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On Hybrid Prescribed-Time Concurrent Learning with Switching Datasets ⋆

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Abstract: We introduce a novel concurrent learning (CL) algorithm designed to solve parameter estimation problems within a user-prescribed time frame and by utilizing alternating datasets during the learning process. The algorithm can tackle applications involving switching data sets (including data sets that are completely uninformative) that are updated in real-time as the algorithm operates. To achieve parameter estimation within a specified time independent of the dataset's richness, the switching algorithm employs dynamic gains. The main result establishes uniform global exponential ultimate boundedness, with an ultimate bound that shrinks to zero as the magnitude of the measurement disturbances decreases. The stability analysis leverages tools from hybrid dynamical systems theory, along with a recently introduced dilation/contraction argument on the hybrid time domains of the solutions. The algorithm and main results are illustrated via a numerical example.

Keywords: Hybrid dynamical systems, Concurrent Learning, System Identification.

1. INTRODUCTION

Concurrent Learning (CL) is a data-driven framework that is suitable for the design of estimation and learning dynamics in a variety of applications where persistence of excitation (PE) conditions are not feasible Kamalapurkar et al. (2017). Examples include parameter estimation problems in batteries Ochoa et al. (2021), exoskeleton robotic systems Casas et al. (2023), extremum seeking Poveda et al. (2021), excavating robots Greene et al. (2021), and reinforcement learning Ochoa and Poveda (2022); Chowdhary and Johnson (2010). In these applications, datasets containing past *recorded* measurements of the relevant signals within the systems are typically available for estimation purposes. When these datasets are considered to be "sufficiently rich", they can be integrated into dynamic estimation algorithms to achieve (uniform) exponential convergence to the unknown parameters in the absence of PE. While these techniques have been recently enhanced via non-smooth tools to achieve finite-time and fixed-time convergence results Ochoa et al. (2021); Ríos et al. (2017); Tatari et al. (2023), most CL techniques suffer from two main limitations that are common in practice, see (Ochoa et al., 2021, Sec. 4),Casas et al. (2023): First, the rate of convergence is directly related to the "level of richness" of the dataset used by the algorithm. This is true in exponentially convergent CL algorithms Chowdhary and Johnson (2010), as well as in finite-time and fixed-time convergence approaches Ochoa et al. (2021); Ríos et al. (2017); Tatari et al. (2023). Second, in many practical applications, CL algorithms do not use a single dataset during the learning process, but rather multiple datasets obtained at different time instants and exhibiting different levels of informativity. Incorporating the dynamic nature of these datasets into the convergence analysis of CL algorithms is thus an essential task to inject confidence in the applicability of CL techniques in practical settings.

In this paper, we introduce a novel CL architecture that aims to simultaneously tackle the above two challenges. The proposed algorithm has two main features: First, instead of relying on momentum Ochoa et al. (2021), non-Lipschitz vector fields Ochoa et al. (2021); Tatari et al. (2023), or resetting techniques to improve transient performance Le and Teel (2022), it makes use of dynamic "blow-up" gains to achieve convergence to the true unknown parameter in a time that is independent of the initial estimate and of the level of richness of the dataset. In this way, the convergence time can be *fully assignable by the user*. The proposed technique is inspired by recent advances in prescribed-time (PT) adaptive control and regulation Orlov (2022); Song et al. (2023), which, however, have remained mostly unexplored in the setting of CL. Yet, unlike standard PT results considered in the literature of continuous-time systems, we incorporate switching datasets into the PT-CL algorithm, thus enabling the use of multiple datasets during the learning process. This is achieved using a *prescribed-time data-querying hybrid automaton* that generates switching signals satisfying suitable "blow-up" average dwell-time and average activation-time constraints, recently introduced in Ochoa et al. (2023) for a more general class of hybrid systems. In the proposed Switching Prescribed-Time Concurrent Learning (SPT-CL) algorithm, only *some* of the datasets are assumed to be sufficiently rich, and at any given time only *one* dataset can be used by the algorithm. This setting is common in reinforcement learning in the context of mini-batch optimization Stapor et al. (2022), but it has remained unexplored in the context of CL.

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Our main result, presented in Theorem 1, is derived using tools from the hybrid dynamical system's literature Goebel et al. (2012), and a dilation/contraction argument on the hybrid time domains of the solutions, recently introduced in Ochoa et al. (2023) for the analysis of PT-convergence in hybrid systems. To the best of our knowledge, the proposed scheme is the first CL algorithm that achieves PT convergence using switching datasets.

