [10]. To efficiently achieve convergence to the NE in a prescribed time, we introduce *PT high-order NE-seeking dynamics with momentum and resets* (PT-NESmr). The proposed algorithm is modeled as a HDS $\mathcal{H}$ with data (24) and maps f_q and R_q defined as follows:

$$f_q(x,\tau) = \begin{pmatrix} \frac{2}{\eta(\tau)}\left(x_2 - x_1\right) \\ -2\eta(\tau)\mathcal{G}_q(x_1)\end{pmatrix},\quad R_q(x) = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}, \tag{52}$$

where $x := (x_1, x_2) \in \mathbb{R}^{2n}$, and $x_2 := (x_2^1, x_2^2, \dots, x_2^n) \in \mathbb{R}^n$, and where $\eta : [0, N_0] \to [\underline{\eta}, \overline{\eta}]$ is an affine bounded mapping defined as:

$$\eta(\tau) := \tau\frac{\left(\overline{\eta}-\underline{\eta}\right)}{N_0} + \underline{\eta} \tag{53}$$

with $\overline{\eta} > \underline{\eta} > 0$ being tunable parameters. In the context of asymptotic convergence, mappings of the form (52), which incorporate momentum (via the state x_2) and resets (via the update $x_2^+ = x_1$), have been recently shown to improve the transient performance of NE-seeking dynamics in (stable) strongly monotone games [43]. To further make the convergence time independent of both the initial conditions and of the monotonicity properties of the game, we study convergence to the NE in prescribed-time.

For every $q \in \mathcal{Q}$, let $\sigma_q > 0$ be such that $\sigma_{\max}\left(I - \partial\mathcal{G}_q(x)\right) \le \sigma_q$ for all $x \in \mathbb{R}$, where $\partial\mathcal{G}_q$ denotes the Jacobian of the pseudogradient, and where $\sigma_{\max}(\cdot)$ denotes the maximum singular value of its argument. Such σ_q always exists since the pseudo-gradient $\mathcal{G}_q$ is assumed to be globally Lipschitz for all $q \in \mathcal{Q}$. We make the following assumption on the parameters of the game and the selection of the tunable parameters in (52)–(53).

Assumption 4 (*Tuning Guidelines*). There exist $0 \le \underline{\eta} \le \overline{\eta}$, $\delta_\eta > 0$, and $\delta_d > 0$ satisfying $\delta_\eta + \delta_d := \delta \in (0,1)$ and:

$$\overline{\eta}^2 \le \delta_\eta \frac{\min_{q\in\mathcal{Q}} \zeta_q}{\left(\max_{q\in\mathcal{Q}} \sigma_q\right)^2},\qquad \frac{1}{\tau_d} \le \delta_d \frac{N_0}{\overline{\eta}-\underline{\eta}} \min_{q\in\mathcal{Q}} \zeta_q, \tag{54}$$

for some $\tau_d > 0$ and $N_0 \ge 1$, where $\zeta_q := \kappa_q/\ell_q^2$. □

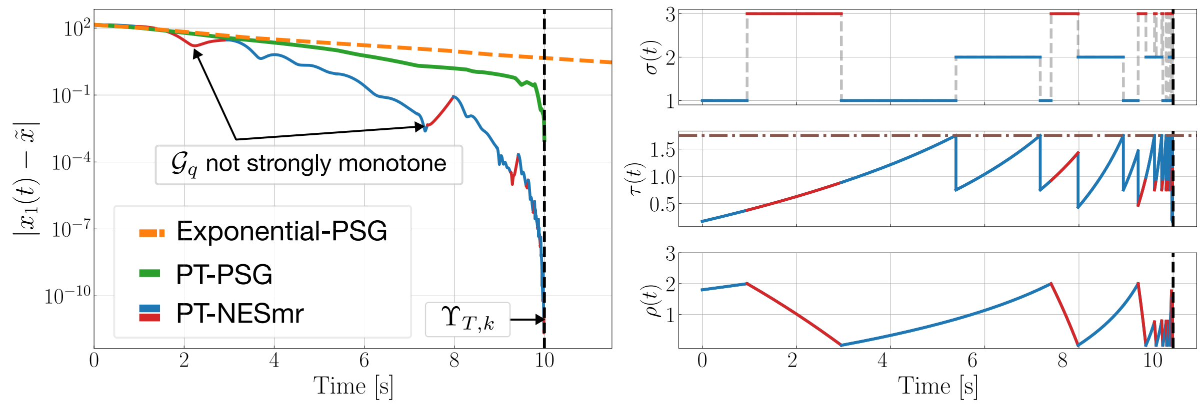

Fig. 5. Comparison between Pseudo-Gradient Flow (PSG) with exponential convergence and PT Nash-Equilibrium Seeking in a Switching Game. Left: Trajectory of the errors to the NE generated by the PT-NESmr, the PT-PSG, and the Exponential PSG dynamics. Right: Trajectories of the switching signal $\sigma(t)$ (top), the dwell-time state τ (middle), and the monitor state $\rho(t)$ (bottom) for the PT-NESmr dynamics.

The stability properties of the states x_1, x_2 are studied with respect to the following set

$$\mathcal{A}_x := \{\tilde{x}\} \times \{\tilde{x}\} \subset \mathbb{R}^n \times \mathbb{R}^n. \tag{55}$$

The following proposition establishes PT-S$_F$ of the set $\mathcal{A}_x$ under the PT-NESmr dynamics.

Proposition 4. *Suppose that Assumption 4 is satisfied. Then, the PT-NESmr dynamics render the set $\mathcal{A}_x \times [0, N_0] \times \mathcal{Q} \times \mathbb{R}_{\geq 1}$ PT-S$_F$, provided*

$$\tau_d > \frac{\max\left\{3, 2\left(\frac{1}{\underline{\kappa}^2} + \overline{\eta}^2\right)\right\} \ln(r)}{4\underline{\eta}\nu}, \tag{56}$$

where $\nu = \frac{(1-\delta_d-\delta_\eta)\overline{\sigma}^2}{\delta_\eta(1-\delta_d)\underline{\zeta}+\overline{\sigma}^2}$, $\overline{\sigma} := \max_{q\in\mathcal{Q}} \sigma_q$, $\underline{\zeta} := \min_{q\in\mathcal{Q}} \zeta_q$, *and* $r = \max\left\{1, \frac{\overline{\ell}^2}{\underline{\kappa}^2}\frac{\eta(N_0-1)^2}{\eta(1)^2} + \frac{1}{2\underline{\kappa}^2\eta(1)^2}\right\}$. □

Proof. We show that, under Assumption 4, a suitable Lyapunov function for the "target" system $\widehat{\mathcal{H}}$ can be used to show that Assumption 3 is satisfied. Indeed, for every $\hat{q} \in \mathcal{Q}$ consider the Lyapunov function

$$V_{\hat{q}}(\hat{x}, \hat{\tau}) = \frac{1}{4}|\hat{x}_2 - x^*|^2 + \frac{1}{4}|\hat{x}_2 - \hat{x}_1|^2 + \frac{\eta(\hat{\tau})^2}{2}\left|\mathcal{G}_{\hat{q}}(\hat{x}_1)\right|^2,$$

which in the flow set and jump set satisfies: $\upsilon_{\hat{q},1}|\hat{x}|^2_{\mathcal{A}_x} \leq V_{\hat{q}}(\hat{x}, \hat{\tau}) \leq \upsilon_{\hat{q},2}|\hat{x}|^2_{\mathcal{A}_x}$, with $\upsilon_{\hat{q},1} := 0.25\min\left\{1, 2\kappa_{\hat{q}}^2\underline{\eta}^2\right\}$, and $\upsilon_{\hat{q},2} := 0.25\max\left\{3,\ 2 + 2\ell_{\hat{q}}^2\overline{\eta}^2\right\}$. Let

$$\mathcal{L}_{(f_{\hat{q}},\rho)}V_{\hat{q}}(\hat{x}, \hat{\tau}) := \left\langle \nabla V_{\hat{q}}(\hat{x}, \hat{\tau}), \begin{pmatrix} f_{\hat{q}}(\hat{x}, \hat{\tau}) \\ \rho \end{pmatrix} \right\rangle \tag{57}$$

