MODULAR CURVES AND NÉRON MODELS OF GENERALIZED JACOBIANS

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To the memory of Bas Edixhoven

Abstract. Let X be a smooth geometrically connected projective curve over the field of fractions of a discrete valuation ring R, and $\mathfrak{m}$ a modulus on X, given by a closed subscheme of X which is geometrically reduced. The generalized Jacobian $J_\mathfrak{m}$ of X with respect to $\mathfrak{m}$ is then an extension of the Jacobian of X by a torus. We describe its Néron model, together with the character and component groups of the special fibre, in terms of a regular model of X over R. This generalizes Raynaud's well-known description for the usual Jacobian. We also give some computations for generalized Jacobians of modular curves $X_0(N)$ with moduli supported on the cusps.

arXiv:2207.13203v2 [math.AG] 28 Sep 2023

Introduction. Let R be a discrete valuation ring, with field of fractions F and residue field k. Let $\mathcal{X}$ be a regular scheme, proper and flat over $S = \operatorname{Spec} R$, whose generic fibre $X = \mathcal{X}_F$ is a smooth curve. In [29] Raynaud describes the relationship between the Néron model of the Jacobian $J = \operatorname{Pic}^0_{X/F}$ of X and the relative Picard functor $P = \operatorname{Pic}_{\mathcal{X}/S}$. The aim of this paper is twofold: first, to extend Raynaud's results to the generalized Jacobian $J_\mathfrak{m}$ of X with respect to a reduced modulus $\mathfrak{m}$. Secondly, to apply these results to compute the component and character groups of the Néron models of generalized Jacobians attached to modular curves and moduli supported on cusps.

Our motivation for this work arises from applications to the arithmetic of modular forms — the point being that just as the arithmetic of cusp forms of weight 2 on a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ is controlled by the Jacobian of the associated complete modular curve, so the arithmetic of the space of holomorphic modular forms on the same group is controlled by a suitable generalized Jacobian. Raynaud's results have been used extensively to study the arithmetic of cusp forms of weight 2 and their associated Galois representations — for example, in [22,24,30,31]. In future work we plan to give arithmetic applications of the results obtained here. We note that generalized modular Jacobians with cuspidal modulus are considered in Gross [13], Yamazaki and Yang [37], Bruinier and Li [5], Wei and Yamazaki [36], and Iranzo [15]. Another point of view, using 1-motives rather than generalized Jacobians (see also Section 1.7 below), has been investigated by Lecouturier [18].

Before describing our main results, we briefly recall from [29] the results of Raynaud on Jacobians. To simplify the discussion, we assume for the rest of this introduction that R is Henselian, k is algebraically closed, and that the greatest common divisor of the multiplicities of the irreducible components of the fibre $\mathcal{X}_s$ at the closed point $s = \operatorname{Spec} k$ is 1. (We review Raynaud's theory in §2.2–3 below in greater detail and under less restrictive hypotheses.) Under these hypotheses, [29, (8.2.1)] shows that P is represented by a smooth group scheme over S, and there is a canonical morphism of group schemes

2020 *Mathematics Subject Classification.* Primary 14K30; Secondary 11G18, 14G35.

Key words and phrases. Néron models, generalized Jacobians, modular curves, cuspidal modulus.

deg: $P \to \mathbb{Z}$, which maps a line bundle to its total degree along the fibres of $\mathcal{X}/S$. The open and closed subgroup scheme $P' = \ker(\deg)$ then has J as its generic fibre.

Let r be the number of irreducible components of $\mathcal{X}_s$. If $r > 1$ then P is not separated over S. Indeed, if $Y \subset \mathcal{X}_s^{\mathrm{red}}$ is an irreducible component, viewed as a reduced divisor on $\mathcal{X}$, then the line bundle $\mathcal{O}_{\mathcal{X}}(Y)$ represents an element of $P'(S)$, nonzero if $r > 1$, whose image in $P'(F)$ vanishes. The closure $E \subset P'$ of the zero section is then an étale (but not separated) S-group scheme, whose generic fibre is trivial, and whose special fibre is isomorphic to $\mathbb{Z}^{r-1}$, generated by the classes of the bundles $\mathcal{O}_{\mathcal{X}}(Y)$ restricted to $\mathcal{X}_s$. Raynaud shows:

i) The maximal separated quotient P'/E is the Néron model $\mathcal{J}$ of J.
ii) The identity component $\mathcal{J}_s^0$ of the special fibre of $\mathcal{J}$ is canonically isomorphic to the Picard scheme $\mathrm{Pic}^0_{\mathcal{X}_s/k}$.
iii) Let $\mathcal{J}_s^{0,\mathrm{lin}}$ be the maximal connected affine subgroup scheme of $\mathcal{J}_s^0$. Its character group $\mathbb{X}(J_s) := \mathrm{Hom}_k(\mathcal{J}_s^{0,\mathrm{lin}}, \mathbb{G}_\mathrm{m})$ is canonically isomorphic to $H_1(\widetilde{\Gamma}_{\mathcal{X}_s}, \mathbb{Z})$, the integral homology of the extended dual graph $\widetilde{\Gamma}_{\mathcal{X}_s}$ of the singular curve $\mathcal{X}_s$ (we recall the definition in §1.2 below).
iv) The component group $\Phi(J) := \mathcal{J}_s/\mathcal{J}_s^0$ is canonically isomorphic to the homology of the complex

$$\mathbb{Z}[C] \to \mathbb{Z}^C \to \mathbb{Z}$$

where C is the set of irreducible components $Y \subset \mathcal{X}_s^{\mathrm{red}}$, the first map is given by the intersection pairing $C \times C \to \mathbb{Z}$ on $\mathcal{X}$, and the second by $(m_Y)_Y \mapsto \sum_Y \delta_Y m_Y$, where δ_Y is the multiplicity of Y in the fibre.

In the special case where $\mathcal{X}_s$ is a reduced divisor on $\mathcal{X}$ with normal crossings, iii) and iv) become:

iii′) $\mathrm{Hom}_k(\mathcal{J}_s^{0,\mathrm{lin}}, \mathbb{G}_\mathrm{m}) \simeq H_1(\Gamma_{\mathcal{X}_s}, \mathbb{Z})$, where $\Gamma_{\mathcal{X}_s}$ is the reduced dual graph of $\mathcal{X}_s$, whose vertex set is C and edge set is $\mathcal{X}_s^{\mathrm{sing}}$.
iv′) $\Phi(J) \simeq \mathrm{coker}\,(\Box\colon \mathbb{Z}[C] \to \mathbb{Z}[C]_0)$, where $\Box = \Box_0$ is the 0-Laplacian (as in [16]) of the graph $\Gamma_{\mathcal{X}_s}$, which is the endomorphism of $\mathbb{Z}[C]$ taking a vertex $v \in C$ to $\sum (v) - (v')$, where the sum is taken over all edges joining v to an adjacent vertex v'.

Now let $\mathfrak{m}$ be a modulus (effective divisor) on X. Then one has [32, 33] the generalized Jacobian $J_\mathfrak{m}$ of X relative to $\mathfrak{m}$, which is an extension of J by a commutative connected linear group H. Assume that $\mathfrak{m} = \sum_{i \in I}(x_i)$ is a sum of distinct points, whose residue fields F_i are all separable over F. This is equivalent to assuming that H is a torus. Then by the results of Raynaud [3, Chapter 10], $J_\mathfrak{m}$ has a Néron model $\mathcal{J}_\mathfrak{m}$, which is a smooth separated group scheme over S, not necessarily of finite type, with generic fibre $J_\mathfrak{m}$ and satisfying the Néron universal property. (In the terminology of [3], $\mathcal{J}_\mathfrak{m}$ is an lft-Néron model.) We obtain results analogous to (i)–(iv′) for $\mathcal{J}_\mathfrak{m}$. Specifically, let R_i be the integral closure of R in F_i, and Σ_s be the disjoint union of the $\mathrm{Spec}(R_i \otimes_R k)$, $i \in I$. The inclusion of the set of points x_i in X gives a morphism $\Sigma_s \to \mathcal{X}_s$. We show:

i) There exists a smooth S-group scheme $P_\mathfrak{m}$, parametrizing equivalence classes of line bundles on $\mathcal{X}$ with a trivialisation at each x_i, and $\mathcal{J}_\mathfrak{m}$ is the maximal separated quotient of $P'_\mathfrak{m} = \ker\,(\deg\colon P_\mathfrak{m} \to \mathbb{Z})$ (Theorems 1.15 and 1.16).
ii) The identity component $\mathcal{J}^0_{\mathfrak{m},s}$ of the special fibre of $\mathcal{J}_\mathfrak{m}$ is canonically isomorphic to $\mathrm{Pic}^0_{(\mathcal{X}_s,\Sigma_s)/k}$, the generalized Picard scheme classifying line bundles on $\mathcal{X}_s$ of degree zero on each irreducible component, together with a trivialisation of the pullback to Σ_s (Corollary 1.18(a)).

iii) The character group $\mathrm{Hom}_k(\mathcal{J}^{0,\mathrm{lin}}_{\mathfrak{m},s}, \mathbb{G}_\mathrm{m})$ is the integral homology of an extended graph $\widetilde{\Gamma}_{\mathcal{X}_s,\Sigma}$, depending only on the combinatorics of the components of $\mathcal{X}_s$ and the reductions $\bar{x}_i \in \mathcal{X}_s$ of the points x_i (Corollary 1.18(a)).

iv) The component group $\Phi(J_\mathfrak{m}) = \mathcal{J}_{\mathfrak{m},s}/\mathcal{J}^0_{\mathfrak{m}.s}$, which is an abelian group of finite type (not necessarily finite), is isomorphic to the homology of the complex (1.6.4)

$$\mathbb{Z}[C] \oplus \mathbb{Z} \to \mathbb{Z}^C \oplus \mathbb{Z}^I \to \mathbb{Z}$$

(Theorem 1.19).

If $\mathcal{X}_s$ is a reduced divisor with normal crossings, and the points x_i are F-rational, then the character and component groups have simple descriptions in terms of the homology and Laplacian of a generalized reduced dual graph (Corollary 1.20).

We then apply these results to a modular curve $X_0(N)$ and a modulus $\mathfrak{m}$ supported on the cusps. If $p > 3$ is a prime exactly dividing N, we compute the character and component groups, together with the action of the Hecke operators on them. In particular, if $N = p$ and $\mathfrak{m} = (\infty) + (0)$ is the sum of the two cusps of $X_0(p)$, then the component group is infinite cyclic, with T_ℓ acting by $\ell + 1$ for $\ell \neq p$, and the representation of the full Hecke algebra on the character group is given by the classical Brandt matrices. We also compute the component group for $N = p^2$, which for the full cuspidal modulus is free of rank 2.

There has been considerable interest in "Jacobians of graphs" — for example, Lorenzini [20, 21], Bacher–de la Harpe–Nagnibeda [1] and Baker–Norine [2]. Our results here on $\Phi(J_\mathfrak{m})$ suggest that there is also a theory of "generalized Jacobians of graphs". We investigate this in the paper [16].

Let us briefly describe the contents of the rest of the paper. In Section 1, we prove our results on Néron models of generalized Jacobians. Although not needed for the applications we have in mind, we decided to work in a very general setting (in particular, there are no conditions imposed on the base discrete valuation ring). Sections 1.1, 1.2 and 1.3 review well-known facts about Néron models, Weil restriction, and Picard schemes of singular curves, as well as some of Raynaud's results from [29].

In §1.4 and 1.5 we describe the structure of the generalized Picard scheme of a singular curve with respect to a modulus, and discuss its functoriality. The main results on the Néron models of generalized Jacobian are contained in §1.6. In the following two sections we explain the relation with 1-motives, and describe some of the behaviour of the Néron model of $J_\mathfrak{m}$ under correspondences.

In Section 2 we apply our results to the modular curves $X_0(N)$ and cuspidal moduli, computing in several cases the component and characters groups of the reduction of the Néron model modulo a prime $p > 3$.

We describe some prior work on these topics. If the points (x_i) are F-rational and their closures in $\mathcal{X}$ are disjoint, then by identifying them, one obtains a singular relative curve $\mathcal{X}/\mathfrak{m}$ which is semifactorial. Some of our results in this case are then subsumed by the works [25, 26, 28] on Picard schemes of semifactorial curves. In [27], Overkamp proves general results on the existence of Néron models of Picard schemes of singular curves. Finally, Suzuki [35] has defined Néron models of 1-motives and studied their duality properties and component groups. We discuss its relation with the present work in Section 1.7.

Notation. Throughout the paper, unless otherwise stated, R will denote a discrete valuation ring with field of fractions F, uniformiser ϖ, and residue field k. Except where stated otherwise, we make no further hypotheses on R or k. We write $p = \max(1, \mathrm{char}(k))$

for the characteristic exponent of k. We put $S = \operatorname{Spec} R$ and denote by s its closed point. Let R^{sh} be a strict henselisation of R, and F^{sh} its field of fractions. Write k^{sep} for the residue field of R^{sh} (a separable closure of k), and $\bar{s}$ for its spectrum. We write (Sm/S) for the category of essentially smooth S-schemes, and $(Sm/S)_{\text{ét}}$ for its étale site. For a scheme X, we write $\kappa(x)$ for the residue field at a point $x \in X$, and if X is irreducible, $\kappa(X)$ for the residue field of the generic point of X. All group schemes considered in this paper will be commutative. We frequently identify étale group schemes over a field with their associated Galois modules.

If S is a finite set we write $\mathbb{Z}[S]$ for the free abelian group on S and $\mathbb{Z}[S]_0$ for the kernel of the degree map $\mathbb{Z}[S] \to \mathbb{Z}$, $s \mapsto 1$ for $s \in S$.

1. Néron models of generalized Jacobians

1.1. **Preliminaries.** In this section we collect together properties of Néron models and Weil restriction of scalars. Most of these may be found in [3], especially Chapter 10.

Recall that if G/F is a smooth group scheme of finite type, then a Néron model for G is a smooth separated group scheme $\mathcal{G}/S$ with generic fibre G, such that for every smooth S-scheme S', the canonical map $\mathcal{G}(S') \to \mathcal{G}(S'_F) = G(S'_F)$ is bijective. If $\mathcal{G}$ exists, it is unique up to unique isomorphism. (In [3] these are called Néron lft-models.) The identity component $\mathcal{G}^0$ of $\mathcal{G}$ is a smooth group scheme of finite type. The formation of Néron models commutes with strict henselisation and completion of the base ring R. If $G \otimes_F \widehat{F}^{\mathrm{sh}}$ does not contain a copy of $\mathbb{G}_{\mathrm{a}}$ then G has a Néron model [3, 10.2 Thm.2]. (More generally, this holds if S is merely a semilocal Dedekind scheme.) We write $\Phi(G)$ for the component group $(\mathcal{G}_s/\mathcal{G}_s^0)(k^{\mathrm{sep}})$. If k is perfect, then by Chevalley's Theorem [7] $\mathcal{G}_s$ has a unique maximal connected affine smooth subgroup scheme $\mathcal{G}_s^{0,\mathrm{lin}}$, and we then write $\mathbb{X}(G)$ for the character group $\operatorname{Hom}(\mathcal{G}_s^{0,\mathrm{lin}} \otimes_k \bar{k}, \mathbb{G}_{\mathrm{m}})$, a finite free $\mathbb{Z}$-module with a continuous action of $\operatorname{Gal}(\bar{k}/k)$.

Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be an exact sequence of smooth connected F-groups which have Néron models $\mathcal{G}_i$. Consider the complexes

$$0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0 \tag{1.1.1}$$

$$0 \to \mathcal{G}_1^0 \to \mathcal{G}_2^0 \to \mathcal{G}_3^0 \to 0 \tag{1.1.2}$$

$$0 \to \Phi(G_1) \to \Phi(G_2) \to \Phi(G_3) \to 0 \tag{1.1.3}$$

The following two exactness results are a restatement of [6, Remark (4.8)(a)], with the same proof, which we give for the reader's convenience.

Lemma 1.1. *Suppose that the induced map $\mathcal{G}_2 \to \mathcal{G}_3$ is a surjection of sheaves for the smooth topology. Then:*

(a) *The sequence* (1.1.1) *is exact.*
(b) *If $\Phi(G_1)$ is torsion-free, then the sequences* (1.1.2) *and* (1.1.3) *are exact.*

Proof. (a) Since locally for the smooth topology the morphism $\mathcal{G}_2 \to \mathcal{G}_3$ of group schemes has a section, it is evidently surjective. Let $\mathcal{G}'$ denote its kernel. By [19, Lemma 4.3(b)], $\mathcal{G}'$ is smooth. The canonical morphism $\mathcal{G}_1 \to \mathcal{G}_2$ factors though a morphism $\gamma\colon \mathcal{G}_1 \to \mathcal{G}'$ which is the identity on generic fibres, and since $\mathcal{G}_1$ is a Néron model, there is a morphism $\delta\colon \mathcal{G}' \to \mathcal{G}_1$ which is the identity on generic fibres. As $\mathcal{G}_1$ and $\mathcal{G}'$ are separated over S, γ and δ are mutually inverse isomorphisms.

(b) The map $\mathcal{G}_2^0 \to \mathcal{G}_3^0$ is surjective, so we have an exact sequence

$$0 \to \mathcal{G}_1 \cap \mathcal{G}_2^0 \to \mathcal{G}_2^0 \to \mathcal{G}_3^0 \to 0$$

in which each term is of finite type over S. Hence $\mathcal{G}^0_{1,s}$ has finite index in $\mathcal{G}_{1,s} \cap \mathcal{G}^0_{2,s}$, and since $\Phi(G_1)$ is torsion-free we have $\mathcal{G}_1 \cap \mathcal{G}^0_2 = \mathcal{G}^0_1$. So (1.1.2) and therefore also (1.1.3) are exact. □

Corollary 1.2. *Suppose that G_1 is a product of tori of the form $\mathcal{R}_{F'/F}T$, where F'/F is finite separable, T is an F'-torus which splits over an unramified extension, and $\mathcal{R}_{F'/F}$ is Weil restriction of scalars. Then* (1.1.1), (1.1.2) *and* (1.1.3) *are exact.*

Proof. Replacing R by R^{sh}, we may assume that each T/F' is split. According to [4, 4.2], [6, (4.5)], one then has $R^1 j_{\mathrm{sm}\,*} G_1 = 0$, where $j_{\mathrm{sm}}\colon (\operatorname{Spec} F)_{\mathrm{sm}} \to S_{\mathrm{sm}}$ is the inclusion of small smooth sites. Therefore $\mathcal{G}_2 \to \mathcal{G}_3$ is surjective as a map of sheaves on S_{sm} . By Proposition 1.4(a) below, $\Phi(G_1)$ is torsion-free, so everything follows from the lemma. □

We will need the following minor generalization of a result from [3].

Proposition 1.3. *Let*

$$0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$$

be an exact sequence of smooth S-group schemes. If $\mathcal{G}_1$ and $\mathcal{G}_3$ are the Néron models of their generic fibres, the same is true for $\mathcal{G}_2$.

Proof. This follows by the same argument as in the proof of §7.5, Proposition 1(b) in [3] (middle of p.185), using the criterion of §10.1, Proposition 2. □

From [3, §7.6] we recall basic properties of Weil restriction. Let Z'/Z be a finite flat morphism of finite presentation. If Y is a quasiprojective Z'-scheme, then the Weil restriction $\mathcal{R}_{Z'/Z}Y$ exists, and is characterised by its functor of points $\mathcal{R}_{Z'/Z}Y(-) = Y(- \times_Z Z')$. If Y is smooth over Z' then $\mathcal{R}_{Z'/Z}Y$ is smooth over Z. If $Y \to X$ is a closed immersion of quasiprojective Z'-schemes, then $\mathcal{R}_{Z'/Z}Y \to \mathcal{R}_{Z'/Z}X$ is a closed immersion. If $Z' \to Z$ is surjective and Y is a quasiprojective Z-scheme, then the canonical map $Y \to \mathcal{R}_{Z'/Z}(Y \times_Z Z')$ is a closed immersion.

Now let k be a field, k' a finite k-algebra, and k'' a finite flat k'-algebra. Let Y be a quasiprojective k-scheme. There is then a canonical map

$$g\colon \mathcal{R}_{k'/k}(Y \otimes_k k') \to \mathcal{R}_{k''/k}(Y \otimes_k k'').$$

We may write $k' = k'_1 \times k'_2$ where $\operatorname{Spec} k'_1 \subset \operatorname{Spec} k'$ is the image of $\operatorname{Spec} k''$ (and k'_2 is possibly zero). The morphism g then factors

$$\mathcal{R}_{k'/k}(Y \otimes_k k') = \mathcal{R}_{k'_1/k}(Y \otimes_k k'_1) \times_{\operatorname{Spec} k} \mathcal{R}_{k'_2/k}(Y \otimes_k k'_2) \xrightarrow{\mathrm{pr}_1} \mathcal{R}_{k'_1/k}(Y \otimes_k k'_1) \to \mathcal{R}_{k'_1/k}\mathcal{R}_{k''/k'_1}(Y \otimes_k k'') = \mathcal{R}_{k''/k}(Y \otimes_k k'')$$

and the second arrow is a closed immersion. In particular, if Y is a smooth k-group, then g is a surjection onto a closed subgroup scheme, and its cokernel is smooth.