The rest of this paper is organized as follows. In Section 2 we present the notation and preliminaries on hybrid dynamical systems. Section 3 presents the main Prescrubed-Time CL algorithm and the main result. Section 4 presents the analysis and the proofs. Section 5 presents a numerical example, and Section 6 ends with the conclusions.

2. PRELIMINARIES

Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, we use $|x|_{\mathcal{A}} := \inf_{s\in\mathcal{A}} \|x - s\|_2$ to denote the distance from x to $\mathcal{A}$. To simplify notation, for two (or more) vectors $u, v \in \mathbb{R}^n$, we write $(u, v) = [u^\top, v^\top]^\top$ to denote their concatenation. Also, given a set $\mathcal{O} \subset \mathbb{R}^n$, we use $\mathbb{I}_{\mathcal{O}}(\cdot)$ to denote the indicator function, which satisfies $\mathbb{I}_{\mathcal{O}}(x) = 1$ if $x \in \mathcal{O}$, and $\mathbb{I}_{\mathcal{O}}(x) = 0$ if $x \notin \mathcal{O}$. In this paper, we will work with hybrid dynamical systems (HDS) aligned with Goebel et al. (2012), given by the inclusions:

$$x \in C, \qquad \dot{x} \in F(x), \tag{1a}$$

$$x \in D, \quad x^+ \in G(x), \tag{1b}$$

where $x \in \mathbb{R}^n$ is the main state, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called the flow map, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called the jump map, $C \subset \mathbb{R}^n$ is called the flow set, and $D \subset \mathbb{R}^n$ is called the jump set. The data of (1) is represented as $\mathcal{H} = (C, F, D, G)$. Solutions to system (1) are parameterized by a continuous-time index $t \in \mathbb{R}_{\geq 0}$, which increases continuously during flows, and a discrete-time index $j \in \mathbb{Z}_{\geq 0}$, which increases by one during jumps. Therefore, solutions to (1) are defined on *hybrid time domains* (HTDs). A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact* HTD if $E = \cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \ldots \leq t_J$. The set E is a HTD if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, \ldots, J\})$ is a compact HTD. For definitions of solutions to HDS, their connections to HTDs, and details on models of the form (1) we refer the reader to (Goebel et al., 2012, Chapter 2).

The following lemma, instrumental for our results, follows by direct integration.

Lemma 1. Consider the following "Blow-Up" ordinary differential equation (BU-ODE): $\dot{\mu} = \frac{1}{\Upsilon}\mu^2, \quad \mu \in \mathbb{R}_{\geq 1}$ where $\Upsilon > 0$. For each $\mu(0) =: \mu_0 \in \mathbb{R}_{\geq 1}$, the unique solution to the BU-ODE is given by:

$$\mu(t) = \frac{\Upsilon \mu_0}{\Upsilon - t \cdot \mu_0}, \quad \forall\, t \in [0, T_{\mu_0}), \;\; T_{\mu_0} := \frac{\Upsilon}{\mu_0}, \tag{2}$$

where $t \mapsto \mu(t)$ is continuous in its domain, strictly increasing, and satisfies $\lim_{t\to\frac{\Upsilon}{\mu_0}} \mu(t) = \infty$. □

3. PT-CONCURRENT LEARNING WITH SWITCHING DATASETS

Consider a parameter estimation problem where the goal is to estimate $\theta^\star \in \mathbb{R}^n$ using real-time and past recorded measurements of the scalar signal

$$\psi(\theta^\star, t) = \phi(t)^\top \theta^\star + d(t), \tag{3}$$

where $d : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is an unknown, bounded, and Lebesgue measurable disturbance function, and $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a known *regressor* that is assumed to be continuous and uniformly bounded.