Since $\mathcal{G}_{\hat{q}}(\cdot)$ is $\kappa_{\hat{q}}$-strongly-monotone and $\ell_{\hat{q}}$–Lipschitz, we have that $\langle x_1 - \tilde{x},\ \mathcal{G}_{\hat{q}}(\hat{x}_1)\rangle \geq \zeta_{\hat{q}}\left|\mathcal{G}_{\hat{q}}(\hat{x}_1)\right|^2$, where $\zeta_{\hat{q}} = \kappa_{\hat{q}}^2/\ell_{\hat{q}}$. During flows:

$$\begin{aligned}
\mathcal{L}_{(f_{\hat{q}},\rho)}V_{\hat{q}}(\hat{x}, \hat{\tau}) &= -\frac{1}{\eta(\hat{\tau})}|\hat{x}_2 - \hat{x}_1|^2 \\
&\quad - 2\eta(\hat{\tau})\left\langle \mathcal{G}_{\hat{q}}(\hat{x}_1),\ \left[I - \partial\mathcal{G}_{\hat{q}}(\hat{x}_1)\right](\hat{x}_2 - \hat{x}_1)\right\rangle \\
&\quad - \eta(\hat{\tau})\left[\left\langle \hat{x}_1 - x^*,\ \mathcal{G}_{\hat{q}}(\hat{x}_1)\right\rangle - \rho\eta'(\hat{\tau})|\mathcal{G}_{\hat{q}}(\hat{x}_1)|^2\right] \\
&\leq -\eta(\hat{\tau})\left\langle \chi_{\hat{q}},\ M_{\zeta_{\hat{q}}}(\hat{x}_1, \hat{\tau})\chi_{\hat{q}}\right\rangle,
\end{aligned} \tag{58}$$

for all $(\hat{x}, \hat{\tau}, \rho) \in \mathbb{R}^{2n} \times [0, N_0] \times [0, \tau_d^{-1}]$, where $\chi_{\hat{q}} := \left(\hat{x}_2 - \hat{x}_1, \mathcal{G}_{\hat{q}}(\hat{x}_1)\right) \in \mathbb{R}^{2n}$, and $M_{\zeta_{\hat{q}}}$ is given by

$$M_{\zeta_{\hat{q}}}(\hat{x}_1, \hat{\tau}) := \begin{pmatrix} \frac{1}{\eta(\hat{\tau})^2}I & I - \partial\mathcal{G}_{\hat{q}}(\hat{x}_1)^\top \\ I - \partial\mathcal{G}_{\hat{q}}(\hat{x}_1) & (\zeta_{\hat{q}} - \rho\eta'(\hat{\tau}))I \end{pmatrix}.$$

Using Lemma 8 in the Appendix, we conclude that $\mathcal{L}_{(f_{\hat{q}},\rho)}V_{\hat{q}}(\hat{x}, \hat{\tau}) \leq -\underline{\eta}\nu_M|\chi_{\hat{q}}|^2$ for all $(\hat{x}_1, \hat{x}_2, \hat{\tau}) \in \mathbb{R}^{2n} \times [0, N_0]$. Hence, by noting that $V_{\hat{q}}(\hat{x}, \hat{\tau}) \leq \frac{1}{4}\max\left\{3, 2\left(\frac{1}{\kappa_{\hat{q}}^2} + \overline{\eta}^2\right)\right\}|\chi_{\hat{q}}|^2$ we obtain:

$$\mathcal{L}_{(f_{\hat{q}},\rho)}V_{\hat{q}}(\hat{x}, \hat{\tau}) \leq -\frac{4\underline{\eta}\nu_M}{\max\left\{3, 2\left(\frac{1}{\kappa_{\hat{q}}^2} + \overline{\eta}^2\right)\right\}}V_{\hat{q}}(\hat{x}, \hat{\tau}). \tag{59}$$

Now, for all $\hat{p}, \hat{q} \in \mathcal{Q}$, let

$$\Delta V_{\hat{p}}^{\hat{q}}(\hat{x}, \hat{\tau}) := V_{\hat{q}}\left(R_{\hat{p}}(\hat{x}), \hat{\tau} - 1\right) - V_{\hat{p}}(\hat{x}, \hat{\tau}), \quad \hat{\tau} \in [1, N_0].$$

During jumps:

$$\begin{aligned}
\Delta V_{\hat{p}}^{\hat{q}}(\hat{x}, \hat{\tau}) &= V_{\hat{q}}\left((\hat{x}_1, \hat{x}_1), \hat{\tau} - 1\right) - V_{\hat{p}}(x, \hat{\tau}) \\
&\leq -\frac{1}{4}|\hat{x}_1 - x^*|^2 - \frac{1}{4}|\hat{x}_1 - \hat{x}_2|^2 + \frac{1}{4\kappa_{\hat{p}}^2}|\mathcal{G}_{\hat{p}}(\hat{x}_1)|^2 \\
&\quad + \frac{1}{2}\left(\eta(N_0 - 1)^2\frac{\ell_{\hat{q}}^2}{\kappa_{\hat{p}}^2} - \eta(1)^2\right)\left|\mathcal{G}_{\hat{p}}(\hat{x}_1)\right|^2 \\
&\leq -\left(1 - \gamma_{\hat{p}}^{\hat{q}}\right)V_{\hat{p}}(\hat{x}, \hat{\tau}),
\end{aligned} \tag{60}$$

where $\gamma_{\hat{p}}^{\hat{q}} := \frac{2\eta(N_0-1)^2\ell_{\hat{q}}^2+1}{2\kappa_{\hat{p}}^2\eta(1)^2}$. The above inequality implies that $V_{\hat{q}}\left(R_{\hat{p}}(\hat{x}), \hat{\tau} - 1\right) \leq \gamma_{\hat{p}}^{\hat{q}}V_{\hat{p}}(\hat{x}, \hat{\tau})$. where $\underline{\ell} := \min_{\hat{q}\in\mathcal{Q}} \ell_{\hat{q}}$, $\overline{\kappa} := \max_{\hat{q}\in\mathcal{Q}} \kappa_{\hat{q}}$, and $\underline{\kappa} := \min_{\hat{q}\in\mathcal{Q}} \kappa_{\hat{q}}$. Thus, noting that $\gamma_{\hat{p}}^{\hat{q}} \leq \frac{\overline{\ell}^2}{\underline{\kappa}^2}\frac{\eta(N_0-1)^2}{\eta(1)^2} + \frac{1}{2\underline{\kappa}^2\eta(1)^2} =: \overline{\gamma}$, we obtain:

$$V_{\hat{q}}\left(R_{\hat{p}}(\hat{x}), \hat{\tau} - 1\right) \leq \overline{\gamma}V_{\hat{p}}(\hat{x}, \hat{\tau}), \tag{61}$$

for all $\tau \in [1, N_0]$, $p, q \in \mathcal{Q}$. By the smoothness properties of $\mathcal{G}_q(\cdot)$ and the differentiability of $\eta(\cdot)$, we obtain that $f_q(x, \tau)$ is locally Lipschitz and, thus, that Assumption 2 also holds. On the other hand, note that via a simple change of coordinates, and without loss of generality, the results of Theorem 1 hold for $\mathcal{A}$ as defined in (28) but with the set $\{0\}$ replaced by the set $\mathcal{A}_x$ in (55). Therefore, the quadratic bounds on the Lyapunov function, together with condition (56), (59), and (61), imply PT-S$_F$ of $\mathcal{A}_x \times [0, N_0] \times \mathcal{Q} \times \mathbb{R}_{\geq 1}$ via Theorem 1-(a). ■

Remark 19 (*PT-NESmr with Non-Monotone $\mathcal{G}_q$*). Unlike [43], the results of Proposition 4 can be directly extended to switching games where some modes lack strong monotonicity in their pseudo-gradients. In this case, we can use the HDS $\mathcal{H}$ with data (38) and leverage Theorem 2, paralleling the approach followed in Section 5.A to study unstable plants. In this case, we obtain conditions on τ_d and τ_a in $\mathcal{H}$, characterizing admissible switching signals under which PT-NESmr dynamics attain prescribed-time stability. This broadens PT-NESmr's applicability to switching games with temporary loss of strong monotonicity. □