Let k be a field and k' a finite k-algebra. Then $\mathcal{R}_{k'/k}\mathbb{G}_{\mathrm{m}}$ is a connected smooth k-group scheme of finite type. It is a torus if and only if k'/k is étale.

We return to Néron models. Recall that the multiplicative group $\mathbb{G}_{\mathrm{m}}/F$ has a Néron model $\mathcal{G}_{\mathrm{m}}/S$, whose special fibre is $\mathbb{G}_{\mathrm{m}} \times \mathbb{Z}$. It fits into an exact sequence of group schemes

$$0 \to \mathbb{G}_{\mathrm{m}} \to \mathcal{G}_{\mathrm{m}} \xrightarrow{v_F} s_*\mathbb{Z} \to 0$$

where on R-points v_F is the normalised valuation $v_F\colon \mathcal{G}_{\mathrm{m}}(R) = F^\times \twoheadrightarrow \mathbb{Z}$.

Let F' be a finite étale F-algebra, $R' \subset F'$ the normalisation of R in F', $S' = \operatorname{Spec} R'$. Let $F' \otimes_F F^{\mathrm{sh}} = \prod_{i \in I} F_i$, where the fields F_i are totally ramified extensions of F^{sh}, of degrees $e_i p^{s_i}$, where p^{s_i} is the degree of the (purely inseparable) residue class extension.

Proposition 1.4.

(a) *The Néron model of $\mathcal{R}_{F'/F}\mathbb{G}_\mathrm{m}$ is $\mathcal{R}_{S'/S}\mathcal{G}_\mathrm{m}$, and the product of the valuations*

$$\mathcal{R}_{S'/S}\mathcal{G}_\mathrm{m}(F^\mathrm{sh}) = \prod_{i\in I} F_i^\times \xrightarrow{(v_{F_i})} \mathbb{Z}^I \tag{1.1.4}$$

induces an isomorphism

$$\Phi(\mathcal{R}_{F'/F}\mathbb{G}_\mathrm{m}) = \pi_0((\mathcal{R}_{S'/S}\mathcal{G}_\mathrm{m})_s) \xrightarrow{\sim} \mathbb{Z}^I. \tag{1.1.5}$$

(b) *The adjunction map $\mathcal{G}_\mathrm{m} \to \mathcal{R}_{S'/S}\mathcal{G}_\mathrm{m}$ is a closed immersion, and its cokernel is the Néron model of $(\mathcal{R}_{F'/F}\mathbb{G}_\mathrm{m})/\mathbb{G}_\mathrm{m}$, inducing a isomorphism*

$$\Phi((\mathcal{R}_{F'/F}\mathbb{G}_\mathrm{m})/\mathbb{G}_\mathrm{m}) \xrightarrow{\sim} \mathrm{coker}(e = (e_i)\colon \mathbb{Z} \to \mathbb{Z}^I).$$

Here we use $\mathcal{G}_\mathrm{m}$ to denote also the Néron model of $\mathbb{G}_\mathrm{m}$ over the semilocal base S'.

Proof. (a) The first statement follows from [3], Propositions 10.1/4 and 6. For the second, replacing F by F^sh we are reduced to the case of a totally ramified field extension F'/F. Then as $\mathcal{G}_{\mathrm{m},s} \simeq \mathbb{G}_{\mathrm{m},s}\times\mathbb{Z}$, we have $(\mathcal{R}_{S'/S}\mathcal{G}_\mathrm{m})_s = \mathcal{R}_{R'\otimes k/k}\mathbb{G}_\mathrm{m}\times\mathcal{R}_{R'\otimes k/k}\mathbb{Z}$. As the first factor is connected, and the second is $\mathbb{Z}$ (since $R'\otimes k/k$ is radicial) we get $\Phi(\mathcal{R}_{F'/F}\mathbb{G}_\mathrm{m}) \simeq \mathbb{Z}$, and the fact that this isomorphism is given by the valuation follows from [3, 1.1/Proposition 7].

(b) By Corollary 1.2, the exact sequence $0 \to \mathbb{G}_\mathrm{m} \to \mathcal{R}_{F'/F}\mathbb{G}_\mathrm{m} \to (\mathcal{R}_{F'/F}\mathbb{G}_\mathrm{m})/\mathbb{G}_\mathrm{m} \to 0$ gives rise to exact sequences of Néron models and component groups. So it is enough to show that the map $\Phi(\mathbb{G}_\mathrm{m}) = \mathbb{Z} \to \Phi(\mathcal{R}_{F'/F}\mathbb{G}_\mathrm{m}) = \mathbb{Z}^I$ is equal to e. Replacing F by F^sh again, we are reduced to the case when F'/F is a totally ramified field extension of degree ep^s with residue degree p^s. Then by (a) we have a commutative square

$$\begin{array}{ccc} \mathcal{G}_\mathrm{m} & \hookrightarrow & \mathcal{R}_{R'/R}\mathcal{G}_\mathrm{m} \\ \downarrow{\scriptstyle v_F} & & \downarrow{\scriptstyle v_{F'}} \\ \mathbb{Z} & \xrightarrow{e} & \mathbb{Z} \end{array}$$

proving the result. □

1.2. **Graphs and Picard schemes of singular curves.** In this section we work over an arbitrary field k. By a curve over k we shall mean a k-scheme X of finite type which is equidimensional of dimension 1 and Cohen-Macaulay (i.e., has no embedded points). Let $\{X_j\}$ be the irreducible components of X, and η_j the generic point of X_j. The local ring $\mathcal{O}_{X,\eta_j}$ is Artinian, and following Raynaud [29, (6.1.1) and (8.1.1)] we write d_j for its length, and δ_j for the total multiplicity of X_j in X. If k'/k is a radicial closure of k, and $\eta'_j \in X \otimes k'$ is the point lying over η_j, then δ_j equals the length of the local ring of η'_j. Moreover $\delta_j = d_j[\kappa(\eta_j)\cap k' : k] = d_jp^{n_j}$ for some $n_j \geq 0$.

Until the end of this section, k denotes an algebraically closed field. We review the well-known description of the toric part of the Picard scheme of a singular curve over k.

Let Y/k be a reduced proper curve, and $Y^\mathrm{sing} \subset Y(k)$ its set of singular points. Write $\phi\colon \widetilde{Y} \to Y$ for its normalisation. Define sets

$$A = Y^\mathrm{sing} \subset Y(k), \quad B = \phi^{-1}(Y^\mathrm{sing}) \subset \tilde{Y}(k), \quad C = \pi_0(\widetilde{Y}).$$

We have maps

$$\phi\colon B \to A, \quad \psi\colon B \to C$$

where ψ maps $x \in B$ to the connected component of $\widetilde{Y}$ containing it.

The *extended graph* $\widetilde{\Gamma}_Y = (\widetilde{V}, \widetilde{E})$ of Y is the graph with vertices $\widetilde{V}$ and edges $\widetilde{E}$, where

- $\widetilde{V} = A \sqcup C$, $\widetilde{E} = B$.

- The endpoints of an edge $b \in B$ are $\phi(b) \in A$ and $\psi(b) \in C$.

The graph $\widetilde{\Gamma}_Y$ is bipartite, and therefore has a canonical structure of directed graph, by directing the edge b so that its source is $\phi(b)$.

Suppose Y only has double points (meaning that if $y \in Y^{\mathrm{sing}}$ then $\phi^{-1}(y)$ has exactly 2 elements). The *reduced graph* $\Gamma_Y = (V, E)$ is the undirected graph (possibly with multiple edges and loops) whose vertex set is $V = \pi_0(\widetilde{Y})$ and edge set is $E = Y^{\mathrm{sing}}$. It is obtained from $\tilde{\Gamma}_Y$ by, for each vertex $v \in A$, deleting v and replacing the two edges incident to v with a single edge. There is a canonical homeomorphism between the geometric realisations of $\widetilde{\Gamma}_Y$ and Γ_Y, under which $v \in A$ is mapped to the midpoint of the replacing edge. If Y/k is a proper curve, not necessarily reduced, we define $\widetilde{\Gamma}_Y = \widetilde{\Gamma}_{Y^{\mathrm{red}}}$, $\Gamma_Y = \Gamma_{Y^{\mathrm{red}}}$, where $Y^{\mathrm{red}} \subset Y$ is the reduced subscheme.

Let $G = \mathrm{Pic}^0_Y$ be the identity component of the Picard scheme of Y. It is a smooth group scheme of finite type over k, classifying line bundles on Y whose restriction to each irreducible component has degree zero. The filtration of G by its linear and unipotent subgroups is described as follows.

Let $Y' \to Y$ be the "seminormalisation" of Y, which is obtained from Y by replacing its singularities with singularities which are étale locally isomorphic to the union of coordinate axes in $\mathbb{A}^N_k$. The normalisation map factors into a pair of finite morphisms $\widetilde{Y} \xrightarrow{\phi'} Y' \to Y$. These give rise to a commutative diagram, whose rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^{\mathrm{unip}} & \longrightarrow & G = \mathrm{Pic}^0_Y & \longrightarrow & \mathrm{Pic}^0_{Y'} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\| & & \big\downarrow{\scriptstyle \phi'^*} & & \\ 0 & \longrightarrow & G^{\mathrm{lin}} & \longrightarrow & G & \longrightarrow & \mathrm{Pic}^0_{\widetilde{Y}} & \longrightarrow & 0 \end{array}$$

(where G^{unip} is the maximal connected unipotent subgroup of G) giving an isomorphism $\ker \phi'^* \simeq G^{\mathrm{tor}} = G^{\mathrm{lin}}/G^{\mathrm{unip}}$ by the snake lemma.

To give a line bundle on Y' is equivalent to giving a line bundle on $\widetilde{Y}$ together with descent data for $\phi' \colon \widetilde{Y} \to Y'$, so the toric part G^{tor} classifies trivial line bundles on $\widetilde{Y}$ equipped with descent data to Y'. For the trivial bundle $\mathcal{O}_{\widetilde{Y}}$, to give such descent data is equivalent to giving, for each singular point $y \in Y$, an element of $(k^\times)^{\phi^{-1}(y)}/k^\times$. The automorphism group of $\mathcal{O}_{\widetilde{Y}}$ is $(k^\times)^{\pi_0(\widetilde{Y})}$. Hence G^{tor} is canonically

$$\mathbb{G}_{\mathrm{m}}^{\pi_0(\widetilde{Y})} \backslash \prod_{y \in Y^{\mathrm{sing}}} \Big(\mathbb{G}_{\mathrm{m}}^{\phi^{-1}(y)} / \mathrm{diag}(\mathbb{G}_{\mathrm{m}}) \Big).$$

Here $\mathbb{G}_{\mathrm{m}}^{\pi_0(\widetilde{Y})}$ acts on $\mathbb{G}_{\mathrm{m}}^{\phi^{-1}(y)}$ by the dual of the map $\phi^{-1}(y) \subset \widetilde{Y}(k) \xrightarrow{\psi} \pi_0(Y)$ associating to $x \in \widetilde{Y}$ the connected component of $\widetilde{Y}$ containing it. The character group of G^{tor} is therefore the kernel of the map

$$\mathbb{Z}[B] \xrightarrow{(\psi,\phi)} \mathbb{Z}[C] \oplus \mathbb{Z}[A]$$

which (after replacing ϕ with $-\phi$) is the chain complex of $\widetilde{\Gamma}_Y$. This gives the formula [8, I.3]

$$\mathrm{Hom}(G^{\mathrm{lin}}, \mathbb{G}_{\mathrm{m}}) = H_1(\widetilde{\Gamma}_Y, \mathbb{Z}). \tag{1.2.1}$$

Suppose now that Y is a proper curve over k, not necessarily reduced. The map $\mathrm{Pic}^0_Y \to \mathrm{Pic}^0_{Y^{\mathrm{red}}}$ is an epimorphism, and its kernel is a connected unipotent group scheme, so (1.2.1) remains valid.

If k is merely assumed to be perfect, (1.2.1) holds as an isomorphism of $\mathrm{Gal}(\bar{k}/k)$-modules.

If Y only has double points with distinct branches, then by the homeomorphism $\widetilde{\Gamma}_Y \to \Gamma_Y$ we obtain the formula

$$\mathrm{Hom}(G^{\mathrm{lin}}, \mathbb{G}_{\mathrm{m}}) = H_1(\Gamma_Y, \mathbb{Z}). \tag{1.2.2}$$

1.3. **The Néron model of J.** In preparation for §1.6, we review in more detail the results of Raynaud. We will follow mainly the notation of [29] (see also [3], where the notations are slightly different).

We consider a proper flat morphism $\mathcal{X} \to S = \operatorname{Spec} R$, satisfying the hypotheses (H1–3) below.

(H1) The generic fibre $X := \mathcal{X}_F$ is a smooth geometrically connected curve over F (in particular, $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = R$).

(H2) $\mathcal{X}$ is regular.

Let the irreducible components of $\mathcal{X}_s$ be indexed by the set C, and for $j \in C$, let $\mathcal{X}_j \subset \mathcal{X}_s$ be the scheme-theoretic closure of the corresponding maximal point of $\mathcal{X}_s$, $\delta_j = p^{n_j} d_j$ its total multiplicity (§1.2), and $Y_j = \mathcal{X}_j^{\mathrm{red}}$. Define $\delta = \gcd\{\delta_j\}$, $d = \gcd\{d_j\}$.

(H3) $(\delta, p) = 1$.

Hypotheses (H1) and (H2) imply that Raynaud's condition $(\mathrm{N})^*$ is satisfied [29, (6.1.4)]. Hypothesis (H1) is not particularly restrictive, since one may always reduce to this case using Stein factorization. In the presence of (H1–2), hypothesis (H3) implies that $\mathcal{X}/S$ is cohomologically flat (equivalently, that $\Gamma(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}) = k$), by [29, (7.2.1)].

Let $J = \mathrm{Pic}^0_{X/F}$ be the Jacobian variety of X, and let $\mathcal{J}$ be the Néron model of J.

The relative Picard functor $P = \mathrm{Pic}_{\mathcal{X}/S}$ is the sheafification (for the fppf topology) of the functor on the category of S-schemes

$$S' \mapsto \mathrm{Pic}(\mathcal{X} \times_S S').$$

There is a morphism of abelian sheaves $\deg \colon P \to \mathbb{Z}$ which takes a line bundle to its total degree along the fibres, and $P' \subset P$ denotes its kernel. By [29, (5.2) and (2.3.2)], P and P' are formally smooth algebraic spaces over S, and the closure $E \subset P$ of the zero section is an étale algebraic space over S, contained in P'. The maximal separated quotient $Q = P/E$ is a smooth separated S-group scheme, and the subgroup $Q' = P'/E$ is the closure in Q of the identity component Q^0 (proof of [29, (8.1.2)(iii)]). One also has the subgroup $Q^\tau \subset Q$, which is the inverse image of the torsion subgroup of Q/Q^0. As $\mathcal{X}$ is regular, condition d) of [29, (8.1.2)] holds, and so Q^τ is closed in Q. By definition $\deg(Q^\tau) = 0$, and therefore $Q' = Q^\tau$. So [29, (8.1.2) and (8.1.4)(b)] imply that $\mathcal{J} = Q' = P'/E$.

(If (H3) is not satisfied, then P is in general not representable, but it still has a maximal separated quotient Q which is a smooth separated S-group scheme [29, (4.1.1)]. If moreover k is perfect, then Q' again equals $\mathcal{J}$ [29, (8.1.4)(a)].)

Let P_s^0 be the identity component of P_s. We have $P_s^0 = \mathrm{Pic}^0_{\mathcal{X}_s/k}$, the identity component of the Picard scheme of $\mathcal{X}_s$. By [29, (6.4.1)(3)], the intersection $P_s^0 \cap E_s$ is a constant group scheme over k, cyclic of order d, generated by the class of the line bundle $\mathcal{L}' = \mathcal{O}(\sum_j (d_j/d) Y_j)$. (Because $\mathcal{X}$ is regular, the integers d and d' [29, (6.1.11)(3)] are equal.) Therefore $\mathcal{J}_s^0$ is canonically isomorphic to $\mathrm{Pic}^0_{\mathcal{X}_s/k} / \langle \mathcal{L}' \rangle$, and in particular, if $d = 1$ then $\mathcal{J}_s^0 = \mathrm{Pic}^0_{\mathcal{X}_s/k}$.

Suppose that k is perfect and $d = 1$. Combining the above with the discussion in §2.2, we then have an isomorphism of $\mathrm{Gal}(\bar{k}/k)$-modules

$$\mathbb{X}(J) := \mathrm{Hom}(\mathcal{J}_s^{0,\mathrm{lin}} \otimes_k \bar{k}, \mathbb{G}_{\mathrm{m}}) = H_1(\tilde{\Gamma}_{\mathcal{X}_s \otimes \bar{k}}, \mathbb{Z}).$$

Finally, we recall the description of the component group. First suppose that R is strictly Henselian (k not necessarily perfect). Then [29, (8.1.2)] shows that the component group $\Phi(J) = \mathcal{J}_s/\mathcal{J}_s^0$ is computed as follows: by the above, $\Phi(J) = Q'_s/Q_s^0$ is the cokernel of the map

$$E_s \to P'_s/P_s^0 = \ker(\deg\colon P_s/P_s^0 \to \mathbb{Z}).$$

One has an isomorphism

$$P_s/P_s^0 \simeq \mathbb{Z}^C, \qquad (\mathcal{L} \in P_s) \mapsto (\deg \mathcal{L}|_{Y_j})_j.$$

Let $D \subset \operatorname{Div} \mathcal{X}$ be the group of Cartier divisors supported in the special fibre, and $D_0 \subset D$ the subgroup of principal divisors. By [29, (6.1.3)] one has $E_s = D/D_0$. As $\mathcal{X}$ is regular and $R = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, D is freely generated by the set of reduced components $\{Y_j\}$, and D_0 is the subgroup generated by the divisor (ϖ) of the special fibre. The complex of [29, (8.1.2)(i)] then becomes

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z}[C] \xrightarrow{a} \mathbb{Z}^C \xrightarrow{b} \mathbb{Z} \to 0 \tag{1.3.1}$$

where the maps are:

$$\begin{aligned} i(1) &= \sum_{j\in C} d_j(j) \\ a(\ell) &= \big(\frac{1}{\delta_j}\deg \mathcal{O}_{\mathcal{X}}(Y_\ell)|_{Y_j}\big)_{j\in C} = \big(\frac{1}{p^{n_j}}(Y_j.Y_\ell)\big)_{j\in C} \qquad (\ell \in C) \\ b(m) &= \sum_{j\in C} \delta_j m_j, \qquad m = (m_j) \in \mathbb{Z}^C \end{aligned} \tag{1.3.2}$$

and $\Phi(J) = \ker(b)/\operatorname{im}(a)$.

If $\mathcal{X}$ is semistable (meaning that $\mathcal{X}_s$ is smooth over k apart from double points with distinct tangents), then both the character group and component group can be described in terms of the reduced graph $\Gamma_{\mathcal{X}_s}$. The character group equals the homology of $\Gamma_{\mathcal{X}_s}$. The map $a\colon \mathbb{Z}[C] \to \ker(b) = \mathbb{Z}^{C,0} \subset \mathbb{Z}^C$ is, after identifying $\mathbb{Z}[C]$ with $\mathbb{Z}^C$, the 0-Laplacian $\square = \square_0$ (as in [16]) of the graph $\Gamma_{\mathcal{X}_s}$, which takes a vertex $v \in C$ to $\sum (v) - (v') \in \mathbb{Z}[C]_0$, the sum taken over all edges joining v to an adjacent vertex v'.

In general, we have an isomorphism of $\operatorname{Gal}(\bar{k}/k)$-modules $\Phi(J) = \ker(b)/\operatorname{im}(a)$, where a, b are the maps in the complex (1.3.1) for the base change $\mathcal{X} \otimes_R R^{\mathrm{sh}}$.

1.4. **Generalized Picard schemes of singular curves.** Let k be a field, and Y/k a proper curve (in the sense of §1.2 above). Write k' for the k-algebra $\Gamma(Y, \mathcal{O}_Y)$. By a *generalized modulus* on Y we mean a morphism of k-schemes $\Sigma \to Y$, where Σ is a finite k-scheme, flat over $\operatorname{Spec} k'$.

Lemma 1.5. *Let $g\colon \Sigma \to Y$ be a generalized modulus. Suppose that $g(\Sigma)$ meets each connected component of Y. Then (Σ, g) is a rigidifier*[1] *of $\operatorname{Pic}_{X/k}$, in the sense of* [29, (2.1.1)]

Proof. For (Σ, g) to be a rigidifier, it is necessary and sufficient that for every k-algebra A, the map $g^*\colon \Gamma(Y \otimes_k A, \mathcal{O}_{Y\otimes A}) \to \Gamma(\Sigma \otimes A, \mathcal{O}_{\Sigma\otimes A})$ is injective. As k is a field it is enough to show this for $A = k$, and this holds since by hypothesis Σ/k' is faithfully flat. □

We define $\operatorname{Pic}_{(Y,\Sigma)/k}$ to be the scheme classifying line bundles on Y together with a trivialisation of the pullback to Σ. Precisely, consider the functor $\mathcal{F}$ which to a k-scheme S associates the set of equivalence classes of pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a line bundle on

[1] "rigidificateur" in [29], "rigidificator" in [3].

