

ORIGINAL ARTICLE

Uniform bounds of families of analytic semigroups and Lyapunov Linear Stability of planar fronts

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Abstract

We study families of analytic semigroups, acting on a Banach space, and depending on a parameter, and give sufficient conditions for existence of uniform with respect to the parameter norm bounds using spectral properties of the respective semigroup generators. In particular, we use estimates of the resolvent operators of the generators along vertical segments to estimate the growth/decay rate of the norm for the family of analytic semigroups. These results are applied to prove the Lyapunov linear stability of planar traveling waves of systems of reaction–diffusion equations, and the bidomain equation, important in electrophysiology.

KEYWORDS

analytic semigroups, bidomain equation, Lyapunov linear stability, planar traveling waves, reaction–diffusion systems, uniform exponential bounds

1 | INTRODUCTION

The problem of finding optimal bounds for the norm of a semigroup of linear operators is very important and wellstudied in the asymptotic theory of semigroups [2, 4–6, 15, 17]. Of particular importance are characterizations that relate the growth/decay bounds to the spectral properties of the generator of the semigroup. We mention here the celebrated Gearhard–Prüss Theorem [7, 18], and refer to [2, section 5.7] for further references or to [9, 10, 14] for more recent results.

For families of semigroups depending on a parameter, the problem of finding estimates for their norms that are *uniform* with respect to the parameter is significantly less studied. In this paper, we aim to find sufficient conditions that guarantee uniform with respect to the parameter exponential decay estimates for families of analytic semigroups in Banach spaces. These results are important in the study of Lyapunov linearized stability of traveling waves in systems of partial differential equations. In many instances, such as parabolic systems of partial differential equations, the linearization along the wave is a *sectorial* operator, hence it generates an *analytic semigroup*. In the case when the linearization along a planar wave is a differential operator on a multidimensional domain, the generator of the semigroup is typically similar to a multiplication operator by an operator-valued function of certain parameter, the dual variable. The Lyapunov linearized stability problem is thus equivalent to the existence of *uniform* with respect to the parameter bounds for a family of analytic semigroups. We present two specific cases, arising in the theory of reaction–diffusion systems and the bidomain equation, illustrating these ideas in Section 4 and Section 5.

It is well known that for any strongly continuous semigroup of linear operators $\{T(t)\}_{t \geq 0}$ on a Banach space $\mathbb{X}$, there exist two constants $L \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Le^{\omega t} \text{ for any } t \geq 0. \quad (1.1)$$

We recall that the infimum (which might not be the minimum) of all ω for which there exists a constant $L \geq 1$ such that (1.1) holds is called the semigroup *growth bound*, and is denoted $\omega_0(T)$. Moreover, in the case of an analytic semigroup one has $\omega_0(T) = s(A) := \sup \operatorname{Re} \sigma(A)$, where $s(A)$ is the *spectral bound* and $\sigma(A)$ is the spectrum of the generator A of the semigroup $\{T(t)\}_{t \geq 0}$. We stress that it is much harder to find a direct formula, or even an estimate, for the constant L in (1.1), unless one imposes additional conditions on the analytic semigroup $\{T(t)\}_{t \geq 0}$ or its generator. A classical example of such condition reads as follows: If A is the generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ and $A - \omega I_{\mathbb{X}}$ is dissipative, then (1.1) holds for $L = 1$, cf., for example, [2, Proposition 3.7.16].

In the case of *families* of semigroups whose generators depend on a parameter α , real or complex, the constants L and ω from (1.1) might depend on α as well. By studying the spectrum of the generator and the properties of its resolvent, one can find a convenient growth rate. However, even if we use more advanced results, such as the celebrated Gearhard–Prüss theorem [7, 18] or later results of Helfer–Sjöstrand [9, 10], the constant $L = L(\alpha)$ is quite often such that $\sup_{\alpha} L(\alpha) = \infty$.

Usually, one aims at finding the best possible decay rate or the smallest growth rate, that is, the smallest ω in (1.1) may be at the expense of making the constant L large. However, in this paper, studying exponential decay and stability of a family of semigroups depending on a parameter, we are not necessarily interested in the optimal decay rate, but rather just in the order of magnitude of the decay rate when the parameter changes. In this context we are willing to give up some decay to get a constant in front of the exponential term, which is uniform with respect to the parameters in the system. In a sense, this philosophy is opposite to what has been used in many papers, for example, in [9, 10, 14].

In this paper, we study uniform stability of a family of semigroups of operators, whose generators are bounded perturbations of a sectorial operator. We start with assumptions on the operator.

Hypothesis (H1). We assume that $A : \operatorname{dom}(A) \subseteq \mathbb{X} \rightarrow \mathbb{X}$ is a *sectorial* linear operator, that is, there exists $a \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$ and $M_0 > 0$ such that

$$\Omega_{a,\theta} := \{\lambda \in \mathbb{C} : \lambda \neq a, |\arg(\lambda) - a| < \theta\} \subseteq \rho(A), \quad \|R(\lambda, A)\| \leq \frac{M_0}{|\lambda - a|} \text{ for any } \lambda \in \Omega_{a,\theta}. \quad (1.2)$$

We recall that there are several concepts of sectorial operators in the literature. In this paper, we use the spectral conditions given above in (1.2). See Remark 2.2 for a more detailed discussion.

It is well known that if A satisfies Hypothesis (H1), then it generates an analytic semigroup that can be evaluated using contour integration along a counterclockwise oriented path surrounding the spectrum of A . These formulas allow us to find estimates of the type (1.1) and find explicit formulas not only for ω , but also for L . We revisit several results known in the literature with the explicit aim of controlling the constant L , which are very useful in the sequel, see Lemma 2.4 and Remark 2.5. Next, we add to Hypothesis (H1) some more assumptions on A .

Hypothesis (H2). We assume the following conditions on the spectrum of the operator A :

- (i) $\sup \operatorname{Re}(\sigma(A) \setminus \{0\}) \leq -\nu$ for some $\nu > 0$;
- (ii) 0 is a *semisimple eigenvalue* of finite multiplicity of A .

If the linear operator A is obtained as the linearization of a partial differential equation (PDE) along a traveling wave, then Hypothesis (H2) is equivalent to what is sometimes called *conditional exponential stability* of the wave. This situation is very common, see examples from Section 4 and Section 5.

Next, we consider a family of operators $A_{\alpha} := A + E(\alpha)$, $\alpha \geq 0$, where the perturbation $E : [0, \infty) \rightarrow \mathcal{B}(\mathbb{X})$ is such that $E(0) = 0$. We denote by $\{T_{\alpha}(t)\}_{t \geq 0}$ the semigroup of linear operators generated by A_{α} , $\alpha \geq 0$. We assume the following conditions on the perturbed family of operators.

Hypothesis (H3). The function $E : [0, \infty) \rightarrow \mathcal{B}(\mathbb{X})$ is continuous in the operator norm topology. Moreover, there exists an increasing function $q : [0, \infty) \rightarrow [0, \infty)$ such that

- (i) there exists $M_1 > 0$ independent of α such that $\sigma(A_{\alpha}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -q(\alpha)\}$ and

$$\|R(\lambda, A_{\alpha})\| \leq \frac{M_1}{\operatorname{Re} \lambda + q(\alpha)} \quad \text{whenever} \quad \operatorname{Re} \lambda > -q(\alpha), \quad \alpha \geq 0; \quad (1.3)$$

- (ii) $q(\alpha) = q_1 \alpha$ for any $\alpha \in [0, q_2]$, for some $q_1, q_2 > 0$;
- (iii) $\ell := \limsup_{\alpha \rightarrow \infty} \frac{\|E(\alpha)\|}{q(\alpha)} < \infty$.

We note that the estimate (1.3) is necessary for the uniform in α exponential stability of the family of semigroups generated by A_α , $\alpha \geq 0$. In Section 4 and Section 5, we present examples of families of operators that satisfy Hypothesis (H3). Moreover, a condition on the spectrum of the perturbed operator A_α , $\alpha \geq 0$, is needed to achieve uniform in α exponential stability of the semigroups $\{T_\alpha(t)\}_{t \geq 0}$, $\alpha \geq 0$. Indeed, in general the spectrum of the perturbed operator A_α might not be stable, even if the space $\mathbb{X}$ is finite dimensional and $E(\alpha) = \alpha W_0$, $\alpha \geq 0$, where W_0 is a constant, bounded, self-adjoint, uniformly negative definite linear operator. See the Appendix for simple counterexamples.

Assuming Hypotheses (H1)–(H3), we show that the family of operators A_α , $\alpha \geq 0$, is uniformly sectorial, that is A_α satisfies (1.2) and all the relevant constants can be chosen independently on α . Using this crucial result, we can prove that the family of semigroups generated by A_α is exponentially stable uniformly for $\alpha \geq \delta$, for any $\delta > 0$. To prove the uniform in α exponential estimate for the norm of the semigroup $\{T_\alpha(t)\}_{t \geq 0}$ for α in a neighborhood of 0, we first find a spectral decomposition of the space $\mathbb{X}$ into two subspaces invariant under A_α . In this decomposition, one spectral subspace is such that the spectrum of the restriction A_α is uniformly bounded away from the imaginary axis, and the other subspace is finite dimensional. To construct such a decomposition, we use a transformation operator borrowed from the classical work of Kato [13, Chapter II, section 4.2] and Daletskii–Krein, [5, Chapter 4, section 1]. To ensure that the spectral and semigroup estimates are uniformly bounded in α , we assume the following.

Hypothesis (H4). The function $E_0 : (0, \infty) \rightarrow \mathcal{B}(\mathbb{X})$ defined by $E_0(\alpha) = \frac{1}{\alpha} E(\alpha)$ has the property

$$\sup_{\alpha \in (0,1]} \|E_0(\alpha)\| < \infty. \quad (1.4)$$

This hypothesis guarantees that there exists a positive number $\varepsilon_0 > 0$, independent of α , such that the circle of radius $\frac{\nu}{4}$ centered at the origin is contained in the resolvent set $\rho(A_\alpha)$, for any $\alpha \in [0, \varepsilon_0]$, where $\nu > 0$ was introduced in Hypothesis (H2) (see Lemma 3.6 for more details). This result is essential in finding the desired spectral decomposition of the space $\mathbb{X}$. Once this step is achieved, we use contour integration and the uniform sectorial property of the family of generators A_α to prove the uniform in α exponential stability of the restriction of the semigroup generated by A_α , $\alpha \in [0, \varepsilon_0]$, to the subspace where its spectrum is bounded away from the imaginary axis uniformly in α .

Next, we turn our attention to the restriction of A_α , $\alpha \in [0, \varepsilon_0]$, to the finite-dimensional subspace where all of its eigenvalues are of order $\mathcal{O}(\alpha)$. The most important step here is to find an expansion of this restriction near $\alpha = 0$, which is so that all the eigenvalues of the leading order term are with negative real part. Moreover, the remainder of the leading order term is of order $o(\alpha)$ and bounded uniformly in α in a neighborhood of 0. To be able to obtain such an expansion, we need the following additional smoothness assumption on the perturbation.

Hypothesis (H5). There exists $r : [0, \infty) \rightarrow [0, \infty)$, a continuous, increasing function such that $r(0) = 0$, $\lim_{\alpha \rightarrow \infty} r(\alpha) = \infty$, and the operator-valued function E_0 defined in (1.4) can be extended continuously at 0 and satisfies the inequality

$$\|E_0(\alpha) - E_0(0)\| \leq r(\alpha) \quad \text{for any } \alpha \geq 0. \quad (1.5)$$

We note that Hypotheses (H4) and (H5) are automatically satisfied if the perturbation E is analytic. The first main result of the paper is the following.

Theorem 1.1. Assume Hypotheses (H1)–(H5). Then, there exists a function $M : (0, 1) \rightarrow (0, \infty)$, independent of the perturbation variable $\alpha \geq 0$, such that the following estimate holds:

$$\|T_\alpha(t)\| \leq M(\kappa) e^{-\kappa q(\alpha)t} \quad \text{for any } t \geq 0, \alpha \geq 0, \kappa \in (0, 1). \quad (1.6)$$

The function M depends on the unperturbed operator A , $E_0(0)$, and the functions $\|E(\cdot)\|$, q , r , and related constants introduced in Hypotheses (H3)–(H5). See Equation (3.100) for a precise formula.

In part, assumption (H3)(i) yields that $\omega_0(T_\alpha) = s(A_\alpha) \leq -q(\alpha)$ as the semigroup $\{T_\alpha(t)\}_{t \geq 0}$ is analytic for any $\alpha \geq 0$. Of course, we cannot replace the decay rate $-\kappa q(\alpha)$, $\kappa \in (0, 1)$, in (1.6) by $\omega_0(T_\alpha)$, and thus by $-q(\alpha)$, unless additional hypotheses are imposed. In many applications, however, we are mostly interested merely in the order of the decay rate, and (1.6) captures this feature.

Next, we turn our attention to the special case when 0 is a *simple* eigenvalue of the unperturbed operator A . The conclusion of Theorem 1.1 can be obtained in this case without assuming Hypothesis (H5). Arguing in the same way as in the general case, we first establish an exponential estimate similar to (1.6) for α away from 0. In addition, when α is close to 0, we can decompose the space $\mathbb{X}$ in the sum of two invariants for A_α subspaces: One is such that the spectrum of the restriction of A_α to the subspace is away from the imaginary axis, and another is a one-dimensional subspace. A modification of Hypothesis (H2) is imposed next.

Hypothesis (H2'). We assume the following conditions on the spectrum of the operator A :

- (i) $\sup \operatorname{Re}(\sigma(A) \setminus \{0\}) \leq -\nu$ for some $\nu > 0$;
- (ii) 0 is a *simple* eigenvalue of A .

Theorem 1.2. Assume Hypotheses (H1), (H2'), (H3) and (H4). Then, there exists a function $N : (0, 1) \rightarrow (0, \infty)$, independent of the perturbation variable $\alpha \geq 0$, such that following estimate holds:

$$\|T_\alpha(t)\| \leq N(\kappa)e^{-\kappa q(\alpha)t} \text{ for any } t \geq 0, \alpha \geq 0, \kappa \in (0, 1). \quad (1.7)$$

The function N depends on the unperturbed operator A and the functions $\|E(\cdot)\|$, q , and related constants introduced in Hypothesis (H3). See Equation (3.107) for a precise formula.

Next, we present applications of Theorem 1.1 and Theorem 1.2 to Lyapunov linear stability of planar traveling waves in reaction–diffusion systems and the bidomain equation. The two models are known to exhibit planar traveling waves. The linearization of each of them along the wave (in the moving frame variables) is similar via the Fourier transform to a multiplication operator by an operator-valued function of a certain parameter (the dual variable). Thus, the Lyapunov linear stability of a planar traveling wave can be obtained by studying the stability of a family of analytic semigroups, which is uniform with respect to the parameter.

The reaction–diffusion system

$$u_t = D\Delta_x u + F(u), \quad t \geq 0, \quad x = (x_1, \dots, x_m)^T \in \mathbb{R}^m, \quad u \in \mathbb{R}^k, \quad (1.8)$$

has exponentially localized planar traveling wave solutions, that is, solutions of the form $u(x, t) = \bar{h}(x_1 - ct)$, under appropriate conditions on the matrix $D \in \mathbb{C}^{k \times k}$ and the vector-valued function $F : \mathbb{R}^k \rightarrow \mathbb{R}^k$. Considering the equation in the moving frame variable $y = x - cte_1$, with $e_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^m$, the linearization along the planar wave $\bar{h}$ is given by $\mathcal{L} = D\Delta_y + cI_k \partial_{y_1} + \mathcal{M}_{F'(\bar{h})}$, considered as a densely defined linear operator on $L^2(\mathbb{R}^m, \mathbb{C}^k)$. Here, $\mathcal{M}_{F'(\bar{h})}$ denotes the operator of multiplication on $L^2(\mathbb{R}^m, \mathbb{C}^k)$ by the bounded, matrix-valued function $F'(\bar{h}(y_1))$. Taking Fourier transform in the variables $(y_2, \dots, y_m) \in \mathbb{R}^{m-1}$, the linear operator $\mathcal{L}$ is unitary equivalent to $\mathcal{M}_{\hat{\mathcal{L}}}$ the operator of multiplication on $L^2(\mathbb{R}^{m-1}, L^2(\mathbb{R}, \mathbb{C}^k))$ by the operator-valued function $\hat{\mathcal{L}} : \mathbb{R}^{m-1} \rightarrow \mathcal{B}(H^2(\mathbb{R}, \mathbb{C}^k), L^2(\mathbb{R}, \mathbb{C}^k))$, defined by $\hat{\mathcal{L}}(\xi) = D\partial_{y_1}^2 + cI_k \partial_{y_1} + \mathcal{M}_{V(\cdot, \xi)}$, where $V(y_1, \xi) = F'(\bar{h}(y_1)) - |\xi|^2 D$. We refer to Section 4 for a more detailed discussion on this topic. Next, we assume the following hypothesis.

Hypothesis (RD). The spectrum of $\hat{\mathcal{L}}(0) = D\partial_{y_1}^2 + cI_k \partial_{y_1} + \mathcal{M}_{F'(\bar{h}(\cdot))}$, the linearization along the one-dimensional problem, satisfies the following conditions:

- (i) $\sup \operatorname{Re}(\sigma(\hat{\mathcal{L}}(0)) \setminus \{0\}) \leq -\nu$ for some $\nu > 0$;
- (ii) 0 is a *semisimple* eigenvalue of finite multiplicity of $\hat{\mathcal{L}}(0)$.

We are now ready to formulate our Lyapunov stability result for the case of reaction–diffusion equations where the diffusion rates of various components of u in (1.8) are close to each other.

Proposition 1.3. Assume Hypothesis (RD) and that the diffusion matrix D is sufficiently close to dI_k , for some $d > 0$, in the sense described in Equation (4.19) below. Then the family of semigroups generated by $\hat{L}(\xi)$ is stable uniformly with respect to $\xi \in \mathbb{R}^{m-1}$. In particular, the front $\bar{h}$ is Lyapunov linearly stable.

Next, we briefly recall the bidomain model in electrophysiology studied by Matano and Mori, see, for example, [16],

$$\begin{cases} u_t = \nabla_x \cdot (A_i \nabla_x u_i) + f(u) \\ \nabla_x \cdot (A_i \nabla_x u_i + A_e \nabla_x u_e) = 0, \quad t \geq 0, x \in \mathbb{R}^2. \\ u = u_i - u_e \end{cases} \quad (1.9)$$

Here, the scalar functions u_i and u_e represent the intracellular and extracellular voltages, $A_i, A_e \in \mathbb{R}^{2 \times 2}$ are symmetric, positive definite matrices. Typically one has

$$A_i = \begin{bmatrix} 1 + \nu_1 + \nu_2 & 0 \\ 0 & 1 + \nu_2 - \nu_1 \end{bmatrix}, \quad A_e = \begin{bmatrix} 1 - \nu_1 - \nu_2 & 0 \\ 0 & 1 - \nu_2 + \nu_1 \end{bmatrix}, \quad \text{with } |\nu_1 \pm \nu_2| < 1. \quad (1.10)$$

The function f is of class $\mathcal{C}^3$ and of bistable type, for example, 0 and 1 are two stable zeros of f , and there exists a unique unstable zero of f in the interval (0,1). For a very detailed discussion regarding the applications and importance of the bidomain Allen–Cahn model to cardiac electrophysiology, we refer to [16] and the references therein.

The bidomain model (1.9) has planar wave solutions of the form

$$(u, u_i, u_e)(x, t) = (\bar{w}, \bar{w}_i, \bar{w}_e)(x_1 \cos \gamma + x_2 \sin \gamma - ct), \quad x = (x_1, x_2) \in \mathbb{R}^2, t \geq 0, c, \gamma \in \mathbb{R}.$$

Next, we pass to the moving frame coordinate system $(y_1, y_2) \in \mathbb{R}^2$, where the new axes are chosen such that the wave travels in the direction of y_1 . Moreover, we note that last two equations of the system (1.9) are linear. Linearizing the system along the traveling wave solution and eliminating the variables u_i and u_e , we obtain that the linearization is given by $\mathcal{A} = -\mathcal{L}_\gamma + c\partial_{y_1} + \mathcal{M}_{f'(\bar{w})}$. The linear operator $\mathcal{L}_\gamma : H^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ is given as the Fourier multiplier $\mathcal{L}_\gamma = \mathcal{F}^{-1} \mathcal{M}_{Q_\gamma} \mathcal{F}$. The function Q_γ is a rational function whose coefficients depend on ν_1, ν_2 , and γ only. Moreover, it can be represented as follows:

$$Q_\gamma(\xi_1, \xi_2) = \begin{cases} \xi_2^2 \left(p\left(\frac{\xi_1}{\xi_2}\right) + g\left(\frac{\xi_1}{\xi_2}\right) \right), & \xi_1 \in \mathbb{R}, \quad \xi_2 \in \mathbb{R} \setminus \{0\}, \\ N_0^2 \xi_1^2, & \xi_1 \in \mathbb{R}, \quad \xi_2 = 0, \end{cases} \quad \text{where} \quad (1.11)$$

$$p(s) = N_0^2(s - \eta_1)^2 + \eta_0, \quad g(s) = \frac{\beta_1 s + \beta_0}{s^2 + 1}. \quad (1.12)$$

The constants $N_0, \beta_0, \beta_1, \eta_0$, and η_1 depend on ν_1 and ν_2 and γ only. Taking Fourier transform $\mathcal{F}_2$ with respect to $y_2 \in \mathbb{R}$, we note that the linear operator $\mathcal{A}$ is unitary equivalent to $\mathcal{M}_{\hat{A}}$, the operator of multiplication on $L^2(\mathbb{R}, L^2(\mathbb{R}))$ by the operator-valued function $\hat{A} : \mathbb{R} \rightarrow \mathcal{B}(H^2(\mathbb{R}), L^2(\mathbb{R}))$, $\hat{A}(\xi_2) = -\mathcal{F}_1^{-1} \mathcal{M}_{Q_\gamma(\cdot, \xi_2)} \mathcal{F}_1 + c\partial_{y_1} + \mathcal{M}_{f'(\bar{w})}$. Here, $\mathcal{F}_1$ denotes the Fourier transform with respect to the variable $y_1 \in \mathbb{R}$. For more details, we refer to Section 5 and [16].