When the regressor ϕ satisfies a persistence of excitation (PE) condition, the typical approach to estimate θ relies on a "standard" recursive least-squares method Narendra and Annaswamy (1987). On the other hand, when ϕ does not satisfy a PE condition, but the practitioner has access to "sufficiently rich" past recorded data of ϕ, concurrent learning (CL) can be used to estimate the true parameter θ^* recursively. In general, depending on their transient performance, there are three types of CL algorithms:

(1) Classic exponentially stable CL algorithms, introduced in Chowdhary and Johnson (2010), and which achieve exponential rates of convergence proportional to the level of richness of the data. This approach extends the classic gradient-based parameter estimation algorithm using a "data-driven" or "batch-based" term that incorporates the recorded data. Other recent approaches that use initial excitation conditions (IEC) Basu Roy and Bhasin (2019) and integral CL achieve similar properties Le et al. (2022).
(2) High-order exponentially stable CL algorithms, studied in Ochoa et al. (2021); Le and Teel (2022), which incorporate momentum and sometimes resets to improve transient performance. Such algorithms can achieve exponential rates of convergence proportional to the square root of the level of richness of the data, which is advantageous whenever the level of richness of the recorded data is "low".
(3) Fixed-time stable CL algorithms (which generalized finite-time convergence methods), studied in Ríos et al. (2017); Ochoa et al. (2021) and Tatari et al. (2023), which use non-smooth dynamics to achieve exact convergence to the true parameter before a fixed-time that is independent of the initial conditions of the algorithm, but inversely proportional to the level of richness of the data.

In this section, we introduce a novel CL architecture that complements the existing methods by offering a new property that has not been studied before in the context of CL: Prescribed-Time stability Song et al. (2023). Such property induces convergence to the true parameter as time approaches a prescribed time that is *independent* of the richness of the data used by the algorithm. We show that such property can be achieved even when the algorithm uses, during bounded periods of time, data sets that are completely uninformative. In practice, this situation emerges when the recorded data is persistently updated during the implementation of the CL algorithm, leading to different batches of available data for the algorithm. Since the practitioner cannot anticipate whether or not the new data is "sufficiently rich", establishing stability guarantees under sporadic implementations of uninformative data can inject confidence in the implementation of the techniques.

3.1 The PT-Estimation Problem

To estimate $\theta^\star$, we consider the algorithm shown in Figure 1, where $\theta \in \mathbb{R}^n$ is the estimate of the parameter; $q \in \mathbb{Z}_{\geq 1}$ is a logic switching state kept constant between switchings

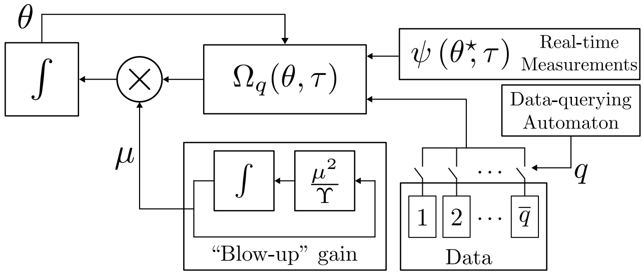

Fig. 1. Block diagram of the proposed Switching Prescribed-Time Concurrent Learning algorithm.

of the algorithm; $\mu \in \mathbb{R}$ is a dynamic gain, and $\tau \in \mathbb{R}_{\geq 0}$ is used to model any explicit dependence on the time t. Between switching times, the states (θ, μ, q, τ) evolve according to the following continuous-time dynamics:

$$\dot{\theta} = \mu \cdot \Omega_q(\theta, \tau), \quad \dot{\mu} = \frac{1}{\Upsilon}\mu^2, \quad \dot{q} = 0, \quad \dot{\tau} = 1, \tag{4}$$

where $\Upsilon > 0$ is a tunable gain, and Ω_q is a mode-dependent mapping to be characterized. By Lemma 1, in this dynamics the state μ will always exhibit finite escape times at the time T_{μ_0}. In our algorithm, this is a desirable feature since, by Lemma 1, these finite escape times are "controlled" by Υ and $\mu(0)$. In practice, the algorithm is usually stopped before T_{μ_0}, inducing a small residual error. Thus, borrowing similar terminology used in the PT-literature Orlov (2022); Song et al. (2023), we will refer to μ as a *blow-up* gain, and to T_{μ_0} as the *prescribed time* (PT). In the literature on PT regulation for ODEs Orlov (2022), μ_0 is typically set to 1. However, for generality, in this paper, we express our results in terms of $\mu_0 \in \mathbb{R}_{\geq 1}$ to characterize how the initial conditions of μ impact the performance of the CL dynamics.