To illustrate the previous discussion, let $\mathcal{Q} = \{1,2,3\}$ and $\mathcal{G}_q(x_1) = \vartheta A_q(x_1 - \tilde{x})$, with $\tilde{x} = (1,1)$, $A_1 = [6,-1.5;-1.5,6]$, $A_2 = [8,-2;2,8]$, $A_3 = [4,6;5;2]$, and $\vartheta = 5\times10^{-2}$. The pseudo-gradient $\mathcal{G}_q(\cdot)$ is κ_q-strongly monotone only for $q \in \{1,2\} =: \mathcal{Q}_s$ and ℓ_q-globally Lipschitz for all $q \in \mathcal{Q}$. Using $k = 1$, $\tau_d = \tau_a = 2.5$, $N_0 = 1.75$, $T_0 = 2$ we simulate the system using a switching signal $\sigma \in \Sigma_{\text{BU-ADT}}(\tau_d, N_0, T, \mu_0)$ with $T = 10$. We compare our results with the continuous-time prescribed-time pseudo-gradient-flows (PT-PSG), recently introduced in [48], and given by $\dot{x}_1 = \mu_1(t)\mathcal{G}_{\sigma(t)}(x_1)$. The resulting trajectories are shown in Fig. 5. As shown in the figure, under the PT-NESmr and the PT-PSG dynamics, the state x_1 rapidly approaches zero as $t \to \Upsilon_{T,1}$ and converges faster than using the standard pseudo-gradient flows with exponential convergence guarantees (Exponential-PSG). Also, note that the synergistic incorporation of momentum, resets, and PT techniques leads to an improvement compared to the continuous-time PT-PSG algorithm under the same switching signal. The overshoots occur when the Nash-equilibrium seeking algorithms operate with a pseudo-gradient that is not κ-strongly monotone, or equivalently when $q \in \mathcal{Q}_u = \mathcal{Q} \setminus \mathcal{Q}_s$.

6. Conclusions

The property of prescribed-time stability was studied and extended for a class of hybrid dynamical systems incorporating switching nonlinear vector fields with time-varying increasing gains, exogenous inputs, and resets. Novel switching conditions that preserve the prescribed-time stability properties of the system were derived using tools from hybrid dynamical systems theory and under a suitable contraction/dilation of the hybrid time domains. The switching conditions allow the incorporation of unstable modes. The results were illustrated in two applications in the context of control and decision-making. Future applications will include prescribed-time concurrent learning and prescribed-time switching extremum seeking. Future work will also include studying the synergies between non-smooth and prescribed-time tools, as well as consistent discretization mechanisms for HDS, similar to [49].

CRediT authorship contribution statement

Daniel E. Ochoa: Investigation, Formal analysis, Conceptualization. **Nicolas Espitia:** Investigation, Formal analysis, Conceptualization. **Jorge I. Poveda:** Supervision, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

Appendix

We present detailed proofs of all the auxiliary lemmas and propositions used in the paper.

A.1. Proofs of Section 3

The results below follow directly by computations and/or straightforward extensions or specializations of existing results in the literature.

Proof of Lemma 1. By direct integration, we have that:

$$\int_{\mu_0}^{\mu_k(t)} \frac{d\mu_k}{\mu_k^{1+\frac{1}{k}}} = \int_0^t \frac{k}{T} dt \implies -k\mu_k^{\frac{-1}{k}}\Big|_{\mu_0}^{\mu_k(t)} = \frac{k}{T}t.$$

Thus, it follows that $k\left(-\mu_k(t)^{\frac{-1}{k}} + \mu_k(0)^{\frac{-1}{k}}\right) = \frac{k}{T}t$, and:

$$\frac{1}{\mu_k(t)^{\frac{1}{k}}} = \frac{1}{\mu_k(0)^{\frac{1}{k}}} - \frac{t}{T} = \frac{T - t\mu_k(0)^{\frac{1}{k}}}{T\mu_k(0)^{\frac{1}{k}}},$$

from which we obtain the result. ■

Proof of Lemma 3. By direct integration, we have that:

$$\int_{\mu_0}^{\hat{\mu}_k(t)} \frac{d\hat{\mu}_k}{\hat{\mu}_k^{\frac{1}{k}}} = \int_0^t \frac{k}{T} dt \implies \frac{1}{1-\frac{1}{k}}\hat{\mu}_k^{1-\frac{1}{k}}\Big|_{\mu_0}^{\hat{\mu}_k(t)} = \frac{k}{T}t.$$

Therefore, we obtain $\frac{k}{k-1}\left(\hat{\mu}_k^{1-\frac{1}{k}}(t) - \mu_0^{1-\frac{1}{k}}\right) = \frac{k}{T}t$, and:

$$\hat{\mu}_k(t) = \left(\frac{k-1}{T}t + \mu_0^{\frac{k-1}{k}}\right)^{\frac{k}{k-1}}.$$

This obtains the result. ■

Proof of Proposition 1.

(P1) Follows by the monotonicity of $\omega_k(\cdot,\cdot)$ in its first argument, combined with the limit $\lim_{t\to\Upsilon_{T,k}} \mu_k(t) = \infty$.

(P2) For $k > 1$, the result follows by direct computation. For $k = 1$, the result is obtained by the properties of the logarithm.

(P3) By definition, the equality $\mathcal{T}_k(0) = 0$ holds for all $k \in \mathbb{R}_{\geq 1}$. For $k = 1$, by direct computation, we have: $\frac{d\mathcal{T}_1(t)}{dt} = \frac{T}{\mu_1(t)}\dot{\mu}_1(t) = \mu_1(t)$. For $k > 1$, by the chain rule, we obtain:

$$\frac{d\mathcal{T}_k(t)}{dt} = \frac{\partial\omega_k(b,\mu_k(0))}{\partial b}\Big|_{b=\mu_k(t)}\dot{\mu}_k = \mu_k(t).$$

(P4) For $k = 1$, we have that $\mu_1(t) = \frac{\mu_0 T}{T-\mu_0 t}$. It then follows that $s = \left(\mathcal{T}_1 \circ \mathcal{T}_1^{-1}\right)(s) = T\ln\left(\frac{\mu_1(\mathcal{T}^{-1}(s))}{\mu_0}\right)$. Solving for $\mathcal{T}_1^{-1}(s)$ leads to $\mathcal{T}_1^{-1}(s) = \Upsilon_{T,1}\left(1 - e^{-\frac{s}{T}}\right)$. For $k > 1$, let $y_k := \mathcal{T}_k^{-1}$. By using (5), and the inverse function theorem, we obtain that $\frac{dy_k}{ds} = \frac{(\Upsilon_{T,k}-y)^k}{T^k}$. Then, by direct integration and using the fact that $y_k(0) = 0$, we obtain the following equality $\Upsilon_{T,k} - y_k(s) = \left(\frac{(k-1)s}{T^k} + \Upsilon_{T,k}^{1-k}\right)^{\frac{1}{1-k}}$. Solving for $\mathcal{T}_k^{-1}(s)$, we obtain that $\mathcal{T}_k^{-1}(s) = \Upsilon_{T,k} - \Upsilon_{T,k}\left(1 + \frac{(k-1)s}{\Upsilon_{T,k}\mu_0}\right)^{\frac{1}{1-k}}$.

(P5) Follows directly by the inverse function theorem.

(P6) For $k = 1$, using the equality $\ln(1-x) = \sum_{l=1}^{\infty} \frac{-1}{l}x^l$, $|x| < 1$, we obtain that $\mathcal{T}_1(t) = \mu_0 t + \sum_{l=2}^{\infty} \frac{1}{l}\mu_0^l t^l T^{1-l}$, for all $t\mu_0 < T$. Letting $T \to \infty$, the second term in this expression vanishes, and we obtain that the equality $\lim_{T\to\infty} \mathcal{T}_1(t) = \mu_0 t$ holds for all $(t,\mu_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 1}$. For $k > 1$, from Remark 5 it follows that

$$\mathcal{T}_k(t) = \frac{T\mu_0^{\frac{k-1}{k}}}{k-1}\left(\left(1 - \frac{t\mu_0^{\frac{1}{k}}}{T}\right)^{1-k} - 1\right). \tag{62}$$

Now, using the binomial theorem we have that

$$\left(1 - \frac{t\mu_0^{\frac{1}{k}}}{T}\right)^{1-k} - 1 = \frac{(k-1)t\mu_0^{\frac{1}{k}}}{T} + \sum_{l=2}^{\infty} g_{k,l}\left(\frac{t\mu_0^{\frac{1}{k}}}{T}\right)^l,$$

for all $t\mu_0^{\frac{1}{k}} < T$, and where $g_{k,l} = \frac{(k-1)k(k+1)\cdots(k+l-2)}{l!}$. Thus, for all $t\mu_0^{\frac{1}{k}} < T$, equality (62) can be written as $\mathcal{T}_k(t) = \mu_0^{\frac{k-1}{k^2}} t + \sum_{l=2}^{\infty} \frac{g_{k,l}}{k-1} t^l \mu_0^{\frac{(k-1)l}{k^2}} T^{1-l}$.