From (1.11) and (1.12), it follows that $\hat{A}(0)$ is a second-order differential operator and there exists $M_b > 0$ such that

$$\|R(\lambda, \hat{A}(0))\| \leq \frac{M_b}{|\lambda|} \quad \text{whenever } \operatorname{Re} \lambda \geq 0, \lambda \neq 0. \quad (1.13)$$

We recall the following result from [16, Corollary 3.3]:

$$\bar{w} \text{ is spectrally stable provided that } \eta_0 > M_b g_\Delta - \bar{g}. \quad (1.14)$$

Here, $g_{\inf} = \inf_{s \in \mathbb{R}} g(s)$, $g_{\sup} = \sup_{s \in \mathbb{R}} g(s)$, $\bar{g} = \frac{g_{\inf} + g_{\sup}}{2}$, $g_{\Delta} = \frac{g_{\sup} - g_{\inf}}{2}$, while η_0 was introduced in (1.12) and M_b was introduced in (1.13). In [16], the authors show that (1.14) is met for certain values of the parameters. In this paper, we prove that the sufficient condition of (1.14) guarantees that the planar front $\bar{w}$ is Lyapunov linearly stable.

Proposition 1.4. *Assume that $\eta_0 > M_b g_{\Delta} - \bar{g}$. Then, the family of semigroups generated by $\hat{A}(\xi_2)$ is stable uniformly with respect to $\xi_2 \in \mathbb{R}$. In particular, the front $\bar{w}$ is Lyapunov linearly stable.*

Plan of the paper. The paper is organized as follows. In Section 2, we discuss the two most common concepts of sectorial operators, and analyze several exponential bounds for analytic semigroups. In Section 3, we prove Theorem 1.1 and Theorem 1.2. Proposition 1.3 and Proposition 1.4 are proved in Section 4 and Section 5, respectively.

A glossary of notation. $L^p(\mathbb{R}^m, \mathbb{X})$, $p \geq 1$, denotes the usual Lebesgue space on $\mathbb{R}^m$ with values in a Banach space $\mathbb{X}$, associated with the Lebesgue measure dx on $\mathbb{R}^m$. $H^s(\mathbb{R}^m, \mathbb{X})$, $s > 0$, is the usual Sobolev space of $\mathbb{X}$ -valued functions. The open disc in $\mathbb{C}$ centered at a of radius $\varepsilon > 0$ is denoted by $D(a, \varepsilon)$. The identity operator on a Banach space $\mathbb{X}$ is denoted by $I_{\mathbb{X}}$. The set of bounded linear operators from a Banach space $\mathbb{X}$ to itself is denoted by $\mathcal{B}(\mathbb{X})$. For an operator B on a Banach space $\mathbb{X}$, we use $\text{dom}(B)$, $\ker B$, $\text{im} B$, $\sigma(B)$, and $B|_{\mathbb{W}}$ to denote the domain, kernel, range, spectrum, adjoint, and the restriction of B to a subspace $\mathbb{W}$ of $\mathbb{X}$. In the case when the space is a Hilbert space, B^* denotes the adjoint operator. We denote by $\sigma_{\text{disc}}(B)$ the set of isolated eigenvalues of finite algebraic multiplicity of the linear operator B , and by $\sigma_{\text{ess}}(B)$ its complement in the spectrum of B . The direct sum of two subspaces $\mathbb{W}_1$ and $\mathbb{W}_2$ is denoted by $\mathbb{W}_1 \oplus \mathbb{W}_2$. The operator of multiplication by a function g is denoted by $\mathcal{M}_g$. We use $\omega_0(T)$ or $\omega_0(A)$ to denote the growth bound of a semigroup $\{T(t)\}_{t \geq 0}$ with generator A . The spectral bound of the generator A is defined by $s(A) = \sup \text{Re } \sigma(A)$.

2 | NORM ESTIMATES OF SEMIGROUPS GENERATED BY SECTORIAL OPERATORS

In this section, we assume that $\mathbb{X}$ is a Banach space and $A : \text{dom}(A) \subseteq \mathbb{X} \rightarrow \mathbb{X}$ is a sectorial operator generating an analytic semigroup of linear operators denoted $\{T(t)\}_{t \geq 0}$. There are various concepts of sectorial operators relevant to our setup that we are going to briefly discuss below.

First, we recall that for any $a \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$, we defined the sector with vertex at a of angle θ the set

$$\Omega_{a,\theta} = \{\lambda \in \mathbb{C} : \lambda \neq a, |\arg(\lambda) - a| < \theta\}. \quad (2.1)$$

One can readily check that

$$\mathbb{C} \setminus \Omega_{a,\theta} = \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq a, |\text{Im } \lambda| \leq (\text{Re } \lambda - a) \tan \theta\}. \quad (2.2)$$

In the literature on semigroup of linear operators (see, e.g., [6, 15, 17]), the definition of a sectorial operator is given using its spectral properties, see (1.2). In the case of linear operators on a Hilbert space, in particular differential operators, several classical texts [20] define sectorial operators using the numerical range. We recall that if $\mathbb{H}$ is a Hilbert space and $B : \text{dom}(B) \subseteq \mathbb{H} \rightarrow \mathbb{H}$ is a closed, densely defined linear operator, the numerical range of B is defined by

$$W(B) = \{\langle Bh, h \rangle : h \in \text{dom}(B), \|h\| = 1\}. \quad (2.3)$$

Definition 2.1. The operator $B : \text{dom}(B) \subseteq \mathbb{H} \rightarrow \mathbb{H}$ is said to be *numerical range sectorial* if there exists $a \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$ such that $W(B) \subseteq \mathbb{C} \setminus \Omega_{a,\theta}$.

Remark 2.2. From [3, Theorem 1.4], one can readily check that any *numerical range sectorial* operator is *sectorial*, but not vice versa. Indeed, if $W(B) \subseteq \mathbb{C} \setminus \Omega_{a,\theta}$ for some $a \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$, since $\Omega_{a,\theta}$ is open, we have $\sigma(B) \subseteq \overline{W(B)} \subseteq \mathbb{C} \setminus \Omega_{a,\theta}$. In addition,

$$\|R(\lambda, B)\| \leq \frac{1}{\text{dist}(\lambda, S_{a,\theta})} = \frac{1}{\text{dist}(\lambda - a, S_{0,\theta})}. \quad (2.4)$$

Another simple computation shows that

$$\text{dist}(z, S_{0,\theta}) = \begin{cases} |z|, & |\arg z| < \theta - \frac{\pi}{2}, \\ |z| \sin(\theta - |\arg z|), & \theta - \frac{\pi}{2} \leq |\arg z| < \theta, \end{cases} \quad (2.5)$$

for any $z \in \Omega_{0,\theta}$. From (2.4) and (2.5), we infer that

$$\|R(\lambda, B)\| \leq \frac{\csc(\theta - \varphi)}{|\lambda - a|} \text{ for any } \lambda \in \Omega_{a,\varphi} \text{ and any } \varphi \in \left(\frac{\pi}{2}, \theta\right), \quad (2.6)$$

and so B is sectorial.

It is well known that if the linear operator $A : \text{dom}(A) \subseteq \mathbb{X} \rightarrow \mathbb{X}$ satisfies Hypothesis (H1), then $\|T(t)\| \leq Le^{at}$ for any $t \geq 0$, for some $L > 0$. Our first task is to revisit the proof of this result and for each $\varphi \in (\frac{\pi}{2}, \theta)$ find a constant $L = L(\varphi)$, not necessarily optimal, satisfying the estimate that depends only on φ and M_0 .

Lemma 2.3. *Assume Hypothesis (H1). Then,*

$$\|T(t)\| \leq \frac{M_0(e\varphi - \sec \varphi)}{\pi} e^{at} \text{ for any } t \geq 0 \text{ and any } \varphi \in \left(\frac{\pi}{2}, \theta\right). \quad (2.7)$$

Proof. We fix $\varphi \in (\frac{\pi}{2}, \theta)$, $t > 0$, and $\varepsilon > 0$. We introduce the curves in the complex plane given by

$$\begin{aligned} \Lambda_{a,\varphi,\varepsilon}^{\pm} &= \{\lambda \in \mathbb{C} : \arg(\lambda - a) = \pm\varphi, |\lambda - a| \geq \varepsilon\} = \{a + se^{\pm i\varphi} : s \geq \varepsilon\}, \\ \Lambda_{a,\varphi,\varepsilon}^c &= \{\lambda \in \mathbb{C} : |\arg(\lambda - a)| \leq \varphi, |\lambda - a| = \varepsilon\} = \{a + \varepsilon e^{i\zeta} : -\varphi \leq \zeta \leq \varphi\}. \end{aligned} \quad (2.8)$$

The path $\Lambda_{a,\varphi,\varepsilon}$ is defined as the union $\Lambda_{a,\varphi,\varepsilon}^- \cup \Lambda_{a,\varphi,\varepsilon}^c \cup \Lambda_{a,\varphi,\varepsilon}^+$ oriented counterclockwise. Since $\Lambda_{a,\varphi,\frac{\varepsilon}{t}} \subseteq \Omega_{a,\theta}$ for any $t > 0$ and $\varepsilon > 0$, we have (see [15, Chapter 2])

$$T(t) = \frac{1}{2\pi i} \int_{\Lambda_{a,\varphi,\frac{\varepsilon}{t}}} e^{\lambda t} R(\lambda, A) d\lambda \text{ for any } t > 0, \varepsilon > 0. \quad (2.9)$$

Next, we estimate the contour integrals above using (1.2). Changing variables we have

$$\begin{aligned} \int_{\Lambda_{a,\varphi,\frac{\varepsilon}{t}}^{\pm}} e^{\lambda t} R(\lambda, A) d\lambda &= \int_{\frac{\varepsilon}{t}}^{\infty} e^{(a+se^{\pm i\varphi})t} R(a + se^{\pm i\varphi}, A) e^{\pm i\varphi} ds \\ &= \frac{e^{at \pm i\varphi}}{t} \int_{\varepsilon}^{\infty} e^{\xi e^{\pm i\varphi}} R\left(a + \frac{\xi}{t} e^{\pm i\varphi}, A\right) d\xi. \end{aligned} \quad (2.10)$$

Since $\Lambda_{a,\varphi,\frac{\varepsilon}{t}}^{\pm} \subseteq \Omega_{a,\theta}$ for any $t > 0$ and $\varepsilon > 0$, from (1.2) and (2.10), we obtain that

$$\left\| \int_{\Lambda_{a,\varphi,\frac{\varepsilon}{t}}^{\pm}} e^{\lambda t} R(\lambda, A) d\lambda \right\| \leq \frac{e^{at}}{t} \int_{\varepsilon}^{\infty} e^{\xi \cos \varphi} \frac{M_0}{|\frac{\xi}{t} e^{\pm i\varphi}|} d\xi = M_0 e^{at} \int_{\varepsilon}^{\infty} \frac{e^{\xi \cos \varphi}}{\xi} d\xi. \quad (2.11)$$

Similarly, since $\Lambda_{a,\varphi,\frac{\varepsilon}{t}}^c \subseteq \Omega_{a,\theta}$ for any $t > 0$ and $\varepsilon > 0$, one can readily check that

$$\left\| \int_{\Lambda_{a,\varphi,\frac{\varepsilon}{t}}^c} e^{\lambda t} R(\lambda, A) d\lambda \right\| = \left\| \frac{\varepsilon i e^{at}}{t} \int_{-\varphi}^{\varphi} e^{\varepsilon e^{i\zeta}} R\left(a + \frac{\varepsilon}{t} e^{i\zeta}, A\right) d\zeta \right\| \leq M_0 e^{at} \int_{-\varphi}^{\varphi} e^{\varepsilon \cos \zeta} d\zeta. \quad (2.12)$$

From (2.9), (2.11), and (2.12) we conclude that

$$\|T(t)\| \leq M_0 C_0(\varphi) e^{at} \text{ for any } t \geq 0, \text{ where } C_0(\varphi) := \frac{1}{\pi} \inf_{\varepsilon > 0} \left(\int_{\varepsilon}^{\infty} \frac{e^{\xi \cos \varphi}}{\xi} d\xi + \frac{1}{2} \int_{-\varphi}^{\varphi} e^{\varepsilon \cos \zeta} d\zeta \right). \quad (2.13)$$

Finally, we note that

$$C_0(\varphi) \leq \frac{1}{\pi} \left(\int_1^{\infty} \frac{e^{\xi \cos \varphi}}{\xi} d\xi + \frac{1}{2} \int_{-\varphi}^{\varphi} e^{\cos \zeta} d\zeta \right) \leq \frac{1}{\pi} \left(\int_1^{\infty} e^{\xi \cos \varphi} d\xi + e\varphi \right) = \frac{e\varphi - \sec \varphi}{\pi}, \quad (2.14)$$

proving the lemma. $\square$

Next, we discuss how to improve estimate (2.7), provided that the sectorial semigroup generator A is such that $\sup \operatorname{Re} \sigma(A) < a$, where a is the vertex of the sector. Such an estimate is important since for many second-order elliptic differential operators, one can immediately prove they are sectorial using Gårding inequality, but typically the vertex is positive. To formulate our result, we introduce the operator-valued function

$$\mathcal{V}_A : (0, \infty) \times (s(A), \infty) \times \left(\frac{\pi}{2}, \theta \right) \rightarrow \mathcal{B}(\mathbb{X}), \quad \mathcal{V}_A(t, \mu, \varphi) = \int_{-(a-\mu)|\tan \varphi|}^{(a-\mu)|\tan \varphi|} e^{ist} R(\mu + is, A) ds. \quad (2.15)$$

Lemma 2.4. Assume Hypothesis (H1) and that the vertex a from (1.2) is such that $a > s(A) = \sup \operatorname{Re} \sigma(A)$. Then,

$$\|T(t)\| \leq \frac{M_0 e^{\mu t}}{\pi(a-\mu)t} + \frac{e^{\mu t}}{2\pi} \sup_{t>0} \|\mathcal{V}_A(t, \mu, \varphi)\| \text{ for any } t > 0, \mu \in (s(A), a), \varphi \in \left(\frac{\pi}{2}, \theta \right). \quad (2.16)$$

Proof. The proof uses the typical contour integral representation for sectorial operators. First we fix $\mu > s(A)$ and $\varphi \in (\frac{\pi}{2}, \theta)$ and set $b = (a - \mu)|\tan \varphi|$ and $c = (a - \mu)|\sec \varphi|$. Next, we introduce the curves in the complex plane given by

$$\Gamma_{a,\mu,\varphi}^{\pm} = \{a + se^{\pm i\varphi} : s \geq c\}, \quad \Gamma_{a,\mu,\varphi}^c = \{\mu + is : -b \leq s \leq b\}. \quad (2.17)$$

The path $\Gamma_{a,\varphi,\varepsilon}$ is defined as the union $\Gamma_{a,\varphi,\varepsilon}^{-} \cup \Gamma_{a,\varphi,\varepsilon}^c \cup \Gamma_{a,\varphi,\varepsilon}^{+}$ oriented counterclockwise. We note that this path is contained in the resolvent set of A and surrounds, the spectrum of A counterclockwise. It follows that (see, e.g., [15, Chapter 2])

$$T(t) = \frac{1}{2\pi i} \int_{\Gamma_{a,\mu,\varphi}} e^{\lambda t} R(\lambda, A) d\lambda \text{ for any } t > 0. \quad (2.18)$$

Since $\Gamma_{a,\mu,\varphi}^c \subseteq \Omega_{a,\theta}$, $a = \mu - c \cos \varphi$ from (1.2) and (2.17) (Figure 1), we obtain that

$$\begin{aligned} \left\| \int_{\Gamma_{a,\mu,\varphi}^{\pm}} e^{\lambda t} R(\lambda, A) d\lambda \right\| &= \left\| \int_c^{\infty} e^{(a+se^{\pm i\varphi})t} R(a + se^{\pm i\varphi}, A) e^{\pm i\varphi} ds \right\| \leq \int_c^{\infty} e^{(a+s \cos \varphi)t} \frac{M_0}{s} ds \\ &= \int_c^{\infty} e^{(\mu+(s-c) \cos \varphi)t} \frac{M_0}{s} ds = M_0 e^{\mu t} \int_c^{\infty} \frac{e^{(s-c) \cos \varphi t}}{s} ds \\ &= M_0 e^{\mu t} \int_0^{\infty} \frac{e^{-\xi t} |\sec \varphi|}{c + |\sec \varphi| \xi} d\xi = M_0 e^{\mu t} \int_0^{\infty} \frac{e^{-\xi t}}{a - \mu + \xi} d\xi \\ &\leq M_0 e^{\mu t} \int_0^{\infty} \frac{e^{-\xi t}}{a - \mu} d\xi = \frac{M_0 e^{\mu t}}{(a - \mu)t}. \end{aligned} \quad (2.19)$$

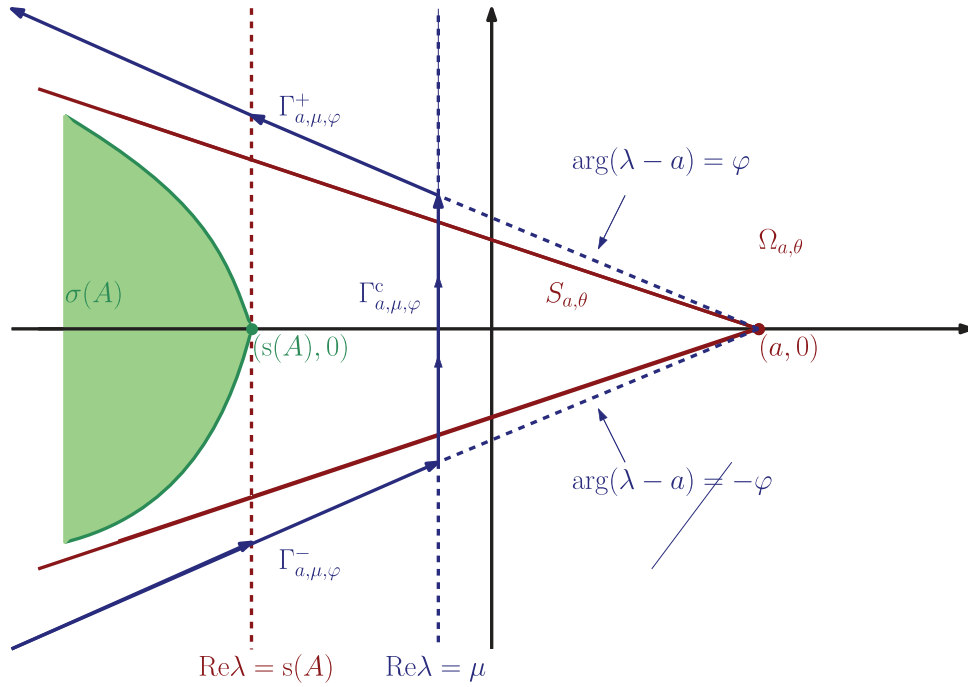


FIGURE 1 A plot of the spectrum of A , the sector $\Omega_{a,\theta}$, and the path $\Gamma_{a,\mu,\varphi}$ in the case when the vertex a is positive.

Unlike $\Lambda_{a,\varphi,\varepsilon}^c$ used in Lemma 2.3 above, $\Gamma_{a,\mu,\varphi}^c$ is not contained in the sector $\Omega_{a,\theta}$. To finish the proof of the lemma, we note that

$$\left\| \int_{\Gamma_{a,\mu,\varphi}^c} e^{\lambda t} R(\lambda, A) d\lambda \right\| = \left\| i e^{\mu t} \int_{-b}^b e^{is t} R(\mu + is, A) ds \right\| \leq e^{\mu t} \sup_{t>0} \|\mathcal{V}_A(t, \mu, \varphi)\|. \quad (2.20)$$

The estimate (2.16) follows shortly from (2.18), (2.19), and (2.20). $\square$

We note that surprisingly the estimate (2.16) is more useful in the case when $a > 0$. The main reason is that if $a > 0$ and $\mu = -\varkappa q(\alpha)$, with the function q as in Hypothesis (H3) and $\varkappa \in (0, 1)$, then the constant $\frac{M_0}{\pi(a-\mu)}$ from (2.16) is uniformly bounded by $\frac{M_0}{\pi a}$. If $a = 0$ and $\mu = -\varkappa q(\alpha)$ with α close to 0, the constant $\frac{M_0}{\pi(a-\mu)}$ is of order $\mathcal{O}(\frac{1}{\alpha})$, making the estimate (2.16) not that useful.

Remark 2.5. Assuming Hypothesis (H1) and that $a > s(A)$, one can readily check that

$$\sup_{t>0} \|\mathcal{V}_A(t, \mu, \varphi)\| \leq 2(a - \mu) |\tan \varphi| \sup\{\|R(\mu + is, A)\| : |s| \leq (a - \mu) |\tan \varphi|\} < \infty \quad (2.21)$$

for any $\mu \in (s(A), a)$ and $\varphi \in (\frac{\pi}{2}, \theta)$. From Lemma 2.4, it follows that

$$\|T(t)\| \leq \frac{M_0 e^{\mu t}}{\pi(a - \mu)t} + \frac{(a - \mu) |\tan \varphi| e^{\mu t}}{\pi} \sup\{\|R(\mu + is, A)\| : |s| \leq (a - \mu) |\tan \varphi|\} \quad (2.22)$$

for any $t > 0$ and any $\mu > s(A)$, $\varphi \in (\frac{\pi}{2}, \theta)$. This result for the special case of analytic semigroups on Banach spaces resembles the famous Gearhart–Prüss result for semigroups on Hilbert spaces [7, 18], which was optimized by Helffer and Sjöstrand [9, 10]. Indeed, the Gearhart–Prüss theorem says that a strongly continuous semigroup is exponentially stable provided its generator has no spectrum in the right half plane and its resolvent operator is bounded along the whole imaginary axis. In case of *analytic* semigroups, however, it is enough to estimate in (2.22) the resolvent operator of A

along a *bounded* vertical segment, not the entire vertical line as it is needed for C_0 -semigroups that are not necessarily analytic.