$Y \times S$ and $\alpha\colon \mathcal{O}_{\Sigma\times S} \xrightarrow{\sim} (g \times \mathrm{id}_S)^*\mathcal{L}$ is a trivialisation, and where pairs $(\mathcal{L}, \alpha)$, $(\mathcal{L}', \alpha')$ are equivalent if there exists an isomorphism $\sigma\colon \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $\alpha' = g^*(\sigma) \circ \alpha$.

Let $Y = Y_1 \sqcup Y_2$ where $g(\Sigma)$ is disjoint from Y_2 and meets each connected component of Y_1. If $Y_2 = \emptyset$ then by Lemma 1.5 we are in the situation of [29, §2], and $\mathcal{F}$ is a sheaf for the fppf topology which we denote $\mathrm{Pic}_{(Y,\Sigma)/k}$. In general, we define $\mathrm{Pic}_{(Y,\Sigma)/k}$ to be the sheafification of $\mathcal{F}$ for the fppf topology. Obviously $\mathrm{Pic}_{(Y,\Sigma)/k} = \mathrm{Pic}_{(Y_1,\Sigma)/k} \times_k \mathrm{Pic}_{Y_2/k}$. Put $k' = k_1 \times k_2$ where $k_i = \Gamma(Y_i, \mathcal{O}_{Y_i})$. From [29] we then obtain:

Proposition 1.6. *The functor* $\mathrm{Pic}_{(Y,\Sigma)/k}$ *is represented by a smooth k-group scheme, and there is an exact sequence of smooth group schemes*

$$0 \to H \to \mathrm{Pic}_{(Y,\Sigma)/k} \to \mathrm{Pic}_{Y/k} \to 0 \tag{1.4.1}$$

where

$$H = H_\Sigma := \mathrm{coker}\left(\mathcal{R}_{k'/k}(\mathbb{G}_\mathrm{m}) \to \mathcal{R}_{\Sigma/k}(\mathbb{G}_\mathrm{m})\right).$$

Proof. From (2.1.2), (2.4.1) and (2.4.3) of [29] we get the representability of $\mathrm{Pic}_{(Y_1,\Sigma)/k}$ along with an exact sequence

$$0 \to \mathcal{R}_{k_1/k}\mathbb{G}_\mathrm{m} \to \mathcal{R}_{\Sigma/k}\mathbb{G}_\mathrm{m} \to \mathrm{Pic}_{(Y_1,\Sigma)/k} \to \mathrm{Pic}_{Y_1/k} \to 0$$

of smooth group schemes (since, in this setting, Raynaud's Γ^*_X and Γ^*_R are just $\mathcal{R}_{k_1/k}\mathbb{G}_\mathrm{m}$ and $\mathcal{R}_{\Sigma/k}\mathbb{G}_\mathrm{m}$). By §1.1 above, the quotient H is a smooth group scheme, and taking products with $\mathrm{Pic}_{Y_2/k}$ gives the result. ☐

Let $\mathrm{Pic}^0_{(Y,\Sigma)/k}$ denote the inverse image of $\mathrm{Pic}^0_{Y/k}$ (classifying line bundles which are of degree zero on every component of Y). Then $\mathrm{Pic}^0_{(Y,\Sigma)/k}$ is a smooth connected k-group scheme of finite type.

Example 1.7. Suppose Y is smooth over k and absolutely irreducible, and that $\Sigma \to Y$ is a closed immersion. Then the image of Σ is an effective divisor $\mathfrak{m} = \sum m_i(y_i)$ for points $y_i \in Y(k)$. In this case $\mathrm{Pic}^0_{(Y,\Sigma)/k}$ is none other than the classical [32, 33] generalized Jacobian $J_\mathfrak{m}(Y)$ of Y. The isomorphism $J_\mathfrak{m}(Y) \xrightarrow{\sim} \mathrm{Pic}^0_{(Y,\Sigma)/k}$ is given on k-points by mapping the class of a divisor $D \in \mathrm{Div}^0(Y \smallsetminus \Sigma)$ to the class of the pair $(\mathcal{O}_Y(D), \alpha_\mathrm{triv})$, where α_triv is the canonical trivialisation $\mathcal{O}_\Sigma \xrightarrow{\sim} \mathcal{O}_Y|_\Sigma = \mathcal{O}_Y(D)|_\Sigma$.

Let k be perfect. Then $\mathrm{Pic}^0_{(Y,\Sigma)/k}$ has a maximal connected affine subgroup $\mathrm{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k}$ which is a linear group, and its character group has the following combinatorial description, generalizing §1.2 above.

First suppose that k is algebraically closed, and that Y, Σ are reduced. As in §1.2, let $\phi\colon \widetilde{Y} \to Y$ be the normalisation, and define $A = Y^\mathrm{sing}$, $B = \phi^{-1}(A)$, $C = \pi_0(\widetilde{Y})$. Decompose $\Sigma = \Sigma^\mathrm{sing} \sqcup \Sigma^\mathrm{reg}$, where $z \in \Sigma^\mathrm{sing}$ (resp. Σ^reg) if $g(z)$ is a singular (resp. smooth) point of Y. There are maps

$$\begin{array}{ccccc} B & \xrightarrow{\ \psi\ } & C & \xleftarrow{\ \theta\ } & \Sigma^\mathrm{reg} \\ \big\downarrow{\scriptstyle\phi} & & & & \\ A & \xleftarrow{\ \lambda\ } & \Sigma^\mathrm{sing} & & \end{array}$$

where ϕ, ψ are as before, λ is the restriction of g to Σ^sing, and $\theta(z)$ is the component of $\widetilde{Y}$ containing $g(z)$.

Define the *extended graph of* (Y, Σ) to be the directed graph $\widetilde{\Gamma}_{Y,\Sigma}$ obtained by adding to the graph $\widetilde{\Gamma}_Y$

- a single vertex v_0

- for each $z \in \Sigma^{\mathrm{sing}}$, an edge from v_0 to the vertex $\lambda(z) \in A \subset V(\widetilde{\Gamma}_Y)$
- for each $z \in \Sigma^{\mathrm{reg}}$, an edge from v_0 to the vertex $\theta(z) \in C \subset V(\widetilde{\Gamma}_Y)$.

If Y only has double points and $\Sigma = \Sigma^{\mathrm{reg}}$, then we may likewise define the *reduced graph* $\Gamma_{Y,\Sigma}$, which is the undirected graph obtained by adding to Γ_Y a single vertex v_0 and, for each $z \in \Sigma$, an edge joining v_0 to $\theta(z) \in C = V(\Gamma_Y)$. As before, the geometric realisations of $\widetilde{\Gamma}_{Y,\Sigma}$ and $\Gamma_{Y,\Sigma}$ are canonically homeomorphic.

For arbitrary perfect k and proper curve Y, we define $\widetilde{\Gamma}_{Y,\Sigma}$, $\Gamma_{Y,\Sigma}$ to be the graphs attached to the curve with modulus $(Y^{\mathrm{red}} \otimes \bar{k}, \Sigma^{\mathrm{red}} \otimes \bar{k})$, which are graphs with a continuous action of $\mathrm{Gal}(\bar{k}/k)$.

Proposition 1.8.

(a) *The character group* $\mathrm{Hom}(\mathrm{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k}, \mathbb{G}_{\mathrm{m}})$ *is canonically isomorphic to* $H_1(\widetilde{\Gamma}_{Y,\Sigma}, \mathbb{Z})$*, as* $\mathrm{Gal}(\bar{k}/k)$*-module.*

(b) *If* Y^{red} *has only double points, then* $\mathrm{Hom}(\mathrm{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k}, \mathbb{G}_{\mathrm{m}}) \simeq H_1(\Gamma_{Y,\Sigma}, \mathbb{Z})$.

Proof. We may assume that k is algebraically closed; the Galois equivariance of the isomorphisms will be clear from the construction. By the homeomorphism between the extended and reduced graphs, it suffices to prove (a).

The map $g^{\mathrm{red}} \colon \Sigma^{\mathrm{red}} \to Y^{\mathrm{red}}$ is a reduced modulus, and the obvious morphism induced by pullback

$$\mathrm{Pic}^0_{(Y,\Sigma)/k} \to \mathrm{Pic}^0_{(Y^{\mathrm{red}},\Sigma^{\mathrm{red}})/k}$$

has unipotent kernel, since the same is true for the maps $\mathcal{R}_{\Sigma/k}\mathbb{G}_{\mathrm{m}} \to \mathcal{R}_{\Sigma^{\mathrm{red}}/k}\mathbb{G}_{\mathrm{m}}$ and $\mathrm{Pic}^0_{Y/k} \to \mathrm{Pic}^0_{Y^{\mathrm{red}}/k}$. So the character group of $\mathrm{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k}$ is unchanged by passing to reduced subschemes; hence we may assume that both Y and Σ are reduced. Next, let $Y' \to Y$ be the seminormalisation. Then as Σ is reduced, $\Sigma \to Y$ factors uniquely through Y', and the resulting map

$$\mathrm{Pic}^0_{(Y,\Sigma)/k} \to \mathrm{Pic}^0_{(Y',\Sigma)/k}$$

has unipotent kernel. So we may assume in addition that Y is seminormal. Finally, normalisation induces an exact sequence

$$0 \to G \to \mathrm{Pic}^0_{(Y,\Sigma)/k} \xrightarrow{\phi^*} \mathrm{Pic}^0_{\tilde{Y}/k} \to 0 \tag{1.4.2}$$

where G classifies equivalence classes of pairs $(\mathcal{L}, \beta)$, where $\mathcal{L}$ is a line bundle on Y whose pullback to $\tilde{Y}$ is trivial, and β is a trivialisation of the pullback of $\mathcal{L}$ to Σ. There is a surjective map

$$\mathbb{G}_{\mathrm{m}}^B \times \mathbb{G}_{\mathrm{m}}^{\Sigma^{\mathrm{sing}}} \times \mathbb{G}_{\mathrm{m}}^{\Sigma^{\mathrm{reg}}} \to G \tag{1.4.3}$$

given as follows: a tuple

$$((a_x)_{x\in B}, (b_z)_{z\in\Sigma^{\mathrm{sing}}}, (c_z)_{z\in\Sigma^{\mathrm{reg}}}) \in (\mathbb{G}_{\mathrm{m}}^B \times \mathbb{G}_{\mathrm{m}}^{\Sigma^{\mathrm{sing}}} \times \mathbb{G}_{\mathrm{m}}^{\Sigma^{\mathrm{reg}}})(k)$$

determines:

(i) for every $y \in A$, and any $x, x' \in \phi^{-1}(y)$, isomorphisms $a_x^{-1}a_{x'} \colon x^*\mathcal{O}_{\tilde{Y}} = k \xrightarrow{\sim} k = x'^*\mathcal{O}_{\tilde{Y}}$ satisfying the cocycle condition, and thus a descent of $\mathcal{O}_{\tilde{Y}}$ to a line bundle $\mathcal{L}$ on Y

(ii) for every $z \in \Sigma^{\mathrm{sing}}$, and every $x \in \phi^{-1}(g(z))$, a trivialisation $b_z a_x^{-1} \colon k \xrightarrow{\sim} k = x^*\mathcal{O}_{\tilde{Y}}$. These trivialisations are compatible with the descent data (i) and therefore give trivialisations $k \xrightarrow{\sim} z^*\mathcal{L}$ for every $z \in \Sigma^{\mathrm{sing}}$.

(iii) for every $z \in \Sigma^{\mathrm{reg}}$, a trivialisation $k \xrightarrow{\sim} z^*\mathcal{L} = k$ given by multiplication by c_z.

What is the kernel of the map (1.4.3)? Fix $y \in A$. Then multiplying a_x, for $x \in \phi^{-1}(y)$, and b_z, for $z \in \Sigma^{\mathrm{sing}}$ such that $g(z) = y$, by a common element of $k^\times$ does not change the descent data (i) or the trivialisation (ii), so we obtain the same $(\mathcal{L}, \beta)$. The equivalence relation on pairs is realised by the automorphism group $\mathbb{G}_{\mathrm{m}}^C$ of $\phi^*\mathcal{L} = \mathcal{O}_{\tilde{Y}}$, which acts on tuples by

$$(d_j)_{j\in C}\colon ((a_x)_{x\in B}, (b_z)_{z\in\Sigma^{\mathrm{sing}}}, (c_x)_{x\in\Sigma^{\mathrm{reg}}}) \mapsto ((d_{\psi(x)}a_x), (b_z), (d_{\theta(z)}c_z)).$$

Therefore G is the torus whose character group is the kernel of the map

$$\mathbb{Z}[B] \oplus \mathbb{Z}[\Sigma^{\mathrm{sing}}] \oplus \mathbb{Z}[\Sigma^{\mathrm{reg}}] \to \mathbb{Z}[C] \oplus \mathbb{Z}[A] \tag{1.4.4}$$

with matrix

$$\begin{bmatrix} \psi & 0 & \theta \\ \phi & \lambda & 0 \end{bmatrix}.$$

The homology complex of $\widetilde{\Gamma}_{Y,\Sigma}$ is

$$\mathbb{Z}[B] \oplus \mathbb{Z}[\Sigma^{\mathrm{sing}}] \oplus \mathbb{Z}[\Sigma^{\mathrm{reg}}] \to \mathbb{Z}[C] \oplus \mathbb{Z}[A] \oplus \mathbb{Z} \tag{1.4.5}$$

with differential given by the matrix

$$\begin{bmatrix} \psi & 0 & \theta \\ -\phi & \lambda & 0 \\ 0 & -\varepsilon & -\varepsilon \end{bmatrix}$$

where $\varepsilon\colon \mathbb{Z}[\Sigma^?] \to \mathbb{Z}$ is the augmentation $z \mapsto 1$, for $z \in \Sigma^?$, $? \in \{\mathrm{reg}, \mathrm{sing}\}$. There is an obvious map from the complex (1.4.5) to the complex (1.4.4) which induces an isomorphism on kernels. Since G is by (1.4.2) the maximal multiplicative quotient of $\mathrm{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k}$, this gives the isomorphism (a). The construction is Galois equivariant by transport of structure. □

1.5. **Functoriality I.** Let $g\colon \Sigma \to Y$, $g'\colon \Sigma' \to Y'$ be generalized moduli on proper curves over k as in the previous section, and suppose we have finite morphisms f, f_Σ fitting into a commutative diagram

$$\begin{array}{ccc} \Sigma' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f_\Sigma} & & \downarrow{\scriptstyle f} \\ \Sigma & \xrightarrow{g} & Y \end{array} \tag{1.5.1}$$

Then there is an associated pullback morphism

$$(f, f_\Sigma)^*\colon \mathrm{Pic}_{(Y,\Sigma)/k} \to \mathrm{Pic}_{(Y',\Sigma')/k}$$

taking a pair $(\mathcal{L}, \alpha\colon \mathcal{O}_\Sigma \xrightarrow{\sim} g^*\mathcal{L})$ to the pair

$$\big(f^*\mathcal{L},\ f_\Sigma^*\alpha\colon \mathcal{O}_{\Sigma'} \xrightarrow{\sim} f_\Sigma^* g^*\mathcal{L} = g'^*(f^*\mathcal{L})\big),$$

which we will simply denote by f^* if no confusion can arise.

To define pushforward, consider the commutative diagram

$$\begin{array}{ccccc} \Sigma' & \xrightarrow{h} & \Sigma \times_Y Y' & \xrightarrow{\mathrm{pr}_2} & Y' \\ & \searrow^{f_\Sigma} & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle f} \\ & & \Sigma & \xrightarrow{g} & Y \end{array}$$

We assume that f is flat, and that h is a closed immersion whose ideal sheaf $\mathcal{I}$ is nilpotent and satisfies

$$\mathcal{N}_{\Sigma\times_Y Y'/\Sigma}(1 + \mathcal{I}) = \{1\}. \tag{1.5.2}$$

We may then define a morphism $f_* = (f,\Sigma)_*\colon \mathrm{Pic}_{(Y',\Sigma')/k} \to \mathrm{Pic}_{(Y,\Sigma)/k}$ by $f_*\colon (\mathcal{L}',\alpha') \mapsto (\mathcal{L},\alpha)$, where $\mathcal{L} = \mathcal{N}_f(\mathcal{L}')$, the norm of $\mathcal{L}'$ [14, 6.5] and α is given as follows: if $\mathcal{I} = 0$ is zero, then (1.5.1) is Cartesian, and α is the composite

$$\alpha\colon \mathcal{O}_\Sigma \xrightarrow[\mathcal{N}_f(\alpha')]{\sim} \mathcal{N}_f(g'^*\mathcal{L}') \xrightarrow{\sim} g^*\mathcal{L}$$

(the second isomorphism given by [14, (6.5.8)]). In general, $\alpha'\colon \mathcal{O}_{\Sigma'} \xrightarrow{\sim} g'^*\mathcal{L}'$ can at least locally be extended to an isomorphism $\alpha''\colon \mathcal{O}_{\Sigma\times_Y Y'} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{L}'$, well-defined up to local sections of $1+\mathcal{I}$. Taking norms, we then get a well-defined global isomorphism $\alpha = \mathcal{N}_{\Sigma\times_Y Y'/\Sigma}(\alpha'')\colon \mathcal{O}_\Sigma \xrightarrow{\sim} g^*\mathcal{L}$.

The maps f^*, f_* preserve Pic^0 in all cases.

Example 1.9. Suppose that Y, Y' are smooth over k and absolutely irreducible, and that $\Sigma \subset Y$, $\Sigma' \subset Y'$ are closed subschemes defined by reduced moduli $\mathfrak{m}$, $\mathfrak{m}'$. Let $J_\mathfrak{m} = \mathrm{Pic}^0_{(Y,\Sigma)/k}$, $J'_{\mathfrak{m}'} = \mathrm{Pic}^0_{(Y',\Sigma')/k}$, be the associated generalized Jacobians. Let $f\colon Y' \to Y$ be a finite morphism with $f^{-1}(\Sigma)^{\mathrm{red}} = \Sigma'$. Then (1.5.2) holds, and therefore we get morphisms

$$f^*\colon J_\mathfrak{m} \to J'_{\mathfrak{m}'}, \quad f_*\colon J'_{\mathfrak{m}'} \to J_\mathfrak{m}.$$

If $f'\colon Y' \to Y$ is another finite morphism with $f'^{-1}(\Sigma) \supset \Sigma'$, then we get an induced endomorphism $f_*f'^*\colon J_\mathfrak{m} \to J_\mathfrak{m}$, compatible with the usual correspondence action on J (pullback along f' followed by norm with respect to f). For later reference, we will say that the modulus $\mathfrak{m}$ is *stable* under the correspondence $f_*f'^*$.

Returning to the general case, assume that k is algebraically closed, that f is flat and that (1.5.2) holds. Write

$$\mathbb{X} = \mathrm{Hom}(\mathrm{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k}, \mathbb{G}_\mathrm{m}), \quad \mathbb{X}' = \mathrm{Hom}(\mathrm{Pic}^{0,\mathrm{lin}}_{(Y',\Sigma')/k}, \mathbb{G}_\mathrm{m})$$

for the character groups of the linear parts of the generalized Picard schemes. Then f^*, f_* induce by functoriality homomorphisms

$$\mathbb{X}(f^*)\colon \mathbb{X}' \to \mathbb{X}, \quad \mathbb{X}(f_*)\colon \mathbb{X} \to \mathbb{X}'. \tag{1.5.3}$$

By Proposition 1.8 and (1.4.4) we have canonical isomorphisms

$$\begin{aligned}\mathbb{X} &= \ker\left(\begin{bmatrix}\psi & 0 & \theta\\ \phi & \lambda & 0\end{bmatrix} : \mathbb{Z}[B]\oplus\mathbb{Z}[\Sigma^{\mathrm{sing}}]\oplus\mathbb{Z}[\Sigma^{\mathrm{reg}}] \to \mathbb{Z}[C]\oplus\mathbb{Z}[A]\right)\\ &= H_1(\widetilde{\Gamma}_{Y,\Sigma},\mathbb{Z})\end{aligned}$$

where $A = (Y^{\mathrm{red}})^{\mathrm{sing}}$, $B = \phi^{-1}(A)$, $C = \pi_0(\widetilde{Y})$, and similarly for $\mathbb{X}'$. We now describe the maps (1.5.3) combinatorially, under further hypotheses. Let A', B', C' denote the corresponding sets for Y', and assume the following.

Hypotheses 1.10.

(i) $f^{-1}(A) = A'$, and f is étale at each point of A'.
(ii) $\Sigma^{\mathrm{sing}} = \emptyset = \Sigma'^{\mathrm{sing}}$, and Σ, Σ' are reduced.
(iii) $\Sigma' = f^{-1}(\Sigma)^{\mathrm{red}}$.