To define the mapping Ω_q in (4), we let $\chi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be given by

$$\chi(\theta, t) := \phi(t) \cdot \left(\phi(t)^\top \theta - \psi(\theta^\star, t)\right), \tag{5}$$

where ϕ and ψ come from (3). Note that $\chi(\theta, t)$ is a signal available to the practitioner at any time t. Indeed, in the context of concurrent learning, it is typically assumed that (ϕ, χ) can be measured in real-time, and that there exists a sequence of *recorded* measurements of $(\phi(\cdot), \chi(\theta, \cdot))$, taken at times $\{t_k\}$, $k \in \{1, 2, \ldots, \bar{k}\}$, that is available to the algorithm. We are interested in CL applications where the recorded data can be segmented among multiple datasets, some of which might be "uniformative".

Remark 1. In this paper, the use of the term "recorded data" should be considered in a more broad sense compared to traditional CL. In particular, since we will assume that the algorithm switches between multiple data sets $\mathcal{D}_q$, a given data set implemented in the future after $T > 0$ seconds could be obtained after recording the data during an initial window of time $[0, T]$. This situation cannot occur in standard CL, where all recorded data must be obtained before running the algorithm. However, to make our analysis tractable, we will impose *a priori* assumptions on the data sets $\mathcal{D}_q$, including situations where such data sets are uninformative. □

To formalize the model of the PT-Estimation algorithm, we associate each dataset with a different value of the state $q \in \mathcal{Q} := \{1, 2, \ldots, \overline{q}\}$, with $\overline{q} \in \mathbb{Z}_{\geq 1}$. Therefore, we use $\mathcal{Q}$ to represent the set of indices of the multiple datasets, each dataset of the form $\mathcal{D}_q := \{\phi(t_{q,k}), \psi(\theta^\star, t_{q,k})\}_{k=1}^{\bar{k}_q}$, where $\bar{k}_q \in \mathbb{Z}_{\geq 1}$, and $\{t_{q,k}\}_{k=1}^{\bar{k}_q}$ is a collection of times. The following definition, which is standard in the CL literature Chowdhary and Johnson (2010), formalizes the informativity properties of these collections.

Definition 1. A dataset $\mathcal{D}_q$ is said to be *sufficiently rich* (SR) if there exists $\alpha_q > 0$ such that

$$\sum_{k=1}^{\bar{k}_q} \phi(t_{q,k})\phi(t_{q,k})^\top \succeq \alpha_q I, \tag{6}$$

where α_q is called the "level of richness" of $\mathcal{D}_q$ (a measure of informativity). If (6) is not satisfied, $\mathcal{D}_q$ is said to be *insufficiently rich* (IR). □

To differentiate the informative datasets from the uninformative ones, we partition $\mathcal{Q}$ as follows:

$$\mathcal{Q} = \mathcal{Q}_s \cup \mathcal{Q}_i, \quad \mathcal{Q}_s \cap \mathcal{Q}_i = \emptyset, \quad \mathcal{Q}_s, \mathcal{Q}_s, \subset \mathbb{Z}_{\geq 1},$$

where the modes in $\mathcal{Q}_s$ correspond to datasets that are SR, while the modes in $\mathcal{Q}_i$ are IR. Additionally, we let:

$$\Omega_q(\theta, \tau) := -k_{\mathrm{t}}\chi(\theta, \tau) - k_{\mathrm{r}} \sum_{k=1}^{\bar{k}_q} \chi(\theta, t_{q,k}), \tag{7}$$

where $k_{\mathrm{t}}, k_{\mathrm{r}} > 0$ are tunable weights. System (4) with Ω_q describes the continuous-time dynamics of the PT algorithm.

To guarantee that (4) can solve the parameter estimation problem we need to impose suitable conditions on the switching signals q. Namely, our goal is to characterize how frequently the PT-algorithm can switch between data sets while preserving stability and convergence. To answer this question we introduce a PT-"Data-Querying" automaton that will characterize the family of switching signals that lead to stable closed-loop systems.