Next, we collect a couple of other properties of the operator-valued function $\mathcal{V}_A$. In particular, we are interested to find an alternative formula involving convolutions.

Remark 2.6. Setting $b = (a - \mu)|\tan \varphi|$ and using the fact that the resolvent operator of A is the Laplace Transform of the analytic semigroup $\{T(t)\}_{t \geq 0}$, it follows that

$$\begin{aligned} \mathcal{V}_A(t, \mu, \varphi) &= \int_{-b}^b e^{ist} \int_0^\infty e^{-(\mu+is)\xi} T(\xi) d\xi ds = \int_{-b}^b \int_0^\infty e^{i(t-\xi)s} e^{-\mu\xi} T(\xi) d\xi ds \\ &= \int_0^\infty \left(\int_{-b}^b e^{i(t-\xi)s} ds \right) e^{-\mu\xi} T(\xi) d\xi = \int_0^\infty \frac{\sin((a-\mu)(t-\xi)|\tan \varphi|)}{t-\xi} e^{-\mu\xi} T(\xi) d\xi \end{aligned} \quad (2.23)$$

for any $\mu > s(A)$ and $\varphi \in (\frac{\pi}{2}, \theta)$. This representation as a convolution shows that the estimate (2.16) is somewhat similar to convolution-type results of Latushkin and Yurov for the case of general semigroups on Banach spaces [14, Theorem 1.2].

Remark 2.7. From Hypothesis (H2)(ii) we infer that 0 is a pole of order 1 of $R(\cdot, A)$, hence $\lim_{\lambda \rightarrow 0} \lambda R(\lambda, A)$ does exist in the operator norm. We infer that the operator-valued function

$$\lambda \rightarrow \lambda R(\lambda, A) : \Omega_{a,\theta} \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\nu\} \setminus \{0\} \rightarrow \mathcal{B}(\mathbb{X}) \quad (2.24)$$

can be extended analytically to $\Omega_{a,\theta} \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\nu\}$. For simplicity, we will use the same notation for this function and set $(\lambda R(\lambda, A))|_{\lambda=0} = \lim_{\lambda \rightarrow 0} \lambda R(\lambda, A)$.

Next, we recall a sufficient condition that guarantees that a linear operator is sectorial due to A. Lunardi [15].

Lemma 2.8. Assume that $A : \operatorname{dom}(A) \subseteq \mathbb{X} \rightarrow \mathbb{X}$ is a linear operator and $r \in \mathbb{R}$ such that

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \omega\} \subseteq \rho(A) \quad \text{and} \quad \|\lambda R(\lambda, A)\| \leq N_0 \quad \text{whenever} \quad \operatorname{Re} \lambda \geq \omega. \quad (2.25)$$

Then, the operator A is sectorial. More precisely,

$$\Omega_{\omega,\tau} \subseteq \rho(A) \quad \text{and} \quad \|R(\lambda, A)\| \leq \frac{\sqrt{4N_0^2 + 1}}{|\lambda - \omega|} \quad \text{for any} \quad \lambda \in \Omega_{\omega,\tau}, \quad (2.26)$$

where $\tau = \pi - \arctan(2N_0)$.

It is well known that any bounded perturbation of a sectorial operator is also sectorial. To finish this section, we discuss a sufficient condition that guarantees that the angle of the sector remains the same.

Lemma 2.9. Assume Hypothesis (H1) and $W \in \mathcal{B}(\mathbb{X})$. Then,

$$\Omega_{\bar{a},\theta} \subseteq \rho(A + W), \quad \|R(\lambda, A + W)\| \leq \frac{2M_0(1 + \csc \theta)}{|\lambda - \bar{a}|} \quad \text{for any} \quad \lambda \in \Omega_{\bar{a},\theta}, \quad (2.27)$$

where $\bar{a} = a + 2M_0\|W\| \csc \theta$.

Proof. Fix $\lambda \in \Omega_{\bar{a},\theta}$. Since $\bar{a} > a$, we have $\lambda \in \Omega_{a,\theta} \subseteq \rho(A)$. Moreover, from Hypothesis (H1) we obtain that

$$|\lambda - a| \geq \operatorname{dist}(a, \Omega_{\bar{a},\theta}) = (\bar{a} - a) \sin \theta = 2M_0\|W\|, \quad \text{and thus} \quad \|WR(\lambda, A)\| \leq \frac{M_0\|W\|}{|\lambda - a|} \leq \frac{1}{2}. \quad (2.28)$$

It follows that $I_{\mathbb{X}} - WR(\lambda, A)$ is invertible with bounded inverse and $\|(I_{\mathbb{X}} - WR(\lambda, A))^{-1}\| \leq 2$. Since $\lambda I_{\mathbb{X}} - A - W = (I_{\mathbb{X}} - WR(\lambda, A))(\lambda I_{\mathbb{X}} - A)$, from (2.28) we infer that $\lambda \in \rho(A + W)$ and

$$\begin{aligned} \|R(\lambda, A + W)\| &\leq \|R(\lambda, A)\| \|(I_{\mathbb{X}} - WR(\lambda, A))^{-1}\| \leq \frac{2M_0}{|\lambda - a|} = \frac{2M_0|\lambda - a - 2M_0\|W\| \csc \theta|}{|\lambda - \bar{a}| |\lambda - a|} \\ &\leq \frac{2M_0}{|\lambda - \bar{a}|} \left(1 + \frac{2M_0\|W\| \csc \theta}{|\lambda - a|}\right) \leq \frac{2M_0(1 + \csc \theta)}{|\lambda - \bar{a}|}, \end{aligned} \quad (2.29)$$

proving the lemma. $\square$

3 | NORM ESTIMATES OF FAMILIES OF ANALYTIC SEMIGROUPS

In this section, we discuss the uniform exponential stability of families of analytic semigroups whose generators are bounded perturbations of a sectorial operator satisfying Hypotheses (H1) and (H2). Throughout this section, we assume that $E : [0, \infty) \rightarrow \mathcal{B}(\mathbb{X})$ is such that $E(0) = 0$, and the family of operators denoted by $A_\alpha := A + E(\alpha)$, $\alpha \geq 0$, satisfies Hypothesis (H3). In addition, we recall that $\{T_\alpha(t)\}_{t \geq 0}$ denotes the semigroup of linear operators generated by A_α , $\alpha \geq 0$.

3.1 | Basic properties of families of analytic semigroups

First, we will show that the operator A_α , $\alpha \geq 0$, is sectorial and all the constants from (1.2) can be chosen independent of α .

Lemma 3.1. *Assume Hypotheses (H1)–(H3). Then, there exist $\tilde{a} > 0$, $\tilde{\theta} \in (\frac{\pi}{2}, \pi)$, and $\tilde{M} > 0$, independent of $\alpha \geq 0$, such that*

$$\Omega_{\tilde{a}, \tilde{\theta}} \subseteq \rho(A_\alpha), \quad \|R(\lambda, A_\alpha)\| \leq \frac{\tilde{M}}{|\lambda - \tilde{a}|} \quad \text{for any } \lambda \in \Omega_{\tilde{a}, \tilde{\theta}}, \alpha \geq 0. \quad (3.1)$$

The constants $\tilde{a}$, $\tilde{\theta}$, and $\tilde{M}$ depend on the unperturbed operator A and the functions $\|E(\cdot)\|$ and q and related constants in Hypotheses (H3).

Proof. Without loss of generality we can assume that $a > 0$, where a is the vertex of the sector from Hypothesis (H1). From Hypothesis (H3)(iii) we have there exists $\alpha_0 > 0$ such that $\|E(\alpha)\| \leq (\ell + 1)q(\alpha)$ for any $\alpha \geq \alpha_0$. From Hypotheses (H2) and (H3), respectively, it is clear that $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq a + 1\} \subseteq \rho(A) \cap \rho(A_\alpha)$ for any $\alpha \geq \alpha_0$. Moreover,

$$R(\lambda, A_\alpha) - R(\lambda, A) = R(\lambda, A_\alpha)E(\alpha)R(\lambda, A) \quad \text{whenever } \operatorname{Re} \lambda \geq a + 1, \alpha \geq \alpha_0. \quad (3.2)$$

From (3.2), Hypothesis (H1), and Hypothesis (H3)(i), we obtain that

$$\begin{aligned} \|\lambda R(\lambda, A_\alpha)\| &\leq \|\lambda R(\lambda, A)\| + \|R(\lambda, A_\alpha)\| \|E(\alpha)\| \|R(\lambda, A)\| \leq \frac{M_0|\lambda|}{|\lambda - a|} \left(1 + \frac{M_1\|E(\alpha)\|}{\operatorname{Re} \lambda + q(\alpha)}\right) \\ &\leq \frac{M_0|\lambda|}{|\lambda - a|} \left(1 + \frac{M_1\|E(\alpha)\|}{q(\alpha)}\right) \leq \frac{|\lambda|}{|\lambda - a|} M_0(1 + M_1(\ell + 1)) \\ &\leq M_0(1 + M_1(\ell + 1)) \left(1 + \frac{a}{|\lambda - a|}\right) \leq M_0(1 + a)(1 + M_1(\ell + 1)) \end{aligned} \quad (3.3)$$

whenever $\operatorname{Re} \lambda \geq a + 1$ and $\alpha \geq \alpha_0$. From Hypothesis (H3), we infer that $\sup_{\alpha \in [0, \alpha_0]} \|E(\alpha)\| < \infty$. Lemma 2.9 yields

$$\Omega_{a_1, \theta} \subseteq \rho(A_\alpha), \quad \|R(\lambda, A_\alpha)\| \leq \frac{2M_0(1 + \csc \theta)}{|\lambda - a_1|} \quad \text{for any } \lambda \in \Omega_{a_1, \theta}, \alpha \in [0, \alpha_0], \quad (3.4)$$

where $a_1 = a + 2M_0 \sup_{\alpha \in [0, \alpha_0]} \|E(\alpha)\| \csc \theta \geq a$. It follows that $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq a_1 + 1\} \subseteq \rho(A_\alpha)$ for any $\alpha \in [0, \alpha_0]$ and

$$\|\lambda R(\lambda, A_\alpha)\| \leq \frac{2M_0(1 + \csc \theta)|\lambda|}{|\lambda - a_1|} \leq 2M_0(1 + \csc \theta) \left(1 + \frac{a_1}{|\lambda - a_1|}\right) \leq 2M_0(1 + \csc \theta)(1 + a_1) \quad (3.5)$$

whenever $\operatorname{Re} \lambda \geq a_1 + 1$ and $\alpha \in [0, \alpha_0]$. Since $a_1 \geq a$ from (3.3) and (3.5), it follows that $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq a_1 + 1\} \subseteq \rho(A_\alpha)$ for any $\alpha \geq 0$ and

$$\|\lambda R(\lambda, A_\alpha)\| \leq M_0(1 + a_1)(1 + \max\{\csc \theta, M_1(\ell + 1)\}) \quad \text{whenever} \quad \operatorname{Re} \lambda \geq a_1 + 1, \alpha \geq 0. \quad (3.6)$$

Setting $\tilde{a} = a_1 + 1$, $\tilde{N} = M_0(1 + a_1)(1 + \max\{\csc \theta, M_1(\ell + 1)\})$, $\tilde{M} = \sqrt{4\tilde{N}^2 + 1}$, and $\tilde{\theta} = \pi - \arctan(2\tilde{N})$, from Lemma 2.8 and (3.6), we infer that

$$\Omega_{\tilde{a}, \tilde{\theta}} = \Omega_{a_1+1, \tilde{\theta}} \subseteq \rho(A_\alpha) \text{ and } \|R(\lambda, A_\alpha)\| \leq \frac{\sqrt{4\tilde{N}^2 + 1}}{|\lambda - a_1 - 1|} = \frac{\tilde{M}}{|\lambda - \tilde{a}|} \text{ for any } \lambda \in \Omega_{\tilde{a}, \tilde{\theta}}, \alpha \geq 0, \quad (3.7)$$

proving the lemma. $\square$

In the next lemma, we apply Lemma 2.4 to estimate the norm of the semigroup $\{T_\alpha(t)\}_{t \geq 0}$ for α away from 0.

Lemma 3.2. Assume Hypotheses (H1)–(H3). Then the following estimate holds,

$$\|T_\alpha(t)\| \leq \overline{M}(\kappa, \alpha) e^{-\kappa q(\alpha)t} \quad \text{for any } t \geq 0, \alpha > 0, \kappa \in (0, 1). \quad (3.8)$$

Here the function $\overline{M} : (0, 1) \times (0, \infty) \rightarrow (0, \infty)$ is defined by

$$\overline{M}(\kappa, \alpha) = \frac{1}{\pi} \max \left\{ \frac{\tilde{M} q(\alpha)}{\tilde{a} + \kappa q(\alpha)} + \frac{M_1(\tilde{a} + \kappa q(\alpha)) \left| \tan \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right|}{(1 - \kappa) q(\alpha)}, \right. \\ \left. \frac{\tilde{M} \left(e(2\tilde{\theta} + \pi) - 4 \sec \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right)}{4} e^{\frac{\tilde{a}}{q(\alpha)} + 1} \right\}. \quad (3.9)$$

In particular, for any $\delta > 0$ the family of semigroups $\{T_\alpha(t)\}_{t \geq 0}$ is exponentially stable uniformly for $\alpha \geq \delta$.

Proof. Fix $\kappa \in (0, 1)$, $\alpha > 0$, and let $\mu = -\kappa q(\alpha)$. We set $\varphi = \frac{\tilde{\theta}}{2} + \frac{\pi}{4} \in (\frac{\pi}{2}, \tilde{\theta})$ and $\tilde{b} = (\tilde{a} - \mu) |\tan \varphi|$. From Hypothesis (H3), we have $\mu > s(A_\alpha)$ and

$$\|R(\mu + is, A_\alpha)\| \leq \frac{M_1}{\operatorname{Re}(\mu + is) + q(\alpha)} = \frac{M_1}{\mu + q(\alpha)} = \frac{M_1}{(1 - \kappa)q(\alpha)} \quad \text{for any } s \in \mathbb{R}. \quad (3.10)$$

Using definition (2.15) leads to

$$\|\mathcal{V}_{A_\alpha}(t, \mu, \varphi)\| = \left\| \int_{-\tilde{b}}^{\tilde{b}} e^{ist} R(\mu + is, A_\alpha) ds \right\| \leq \frac{2M_1 \tilde{b}}{(1 - \kappa)q(\alpha)} = \frac{2M_1(\tilde{a} - \mu) \left| \tan \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right|}{(1 - \kappa)q(\alpha)} \quad (3.11)$$

for any $t > 0$. From (3.1), (3.11), and Lemma 2.4, it follows that

$$\begin{aligned} \|T_\alpha(t)\| &\leq \frac{\tilde{M}e^{\mu t}}{\pi(\tilde{a} - \mu)t} + \frac{e^{\mu t}}{2\pi} \cdot \frac{2M_1(\tilde{a} - \mu) \left| \tan\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right) \right|}{(1 - \varkappa)q(\alpha)} \\ &= \frac{e^{-\varkappa q(\alpha)t}}{\pi} \left(\frac{\tilde{M}}{(\tilde{a} + \varkappa q(\alpha))t} + \frac{M_1(\tilde{a} + \varkappa q(\alpha)) \left| \tan\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right) \right|}{(1 - \varkappa)q(\alpha)} \right) \text{ for any } t > 0. \end{aligned} \quad (3.12)$$

If $t \geq \frac{1}{q(\alpha)}$, then $(\tilde{a} + \varkappa q(\alpha))t \geq \frac{\tilde{a}}{q(\alpha)} + \varkappa$, thus from (3.12), we have

$$\|T_\alpha(t)\| \leq \frac{e^{-\varkappa q(\alpha)t}}{\pi} \left(\frac{\tilde{M}q(\alpha)}{\tilde{a} + \varkappa q(\alpha)} + \frac{M_1(\tilde{a} + \varkappa q(\alpha)) \left| \tan\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right) \right|}{(1 - \varkappa)q(\alpha)} \right) \text{ for any } t \geq \frac{1}{q(\alpha)}. \quad (3.13)$$

From Lemma 2.3 and Lemma 3.1, we infer

$$\begin{aligned} \|T_\alpha(t)\| &\leq \frac{\tilde{M}(e\varphi - \sec \varphi)}{\pi} e^{\tilde{a}t} = \frac{\tilde{M} \left(e(2\tilde{\theta} + \pi) - 4 \sec\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right) \right)}{4\pi} e^{(\tilde{a} + \varkappa q(\alpha))t} e^{-\varkappa q(\alpha)t} \\ &\leq \frac{\tilde{M} \left(e(2\tilde{\theta} + \pi) - 4 \sec\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right) \right)}{4\pi} e^{\frac{\tilde{a}}{q(\alpha)} + 1} e^{-\varkappa q(\alpha)t} \end{aligned} \quad (3.14)$$

for any $t \in \left[0, \frac{1}{q(\alpha)}\right)$. The estimate (3.8) follows from (3.13) and (3.14), with $\overline{M}(\varkappa, \alpha)$ given by (3.9). One can readily check that

$$\overline{M}(\varkappa, \alpha) \leq \frac{1}{\pi} \max \left\{ \frac{\tilde{M}}{\varkappa} + \frac{M_1 \left| \tan\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right) \right| \left(\frac{\tilde{a}}{q(\alpha)} + \varkappa \right)}{(1 - \varkappa)}, \frac{\tilde{M} \left(e(2\tilde{\theta} + \pi) - 4 \sec\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right) \right)}{4} e^{\frac{\tilde{a}}{q(\alpha)} + 1} \right\}. \quad (3.15)$$

Since $\inf_{\alpha \geq \delta} q(\alpha) = q(\delta) > 0$ for any $\delta > 0$, by Hypothesis (H3), from (3.8) and (3.15), it follows that for any $\delta > 0$ the family of semigroups $\{T_\alpha(t)\}_{t \geq 0}$ is exponentially stable uniformly for $\alpha \geq \delta$. $\square$

Remark 3.3. In the case when the space $\mathbb{X}$ is a Hilbert space, one can use the results of B. Helffer and J. Sjöstrand [9, 10] to obtain an estimate similar to (3.8). Indeed, from Lemma 2.3 and Lemma 3.1, it follows that

$$\|T_\alpha(t)\| \leq \frac{\tilde{M} \left(e(2\tilde{\theta} + \pi) - 4 \sec\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right) \right)}{4\pi} e^{\tilde{a}t} \text{ for any } t > 0, \alpha > 0. \quad (3.16)$$

From Hypothesis (H3)(i), we have

$$\sup_{\operatorname{Re} \lambda \geq -\varkappa q(\alpha)} \|R(\lambda, A_\alpha)\| \leq \sup_{\operatorname{Re} \lambda \geq -\varkappa q(\alpha)} \frac{M_1}{\operatorname{Re} \lambda + q(\alpha)} \leq \frac{M_1}{(1 - \varkappa)q(\alpha)} \text{ for any } \alpha > 0, \varkappa \in (0, 1). \quad (3.17)$$

From [9, Proposition 2.1], we obtain that

$$\|T_\alpha(t)\| \leq \bar{N}(\kappa, \alpha) e^{-\kappa q(\alpha)t} \quad \text{for any } t > 0, \alpha > 0, \quad (3.18)$$

where the function $\bar{N} : (0, 1) \times (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$\bar{N}(\kappa, \alpha) = \frac{\tilde{M}\left(e(2\tilde{\theta} + \pi) - 4 \sec\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right)\right)}{4\pi} \left(1 + \frac{\tilde{M}\left(e(2\tilde{\theta} + \pi) - 4 \sec\left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4}\right)\right)}{2\pi} \left(1 + \frac{M_1(\tilde{a} + \kappa q(\alpha))}{(1 - \kappa)q(\alpha)}\right)\right). \quad (3.19)$$

Here $\tilde{M}$, $\tilde{\theta}$, and $\tilde{a}$ are obtained in Lemma 3.1 and M_1 is taken from Hypothesis (H3)(i). We note that $\bar{N}(\kappa, \alpha) = \mathcal{O}(\frac{1}{q(\alpha)})$ and $\bar{M}(\kappa, \alpha) = \mathcal{O}(\frac{1}{q(\alpha)})$ as $\alpha \rightarrow 0$, which shows that regardless of what approach is used one can conclude uniform exponential stability of the family of semigroups $\{T_\alpha(t)\}_{t \geq 0}$ for $\alpha \geq \delta$, for any $\delta > 0$, but not for $\alpha \geq 0$. Also, from (3.19) we note that $\sup_{\alpha \geq 0} \bar{N}(\kappa, \alpha) < \infty$ only if $\tilde{a} = 0$, which occurs only when the family of semigroups $\{T_\alpha(t)\}_{t \geq 0}$, $\alpha \geq 0$, is bounded uniformly in $\alpha \geq 0$, a property which is as difficult to establish as the result of exponential decay of the family of semigroups with uniform in α constants.

3.2 | Norm estimates of $\{T_\alpha(t)\}_{t \geq 0}$ when α is in a neighborhood of 0

To estimate the norm of the semigroup $\{T_\alpha(t)\}_{t \geq 0}$ when α is in a neighborhood of 0, we need a different approach. The key part of the argument is to find a spectral decomposition of the Banach space $\mathbb{X}$, into a sum of two closed subspaces invariant under the semigroup $\{T_\alpha(t)\}_{t \geq 0}$, such that the spectrum of the restriction of A_α to the two subspaces is either away from the imaginary axis or made up entirely of eigenvalues. We first look for such a decomposition of the linear operator A . From Hypothesis (H2) we have the spectrum of A has two disjoint parts, separated by the circle $\partial D(0, \frac{\nu}{2})$. We define $P_0 \in \mathcal{B}(\mathbb{X})$ as the spectral projection relative to the spectral subset $\{0\}$ of $\sigma(A)$, given by the formula

$$P_0 = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} R(\lambda, A) d\lambda. \quad (3.20)$$

Remark 3.4. We collect several well-known properties of the spectral projection P_0 , see, for example, [6, 15, 17]:

- (i) $\text{Im} P_0 \subset \text{dom}(A)$;
- (ii) $\text{Im} P_0$ and $\text{Ker} P_0$ are invariant under A and $T(t)$ for any $t \geq 0$;
- (iii) the linear operator $\tilde{A} := A|_{\text{dom}(A) \cap \text{Ker} P_0}$ is the generator of the semigroup $\{\tilde{T}(t)\}_{t \geq 0} = \{T(t)|_{\text{Ker} P_0}\}_{t \geq 0}$ on $\text{Ker} P_0$;
- (iv) $\sigma(\tilde{A}) = \sigma(A) \setminus \{0\}$.