Hypotheses 1.10(ii) and (iii) together imply that (1.5.2) is satisfied. Then f induces maps $A' \to A$, $B' \to B$, $C' \to C$ which we also denote by f. The diagram

$$\begin{array}{ccccccc} A' & \xleftarrow{\phi'} & B' & \xrightarrow{\psi'} & C' & \xleftarrow{\theta'} & \Sigma' \\ \downarrow f & & \downarrow f & & \downarrow f & & \downarrow f \\ A & \xleftarrow{\phi} & B & \xrightarrow{\psi} & C & \xleftarrow{\theta} & \Sigma \end{array} \tag{1.5.4}$$

commutes, and so we have a commutative square

$$
\begin{array}{ccc}
\mathbb{Z}[B'] \oplus \mathbb{Z}[\Sigma'] & \xrightarrow{\left[\begin{smallmatrix}\psi' & \theta' \\ \phi' & 0\end{smallmatrix}\right]} & \mathbb{Z}[C'] \oplus \mathbb{Z}[A'] \\
\Big\downarrow f & & \Big\downarrow f \\
\mathbb{Z}[B] \oplus \mathbb{Z}[\Sigma] & \xrightarrow{\left[\begin{smallmatrix}\psi & \theta \\ \phi & 0\end{smallmatrix}\right]} & \mathbb{Z}[C] \oplus \mathbb{Z}[A]
\end{array}
\tag{1.5.5}
$$

Proposition 1.11. *Assume Hypotheses* 1.10. *The homomorphism* $\mathbb{X}(f^*)$ *is induced by the vertical maps in* (1.5.5).

Proof. We first observe that we may assume in addition that Y and Y' are seminormal (and therefore reduced). Indeed, the descriptions of the character groups $\mathbb{X}$ and $\mathbb{X}'$ is unchanged after replacing the curves by their seminormalisations. It remains to verify that the induced map on seminormalisations $f^{\mathrm{sn}}\colon Y'^{\mathrm{sn}} \to Y^{\mathrm{sn}}$ is flat. But $Y^{\mathrm{sn}} \smallsetminus A = Y^{\mathrm{red}} \smallsetminus A$ is smooth, and so the restriction of f^{sn} to $Y'^{\mathrm{sn}} \smallsetminus A'$ is automatically flat. By hypothesis, there is a neighbourhood $U \subset Y$ of A such that $f\colon U' := f^{-1}(U) \to U$ is étale. Then by [11, Prop. 5.1], we have a Cartesian square

$$
\begin{array}{ccc}
U'^{\mathrm{sn}} & \longrightarrow & U' \\
\Big\downarrow f^{\mathrm{sn}} & & \Big\downarrow f \\
U^{\mathrm{sn}} & \longrightarrow & U
\end{array}
$$

and in particular f^{sn} restricted to U'^{sn} is étale.

We now compute the dual map $f^*\colon \operatorname{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k} \to \operatorname{Pic}^{0,\mathrm{lin}}_{(Y',\Sigma')/k}$ which is a morphism of tori (since we are assuming that Y and Y' are seminormal and Σ, Σ' are reduced). As explained in §1.4, a k-point of $\operatorname{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k}$ is represented by a pair $((a_x)_{x\in B}, (c_z)_{z\in\Sigma}) \in (k^\times)^B \times (k^\times)^\Sigma$, where (a_x) determines descent data for $\mathcal{O}_{\widetilde{Y}}$ with respect to the normalisation morphism $\phi\colon \widetilde{Y} \to Y$, and (c_z) determines trivialisations $\times c_z\colon k \xrightarrow{\sim} k = z^*\mathcal{O}_{\widetilde{Y}}$ which descend to a rigidification along Σ of the descended line bundle.

Let $a'_{x'} = a_{f(x')}$ ($x' \in B'$) and $c'_{z'} = c_{f(z')}$ ($z' \in \Sigma'$). Then if $(\mathcal{L}, \alpha) \in \operatorname{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k}(k)$ is represented by the pair $((a_x), (c_z))$, the pullback $f^*(\mathcal{L}, \alpha)$ is represented by $((a'_{x'}), (c'_{z'}))$. The obvious map

$$\operatorname{Aut}\mathcal{O}_{\widetilde{Y}} = (k^\times)^C \to \operatorname{Aut}\mathcal{O}_{\widetilde{Y}'} = (k^\times)^{C'}$$

is induced by $f\colon C' \to C$, and therefore $\mathbb{X}(f^*)$ is induced by the vertical maps f in (1.5.5) as required. ☐

We now compute $\mathbb{X}(f_*)$. Let

$$f^*\colon \begin{cases} \mathbb{Z}[\Sigma] \to \mathbb{Z}[\Sigma'] \\ \mathbb{Z}[A] \to \mathbb{Z}[A'] \\ \mathbb{Z}[B] \to \mathbb{Z}[B'] \end{cases}$$

be the inverse image maps on divisors. By Hypothesis 1.10(i), this means that if $x \in A$ or $x \in B$, then $f^*\colon (x) \mapsto \sum_{f(x')=x}(x')$, and if $z \in \Sigma$, then

$$f^*\colon (z) \mapsto \sum_{f(z')=z} r_{z'/z}(z')$$

where $r_{z'/z}$is the ramification degree of f at z'. Finally, define $f^*\colon \mathbb{Z}[C] \to \mathbb{Z}[C']$ by

$$f^*\colon (Z) \mapsto \sum_{f(Z')=Z} [\kappa(Z') : \kappa(Z)]\,(Z')$$

where $Z \subset \widetilde{Y}$, $Z' \subset \widetilde{Y}'$ are connected components. These maps fit into the diagram

$$\begin{array}{ccc} \mathbb{Z}[B] \oplus \mathbb{Z}[\Sigma] & \xrightarrow{\left[\begin{smallmatrix}\psi & \theta \\ \phi & 0\end{smallmatrix}\right]} & \mathbb{Z}[C] \oplus \mathbb{Z}[A] \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f^*} \\ \mathbb{Z}[B'] \oplus \mathbb{Z}[\Sigma'] & \xrightarrow{\left[\begin{smallmatrix}\psi' & \theta' \\ \phi' & 0\end{smallmatrix}\right]} & \mathbb{Z}[C'] \oplus \mathbb{Z}[A']\,. \end{array} \tag{1.5.6}$$

Proposition 1.12. *Assume Hypotheses* 1.10*. The diagram* (1.5.6) *is commutative, and the vertical maps induce the homomorphism* $\mathbb{X}(f_*)\colon \mathbb{X} \to \mathbb{X}'$*.*

Proof. As in 1.11, we may assume that Y and Y' are seminormal. Consider again the dual map of tori $f^*\colon \mathrm{Pic}^{0,\mathrm{lin}}_{(Y',\Sigma')/k} \to \mathrm{Pic}^{0,\mathrm{lin}}_{(Y,\Sigma)/k}$. Let $(\mathcal{L}', \alpha') \in \mathrm{Pic}^{0,\mathrm{lin}}_{(Y',\Sigma')/k}(k)$, represented by the pair $((a'_{x'})_{x' \in B'}, (c'_{z'})_{z' \in \Sigma'})$. Since f is étale at B', the normalised map $\tilde{f}\colon \widetilde{Y}' \to \widetilde{Y}$ induces a norm homomorphism

$$\mathcal{N}_{\tilde{f}}\colon \Gamma(\phi'^{-1}(Y'^{\mathrm{sing}}), \mathcal{O}^\times) = (k^\times)^{B'} \to \Gamma(\phi^{-1}(Y^{\mathrm{sing}}), \mathcal{O}^\times) = (k^\times)^{B}$$

which equals the homomorphism $f_!\colon (k^\times)^{B'} \to (k^\times)^B$ given by

$$f_!\colon (a'_{x'})_{x' \in B'} \mapsto (a_x)_{x \in B}, \quad a_x = \prod_{f(x')=x} a'_{x'}.$$

The analogous statement holds for

$$\mathcal{N}_f\colon \Gamma(Y'^{\mathrm{sing}}, \mathcal{O}^\times) = (k^\times)^{A'} \to \Gamma(Y^{\mathrm{sing}}, \mathcal{O}^\times) = (k^\times)^{A}\,.$$

Since f is étale at A', the square

$$\begin{array}{ccc} B' & \xrightarrow{\phi'} & A' \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow{\phi} & A \end{array}$$

is in fact Cartesian, and therefore

$$\phi^* \circ f_! = f_! \circ \phi'^*\colon (k^\times)^{B'} \to (k^\times)^{A}\,.$$

Next, we consider the rigidification $\alpha'\colon \mathcal{O}_{\Sigma'} \xrightarrow{\sim} g'^*\mathcal{L} = g'^*\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_{\Sigma'}$ given by multiplication by $(c'_{z'}) \in (k^\times)^{\Sigma'}$. Let $\Sigma'' = f^{-1}(\Sigma)$ be the scheme-theoretic inverse image of Σ. So $\Sigma'' = \coprod_{z' \in \Sigma'} \tilde{z}'$ say, where $\tilde{z}' \simeq \operatorname{Spec} k[t]/(t^{r_{z'/z}})$. According to (ii) above, to compute $f_*(\mathcal{L}', \alpha')$ we need to extend α' to a rigidification

$$\alpha''\colon \mathcal{O}_{\Sigma''} \xrightarrow{\sim} \mathcal{L}'|_{\Sigma''} = \mathcal{O}_{\Sigma''}$$

and we may as well take α'' to be the sum of the maps $\mathcal{O}_{\tilde{z}'} \xrightarrow{\sim} \mathcal{O}_{\tilde{z}'}$ given by multiplication by $c'_{z'}$. Then $\mathcal{N}(\alpha'')\colon \mathcal{O}_\Sigma \xrightarrow{\sim} \mathcal{O}_\Sigma$ is multiplication by $(c_z) = \hat{f}_!(c'_{z'})$, where $\hat{f}_!\colon (k^\times)^{\Sigma'} \to (k^\times)^{\Sigma}$ is the map

$$\hat{f}_!\colon (c'_{z'}) \mapsto (c_z), \quad c_z = \prod_{f(z')=z} (c'_{z'})^{r_{z'/z}}$$

whose dual is the map $f^*\colon \mathbb{Z}[\Sigma'] \to \mathbb{Z}[\Sigma]$ defined above.

Finally we need to compute the action of $\operatorname{Aut}\mathcal{O}_{\widetilde{Y}'} = (k^\times)^{C'}$. From §1.4 we know that $d' \in (k^\times)^{C'}$ maps $((a'_{x'})_{x'\in B'}, (c'_{z'})_{z'\in\Sigma'})$ to $((d'_{\psi'(x')}a'_{x'}), (d'_{\theta'(z')}c'_{z'}))$, which under the norm maps to

$$\Big(\Big(\prod_{\tilde f(x')=x} d'_{\psi'(x')}a'_{x'}\Big)_{x\in B}, \Big(\prod_{\tilde f(z')=z} (d'_{\theta'(z')}c'_{z'})^{r_{z'/z}}\Big)_{z\in\Sigma}\Big). \tag{1.5.7}$$

Let $x \in B$ be fixed. Then if $Z = \psi(x) \in C$ is the component containing x, and $Z' \in C'$ is a component of $\widetilde{Y}'$ lying over Z, the set $f^{-1}(x) \cap Z'$ has cardinality $[\kappa(Z') : \kappa(Z)]$, since f is étale at $f^{-1}(x)$. Therefore

$$\prod_{\tilde f(x')=x} d'_{\psi'(x')} = \prod_{\substack{Z'\in C'\\ f(Z')=\psi(x)}} (d'_{Z'})^{[\kappa(Z'):\kappa(Z)]}.$$

Similarly, let $z \in \Sigma$ be fixed, and $Z = \theta(z) \in C$ the component of $\widetilde{Y}$ containing it. Then if $Z' \in C'$ is a component of $\widetilde{Y}'$ lying over Z,

$$\sum_{z'\in f^{-1}(z)\cap Z'} r_{z'/z} = [\kappa(Z') : \kappa(Z)]$$

and therefore

$$\prod_{\tilde f(z')=z} (d'_{\theta'(z')})^{r_{z'/z}} = \prod_{\substack{Z'\in C'\\ f(Z')=\theta(z)}} (d'_{Z'})^{[\kappa(Z'):\kappa(Z)]}.$$

In other words, the pair (1.5.7) equals

$$(d_{\psi(x)}(f_!a')_x, d_{\theta(Z)}(\hat f_!c')_Z)$$

where

$$d_Z = \prod_{\substack{Z'\in C'\\ f(Z')=Z}} (d'_{Z'})^{[\kappa(Z'):\kappa(Z)]}.$$

The dual of this map $d' \mapsto d$ is therefore the homomorphism $f^*\colon \mathbb{Z}[C] \to \mathbb{Z}[C']$ defined above. □

1.6. **Generalized Jacobians over DVRs.** We resume the notations and hypotheses of §1.3. Let $(x_i)_{i\in I}$ be a nonempty finite family of distinct closed points of X, whose residue fields F_i are separable over F. Let $\mathfrak{m} = \sum_{i\in I}(x_i)$ be the associated modulus on X, and $J_\mathfrak{m} = \operatorname{Pic}^0_{(X,\mathfrak{m})/F}$ the generalized Jacobian of X with respect to $\mathfrak{m}$. The semiabelian variety $J_\mathfrak{m}$ is an extension of J by the torus

$$T_\mathfrak{m} = \Big(\prod_{i\in I}\mathcal{R}_{F_i/F}\mathbb{G}_\mathrm{m}\Big)/\mathbb{G}_\mathrm{m}.$$

Write $\mathcal{J}_\mathfrak{m}$ for the Néron model of $J_\mathfrak{m}$.

Let R_i be the integral closure of R in F_i. Then the inclusion of the points (x_i) in X extends to a unique morphism

$$\Sigma := \coprod_{i\in I}\operatorname{Spec} R_i \xrightarrow{g} \mathcal{X}.$$

As $\Gamma(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}) = k$, the special fibre $g_s\colon \Sigma_s \to \mathcal{X}_s$ is a generalized modulus, in the sense of the previous section. By Proposition 1.4(b) the Néron model $\mathcal{T}_\mathfrak{m}$ of $T_\mathfrak{m}$ equals $(\mathcal{R}_{\Sigma/S}\mathcal{G}_\mathrm{m})/\mathcal{G}_\mathrm{m}$, and its identity subgroup is $\mathcal{T}^0_\mathfrak{m} = (\mathcal{R}_{\Sigma/S}\mathbb{G}_\mathrm{m})/\mathbb{G}_\mathrm{m}$.

Lemma 1.13. *The pair (Σ, g) is a rigidifier* [29, (2.1.1)] *of* $\operatorname{Pic}_{\mathcal{X}/S}$.

Proof. Let S' be any S-scheme. Since $\mathcal{X}/S$ is cohomologically flat and Σ is flat over S, we have

$$\Gamma(X \times_S S', \mathcal{O}_{\mathcal{X}\times_S S'}) = \Gamma(S', \mathcal{O}_{S'}) \quad \text{and}$$
$$\Gamma(\Sigma \times_S S', \mathcal{O}_{\Sigma\times_S S'}) = \Gamma(\Sigma, \mathcal{O}_\Sigma) \otimes_R \Gamma(S', \mathcal{O}_{S'}).$$

As Σ is nonempty, $R \to \Gamma(\Sigma, \mathcal{O}_\Sigma)$ is a split injection of R-modules, and therefore $\Gamma(\mathcal{X} \times_S S', \mathcal{O}_{\mathcal{X}\times_S S'}) \to \Gamma(\Sigma \times_S S', \mathcal{O}_{\Sigma\times_S S'})$ is injective. □

Let P_Σ denote the rigidified Picard functor of [29, (2.1)]: for any S-scheme S', $P_\Sigma(S')$ is the group of equivalence classes of pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a line bundle on $\mathcal{X} \times_S S'$, and $\alpha\colon \mathcal{O}_{\Sigma\times_S S'} \xrightarrow{\sim} (g \times \mathrm{id}_{S'})^*\mathcal{L}$ is a trivialisation. Pairs $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$ are equivalent if there exists an isomorphism $\sigma\colon \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that $\alpha' = (g \times \mathrm{id}_{S'})^*(\sigma) \circ \alpha$. By [29, (2.3.1–2)], P_Σ is a smooth algebraic space in groups over S, and we have an exact sequence of algebraic spaces in groups [29, (2.4.1)]

$$0 \to \mathcal{T}^0_{\mathfrak{m}} = \mathcal{R}_{\Sigma/S}\mathbb{G}_m/\mathbb{G}_m \to P_\Sigma \xrightarrow{r} P \to 0$$

where r is the "forget the rigidification" functor. (Since $\mathcal{X}/S$ is cohomologically flat and $f_*\mathcal{O}_\mathcal{X} = \mathcal{O}_S$, one has $\Gamma^*_X = \mathbb{G}_m$.) If S is strictly Henselian, P_Σ is a scheme; indeed, P is a scheme, and $\mathcal{T}^0_{\mathfrak{m}}$ is affine, so by flat descent for affine schemes [34, tag 0245], the $\mathcal{T}^0_{\mathfrak{m}}$-torsor P_Σ over P is representable.

Define the sheaf $P_{\mathfrak{m}}$ to be the pushout of fppf sheaves:

(1.6.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}^0_{\mathfrak{m}} & \longrightarrow & P_\Sigma & \xrightarrow{r} & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{T}_{\mathfrak{m}} & \longrightarrow & P_{\mathfrak{m}} & \xrightarrow{r'} & P & \longrightarrow & 0 \end{array}$$

Explicitly, $P_{\mathfrak{m}}$ is the sheafification of the functor on S-schemes

(1.6.2)
$$S' \mapsto \mathcal{T}^0_{\mathfrak{m}}(S')\backslash(P_\Sigma(S') \times \mathcal{T}_{\mathfrak{m}}(S'))$$

where $\mathcal{T}^0_{\mathfrak{m}}(S')$ acts on the product by $a(b, c) = (ab, a^{-1}c)$.

Proposition 1.14. *$P_{\mathfrak{m}}$ is a smooth algebraic space in groups over S. If S is strictly Henselian, $P_{\mathfrak{m}}$ is represented by a smooth S-group scheme.*

Proof. We have an exact sequence $0 \to \mathcal{T}^0_{\mathfrak{m}} \to \mathcal{T}_{\mathfrak{m}} \xrightarrow{\pi} s_*\Phi(T) \to 0$ of S-group schemes. For $h \in \Phi(T) = (s_*\Phi(T))(S)$, let $U_h = \pi^{-1}(h)$, an affine open subscheme of $\mathcal{T}_{\mathfrak{m}}$. Then $\mathcal{T}_{\mathfrak{m}}$ is the union of the U_h, glued along their generic fibres. If $\hat{h} \in \mathcal{T}_{\mathfrak{m}}(S) = T(F)$ is any lift of h, then U_h is the translate of $\mathcal{T}^0_{\mathfrak{m}}$ by $\hat{h}$. Therefore $P_{\mathfrak{m}}$ is the union of copies of P_Σ indexed by $\Phi(T)$, glued along their generic fibres by the isomorphism given by translation by $\hat{h} \in P_\Sigma(F)$, and the result follows from the corresponding statement for P_Σ. □

This result implies that $P_{\mathfrak{m}}$ is determined by its restriction to (Sm/S), the category of essentially smooth S-schemes. We can describe this functor explicitly. Let $\mathcal{F}^*$ be the functor on (Sm/S) whose value on S' is the group of equivalence classes of pairs $(\mathcal{L}, \beta = (\beta_i)_{i\in I})$, where $\mathcal{L}$ is a line bundle on $\mathcal{X} \times_S S'$ and for each $i \in I$, $\beta_i\colon \mathcal{O}_{S'} \otimes_R F_i \xrightarrow{\sim} (x_i \times \mathrm{id}_{S'})^*\mathcal{L}$ is a trivialisation of $\mathcal{L}$ at $x_i \times_S S'$. Pairs $(\mathcal{L}, \beta)$ and $(\mathcal{L}', \beta')$ are equivalent if there exists an isomorphism $\sigma\colon \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ and some $u \in \mathcal{O}^\times(S' \otimes_R F)$ such that for every i

the diagram

$$
\begin{array}{ccc}
\mathcal{O}_{S'\otimes_R F_i} & \xrightarrow{\beta_i} & (x_i \times \mathrm{id}_{S'})^*\mathcal{L} \\
\downarrow{\scriptstyle \times u} & & \downarrow{\scriptstyle \sigma} \\
\mathcal{O}_{S'\otimes_R F_i} & \xrightarrow{\beta_i'} & (x_i \times \mathrm{id}_{S'})^*\mathcal{L}'
\end{array}
\tag{1.6.3}
$$

commutes. Note that u is uniquely determined by σ. If $S' \in (Sm/S)$ is actually an F-scheme, then giving a pair (u, σ) is the same as giving an isomorphism $(\mathcal{L}, \beta) \xrightarrow{\sim} (\mathcal{L}', \beta')$, since we can absorb u into σ, and therefore the restrictions of $\mathcal{F}^*$ and P_Σ to (Sm/F) are equal.