Letting $T \to \infty$, the second term in this expression vanishes. Thus, it follows that the limit $\lim_{T\to\infty} \mathcal{T}_k(t) = \mu_0^{\frac{k-1}{k^2}} t$ holds for all $(t, \mu_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 1}$. ∎

A.2. Proofs of Section 4

In this section, we present the proofs of Section 4.

Proof of Lemma 4. Let $\tau_d > 0$, $N_0 \geq 1$, and $\sigma(t) \in \Sigma_{\text{ADT}}(\tau_d, N_0)$. Then, it follows that

$$N(t_2, t_1) \leq \frac{1}{\tau_d}(t_2 - t_1) + N_0, \tag{63}$$

for all $t_1 \leq t_2$. We prove that expression (63) can be upper bounded by the right-hand side of (22).

Case $k = 1$*:* Assume that $t_1, t_2 \in [0, Y_{T,k})$ and define $X := \frac{Y_{T,k}-t_1}{Y_{T,k}-t_2}$, where $Y_{T,1} = T\mu_0^{-1}$ and $\mu_0 \geq 1$ fixed. Then, $X \geq 1$ and:

$$t_2 - t_1 = (Y_{T,1} - t_1)\left(1 - \frac{1}{X}\right).$$

Now, fix t_1 and define $f(X) := \ln(X) - T^{-1}(Y_{T,1} - t_1)\left(1 - \frac{1}{X}\right)$. Since t_1 satisfies $t_1 \leq Y_{T,1} \leq T$ by assumption, it follows that there exists $\delta_{t_1} \in [0, 1]$ such that:

$$f(X) = \ln(X) - \delta_{t_1}\left(1 - \frac{1}{X}\right).$$

By noting that $f(1) = 0$, and since $X \geq 1$, it follows that the derivative of f satisfies:

$$f'(X) = \frac{1}{X}\left(1 - \frac{\delta_{t_1}}{X}\right) \geq 0,$$

for all $\delta_{t_1} \in [0, 1]$. Thus, $f(X) \geq 0$ for all $X \geq 1$ and $t_1 \leq Y_{T,1}$. Equivalently, by using the definition of X, it follows that:

$$T \ln\left(\frac{Y_{T,1} - t_1}{Y_{T,1} - t_2}\right) - (t_2 - t_1) \geq 0,$$

for all $0 \leq t_1 \leq t_2 < Y_{T,1}$, where we have used the definition of X. Using this bound in (63) yields:

$$N(t_2, t_1) \leq \frac{T}{\tau_d} \ln\left(\frac{Y_{T,1} - t_1}{Y_{T,1} - t_2}\right) + N_0,$$

for all $0 \leq t_1 \leq t_2 < Y_{T,1}$, which implies that $\sigma(t)$, when restricted to $[0, Y_{T,1})$, satisfies the bound (22) for $k = 1$.

Case $k > 1$*:* Assume that $t_1, t_2 \in [0, Y_{T,k})$, with $Y_{T,k} = T\mu_0^{-\frac{1}{k}}$, $T > 0$ and $\mu_0 \geq 1$. Let $\Delta = t_2 - t_1$, and define

$$f(\Delta) = \mathcal{T}_k(t_1 + \Delta) - \mathcal{T}_k(t_1) - \Delta, \quad \Delta \in [0, Y_{T,k}).$$

Then, by using the result of Proposition 1-(P3) the derivative of f satisfies:

$$f'(\Delta) = \mu_k(t_1 + \Delta) - 1,$$

for all $t_1, \Delta \in [0, Y_{T,k})$. Since $\mu_k(t) \geq 1$ for all $t \in \mathbb{R}_{\geq 0}$, the previous equality implies that $f'(\Delta) \geq 0$. This result, together with the fact that $f(0) = 0$, implies that $f(\Delta) \geq 0$ for all $t_1, \Delta \in [0, Y_{T,k})$. Equivalently, by using the definition of Δ we obtain:

$$\begin{aligned} 0 &\leq \mathcal{T}_k(t_2) - \mathcal{T}_k(t_1) - (t_1 - t_2) \\ &\implies (t_1 - t_2) \leq \omega_k(\mu_k(t_2), \mu_k(t_1)), \end{aligned}$$

where the implication follows from the result of Proposition 1-(P2). Using this bound in (63) yields:

$$N(t_2, t_1) \leq \frac{1}{\tau_d}\omega_k(\mu_k(t_2), \mu_k(t_1)) + N_0,$$

for all $0 \leq t_1 \leq t_2 < Y_{T,1}$, which implies that $\sigma(t)$, when restricted to $[0, Y_{T,1})$, satisfies the bound (22) for $k \in \mathbb{Z}_{\geq 1}$. ∎

Proof of Lemma 5. The case $k = 1$ follows directly by the definition of $\mathcal{T}_1$ and Remark 3. For $k > 1$, consider expanding the right-hand side of (22):

$$\begin{aligned} N(t_2, t_1) &\leq \frac{T}{\tau_d}\left(\frac{\mu_k(t_2)^{\frac{k-1}{k}}}{k-1} - \frac{\mu_k(t_1)^{\frac{k-1}{k}}}{k-1}\right) + N_0 \\ &= \frac{T^k}{\tau_d(k-1)}\left(\frac{\left(Y_{T,k} - t_1\right)^{k-1} - \left(Y_{T,k} - t_2\right)^{k-1}}{\left(\left(Y_{T,k} - t_2\right)\left(Y_{T,k} - t_1\right)\right)^{k-1}}\right) + N_0. \end{aligned}$$

Taking the limit as $k \to 1$, one obtains (23), see also Remark 3. On the other hand, when $k \in \mathbb{Z}_{>1}$, the Binomial theorem can be used to write $(Y_{T,k} - t_i)^{k-1} = \sum_{\ell=0}^{k-1} b_{k,l} Y_{T,k}^{k-1-\ell}(-t_i)^\ell$, for $i \in \{1, 2\}$, where $b_{k,l} := \frac{(k-1)!}{\ell!(k-\ell-1)!}$ are the so-called Binomial coefficients. Let

$$\begin{aligned} S &:= \sum_{\ell=0}^{k-1} b_{k,l} Y_{T,k}^{k-1-\ell}(-t_1)^\ell - \sum_{\ell=0}^{k-1} b_{k,l} Y_{T,k}^{k-1-\ell}(-t_2)^\ell \\ &= \sum_{\ell=1}^{k-1} b_{k,l} Y_{T,k}^{k-1-\ell}(-t_1)^\ell - \sum_{\ell=1}^{k-1} b_{k,l} Y_{T,k}^{k-1-\ell}(-t_2)^\ell \\ &= b_{k,1} Y_{T,k}^{k-2}(t_2 - t_1) + \sum_{\ell=2}^{k-1} b_{k,l} Y_{T,k}^{k-1-\ell}\left((-t_1)^\ell - (-t_2)^\ell\right) \\ &= b_{k,1} Y_{T,k}^{k-2}(t_2 - t_1) + \sum_{\ell=2}^{k-1} (-1)^{\ell+1} b_{k,l} Y_{T,k}^{k-1-\ell}\left(t_2^\ell - t_1^\ell\right). \end{aligned}$$

Therefore, the BU_k-ADT bound can be written as

$$\begin{aligned} N(t_2, t_1) &\leq \frac{T^k}{\tau_d(k-1)}\left(\frac{S}{\left((Y_{T,k} - t_2)(Y_{T,k} - t_1)\right)^{k-1}}\right) + N_0 \\ &= \frac{\gamma_k(t_1, t_2)}{\tau_d}\left[(t_2 - t_1) + \sum_{\ell=2}^{k-1} \tilde{c}_{\ell,k}\left(t_2^\ell - t_1^\ell\right)\right] + N_0, \end{aligned}$$

where

$$\tilde{c}_{\ell,k} = (-1)^{\ell+1} b_{k,l} Y_{T,k}^{k-1-\ell}\left(b_{k,1} Y_{T,k}^{k-2}\right)^{-1} = (-1)^{\ell+1}\frac{b_{k,l}}{b_{k,1}} Y_{T,k}^{1-\ell},$$

and

$$\begin{aligned} \gamma_k(t_1, t_2) &= \frac{b_{k,1} T^k Y_{T,k}^{k-2}}{(k-1)}\left(\frac{1}{(Y_{T,k} - t_2)(Y_{T,k} - t_1)}\right)^{k-1} \\ &= \frac{T^k}{Y_{T,k}}\left[\frac{Y_{T,k}}{\left(Y_{T,k} - t_2\right)\left(Y_{T,k} - t_1\right)}\right]^{k-1} \\ &= \mu_0\left[\frac{Y_{T,k}^2}{\left(Y_{T,k} - t_2\right)\left(Y_{T,k} - t_1\right)}\right]^{k-1} \end{aligned}$$

where we have used the fact that $b_{k,1} = k - 1$. ∎

A.3. Auxiliary results of Section 5

The following Lemma is instrumental in studying the stability properties of the HDS with data (52).