In the next lemma, we aim to refine the properties of P_0 by making use of Hypotheses (H1)–(H2).

Lemma 3.5. Assume Hypotheses (H1)–(H2). Then, the following assertions hold true:

- (i) $\dim \text{Im} P_0 < \infty$ and $A|_{\text{Im} P_0} \equiv 0$;
- (ii) The operator $\tilde{A}$ is sectorial, and thus it generates an analytic semigroup. Moreover, there exists $M_2 > 0$ such that¹

$$\|\tilde{T}(t)\| \leq M_2 e^{-\frac{7\nu}{8}t} \quad \text{for any } t > 0; \quad (3.21)$$

- (iii) The linear operators A and $T(t)$, $t \geq 0$, respectively, have the representations

$$A = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A} \end{bmatrix}, \quad T(t) = \begin{bmatrix} I_{\text{Im} P_0} & 0 \\ 0 & \tilde{T}(t) \end{bmatrix}, \quad t \geq 0, \quad (3.22)$$

with respect to the decomposition $\mathbb{X} = \text{Im} P_0 \oplus \text{Ker} P_0$ (direct sum, not necessarily orthogonal).

Proof. (i) Since 0 is a semisimple eigenvalue of finite multiplicity of A we infer that $\dim \operatorname{Im} P_0 < \infty$. Moreover, from Remark 3.4(i) and since A is a closed linear operator, we have

$$AP_0 = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} AR(\lambda, A) d\lambda = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} (\lambda R(\lambda, A) - I_{\mathbb{X}}) d\lambda. \quad (3.23)$$

By Remark 2.7 we know that the function $\lambda \rightarrow \lambda R(\lambda, A) : \Omega_{a,\theta} \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\nu\} \setminus \{0\} \rightarrow \mathcal{B}(\mathbb{X})$ can be extended analytically to $\Omega_{a,\theta} \cup \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > -\nu\}$, therefore from (3.23), we obtain that $AP_0 \equiv 0$, hence $A|_{\operatorname{Im} P_0} \equiv 0$.

(ii) From Remark 3.4(iv) and Hypothesis (H1), we have $\Omega_{a,\theta} \subset \rho(A) \subset \rho(\tilde{A})$ and $R(\lambda, \tilde{A}) = R(\lambda, A)|_{\operatorname{Ker} P_0}$ for any $\lambda \in \Omega_{a,\theta}$, hence

$$\|R(\lambda, \tilde{A})\| \leq \|R(\lambda, A)\| \leq \frac{M}{|\lambda - a|} \quad \text{for any } \lambda \in \Omega_{a,\theta}. \quad (3.24)$$

We conclude that the linear operator $\tilde{A}$ is sectorial, therefore the semigroup $\{\tilde{T}(t)\}_{t \geq 0}$ is analytic. From Hypothesis (H2)(i), it follows that

$$\omega_0(\tilde{A}) = \sup \operatorname{Re} \sigma(\tilde{A}) = \sup \operatorname{Re}(\sigma(A) \setminus \{0\}) \leq -\nu < -\frac{7\nu}{8}. \quad (3.25)$$

From the definition of the growth rate of a semigroup, one immediately concludes that

$$M_2 = \sup_{t \geq 0} \left(e^{\frac{7\nu}{8}t} \|\tilde{T}(t)\| \right) < \infty, \quad (3.26)$$

proving (ii). Assertion (iii) follows shortly from (i) and Remark 3.4(ii) and (iii). $\square$

To construct a representation of A_α similar to (3.22) for the case when α is in the neighborhood of 0, we utilize the standard transformation/conjugation operators, see [13, Chapter II, section 4.2] and [5, Chapter 4, section 1]. First, we need to show that the circle $\partial D(0, \frac{\nu}{2})$ separates $\sigma(A_\alpha)$ into two disjoint spectral subsets for any α in a neighborhood of 0. For the remainder of the section, we need to assume Hypothesis (H4) in order to ensure that all the spectral and semigroup bounds are α -independent. Moreover, we recall that from (1.4) and Hypothesis (H3), we can immediately infer that E_0 is continuous on $(0, \infty)$ in $\mathcal{B}(\mathbb{X})$.

Lemma 3.6. *Assume Hypotheses (H1)–(H4). Then,*

$$\mathcal{E}_\nu := \left\{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -\frac{3\nu}{4}, |\lambda| \geq \frac{\nu}{4} \right\} \subset \rho(A_\alpha) \text{ and } \|R(\lambda, A_\alpha)\| \leq \frac{16M_2}{\nu} \quad (3.27)$$

for any $\lambda \in \mathcal{E}_\nu$, $\alpha \in [0, \varepsilon_0]$, where

$$\varepsilon_0 := \min \left\{ \frac{\nu}{16M_2 \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| + 1}, 1 \right\} \quad (3.28)$$

and M_2 is defined in (3.26).

Proof. First, we collect some properties of the linear operator $\tilde{A}$. From Remark 3.4(iv) and Lemma 3.5(ii), respectively, we have

$$\sigma(\tilde{A}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -\nu\} \text{ and } \|R(\lambda, \tilde{A})\| \leq \frac{M_2}{\operatorname{Re} \lambda + \frac{7\nu}{8}} \text{ whenever } \operatorname{Re} \lambda > -\frac{7\nu}{8}. \quad (3.29)$$

In addition, from Lemma 3.5(iii) it follows that the resolvent operator of A has the representation

$$R(\lambda, A) = \begin{bmatrix} \frac{1}{\lambda} I_{\text{Im}P_0} & 0 \\ 0 & R(\lambda, \tilde{A}) \end{bmatrix} \quad \text{whenever } \text{Re}\lambda > -\nu, \lambda \neq 0 \quad (3.30)$$

with respect to the decomposition $\mathbb{X} = \text{Im}P_0 \oplus \text{Ker}P_0$. From (3.29) and (3.30), it follows that

$$\|R(\lambda, A)\| \leq \max \left\{ \frac{1}{|\lambda|}, \frac{M_2}{\text{Re}\lambda + \frac{7\nu}{8}} \right\} \leq \max \left\{ \frac{4}{\nu}, \frac{8M_2}{\nu} \right\} = \frac{8M_2}{\nu} \quad \text{for any } \lambda \in \mathcal{E}_\nu. \quad (3.31)$$

Fix $\lambda \in \mathcal{E}_\nu$ and $\alpha \in [0, \varepsilon_0]$, where ε_0 is defined in (3.28). Then,

$$\lambda I_{\mathbb{X}} - A_\alpha = \lambda I_{\mathbb{X}} - A - E(\alpha) = (I_{\mathbb{X}} - E(\alpha)R(\lambda, A))(\lambda I_{\mathbb{X}} - A). \quad (3.32)$$

Moreover, from (1.4) and (3.31), we obtain that

$$\|E(\alpha)R(\lambda, A)\| \leq \frac{8M_2\alpha}{\nu} \|E_0(\alpha)\| \leq \frac{8M_2\varepsilon_0}{\nu} \sup_{\alpha \in [0, \varepsilon_0]} \|E_0(\alpha)\| \leq \frac{1}{2}, \quad (3.33)$$

which implies that $I_{\mathbb{X}} - E(\alpha)R(\lambda, A)$ is invertible and $\|(I_{\mathbb{X}} - E(\alpha)R(\lambda, A))^{-1}\| \leq 2$. From (3.31) and (3.32), we infer that

$$\mathcal{E}_\nu \subset \rho(A_\alpha) \text{ and } \|R(\lambda, A_\alpha)\| \leq \frac{16M_2}{\nu} \text{ for any } \lambda \in \mathcal{E}_\nu, \alpha \in [0, \varepsilon_0],$$

proving the lemma. □

Next, we note that assertion (3.27) allows us to define the spectral projection

$$P_\alpha = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} R(\lambda, A_\alpha) d\lambda, \quad \alpha \in [0, \varepsilon_0]. \quad (3.34)$$

Remark 3.7. Clearly, P_α satisfies the following properties:

- (i) $\text{Im}P_\alpha$ and $\text{Ker}P_\alpha$ are invariant under A_α and $T_\alpha(t)$ for any $t \geq 0, \alpha \in [0, \varepsilon_0]$;
- (ii) $\text{Im}P_\alpha \subset \text{dom}(A)$ for any $\alpha \in [0, \varepsilon_0]$;
- (iii) The linear operator $\tilde{A}_\alpha := (A_\alpha)|_{\text{dom}(A) \cap \text{Ker}P_\alpha}$ is the generator of the semigroup $\{\tilde{T}_\alpha(t)\}_{t \geq 0} = \{T(t)|_{\text{Ker}P_\alpha}\}_{t \geq 0}$ on $\text{Ker}P_\alpha$ for any $\alpha \in [0, \varepsilon_0]$;
- (iv) $\sigma(\tilde{A}_\alpha) = \sigma(A_\alpha) \setminus D(0, \frac{\nu}{2})$ for any $\alpha \in [0, \varepsilon_0]$.

In the next lemma, we show that the family of spectral projections $\{P_\alpha\}_{\alpha \in [0, \varepsilon_0]}$ is continuous in the operator norm.

Lemma 3.8. *Assume Hypotheses (H1)–(H4). Then, the function $\alpha \rightarrow P_\alpha : [0, \varepsilon_0] \rightarrow \mathcal{B}(\mathbb{X})$ is continuous in the operator norm. Moreover, the following estimate holds:*

$$\|P_{\alpha_1} - P_{\alpha_2}\| \leq \frac{128M_2^2}{\nu} \|E(\alpha_1) - E(\alpha_2)\| \quad \text{for any } \alpha_1, \alpha_2 \in [0, \varepsilon_0]. \quad (3.35)$$

Proof. Since $A_\alpha = A + E(\alpha)$ for any $\alpha \geq 0$, from (3.27), we immediately infer that

$$R(\lambda, A_{\alpha_1}) - R(\lambda, A_{\alpha_2}) = R(\lambda, A_{\alpha_1})(E(\alpha_1) - E(\alpha_2))R(\lambda, A_{\alpha_2}) \quad (3.36)$$

for any $\lambda \in \mathcal{E}_\nu$, $\alpha_1, \alpha_2 \in [0, \varepsilon_0]$. From (3.27), (3.34), and (3.36), it follows that

$$\begin{aligned} \|P_{\alpha_1} - P_{\alpha_2}\| &= \frac{1}{2\pi} \left\| \int_{\partial D(0, \frac{\nu}{2})} R(\lambda, A_{\alpha_1})(E(\alpha_1) - E(\alpha_2))R(\lambda, A_{\alpha_2})d\lambda \right\| \\ &\leq \frac{\text{length}\left(\partial D(0, \frac{\nu}{2})\right)}{2\pi} \sup_{|\lambda|=\frac{\nu}{2}} \|R(\lambda, A_{\alpha_1})\| \|E(\alpha_1) - E(\alpha_2)\| \sup_{|\lambda|=\frac{\nu}{2}} \|R(\lambda, A_{\alpha_2})\| \\ &\leq \frac{128M_2^2}{\nu} \|E(\alpha_1) - E(\alpha_2)\| \quad \text{for any } \alpha_1, \alpha_2 \in [0, \varepsilon_0]. \end{aligned} \quad (3.37)$$

The lemma follows shortly from (3.37) and since E is continuous on $[0, \infty)$ in $\mathcal{B}(\mathbb{X})$. $\square$

Following [5, Chapter 4], we introduce the operator transformation/conjugation function associated to the family of projections $\{P_\alpha\}_{\alpha \in [0, \varepsilon_0]}$ as follows:

$$U : [0, \varepsilon_0] \rightarrow \mathcal{B}(\mathbb{X}), \quad U(\alpha) = P_\alpha P_0 + (I_\mathbb{X} - P_\alpha)(I_\mathbb{X} - P_0). \quad (3.38)$$

The properties of the operator transformation/conjugation function U defined above are collected in the following lemma.

Lemma 3.9. *Assume Hypotheses (H1)–(H4). Then, the following assertions hold true:*

- (i) *The function U is continuous on $[0, \varepsilon_0]$ in the $\mathcal{B}(\mathbb{X})$ operator norm;*
- (ii) *$U(\alpha)$ is an invertible operator for any $\alpha \in [0, \varepsilon_1]$, where*

$$\varepsilon_1 := \min \left\{ \frac{\nu}{256M_2^2(16M_2 + 1) \sup_{\alpha \in [0, 1]} \|E_0(\alpha)\| + 1}, 1 \right\} \quad (3.39)$$

and M_2 is defined in (3.26);

- (iii) *The function $U(\cdot)^{-1}$ is continuous on $[0, \varepsilon_1]$ in the $\mathcal{B}(\mathbb{X})$ operator norm;*
- (iv) *$P_\alpha = U(\alpha)P_0U(\alpha)^{-1}$ for any $\alpha \in [0, \varepsilon_1]$.*

Proof. Assertion (i) follows immediately from (3.38) and Lemma 3.8.

(ii) Since P_0 is a projection, one can readily check that

$$\begin{aligned} U(\alpha) &= I_\mathbb{X} + P_\alpha P_0 + (I_\mathbb{X} - P_\alpha)(I_\mathbb{X} - P_0) - P_0^2 - (I_\mathbb{X} - P_0)^2 \\ &= I_\mathbb{X} + (P_\alpha - P_0)P_0 + ((I_\mathbb{X} - P_\alpha) - (I_\mathbb{X} - P_0))(I_\mathbb{X} - P_0) \\ &= I_\mathbb{X} + (P_\alpha - P_0)(2P_0 - I_\mathbb{X}) \quad \text{for any } \alpha \in [0, \varepsilon_0]. \end{aligned} \quad (3.40)$$

Moreover, from (3.20) and Lemma 3.6, we have

$$\|P_\alpha\| = \frac{1}{2\pi} \left\| \int_{\partial D(0, \frac{\nu}{2})} R(\lambda, A_\alpha)d\lambda \right\| \leq \frac{\text{length}\left(\partial D\left(0, \frac{\nu}{2}\right)\right)}{2\pi} \sup_{|\lambda|=\frac{\nu}{2}} \|R(\lambda, A_\alpha)\| \leq 8M_2 \quad (3.41)$$

for any $\alpha \in [0, \varepsilon_0]$. Since $E(0) = 0$ and $\varepsilon_1 \leq 1$ from (1.4), (3.35), and (3.41), we obtain that

$$\|(P_\alpha - P_0)(2P_0 - I_\mathbb{X})\| \leq \frac{128M_2^2}{\nu} (16M_2 + 1) \|E(\alpha)\| \leq \alpha \frac{128M_2^2(16M_2 + 1)}{\nu} \sup_{\alpha \in [0, 1]} \|E_0(\alpha)\| \leq \frac{1}{2} \quad (3.42)$$

for any $\alpha \in [0, \varepsilon_1]$. From (3.40) and (3.42), we obtain that $U(\alpha)$ is invertible for any $\alpha \in [0, \varepsilon_0]$. Moreover,

$$\|U(\alpha)\| \leq \frac{3}{2}, \quad \|U(\alpha)^{-1}\| \leq 2 \quad \text{for any } \alpha \in [0, \varepsilon_1]. \quad (3.43)$$

(iii) From (3.43), we immediately infer that

$$\|U(\alpha_1)^{-1} - U(\alpha_2)^{-1}\| \leq \|U(\alpha_1)^{-1}\| \|U(\alpha_1) - U(\alpha_2)\| \|U(\alpha_2)^{-1}\| \leq 4\|U(\alpha_1) - U(\alpha_2)\| \quad (3.44)$$

for any $\alpha_1, \alpha_2 \in [0, \varepsilon_1]$. Assertion (iii) follows from (i) and (3.44).

(iv) Since P_α is a projection for any $\alpha \in [0, \varepsilon_1]$ from (3.38), we have

$$U(\alpha)P_0 = P_\alpha P_0 = P_\alpha U(\alpha) \quad \text{for any } \alpha \in [0, \varepsilon_1]. \quad (3.45)$$

Assertion (iv) follows from (iii) and (3.45). $\square$

When $\mathbb{X}$ is a Hilbert space, the operator transformation/conjugation function U can be chosen such that $U(\alpha)$ is an unitary operator, see [13, Formula (4.18), p. 102]. In this case, the estimates (3.43) are simpler. However, since our goal is to find uniform bounds, not necessarily optimal bounds, not having $U(\alpha)$ and its inverse of norm 1 is not an inconvenience.

We introduce the linear operators $B_\alpha : \text{dom}(B_\alpha) \subseteq \mathbb{X} \rightarrow \mathbb{X}, \alpha \in [0, \varepsilon_1]$ by

$$\text{dom}(B_\alpha) = U(\alpha)^{-1} \text{dom}(A), \quad B_\alpha = U(\alpha)^{-1} A_\alpha U(\alpha), \quad \alpha \in [0, \varepsilon_1]. \quad (3.46)$$

Next, we collect some of the properties of the family of linear operators $B_\alpha, \alpha \in [0, \varepsilon_1]$.

Remark 3.10. Since $U(\alpha)$ is invertible by Lemma 3.9(ii) for any $\alpha \in [0, \varepsilon_1]$ from (3.46), we infer that

$$\sigma(B_\alpha) = \sigma(A_\alpha), \quad \sigma_{\text{disc}}(B_\alpha) = \sigma_{\text{disc}}(A_\alpha) \quad \text{for any } \alpha \in [0, \varepsilon_1]. \quad (3.47)$$

Therefore, the circle $\partial D(0, \frac{\nu}{2})$ separates $\sigma(B_\alpha)$ into two disjoint spectral subsets for any $\alpha \in [0, \varepsilon_1]$. Moreover, from (3.46), we have

$$R(\lambda, B_\alpha) = U(\alpha)^{-1} R(\lambda, A_\alpha) U(\alpha) \quad \text{for any } \lambda \in \rho(A_\alpha) = \rho(B_\alpha), \alpha \in [0, \varepsilon_1]. \quad (3.48)$$

From Lemma 3.1, (3.47), and (3.48), we obtain

$$\Omega_{\tilde{\alpha}, \tilde{\theta}} \subseteq \rho(B_\alpha), \quad \|R(\lambda, B_\alpha)\| \leq \|U(\alpha)^{-1}\| \|R(\lambda, A_\alpha)\| \|U(\alpha)\| \leq \frac{3\tilde{M}}{|\lambda - \tilde{\alpha}|} \quad (3.49)$$

for any $\lambda \in \Omega_{\tilde{\alpha}, \tilde{\theta}}, \alpha \in [0, \varepsilon_1]$.

Lemma 3.11. Assume Hypotheses (H1)–(H4). Then, the following assertions hold true:

- (i) P_0 is the spectral projection of B_α associated to the spectral subset of $\sigma(B_\alpha)$ contained in $D(0, \frac{\nu}{2})$ for any $\alpha \in [0, \varepsilon_1]$.
- (ii) $\text{Im } P_0 \subseteq \text{dom}(B_\alpha)$ for any $\alpha \in [0, \varepsilon_1]$.
- (iii) $\text{Im } P_0$ and $\text{Ker } P_0$ are invariant under B_α for any $\alpha \in [0, \varepsilon_1]$.

Proof. (i) Since $\partial D(0, \frac{\nu}{2}) \subset \mathcal{E}_\nu \subset \rho(A_\alpha) = \rho(B_\alpha)$ for any $\alpha \in [0, \varepsilon_1] \subset [0, \varepsilon_0]$ by Lemma 3.6, from (3.48) and Lemma 3.9(iv), it follows that the spectral projection of B_α associated to the spectral subset of $\sigma(B_\alpha)$ contained in $D(0, \frac{\nu}{2})$ is given by

$$\frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} R(\lambda, B_\alpha) d\lambda = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} U(\alpha)^{-1} R(\lambda, A_\alpha) U(\alpha) d\lambda = U(\alpha)^{-1} P_\alpha U(\alpha) = P_0 \quad (3.50)$$

for any $\alpha \in [0, \varepsilon_1]$. Assertions (ii) and (iii) follow shortly, cf. [6, 15, 17]. $\square$

We define the linear operators $\tilde{B}_\alpha : \text{dom}(\tilde{B}_\alpha) \subseteq \text{Ker } P_0 \rightarrow \text{Ker } P_0$ and $K_\alpha : \text{Im } P_0 \rightarrow \text{Im } P_0$ by

$$\text{dom}(\tilde{B}_\alpha) = \text{dom}(B_\alpha) \cap \text{Ker } P_0, \quad \tilde{B}_\alpha = (B_\alpha)|_{\text{dom}(B_\alpha) \cap \text{Ker } P_0}, \quad K_\alpha = (B_\alpha)|_{\text{Im } P_0}, \quad \alpha \in [0, \varepsilon_1]. \quad (3.51)$$

In addition, we denote by $\{S_\alpha(t)\}_{t \geq 0}$ and $\{\tilde{S}_\alpha(t)\}_{t \geq 0}$ the semigroups generated by B_α and $\tilde{B}_\alpha$, $\alpha \in [0, \varepsilon_1]$, respectively. The advantage of working with the family of operators B_α , $\alpha \in [0, \varepsilon_1]$, is that its spectral projection P_0 associated to the spectral subset of $\sigma(B_\alpha)$ contained in $D(0, \frac{\nu}{2})$ is independent on $\alpha \in [0, \varepsilon_1]$. From (3.49) we see that the semigroups $\{S_\alpha(t)\}_{t \geq 0}$ and $\{\tilde{S}_\alpha(t)\}_{t \geq 0}$ are analytic, moreover, the following representation holds:

$$T_\alpha(t) = U(\alpha)S_\alpha(t)U(\alpha)^{-1}, \quad S_\alpha(t) = \begin{bmatrix} e^{tK_\alpha} & 0 \\ 0 & \tilde{S}_\alpha(t) \end{bmatrix}, \quad \text{for any } t \geq 0, \alpha \in [0, \varepsilon_1], \quad (3.52)$$

with respect to the decomposition $\mathbb{X} = \text{Im } P_0 \oplus \text{Ker } P_0$.