Theorem 1.15. *The restriction of $P_\mathfrak{m}$ to $(Sm/S)_{\text{ét}}$ is the sheafification for the étale topology of the presheaf $\mathcal{F}^*$.*

Proof. We have an exact sequence of fppf sheaves

$$0 \to \mathbb{G}_\mathrm{m} \xrightarrow{\text{diag}} \mathcal{R}_{\Sigma/S}\mathbb{G}_\mathrm{m} \times \mathcal{G}_\mathrm{m} \xrightarrow{\psi} P_\Sigma \times \mathcal{R}_{\Sigma/S}\mathcal{G}_\mathrm{m}$$

where the map ψ on S'-valued points is given by

$$\psi\colon (a, b) \mapsto ((\mathcal{O}_{\mathcal{X}\times_S S'}, a\cdot \mathrm{id}_{\mathcal{O}_{\Sigma\times S'}}), a^{-1}b) \in P_\Sigma(S') \times \mathcal{G}_\mathrm{m}(\Sigma \times_S S').$$

By definition, $P_\mathfrak{m}$ is the cokernel of ψ in the category of fppf sheaves. As the coimage of ψ is a smooth S-group scheme, $P_\mathfrak{m}$ is also the cokernel of ψ in the category of étale sheaves. Let $S' \in (Sm/S)$ and consider the map

$$\phi_{S'}\colon P_\Sigma(S') \times \mathcal{R}_{\Sigma/S}\mathcal{G}_\mathrm{m}(S') = P_\Sigma(S') \times \mathbb{G}_\mathrm{m}(\Sigma_F \times_S S') \to \mathcal{F}^*(S')$$

given as follows: let $(\mathcal{L}, \alpha)$ represent an element of $P_\Sigma(S')$ and $v \in \mathbb{G}_\mathrm{m}(\Sigma_F \times_S S')$. We map the pair $((\mathcal{L}, \alpha), v)$ to the equivalence class of $(\mathcal{L}, \beta)$, where $\beta = \alpha \otimes v\colon \mathcal{O}_{\Sigma\times S'\otimes F} \xrightarrow{\sim} (g \times \mathrm{id}_{S'\otimes F})^*\mathcal{L}$. It is easy to see that this is well-defined and functorial, and that the resulting sequence of presheaves on (Sm/S)

$$\mathcal{R}_{\Sigma/S}\mathbb{G}_\mathrm{m} \times \mathcal{G}_\mathrm{m} \xrightarrow{\psi} P_\Sigma \times \mathcal{R}_{\Sigma/S}\mathcal{G}_\mathrm{m} \xrightarrow{\phi} \mathcal{F}^*$$

is exact. Moreover, for any $(\mathcal{L}, \beta) \in \mathcal{F}^*(S')$, there exists a Zariski cover $S'' \to S'$ such that $(\mathcal{L}, \beta)|_{S''}$ is in the image of $\phi_{S''}$. The result follows. □

Let $E_\mathfrak{m}$ denote the closure in $P_\mathfrak{m}$ of the zero section. It is contained in

$$P'_\mathfrak{m} = \ker(\deg\colon P_\mathfrak{m} \to P \to \mathbb{Z}).$$

Theorem 1.16.
(a) *The map r'* (1.6.1) *induces an isomorphism $E_\mathfrak{m} \xrightarrow{\sim} E$.*
(b) *The quotient $P'_\mathfrak{m}/E_\mathfrak{m}$ is represented by the Néron model $\mathcal{J}_\mathfrak{m}$ of $J_\mathfrak{m}$.*
(c) *There is an exact sequence of Néron models*

$$0 \to \mathcal{T}_\mathfrak{m} \to \mathcal{J}_\mathfrak{m} \to \mathcal{J} \to 0.$$

(d) *Assume that S is strictly Henselian. Then there is a canonical isomorphism*

$$P_{\mathfrak{m},s}/P^0_{\mathfrak{m},s} \xrightarrow{\sim} \mathbb{Z}^C \oplus \mathbb{Z}^I/e\mathbb{Z}$$

where $e = (e_i)\colon \mathbb{Z} \to \mathbb{Z}^I$ is as in Proposition 1.4.

The analogue of (a) need not hold for P_Σ — see Example 1.17 after the proof.

Proof. (a) By [29, (3.3.5)] and Proposition 1.14, $E_\mathfrak{m}$ is an étale algebraic space in groups over S. So we may compute it by restriction to $(Sm/S)_{\text{ét}}$, using the description of Theorem 1.15, and we may also assume that S is strictly Henselian. In this case, from §1.3 we have that $E(S)$ is generated by the classes of the line bundles $\mathcal{O}_\mathcal{X}(Y_j)$. Let $\beta_{\text{triv}} = (\beta_{\text{triv},i})$ be the trivial rigidification of the generic fibre $\mathcal{O}_\mathcal{X}(Y_j)_F = \mathcal{O}_X$ at (x_i). Then $E_\mathfrak{m}$ is generated by the equivalence classes of pairs $(\mathcal{O}_\mathcal{X}(Y_j), \beta_{\text{triv}})$, and therefore $E_\mathfrak{m} \simeq E$.

(b),(c) We now have an exact sequence

$$0 \to \mathcal{T}_\mathfrak{m} \to P'_\mathfrak{m}/E_\mathfrak{m} \to P'/E \to 0$$

of smooth separated S-algebraic spaces in groups, which are therefore separated S-group schemes [29, (3.3.1)], whose generic fibre is the sequence $0 \to T_\mathfrak{m} \to J_\mathfrak{m} \to J \to 0$. As $\mathcal{T}_\mathfrak{m}$ and P'/E are the Néron models of T and J, the result follows from Proposition 1.3.

(d) From §1.1 above, $\mathcal{T}_{\mathfrak{m},s}/\mathcal{T}^0_{\mathfrak{m},s} \simeq \operatorname{coker}(e\colon \mathbb{Z} \to \mathbb{Z}^I)$. We then have a commutative diagram of étale sheaves on (Sm/S)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{T}_\mathfrak{m} & \longrightarrow & P_\mathfrak{m} & \longrightarrow & P & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & s_*(\mathbb{Z}^I/e\mathbb{Z}) & \longrightarrow & P_\mathfrak{m}/P^0_\mathfrak{m} & \longrightarrow & P/P^0 & \longrightarrow & 0
\end{array}$$

whose rows are exact (since π_0 is right exact). For S'/S smooth, and $(\mathcal{L}, \beta = (\beta_i))$ representing an element of $\mathcal{F}^*(S')$, $\beta_i(1)$ is a rational section of $(g_i \times \mathrm{id}_{S'})^*\mathcal{L}$ so has a well-defined order along the special fibre $\operatorname{ord}_\mathcal{L} \beta_i(1) \in \Gamma(S', s_*\mathbb{Z})$. If $(\mathcal{L}', \beta')$ is equivalent to $(\mathcal{L}, \beta)$ then $(\operatorname{ord}_\mathcal{L} \beta'_i(1) - \operatorname{ord}_{\mathcal{L}'} \beta_i(1))_i \in \Gamma(S', s_*(e\mathbb{Z}))$, which gives a splitting of the bottom row in the diagram (which is therefore also exact on the left). ☐

Example 1.17. Let's work out the simplest nontrivial example: assume that $\operatorname{char}(F) \neq 2$, and let $\mathcal{X}$ be the closed subscheme of $\mathbb{P}^2_R$ given by the equation $T_1T_2 = \varpi T_0^2$. Then $X = \mathcal{X}_F$ is a smooth conic, split over F, and $\mathcal{X}_s$ is the line pair $T_1T_2 = 0$. Hypotheses (H1–3) of Section 1.3 are all satisfied. Let x_0, $x_1 \in X(F) = \mathcal{X}(S)$ be distinct points. Let $\mathcal{X}_s = Y \cup Y'$, where the components are labelled in such a way that x_0 meets Y'. We consider the generalized Jacobian $J_\mathfrak{m}$ with $\mathfrak{m} = (x_0) + (x_1)$. The relative Picard space $P = \operatorname{Pic}_{\mathcal{X}/S}$ is a scheme, and is the union of its sections over S. We have $P(F) = \mathbb{Z}$, generated by the class of $\mathcal{O}_X(x_0)$, and $P(R) = P_s(k) = \mathbb{Z}^2$, generated by the classes of $\mathcal{O}_\mathcal{X}(Y) \simeq \mathcal{O}_\mathcal{X}(-Y')$ and $\mathcal{O}_\mathcal{X}(x_0)$. The restriction map $P(S) \to P(F)$ is the second projection $\mathbb{Z}^2 \to \mathbb{Z}$, and equals the degree map. Therefore $P' = E$ is the "skyscraper scheme" $s_*\mathbb{Z}$, obtained by gluing copies of S indexed by $\mathbb{Z}$ along their generic points, and $P'(S)$ is generated by the class of $\mathcal{O}_\mathcal{X}(Y)$.

There is an isomorphism $\mathbb{G}_\mathrm{m} \xrightarrow{\sim} J_\mathfrak{m} = P'_\Sigma \otimes F$, which on F-points takes $a \in F^\times$ to the equivalence class of the pair $(\mathcal{O}_\mathcal{X}, \alpha = (\alpha_0, \alpha_1))$, where $\alpha_i\colon F \to x_i^*\mathcal{O}_\mathcal{X} = F$ is the identity for $i = 0$ and multiplication by a for $i = 1$. As x_0 doesn't meet Y we also have $x_0^*\mathcal{O}_\mathcal{X}(Y) = \mathcal{O}_S$. We now have two cases:

- If x_1 meets Y', then $x_1^*\mathcal{O}_\mathcal{X}(Y) = \mathcal{O}_S$ as well. So there is a canonical rigidification (α_i) of $\mathcal{O}_\mathcal{X}(Y)$ along Σ, for which each α_i is the identity map on $\mathcal{O}_S$, and therefore $P_\Sigma \simeq \mathbb{G}_\mathrm{m} \times P$ splits (and is not separated). Likewise, $P'_\Sigma \simeq \mathbb{G}_\mathrm{m} \times s_*\mathbb{Z}$. The pushout $P'_\mathfrak{m}$ is simply the product $\mathcal{G}_\mathrm{m} \times s_*\mathbb{Z}$.
- If x_1 meets Y, then $x_1^*\mathcal{O}_\mathcal{X}(Y) = \mathcal{O}_S(s) = \varpi^{-1}\mathcal{O}_S$. So there is a bijection $\mathbb{Z} \times R^\times \xrightarrow{\sim} P'_\Sigma(S)$ which takes (n, a) to the line bundle $\mathcal{O}_\mathcal{X}(nY)$ with rigidification $\alpha_0 = \mathrm{id}$, $\alpha_1(1) = \varpi^{-n}$. Its composition with restriction to the generic fibre is the bijection

$\mathbb{Z} \times R^\times \to P_\Sigma(F) = F^\times$ given by $(n,a) \mapsto \varpi^{-1}a$. So P'_Σ is separated, and is isomorphic to the Néron model $\mathcal{G}_{\mathrm{m}}$. The pushout $P'_{\mathfrak{m}}$ is then the tautological splitting of the extension $\mathbb{G}_{\mathrm{m}} \to \mathcal{G}_{\mathrm{m}} \to s_*\mathbb{Z}$ after pushing out through $\mathbb{G}_{\mathrm{m}} \to \mathcal{G}_{\mathrm{m}}$, so is isomorphic to $\mathcal{G}_{\mathrm{m}} \times s_*\mathbb{Z}$ in this case as well.

We return to the general case. From Section 1.3, $P^0_s \cap E_s$ is finite constant and cyclic of order d, generated by the class of the line bundle $\mathcal{L}'$. Therefore $P^0_{\mathfrak{m},s} \cap E_{\mathfrak{m},s}$ is finite constant and cyclic of order dividing d. Applying the results of Section 1.4, we obtain:

Corollary 1.18. *Assume that $d = 1$. Then:*
(a) $\mathcal{J}^0_{\mathfrak{m},s} = \mathrm{Pic}^0_{(\mathcal{X}_s,\Sigma_s)/k}$.
(b) *If k is perfect, there is a canonical isomorphism of* $\mathrm{Gal}(\bar{k}/k)$*-modules*

$$\mathrm{Hom}(\mathcal{J}^{0,\mathrm{lin}}_{\mathfrak{m},s} \otimes_k \bar{k}, \mathbb{G}_{\mathrm{m}}) = H_1(\widetilde{\Gamma}_{\mathcal{X}_{\bar{s}},\Sigma_{\bar{s}}}, \mathbb{Z})$$

where the graph $\widetilde{\Gamma}_{\mathcal{X}_{\bar{s}},\Sigma_{\bar{s}}}$ is as in Section 1.4.

Finally we compute the component group $\Phi(J_{\mathfrak{m}})$.

Theorem 1.19. *Suppose that R is strictly Henselian. Then $\Phi(J_{\mathfrak{m}})$ is canonically isomorphic to the homology of the complex*

$$\mathbb{Z}[C] \xrightarrow{(a,h)} \mathbb{Z}^C \oplus \mathbb{Z}^I/e\mathbb{Z} \xrightarrow{b\oplus 0} \mathbb{Z} \tag{1.6.4}$$

where a and b are as in (1.3.2)*, and $h\colon \mathbb{Z}[C] \to \mathbb{Z}^I/e\mathbb{Z}$ is induced by the map*

$$\begin{aligned} C \times I &\to \mathbb{Z} \\ (j,i) &\mapsto h_{ij} := \mathrm{ord}_{\mathcal{O}_{\mathcal{X}}(Y_j)}\, \beta_{\mathrm{triv},i}(1). \end{aligned}$$

(Equivalently, h_{ij} is the degree of the divisor $g_i^*Y_j$ on $\operatorname{Spec} R_i$.)

Proof. By Theorem 1.16, $\Phi(J_{\mathfrak{m}})$ is the group of connected components of the quotient $P'_{\mathfrak{m},s}/E_{\mathfrak{m},s}$, hence is the homology of the complex $E_{\mathfrak{m}}(k) \to \pi_0(P_{\mathfrak{m},s}) \xrightarrow{\deg} \mathbb{Z}$. By Theorem 1.16(d), we may rewrite this complex as (1.6.4). What remains is to identify the map h. By the proof of 1.16(a), $E_{\mathfrak{m}}(k)$ is generated by the equivalence class of pairs $(\mathcal{O}_{\mathcal{X}}(Y_j), \beta_{\mathrm{triv}})$, and the proof of Theorem 1.16(d) then gives the desired formula for h. □

For general S we have $\Sigma \times_S S^{\mathrm{sh}} = \coprod_{\tilde{i}\in\widetilde{I}} S_{\tilde{i}}$, where $S_{\tilde{i}}$ is the spectrum of a DVR finite over R^{sh}. Let $\widetilde{C}$ be the set of irreducible components of $\mathcal{X} \otimes k^{\mathrm{sep}}$. Then $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on $\widetilde{I}$ and $\widetilde{C}$ through its quotient $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, and the above gives a $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-equivariant isomorphism between $\Phi(J_{\mathfrak{m}})$ and the homology of the complex

$$\mathbb{Z}[\widetilde{C}] \xrightarrow{(a,h)} \mathbb{Z}^{\widetilde{C}} \oplus \mathbb{Z}^{\widetilde{I}}/e\mathbb{Z} \xrightarrow{b\oplus 0} \mathbb{Z} \tag{1.6.5}$$

attached to $\mathcal{X} \times_S S^{\mathrm{sh}}$.

In the semistable case we can describe both the character and component groups in terms of the reduced extended graph.

Corollary 1.20. *Suppose that $\mathcal{X} \otimes R^{\mathrm{sh}}$ is semistable and $I = I^{\mathrm{reg}}$. Then*
(a) *If k is perfect, there is a canonical isomorphism of* $\mathrm{Gal}(\bar{k}/k)$*-modules*

$$\mathrm{Hom}(\mathcal{J}^{0,\mathrm{lin}}_{\mathfrak{m},s} \otimes_k \bar{k}, \mathbb{G}_{\mathrm{m}}) = H_1(\Gamma_{\mathcal{X}_{\bar{s}},\Sigma_{\bar{s}}}, \mathbb{Z})$$

where the reduced extended graph $\Gamma_{\mathcal{X}_{\bar{s}},\Sigma_{\bar{s}}}$ is as in Section 1.4.

(b) *Assume that R is strictly Henselian. There is a canonical isomorphism*

$$\Phi(J_{\mathfrak{m}}) = \operatorname{coker}\big((\Box, \theta^*) \colon \mathbb{Z}[C] \to \mathbb{Z}[C]_0 \oplus \mathbb{Z}^I\big)$$

where $\Box$ is the Laplacian of the reduced graph $\Gamma_{\mathcal{X}_s}$, and $\theta \colon I \to C$ is the map from Section 1.4.

Note that θ depends only on the labelled graph $(\Gamma_{\mathcal{X}_s,\Sigma_s}, v_0)$. The hypothesis $I = I^{\mathrm{reg}}$ is satisfied if for example $\{x_i\} \subset X(F^{\mathrm{sh}})$.

Proof. (a) follows immediately from Corollary 1.18(b) and the fact that the geometric realisations of $\Gamma_{\mathcal{X}_s,\Sigma_s}$ and $\widetilde{\Gamma}_{\mathcal{X}_s,\Sigma_s}$ are homeomorphic. For (b), it is enough to observe that $(\Box, \theta^*)$ maps $1 \in \mathbb{Z}[C]$ to $(0, 1) \in \mathbb{Z}[C]_0 \oplus \mathbb{Z}^I$, and so the result follows from Theorem 1.19. $\Box$

1.7. **Description via Néron models of 1-motives** [35]**.** An alternative approach to the determination of the component group $\Phi(J_{\mathfrak{m}})$ is via duality and the theory of Néron models of 1-motives developed in [35]. We recall some of the notions and results of that paper. Recall that a 1-motive over F is a two-term complex of group schemes over F

$$M = [\, L \xrightarrow{f} G \,]$$

where L is étale, free and finitely generated (i.e. $L \otimes_F F^{\mathrm{sep}} \simeq \mathbb{Z}^r$), and G/F is a semiabelian variety. Let $T \subset G$ be its toric part, and $A = G/T$ the abelian variety quotient. We assume here that L and T split over an extension of F in which R is unramified. Then L extends to a local system Λ on S. Let $\mathcal{G}$ be the Néron model of G. By the Néron property, f extends to a morphism $f_S \colon \Lambda \to \mathcal{G}$ of S-group schemes, and by definition, the Néron model of M is the complex of S-group schemes

$$\mathcal{M} = [\, \Lambda \xrightarrow{f_S} \mathcal{G} \,].$$

Its component complex is the complex of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-modules

$$\Phi(M) = [\, \Lambda_{\bar{s}} \to \Phi(G) \,]$$

in degrees -1 and 0. (In [35] this complex is denoted $\mathcal{P}(\mathcal{M})$.)

Let M' be the 1-motive dual to M. So

$$M' = [\, L' \xrightarrow{f'} G' \,]$$

where $L' = \mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$ is the character group of T, and G' is an extension $T' \to G' \to A'$, where A' is the dual abelian variety of A, and T' is the torus with character group L. Then [35, Theorem B] shows that if either

(i) A has semistable reduction, or
(ii) k is perfect

there is a canonical isomorphism, in the derived category of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-modules, between $\Phi(M')$ and $\mathrm{RHom}(\Phi(M), \mathbb{Z})[1]$.

Now let $\mathcal{X}/S$ be as in Section 1.6. We will assume that R is strictly Henselian. Suppose that all n_j are zero (which holds, for example, if k is perfect), that $\delta = 1$, and that the points $(x_i)_{i \in I}$ are all F-rational.

Since J is autodual, the dual 1-motive to $J_{\mathfrak{m}}$ is the 1-motive

$$M = [\, \mathbb{Z}[I]_0 \to J \,], \qquad i \mapsto \mathcal{O}_X(x_i)$$

whose component complex $\Phi(M)$ is the complex $[\, \mathbb{Z}[I]_0 \to \Phi(J) \,]$ of abelian groups, concentrated in degrees -1 and 0. Using the description (1.3.1) of $\Phi(J)$, this is isomorphic to the complex $[\, \mathbb{Z}[I]_0 \to \mathbb{Z}^{C,0}/a(\mathbb{Z}^C) \,]$.

As $\delta = 1$, by [29, (8.1.2)] the complex (1.3.1) has only one nonzero homology group, namely $\ker(b)/\operatorname{im}(a) = \Phi(J)$, and the map a is given by the intersection pairing on the components of the special fibre. Therefore $\Phi(M)$ is quasi-isomorphic to the complex

$$\mathbb{Z} \xrightarrow{(i,0)} \mathbb{Z}[C] \oplus \mathbb{Z}[I]_0 \xrightarrow{a \oplus {}^t h} \mathbb{Z}^C \xrightarrow{\ b\ } \mathbb{Z}. \tag{1.7.1}$$

Here ${}^t h\colon \mathbb{Z}[I]_0 \to \mathbb{Z}^C$ is the transpose of h, and the term $\mathbb{Z}^C$ is in degree 0. The dual of (1.7.1) is

$$\mathbb{Z} \xrightarrow{{}^t b} \mathbb{Z}[C] \xrightarrow{({}^t a, h)} \mathbb{Z}^C \oplus \mathbb{Z}^I/\mathbb{Z} \xrightarrow{{}^t i \oplus 0} \mathbb{Z}$$

The assumption $n_j = 0$ ensures that a is symmetric, and that i and b are transposes of one another, by (1.3.2). Assuming that one of (i), (ii) above holds, we then recover the description of $\Phi(J_\mathfrak{m})$ as the homology of (1.6.4).

