Stabilization Under Arbitrary Tight and One Sided Control Constraints: A Variational Equations Approach

Ilya Kolmanovsky[1] and Emanuele Garone[2]

***Abstract*— Stabilization of a linear system under control constraints is approached by combining the classical variation of parameters method for solving ODEs and a straightforward construction of a feedback law for the variational system based on a quadratic Lyapunov function. Sufficient conditions for global closed-loop stability under control constraints with zero in the interior and zero on the boundary of the control set are derived, and several examples are reported. The extension of the method to nonlinear systems with control constraints is described.**

I. Introduction

Constrained/bounded feedback stabilization of linear systems has been of interest for several decades [5], [9], [10]. Exponentially unstable linear time invariant systems cannot be globally stabilized by bounded feedback but stabilizable systems with no eigenvalues in the open right half plane, such as chains of integrators, can be [9], [11]. Semi-global and global stabilization techniques for systems with control constraints have been developed [3], [8], [12] for continuous and discrete-time systems. The actual construction of the control laws can be quite involved, as exemplified by [13] for spacecraft relative motion stabilization.

In this paper, we first consider the case of linear time-invariant systems in which the system matrix can have eigenvalues on the imaginary axis that are of the same algebraic and geometric multiplicity (zero defect) and possibly other eigenvalues in the open left half plane. Under a stabilizability assumption, we present an intuitive and straightforward construction of globally stabilizing feedback laws for generating control signals not exceeding specified bounds or, under the assumption of no zero eigenvalue, even satisfying "one-sided" control constraints.

Our approach in Section II-A for systems with eigenvalues only on the imaginary axis is based on the classical variation of parameters method for solving ODEs [7] due to Euler and Lagrange. Applying the variation of parameters approach, we obtain a variational system which is time-varying. We then define a quadratic Lyapunov function candidate for this variational system, from which a feedback law follows in a very straightforward way. Finally, we appeal to the LaSalle's Theorem [6] to demonstrate closed-loop stability.

Since the variational system is time-dependent, the application of LaSalle's theorem requires special care and involves re-writing the closed-loop system as an interconnection of an autonomous generator subsystem and a nonlinear time-invariant subsystem. With the additional restriction that there are no eigenvalues at the origin, we also show that the system can be globally stabilized with the "one-sided" control constraints, i.e., when the set of allowed controls has zero on the boundary rather than in the interior. Such results on the "on-sided" global stabilization do not appear to be readily available in the bounded feedback stabilization literature, to the best of the author's knowledge.

The construction of the control laws is then extended in Section II-B to a more general case of linear time-invariant systems which can have zero defect eigenvalues on the imaginary axis and other eigenvalues in the open-left half plane. Simulation examples are reported in Section III.

Remarkably, our approach extends to bounded feedback stabilization of nonlinear systems. We discuss this extension in Section IV and illustrate it with examples in Section V, including spacecraft stabilization to L_4 Lagrange point for nonlinear Circular Restricted Three Body Problem (CR3BP) dynamics [4]. Finally, Section VI contains concluding remarks.

II. Stabilization of a linear time-invariant system

Consider a linear time-invariant system represented by the model,

$$\dot{x} = Ax + Bu, \tag{1}$$

which is to be stabilized to the origin subject to the control constraints,

$$u(t) \in \mathcal{U},\ t \geq 0. \tag{2}$$

Here $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}^{n_u}$.

A. Eigenvalues on the imaginary axis

We make the following assumptions.

Assumption 1: The eigenvalues of A are on the $j\omega$-axis (inclusive of 0) of the complex plane and the Jordan canonical form of A is diagonal.

Assumption 2: The pair (A, B) is controllable.

Assumption 3: The set $\mathcal{U}$ is closed, convex and $0 \in \operatorname{Int}\mathcal{U}$.