In the next lemma, we study the spectrum of the linear operator $\tilde{B}_\alpha$, in particular, we estimate $\sup \text{Re} \sigma(\tilde{B}_\alpha)$ for $\alpha \in [0, \varepsilon_1]$.

Lemma 3.12. *Assume Hypotheses (H1)–(H4). Then, the following assertions hold true:*

- (i) $\sigma(\tilde{B}_\alpha) \subseteq \{\lambda \in \mathbb{C} : \text{Re} \lambda \leq -\frac{\nu}{2}\}$ for any $\alpha \in [0, \varepsilon_1]$;
- (ii) $\|R(\lambda, \tilde{B}_\alpha)\| \leq \frac{96M_2}{\nu}$ whenever $\text{Re} \lambda \geq -\frac{\nu}{2}$, $\alpha \in [0, \varepsilon_1]$.

Proof. (i) From Lemma 3.6, Lemma 3.11, (3.51), and (3.47), we have

$$\sigma(\tilde{B}_\alpha) = \sigma(B_\alpha) \setminus D\left(0, \frac{\nu}{2}\right) = \sigma(A_\alpha) \setminus D\left(0, \frac{\nu}{2}\right) \subseteq (\mathbb{C} \setminus \mathcal{E}_\nu) \setminus D\left(0, \frac{\nu}{2}\right) \subseteq \left\{\lambda \in \mathbb{C} : \text{Re} \lambda \leq -\frac{\nu}{2}\right\}. \quad (3.53)$$

for any $\alpha \in [0, \varepsilon_1]$, proving (i).

(ii) First, we note that from Lemma 3.6, (3.43), and (3.48), it follows that

$$\|R(\lambda, \tilde{B}_\alpha)\| \leq \|R(\lambda, B_\alpha)\| = \|U(\alpha)^{-1}R(\lambda, A_\alpha)U(\alpha)\| \leq 3\|R(\lambda, A_\alpha)\| \leq \frac{48M_2}{\nu} \quad (3.54)$$

for any $\lambda \in \mathcal{E}_\nu$, $\alpha \in [0, \varepsilon_1]$. To prove the lemma, we need to prove the estimate from (ii) for the case when $\lambda \in D(0, \frac{\nu}{4})$. From Lemma 3.9(iv), one can readily check that

$$\text{Ker } P_\alpha = U(\alpha)\text{Ker } P_0 \text{ for any } \alpha \in [0, \varepsilon_1]. \quad (3.55)$$

We infer that the linear operator $\tilde{U}(\alpha) := U(\alpha)|_{\text{Ker } P_0}$ is bounded, invertible from $\text{Ker } P_0$ to $\text{Ker } P_\alpha$, with bounded inverse $\tilde{U}(\alpha)^{-1} = U(\alpha)^{-1}|_{\text{Ker } P_\alpha}$, for any $\alpha \in [0, \varepsilon_1]$. From Lemma 3.9(i), (3.46), and (3.55), we obtain that

$$U(\alpha)(\text{dom}(B_\alpha) \cap \text{Ker } P_0) = \text{dom}(A_\alpha) \cap \text{Ker } P_\alpha = \text{dom}(A) \cap \text{Ker } P_\alpha \text{ for any } \alpha \in [0, \varepsilon_1], \quad (3.56)$$

which implies that

$$\tilde{B}_\alpha = \tilde{U}(\alpha)^{-1}\tilde{A}_\alpha\tilde{U}(\alpha) \text{ for any } \alpha \in [0, \varepsilon_1]. \quad (3.57)$$

Moreover, from (3.43) and (3.57), we have

$$\begin{aligned} \|R(\lambda, \tilde{B}_\alpha)\| &= \|\tilde{U}(\alpha)^{-1}R(\lambda, \tilde{A}_\alpha)\tilde{U}(\alpha)\| \leq \|U(\alpha)^{-1}|_{\text{Ker } P_\alpha}\| \|R(\lambda, \tilde{A}_\alpha)\| \|\tilde{U}(\alpha)\| \\ &\leq \|U(\alpha)^{-1}\| \|R(\lambda, \tilde{A}_\alpha)\| \|U(\alpha)\| \leq 3\|R(\lambda, \tilde{A}_\alpha)\| \end{aligned} \quad (3.58)$$

for any $\lambda \in \rho(\tilde{B}_\alpha) = \rho(\tilde{A}_\alpha)$ and $\alpha \in [0, \varepsilon_1]$. To estimate $\|R(\lambda, \tilde{A}_\alpha)\|$ for $\lambda \in D(0, \frac{\nu}{4})$ and $\alpha \in [0, \varepsilon_1]$, we need to find a contour integral representation of $R(\lambda, \tilde{A}_\alpha)$. We consider the function $\tilde{F} : \mathbb{C} \setminus \partial D(0, \frac{\nu}{2}) \times [0, \varepsilon_1] \rightarrow \mathcal{B}(\mathbb{X})$ defined by

$$\tilde{F}(\lambda, \alpha) := \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \frac{1}{\lambda - \zeta} R(\zeta, A_\alpha) d\zeta, \quad (3.59)$$

cf. [13, section III.6.5]. From Remark 3.7(i), we have $P_\alpha R(\zeta, A_\alpha) = R(\zeta, A_\alpha) P_\alpha$ for any $\zeta \in \partial D(0, \frac{\nu}{2})$ and $\alpha \in [0, \varepsilon_1]$, which implies that

$$P_\alpha \tilde{F}(\lambda, \alpha) = \tilde{F}(\lambda, \alpha) P_\alpha \text{ for any } \lambda \in \mathbb{C} \setminus \partial D(0, \frac{\nu}{2}), \alpha \in [0, \varepsilon_1]. \quad (3.60)$$

Since $A_\alpha R(\zeta, A_\alpha) = \zeta R(\zeta, A_\alpha) - I_\mathbb{X}$ for any $\zeta \in \partial D(0, \frac{\nu}{2})$ and A_α is a closed linear operator, we obtain that $\text{Im } \tilde{F}(\lambda, \alpha) \subseteq \text{dom}(A_\alpha)$ and

$$\begin{aligned} A_\alpha \tilde{F}(\lambda, \alpha) &= \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \frac{1}{\lambda - \zeta} (\zeta R(\zeta, A_\alpha) - I_\mathbb{X}) d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \left(\frac{\lambda}{\lambda - \zeta} - 1 \right) R(\zeta, A_\alpha) d\zeta + \frac{1}{2\pi i} \left(\int_{\partial D(0, \frac{\nu}{2})} \frac{d\zeta}{\zeta - \lambda} \right) I_\mathbb{X} \\ &= \lambda \tilde{F}(\lambda, \alpha) - P_\alpha + \frac{1}{2\pi i} \left(\int_{\partial D(0, \frac{\nu}{2})} \frac{d\zeta}{\zeta - \lambda} \right) I_\mathbb{X} \end{aligned} \quad (3.61)$$

for any $\lambda \in \mathbb{C} \setminus \partial D(0, \frac{\nu}{2})$ and $\alpha \in [0, \varepsilon_1]$. From (3.60) and (3.61), it follows that $\tilde{F}(\lambda, \alpha) \text{Ker } P_\alpha \subseteq (\text{dom}(A_\alpha) \cap \text{Ker } P_\alpha) = \text{dom}(\tilde{A}_\alpha)$ and

$$\tilde{A}_\alpha \tilde{F}(\lambda, \alpha)x = A_\alpha \tilde{F}(\lambda, \alpha)x = \lambda \tilde{F}(\lambda, \alpha)x - P_\alpha x + \frac{1}{2\pi i} \left(\int_{\partial D(0, \frac{\nu}{2})} \frac{d\zeta}{\zeta - \lambda} \right) x = \lambda \tilde{F}(\lambda, \alpha)x + x \quad (3.62)$$

for any $x \in \text{Ker } P_\alpha$, $\lambda \in D(0, \frac{\nu}{4})$, and $\alpha \in [0, \varepsilon_1]$. Since $D(0, \frac{\nu}{4}) \subset D(0, \frac{\nu}{2}) \subseteq \rho(\tilde{A}_\alpha)$ for any $\alpha \in [0, \varepsilon_1]$, by Remark 3.7(iv), we conclude that $R(\lambda, \tilde{A}_\alpha) = -\tilde{F}(\lambda, A_\alpha)|_{\text{Ker } P_\alpha}$ for any $\lambda \in D(0, \frac{\nu}{4})$ and $\alpha \in [0, \varepsilon_1]$. Since $D(0, \frac{\nu}{2}) \subset \mathcal{E}_\nu$ and $|\lambda - \zeta| \geq |\zeta| - |\lambda| \geq \frac{\nu}{4}$ for any $\lambda \in D(0, \frac{\nu}{4})$ and $\zeta \in \partial D(0, \frac{\nu}{2})$, from Lemma 3.6 and (3.59), we infer that

$$\begin{aligned} \|R(\lambda, \tilde{A}_\alpha)\| &= \|\tilde{F}(\lambda, A_\alpha)|_{\text{Ker } P_\alpha}\| \leq \|\tilde{F}(\lambda, \alpha)\| = \left\| \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \frac{1}{\lambda - \zeta} R(\zeta, A_\alpha) d\zeta \right\| \\ &\leq \frac{\text{length}\left(\partial D\left(0, \frac{\nu}{2}\right)\right)}{2\pi} \sup_{|\zeta|=\frac{\nu}{2}} \|R(\zeta, A_\alpha)\| \sup_{|\zeta|=\frac{\nu}{2}} \frac{1}{|\lambda - \zeta|} \leq \frac{32M_2}{\nu} \end{aligned} \quad (3.63)$$

for any $\lambda \in D(0, \frac{\nu}{4})$ and $\alpha \in [0, \varepsilon_1]$. Finally, from (3.58) and (3.63), we conclude that

$$\|R(\lambda, \tilde{B}_\alpha)\| \leq \frac{96M_2}{\nu} \text{ for any } \lambda \in D\left(0, \frac{\nu}{4}\right), \alpha \in [0, \varepsilon_1], \quad (3.64)$$

proving the lemma. $\square$

We are now ready to prove that the semigroup $\{\tilde{S}_\alpha(t)\}_{t \geq 0}$, generated by $\tilde{B}_\alpha$, is uniformly exponentially stable for $\alpha \in [0, \varepsilon_1]$.

Lemma 3.13. Assume Hypotheses (H1)–(H4). Then, the following estimate holds:

$$\|\tilde{S}_\alpha(t)\| \leq M_3 e^{-\frac{\nu}{2}t} \quad \text{for any } t \geq 0, \alpha \in [0, \varepsilon_1], \text{ where} \quad (3.65)$$

$$M_3 = \max \left\{ \left(\frac{3\tilde{M}}{\pi\tilde{a}} + \frac{96M_2}{\pi\nu} \left(\tilde{a} + \frac{\nu}{2} \right) \left| \tan \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right| \right), \frac{\tilde{M} \left(e(6\tilde{\theta} + 3\pi) - 12 \sec \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right)}{4\pi} e^{(\tilde{a} + \frac{\nu}{2})} \right\}, \quad (3.66)$$

and M_2 is defined in (3.26).

Proof. The proof of the lemma is similar to the proof of Lemma 3.2, the main tool is the estimate (2.16) of Lemma 2.4. We fix $\alpha \in [0, \varepsilon_1]$ and $\mu \in (-\frac{\nu}{2}, 0)$. Hence, $\mu > s(\tilde{B}_\alpha)$ by Lemma 3.12. Also, we set $\varphi = \frac{\tilde{\theta}}{2} + \frac{\pi}{4} \in (\frac{\pi}{2}, \tilde{\theta})$ and $\tilde{b} = (\tilde{a} - \mu) |\tan \varphi|$.

First, from (3.49) and (3.51), we immediately conclude that $\tilde{B}_\alpha$ is sectorial, moreover

$$\Omega_{\tilde{a}, \tilde{\theta}} \subseteq \rho(B_\alpha) \subseteq \rho(\tilde{B}_\alpha), \quad \|R(\lambda, \tilde{B}_\alpha)\| \leq \|R(\lambda, B_\alpha)\| \leq \frac{3\tilde{M}}{|\lambda - \tilde{a}|} \quad \text{for any } \lambda \in \Omega_{\tilde{a}, \tilde{\theta}}. \quad (3.67)$$

In addition, from (2.15) and Lemma 3.12(ii), we obtain

$$\|\mathcal{V}_{\tilde{B}_\alpha}(t, \mu, \varphi)\| = \left\| \int_{-\tilde{b}}^{\tilde{b}} e^{ist} R(\mu + is, \tilde{B}_\alpha) ds \right\| \leq \frac{192M_2\tilde{b}}{\nu} = \frac{192M_2(\tilde{a} - \mu) \left| \tan \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right|}{\nu} \quad (3.68)$$

for any $t > 0$. Applying Lemma 2.4, from (3.67) and (3.68), we infer that

$$\|\tilde{S}_\alpha(t)\| \leq \frac{3\tilde{M}e^{\mu t}}{\pi(\tilde{a} - \mu)t} + \frac{96M_2(\tilde{a} - \mu) \left| \tan \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right| e^{\mu t}}{\pi\nu} \leq \left(\frac{3\tilde{M}}{\pi\tilde{a}} + \frac{96M_2}{\pi\nu} \left(\tilde{a} + \frac{\nu}{2} \right) \left| \tan \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right| \right) e^{\mu t} \quad (3.69)$$

for any $t \geq 1$. From Lemma 2.3, Lemma 3.1, (3.43), and (3.52), it follows that

$$\begin{aligned} \|\tilde{S}_\alpha(t)\| &\leq \|S_\alpha(t)\| = \|U(\alpha)^{-1}T_\alpha(t)U(\alpha)\| \leq \|U(\alpha)^{-1}\| \|T_\alpha(t)\| \|U(\alpha)\| \leq 3\|T_\alpha(t)\| \\ &\leq \frac{3\tilde{M}(e\varphi - \sec \varphi)}{\pi} e^{\tilde{a}t} = \frac{\tilde{M} \left(e(6\tilde{\theta} + 3\pi) - 12 \sec \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right)}{4\pi} e^{(\tilde{a} + \frac{\nu}{2})t} e^{-t\frac{\nu}{2}} \\ &\leq \frac{\tilde{M} \left(e(6\tilde{\theta} + 3\pi) - 12 \sec \left(\frac{\tilde{\theta}}{2} + \frac{\pi}{4} \right) \right)}{4\pi} e^{(\tilde{a} + \frac{\nu}{2})t} e^{-t\frac{\nu}{2}} \quad \text{for any } t \in [0, 1]. \end{aligned} \quad (3.70)$$

Passing to the limit as $\mu \rightarrow -\frac{\nu}{2}$ in (3.69), we obtain the estimate (3.65) from (3.69) and (3.70). $\square$

Next, we focus our attention on the family of operators $K_\alpha = (B_\alpha)|_{\text{Im}P_0}$, $\alpha \in [0, \varepsilon_1]$. First, we show that it depends continuously on $\alpha \in [0, \varepsilon_1]$. Moreover, we look for a representation of K_α in a neighborhood of 0.

Lemma 3.14. Assume Hypotheses (H1)–(H4). Then, the following assertions hold true:

(i) The operator-valued function $\alpha \rightarrow K_\alpha : [0, \varepsilon_1] \rightarrow \mathcal{B}(\text{Im}P_0)$ is continuous;

(ii) $K_\alpha = \alpha (P_0 U(\alpha)^{-1} G(\alpha) U(\alpha))_{|\text{Im} P_0}$ for any $\alpha \in [0, \varepsilon_1]$, where $G : [0, \varepsilon_1] \rightarrow \mathcal{B}(\mathbb{X})$ defined by

$$G(\alpha) = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \lambda R(\lambda, A_\alpha) E_0(\alpha) R(\lambda, A) d\lambda; \quad (3.71)$$

(iii) The function G is bounded. Moreover,

$$\|G(\alpha)\| \leq 64M_2^2 \sup_{\alpha \in [0, 1]} \|E_0(\alpha)\| \quad \text{for any } \alpha \in [0, \varepsilon_1]. \quad (3.72)$$

Proof. (i) From Lemma 3.9(iv) and (3.51), we have

$$K_\alpha x = P_0 K_\alpha x = P_0 U(\alpha)^{-1} A_\alpha U(\alpha) x = P_0 U(\alpha)^{-1} A_\alpha U(\alpha) P_0 x = P_0 U(\alpha)^{-1} A_\alpha P_\alpha U(\alpha) x \quad (3.73)$$

for any $x \in \text{Im} P_0$ and $\alpha \in [0, \varepsilon_1]$. Since $A_\alpha R(\lambda, A_\alpha) = \lambda R(\lambda, A_\alpha) - I_\mathbb{X}$ for any $\lambda \in \rho(A_\alpha)$, $\text{Im} P_\alpha \subset \text{dom}(A_\alpha)$, by Remark 3.7(ii), from (3.34), we obtain that

$$\begin{aligned} A_\alpha P_\alpha &= \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} A_\alpha R(\lambda, A_\alpha) d\lambda = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} (\lambda R(\lambda, A_\alpha) - I_\mathbb{X}) d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \lambda R(\lambda, A_\alpha) d\lambda \end{aligned} \quad (3.74)$$

for any $\alpha \in [0, \varepsilon_1]$. Since the operator-valued function E is continuous on $[0, \varepsilon_1]$, by Hypothesis (H3), from Lemma 3.6, we infer that the function

$$(\lambda, \alpha) \rightarrow R(\lambda, A_\alpha) : \mathcal{E}_\nu \times [0, \varepsilon_1] \rightarrow \mathcal{B}(\mathbb{X}) \text{ is continuous.} \quad (3.75)$$

Since $\partial D(0, \frac{\nu}{2}) \subset \mathcal{E}_\nu$, by (3.27), from (3.74) and (3.75), we conclude that

$$\alpha \rightarrow A_\alpha P_\alpha : [0, \varepsilon_1] \rightarrow \mathcal{B}(\mathbb{X}) \text{ is continuous.} \quad (3.76)$$

Assertion (i) follows shortly from Lemma 3.9(i) and (iii), (3.73), and (3.76).

(ii) We recall that from Lemma 3.5(i) and (3.23), we have

$$0 = AP_0 = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} (\lambda R(\lambda, A) - I_\mathbb{X}) d\lambda = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \lambda R(\lambda, A) d\lambda. \quad (3.77)$$

From (1.4), (3.74), and (3.77), it follows that

$$\begin{aligned} A_\alpha P_\alpha &= \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \lambda R(\lambda, A_\alpha) d\lambda = \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \lambda (R(\lambda, A_\alpha) - R(\lambda, A)) d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \lambda R(\lambda, A_\alpha) E(\alpha) R(\lambda, A) d\lambda = \frac{\alpha}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \lambda R(\lambda, A_\alpha) E_0(\alpha) R(\lambda, A) d\lambda \end{aligned} \quad (3.78)$$

for any $\alpha \in [0, \varepsilon_1]$, proving (ii).

(iii) From (3.27), one can readily check that

$$\begin{aligned}
 \|G(\alpha)\| &= \left\| \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \lambda R(\lambda, A_\alpha) E_0(\alpha) R(\lambda, A) d\lambda \right\| \\
 &\leq \frac{\text{length}\left(\partial D\left(0, \frac{\nu}{2}\right)\right)}{2\pi} \sup_{|\lambda|=\frac{\nu}{2}} \|\lambda R(\lambda, A_\alpha)\| \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| \sup_{|\lambda|=\frac{\nu}{2}} \|R(\lambda, A)\|; \\
 &\leq 64M_2^2 \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| \quad \text{for any } \alpha \in [0, \varepsilon_1],
 \end{aligned} \tag{3.79}$$

proving the lemma. $\square$

The previous lemma shows that K_α is of order $\mathcal{O}(\alpha)$ in a neighborhood of 0. In addition, we note that the operator-valued function G is not necessarily continuous at 0. Therefore, to prove our main result, we assume Hypothesis (H5).

Lemma 3.15. *Assume Hypotheses (H1)–(H5). Then, the following assertions hold true:*

- (i) *The operator-valued function G defined in (3.71) is continuous in the $\mathcal{B}(\mathbb{X})$ operator norm;*
- (ii) $P_0 G(0)|_{\text{Im} P_0} = P_0 E_0(0)|_{\text{Im} P_0}$.
- (iii) $\|K_\alpha - \alpha P_0 E_0(0)|_{\text{Im} P_0}\| \leq \frac{8M_2^2 \nu}{\varepsilon_1^2} \alpha^2 + 768M_2^2 \alpha r(\alpha)$ for any $\alpha \in [0, \varepsilon_1]$.

Proof. From (1.4) and (1.5) and since E is continuous in the $\mathcal{B}(\mathbb{X})$ operator norm by Hypothesis (H3), we infer that E_0 is continuous on $[0, \infty)$ in the $\mathcal{B}(\mathbb{X})$ operator norm. Moreover, U and U^{-1} are continuous on $[0, \varepsilon_1]$, by Lemma 3.9. Assertion (i) follows shortly from (3.73) and (3.76).