Lemma 8. *Consider the matrix*

$$M_{\zeta_q}(x_1, \tau) := \begin{pmatrix} \frac{1}{\eta(\tau)^2} I & I - \partial\mathcal{G}_q(x_1)^\top \\ I - \partial\mathcal{G}_q(x_1) & (\zeta_q - \rho\eta'(\tau))I \end{pmatrix}, \tag{64}$$

where $q \in \mathcal{Q}$, $\tau \in [0, N_0]$, $\eta(\tau) \in [\underline{\eta}, \overline{\eta}]$, $\rho \in [0, 1/\tau_d]$, *and* $\eta'(\tau) := \frac{d\eta}{d\tau}(\tau)$, $\mathcal{G}_q(\cdot)$, *and* ζ_q *are as introduced in Section 5.2. Suppose that Assumption 4 is satisfied. Then,*

$$M_{\zeta_q}(x_1, \tau) \geq \nu_M I, \quad \forall\, \tau \in [0, N_0],\ x_1 \in \mathbb{R}^n \tag{65}$$

where $\nu_M := \frac{(1-\delta_d-\delta_\eta)\overline{\sigma}^2}{\delta_\eta(1-\delta_d)\underline{\zeta}+\overline{\sigma}^2}$, *with* $\underline{\zeta} := \min_{q\in\mathcal{Q}} \zeta_q$ *and* $\overline{\sigma} := \max_{q\in\mathcal{Q}} \sigma_q$. □

Proof.

First we show that matrix-valued function $M_{\zeta_q}(\cdot,\cdot)$ is positive-definite uniformly over $\rho \in [0, \tau_d^{-1}]$, $x_1 \in \mathbb{R}^n$, and $\tau \in [0, N_0]$. To this end, we decompose the matrix $M_{\zeta_q}(x_1, \tau)$ as follows:

$$M_{\zeta_q}(x_1,\tau) = U_q(x_1,\tau)W_q(\tau,x_1)U_q(x_1,\tau)^\top, \tag{66a}$$

$$W_q(\tau,x_1) := \begin{pmatrix} \frac{I}{\eta(\tau)^2} & 0 \\ 0 & \varrho_q(\tau)I - \eta^2(\tau)\Sigma_q(x_1)\Sigma_q(x_1)^\top \end{pmatrix}, \tag{66b}$$

$$\varrho_q(\tau) := \zeta_q - \rho\eta'(\tau), \quad \Sigma_q(x_1) := I - \partial\mathcal{G}_q(x_1), \tag{66c}$$

$$U_q(x_1,\tau) := \begin{pmatrix} I & 0 \\ \eta^2(\tau)\Sigma_q(x_1) & I \end{pmatrix}. \tag{66d}$$

By the fact that $\eta(\tau) \in [\underline{\eta}, \overline{\eta}]$ for all $\tau \in [0, N_0]$ it follows that

$$\frac{1}{\eta(\tau)^2}I \geq \frac{1}{\overline{\eta}^2}I. \tag{67}$$

Also, by Assumptions 4, we have that

$$\begin{aligned} \varrho_q(\tau)I - \eta(\tau)^2\Sigma_q(x_1)\Sigma_q(x_1) &\geq \left(\varrho_q(\tau) - \overline{\eta}^2\sigma_q^2\right)I \\ &\geq \left(\zeta_q - \frac{\overline{\eta}-\underline{\eta}}{\tau_d N_0} - \overline{\eta}^2\sigma_q^2\right)I \\ &\geq \tilde{\delta}I, \end{aligned} \tag{68}$$

where $\tilde{\delta} := (1-\delta)\underline{\zeta}$, with $\underline{\zeta} := \min_{q\in Q}\zeta_q$. Therefore, via [50, Theorem 7.7.7], the matrix $M_{\zeta_q}(x_1,\tau)$ is positive definite for all $x_1 \in \mathbb{R}^n$ and $\tau \in [0, N_0]$. Now, we establish the matrix inequality (65). To do so, we use (67) and (68) in (66a) to obtain that

$$\begin{aligned} M_{\zeta_q}(x_1,\tau) &\geq U_q(x_1,\tau)\begin{pmatrix} \frac{1}{\overline{\eta}^2}I & 0 \\ 0 & \tilde{\delta}I \end{pmatrix}U_q^\top(x_1,\tau) \\ &\geq Z_q(x_1,\tau)Z_q(x_1,\tau)^\top, \end{aligned} \tag{69}$$

where $Z_q(x_1,\tau)^\top$ is the upper block triangular matrix

$$Z_q(x_1,\tau)^\top := \begin{pmatrix} \frac{1}{\overline{\eta}}I & \frac{\eta(\tau)^2}{\overline{\eta}}\Sigma_q(x_1,\tau)^\top \\ 0 & \sqrt{\tilde{\delta}}I \end{pmatrix}.$$

By applying [51, Lemma 9], and using (69) together with the fact that $Z_q(x_1,\tau)$ has full column rank for all $x_1 \in \mathbb{R}^n$ and $\tau \in [0, N_0]$ and thus that $\sigma_{\min}(Z_q(x_1,\tau)Z_q(x_1,\tau)^\top) \geq \sigma_{\min}(Z_q(x_1,\tau))\sigma_{\min}(Z_q(x_1,\tau)^\top) = \sigma^2_{\min}(Z_q(x_1,\tau)^\top)$, we obtain

$$\begin{aligned} M_{\zeta_q}(x_1,\tau) &\geq \frac{1}{\overline{\eta}^2\left(1 + \frac{\overline{\eta}^2}{\tilde{\delta}}\|\Sigma_q(x_1)^2\|\right) + \frac{1}{\tilde{\delta}}}I \\ &\geq \frac{(1-\delta_d-\delta_\eta)\overline{\sigma}^2}{\delta_\eta(1-\delta_d)\underline{\zeta} + \overline{\sigma}^2}I, \end{aligned}$$

where in the last two steps we used Assumption 4. This completes the proof. ■

A.4. Lyapunov conditions for exponential-ISS of hybrid dynamical systems

The following lemma is a specialization of [33, Prop. 2.7] for the case when the system is exponentially ISS. We present the complete proof here only for the purpose of completeness.