3.2 The PT-"Data-Querying" Automaton

To model the switching dynamics of q, while simultaneously considering the goal of achieving prescribed-time (PT) convergence to the true parameter $\theta^\star$, we consider a PT-data-querying hybrid automaton that can be seen as an extension of the standard hybrid automaton that induces average dwell-time constraints on the switching signals Cai et al. (2008). The data-querying automaton can be modeled as a hybrid dynamical system of the form (1), with state $y = (q, \rho_d, \rho_a, \mu) \in \mathbb{Z}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 1}$ and set-valued dynamics:

$$y \in C_\sigma, \qquad \dot{y} \in \mu \cdot F_\sigma(y), \tag{8a}$$

$$y \in D_\sigma, \qquad y^+ \in G_\sigma(y), \tag{8b}$$

where the elements in (8a) are given by

$$C_\sigma := \mathcal{Q} \times [0, N_0] \times [0, T_0] \times \mathbb{R}_{\geq 1} \tag{9a}$$

$$F_\sigma(y) := \{0\} \times \left[0, \frac{1}{\tau_d}\right] \times \left(\left[0, \frac{1}{\tau_a}\right] - \mathbb{I}_{\mathcal{Q}_i}(q)\right) \times \{0\} \tag{9b}$$

and the elements of (8b) are given by

$$D_\sigma := \mathcal{Q} \times [1, N_0] \times [0, T_0] \times \mathbb{R}_{\geq 1} \tag{9c}$$

$$G_\sigma(y) := \mathcal{Q} \setminus \{q\} \times \{\rho_d - 1\} \times \{\rho_a\} \times \{\mu\}, \tag{9d}$$

where $T_0 > 0$, $N_0 \geq 1$, $\tau_d > 0$, and $\tau_a > 1$ are tunable parameters.

When $\mu \equiv 1$, HDS of the form (8) are common in the study of asymptotic stability properties of switching systems under average dwell-time and average activation time constraints on the switching signals, see for example Poveda and Teel (2017); Liu et al. (2022). Yet, incorporating the *dynamic* gain μ in the continuous-time dynamics generates switching signals that differ from traditional average dwell-time signals. For example, as $t \to T_{\mu_0}$, system (8) allows for signals that switch at a faster rate.

The following Lemma provides some useful properties of the solutions to (8). The proof follows as a particular case of (Ochoa et al., 2023, Lemma 6) which covers a more general class of hybrid systems, and can be found in the extended manuscript Ochoa and Poveda (2024).

Lemma 2. Let y be a maximal solution to the HDS (8), with $\mu(0,0) = \mu_0$. Then, the following holds:

(a) The total amount of flow time is bounded by T_{μ_0}, i.e.,
$$\sup\{t \geq 0 : \exists\, j \in \mathbb{Z}_{\geq 0}, (t,j) \in \mathrm{dom}(y)\} \leq T_{\mu_0}. \qquad (10)$$
(b) For all $(t_1, j_1), (t_2, j_2) \in \mathrm{dom}(y)$ such that $t_2 \geq t_1$, the following "Blow-Up" Average Dwell-Time (BU-ADT) condition is satisfied:
$$j_2 - j_1 \leq \frac{\Upsilon}{\tau_d} \ln\left(\frac{\Upsilon - t_1\mu_0}{\Upsilon - t_2\mu_0}\right) + N_0, \qquad (11)$$
(c) For all $(t_1, j_1), (t_2, j_2) \in \mathrm{dom}(y)$ such that $t_2 \geq t_1$, the following "Blow-Up" Average Activation Time (BU-AAT) condition is satisfied:
$$\int_{t_1}^{t_2} \frac{\mathbb{I}_{\mathcal{Q}_i}(q(t,\underline{j}(t)))}{\Upsilon - t\mu_0} dt \leq \frac{1}{\tau_a \mu_0} \ln\left(\frac{\Upsilon - t_1\mu_0}{\Upsilon - t_2\mu_0}\right) + T_0, \qquad (12)$$
where $\underline{j}(t) = \min\{j \in \mathbb{Z}_{\geq 1} : (t,j) \in \mathrm{dom}(y)\}$.
(d) For every hybrid arc $q : \mathrm{dom}(q) \to \mathcal{Q}$ satisfying (11) and (12) there exists a maximal solution y to the HDS (8) having the same hybrid time domain. □

Remark 2. Item (a) of Lemma 2 states that every solution to the HDS (8) can flow for at most T_{μ_0} amount of time. Such time domains are typical in the context of PT- control Orlov (2022); Todorovski and Krstić (2023); Song et al. (2023), where the behaviors of the algorithms are mostly of interest during an initial finite window of time. □

Remark 3. In (8), the state ρ_d acts as a timer that regulates how frequently q switches between modes. Since in (8a) the flow map is multiplied by μ, the rate of change of ρ_d during flows is allowed to increase as $t \to T_{\mu_0}$. This leads to the BU-ADT condition (11), where the right hand side of the inequality grows to infinity as $t \to T_{\mu_0}$, thus allowing for more frequent switching as $t \to T_{\mu_0}$. □