(ii) First, we note that from Lemma 3.5(iii) we have $(R(\lambda, A))|_{\text{Im} P_0} = \frac{1}{\lambda} I_{\text{Im} P_0}$ for any $\lambda \in \partial D(0, \frac{\nu}{2})$. Hence, from (3.71), it follows that

$$\begin{aligned}
 P_0 G(0)|_{\text{Im} P_0} &= \frac{1}{2\pi i} P_0 \int_{\partial D(0, \frac{\nu}{2})} R(\lambda, A) E_0(0) (\lambda R(\lambda, A)|_{\text{Im} P_0}) d\lambda \\
 &= \frac{1}{2\pi i} P_0 \int_{\partial D(0, \frac{\nu}{2})} R(\lambda, A) E_0(0)|_{\text{Im} P_0} d\lambda = P_0 E_0(0)|_{\text{Im} P_0}.
 \end{aligned} \tag{3.80}$$

To prove (iii), we need a long but standard series of estimates of all the functions involved in formula (3.71). For completeness we give the details below. Since $E(0) = 0$ from (1.4), (3.27), (3.36), and (3.39), we have

$$\|R(\lambda, A_\alpha) - R(\lambda, A)\| \leq \|R(\lambda, A_\alpha)\| \|E(\alpha)\| \|R(\lambda, A)\| \leq \alpha \frac{128M_2^2}{\nu^2} \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| \leq \frac{\alpha}{2\varepsilon_1 \nu} \tag{3.81}$$

for any $\lambda \in \mathcal{E}_\nu$, $\alpha \in [0, \varepsilon_1]$. From (1.5), (3.27), and (3.81), we obtain

$$\begin{aligned}
 \|R(\lambda, A_\alpha) E_0(\alpha) - R(\lambda, A) E_0(0)\| &\leq \|(R(\lambda, A_\alpha) - R(\lambda, A)) E_0(\alpha)\| + \|R(\lambda, A) (E_0(\alpha) - E_0(0))\| \\
 &\leq \frac{\alpha}{2\varepsilon_1 \nu} \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| + \frac{16M_2}{\nu} r(\alpha)
 \end{aligned} \tag{3.82}$$

for any $\lambda \in \mathcal{E}_\nu$, $\alpha \in [0, \varepsilon_1]$. Using again (3.27), from (3.71) and (3.82), it follows that

$$\begin{aligned} \|G(\alpha) - G(0)\| &= \left\| \frac{1}{2\pi i} \int_{\partial D(0, \frac{\nu}{2})} \lambda(R(\lambda, A_\alpha)E_0(\alpha) - R(\lambda, A)E_0(0))R(\lambda, A)d\lambda \right\| \\ &\leq \frac{\text{length}\left(\partial D(0, \frac{\nu}{2})\right)}{2\pi} \sup_{|\lambda|=\frac{\nu}{2}} \|\lambda R(\lambda, A)\| \sup_{|\lambda|=\frac{\nu}{2}} \|R(\lambda, A_\alpha)E_0(\alpha) - R(\lambda, A)E_0(0)\| \\ &\leq \frac{2M_2\alpha}{\varepsilon_1} \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| + 64M_2^2r(\alpha) \quad \text{for any } \alpha \in [0, \varepsilon_1]. \end{aligned} \quad (3.83)$$

Moreover, from (3.40), (3.42), and (3.44), we have

$$\begin{aligned} \|U(\alpha) - I_\mathbb{X}\| &= \|(P_\alpha - P_0)(2P_0 - I_\mathbb{X})\| \leq \alpha \frac{128M_2^2(16M_2 + 1)}{\nu} \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| \leq \frac{\alpha}{2\varepsilon_1} \\ \|U(\alpha)^{-1} - I_\mathbb{X}\| &\leq 2\|U(\alpha) - I_\mathbb{X}\| \leq \frac{\alpha}{\varepsilon_1} \quad \text{for any } \alpha \in [0, \varepsilon_1]. \end{aligned} \quad (3.84)$$

From (3.39), (3.43), (3.79), and (3.81)–(3.84), we infer that

$$\begin{aligned} \|U(\alpha)^{-1}G(\alpha)U(\alpha) - G(0)\| &\leq \|(U(\alpha)^{-1} - I_\mathbb{X})G(\alpha)U(\alpha)\| + \|G(\alpha)U(\alpha) - G(0)\| \\ &\leq \frac{96M_2^2\alpha}{\varepsilon_1} \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| + \|(G(\alpha) - G(0))U(\alpha)\| + \|G(0)(U(\alpha) - I_\mathbb{X})\| \\ &\leq \frac{96M_2^2\alpha}{\varepsilon_1} \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| + \frac{3M_2\alpha}{\varepsilon_1} \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| + 96M_2^2r(\alpha) + \frac{32M_2^2\alpha}{\varepsilon_1} \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| \\ &= \alpha \frac{128M_2^2 + 3M_2}{\varepsilon_1} \sup_{\alpha \in [0,1]} \|E_0(\alpha)\| + 96M_2^2r(\alpha) \leq \frac{\alpha\nu}{\varepsilon_1^2} + 96M_2^2r(\alpha) \quad \text{for any } \alpha \in [0, \varepsilon_1]. \end{aligned} \quad (3.85)$$

Assertion (iii) follows from (i), (3.41), (3.80), and (3.85). $\square$

Lemma 3.16. Assume Hypotheses (H1)–(H5). Then,

$$\sup \text{Re}\sigma(P_0E_0(0)|_{\text{Im}P_0}) \leq -q_1, \quad (3.86)$$

where the constant $q_1 > 0$ was introduced in Hypothesis (H3)(ii).

Proof. We denote by $G_0 : [0, \varepsilon_1] \rightarrow \mathcal{B}(\text{Im}P_0)$ the function defined by

$$G_0(\alpha) = (P_0U(\alpha)^{-1}G(\alpha)U(\alpha))|_{\text{Im}P_0}, \quad (3.87)$$

and introduced in Lemma 3.15(i). Fix $\lambda_0 \in \sigma(P_0E_0(0)|_{\text{Im}P_0})$. From Lemma 3.15(ii), we have $\lambda_0 \in \sigma(P_0G(0)|_{\text{Im}P_0}) = \sigma(G_0(0))$. Since G is continuous on $[0, \varepsilon_1]$, we conclude that G_0 is continuous on $[0, \varepsilon_1]$. Using the semicontinuity property of the spectrum of bounded linear operators in finite-dimensional spaces, there exist two sequences $\{\alpha_n\}_{n \geq 1}$ and $\{\tilde{\lambda}_n\}_{n \geq 1}$ such that

$$\alpha_n \rightarrow 0, \quad \tilde{\lambda}_n \rightarrow \lambda_0 \quad \text{as } n \rightarrow \infty, \quad \alpha_n \in (0, \varepsilon_1), \quad \tilde{\lambda}_n \in \sigma(G_0(\alpha_n)) \quad \text{for any } n \geq 1. \quad (3.88)$$

Since $K_\alpha = \alpha G_0(\alpha)$ for any $\alpha \in [0, \varepsilon_1]$, by Lemma 3.15(i), from Hypothesis (H3)(i), (3.47), (3.51), and (3.88), it follows that

$$\alpha_n \tilde{\lambda}_n \in \sigma(K_{\alpha_n}) \subseteq \sigma(B_{\alpha_n}) = \sigma(A_{\alpha_n}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -q(\alpha_n)\} \quad \text{for any } n \geq 1. \quad (3.89)$$

From (3.88), we have there exists $n_0 \geq 1$ such that $\alpha_n \in (0, q_2)$ for any $n \geq n_0$. Since $q(\alpha) = q_1 \alpha$ for any $\alpha \in [0, q_2]$, from (3.89), we obtain

$$\operatorname{Re} \tilde{\lambda}_n \leq -q_1 \quad \text{for any } n \geq n_0. \quad (3.90)$$

Passing to the limit as $n \rightarrow \infty$ yields $\operatorname{Re} \lambda_0 \leq -q_1$, proving the lemma. $\square$

We are now ready to estimate the norm of the semigroup generated by K_α for α in a neighborhood of 0. To formulate the result, we need to point out a couple of immediate consequences of Hypothesis (H5) and Lemma 3.16.

Remark 3.17. We assumed in Hypothesis (H5) that the function $r : [0, \infty) \rightarrow [0, \infty)$ is continuous, increasing, and $r(0) = 0$. Hence, it is one-to-one and its inverse $r^{-1} : [0, \infty) \rightarrow [0, \infty)$ is continuous and increasing. Moreover, since $G_0(0) = P_0 E_0(0)|_{\operatorname{Im} P_0} \in \mathcal{B}(\operatorname{Im} P_0)$ by (3.80) and (3.87), from Lemma 3.15(iii), it follows that

$$\omega_0(G_0(0)) = \sup \operatorname{Re} \sigma(G_0(0)) \leq -q_1. \quad (3.91)$$

By the definition of the growth rate of a semigroup, one may define $M_4 : (0, 1) \rightarrow [1, \infty)$ such that

$$M_4(\kappa) = \sup_{t \geq 0} \left(e^{\frac{1+\kappa}{2} q_1 t} \|e^{t G_0(0)}\| \right) < \infty. \quad (3.92)$$

Lemma 3.18. Assume Hypotheses (H1)–(H5). Then the following estimates hold true,

(i) $\|e^{t K_\alpha}\| \leq M_4(\kappa) e^{-\kappa q(\alpha)t}$ for any $t \geq 0$, $\alpha \in [0, \varepsilon_2(\kappa)]$, $\kappa \in (0, 1)$, where $\varepsilon_2 : (0, 1) \rightarrow (0, \infty)$ is the function defined by

$$\varepsilon_2(\kappa) := \min \left\{ \varepsilon_1, q_2, \frac{(1-\kappa)q_1 \varepsilon_1^2}{32M_2 \nu}, r^{-1} \left(\frac{(1-\kappa)q_1}{3072M_2^2} \right) \right\} > 0. \quad (3.93)$$

(ii) $\|T_\alpha(t)\| \leq 3(8M_2 + 1)(M_3 + M_4(\kappa)) e^{-\kappa q(\alpha)t}$ for any $t \geq 0$, $\alpha \in [0, \varepsilon_3(\kappa)]$, $\kappa \in (0, 1)$, where $\varepsilon_3 : (0, 1) \rightarrow (0, \infty)$ is the function defined by

$$\varepsilon_3(\kappa) := \min \left\{ \varepsilon_2(\kappa), \frac{\nu}{2q_1} \right\} = \min \left\{ \varepsilon_1, q_2, \frac{\nu}{2q_1}, \frac{(1-\kappa)q_1 \varepsilon_1^2}{32M_2 \nu}, r^{-1} \left(\frac{(1-\kappa)q_1}{3072M_2^2} \right) \right\} > 0, \quad (3.94)$$

and the constant M_3 and the function M_4 are defined in (3.66) and (3.92), respectively.

Proof. (i) Fix $\kappa \in (0, 1)$. First, we note that the estimate from Lemma 3.15(ii) is equivalent to

$$\|G_0(\alpha) - G_0(0)\| \leq \frac{8M_2 \nu}{\varepsilon_1^2} \alpha + 768M_2^2 r(\alpha) \quad \text{for any } \alpha \in [0, \varepsilon_1]. \quad (3.95)$$

From Remark 3.17, (3.92), Gronwall's inequality, and since $\alpha \leq \frac{(1-\kappa)q_1 \varepsilon_1^2}{32M_2 \nu}$ and $r(\alpha) \leq \frac{(1-\kappa)q_1}{3072M_2^2}$ for any $\alpha \in [0, \varepsilon_2(\kappa)]$, we infer that

$$\|e^{t G_0(\alpha)}\| \leq M_4(\kappa) e^{(-\frac{1+\kappa}{2} q_1 + \frac{8M_2 \nu}{\varepsilon_1^2} \alpha + 768M_2^2 r(\alpha))t} \leq M_4(\kappa) e^{(-\frac{1+\kappa}{2} q_1 + \frac{1-\kappa}{2} q_1)t} = M_4(\kappa) e^{-\kappa q_1 t} \quad (3.96)$$

for any $t \geq 0, \alpha \in [0, \varepsilon_2(\kappa)]$. Since $K_\alpha = \alpha G_0(\alpha)$ by Lemma 3.14 and $q(\alpha) = q_1 \alpha$ for any $\alpha \in [0, \varepsilon_2(\kappa)]$ by Hypothesis (H3), using Lemma 3.15(i), we conclude from (3.96) that

$$\|e^{tK_\alpha}\| = \|e^{(t\alpha)G_0(\alpha)}\| \leq M_4(\kappa)e^{-\kappa q_1 \alpha t} = M_4(\kappa)e^{-\kappa q(\alpha)t} \text{ for any } t \geq 0, \alpha \in [0, \varepsilon_2(\kappa)]. \quad (3.97)$$

(ii) Fix again $\kappa \in (0, 1)$. From Lemma 3.13 and $\varepsilon_3(\kappa) \leq \min\{q_2, \frac{\nu}{2q_1}\}$, so that $q(\alpha) = q_1 \alpha$ for any $\alpha \in [0, \varepsilon_3(\kappa)]$, we have

$$\|\tilde{S}_\alpha(t)\| \leq M_3 e^{-\frac{\nu}{2}t} = M_3 e^{(-\frac{\nu}{2} + q_1 \alpha)t} e^{-q(\alpha)t} \leq M_3 e^{-q(\alpha)t} \text{ for any } t \geq 0, \alpha \in [0, \varepsilon_3(\kappa)]. \quad (3.98)$$

From (3.41), (3.52), (3.98), and (i) we conclude that

$$\begin{aligned} \|T_\alpha(t)\| &= \|U(\alpha)S_\alpha(t)U(\alpha)^{-1}\| \leq \|U(\alpha)\| \|S_\alpha(t)\| \|U(\alpha)^{-1}\| \leq 3\|S_\alpha(t)\| \\ &\leq 3(\|e^{tK_\alpha}\| \|P_0\| + \|\tilde{S}_\alpha(t)\| \|I_\infty - P_0\|) \leq 3(8M_2 + 1)(M_3 + M_4(\kappa))e^{-\kappa q(\alpha)t} \end{aligned} \quad (3.99)$$

for any $t \geq 0, \alpha \in [0, \varepsilon_3(\kappa)]$, proving the lemma. $\square$

We conclude this subsection by proving one of our main results, the uniform in α exponential stability of the family of semigroups $\{T_\alpha(t)\}_{t \geq 0, \alpha \geq 0}$.

Proof of Theorem 1.1. From Lemma 3.2 and Lemma 3.18(ii), we derive

$$\|T_\alpha(t)\| \leq \max\{\overline{M}(\kappa, \varepsilon_3(\kappa)), 3(8M_2 + 1)(M_3 + M_4(\kappa))\}e^{-\kappa q(\alpha)t} \quad (3.100)$$

for any $t \geq 0, \alpha \geq 0, \kappa \in (0, 1)$. Here, $\overline{M}(\cdot, \cdot)$, M_2 , M_3 , $M_4(\cdot)$, and $\varepsilon_3(\cdot)$ are defined in (3.9), (3.26), (3.66), (3.92), and (3.94), respectively. Tracing back the dependence of each of these quantities, we conclude that estimate (1.6) holds and that the function M depends on the unperturbed operator A , $E_0(0)$, and the functions $\|E(\cdot)\|$, q and r , and the relevant constants in Hypotheses (H3)–(H5). $\square$

3.3 | The special case when 0 is a simple eigenvalue

In this subsection, we show that in the case when 0 is a simple eigenvalue, we can prove the main estimate (3.100) without using Hypothesis (H5). Throughout this subsection, we assume Hypotheses (H1), (H3), (H4), and Hypothesis (H2'). In this case, $\dim \operatorname{Im} P_0 = 1$. This fact makes estimating $\|e^{tK_\alpha}\|$ significantly simpler. In particular, the function M_4 in Lemma 3.18 can be replaced by 1, see (3.104) below.

Lemma 3.19. *Assume Hypotheses (H1), (H2'), (H3) and (H4). Then,*

$$\|T_\alpha(t)\| \leq 3(M_3 + 1)(8M_2 + 1)e^{-q(\alpha)t} \text{ for any } t \geq 0, \alpha \in [0, \varepsilon_4], \quad (3.101)$$

where $\varepsilon_4 := \min\{\varepsilon_1, q_2, \frac{\nu}{2q_1}\}$, M_3 is taken from Lemma 3.13, ε_1 from Lemma 3.9, q_1 and q_2 from Hypothesis (H3).

Proof. Since $\dim \operatorname{Im} P_0 = 1$, there exists a function $k : [0, \varepsilon_1] \rightarrow \mathbb{C}$ such that

$$K_\alpha = k(\alpha)I_{\operatorname{Im} P_0} \text{ for any } \alpha \in [0, \varepsilon_1]. \quad (3.102)$$

Since $K_\alpha = (B_\alpha)_{|\operatorname{Im} P_0}$, from (3.47), (3.51), and (3.102), it follows that $k(\alpha) \in \sigma(B_\alpha) = \sigma(A_\alpha)$ for any $\alpha \in [0, \varepsilon_1]$. From Hypothesis (H3)(i), we infer that

$$\operatorname{Re} k(\alpha) \leq -q(\alpha) \text{ for any } \alpha \in [0, \varepsilon_1]. \quad (3.103)$$

From (3.102) and (3.103), we immediately conclude that

$$\|e^{tK_\alpha}\| = e^{\operatorname{Re} k(\alpha)t} \leq e^{-tq(\alpha)} \quad \text{for any } t \geq 0, \alpha \in [0, \varepsilon_1]. \quad (3.104)$$

Since $\varepsilon_4 \leq \min \left\{ q_2, \frac{\nu}{2q_1} \right\}$, and so $q(\alpha) = q_1\alpha$ for any $\alpha \in [0, \varepsilon_4]$, it follows from Lemma 3.13 that

$$\|\tilde{S}_\alpha(t)\| \leq M_3 e^{-\frac{\nu}{2}t} = M_3 e^{(-\frac{\nu}{2}+q_1\alpha)t} e^{-q(\alpha)t} \leq M_3 e^{-q(\alpha)t} \quad \text{for any } t \geq 0, \alpha \in [0, \varepsilon_4]. \quad (3.105)$$

From Lemma 3.13, (3.52), and (3.104), we infer

$$\begin{aligned} \|T_\alpha(t)\| &= \|U(\alpha)S_\alpha(t)U(\alpha)^{-1}\| \leq \|U(\alpha)\| \|S_\alpha(t)\| \|U(\alpha)^{-1}\| \leq 3\|S_\alpha(t)\| \\ &\leq 3(\|e^{tK_\alpha}\| \|P_0\| + \|\tilde{S}_\alpha(t)\| \|I_\infty - P_0\|) \leq 3(M_3 + 1)(8M_2 + 1)e^{-q(\alpha)t} \end{aligned} \quad (3.106)$$

for any $t \geq 0, \alpha \in [0, \varepsilon_1]$, proving the lemma. $\square$

Proof of Theorem 1.2. From Lemma 3.2 and Lemma 3.19, it follows that

$$\|T_\alpha(t)\| \leq \max\{\overline{M}(\varkappa, \varepsilon_4), 3(M_3 + 1)(8M_2 + 1)\}e^{-\varkappa q(\alpha)t} \quad \text{for any } t \geq 0, \alpha \geq 0, \varkappa \in (0, 1). \quad (3.107)$$

Here, $\overline{M}(\cdot)$, M_2 , M_3 , and ε_1 are defined in (3.9), (3.26), (3.66), and Lemma 3.19, respectively. Arguing the same way as in the proof of Theorem 1.1, we conclude that estimate (1.7) holds and that the function N depends on the unperturbed operator A and the functions $\|E(\cdot)\|$ and q and the relevant constants in Hypotheses (H3)–(H4). $\square$

4 | APPLICATIONS TO LINEAR STABILITY OF PLANAR TRAVELING WAVES IN REACTION–DIFFUSION SYSTEMS

In this section, we give an application of our results to the case of families of analytic semigroups obtained by linearizing a reaction–diffusion system along a planar traveling wave (front). In particular, we give sufficient conditions for Lyapunov linear stability of such fronts, proving Proposition 1.3. First, we recall the reaction–diffusion system (1.8), which reads as follows:

$$u_t = D\Delta_x u + F(u), \quad t \geq 0, \quad x = (x_1, \dots, x_m)^T \in \mathbb{R}^m.$$

Here, $F : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is a function of class at least $\mathcal{C}^3$ and $D \in \mathbb{C}^{k \times k}$ is a matrix satisfying the condition

$$\inf \operatorname{Re} \sigma(D) > 0. \quad (4.1)$$

We recall that a planar traveling wave of (1.8) is a solution of (1.8) of the form $u(x, t) = \bar{h}(x_1 - ct)$, where $c \in \mathbb{R}$ and $\bar{h} : \mathbb{R} \rightarrow \mathbb{R}^k$ is a smooth function exponentially convergent at $\pm\infty$ to the limit values $\bar{h}_\pm$. One can readily check that the profile $\bar{h}$ satisfies the nonlinear system of equations

$$D\bar{h}'' + c\bar{h}' + F(\bar{h}) = 0. \quad (4.2)$$

Making the change of variables $y = x - ct\mathbf{e}_1$, where $\mathbf{e}_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^m$, we notice that Equation (1.8) is equivalent to

$$u_t = D\Delta_y u + c\partial_{y_1} u + F(u), \quad t \geq 0, \quad y = (y_1, \dots, y_m) \in \mathbb{R}^m. \quad (4.3)$$

We note that $\bar{h}$ is a standing wave solution of (4.3) depending only on y_1 . The linearization of (4.3) along $\bar{h}$ reads as follows,

$$u_t = \mathcal{L}u, \quad t \geq 0, \quad \text{where} \quad \mathcal{L} = D\Delta_y + cI_k \partial_{y_1} + \mathcal{M}_{F'(\bar{h})}. \quad (4.4)$$

We recall that $\mathcal{M}_{F'(\bar{h})}$ denotes the operator of multiplication on $L^2(\mathbb{R}^m, \mathbb{C}^k)$ by the bounded, matrix-valued function $F'(\bar{h}(y_1))$, while $\mathcal{L}$ is considered as a closed, densely defined linear operator on $L^2(\mathbb{R}^m, \mathbb{C}^k)$ with domain $H^2(\mathbb{R}^m, \mathbb{C}^k)$. Moreover, as mentioned in the introduction, by taking Fourier transform in the variables $(y_2, \dots, y_m) \in \mathbb{R}^{m-1}$, we infer that the linear operator $\mathcal{L}$ is unitary equivalent to $\mathcal{M}_{\hat{L}}$, the operator of multiplication acting on $L^2(\mathbb{R}^{m-1}, L^2(\mathbb{R}, \mathbb{C}^k))$ by the operator-valued function

$$\hat{L} : \mathbb{R}^{m-1} \rightarrow \mathcal{B}(H^2(\mathbb{R}, \mathbb{C}^k), L^2(\mathbb{R}, \mathbb{C}^k)), \quad \hat{L}(\xi) = D\partial_{y_1}^2 + cI_k \partial_{y_1} + \mathcal{M}_{V(\cdot, \xi)}, \quad (4.5)$$

where $V : \mathbb{R}^m \rightarrow \mathbb{R}^k$ is defined by

$$V(y_1, \xi) = F'(\bar{h}(y_1)) - |\xi|^2 D. \quad (4.6)$$

For any $\xi \in \mathbb{R}^{m-1}$ the linear operator $\hat{L}(\xi)$ can be considered as a closed, densely defined linear operator on $L^2(\mathbb{R}, \mathbb{C}^k)$ with domain $H^2(\mathbb{R}, \mathbb{C}^k)$.