Lemma 9. *Consider the HDS (3), and a closed set $\mathcal{A} \subset \mathbb{R}^m$. Suppose there exist constants $\underline{\alpha}, \overline{\alpha}, \rho, p > 0$, $\lambda \in (0,1)$, and a smooth function $V : C \cup D \to \mathbb{R}_{\geq 0}$, such that the following inequalities hold:*

$$\begin{aligned} \underline{\alpha}|z|^p_{\mathcal{A}} \leq V(z) \leq \overline{\alpha}|z|^p_{\mathcal{A}}, &\quad \forall\, z \in C \cup D \cup G(D), \\ \langle \nabla V(z), F(z,u)\rangle \leq -\lambda V(z) + \rho|u|^p, &\quad \forall\,(z,u) \in C \times \mathbb{R}^m, \\ V(G(z)) - V(z) \leq -\lambda V(z) + \rho|u|^p, &\quad \forall\,(z,u) \in D \times \mathbb{R}^m. \end{aligned}$$

Then, every solution of (3) satisfies

$$|z(s,j)|_{\mathcal{A}} \leq \kappa_1 e^{-\kappa_2(s+j)}|z(0,0)|_{\mathcal{A}} + \kappa_3 \sup_{0\leq\tau\leq s}|u(\tau)|, \tag{70}$$

for all $(s,j) \in dom(z)$, and where $\kappa_1 = \left(\overline{\alpha}/\underline{\alpha}\right)^p$, $\kappa_2 = \lambda/2p$, and $\kappa_3 = \left(\frac{2\rho}{\lambda\underline{\alpha}}\right)^{1/p}$. □

Proof. We follow similar ideas as in the proof of [33, Prop. 2.7], but considering set-valued flow and jump maps. The proof has four main steps:

Step 1: First, note that for all $(z,u) \in (C \cup D) \times \mathbb{R}^m$:

$$-\lambda V(z) + \rho|u|^p \leq -\frac{\lambda}{2}V(z), \quad \text{if } V(z) \geq \frac{2\rho}{\lambda}|u|^p. \tag{71}$$

Therefore, whenever $V(z) \geq \frac{2\rho}{\lambda}|u|^p$ we have that

$$\begin{aligned} &\langle \nabla V(z), F(z,u)\rangle \leq -\tilde{\lambda}V(z), \ \forall (z,u) \in C \times \mathbb{R}^m, \\ &V(G(z)) - V(z) \leq -\tilde{\lambda}V(z), \ \forall (z,u) \in D \times \mathbb{R}^m, \end{aligned}$$

where $\tilde{\lambda} := \lambda/2$.

Step 2: For any $r \geq 0$, define $\gamma_{c_4}(r,s,j) = e^{-\tilde{\lambda}(s+j)}r$. We first show that when $V(z) \geq \frac{2\rho}{\lambda}|u|^p$, the function V evaluated along the solutions of (3) satisfies

$$V(z(s,j)) \leq \gamma_\lambda(V(z(0,0)),s,j), \quad \forall\,(s,j) \in \mathrm{dom}(z). \tag{72}$$

To establish this property, note that since $V(z(\cdot,\cdot))$ is not increasing during flows and jumps, if there is $(s',j') \in \mathrm{dom}(z)$ with $0 < s'+j' < t+j$ and such that $V(z(s',j')) = 0$, then we necessarily must have $V(z(\tilde{s},\tilde{j})) = 0$ for all $(\tilde{s},\tilde{j}) \in \mathrm{dom}(z)$ such that $s'+j' \leq \tilde{s}+\tilde{j} \leq s+j$, and (72) would hold for such times $(\tilde{s},\tilde{j})$. Suppose there is no $(s',j') \in \mathrm{dom}(z)$ with $0 < s'+j' < t+j$ such that $V(z(s',j')) = 0$. For each $(s,j) \in \mathrm{dom}(z)$, we partition the hybrid time domain of z up to time (s,j) as $\mathrm{dom}(z) = \bigcup_{n=0}^{j}[s_n, s_{n+1}] \times \{n\}$, with $s_0 = 0$ and $s_{j+1} = s$. For any $n \in \{0,1,\dots,j\}$, V satisfies

$$\int_{s_n}^{s_{n+1}} \frac{\overbrace{V(z(\tau,n))}^{\cdot}}{\tilde{\lambda}V(z(\tau,n))}d\tau \leq -\int_{s_n}^{s_{n+1}} d\tau = -(s_{n+1}-s_n).$$

Using the new variable $\varrho = V(z(\tau,n))$, we obtain $d\varrho = \dot{V}d\tau$ and the above integral can be written as

$$\int_{V(z(s_n,n))}^{V(z(s_{n+1},n))} \frac{d\varrho}{\tilde{\lambda}\varrho} \leq -(s_{n+1}-s_n). \tag{73}$$

Similarly, note that

$$\begin{aligned} \int_{V(z(s_{n+1},n))}^{V(z(s_{n+1},n+1))} \frac{d\varrho}{\tilde{\lambda}\varrho} &\leq \int_{V(z(s_{n+1},n))}^{V(z(s_{n+1},n+1))} \frac{d\varrho}{\tilde{\lambda}V(z(s_{n+1},n))} \\ &\leq -1, \end{aligned}$$

where the last inequality follows by the inequality $V(z(s,j+1)) - V(z(s,j)) \leq -\tilde{\lambda}V(z(s,j))$. Combining the above two inequalities, we obtain

$$\begin{aligned} \int_{V(z(0,0))}^{V(z(s,j))} \frac{d\varrho}{\tilde{\lambda}\varrho} &= \sum_{n=0}^{j}\int_{V(z(s_n,n))}^{V(z(s_{n+1},n))} \frac{d\varrho}{\tilde{\lambda}\varrho} \\ &\quad + \sum_{n=1}^{j}\int_{V(z(s_{n+1},n))}^{V(z(s_{n+1},n+1))} \frac{d\varrho}{\tilde{\lambda}\varrho} \\ &\leq -\left(\sum_{n=0}^{j}(s_{n+1}-s_n) + \sum_{n=1}^{j}1\right) \\ &= -(s_{j+1} - s_0 + j) = -(s+j). \end{aligned} \tag{74}$$

Integrating the left-hand side, we obtain $\frac{1}{\tilde{\lambda}}\ln\left(\frac{V(z(s,j))}{V(z(0,0))}\right) \leq -(s+j)$, from which we directly get

$$V(z(s,j)) \leq V(z(0,0))e^{-\frac{\lambda}{2}(s+j)} \tag{75}$$

Step 3: Let (z,u) be a maximal solution pair of (3). Define the set

$$\Omega := \left\{ z \in \mathbb{R}^n : V(z) \leq \frac{2\rho}{\lambda}|u|_\infty^p \right\}. \tag{76}$$

For each $z_0 \in \mathbb{R}^n$, let

$$T_{z,u,z_0} := \sup\Big\{ \tau \in \mathbb{R}_{\geq 0} : z(s,j) \notin \Omega,\ z(0,0) = z_0,\ \forall\,(s,j) \in \mathrm{dom}(z),\ 0 \leq s+j \leq \tau \Big\}.$$

It follows that for all solutions of (3) with $z(0,0) = z_0$ and $(s,j) \in \mathrm{dom}(z)$ such that $0 \leq s+j < T_{z,u,z_0}$ we have that $V(z) > \frac{2\rho}{\lambda}|u|_\infty^p$, which, by Step 2, implies that V satisfies (75). Using the quadratic upper and lower bounds on V, we obtain:

$$|z(s,j)|_{\mathcal{A}} \leq \left(\frac{\overline{\alpha}}{\underline{\alpha}}\right)^{\frac{1}{p}} |z(0,0)|_{\mathcal{A}} e^{-\frac{\lambda}{2p}(s+j)}, \tag{77}$$

which holds for all $(s,j) \in \mathrm{dom}(z)$ such that $0 \leq s+j < T_{z,u,z_0}$.

Step 4: The last step is to prove forward invariance of Ω. Suppose there exist $(s',j') \in \mathrm{dom}(z)$ such that $z(s',j') \in \Omega$ and $(s',j'+1) \in \mathrm{dom}(z)$. Since $\tilde{\lambda} < \lambda$, V satisfies

$$\begin{aligned} V(z(s',j'+1)) &\leq (1-\tilde{\lambda})V(z(s',j')) + \rho|u|_\infty^p, \\ &\leq \left(1 - \frac{\lambda}{2}\right)\frac{2\rho}{\lambda}|u|_\infty^p + \rho|u|_\infty^p = \frac{2\rho}{\lambda}|u|_\infty^p. \end{aligned}$$