Remark 4. The state ρ_a in (8) can be seen as a timer that regulates the amount of time spent by q in the set $\mathcal{Q}_i$ compared to the time spent on Q_s. By construction, any maximal solution to this timer will induce the BU-AAT bound (8), thus limiting the proportion of time that the system spends in the uninformative modes. □

Remark 5. In the context of PT-Stable algorithms, implementations are carried out by stopping the algorithm "slightly" before the prescribed time to avoid any singularity. We refer the reader to the recent works Orlov (2022); Todorovski and Krstić (2023); Song et al. (2023) for a comprehensive discussion on practical applications and implementation strategies. □

Remark 6. We stress that the main goal of the Data-Querying automaton (8) is to provide a mathematical model to capture a family of switching signals under which the dynamics (4) achieve PT-convergence to the true parameter, i.e., it is used mainly for the purpose of analysis. Such signals might be *unknown* to the practitioner who, even though might control the switching times, is aware a priori of the level of richness of the next data set $\mathcal{D}_{q^+}$. Again, this situation is common in practical applications where the data used by the CL algorithm is "updated" after a finite amount of time. Our hybrid model aims to capture such types of heuristics, which, to the best of our knowledge, have not been rigorously studied before, let alone in the context of PT stability. □

3.3 Closed-Loop System and Main Result

By leveraging Lemma 2, we study the stability properties of the complete dynamics that describe the Switching Prescribed-Time Concurrent Learning (SPT-CL) algorithm shown in Figure 1.

The continuous-time dynamics of the overall system evolve in the flow set: $C := \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times C_\sigma$ via the following set-valued flow map
$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ \dot{\tau} \\ \dot{y} \end{pmatrix} \in F(x) := \begin{pmatrix} \mu \cdot \Omega_q(\theta, \tau) \\ 1 \\ \mu \cdot F_\sigma(y) \end{pmatrix}. \qquad (13a)$$
The discrete-time dynamics evolve in the following jump set: $D := \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times D_\sigma$, via the set-valued jump map
$$x^+ = \begin{pmatrix} \theta^+ \\ \tau^+ \\ y^+ \end{pmatrix} \in G(x) := \begin{pmatrix} \theta \\ \tau \\ G_\sigma(y) \end{pmatrix}. \qquad (13b)$$
We can now state the main result of this paper, which provides stability guarantees for the PT-CL algorithm with switching datasets. Its proof, which can be found in the extended manuscript Ochoa and Poveda (2024), relies upon a Lyapunov-based approach and leverages the results of Lemma 2.

Theorem 1. Consider the parameter estimation problem characterized by (3), and suppose that:

(a) The set $\mathcal{Q}_s$ is not empty.
(b) There exists $\delta \geq 0$ such that $|d(t)| \leq \delta$ for all $t \geq 0$.
(c) The BU-ADT and BU-AAT parameters satisfy $\tau_d > 0$ and $\tau_a > 1 + \frac{1}{k_r \underline{\alpha}}$, where $\underline{\alpha} = \min_{q \in \mathcal{Q}_s} \alpha_q$.

Then, there exist $\kappa_1, \kappa_2 > 0$ such that for every solution $x = (\theta, \tau, y)$ to (13), the estimate θ satisfies:
$$|\theta(t,j) - \theta^\star| \leq \kappa_1 \left(1 - \frac{t}{T_{\mu_0}}\right)^{\kappa_2 \Upsilon} |\vartheta_0|\, e^{-\kappa_2 j} + \gamma(\delta), \quad (14)$$
for all $(t,j) \in \mathrm{dom}(x)$, where $\vartheta_0 = \theta(0,0) - \theta^\star$ and γ is a positive definite function. □

Remark 7. Bound (14) in Theorem 1 establishes a prescribed time ultimate boundedness result with respect to the true parameter $\theta^\star$ for the PT-CL algorithm with switching datasets. Since $1 - \frac{t}{T_{\mu_0}} \to 0^+$ as $t \to T_{\mu_0}$, the convergence to the ultimate bound $\gamma(\delta)$ is achieved in the prescribed-time T_{μ_0} assigned by the user via the choice of $\Upsilon > 0$ and $\mu_0 \geq 1$ in (10). To the authors' best knowledge, this is the first PT-result in the context of CL, let alone