It is well known that elliptic operators generate analytic semigroups, see, for example, [1, 2, 11, 15, 17, 21, 22]. In the next lemma, we show that condition (4.1) is enough to infer the analyticity of the semigroup generated by $\hat{L}(0)$.

Lemma 4.1. *If $\inf \operatorname{Re} \sigma(D) > 0$, then the linear operator $\hat{L}(\xi)$, defined in (4.5), is sectorial, for any $\xi \in \mathbb{R}^{m-1}$, hence it generates an analytic semigroup. In particular, $\hat{L}(0)$ satisfies Hypothesis (H1).*

Proof. Let $d_0 = \inf \operatorname{Re} \sigma(D) > 0$. First, we show that the linear operator $D\partial_{y_1}^2$ is sectorial. Denoting by $\mathcal{F}_1$ the Fourier transform with respect to the variable $y_1 \in \mathbb{R}$, one can readily check that

$$D\partial_{y_1}^2 = \mathcal{F}_1^{-1} M_{\hat{D}} \mathcal{F}_1, \quad \text{where } \hat{D} : \mathbb{R} \rightarrow \mathbb{C}^{k \times k} \text{ is defined by } \hat{D}(\xi_1) = -\xi_1^2 D. \quad (4.7)$$

Since $D \in \mathbb{C}^{k \times k}$, we have

$$\omega_0(-D) = \sup \operatorname{Re} \sigma(-D) = -\inf \operatorname{Re} \sigma(D) = -d_0 < 0. \quad (4.8)$$

From the definition of the growth rate of a semigroup, we obtain that there exists $C_0 > 0$ such that

$$\|e^{-tD}\| \leq C_0 e^{-\frac{d_0}{2}t} \quad \text{for any } t \geq 0. \quad (4.9)$$

Since $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -\frac{d_0}{2}\} \subset \rho(-D)$, we have

$$\lambda(\lambda I_k + D)^{-1} = I_k - D(\lambda I_k + D)^{-1} = I_k - D \int_0^\infty e^{-\lambda t} e^{-tD} dt \quad \text{whenever } \operatorname{Re} \lambda > -\frac{d_0}{2}. \quad (4.10)$$

Therefore, from (4.9) and (4.10), we immediately obtain

$$\sup_{\operatorname{Re} \lambda \geq 0} \|\lambda(\lambda I_k + D)^{-1}\| \leq 1 + \frac{2\|D\|C_0}{d_0} < \infty. \quad (4.11)$$

From Lemma 2.8, it follows that there exists $\tilde{C}_0 \geq 1$ and $\varphi_0 \in (\frac{\pi}{2}, \pi)$ such that

$$\Omega_{0, \varphi_0} \subset \rho(-D), \quad \|(\lambda I_k + D)^{-1}\| \leq \frac{\tilde{C}_0}{|\lambda|} \quad \text{for any } \lambda \in \Omega_{0, \varphi_0}. \quad (4.12)$$

Since $\frac{\lambda}{\xi_1^2} \in \Omega_{0,\varphi_0}$ for any $\lambda \in \Omega_{0,\varphi_0}$ and $\xi_1 \in \mathbb{R} \setminus \{0\}$, from (4.7) and (4.12), we conclude that

$$\Omega_{0,\varphi_0} \subset \rho(\widehat{D}(\xi_1)), \quad \|(\lambda I_k - \widehat{D}(\xi_1))^{-1}\| \leq \frac{\widetilde{C}_0}{|\lambda|} \quad \text{for any } \lambda \in \Omega_{0,\varphi_0}, \xi_1 \in \mathbb{R}, \quad (4.13)$$

which is equivalent to

$$\Omega_{0,\varphi_0} \subset \rho(M_{\widehat{D}}), \quad \|R(\lambda, M_{\widehat{D}})\| \leq \frac{\widetilde{C}_0}{|\lambda|} \quad \text{for any } \lambda \in \Omega_{0,\varphi_0}. \quad (4.14)$$

From (4.14), we infer that the linear operator $M_{\widehat{D}}$, and hence $D\partial_{y_1}^2$, is a sectorial operator. Since $D \in \mathbb{C}^{k \times k}$ is an invertible matrix, one can readily check that

$$\|\partial_{y_1} u\|_2^2 = -\langle \partial_{y_1}^2 u, u \rangle_{L^2(\mathbb{R}, \mathbb{C}^k)} = -\langle D\partial_{y_1}^2 u, (D^{-1})^* u \rangle_{L^2(\mathbb{R}, \mathbb{C}^k)} \leq \|D^{-1}\| \|D\partial_{y_1}^2 u\|_2 \|u\|_2 \quad (4.15)$$

for any $u \in H^2(\mathbb{R}, \mathbb{C}^k)$, which implies that

$$\|\partial_{y_1} u\|_2 \leq \|D^{-1}\|^{1/2} \|D\partial_{y_1}^2 u\|_2^{1/2} \|u\|_2^{1/2} \leq \|D^{-1}\|^{1/2} \|u\|_{\text{dom}(D\partial_{y_1}^2)}^{1/2} \|u\|_2^{1/2} \quad (4.16)$$

for any $u \in \text{dom}(D\partial_{y_1}^2) = H^2(\mathbb{R}, \mathbb{C}^k)$. Applying the results from [15, Chapter 2], we infer that $D\partial_{y_1}^2 + cI_k\partial_{y_1}$ is also a sectorial operator. Since $\bar{h}$, and thus V , are bounded functions, from Lemma 2.9 and (4.5), we conclude that $\widehat{L}(\xi)$ is a sectorial operator for any $\xi \in \mathbb{R}^{m-1}$, proving the lemma. $\square$

Next, we note that the operator-valued function $\widehat{L}$ has the representation

$$\widehat{L}(\xi) = \widehat{L}(0) + E_{\text{RD}}(|\xi|^2) \quad \text{for any } \xi \in \mathbb{R}^{m-1}, \quad (4.17)$$

where $E_{\text{RD}} : [0, \infty) \rightarrow \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^k))$, is defined by $E_{\text{RD}}(\alpha) = -\alpha \mathcal{M}_D$.

Remark 4.2. Assuming Hypothesis (RD) from the Introduction and (4.1), from Lemma 4.1 we derive that $\widehat{L}(0)$ satisfies Hypotheses (H1)–(H2). Hence, it follows from Lemma 3.5 that the semigroup generated by $\widehat{L}(0)$ has a block representation of the form (3.22) and satisfies the estimate (3.21). We infer that the semigroup generated by $\widehat{L}(0)$ is bounded, that is, $\widetilde{M}_0 = \sup_{t \geq 0} \|e^{t\widehat{L}(0)}\| < \infty$, which implies that

$$\|R(\lambda, \widehat{L}(0))\| \leq \frac{\widetilde{M}_0}{\text{Re } \lambda} \quad \text{whenever } \text{Re } \lambda > 0. \quad (4.18)$$

In many applications, the diffusion rates of various components of the vector-valued function u in (1.8) are close to each other. In this case, we can prove the Lyapunov linear stability of the front $\bar{h}$ using the results of Section 3. First, we prove the following lemma.

Lemma 4.3. Assume Hypothesis (RD) and that the matrix $D \in \mathbb{C}^{k \times k}$ is sufficiently close to a diagonal matrix in the sense that there exists some $d > 0$ such that for $\widetilde{M}_0$ from (4.18), one has

$$\|D - dI_k\| < \frac{d}{\widetilde{M}_0}. \quad (4.19)$$

Then, the family of operators $A_{\text{RD}}(\alpha) := \widehat{L}(0) - \alpha \mathcal{M}_D$, $\alpha \geq 0$, satisfies Hypotheses (H1)–(H5).

Proof. Denoting by $J = D - dI_k \in \mathbb{C}^{k \times k}$, we have $\|J\| < \frac{d}{\tilde{M}_0}$. One can readily check that $\sigma(J) \subset D(0, \frac{d}{\tilde{M}_0})$. Since $\tilde{M}_0 = \sup_{t \geq 0} \|e^{t\hat{L}(0)}\| \geq 1$, we have $\inf \operatorname{Re} \sigma(J) > -d$, which implies that $\inf \operatorname{Re} \sigma(D) > 0$. From Remark 4.2, we conclude that the linear operator $\hat{L}(0)$ satisfies Hypotheses (H1) and (H2).

Next, we study the spectrum of $A_{\text{RD}}(\alpha)$. Since $\frac{\|J\|\tilde{M}_0}{d} < 1$, there exists $\delta_0 \in (0, 1)$ such that $\frac{\|J\|\tilde{M}_0}{d} < (1 - \delta_0)^2$. Next, we fix $\alpha \geq 0$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > -\delta_0 \alpha d$. Then, from Hypothesis (RD) we note that $\lambda + \alpha d \in \rho(\hat{L}(0))$. Moreover, since $D = dI_k + J$, we have

$$\begin{aligned} \lambda I_{L^2(\mathbb{R}, \mathbb{C}^k)} - A_{\text{RD}}(\alpha) &= (\lambda + \alpha d) I_{L^2(\mathbb{R}, \mathbb{C}^k)} - \hat{L}(0) + \alpha \mathcal{M}_J \\ &= (I_{L^2(\mathbb{R}, \mathbb{C}^k)} + \alpha \mathcal{M}_J R(\lambda + \alpha d, \hat{L}(0))) ((\lambda + \alpha d) I_{L^2(\mathbb{R}, \mathbb{C}^k)} - \hat{L}(0)). \end{aligned} \quad (4.20)$$

From (4.18), we obtain

$$\|\alpha \mathcal{M}_J R(\lambda + \alpha d, \hat{L}(0))\| \leq \frac{\alpha \|J\| \tilde{M}_0}{\operatorname{Re} \lambda + \alpha d} \leq \frac{\|J\| \tilde{M}_0}{(1 - \delta_0)d} \leq 1 - \delta_0 < 1, \quad (4.21)$$

which implies that $I_{L^2(\mathbb{R}, \mathbb{C}^k)} + \alpha \mathcal{M}_J R(\lambda + \alpha d, \hat{L}(0))$ is invertible and

$$\left\| (I_{L^2(\mathbb{R}, \mathbb{C}^k)} + \alpha \mathcal{M}_J R(\lambda + \alpha d, \hat{L}(0)))^{-1} \right\| \leq \frac{1}{\delta_0}. \quad (4.22)$$

From (4.20) and (4.22), we infer that $\lambda \in \rho(A_{\text{RD}}(\alpha))$ and

$$\begin{aligned} \|R(\lambda, A_{\text{RD}}(\alpha))\| &\leq \|R(\lambda + \alpha d, \hat{L}(0))\| \left\| (I_{L^2(\mathbb{R}, \mathbb{C}^k)} + \alpha \mathcal{M}_J R(\lambda + \alpha d, \hat{L}(0)))^{-1} \right\| \\ &\leq \frac{\tilde{M}_0}{\delta_0(\operatorname{Re} \lambda + \alpha d)} \leq \frac{\tilde{M}_0}{\delta_0(\operatorname{Re} \lambda + \delta_0 \alpha d)}. \end{aligned} \quad (4.23)$$

We conclude that

$$\sigma(A_{\text{RD}}(\alpha)) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -q(\alpha)\} \text{ and } \|R(\lambda, A_{\text{RD}}(\alpha))\| \leq \frac{\tilde{M}_0}{\delta_0(\operatorname{Re} \lambda + q(\alpha))} \quad (4.24)$$

whenever $\operatorname{Re} \lambda > -q(\alpha)$, where the function $q : [0, \infty) \rightarrow [0, \infty)$ is defined by $q(\alpha) = \delta_0 \alpha d$. Since $\lim_{\alpha \rightarrow \infty} \frac{\|E_{\text{RD}}(\alpha)\|}{q(\alpha)} = \frac{\|D\|}{\delta_0 d}$, Hypothesis (H3) is satisfied. From (4.17), one can readily check that Hypotheses (H4) and (H5) are satisfied. $\square$

We are now ready to prove the main result of this section, Proposition 1.3, assuming that $D \in \mathbb{C}^{k \times k}$ is sufficiently close to dI_k in the sense that $\|D - dI_k\| < \frac{d}{\tilde{M}_0}$

Proof of Proposition 1.3. First, we recall that $\tilde{M}_0 = \sup_{t \geq 0} \|e^{t\hat{L}(0)}\|$. From Lemma 4.3 and Theorem 1.1, it follows that the family of semigroups generated by $A_{\text{RD}}(\alpha)$, $\alpha \geq 0$, is uniformly stable. Setting $\varkappa = \frac{1}{2}$ in Theorem 1.1, we obtain that there exists a constant $\hat{M}_0 > 0$ such that

$$\|e^{tA_{\text{RD}}(\alpha)}\| \leq \hat{M}_0 e^{-\frac{\delta_0 \alpha d}{2} t} \text{ for any } t \geq 0, \alpha \geq 0. \quad (4.25)$$

Using the identity $\hat{L}(\xi) = A_{\text{RD}}(|\xi|^2)$ for any $\xi \in \mathbb{R}^{m-1}$, from (4.25), we infer that

$$\|e^{t\hat{L}(\xi)}\| \leq \hat{M}_0 e^{-\frac{\delta_0 |\xi|^2 d}{2} t} \text{ for any } t \geq 0, \xi \in \mathbb{R}^{m-1}. \quad (4.26)$$

Since $\mathcal{L}$, the linearization along the front $\bar{h}$, is unitary equivalent to $\mathcal{M}_{\hat{L}}$, from (4.26) we see that the semigroup generated by $\mathcal{L}$ is bounded, proving that the planar front $\bar{h}$ is Lyapunov linearly stable. $\square$

5 | APPLICATIONS TO LINEAR STABILITY OF PLANAR FRONTS IN THE BIDOMAIN EQUATION

In this section, we give yet another application of our results to the Lyapunov linear stability of planar traveling waves (fronts) in the bidomain Allen–Cahn model (1.9). In [16], the authors show that the model has *spectrally stable* planar traveling waves (fronts). Our goal is to show that the same condition also guarantees the Lyapunov linear stability of such fronts. Following [16], we consider Equation (1.9) on $\mathbb{R}^2$. It is well known that (1.9) has planar traveling waves, that is, solutions of the form

$$(u, u_i, u_e)(x, t) = (\bar{w}, \bar{w}_i, \bar{w}_e)(x_1 \cos \gamma + x_2 \sin \gamma - ct), \quad x = (x_1, x_2) \in \mathbb{R}^2, t \geq 0, \quad (5.1)$$

for some $c, \gamma \in \mathbb{R}$. Moreover, the profile $\bar{w}$ can be chosen such that

$$\bar{w} \text{ is decreasing, } \lim_{s \rightarrow -\infty} e^{-\rho_0 s} (\bar{w}(s) - 1) = \lim_{s \rightarrow \infty} e^{\rho_0 s} \bar{w}(s) = 0, \text{ for some } \rho_0 > 0. \quad (5.2)$$

Making in (1.9) the change of variables

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := R_{-\gamma} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} ct \\ 0 \end{pmatrix}, \quad \text{where } R_\gamma = \begin{bmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{bmatrix}, \quad (5.3)$$

we obtain the system

$$\begin{cases} u_t = \nabla_y \cdot (A_{i,\gamma} \nabla_y u_i) + c \partial_{y_1} u + f(u), \\ \nabla_y \cdot (A_{i,\gamma} \nabla_y u_i + A_{e,\gamma} \nabla_y u_e) = 0, \\ u = u_i - u_e. \end{cases} \quad t \geq 0, y \in \mathbb{R}^2. \quad (5.4)$$

Here, the symmetric, positive definite matrices $A_{i,\gamma}$ and $A_{e,\gamma}$ are defined by

$$A_{i,\gamma} = R_\gamma A_i R_{-\gamma}, \quad A_{e,\gamma} = R_\gamma A_e R_{-\gamma}. \quad (5.5)$$

In the new coordinate system, the front $(\bar{w}, \bar{w}_i, \bar{w}_e)$ travels along the y_1 -axis and it is a standing wave solution of (5.4) depending only on y_1 . In addition, the second and third equations of (5.4) are linear equations. Therefore, the linearization of (5.4) along the front $(\bar{w}, \bar{w}_i, \bar{w}_e)$ is given by

$$\begin{cases} v_t = \nabla_y \cdot (A_{i,\gamma} \nabla_y v_i) + c \partial_{y_1} v + f'(\bar{w}(y_1))v, \\ \nabla_y \cdot (A_{i,\gamma} \nabla_y v_i + A_{e,\gamma} \nabla_y v_e) = 0, \\ v = v_i - v_e. \end{cases} \quad t \geq 0, y \in \mathbb{R}^2 \quad (5.6)$$

Following [16], we consider this system in $L^2(\mathbb{R}^2, \mathbb{C}^3)$. Taking Fourier transform in $y = (y_1, y_2)$, denoted by $\mathcal{F}$, we can eliminate the variables v_i and v_e from (5.6) to obtain the following equation:

$$v_t = \mathcal{A}v, \quad t \geq 0, \quad \text{where } \mathcal{A} = -\mathcal{L}_\gamma + c \partial_{y_1} + \mathcal{M}_{f'(\bar{w})}. \quad (5.7)$$

The linear operator $\mathcal{L}_\gamma : H^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ is defined as the Fourier multiplier

$$\mathcal{L}_\gamma = \mathcal{F}^{-1} \mathcal{M}_{Q_\gamma} \mathcal{F}, \quad Q_\gamma(\xi) = \frac{Q_{i,\gamma}(\xi) Q_{e,\gamma}(\xi)}{Q_{i,\gamma}(\xi) + Q_{e,\gamma}(\xi)}, \quad Q_{i/e,\gamma}(\xi) = \xi^T \cdot A_{i/e,\gamma} \xi, \quad \xi \in \mathbb{R}^2. \quad (5.8)$$

We recall that the wave $\bar{w}$ is called Lyapunov linearly stable if the semigroup generated by $\mathcal{A}$ is bounded.

Taking Fourier transform with respect to $y_2 \in \mathbb{R}$, denoted by $\mathcal{F}_2$, we note that the linear operator $\mathcal{A}$ is unitary equivalent to $\mathcal{M}_{\hat{A}}$, the operator of multiplication on $L^2(\mathbb{R}, L^2(\mathbb{R}))$ by the operator-valued function

$$\hat{A} : \mathbb{R} \rightarrow \mathcal{B}(H^2(\mathbb{R}), L^2(\mathbb{R})), \quad \hat{A}(\xi_2) = -\mathcal{F}_1^{-1} \mathcal{M}_{Q_\gamma(\cdot, \xi_2)} \mathcal{F}_1 + c \partial_{y_1} + \mathcal{M}_{f'(\bar{w})}. \quad (5.9)$$

Here, $\mathcal{F}_1$ denotes the Fourier transform with respect to the variable $y_1 \in \mathbb{R}$. For more details, we refer to [16, section 2].

To study the Lyapunov linear stability of the front $\bar{w}$, we study the family of semigroups generated by $\hat{A}(\xi_2)$, $\xi_2 \in \mathbb{R}$. We recall that the function Q_γ has the representation:

$$Q_\gamma(\xi_1, \xi_2) = \begin{cases} \xi_2^2 \left(p\left(\frac{\xi_1}{\xi_2}\right) + g\left(\frac{\xi_1}{\xi_2}\right) \right), & \xi_1 \in \mathbb{R}, \xi_2 \in \mathbb{R} \setminus \{0\}, \\ N_0^2 \xi_1^2, & \xi_1 \in \mathbb{R}, \xi_2 = 0, \end{cases} \quad \text{where}$$

$$p(s) = N_0^2(s - \eta_1)^2 + \eta_0, \quad g(s) = \frac{\beta_1 s + \beta_0}{s^2 + 1}$$

as pointed out in (1.11) and (1.12). Also, we recall that the constants $N_0, \beta_0, \beta_1, \eta_0$, and η_1 depend on ν_1, ν_2 , and γ only. For the exact formulas, we refer to [16, Formula (2.20)]. From (1.11), one can readily check that

$$\hat{A}(0) = N_0^2 \partial_{y_1}^2 + c \partial_{y_1} + \mathcal{M}_{f'(\bar{w})}. \quad (5.10)$$

The linear operator $\hat{A}(0)$ is a second-order differential operator, therefore we can determine the properties of its spectrum, see, for instance, [12, 19, 20, 23]. In the next lemma, we summarize the most relevant properties of the spectrum of $\hat{A}(0)$ given in [16].

Lemma 5.1 [16, Proposition 3.1]. *The operator $\hat{A}(0)$ satisfies the following properties:*

- (i) *There exists $\nu > 0$ such that $\sup \operatorname{Re} \left(\sigma(\hat{A}(0)) \setminus \{0\} \right) \leq -\nu$;*
- (ii) *0 is a simple eigenvalue of $\hat{A}(0)$. Moreover, $\operatorname{Ker} \hat{A}(0) = \operatorname{Span}\{\bar{w}'\}$;*
- (iii) *There exists $M_b > 0$ such that*

$$\|R(\lambda, \hat{A}(0))\| \leq \frac{M_b}{|\lambda|} \text{ whenever } \operatorname{Re} \lambda \geq 0, \lambda \neq 0.$$

Remark 5.2. From Lemma 2.8 and Lemma 5.1(ii) and (iii), we infer that $\hat{A}(0)$ is a sectorial operator, hence $\hat{A}(0)$ satisfies Hypothesis (H1). Moreover, from Lemma 5.1(i) and (ii), it follows that $\hat{A}(0)$ satisfies Hypothesis (H2').