1.8. **Functoriality II.** We will need to understand the action of correspondences on generalized Jacobians and their Néron models.

Suppose that we have two smooth geometrically connected curves X, X' over F, with regular models $\mathcal{X}$, $\mathcal{X}'$ satisfying the hypotheses of §1.3. Let $\mathfrak{m} = \sum_{i\in I}(x_i)$, $\mathfrak{m}' = \sum_{j\in I'}(x'_j)$, be nonzero moduli on X, X'. As in §1.6 we assume that the points x_i, x'_j are distinct, and that their residue fields

$$F_i = \kappa(x_i), \quad F'_j = \kappa(x'_j)$$

are separable over F. Let R_i, R'_j denote the integral closures of R in F_i, F'_j, and $\Sigma = \coprod_{i\in I} \operatorname{Spec} R_i$, $\Sigma' = \coprod_{j\in I'} \operatorname{Spec} R'_j$. Let $J_\mathfrak{m}$, $J'_{\mathfrak{m}'}$ be the associated generalized Jacobians.

Let $f\colon X' \to X$ be a finite morphism such that $f^{-1}(\Sigma_F) = \Sigma'_F$ as sets. We write $f\colon I' \to I$ also for the induced surjective map on index sets. For $j \in I'$, denote by r_j the ramification degree of f at x'_j.

The discussion in §1.5 applies, and since f^*, f_* preserve line bundles of degree zero, we obtain morphisms

$$f^*\colon J_\mathfrak{m} \to J'_{\mathfrak{m}'}, \quad f_*\colon J'_{\mathfrak{m}'} \to J_\mathfrak{m}\,.$$

By the universal Néron property, they extend uniquely to morphisms f^*, f_* of the Néron models $\mathcal{J}_\mathfrak{m}$, $\mathcal{J}'_{\mathfrak{m}'}$. Let the induced homomorphisms of character groups be $\mathbb{X}(f^*)$, $\mathbb{X}(f_*)$ and of component groups $\Phi(f^*)$, $\Phi(f_*)$. In the next section we will need to know explicitly the restriction of these maps to the tori $\mathcal{T}_\mathfrak{m}$, $\mathcal{T}'_{\mathfrak{m}'}$. For (a) and (b) below, recall that we have a canonical isomorphism

$$\operatorname{Hom}(T_\mathfrak{m} \otimes F^{\mathrm{sep}}, \mathbb{G}_\mathrm{m}) \xrightarrow{\sim} \mathbb{Z}[\Sigma(F^{\mathrm{sep}})]^{\deg=0}$$

and similarly for $T'_{\mathfrak{m}'}$.

Proposition 1.21.

(a) *The map*

$$f^*\colon T_\mathfrak{m} = \Big(\prod_{i\in I} R_{F_i/F}\mathbb{G}_\mathrm{m}\Big)/\mathbb{G}_\mathrm{m} \to T'_{\mathfrak{m}'} = \Big(\prod_{j\in I'} R_{F'_j/F}\mathbb{G}_\mathrm{m}\Big)/\mathbb{G}_\mathrm{m}$$

is induced by the inclusions $f^*\colon F_i \hookrightarrow F'_j$, $i = f(j)$. Its transpose is the homomorphism

$$f_*\colon \mathbb{Z}[\Sigma'(F^{\mathrm{sep}})]^{\deg=0} \to \mathbb{Z}[\Sigma(F^{\mathrm{sep}})]^{\deg=0}$$

given by pushforward of divisors of degree zero.

(b) *The map* $f_*\colon T'_{\mathfrak{m}'} \to T_{\mathfrak{m}}$ *is given by the morphisms of tori, for* $i = f(j)$,

$$\begin{aligned}\mathcal{R}_{F'_j/F_i}\mathbb{G}_{\mathrm{m}} &\to \mathcal{R}_{F_i/F}\mathbb{G}_{\mathrm{m}}\\ t &\mapsto (N_{F'_j/F_i}t)^{r_j}\ .\end{aligned}$$

Its transpose is the homomorphism

$$f^*\colon \mathbb{Z}[\Sigma(F^{\mathrm{sep}})]^{\deg=0} \to \mathbb{Z}[\Sigma'(F^{\mathrm{sep}})]^{\deg=0}$$

given by pullback of divisors.

(c) *Assume that* $R = R^{\mathrm{sh}}$. *Then the induced maps between component groups* $\Phi(T_{\mathfrak{m}})$, $\Phi(T'_{\mathfrak{m}'})$ *are*

$$\begin{aligned}\Phi(f^*)\colon\ \Phi(T_{\mathfrak{m}}) = \mathbb{Z}^I/e\mathbb{Z} &\longrightarrow \Phi(T'_{\mathfrak{m}'}) \simeq \mathbb{Z}^{I'}/e'\mathbb{Z}\\ (n_i)_{i\in I} &\longmapsto \big((e'_j/e_{f(j)})n_{f(j)}\big)_{j\in I'}\end{aligned}$$

$$\begin{aligned}\Phi(f_*)\colon\qquad \mathbb{Z}^{I'}/e'\mathbb{Z} &\longrightarrow \mathbb{Z}^I/e\mathbb{Z}\\ (n_j)_{j\in I'} &\longmapsto \Big(\sum_{j\in f^{-1}(\{i\})} r_j n_j\Big)_{i\in I}\end{aligned}$$

(d) *Assume that* $R = R^{\mathrm{sh}}$, *and that* k *is algebraically closed (so that* $\Sigma(k) \simeq I$, $\Sigma'(k) \simeq I'$*). Then the induced maps on character groups of Néron models are*

$$\begin{array}{rccc}\mathbb{X}(f^*)\colon & \mathbb{X}(T'_{\mathfrak{m}'}) & \longrightarrow & \mathbb{X}(T_{\mathfrak{m}})\\ & \| & & \| \\ & \mathbb{Z}[I']^{\deg=0} & \xrightarrow{\ f\ } & \mathbb{Z}[I]^{\deg 0}\end{array}$$

$$\begin{aligned}\mathbb{X}(f_*)\colon\ \mathbb{Z}[I]^{\deg=0} &\xrightarrow{\ f\ } \mathbb{Z}[I']^{\deg 0}\\ r(i) &\longmapsto \sum_{j\in f^{-1}(\{i\})} r_j[F'_j : F_i](j)\end{aligned}$$

Proof. For (a) and (b), it suffices to compute the map on character groups. The formulae are then special cases of Propositions 1.11 and 1.12 with $A = B = \Sigma^{\mathrm{sing}} = \emptyset$, $C = \{*\}$.

Combining these with Proposition 1.4 then gives the remaining parts. □

Remark. From (a) and (b) we see that if f, $f'\colon X' \to X$ are finite morphisms and $\mathfrak{m}$ is a reduced modulus on X which is stable under the correspondence $A = f_*f'^*$ (in the sense of Example 1.9), then the induced endomorphism ${}^tA$ of the character group $\mathrm{Hom}(T_{\mathfrak{m}} \otimes F^{\mathrm{sep}}, \mathbb{G}_{\mathrm{m}})$ equals the map $D \mapsto f'_*f^*D$ on divisors of degree zero.

2. Generalized Jacobians of modular curves

2.1. **Generalities on modular curves.** For an integer $N \geq 1$, let $X_0(N)_{\mathbb{Q}}$ denote the usual complete modular curve over $\mathbb{Q}$. Its non-cuspidal points parametrize pairs (E, C), where E is an elliptic curve (over some $\mathbb{Q}$-scheme) and $C \subset E$ is a subgroup scheme which is cyclic of order N. We write $X_0(N)_{\mathbb{Z}}$ for the integral model constructed by Katz and Mazur [17, Ch. 8], which they denote $\overline{M}([\Gamma_0(N)])$. Its non-cuspidal points parametrize pairs (E, C), where $C \subset E$ is a subgroup scheme of rank N which is cyclic in the sense of *loc. cit.*, §6.1 — see also [9, §1.1].

For every prime ℓ we have a Hecke correspondence $T_\ell = v_* u^*$, where the finite morphisms $u = u_\ell$, $v = v_\ell$ are given by:

$$\begin{array}{ccc} & X_0(N\ell) & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle v} \\ X_0(N) & & X_0(N) \end{array} \qquad \begin{array}{ccc} & (E,C) & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle v} \\ (E,\ell C) & & (E/NC, C/NC) \end{array}$$

For $\ell \nmid N$ (resp. $\ell \mid N$), the morphisms u, v are of degree $\ell + 1$ (resp. ℓ). For $p \mid N$ we also have the Atkin-Lehner involution $W_p \colon X_0(N) \to X_0(N)$. If $v_p(N) = r \geq 1$ then

$$W_p \colon (E, C) \mapsto (E/(C \cap E[p^r]), (C + E[p^r])/(C \cap E[p^r])).$$

When $\ell \nmid N$, $u = v \circ W_\ell$. and the correspondence T_ℓ is symmetric. When $\ell \mid N$, T_ℓ is no longer symmetric, and what we call T_ℓ is often elsewhere defined to be the transpose of T_ℓ (and also often written U_ℓ). We have chosen our normalisations so that the endomorphism $T_\ell = v_* u^*$ of the Jacobian $J_0(N)_\mathbb{Q}$ agrees with the Hecke operator in [30, p. 445] defined by "Picard functoriality".

Write $X_0(N)^\infty_\mathbb{Q} \subset X_0(N)_\mathbb{Q}$ for the cuspidal subscheme. It is classical that $X_0(N)^\infty_\mathbb{Q}$ is the disjoint union, over positive divisors $d \mid N$, of schemes $z_d \simeq \operatorname{Spec} \mathbb{Q}(\mu_{(d,N/d)})$ (where here $(d, N/d)$ denotes greatest common divisor). We recall (e.g. from [8, IV.4.11–13]) that the cusps of $X_0(N)_\mathbb{Q}$ can be conveniently described using generalized elliptic curves. Suppose that $d \mid N$, and let Nér_d denote the standard Néron polygon over $\mathbb{Q}$ with d sides [8, II.1.1], whose smooth locus $\text{Nér}_d^{\text{reg}}$ equals $\mathbb{G}_\text{m} \times \mathbb{Z}/d$. For a primitive N-th root of unity $\zeta_N \in \overline{\mathbb{Q}}$, let C_{d,ζ_N} denote the cyclic subgroup scheme

$$C_{d,\zeta_N} = \langle (\zeta_N, 1) \rangle \subset \text{Nér}_d^{\text{reg}}.$$

Then the pair $(\text{Nér}_d, C_{d,\zeta_N})$ determines a $\overline{\mathbb{Q}}$-point of $X_0(N)^\infty$. If $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_d))$, then $C_{d,\sigma\zeta_N} = C_{d,\zeta_N}$, and if $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{(d,N/d)}))$ then the pairs $(\text{Nér}_d, C_{d,\sigma\zeta_N})$ and $(\text{Nér}_d, C_{d,\zeta_N})$ are isomorphic (and the isomorphism is unique if it is required to be the identity on the identity component of Nér_d). Therefore the isomorphism class of $(\text{Nér}_d, C_{d,\zeta_N})$ over $\overline{\mathbb{Q}}$ is determined by the pair $(d, \zeta_N^{N/(d,N/d)})$ and gives rise to a closed point $z_d = z_{N,d} \simeq \operatorname{Spec} \mathbb{Q}(\mu_{(d,N/d)})$ of $X_0(N)^\infty_\mathbb{Q}$.

In particular, the (rational) cusps $\infty = z_{N,1}$ and $0 = z_{N,N}$ correspond to the pairs $(\text{Nér}_1, \mu_N)$ and $(\text{Nér}_N, \{1\} \times \mathbb{Z}/N)$, respectively. We also know that the scheme-theoretic closure of $X_0(N)^\infty_\mathbb{Q}$ in $X_0(N)_\mathbb{Z}$ is the disjoint union of copies of $\operatorname{Spec} \mathbb{Z}(\mu_{(d,N/d)})$ (this follows from [9, Thm. 1.2.2.1]).

Now let $\mathfrak{m}$ be a reduced modulus on $X_0(N)_\mathbb{Q}$, whose support is contained in $X_0(N)^\infty_\mathbb{Q}$. Let ℓ be any prime such that the support of $\mathfrak{m}$ is stable under T_ℓ, in the sense of Example 1.9. Then T_ℓ determines an endomorphism $T_\ell = v_* u^*$ of $J_\mathfrak{m}$. Let $p \mid N$, and let $\mathcal{J}_\mathfrak{m}$ be the Néron model of $J_\mathfrak{m}$ over $\mathbb{Z}_{(p)}$. By the universal Néron property, T_ℓ extends to an endomorphism of $\mathcal{J}_\mathfrak{m}$, and therefore induces endomorphisms

$$T_\ell \colon \Phi(J_\mathfrak{m}) \to \Phi(J_\mathfrak{m})$$
$${}^t T_\ell \colon \mathbb{X}(J_\mathfrak{m}) \to \mathbb{X}(J_\mathfrak{m}).$$

In order to compute these endomorphisms combinatorially, we need to compute the action of T_ℓ on the torus $\mathcal{T}_\mathfrak{m}$, using the formulae of Proposition 1.21. In other words, we need to compute the restrictions of $u = u_\ell$, $v = v_\ell$ to the cusps, along with the ramification degrees.

Let $\zeta_{N\ell} \in \overline{\mathbb{Q}}$ be a primitive $N\ell$-th root of unity, and for $L \mid N\ell$, $\zeta_L = \zeta_{N\ell}^{N\ell/L}$. Let $z = (\text{Nér}_d, C_{d,\zeta_{N\ell}}) \in X_0(N\ell)^\infty(\overline{\mathbb{Q}})$ be a cusp. Then $u(z) \in X_0(N)^\infty(\overline{\mathbb{Q}})$ is obtained as follows: replace $C_{d,\zeta_{N\ell}}$ by $\ell C_{d,\zeta_{N\ell}} = \langle(\zeta_N, \ell)\rangle \subset \mathbb{G}_\mathrm{m} \times \mathbb{Z}/d$, and then contract any components of Nér_d which do not meet it [8, IV.1.2]. Similarly, we obtain $v(z)$ as the quotient of $(\text{Nér}_d, C_{d,\zeta_{N\ell}})$ by the rank-ℓ group scheme $NC_{d,\zeta_{N\ell}} = \langle(\zeta_\ell, N)\rangle \subset \mathbb{G}_\mathrm{m} \times \mathbb{Z}/d$.

Explicitly, suppose that $N = M\ell^k$, $(\ell, M) = 1$, and that $d \mid N\ell$. Let a, $b \in \mathbb{Z}$ with $a\ell + bM = 1$ and $a \equiv 1 \pmod{\ell^k}$. Then if $\ell \nmid d$,

$$(\text{Nér}_d, \langle(\zeta_N, \ell)\rangle) = (\text{Nér}_d, C_{d,\zeta_N^a})$$

but if $\ell \mid d$, the subgroup $\langle(\zeta_N, \ell)\rangle)$ does not meet the components $\mathbb{G}_\mathrm{m} \times \{i\}$ with $(i, \ell) = 1$. The map $\text{Nér}_d \to \text{Nér}_{d/\ell}$ contracting them takes $\langle(\zeta_N, \ell)\rangle$ to C_{d,ζ_N}, and so

$$u\colon (\text{Nér}_d, C_{d,\zeta_{N\ell}}) \mapsto \begin{cases} (\text{Nér}_d, C_{d,\zeta_N^a}) & \text{if } \ell \nmid d \\ (\text{Nér}_{d/\ell}, C_{d/\ell,\zeta_N}) & \text{otherwise.} \end{cases}$$

On closed points of $X_0(N)^\infty_{\mathbb{Q}}$ we then have

$$u^{-1}(z_{N,d}) = \begin{cases} \{z_{N\ell,d\ell}\} & \text{if } \ell | d \\ \{z_{N\ell,d},\ z_{N\ell,d\ell}\} & \text{otherwise.} \end{cases} \tag{2.1.1}$$

If $\ell \not| N$ then u has degree $\ell + 1$ and

$$\deg z_{N,d} = \deg z_{N\ell,d} = \deg z_{N\ell,d\ell} = \phi((d, N/d)).$$

It is well known (and follows, for example, from the Eichler-Shimura congruence relation) that u is étale at $z_{N\ell,d}$, and so has ramification degree ℓ at $z_{N\ell,d\ell}$.

Suppose now that $k \geq 1$ and $d_0 | M$, $d = d_0\ell^s$. Then u has degree ℓ and

$$\begin{aligned} \deg z_{N\ell,d} &= \phi(\ell^{\min(s,k+1-s)})\phi((d_0, M/d_0)) \\ \deg z_{N\ell,d/\ell} &= \phi(\ell^{\min(s-1,k+1-s)})\phi((d_0, M/d_0)) \quad \text{if } s \geq 1 \end{aligned}$$

so by (2.1.1), the ramification degree of u at $z_{N\ell,d}$ equals

$$\begin{aligned} 1 &\quad \text{if } 1 < s \leq (k+1)/2, \text{ and} \\ \ell &\quad \text{if } (k+1)/2 < s \leq k+1. \end{aligned}$$

Moreover, since

$$\begin{aligned} \deg z_{N\ell,d_0\ell} &= (\ell - 1)\phi((d_0, M/d_0)) \\ \deg z_{N\ell,d_0} = \deg z_{N,d_0} &= \phi((d_0, M/d_0)) \end{aligned}$$

the ramification degree equals 1 also for $s \in \{0, 1\}$.

Similarly, if $d \mid N$, then the subgroup $NC_{d,\zeta_{N\ell}} \subset \text{Nér}_d^{\text{reg}} = \mathbb{G}_\mathrm{m} \times \mathbb{Z}/d$ equals $\mu_\ell \times \{0\}$, and therefore is the kernel of the endomorphism $(t, i) \mapsto (t^\ell, i)$ of Nér_d. If $d \mid N\ell$ but $d \nmid N$ then $NC_{d,\zeta_{N\ell}} = \langle(\zeta_\ell, N)\rangle$ is the kernel of the map $\text{Nér}_d \to \text{Nér}_{d/\ell}$, $(t, i) \mapsto (t\zeta_{\ell^k+1}^{-bi}, i \bmod d/\ell)$, which maps $(\zeta_{N\ell}, 1)$ to $(\zeta_N^a, 1)$, and therefore

$$v\colon (\text{Nér}_d, C_{d,\zeta_{N\ell}}) \mapsto \begin{cases} (\text{Nér}_d, C_{d,\zeta_N}) & \text{if } d \mid N \\ (\text{Nér}_{d/\ell}, C_{d/\ell,\zeta_N^a}) & \text{otherwise.} \end{cases}$$

A similar computation as for u shows that the ramification degree of v at $z_{N\ell,d}$ is 1 if $v_\ell(d) \geq (k+1)/2$, and ℓ otherwise.

In particular, if $\mathfrak{m}$ is any reduced modulus supported on $X_0(N)^\infty_{\overline{\mathbb{Q}}}$, then for every $\ell \nmid N$, $\mathfrak{m}$ is stable under T_ℓ (in the sense of Example 1.9) and therefore we obtain an endomorphism $T_\ell = v_* u^*$ of the generalized Jacobian $J_\mathfrak{m} = J_0(N)_\mathfrak{m}$. If $\mathfrak{m}$ is the full cuspidal modulus (i.e. the reduced modulus whose support is $X_0(N)^\infty_{\overline{\mathbb{Q}}}$) then $\mathfrak{m}$ is stable under T_ℓ for every ℓ. Using the formulae from Proposition 1.21 together with the fact that $(\mathrm{Nér}_d, C_{d,\zeta_N})$ depends only on $(d, \zeta_N^{N/(d,N/d)})$, we can compute the induced endomorphism ${}^tT_\ell$ of the character group

$$\mathrm{Hom}(T_\mathfrak{m} \otimes \overline{\mathbb{Q}}, \mathbb{G}_\mathrm{m}) = \mathbb{Z}[X_0(N)^\infty(\overline{\mathbb{Q}})]^{\deg=0}$$

which is the restriction of $u_* v^*$ to divisors of degree zero.