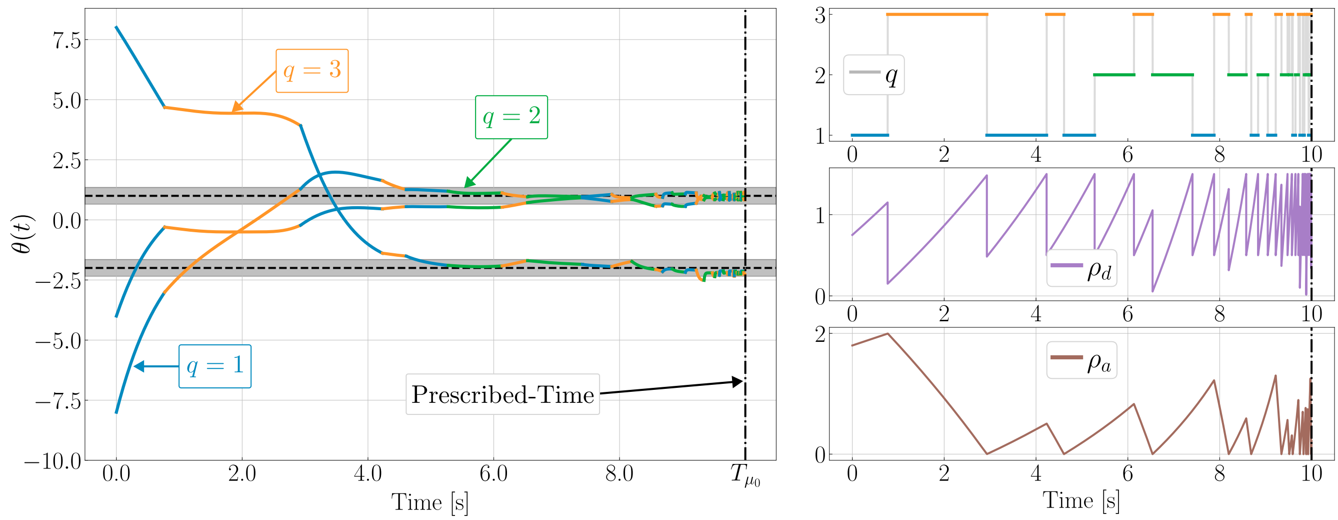

Fig. 2. Switching Prescribed-Time Concurrent Learning with informative data $\mathcal{Q}_s = \{1, 2\}$, uninformative data $\mathcal{Q}_i = \{3\}$, prescribed time $T_{\mu_0} = 10$, $N_0 = 1.5$, $T_0 = 2$, $\tau_d = 1$, $\tau_a = 2$, and true parameter $\theta^\star = (1, -2, 1)$.

with switching datasets containing informative and uninformative data. In particular, Theorem 1 can be seen as the PT (and switching) counterpart of the recent *fixed-time* stability results for CL algorithms established in Ochoa et al. (2021); Ríos et al. (2017) and Tatari et al. (2023). When $\delta = 0$, (14) establishes exact PT convergence. □

Remark 8. Since by Lemma 2 every solution to (13) satisfies the BU-ADT and the BU-AAT conditions, the result of Theorem 1 can also be interpreted as a PT-convergence result for a switching system having a switching signal $q : \mathbb{R}_{\geq 0} \to \mathcal{Q}$ that satisfies (11) and (12), where the left hand side of (11) corresponds to the total number of switches of q in the interval $[t_1, t_2]$. □

Remark 9. Theorem 1 can also be interpreted as a robustness result concerning uninformative data sets. Namely, if the PT-algorithm implements uninformative data sets $\mathcal{D}_q$ during periods of time that satisfy the Blow-Up Average Activation Time condition (12), then PT convergence to the true parameter will still be achieved. □

Remark 10. As discussed in Orlov (2022); Todorovski and Krstić (2023); Song et al. (2023), one of the limitations of PT techniques that rely on dynamic gains is the need for "stopping" the algorithm before the prescribed time. The bound (14) provides some information on the residual bounds one can expect to obtain in such scenarios. □

Remark 11. (On the use of PT-Algorithms in Estimation Problems) Prescribed-Time stable algorithms were initially developed for regulation problems where the magnitude of the regulation error can be used as a "signal" to stop the PT routine. The lack of access to such error signals in parameter estimation problems is not problematic for the successful experimental implementation of PT-estimation dynamics, as shown in Pan et al. (2022) for energy systems (e.g., batteries), the numerical study of Schrodinger systems Steeves et al. (2020), PT-estimation problems in quadrotor UAVS Gong et al. (2023), etc. In most scenarios, the proximity to the prescribed-time T_{μ_0} is used as a signal to clip the algorithm action. □

We finish this section by discussing the case when only one (SR) dataset is available for the estimation of $\theta^\star$, the following corollary can be directly obtained. This result follows by simply taking the states (q, ρ_d, ρ_a) as fixed constants in the HDS (13).