Moreover, if $(s',j'+1) \in \mathrm{dom}(z)$, then z cannot leave Ω via flows because $\dot{V} \leq 0$ if $V(z) \geq \frac{2\rho}{\lambda}|u|_\infty^p$. It follows that for all $(s,j) \in \mathrm{dom}(z)$ such that $s+j \geq T_{z,u,z_0}$ the solution z satisfies:

$$\underline{\alpha}|z(s,j)|_{\mathcal{A}}^p \leq V(z(s,j)) \leq \frac{2\rho}{\lambda}|u|_\infty^p, \tag{78}$$

that is, $|z(s,j)|_{\mathcal{A}} \leq \left(\frac{2\rho}{\lambda\underline{\alpha}}\right)^{\frac{1}{p}}|u|_\infty$, for all $s+j \geq T_{z,u,z_0}$. Combining this bound with (77) we obtain

$$|z(s,j)|_{\mathcal{A}} \leq \max\left\{ \left(\frac{\overline{\alpha}}{\underline{\alpha}}\right)^{\frac{1}{p}} |z(0,0)| e^{-\frac{\lambda}{2p}(s+j)}, \left(\frac{2\rho}{\lambda\underline{\alpha}}\right)^{\frac{1}{p}} |u|_\infty \right\}, \tag{79}$$

for all $(s,j) \in \mathrm{dom}(z)$. Since $\max\{a,b\} \leq a+b$, we obtain

$$|z(s,j)|_{\mathcal{A}} \leq \kappa_1 |z(0,0)| e^{-\kappa_2(s+j)} + \kappa_3|u|_\infty, \tag{80}$$

with $\kappa_1 = \left(\frac{\overline{\alpha}}{\underline{\alpha}}\right)^{\frac{1}{p}}$, $\kappa_2 = \frac{\lambda}{2p}$ and $\kappa_3 = \left(\frac{2\rho}{\lambda\underline{\alpha}}\right)^{\frac{1}{p}}$. The result follows from the above inequality by time-invariance and causality. ■

The following result relaxes the third condition in Lemma 9 under a standard average dwell-time condition on the jumps.

Lemma 10. *Consider the HDS (3), and suppose that: (a) every solution satisfies the ADT constraint (21); (b) there exist constants $\underline{\alpha}, \overline{\alpha}, \rho, p > 0$, $\lambda \in (0,1)$, and a smooth function $V : C \cup D \to \mathbb{R}_{\geq 0}$, such that the following inequalities hold:*

$$\begin{aligned} \underline{\alpha}|z|_{\mathcal{A}}^p \leq V(z) &\leq \overline{\alpha}|z|_{\mathcal{A}}^p, && \forall\, z \in C \cup D \cup G(D), \\ \langle \nabla V(z), F(z,u)\rangle &\leq -\lambda V(z) + \rho|u|^p, && \forall\,(z,u) \in C \times \mathbb{R}^m, \\ V(G(z)) - V(z) &\leq 0, && \forall\, z \in D. \end{aligned}$$

Then, every solution of (3) satisfies

$$|z(s,j)|_{\mathcal{A}} \leq \kappa_1 e^{-\kappa_2(s+j)}|z(0,0)|_{\mathcal{A}} + \kappa_3 \sup_{0\leq\tau\leq s}|u(\tau)|, \tag{81}$$

for all $(s,j) \in \mathrm{dom}(z)$, where $\kappa_i > 0$, for $i \in \{1,2,3\}$. □

Proof. The proof follows similar steps as the proof of Lemma 9. In particular, inequality (73) still holds. On the other hand, during jumps, we now have

$$V(z(s_{n+1},n+1)) - V(z(s_{n+1},n)) \leq 0 \tag{82}$$

Dividing both sides by $\tilde{\lambda}V(z(s_{n+1},n))$, we obtain

$$\begin{aligned} 0 &\geq \frac{V(z(s_{n+1},n+1)) - V(z(s_{n+1},n))}{\tilde{\lambda}V(z(s_{n+1},n))} \\ &= \int_{V(z(s_{n+1},n))}^{V(z(s_{n+1},n+1))} \frac{d\varrho}{\tilde{\lambda}V(z(s_{n+1},n))}. \end{aligned}$$

It follows that inequality (74) now becomes $\int_{V(z(0,0))}^{V(z(s,j))} \frac{d\varrho}{\tilde{\lambda}\varrho} \leq -s$, from which we obtain after integration:

$$V(z(s,j)) \leq V(z(0,0))e^{-\frac{\lambda}{2}s} \tag{83}$$

Finally, the ADT condition (27) guarantees that $j \leq \frac{1}{\tau_d}s + N_0$ for any $(s,j) \in \mathrm{dom}(\hat{z})$, which implies that $s+j \leq (\frac{1}{\tau_d}+1)s + N_0$. In turn, this inequality can be written as $s \geq \frac{\tau_d}{1+\tau_d}(s+j) - \frac{\tau_d}{1+\tau_d}N_0$, so that (83) can be upper-bounded as follows:

$$V(z(s,j)) \leq \kappa_7 e^{-\kappa_8(s+j)}V(z(0,0)), \tag{84}$$

where $\kappa_7 := e^{\frac{\lambda}{2}\frac{\tau_d}{1+\tau_d}N_0}$ and $\kappa_8 := \frac{\lambda}{2}\frac{\tau_d}{1+\tau_d}$. From here the proof follows the same Steps 3–4 from the proof of Lemma 9. In particular, the inequality (80) now becomes

$$|z(s,j)|_{\mathcal{A}} \leq \tilde{\kappa}_1|z(0,0)|e^{-\tilde{\kappa}_2(s+j)} + \tilde{\kappa}_3|u|_\infty,$$

with $\tilde{\kappa}_1 := \left(\frac{\overline{\alpha}}{\underline{\alpha}}\right)^{\frac{1}{p}} e^{\frac{\lambda}{2p}\frac{\tau_d}{1+\tau_d}N_0}$, $\tilde{\kappa}_2 := \frac{\lambda}{2p}\frac{\tau_d}{1+\tau_d}$, and $\kappa_3 = \left(\frac{2\rho}{\lambda\underline{\alpha}}\right)^{\frac{1}{p}}$. ■

Corollary 2. *Consider the normalized-by-μ_k BU-ODE of Lemma 3, $\frac{d\hat{\mu}_k}{ds} = \frac{k}{T}\hat{\mu}_k^{\frac{1}{k}}$. Then, for any $\ell > 0$ and any solution $\hat{\mu}_k$ to the ODE satisfying $\mu_k(0) = \mu_0 \geq 1$ the following bound holds:*

$$\mu_1^{-\ell}(s) \leq e^{-\ell\frac{s}{T}}, \quad \forall s \in \mathbb{R}_{\geq 1},$$

when $k = 1$, and

$$\mu_k^{-\ell}(s) \leq \left(\frac{k-1}{T}s + 1\right)^{-\ell\frac{k}{k-1}}, \quad \forall s \in \mathbb{R}_{\geq 1},$$

when $k \in \mathbb{Z}_{\geq 2}$. □

Proof. We divide the proof into two cases.

Case $k = 1$: From Lemma 3, for $k = 1$, the solution to the normalized-by-μ_k BU-ODE is given by:

$$\hat{\mu}_k(s) = \mu_0 e^{\frac{s}{T}}.$$

It follows that $\mu_k^{-\ell}(s) = \mu_0^{-\ell}e^{-\frac{\ell}{T}s} \leq e^{-\frac{\ell}{T}s}$ for all $s \geq 0$, where we have used the fact that $\mu_0^{-\ell} \leq 1$ since $\mu_0 \geq 1$ and $\ell > 0$ by assumption.

Case $k > 1$: From Lemma 3, for $k > 1$, the solution to the normalized-by-μ_k BU-ODE is given by:

$$\hat{\mu}_k(s) = \left(\frac{k-1}{T}s + \mu_0^{\frac{k-1}{k}}\right)^{\frac{k}{k-1}}.$$

Using the fact that $\mu_0 \geq 1$ and that $(\cdot)^{\frac{k}{k-1}}$ is monotonically increasing in $\mathbb{R}_{\geq 0}$ for any $k > 1$, and thus preserves the order in $\mathbb{R}_{\geq 0}$, it follows that $\hat{\mu}_k(s) \geq \left(\frac{k-1}{T}s + 1\right)^{\frac{k}{k-1}}$. Therefore, we obtain: $\hat{\mu}_k^{-\ell}(s) \leq \left(\frac{k-1}{T}s + 1\right)^{-\ell\frac{k}{k-1}}$ for all $s \geq 0$.