Proposition 2.1. (a) *If $(\ell, N) = 1$, then*

$${}^tT_\ell(\mathrm{Nér}_d, C_{d,\zeta_N}) = (\mathrm{Nér}_d, C_{d,\zeta_N^\ell}) + \ell(\mathrm{Nér}_d, C_{d,\zeta_N^a}).$$

(b) *If $N = M\ell^k$ with $(M, \ell) = 1$ and $k > 0$, and $v_\ell(d) = i$, then let $d = d_0\ell^i$, $e_0 = (d_0, M/d_0)$, $\Gamma_i = \mathrm{Gal}(\mathbb{Q}(\mu_{e_0\ell^{k+1-i}})/\mathbb{Q}(\mu_{e_0\ell^{k-i}}))$. Then*

$${}^tT_\ell(\mathrm{Nér}_d, C_{d,\zeta_N}) = \begin{cases} \ell(\mathrm{Nér}_d, C_{d,\zeta_N^a}) & i = 0 \\ \ell(\mathrm{Nér}_{d/\ell}, C_{d/\ell,\zeta_N}) & 0 < i < (k+1)/2 \\ \sum_{\sigma\in\Gamma_i} \sigma(\mathrm{Nér}_{d/\ell}, C_{d/\ell,\zeta_N}) & (k+1)/2 \le i < k \\ \sum_{\sigma\in\Gamma_k} \sigma(\mathrm{Nér}_{d/\ell}, C_{d/\ell,\zeta_N}) + (\mathrm{Nér}_d, C_{d,\zeta_N^q}) & i = k \end{cases}$$

where a, b are as above, and $aq \equiv 1 \pmod N$.

(In (b) $\Gamma_i \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times$ if $i = k$ and $\mathbb{Z}/\ell\mathbb{Z}$ otherwise, so consistent with $\deg {}^tT_\ell = \ell$.)

Proof. First note that if $v_\ell(d) = k+1$ then

$$v\colon (\mathrm{Nér}_d, C_{d,\zeta_{N\ell}^q}) \mapsto (\mathrm{Nér}_{d/\ell}, C_{d/\ell,\zeta_N}).$$

$\boxed{k = 0}$ Then

$$v^*(\mathrm{Nér}_d, C_{d,\zeta_N}) = \ell(\mathrm{Nér}_d, C_{d,\zeta_{N\ell}}) + (\mathrm{Nér}_{d\ell}, C_{d\ell,\zeta_{N\ell}^q})$$

hence (since $q \equiv \ell \bmod N$ when $k = 0$)

$${}^tT_\ell(\mathrm{Nér}_d, C_{d,\zeta_N}) = \ell(\mathrm{Nér}_d, C_{d,\zeta_N^a}) + (\mathrm{Nér}_d, C_{d\ell,\zeta_N^\ell}).$$

$\boxed{k > 0}$ Then if $v_\ell(d) < (k+1)/2$,

$$v^*(\mathrm{Nér}_d, C_{d,\zeta_N}) = \ell(\mathrm{Nér}_d, C_{d,\zeta_{N\ell}})$$

and applying u_* to this gives $\ell(\mathrm{Nér}_d, C_{d,\zeta_N^a})$.

If $(k+1)/2 \le v_\ell(d) < k$ then the inverse image of the cusp $(\mathrm{Nér}_d, C_{d,\zeta_N})$ is the union of ℓ cusps conjugate to $(\mathrm{Nér}_d, C_{d,\zeta_{N\ell}})$, namely

$$v^*(\mathrm{Nér}_d, C_{d,\zeta_N}) = \sum_{\sigma\in\Gamma_i} \sigma(\mathrm{Nér}_d, C_{d,\zeta_{N\ell}}).$$

Finally if $v_\ell(d) = k$ then

$$v^*(\mathrm{Nér}_d, C_{d,\zeta_N}) = \sum_\sigma \sigma(\mathrm{Nér}_d, C_{d,\zeta_{N\ell}}) \qquad (\ell - 1 \text{ terms})$$
$$+ (\mathrm{Nér}_{d\ell}, C_{d\ell,\zeta_{N\ell}^q})$$

Apply u_* to this and we get the claimed formula. □

Example 2.2. Set $D = (0) - (\infty)$. Here are particular cases we will need:

(a) $(\ell,N)=1$, $\mathfrak{m}=(\infty)+(0)=(\text{Nér}_1,\mu_N)+(\text{Nér}_N,\mathbb{Z}/N)$. Then

$$^tT_\ell\colon D\mapsto(\ell+1)D.$$

(b) $N=p$ prime, $\mathfrak{m}=(\infty)+(0)$. Then $^tT_p\colon D\mapsto D$.

(c) $N=p^2$. There are $(p+1)$ elements of $X_0(p^2)^\infty(\overline{\mathbb{Q}})$:

$$(\infty)=(\text{Nér}_1,C_{1,\zeta_{p^2}}),\ (0)=(\text{Nér}_{p^2},C_{p^2,\zeta_{p^2}}),\ (\zeta_p)=(\text{Nér}_p,C_{p,\zeta_{p^2}})\ (1\neq\zeta_p\in\mu_p).$$

Then if $\ell\neq p$,

$$\begin{aligned}^tT_\ell\colon\ D&\mapsto(\ell+1)D\\ (\zeta_p)-(\infty)&\mapsto\ell(\zeta_p^{1/\ell})+(\zeta_p^\ell)-(\ell+1)(\infty)\end{aligned}$$

and

$$\begin{aligned}^tT_p\colon\ D&\mapsto\sum_{1\neq\zeta_p\in\mu_p}(\zeta_p)+(0)-p(\infty)\\ (\zeta_p)-(\infty)&\mapsto 0.\end{aligned}$$

2.2. **Character groups.** Assume that $N=pM$, with $p>3$ prime and $(p,M)=1$. Let SS_M be the set of supersingular points of $X_0(M)(\overline{\mathbb{F}}_p)$, which is the set of isomorphism classes of pairs (E,C), where $E/\overline{\mathbb{F}}_p$ is a supersingular elliptic curve and $C\subset E$ is a cyclic subgroup scheme of order M.

For $\ell\nmid M$, we have the Hecke operator

$$\begin{aligned}T_\ell\colon\mathbb{Z}[\mathsf{SS}_M]&\to\mathbb{Z}[\mathsf{SS}_M]\\ (E,C)&\mapsto\sum_{D\subset E,\ \#D=\ell}(E/D,(C+D)/D).\end{aligned}$$

Theorem 2.3. *Let $\mathfrak{m}=(\infty)+(0)$ and $J=J_0(N)$ with $N=pM$ as above. Then there is a canonical isomorphism*

$$\mathbb{X}(J_\mathfrak{m})\xrightarrow{\sim}\mathbb{Z}[\mathsf{SS}_M]$$

taking $^tT_\ell$ to T_ℓ for every $\ell\nmid N$. Its restriction to $\mathbb{X}(J)\hookrightarrow\mathbb{X}(J_\mathfrak{m})$ is an isomorphism $\mathbb{X}(J)\xrightarrow{\sim}\mathbb{Z}[\mathsf{SS}_M]_0$.

(The second isomorphism is of course well known: see [30, Prop. 3.1].)

Proof. We work over S, the strict henselisation of $\operatorname{Spec}\mathbb{Z}_{(p)}$, and use the notations from §2, so that $k=\overline{\mathbb{F}}_p$. Let $\mathcal{X}'$ denote the Deligne-Rapoport model of $X_0(N)$ over S. Since p exactly divides N, $\mathcal{X}'$ is regular apart from possible A_2 or A_3 singularities at supersingular points in the special fibre where $j=0$ or 1728. Let $\mathcal{X}\to\mathcal{X}'$ be its minimal desingularisation. The special fibre $\mathcal{X}'_s$ is the union of two copies of the modular curve $X_0(M)_{\overline{\mathbb{F}}_p}$ meeting transversally at the supersingular points. The cusp ∞ (resp. 0) meets the component of $\mathcal{X}'_s$ parametrizing (E,C) where C contains the kernel of Frobenius (resp. Verschiebung). Let us refer to these as the ∞-component Z_∞ and 0-component Z_0 of $\mathcal{X}_s$.

First we assume that $\mathcal{X}'=\mathcal{X}$ is regular (which holds, for example, if M is divisible by some prime $q\equiv-1\pmod{12}$ or by 36 — see the second table in [10, 4.1.1]). Since $\mathcal{X}_s$ has an irreducible component of multiplicity one, the hypotheses (H1–3) of Section 1.3 are satisfied. Let $\Sigma\to\mathcal{X}$ be the morphism induced by $\mathfrak{m}$, so that Σ is the disjoint union of two sections of $\mathcal{X}$ over S. Then

$$J^0_{\mathfrak{m}/\overline{\mathbb{F}}_p}\simeq\operatorname{Pic}^0_{(\mathcal{X}_s,\Sigma_s)/k}\tag{2.2.1}$$

and since $\Sigma_s \subset \mathcal{X}_s^{\mathrm{reg}}$, by (1.4.4) we have

$$\mathbb{X}(J_{\mathfrak{m}}) = \ker\Big[\mathbb{Z}[\widetilde{\mathsf{SS}}] \oplus \mathbb{Z}[\Sigma_s] \xrightarrow{\left[\begin{smallmatrix}\psi & \theta\\ \phi & 0\end{smallmatrix}\right]} \mathbb{Z}[C] \oplus \mathbb{Z}[\mathsf{SS}]\Big] \tag{2.2.2}$$

where $C = \pi_0(\widetilde{\mathcal{X}_s})$, $\mathsf{SS} = \mathsf{SS}_M \simeq \mathcal{X}_s^{\mathrm{sing}}$ and $\widetilde{\mathsf{SS}}$ is the inverse image of SS in the normalisation $\widetilde{\mathcal{X}_s}$ of $\mathcal{X}_s$.

The map $\theta\colon \mathbb{Z}[\Sigma_s] \to \mathbb{Z}[C]$ is a bijection since the cusps meets different components, and $\mathcal{X}_s$ has only ordinary double points, so the vertical maps between the three 2-term complexes

$$\begin{array}{ccc}
\mathbb{Z}[\widetilde{\mathsf{SS}}] \oplus \mathbb{Z}[\Sigma_s] & \longrightarrow & \mathbb{Z}[C] \oplus \mathbb{Z}[\mathsf{SS}] \\
\downarrow & & \downarrow \\
\mathbb{Z}[\widetilde{\mathsf{SS}}] & \longrightarrow & \mathbb{Z}[\mathsf{SS}] \\
{\scriptstyle i}\uparrow & & \uparrow \\
\mathbb{Z}[\mathsf{SS}] & \longrightarrow & 0
\end{array}$$

are quasi-isomorphisms. Here i is the map taking $x \in \mathsf{SS}$ to $x^{(\infty)} - x^{(0)}$, with $x^{(\infty)}$, $x^{(0)} \in \mathcal{X}_s$ being the supersingular points above x lying in the components containing ∞, 0 respectively. These quasi-isomorphisms then induce the isomorphism $\mathbb{X}(J_{\mathfrak{m}}) \simeq \mathbb{Z}[\mathsf{SS}]$. To get $\mathbb{X}(J)$ we drop the factor $\mathbb{Z}[\Sigma_s]$ from (2.2.2), and then the kernel becomes $\mathbb{Z}[\mathsf{SS}]_0$.

Still assuming that $\mathcal{X}'$ is regular, let $\ell \neq p$ be prime. Then the Deligne-Rapoport model for $X_0(N\ell)$ over S is also regular. Let us denote it $\mathcal{X}^{(\ell)}$. Then maps u, v extend to finite morphisms $\mathcal{X}^{(\ell)} \to \mathcal{X}$ which are therefore also flat. Therefore the endomorphism T_ℓ of $\mathcal{J}^0_{\mathfrak{m},s}$ is, under the isomorphism (2.2.1), identified with the endomorphism $T_\ell = u_* v^*$ of $\mathrm{Pic}^0_{(\mathcal{X}_s,\Sigma_s)/k}$. Now the maps u, $v\colon \mathcal{X}_s^{(\ell)} \to \mathcal{X}_s$ map the ∞- and 0-component of $\mathcal{X}_s^{(\ell)}$ to the ∞- and 0-component, respectively, of $\mathcal{X}_s$, and on each of these, they are just the maps u, $v\colon X_0(M\ell) \to X_0(M)$. So $u_* v^*$ induces the map T_ℓ on $\mathbb{Z}[\mathsf{SS}_M]$.

In general, choose a multiple $N' = nN = pM'$ of N with $(p, n) = 1$ such that the Deligne-Rapoport model of $X_0(N')$ over S is regular. Let $f\colon X_0(N') \to X_0(N)$ be the map $(E, C) \mapsto (E, nC)$, and $\mathfrak{m}'$ the reduced modulus $f^{-1}((\infty) + (0))^{\mathrm{red}}$ on $X_0(N')$. Then $f^*\colon J_0(N)_{\mathfrak{m}} \to J_0(N')_{\mathfrak{m}'}$ induces a surjection

$$\begin{array}{ccc}
{}^t f^*\colon \mathbb{X}(J_0(N')_{\mathfrak{m}'}) & \longrightarrow & \mathbb{X}(J_0(N)_{\mathfrak{m}}) \\
\wr\uparrow & & \wr\uparrow \\
\mathbb{Z}[\mathsf{SS}_{M'}] & & \mathbb{Z}[\mathsf{SS}_M]
\end{array}$$

which is equivariant with respect to ${}^tT_\ell$ for all $\ell \nmid N'$. According to 1.11 this is induced by the map $f\colon \mathsf{SS}_{M'} \to \mathsf{SS}_M$, hence commutes with the maps T_ℓ on $\mathbb{Z}[\mathsf{SS}_{M'}]$ and $\mathbb{Z}[\mathsf{SS}_M]$. $\square$

Remark. Restricting to the case when p exactly divides N is rather natural, since the toric part of the special fibre of the Néron model of $J_0(p^r M)$, $r > 1$, is a product of copies of the toric part for $J_0(pM)$.

In the case $N = p$ we may describe everything (including T_p) in terms of the classical Brandt matrices, whose definition we now recall [12]. Let $\{E_i \mid 1 \leq i \leq h\}$ be representatives of the isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$ (so that h is the class number of the definite quaternion algebra $\mathrm{End}(E_i) \otimes \mathbb{Q}$). Let $\mathrm{Hom}(E_i, E_j)_n$

be the set of isogenies from E_i to E_j of degree n. Define an equivalence relation $\sim$ on $\mathrm{Hom}(E_i, E_j)_n$ by

$$f \sim g \iff \ker f = \ker g \iff f = \alpha g \text{ for an automorphism } \alpha \text{ of } E_j$$

and set $\overline{\mathrm{Hom}}(E_i, E_j)_n = \mathrm{Hom}(E_i, E_j)/\sim$. We then define the $h \times h$ Brandt matrix $B(n)$ for $n \geq 1$ by

$$B(n)_{ij} = \#\overline{\mathrm{Hom}}(E_i, E_j)_n. \tag{2.2.3}$$

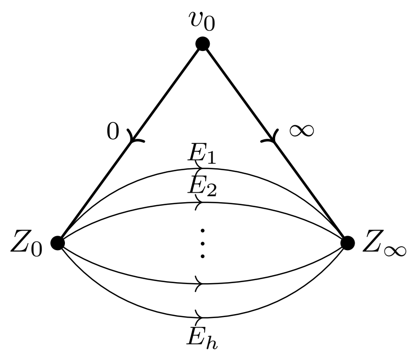
The matrices $B(n)$ for $n \geq 1$ commute. They are constant row-sum matrices, with the sum of the entries in any row of $B(n)$ equal to

$$\sigma'(n) = \sum_{d|n,\ (p,d)=1} d$$

for $n \geq 1$.

Theorem 2.4. *Let $N = p$ and $\mathfrak{m} = (\infty) + (0)$. The isomorphism $\mathbb{X}(J_\mathfrak{m}) \xrightarrow{\sim} \mathbb{Z}[\mathsf{SS}_1]$ of Theorem 2.3 takes ${}^tT_\ell$ to the transpose ${}^tB(\ell)$ of the Brandt matrix, for every prime ℓ (including $\ell = p$).*

Proof. For $\ell \neq p$ this follows immediately from the definition of the Brandt matrix $B(\ell)$. For $\ell = p$, we first note that the endomorphism $T_p + W_p$ of $J_0(p)_\mathfrak{m}$ is zero. Indeed, on the quotient $J_0(p)$ it is zero, by [30, Proposition 3.7], and since W_p interchanges the two cusps and ${}^tT_p$ fixes $(0) - (\infty)$, it is zero on the torus $T_\mathfrak{m} = \mathbb{G}_\mathrm{m} \subset J_0(p)_\mathfrak{m}$. So as any morphism from $J_0(p)$ to $\mathbb{G}_\mathrm{m}$ is constant, $T_p + W_p$ is zero on $J_0(p)_\mathfrak{m}$. Therefore it is enough to compute the action of W_p in $\mathbb{X}(J_0(p)_\mathfrak{m})$. For this it is convenient to compute using the extended reduced graph $\Gamma_{\mathcal{X}'_s,\Sigma}$ defined in §1.4, with $\Sigma = X_0(p)_s^\infty = \{\infty, 0\}$ (where we have fixed an orientation):



As the regular model $\mathcal{X}$ is obtained by replacing the A_2- and A_3-singularities by chains of lines, the extended graphs of $\mathcal{X}_s$ and $\mathcal{X}'_s$ are homotopy equivalent, and so we may restrict to $\mathcal{X}'_s$. On the special fibre $\mathcal{X}'_s$, W_p interchanges the two irreducible components. Recall also that the supersingular points SS_1 are $\mathbb{F}_{p^2}$-rational, and if $x \in \mathsf{SS}_1$ is a supersingular point, corresponding to the class of a supersingular elliptic curve $E/\overline{\mathbb{F}}_p$, then $W_p(x) = x^{(p)}$ is the point corresponding to $E^{(p)} = E/\ker(F)$. So the automorphism W_p extends to to an automorphism of the graph, fixing v_0 and interchanging Z_0 and Z_∞, and mapping the edges labelled E_i to $E_i^{(p)}$. The homology $H_1(\Gamma_{\mathcal{X}'_s,\Sigma}, \mathbb{Z})$ is freely generated by the cycles $\gamma_i = (0) + (E_i) - (\infty)$, and $W_p \colon \gamma_i \mapsto -\gamma_i^{(p)} = -(0) - (E_i^{(p)}) + (\infty)$. Now the only element of $\overline{\mathrm{Hom}}(E_i, E_j)_p$ is the Frobenius $E_i \to E_i^{(p)} = E_j$, and therefore the matrix of $T_p = -W_p$ equals ${}^tB(p)$. $\square$

2.3. **Component groups.** Throughout this section, we assume that $p > 3$. Let $N = p^r M$, with $(p, M) = 1$ and $r \geq 1$. As in the previous section, work over S, the strict henselisation of $\operatorname{Spec} \mathbb{Z}_{(p)}$. Let $\mathfrak{m}$ be a reduced modulus on $X_0(N)$ supported at the cusps, and $\mathbb{T}$ a subalgebra of the Hecke algebra $\mathbb{Z}[\{T_\ell\}]$ which preserves the support of $\mathfrak{m}$. Let $J_\mathfrak{m}$ be the generalized Jacobian of $X_0(N)$ for the modulus $\mathfrak{m}$. Then $\mathbb{T}$ acts on $J_\mathfrak{m}$, stabilising the torus $T_\mathfrak{m}$. It therefore acts on the extension of component groups

$$0 \to \Phi(T_\mathfrak{m}) \to \Phi(J_\mathfrak{m}) \to \Phi(J) \to 0$$

and the action commutes with the action of $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. For the action of $\mathbb{T}$ on $\Phi(J)$ we have the following result, proved by Edixhoven [10], generalizing Ribet [31] who treated the case of N squarefree.

Theorem 2.5. *For every $\ell \nmid N$, T_ℓ acts on $\Phi(J)$ as multiplication by $\ell + 1$.*

Corollary 2.6. *Assume that M is squarefree and $p > 3$. Then for every $\ell \nmid N$, T_ℓ acts on $\Phi(J_\mathfrak{m})$ as multiplication by $\ell + 1$.*

Proof. Let $x \simeq \operatorname{Spec} \mathbb{Q}(\mu_{(d,N/d)}) \subset X_0(N)^\infty$ be a cusp, where $d \mid N$. As M is squarefree, $(d, N/d)$ is a power of p. Therefore $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially on $\Phi(T_\mathfrak{m})$ by (1.1.5). So T_ℓ acts on $\Phi(T_\mathfrak{m})$ as multiplication by $\ell+1$. So the endomorphism $T_\ell - \ell - 1$ of $\Phi(J_\mathfrak{m})$ factors through a map $\Phi(J) \to \Phi(T_\mathfrak{m})$, which is zero as $\Phi(J)$ is finite and $\Phi(T_\mathfrak{m})$ is free. □

Remark. Similarly, let N be arbitrary, and $\mathfrak{m}$ the reduced modulus on $X_0(N)$ which is the sum of all the cusps. Write $T_\mathfrak{m} \to T_{p\text{-spl}}$ for the maximal quotient which is split over $\mathbb{Q}(\mu_{p^r})$. Let $J_{p\text{-spl}}$ be the corresponding quotient of $J_\mathfrak{m}$. Then by Corollary 1.2 the sequence of Néron models

$$0 \to \mathcal{T}_{p\text{-spl}} \to \mathcal{J}_{p\text{-spl}} \to \mathcal{J} \to 0$$

is exact, and the same argument shows that $T_\ell = \ell + 1$ on $\Phi(\mathcal{J}_{p\text{-spl}})$.