Corollary 2. Consider the parameter estimation problem characterized by (3), suppose that items (a)-(b) of Theorem 1 hold and that $\mathcal{Q} = \{1\}$. Then, there exist $\kappa_1, \kappa_2 > 0$ such that for every solution $x = (\theta, \tau, y)$ to (13), the estimate θ satisfies (14) for all $(t, j) \in \text{dom}(x)$. □

4. NUMERICAL EXAMPLE

We numerically illustrate Theorem 1, by considering the scalar-valued signal: $\psi(\theta^\star, \tau) = (\sin(\tau) - 1)^2 + d(\tau)$, with $d(\tau) = \frac{1}{4}\tanh(\tau)$. In this case, ψ takes the form of (3) with $\theta^\star = (1, -2, 1)$, and $\phi(\tau) = (1, \sin(\tau), \sin(\tau)^2)$. Consider three datasets $\{\mathcal{D}_q\}_{q=1}^3$ recorded at the sequences of times: $\{t_{1,k}\}_{k=1}^{\bar{k}_1} = \{0, -\frac{\pi}{2}, -\frac{3\pi}{2}\}$, $\{t_{2,k}\}_{k=1}^{\bar{k}_2} = \{0, -\frac{\pi}{4}, -\frac{7\pi}{4}\}$, and $\{t_{3,k}\}_{k=1}^{\bar{k}_3} = \{0, -\pi, -2\pi\}$. The datasets $\mathcal{D}_1$ and $\mathcal{D}_2$ are SR with levels of richness $\alpha_1 = 0.44$ and $\alpha_2 = 0.15$, respectively, while $\mathcal{D}_3$ is IR. Using these datasets, we implement the PT-CL dynamics with $T_{\mu_0} = 10$. To generate a switching signal q that satisfies the BU-ADT and BU-AAT conditions (11) and (12) we use the PT-hybrid automaton (8) with $T_0 = 1$ and $N_0 = 2$, $\tau_d = 1$ and $\tau_a = 2$. With this choice of parameters, all the assumptions of Theorem 1 are satisfied. Figure 2 displays the trajectories of each of the components of the state $\theta \in \mathbb{R}^3$, the resulting switching sequence q, and the associated average dwell-time and average activation time states ρ_d and ρ_a. In gray, we show the uniform ultimate bound $\gamma(\delta)$. As shown in the figure, the state θ rapidly approaches a neighborhood of $\theta^\star$ as t approaches the prescribed-time T_{μ_0}, even when the algorithm repeatedly selects the IR dataset $\mathcal{D}_3$. This result illustrates a common scenario that emerges in many practical applications of data-enabled adaptive dynamics, where the dataset used by the CL algorithm is dynamically changed as the algorithm evolves in time. On the other hand, the behavior shown in Figure 2 can also be interpreted as a robustness result of prescribed-time CL with respect to intermittent and uninformative dynamic data, and/or switching regressors.

5. CONCLUSIONS

The main contribution of this paper is to present a novel CL algorithm that achieves *prescribed-time* (PT) convergence under switching datasets, thus offering another potential alternative that contributes to the catalog of existing CL approaches. However, unlike existing results, it was shown that the PT convergence properties are retained even when the algorithm uses recorded data that is not sufficiently rich during bounded periods. Our convergence analysis relies on using tools from set-valued hybrid dynamical systems to model the switching signals as solutions of an autonomous system (with finite escape times). The analysis also leverages a dilation/contraction transformation on the hybrid time domains of the solutions, which, to the best of our knowledge has not been used before in the context of switching parameter estimation algorithms. Future research directions will focus on experimental implementations that will test the validity of the theoretical results presented in this paper under different realistic scenarios, e.g., noisy sensors, limited computation, saturated actuators, etc.

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