We recall the following notation needed to formulate (1.14), a sufficient condition for spectral stability of the planar front $\bar{w}$,

$$g_{\inf} = \inf_{s \in \mathbb{R}} g(s), \quad g_{\sup} = \sup_{s \in \mathbb{R}} g(s), \quad \bar{g} = \frac{g_{\inf} + g_{\sup}}{2}, \quad g_\Delta = \frac{g_{\sup} - g_{\inf}}{2}. \quad (5.11)$$

A crucial ingredient in the proof of (1.14) is the identity

$$\hat{A}(\xi_2) = \mathcal{M}_{e^{i\eta_1 \xi_2}} (\hat{A}(0) + H(\xi_2)) \mathcal{M}_{e^{-i\eta_1 \xi_2}}, \quad \text{for any } \xi_2 \in \mathbb{R}, \quad (5.12)$$

where $H : \mathbb{R} \rightarrow \mathcal{B}(L^2(\mathbb{R}))$ is defined by

$$H(\xi_2) = \begin{cases} -\xi_2^2 \mathcal{F}_1^{-1} \mathcal{M}_{g(\frac{\cdot}{\xi_2} + \eta_1)} \mathcal{F}_1 - (\eta_0 \xi_2^2 - c i \eta_1 \xi_2) I_{L^2(\mathbb{R})}, & \xi_2 \in \mathbb{R} \setminus \{0\}, \\ 0, & \xi_2 = 0. \end{cases} \quad (5.13)$$

Remark 5.3. From [16, Lemma 4.1], we see that the operator-valued function H is continuous on $\mathbb{R}$ in $\mathcal{B}(L^2(\mathbb{R}))$ -norm. Moreover, assuming that $\eta_0 > M_b g_\Delta - \bar{g}$ and using the same argument as in [16, Theorem 3.2], we can prove that

$$\sigma(\hat{A}(0) + H(\xi_2)) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -(\eta_0 - M_b g_\Delta + \bar{g})\xi_2^2\} \text{ for any } \xi_2 \in \mathbb{R}; \quad (5.14)$$

$$\|R(\hat{A}(0) + H(\xi_2))\| \leq \frac{M_b}{\operatorname{Re} \lambda + (\eta_0 - M_b g_\Delta + \bar{g})\xi_2^2} \text{ whenever } \operatorname{Re} \lambda > -(\eta_0 - M_b g_\Delta + \bar{g})\xi_2^2. \quad (5.15)$$

To check the two properties above, we first define $\tilde{H} : \mathbb{R} \rightarrow \mathcal{B}(L^2(\mathbb{R}))$ by

$$\tilde{H}(\xi_2) := H(\xi_2) + ((\eta_0 + \bar{g})\xi_2^2 - ci\eta_1\xi_2)I_{L^2(\mathbb{R})}. \quad (5.16)$$

We note that (5.14) and (5.15) hold true if $\xi_2 = 0$. Next, we fix $\xi_2 \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > -(\eta_0 - M_b g_\Delta + \bar{g})\xi_2^2$ and we set $\tilde{\lambda} := \lambda + (\eta_0 + \bar{g})\xi_2^2 - ci\eta_1\xi_2$. Using (5.11), one can readily check that

$$\operatorname{Re} \tilde{\lambda} = \operatorname{Re} \lambda + (\eta_0 + \bar{g})\xi_2^2 > M_b g_\Delta \xi_2^2 > 0 \text{ and } \|\tilde{H}(\xi_2)\| \leq \xi_2^2 \left\| \bar{g} - g\left(\frac{\cdot}{\xi_2} + \eta_1\right) \right\|_\infty \leq \xi_2^2 g_\Delta. \quad (5.17)$$

From (5.17) and Lemma 5.1, we have $\tilde{\lambda} \in \rho(\hat{A}(0))$ and $\|R(\tilde{\lambda}, \hat{A}(0))\| \leq \frac{M_b}{|\tilde{\lambda}|} \leq \frac{M_b}{\operatorname{Re} \tilde{\lambda}}$, which implies

$$\|\tilde{H}(\xi_2)R(\tilde{\lambda}, \hat{A}(0))\| \leq \|\tilde{H}(\xi_2)\| \|R(\tilde{\lambda}, \hat{A}(0))\| \leq \xi_2^2 g_\Delta \frac{M_b}{\operatorname{Re} \tilde{\lambda}} < 1. \quad (5.18)$$

Using elementary spectral theory, from (5.18) we obtain that $\tilde{\lambda} \in \rho(\hat{A}(0) + \tilde{H}(\xi_2))$, hence $\lambda \in \rho(\hat{A}(0) + H(\xi_2))$. Moreover,

$$\begin{aligned} \|R(\lambda, \hat{A}(0) + H(\xi_2))\| &= \|R(\tilde{\lambda}, \hat{A}(0) + \tilde{H}(\xi_2))\| \leq \frac{\|R(\tilde{\lambda}, \hat{A}(0))\|}{1 - \|\tilde{H}(\xi_2)R(\tilde{\lambda}, \hat{A}(0))\|} \\ &\leq \frac{M_b}{\operatorname{Re} \tilde{\lambda} - M_b g_\Delta \xi_2^2} = \frac{M_b}{\operatorname{Re} \lambda + (\eta_0 - M_b g_\Delta + \bar{g})\xi_2^2}, \end{aligned} \quad (5.19)$$

proving (5.14) and (5.15).

To prove uniform estimates for the family of semigroups generated by $\hat{A}(\xi_2)$, $\xi_2 \in \mathbb{R}$, we introduce the operator-valued functions $E_\pm : [0, \infty) \rightarrow \mathcal{B}(L^2(\mathbb{R}))$ defined by

$$E_\pm(\alpha) := H(\pm\sqrt{\alpha}) \mp ci\eta_1\sqrt{\alpha}I_{L^2(\mathbb{R})}. \quad (5.20)$$

Lemma 5.4. Assume that $\eta_0 > M_b g_\Delta - \bar{g}$. Then, the family of operators $\tilde{A}_\pm(\alpha) := \hat{A}(0) + E_\pm(\alpha)$, $\alpha \geq 0$, satisfies Hypotheses (H3) and (H4).

Proof. From Remark 5.3 and (5.20), we immediately conclude that the operator-valued function $E_\pm$ is continuous on $[0, \infty)$ in $\mathcal{B}(L^2(\mathbb{R}))$ -norm. It follows from (5.20) that

$$\begin{aligned} \sigma(\hat{A}(0) + E_\pm(\alpha)) &= \sigma(\hat{A}(0) + H(\pm\sqrt{\alpha}) \mp ci\eta_1\sqrt{\alpha}) \\ R(\lambda, \hat{A}(0) + E_\pm(\alpha)) &= R(\lambda \pm i\eta_1\sqrt{\alpha}, \hat{A}(0) + H(\pm\sqrt{\alpha})) \end{aligned} \quad (5.21)$$

for any $\alpha \geq 0$ and any $\lambda \in \rho(\hat{A}(0) + E_\pm(\alpha))$. From (5.14), (5.15), and (5.21), we infer that

$$\sigma(\hat{A}(0) + E_\pm(\alpha)) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -(\eta_0 - M_b g_\Delta + \bar{g})\alpha\} \text{ for any } \alpha \geq 0; \quad (5.22)$$

$$\|R(\hat{A}(0) + E_\pm(\alpha))\| \leq \frac{M_b}{\operatorname{Re} \lambda + (\eta_0 - M_b g_\Delta + \bar{g})\alpha} \text{ whenever } \operatorname{Re} \lambda > -(\eta_0 - M_b g_\Delta + \bar{g})\alpha. \quad (5.23)$$

Let $q : [0, \infty) \rightarrow [0, \infty)$ be the function defined by $q(\alpha) = (\eta_0 - M_b g_\Delta + \bar{g})\alpha$. From (5.22) and (5.23), we see that conditions (i) and (ii) of Hypothesis (H3) are satisfied. Since $g \in L^\infty(\mathbb{R})$, by (1.12), from (5.13) and (5.20), we obtain

$$\|E_\pm(\alpha)\| \leq \alpha(\|g\|_\infty + \eta_0) \text{ for any } \alpha \geq 0, \quad (5.24)$$

which implies that $\limsup_{\alpha \rightarrow \infty} \frac{\|E_\pm(\alpha)\|}{q(\alpha)} \leq \|g\|_\infty + \eta_0 < \infty$. Hence, condition (iii) of Hypothesis (H3) is satisfied. Finally, it follows from (5.24) that Hypothesis (H4) is satisfied, proving the lemma. $\square$

We are now ready to prove the main result of this section, the Lyapunov linear stability of the planar front $\bar{w}$.

Proof of Proposition 1.4. From Remark 5.2 and Lemma 5.4, the family of operators $\tilde{A}_\pm(\alpha) = \hat{A}(0) + E_\pm(\alpha)$, $\alpha \geq 0$, satisfies Hypotheses (H1), (H2'), (H3), and (H4). Hence, from Theorem 1.2 we conclude that the family of analytic semigroups generated by $\tilde{A}_\pm(\alpha)$ is uniformly exponentially stable for $\alpha \geq 0$. Setting $\varkappa = \frac{1}{2}$ in Theorem 1.2, we see that there exists a constant $\tilde{M}_b > 0$ such that

$$\|e^{t\tilde{A}_\pm(\alpha)}\| \leq \tilde{M}_b e^{-\frac{(\eta_0 - M_b g_\Delta + \bar{g})\alpha}{2}t} \text{ for any } t \geq 0, \alpha \geq 0. \quad (5.25)$$

From (5.20), we have

$$H(\xi_2) = E_\sigma(\xi_2^2) + ci\eta_1 \xi_2 I_{L^2(\mathbb{R})} \text{ for any } \xi_2 \in \mathbb{R}, \text{ where } \sigma = \begin{cases} +, & \xi_2 \geq 0, \\ -, & \xi_2 < 0. \end{cases} \quad (5.26)$$

From (5.12) and (5.26) one can readily check that

$$e^{t\hat{A}(\xi_2)} = \mathcal{M}_{e^{i\eta_1 \xi_2}} \cdot e^{t(\hat{A}(0) + H(\xi_2))} \mathcal{M}_{e^{-i\eta_1 \xi_2}} = e^{ci\eta_1 \xi_2 t} \mathcal{M}_{e^{i\eta_1 \xi_2}} \cdot e^{t\tilde{A}_\sigma(\xi_2^2)} \mathcal{M}_{e^{-i\eta_1 \xi_2}}. \quad (5.27)$$

for any $t \geq 0$ and $\xi_2 \in \mathbb{R}$. From (5.25) and (5.27), we conclude that

$$\|e^{t\hat{A}(\xi_2)}\| \leq \tilde{M}_b e^{-\frac{(\eta_0 - M_b g_\Delta + \bar{g})\xi_2^2}{2}t} \text{ for any } t \geq 0, \xi_2 \in \mathbb{R}. \quad (5.28)$$

Since $\mathcal{A}$, the linearization along the front $\bar{w}$, is unitary equivalent to $\mathcal{M}_{\hat{A}}$, from (5.28), we see that the semigroup generated by $\mathcal{A}$ is bounded, proving that the front $\bar{w}$ is Lyapunov linearly stable. $\square$

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ENDNOTE

¹In (3.21), we can replace $-\frac{7\nu}{8}$ by $-\varkappa\nu$ for any $\varkappa \in (0, 1)$, but not by $-\nu$. However, such a change would not enhance our main result, cf. Theorem 1.1, since for α close to 0 under Hypothesis (H3), the decay rate $-\frac{7\nu}{8}$ is stronger than $-\varkappa q(\alpha)$, the decay rate in the main result.

REFERENCES

- [1] S. Agmon, *On the eigenfunctions and the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.
- [2] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics, 2nd ed., vol. 96, Birkhäuser/Springer Basel AG, Basel, 2011.
- [3] W. Auzinger, *Sectorial operators and normalized numerical range*, Appl. Numer. Math. **45** (2003), no. 4, 367–388.
- [4] C. Chicone and Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Math. Surv. Monogr., vol. 70, AMS, Providence, RI, 1999.
- [5] Y. L. Daletskii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Spaces*, Amer. Math. Soc. Transl., vol. 43, Amer. Math. Soc., Providence, RI, 1974.
- [6] K. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer-Verlag, New York, 2000.
- [7] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert spaces*, Trans. Amer. Math. Soc. **236** (1978), 385–394.

- [8] V. Hardt and E. Wagenfuhrer, *Spectral properties of a multiplication operator*, Math. Nachr. **178** (1996), 135–156.
- [9] B. Helffer and J. Sjöstrand, *From resolvent bounds to semigroup bounds*, arXiv:1001.4171v1, 2010, math. FA. <https://doi.org/10.48550/arXiv.1001.4171>
- [10] B. Helffer and J. Sjöstrand, *Improving semigroup bounds with resolvent estimate*, Integral Equations Operator Theory **93** (2021), no. 3, Paper No. 36, 41 pp.
- [11] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, vol. 840, Springer, New York, 1981.
- [12] T. Kapitula and K. Promislow, *Spectral and dynamical stability of nonlinear waves*, Springer, New York, 2013.
- [13] T. Kato, *Perturbation theory for linear operators*, Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
- [14] Y. Latushkin and V. Yurov, *Stability estimates for semigroups on Banach spaces*, Discrete Contin. Dyn. Syst. **33** (2013), no. 11–12, 5203–5216.
- [15] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Progress in Nonlinear Differential Equations and their Applications, vol. 16, Birkhäuser Verlag, Basel, 1995.
- [16] Y. Mori and H. Matano, *Stability of front solutions of the bidomain equation*, Comm. Pure Appl. Math. **LXIX** (2016), 2364–2426.
- [17] A. Pazy, *Semigroups of linear operators and applications to partial differential operators*, Appl. Math. Sci., vol. 44, Springer, New York, 1983.
- [18] J. Prüss, *On the spectrum of C_0 -semigroups*, Trans. Amer. Math. Soc. **284** (1984), 847–857.
- [19] M. Reed and B. Simon, *Methods of modern mathematical physics.II: Fourier analysis, self-adjointness*, Academic Press, New York, 1975.
- [20] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*, Graduate Texts in Mathematics, vol. 265, Springer, Dordrecht, 2012.
- [21] H. B. Stewart, *Generation of analytic semigroups by strongly elliptic operators*, Trans. Amer. Math. Soc. **199** (1974), 141–162.
- [22] H. B. Stewart, *Generation of analytic semigroups by strongly elliptic operators under general boundary conditions*, Trans. Amer. Math. Soc. **259** (1980), 299–310.
- [23] A. I. Volpert, V. A. Volpert, and V. A. Volpert, *Traveling wave solutions of parabolic systems*, Translations of Mathematical Monographs, vol. 140, American Mathematical Society, Providence, RI, 1994.

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APPENDIX: SOME EXAMPLES

In this section, we give two examples to show that by perturbing an operator satisfying Hypotheses (H1) and (H2), we might have unstable spectrum, even if the space is finite dimensional and the perturbation is a self-adjoint, bounded, uniformly negative definite operator.

Example A.1. Let $J_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $Z_0 = \begin{bmatrix} 1 & b_0 \\ b_0 & \frac{1}{4} \end{bmatrix}$ with $b_0 \in \left(-\frac{1}{2}, \frac{1-\sqrt{2}}{2}\right)$. One can readily check that

$$\left\langle Z_0 \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{\mathbb{C}^2} = |u|^2 + 2b_0 \operatorname{Re}(u\bar{v}) + \frac{1}{4}|v|^2 \geq (|u| - b_0|v|)^2 + \left(\frac{1}{4} - b_0^2\right)|v|^2 \text{ for any } u, v \in \mathbb{C}. \quad (\text{A.1})$$

From (A.1), we infer that Z_0 is a symmetric, positive definite matrix. Since $b_0 \in \left(-\frac{1}{2}, \frac{1-\sqrt{2}}{2}\right)$, a simple computation shows that $\sigma(J_0 - Z_0) = \{\mu_0^*, \lambda_0^*\}$, where $\mu_0^* = -\frac{5}{8} - \sqrt{\frac{25}{64} - (1 + 4b_0 - 4b_0^2)} < 0$ and $\lambda_0^* = -\frac{5}{8} + \sqrt{\frac{25}{64} - (1 + 4b_0 - 4b_0^2)} > 0$. We obtain

$$b_1 := \frac{1}{4} \min\{\lambda_0^*, m_0\} > 0, \text{ where } m_0 = \min \left\{ \left| Z_0^{-\frac{1}{2}} \begin{pmatrix} u \\ v \end{pmatrix} \right|^2 : \left| \begin{pmatrix} u \\ v \end{pmatrix} \right| = 1 \right\}. \quad (\text{A.2})$$

It follows that $\sigma(J_0 - b_1 I_2) = \{-b_1\}$ is stable, $Y_0 := b_1 I_2 - Z_0$ is a symmetric, uniformly negative definite matrix, and $\lambda_0^* \in \sigma(J_0 - b_1 I_2 + Y_0) = \sigma(J_0 - Z_0)$. We introduce the matrices

$$A_0 = \begin{bmatrix} -b_1 & 1 & 0 \\ 0 & -b_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } W_0 = \begin{bmatrix} b_1 - 1 & -b_0 & 0 \\ -b_0 & b_1 - \frac{1}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (\text{A.3})$$

One can readily check that 0 is a simple eigenvalue of A_0 and that $\sigma(A_0) = \{0, -b_1\}$. Since any bounded operator is sectorial, we immediately infer that A_0 defined in (A.3) satisfies Hypotheses (H1) and (H2). Since $W_0 = Y_0 \oplus -1$, we have W_0 is a symmetric, uniform negative definite matrix. However, $A_0 + W_0 = (J_0 - Z_0) \oplus -1$, which shows that $\sigma(A_0 + W_0) = (J_0 - Z_0) \cup \{-1\}$, hence $\lambda_0^* \in \sigma(A_0 + W_0) \cap (0, \infty)$.

This example shows that by perturbing an operator that satisfies Hypotheses (H1) and (H2) we might generate unstable point spectrum, even if the perturbation is symmetric, negative definite. Using a similar argument, one can see that the essential spectrum can become unstable under the same type of perturbation.

Example A.2. We set $\mathbb{H} = L^2([1, 2], \mathbb{C}^2) \oplus \mathbb{C}$ and let $\tilde{A}_0, \tilde{W}_0 : \mathbb{H} \rightarrow \mathbb{H}$ be the bounded linear operators defined by

$$\tilde{A}_0 = \mathcal{M}_{J_1} \oplus 0, \quad \tilde{W}_0 = \mathcal{M}_{Y_1} \oplus -1, \quad (\text{A.4})$$

where $\mathcal{M}_{J_1}$ and $\mathcal{M}_{Y_1}$ are the multiplication operators on $L^2([1, 2], \mathbb{C}^2)$ by the matrix-valued functions $J_1, Y_1 : [1, 2] \rightarrow \mathbb{C}^{2 \times 2}$ defined by $J_1(s) = s(J_0 - b_1 I_2)$ and $Y_1(s) = sY_0 = s(b_1 I_2 - Z_0)$. Here, the matrices J_0, Z_0 , and Y_0 were introduced in Example A.1 above. We recall that $\sigma(J_0 - b_1 I_2) = \{-b_1\}$ and $\sigma(J_0 - Z_0) = \{\mu_0^*, \lambda_0^*\}$, with $\mu_0^* < 0 < \lambda_0^*$.

Since the matrix Y_0 is symmetric, uniform negative definite, we immediately infer that $\mathcal{M}_{Y_1}$ is self-adjoint, uniform negative definite on $L^2([1, 2], \mathbb{C}^2)$. Hence, $\tilde{W}_0$ is self-adjoint, uniformly negative definite on $\mathbb{H}$. Moreover, since the functions J_1 and Y_1 are continuous by [8], it follows that

$$\begin{aligned} \sigma(\mathcal{M}_{J_1}) &= \sigma_{\text{ess}}(\mathcal{M}_{J_1}) = \overline{\bigcup_{s \in [1, 2]} \sigma(J_1(s))} = [-2b_1, -b_1], \\ \sigma(\mathcal{M}_{J_1 + Y_1}) &= \sigma_{\text{ess}}(\mathcal{M}_{J_1 + Y_1}) = \overline{\bigcup_{s \in [1, 2]} s\sigma(J_0 - Z_0)} = [-2\mu_0^*, -\mu_0^*] \cup [\lambda_0^*, 2\lambda_0^*]. \end{aligned} \quad (\text{A.5})$$

Since the linear operator A_0 is bounded on $\mathbb{H}$ we have it is sectorial. Moreover, from (A.4) and (A.5), we infer that 0 is a simple eigenvalue of $\tilde{A}_0$ and $\sigma(\tilde{A}_0) \setminus \{0\} = \sigma(\mathcal{M}_{J_1}) = [-2b_1, -b_1]$, thus $\sup \operatorname{Re}(\sigma(\tilde{A}_0) \setminus \{0\}) = -b_1 < 0$. Hence, $\tilde{A}_0$ satisfies Hypotheses (H1) and (H2). Even if $\tilde{W}_0$ is self-adjoint, uniform negative definite on $\mathbb{H}$, from (A.5) we have $\sigma_{\text{ess}}(\tilde{A}_0 + \tilde{W}_0) = \sigma_{\text{ess}}(\mathcal{M}_{J_1 + Y_1}) = [-2\mu_0^*, -\mu_0^*] \cup [\lambda_0^*, 2\lambda_0^*]$, which shows that $\sigma_{\text{ess}}(\tilde{A}_0 + \tilde{W}_0) \cap (0, \infty) = [\lambda_0^*, 2\lambda_0^*]$ is nonempty.