Now we turn to the abelian group structure of $\Phi(J_\mathfrak{m})$.

For $N = pM$, $(p, M) = 1$, the structure of $\Phi(J)$ was determined completely by Deligne, and described by Mazur and Rapoport in [23], using the description of the regular model of $X_0(N)$ given in [8] — see Table 2 on p. 174 and the calculations of §2 in *loc. cit.*, and the corrections to their calculations made by Edixhoven [10, 4.4.1]. We recall these formulae in 2.8 below.

For general N, the minimal desingularisation $\mathcal{X} \to \mathcal{X}'$ was computed by Edixhoven [9] using the description of $\mathcal{X}'$ in [17]. Since the component of $\mathcal{X}'_s$ meeting the cusp ∞ has multiplicity one, $\mathcal{X}$ satisfies hypotheses (H1–3). From this it is in principle an exercise to compute $\Phi(J)$ in any given case, and in [10, 4.4.2] this is done for $N = p^2$.

We will compute $\Phi(J_\mathfrak{m})$ in various cases. First some notation: as in the previous section, let $\mathsf{SS}_M \subset X_0(M)(\overline{\mathbb{F}}_p)$ be the set of supersingular points, and $n = \#\mathsf{SS}_M$. For $j \in \{2, 3\}$ let e_j be the number of elements $(E, C) \in \mathsf{SS}_M$ for which $\#\operatorname{Aut}(E, C) = 2j$.

2.3.1. $X_0(pM)$ *with* $(p, M) = 1$ *and* $\mathfrak{m} = (\infty) + (0)$.

Theorem 2.7. *Let $N = pM$ with $(p, M) = 1$. Let $J_\mathfrak{m}$ be the generalized Jacobian of $X_0(N)$ with respect to the modulus $\mathfrak{m} = (\infty) + (0)$. Then:*

(a) $\Phi(J_\mathfrak{m}) \simeq \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\max(e_2-1,0)} \oplus (\mathbb{Z}/3\mathbb{Z})^{\max(e_3-1,0)}$

(b) *The homomorphism* $\Phi(T_{\mathfrak{m}}) = \mathbb{Z} \to \Phi(J_{\mathfrak{m}})$ *is given in terms of the isomorphism* (a) *by*

$$1 \mapsto \begin{cases} n & \textit{if } e_2 = e_3 = 0 \\ (2n - e_2; 1, \ldots, 1) & \textit{if } e_2 > 0,\ e_3 = 0 \\ (3n - 2e_3; 1, \ldots, 1) & \textit{if } e_2 = 0,\ e_3 > 0 \\ (6n - 3e_2 - 4e_3; 1, \ldots, 1; 1, \ldots, 1) & \textit{otherwise.} \end{cases}$$

Proof. Recall that the special fibre $\mathcal{X}'_s$ of the Deligne–Rapoport model of $X_0(N)$ is the union of two copies of $X_0(M)_{\overline{\mathbb{F}}_p}$, meeting tranversally at the supersingular points. The cusps ∞ and 0 belong to different components. The total space $\mathcal{X}'$ has a type A_j quotient singularity at each point where $\#\operatorname{Aut}(E, C) = 2j \in \{4, 6\}$. Taking their minimal resolution gives the model $\mathcal{X}$. Its special fibre is obtained by replacing each crossing point which is an A_j-singularity with a chain of $(j - 1)$ copies of $\mathbb{P}^1$. In other words, $\mathcal{X}_s$ has $2 + e_2 + 2e_3$ irreducible components:

- Z_∞ and Z_0, the strict transforms of the irreducible components of $\mathcal{X}'_s$, isomorphic to $X_0(M)_{\overline{\mathbb{F}}_p}$, and labelled in such a way that the cusp $\alpha \in \{\infty, 0\}$ belongs to Z_α.
- Components in the fibres of $\mathcal{X}_s \to \mathcal{X}'_s$: denote these as E_i (for $1 \leq i \leq e_2$), and $F_{\infty,i}$, $F_{0,i}$ (for $1 \leq i \leq e_3$), where $F_{\alpha,i}$ intersects Z_α.

Their intersection numbers are

- $(Z_\alpha.Z_\alpha) = -n$, $(Z_\infty.Z_0) = n - e_2 - e_3$
- $(Z_\alpha.E_i) = 1 = (Z_\alpha.F_{\alpha,i})$
- All other intersection numbers are zero.

We now use the formula for $\Phi(J_{\mathfrak{m}})$ from Theorem 1.19. We have $C = \pi_0(\widetilde{\mathcal{X}_s})$, and write $Y^\vee \in \mathbb{Z}^C$ for the basis element dual to $Y \in C$. We also have $I = \{\infty, 0\}$, and e (as in Theorem 1.19) equals $(1, 1)$. Therefore $\mathbb{Z}^I/e\mathbb{Z} = \mathbb{Z}.V$ where V is the image of the dual of ∞ (so that $-V$ is the image of the dual of 0). The map $h\colon \mathbb{Z}[C] \to \mathbb{Z}^I/e\mathbb{Z}$ takes Z_∞ to V and Z_0 to $-V$, and all other elements of C to 0. The map $(a, h)\colon \mathbb{Z}[C] \to \mathbb{Z}^C \oplus \mathbb{Z}^I/e\mathbb{Z}$ is then given by the matrix:

$$\begin{array}{c} \\ Z_\infty^\vee \\ Z_0^\vee \\ E_1^\vee \\ \vdots \\ E_{e_2}^\vee \\ F_{\infty,1}^\vee \\ F_{0,c_1}^\vee \\ \vdots \\ F_{\infty,e_3}^\vee \\ F_{0,e_3}^\vee \\ V \end{array} \begin{array}{c} \begin{array}{cccccccccc} Z_\infty & Z_0 & E_1 & \cdots & E_{e_2} & F_{\infty,1} & F_{0,1} & \cdots & F_{\infty,e_3} & F_{0,e_3} \end{array} \\ \left[\begin{array}{cccccccccc} -n & n - e_2 - e_3 & 1 & \cdots & 1 & 1 & 0 & \cdots & 1 & 0 \\ n - e_2 - e_3 & -n & 1 & \cdots & 1 & 0 & 1 & \cdots & 0 & 1 \\ 1 & 1 & -2 & & & & & & & \\ \vdots & \vdots & \vdots & \ddots & & & & & & \\ 1 & 1 & & & -2 & & & & & \\ 1 & 0 & & & & -2 & 1 & & & \\ 0 & 1 & & & & 1 & -2 & & & \\ \vdots & \vdots & \vdots & & & & & \ddots & & \\ 1 & 0 & & & & & & & -2 & 1 \\ 0 & 1 & & & & & & & 1 & -2 \\ 1 & -1 & 0 & \cdots & & & & & & 0 \end{array}\right] \end{array}$$

As a basis for $\mathbb{Z}^{C,0}$ we take $\overline{Z} = Z_\infty^\vee - Z_0^\vee$, $\overline{E}_i = E_i^\vee - Z_0^\vee$, $\overline{F}_{\alpha,i} = F_{\alpha,i}^\vee - Z_0^\vee$. So $\Phi(J_{\mathfrak{m}})$ is isomorphic to the quotient of the free module generated by $\overline{Z}$, $\{\overline{E}_i\}$, $\{\overline{F}_{\alpha,i}\}$ and V by the submodule of relations

$$\overline{Z} = 2\overline{E}_i = 3\overline{F}_{0,i}, \qquad \overline{F}_{\infty,i} = 2\overline{F}_{0,i}$$
$$V = n\overline{Z} - \sum_{i=1}^{e_2} \overline{E}_i - 2\sum_{i=1}^{e_3} \overline{F}_{0,i}\,.$$

If $e_2 > 1$ then for every $i > 1$, $U_i = \overline{E}_i - \overline{E}_0$ has order 2, and if $e_3 > 1$ then $V_i = \overline{F}_{0,i} - \overline{F}_{0,0}$ has order 3. The subgroup generated by $\overline{Z}$, $\overline{E}_1$ (if $e_1 \geq 1$) and $\overline{F}_{1,0}$ (if $e_3 \geq 1$) is infinite cyclic, with generator

$$\begin{array}{cl} \overline{Z} & \text{if } e_2 = e_3 = 0 \\ \overline{E}_1 & \text{if } e_2 > 0,\ e_3 = 0 \\ \overline{F}_{0,1} & \text{if } e_2 = 0,\ e_3 > 0 \\ \overline{E}_1 - \overline{F}_{0,1} & \text{otherwise} \end{array}$$

This gives (a), and (b) follows since the inclusion $\Phi(T_{\mathfrak{m}}) = \mathbb{Z} \to \Phi(J_{\mathfrak{m}})$ maps 1 to V. □

Since $\Phi(J) = \Phi(J_{\mathfrak{m}})/\Phi(T_{\mathfrak{m}})$, an easy computation gives:

Corollary 2.8. ([23, Table 2]; [10, 4.4.1])

$$\Phi(J) \simeq \mathbb{Z}/P\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\max(e_2-2,0)} \oplus (\mathbb{Z}/3\mathbb{Z})^{\max(e_3-2,0)}$$

where

$$P = 2^{\min(e_2,2)}3^{\min(e_3,2)}\left(n - \frac{e_2}{2} - \frac{2e_3}{3}\right).$$

2.3.2. *$X_0(p)$ with $\mathfrak{m} = (\infty) + (0)$.* In the setting of Theorem 2.7, if e_2 and e_3 are at most 1, then $\Phi(J_{\mathfrak{m}})$ is infinite cyclic and the map $\Phi(T_{\mathfrak{m}}) \simeq \mathbb{Z} \to \Phi(J_{\mathfrak{m}})$ is, up to sign, multiplication by the order of the cyclic group $\Phi(J)$. Therefore the actions of all the Hecke operators T_ℓ (including for $p \mid N$) can be computed from the actions on $\Phi(T_{\mathfrak{m}})$. For example, suppose $N = p$. Then by Example 2.2(b), without appealing to the results of Ribet and Edixhoven we obtain:

Corollary 2.9. *Suppose that $N = p$ and $\mathfrak{m} = (\infty) + (0)$. Then*
- *$\Phi(J_{\mathfrak{m}})$ is infinite cyclic.*
- *$\Phi(J) = \operatorname{coker}(\Phi(T_{\mathfrak{m}}) \to \Phi(J_{\mathfrak{m}}))$ is cyclic of order n, the numerator of $(p-1)/12$.*
- *For $\ell \neq p$, $T_\ell = \ell + 1$ on $\Phi(J_{\mathfrak{m}})$, and $T_p = 1$ on $\Phi(J_{\mathfrak{m}})$.*

2.3.3. *$X_0(pM)$ with $(p, M) = 1$ and $\mathfrak{m}$ a general cuspidal modulus.* Now let $\mathfrak{m}$ be any nonzero reduced modulus supported on the cusps of $X_0(N)$, with p exactly dividing N. Recall that we are working over the strict henselisation R of $\mathbb{Z}_{(p)}$. Then since $p^2 \nmid N$, all the cusps are rational over F so $e = (1, \ldots, 1)$ and

$$\Phi(J_{\mathfrak{m}}) = \operatorname{coker}(\mathbb{Z}[C] \xrightarrow{(a,h)} \mathbb{Z}^{C,0} \oplus \mathbb{Z}^I/\operatorname{diag}(\mathbb{Z}))$$

with $I = \operatorname{supp}(\mathfrak{m}) \subset X_0(N)^\infty(\overline{\mathbb{Q}})$.

Proposition 2.10. *If the closure of the support of $\mathfrak{m}$ meets just one component of the special fibre $\mathcal{X}'_s$, then there is a canonical splitting*

$$\Phi(J_{\mathfrak{m}}) = \Phi(J) \oplus \Phi(T_{\mathfrak{m}}).$$

Otherwise, if $x_0 = \infty$, $x_0 \in \operatorname{supp}(\mathfrak{m})$ meet Z_∞, Z_0 respectively, and $\mathfrak{m}' = (x_\infty) + (x_0)$, then there is a canonical splitting

$$\Phi(J_{\mathfrak{m}}) = \Phi(J_{\mathfrak{m}'}) \oplus \mathbb{Z}^{I \smallsetminus \{x_\infty, x_0\}}.$$

Proof. In the first case, we may assume that the closure of the support of $\mathfrak{m}$ meets only the component Z_∞. Then $h(Y)=0$ if $Y \in C$, $Y \neq Z_\infty$ but $h(Z_\infty)=(1,\dots,1)$, so the composite $h\colon \mathbb{Z}[C] \to \mathbb{Z}^I/\mathrm{diag}(\mathbb{Z})$ is zero.

In the second case, we have $h(Y)=0$ if $Y \notin \{Z_\infty, Z_0\}$, and

$$h(Z_\infty) = (1,\dots,1,0,\dots,0)$$
$$h(Z_0) = (0,\dots,0,1,\dots,1)$$

for a suitable ordering of I. Therefore

$$\mathbb{Z}^I/\mathrm{diag}(\mathbb{Z}) = \mathrm{im}(h) \oplus \{b \in \mathbb{Z}^I \mid b(Z_\infty) = b(Z_0)\}/\mathrm{diag}(\mathbb{Z})$$

giving the splitting. □

2.3.4. $X_0(p^2)$. Finally, let us consider the curve $X_0(p^2)$, $p>3$. The Katz-Mazur model $\mathcal{X}'$ over S has three irreducible components in its special fibre, which we denote Z_i' $(0 \le i \le 2)$. The non-supersingular non-cuspidal points of Z_i' parametrize pairs (E,C), where E is an elliptic curve and C is a cyclic (in the sense of Drinfeld) subgroup scheme of rank p^2, whose étale quotient has rank p^i. The components Z_0', Z_2' have multiplicity 1, and Z_1' has multiplicity $p-1$. They meet at the supersingular points.

The cuspidal divisor $X_0(p^2)^\infty_{\mathbb{Q}}$ consists of three closed points $\infty = z_1 = \operatorname{Spec}\mathbb{Q}$, $z_p = \operatorname{Spec}\mathbb{Q}(\mu_p)$ and $0 = z_{p^2} = \operatorname{Spec}\mathbb{Q}$, in the notation of Section 2.1. For each i, the closure in $\mathcal{X}'$ of the point z_{p^i} meets the component Z_i' in a single point, and the completed local ring at the intersection is computed in [9, Proposition 1.2.2.1] as

$$\begin{array}{ll} \mathbb{Z}_p[[q]] & \text{if } i=0 \\ \mathbb{Z}_p[\mu_p][[q]] & \quad 1 \\ \mathbb{Z}_p[[q^{1/p^2}]] & \quad 2 \end{array} \tag{2.3.1}$$

where q is the usual parameter at infinity on the modular curve of level 1.

The minimal resolution $\pi\colon \mathcal{X} \to \mathcal{X}'$ is described in detail in [9, §1.5]. We summarise the final result. Write $p = 12k+1+4a+6b$, with a, $b \in \{0,1\}$. We again work over the strict henselisation R of $\mathbb{Z}_{(p)}$.

The Katz–Mazur model $\mathcal{X}'$ has exactly two singular points, which are the points x_0, $x_{1728} \in Z_1'$ lying over the points $j=0$, 1728 in the curve $X(1)_{\overline{\mathbb{F}}_p}$. Let $E = \pi^{-1}(x_{1728})^{\mathrm{red}}$, $F = \pi^{-1}(x_0)$. Then $E \simeq F \simeq \mathbb{P}^1$, E has multiplicity $(p-1+2b)/2$ and F has multiplicity $(p-1+2a)/3$.

Let Z_i be the reduced strict transform of Z_i'. The intersection matrix of $\mathcal{X}_s$ is:

$$\begin{array}{c} \\ Z_0 \\ Z_1 \\ Z_2 \\ E \\ F \end{array}\begin{array}{c} \begin{array}{ccccc} Z_0 & Z_1 & Z_2 & E & F \end{array} \\ \begin{bmatrix} -L & k & k & b & a \\ k & -1 & k & 1 & 1 \\ k & k & -L & b & a \\ b & 1 & b & -2 & 0 \\ a & 1 & a & 0 & -3 \end{bmatrix} \end{array} \tag{2.3.2}$$

where $L = (p^2-1)/12 - k$. As a basis for $\ker(\mathbb{Z}^C \xrightarrow{b} \mathbb{Z})$ we take $\overline{Y} = Y^\vee - d_Y Z_2^\vee$, for $Y \in \{Z_0, Z_1, E, F\}$ and where d_Y is the multiplicity of Y. (Since the residue field is perfect, $d_Y = \delta_Y$.)

We first consider the modulus $\mathfrak{m} = X_0(p^2)^\infty_{\mathbb{Q}} = \sum_{0\le i\le 2}(z_{p^i})$ of all cusps. Since the cusp z_p is isomorphic to $\operatorname{Spec}\mathbb{Q}(\mu_p)$, and the other cusps are rational, $e=(1,p-1,1)$. From the description (2.3.1) of the completed local rings at the cusps, we see that $\Sigma \simeq$

Spec $R \sqcup$ Spec $R[\mu_p] \sqcup$ Spec R, and the pullback of the divisor Z_i to the component of Σ which it meets has degree 1. Therefore the matrix (h_{ij}) giving the pairing $C \times I \to \mathbb{Z}$ in Theorem 1.19 is

$$\begin{array}{c} \\ z_1 \\ z_p \\ z_{p^2} \end{array} \begin{array}{c} \begin{matrix} Z_0 & Z_1 & Z_2 & E & F \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

Let $V_i \in \mathbb{Z}^I/e\mathbb{Z}$ be the image of the i-th basis vector (dual to z_{p^i}) of $\mathbb{Z}^I$. We will take $\{V_0, V_1\}$ as basis for $\mathbb{Z}^I/e\mathbb{Z}$.

Next consider the modulus $\mathfrak{m}' = (\infty) + (0) = (z_1) + (z_{p^2})$. Then $e = (1,1)$, and the pairing $C \times I \to \mathbb{Z}$ is given by the same matrix with the z_p-row deleted, and $\mathbb{Z}^I/e\mathbb{Z}$ is generated by $V_0 = -V_2$.

Under the isomorphism of 1.19, the image of $\mathbb{Z}^I/e\mathbb{Z}$ in the homology of the complex (1.3.1) is the subgroup $\Phi(T_\mathfrak{m})$ of $\Phi(J_\mathfrak{m})$. The analogous statement holds for $\mathfrak{m}'$.

Theorem 2.11. (a) *The component group $\Phi(J_\mathfrak{m})$ is isomorphic to $\mathbb{Z}^2$, and (for a suitable choice of isomorphism), the image of the generators V_0, V_1 of $\Phi(T_\mathfrak{m})$ are*

$$(L + (3b - 2a)k - a + b, -6k - 2a - 3b) \quad \textit{and} \quad (-k - b, 1).$$

(b) *The component group $\Phi(J_{\mathfrak{m}'})$ is isomorphic to $\mathbb{Z}$, and (up to sign), the image of the generator V_0 of $\Phi(T_\mathfrak{m})$ is $(p^2 - 1)/24$.*

Remark. (i) From the computation in (b) we recover the result [10, Sect. 4.1, Prop. 2] that $\Phi(J)$ is cyclic of order $(p^2 - 1)/24$.

(ii) In both cases the map $\Phi(T_\mathfrak{m}) \to \Phi(J_\mathfrak{m})$ is an injection of free abelian groups of the same rank, so the action of Hecke operators on $\Phi(T_\mathfrak{m})$ determines that on $\Phi(J_\mathfrak{m})$ and therefore on the quotient $\Phi(J)$, "by pure thought".

Proof. From (2.3.2) we see that $\Phi(J_\mathfrak{m})$ is generated by $\{V_0, V_1, \overline{Z}_0, \overline{Z}_1, \overline{E}, \overline{F}\}$ with relations

$$V_0 = L\overline{Z}_0 - k\overline{Z}_1 - b\overline{E} - a\overline{F}$$
$$V_1 = -k\overline{Z}_0 + \overline{Z}_1 - \overline{E} - \overline{F}$$
$$b\overline{Z}_0 + \overline{Z}_1 - 2\overline{E} = 0 = a\overline{Z}_0 + \overline{Z}_1 - 3\overline{F}$$

and linear algebra then gives an isomorphism $\Phi(J_\mathfrak{m}) \xrightarrow{\sim} \mathbb{Z}^2$ by

$$\begin{aligned} \overline{Z}_0 &\mapsto (1, 0) \\ \overline{Z}_1 &\mapsto (2a - 3b, 6) \\ \overline{E} &\mapsto (a - b, 3) \\ \overline{F} &\mapsto (a - b, 2) \\ V_0 &\mapsto (L - (2k+1)a + (3k+1)b, -6k - 2a - 3b) \\ V_1 &\mapsto (-k - b, 1) \end{aligned}$$

This proves (a). For (b), we compose with the map $\mathbb{Z}^2 \xrightarrow{(1,k+b)} \mathbb{Z}$, whose kernel is the subgroup generated by V_1, and which takes V_0 to

$$L - (2k+1)a + (3k+1)b + (k+b)(-6k - 2a - 3b) = \frac{p^2 - 1}{24}.$$

□

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