Following the classical variation of parameters method for solving nonhomogeneous ODEs [7] consider the representation of the solution to (1) in the form,

$$x(t) = e^{At}C(t), \tag{3}$$

[1]University of Michigan, Ann Arbor, MI 48109 USA ilya@umich.edu. [2]Université Libre de Bruxelles, Brussels 1050, Belgium Emanuele.Garone@ulb.be. The authors acknowledge support for this research by the National Science Foundation award ECCS-1931738 and the Chaire Jaumotte of the Belgian Royal Academy of Science.

where $C(t)$ is a vector of time-varying coefficients. Differentiating both sides with respect to time, it follows that

$$\dot{x} = Ae^{At}C + e^{At}\dot{C}.$$

On the other hand,

$$\dot{x} = Ax + Bu = Ae^{At}C + Bu,$$

leading to

$$e^{At}\dot{C} = Bu,$$

and

$$\dot{C} = e^{-At}Bu, \tag{4}$$

which represents the time-dependent drift-free dynamics for the coefficients.

Define a quadratic Lyapunov function candidate for the variational system as

$$V = \frac{1}{2}C^{\mathsf{T}}PC, \quad P = P^{\mathsf{T}} \succ 0. \tag{5}$$

Then the time rate of change of V along the trajectories of (4) satisfies

$$\dot{V} = C^{\mathsf{T}}Pe^{-At}Bu.$$

Define a control law,

$$u_{\mathtt{nom}}(t) = -KB^{\mathsf{T}}e^{-A^{\mathsf{T}}t}PC, \quad u(t) = \mathtt{Proj}_{\mathcal{U}}\left[u_{\mathtt{nom}}(t)\right], \tag{6}$$

where $K > 0$ is a scalar gain, $\mathtt{Proj}_{\mathcal{U}}$ denotes the minimum 2-norm projection on $\mathcal{U}$ and note substitution

$$C = e^{-At}x \tag{7}$$

permits (6) to be expressed only as a function of x and t. With (6),

$$\dot{V} = -\frac{1}{K}u_{\mathtt{nom}}^{\mathsf{T}}u(t).$$

Since $0 \in \mathcal{U}$ and $\mathcal{U}$ is convex and closed (Assumption 3), while $u(t)$ is a 2-norm projection of $u_{\mathtt{nom}}$ onto $\mathcal{U}$, the necessary conditions for optimality of $u(t)$ for minimizing $g(\zeta) = \frac{1}{2}\|u_{\mathtt{nom}}(t) - \zeta\|^2$ with respect to $\zeta \in \mathcal{U}$ imply

$$g'(u(t))(\zeta - u(t)) = -(u_{\mathtt{nom}}(t) - u(t))^{\mathsf{T}}(\zeta - u(t)) \geq 0$$

for all $\zeta \in \mathcal{U}$. Applying this for $\zeta = 0$ yields

$$(u_{\mathtt{nom}}(t) - u(t))^{\mathsf{T}}(0 - u(t)) \leq 0,$$

or

$$-u_{\mathtt{nom}}^{\mathsf{T}}(t)u(t) \leq -u^{\mathsf{T}}(t)u(t).$$

Thus

$$\dot{V} = -\frac{1}{K}u_{\mathtt{nom}}^{\mathsf{T}}u(t) \leq -\frac{1}{K}u^{\mathsf{T}}(t)u(t) \leq 0. \tag{8}$$

Under Assumption 1, the closed-loop system dynamics,

$$\dot{C} = e^{-At}B\mathtt{Proj}_{\mathcal{U}}\left[-KB^{\mathsf{T}}e^{-A^{\mathsf{T}}t}PC\right], \tag{9}$$

can be re-written in the time-invariant form,

$$\dot{C} = f(C, z), \tag{10}$$

$$\dot{z} = g(z), \tag{11}$$

for suitable defined f and g that are locally Lipschitz functions of C and z, where $z(0) \in \mathcal{Z}$ and $\mathcal{Z}$ is a compact *invariant* set for (11). The subsystem (11) can be constructed by transforming A into the real Jordan canonical form, and defining the states z to represent $\sin(\omega_i t)$, $\cos(\omega_i t)$ terms in e^{-At} for each block corresponding to $\pm j\omega_i$ eigenvalues.