Lemma 11. *Suppose that every solution pair $(\hat{z},\hat{u})$ of the HDS (26) satisfies the bound (30) for all $(s,j) \in \mathrm{dom}(\hat{z})$. Assume that $\Delta(\hat{\mu}_k) = \hat{\mu}_k^{-\ell}$, where $\ell > 0$. Then, $(\hat{z},\hat{u})$ satisfies the inequality*

$$|\hat{z}(s,j)|_{\mathcal{A}} \leq \beta_k\left(\bar{\kappa}_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\bar{\kappa}_2(s+j)} + \bar{\kappa}_3|\hat{u}|_{(s,j)}, s\right),$$

for all $(s,j) \in \mathrm{dom}(\hat{z})$, where $\bar{\kappa}_1 := \kappa_1$, $\kappa_2 := \frac{\kappa_2}{2}$, $\bar{\kappa}_3 := 2\kappa_3$. Here $\beta_k(r,s) \in \mathcal{KL}$ is defined as $\beta_k(r,s) = r\cdot\max\{\kappa_1 e^{-\kappa_2 s},\ \xi_k^{-\ell}(s)\}$, $\xi_k(s) = \left(\frac{k-1}{T}s + 1\right)^{\frac{k}{k-1}}$ for all $k > 1$, and $\xi_1(s) = e^{\frac{s}{T}}$. □

Proof. Consider a complete solution pair $(\hat{z}, \hat{u})$ of the HDS (26) satisfying the bound (30). Then, we have that

$$|\hat{z}(s,j)|_{\mathcal{A}} \le \kappa_1 e^{-\kappa_2(s+j)}|\hat{z}(0,0)|_{\mathcal{A}} + \kappa_3 \cdot \sup_{0\le\zeta\le s}|\hat{\Delta}(\zeta)|, \quad (85)$$

for all $(s,j) \in \mathrm{dom}(\hat{z})$, and where $\hat{\Delta}(s) := \Delta(\mu_k^{-\ell}(s))\hat{u}(s)$. Next, pick an arbitrary time $(\bar{s}, \bar{j}) \in \mathrm{dom}(\hat{z})$, and let $\hat{y}(r,k) := \hat{z}(r+\bar{s}, k+\bar{j})$, and $v(r,k) := \mu_k^{-\ell}(\bar{s}+r)$. Since $\hat{y}$ is also a hybrid arc that is a solution to (26), using the above bound and by time-invariance, it satisfies:

$$\begin{aligned}|\hat{y}(r,k)|_{\mathcal{A}} &\le \kappa_1|\hat{y}(0,0)|e^{-\kappa_2(r+k)} + \kappa_3|\hat{u}|_{(r,k)}|v|_{(r,k)}\\ &= \kappa_1|\hat{z}(\bar{s},\bar{j})|e^{-\kappa_2(r+k)} + \kappa_3|\hat{u}|_{(r,k)}\sup_{0\le\tau\le r}\hat{\mu}^{-\ell}(\bar{s}+\tau)\\ &\le \kappa_1|\hat{z}(\bar{s},\bar{j})|e^{-\kappa_2(r+k)} + \kappa_3|\hat{u}|_{(r,k)}\hat{\mu}_k^{-\ell}(\bar{s}). \end{aligned}\quad (86)$$

Now, using (85) with $s=\bar{s}$ and $j=\bar{j}$, we obtain:

$$|\hat{z}(\bar{s},\bar{j})|_{\mathcal{A}} \le \kappa_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\kappa_2(\bar{s}+\bar{j})} + \kappa_3|\hat{u}|_{(\bar{s},\bar{j})}\sup_{0\le\tau\le r}\hat{\mu}_k^{-\ell}(\tau). \quad (87)$$

Combining (86) and (87), and using Remark 2, we have

$$\begin{aligned}|\hat{y}(r,k)|_{\mathcal{A}} \le \kappa_1\Big(&\kappa_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\kappa_2(\bar{s}+\bar{j})}\\ &+\kappa_3\sup_{0\le\tau\le r}|\hat{u}(\tau)|\sup_{0\le\tau\le r}\hat{\mu}_k^{-\ell}(\tau)\Big)e^{-\kappa_2(r+k)}\\ &+\kappa_3\sup_{0\le\tau\le r}|\hat{u}(\tau)|\hat{\mu}_k^{-\ell}(\bar{s}).\end{aligned}$$

Evaluating the above bound at $r=\bar{s}$ and $\tilde{j}\in\mathbb{Z}_{\ge0}$ such that $(\bar{s},\tilde{j})\in \mathrm{dom}(y)$, we obtain:

$$\begin{aligned}|\hat{y}(\bar{s},\tilde{j})|_{\mathcal{A}} &\le \kappa_1\Big(\kappa_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\kappa_2(\bar{s}+\bar{j})}\\ &\quad+\kappa_3\sup_{0\le\tau\le\bar{s}}|\hat{u}(\tau)|\sup_{0\le\tau\le\bar{s}}\hat{\mu}_k^{-\ell}(\tau)\Big)e^{-\kappa_2(\bar{s}+\tilde{j})}\\ &\quad+\kappa_3\sup_{0\le\tau\le\bar{s}}|\hat{u}(\tau)|\hat{\mu}_k^{-\ell}(\bar{s})\\ &\le \kappa_1\Big(\kappa_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\kappa_2(\bar{s}+\bar{j})}+\kappa_3\sup_{0\le\tau\le\bar{s}}|\hat{u}(\tau)|\Big)e^{-\kappa_2(\bar{s}+\tilde{j})}\\ &\quad+\kappa_3\sup_{0\le\tau\le\bar{s}}|\hat{u}(\tau)|\hat{\mu}_k^{-\ell}(\bar{s})\\ &\le\Big(\kappa_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\kappa_2(\bar{s}+\bar{j}+\tilde{j})}\\ &\quad+2\kappa_3\sup_{0\le\tau\le\bar{s}}|\hat{u}(\tau)|\Big)\max\left\{\kappa_1e^{-\kappa_2\bar{s}},\hat{\mu}_k^{-\ell}(\bar{s})\right\},\end{aligned}$$

where we used the fact that $e^{-\kappa_2\tilde{j}} \le 1$, and $\sup_{0\le\tau\le\bar{s}}\hat{\mu}_k^{-\ell}(\tau) \le \mu_0^{-\ell} \le 1$ since $\mu_0 \ge 1$ and $\ell > 0$. Using the result of Corollary 2 it then follows that

$$|\hat{y}(\bar{s},\tilde{j})|_{\mathcal{A}} \le \Big(\kappa_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\kappa_2(\bar{s}+\bar{j}+\tilde{j})}+2\kappa_3\sup_{0\le\tau\le\bar{s}}|\hat{u}(\tau)|\Big)\eta_k(\bar{s})$$

where $\eta_k(s) := \max\{\kappa_1e^{-\kappa_2 s}, \xi_k^{-\ell}(s)\}$, $\xi_k(s) = \left(\frac{k-1}{T}s+1\right)^{\frac{k}{k-1}}$ for all $k>1$ and $\xi_1(s) = e^{\frac{s}{T}}$. Note that η_k is continuous and satisfies $\eta_k(s)\to 0$ as $s\to\infty$ since $\kappa_1e^{-\kappa_2 s}\to 0$ and $\xi_k(s)\to 0$ as $s\to\infty$. Now, using the definition of $\hat{y}$, and letting $\lambda := 2\bar{s}$, $i := \bar{j}+\tilde{j}$:

$$|\hat{z}(\lambda,i)|_{\mathcal{A}} \le \Big(\kappa_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\kappa_2(\frac{\lambda}{2}+i)}+2\kappa_3\sup_{0\le\tau\le\bar{s}}|\hat{u}(\tau)|\Big)\eta_k(\lambda/2)$$

Since the choice of $(\bar{s},\bar{j})\in\mathrm{dom}(z)$ was arbitrary, z is complete, and the previous inequality holds for all $\tilde{j}\in\mathbb{Z}_{\ge0}$, in particular we can use $s=2\bar{s}$, $j=\bar{j}$, and $\tilde{j}=0$ such that $(s,j)\in\mathrm{dom}(z)$. Thus, from the above inequality and using Remark 2, we obtain that there exists $\beta_k(r,s) := r\cdot\eta_k(s)\in\mathcal{KL}$ such that

$$|\hat{z}(s,j)|_{\mathcal{A}} \le \beta_k\Big(\bar{\kappa}_1|\hat{z}(0,0)|_{\mathcal{A}}e^{-\bar{\kappa}_2(s+j)}+\bar{\kappa}_3|\hat{u}|_{(s,j)}, s\Big),$$

with $\bar{\kappa}_1 := \kappa_1$, $\kappa_2 := \frac{\kappa_2}{2}$, $\bar{\kappa}_3 := 2\kappa_3$. ■

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