For instance, consider

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\omega^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

that has eigenvalues at $\pm j\omega$ and 0. Then,

$$e^{-At} = \begin{bmatrix} \cos(\omega t) & -sin(\omega t)/\omega & 0 \\ \omega\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $e^{-A^{\mathsf{T}}t} = (e^{-At})^{\mathsf{T}}$. Let

$$\dot{z}_1 = \omega z_2, \; \dot{z}_2 = -\omega z_1, \; \dot{z}_3 = 0,$$
$$z_1(0) = 0, \quad z_2(0) = 1, \quad z_3(0) = 1.$$

Then

$$z_1(t) = \sin(\omega t), \; z_2(t) = \cos(\omega t), \; z_3(t) = 1,$$

and

$$e^{-At} = F_1(z_1, z_2, z_3) = \begin{bmatrix} z_2 & -z_1/\omega & 0 \\ \omega z_1 & z_2 & 0 \\ 0 & 0 & z_3 \end{bmatrix},$$

while $e^{-A^{\mathsf{T}}t} = F_2(z_1, z_2, z_3) = (F_1(z_1, z_2, z_3))^{\mathsf{T}}$. The set $\mathcal{Z} = \{z : S(z) = \frac{1}{2}\|z\|^2 \leq 1\}$ is such that $z(0) \in \mathcal{Z}$ and is invariant as $\frac{d}{dt}S(z(t)) = 0$.

Theorem 1: Under Assumptions 1-3, the closed-loop system,

$$\dot{x} = Ax + B\mathtt{Proj}_{\mathcal{U}}\left[-KB^{\mathsf{T}}e^{-A^{\mathsf{T}}t}Pe^{-At}x\right], \tag{12}$$

is globally uniformly asymptotically stable at the origin.

Proof: Let $0 \leq t_0 \leq t$ and $x(t_0)$ be given. Define $C(t_0) = e^{-At_0}x(t_0)$ and $z(t_0) \in \mathcal{Z}$ be such that it generates e^{-At} for $t \geq t_0$. Then, trajectories of $x(t)$ from (12) and from (10), (11), (3) coincide.

Let $V(t_0) = \frac{1}{2}C^{\mathsf{T}}(t_0)PC(t_0)$. The set $\Omega = \{(C, z) : \frac{1}{2}C^{\mathsf{T}}PC \leq V(t_0), \; z \in \mathcal{Z}\}$ is compact and positively invariant for (10)-(11). By LaSalle's Theorem (Theorem 6.4 in [6], Theorem 3.3 in [1]) all trajectories $(C(t), z(t))$ of (10)-(11) emanating from Ω converge to the largest positively invariant set, $\mathcal{M}$ contained in the set $M = \{(C, z) : \dot{V} = 0\}$, i.e., $(C(t), z(t)) \to \mathcal{M}$ as $t \to \infty$.

Consider a trajectory of (10)-(11) on which $\dot{V}(t) = 0$ for all $t \geq t_0$, and hence, from (8), $u(t) = 0$ for $t \geq t_0$. Then, on such a trajectory, by (4), $C(t) = C(t_0)$ for $t \geq t_0$. Since $0 \in \text{Int}\mathcal{U}$, and $u(t)$ is the projection of $u_{\mathtt{nom}}(t)$, $u_{\mathtt{nom}}(t) = 0$ for all $t \geq t_0$. Hence for all $t \geq t_0$,

$$\beta(t) \triangleq \frac{u_{\mathtt{nom}}^T(t)}{K} = -C^{\mathsf{T}}(t)Pe^{-At}B = -C^{\mathsf{T}}(t_0)Pe^{-At}B = 0.$$

Furthermore, $\frac{d^k}{dt}\beta(t) = 0$ which implies

$$C^\mathsf{T}(t_0)Pe^{-At}A^kB = 0,\ k = 1, \cdots, n-1.$$

Thus at $t = t_0$,

$$C^\mathsf{T}(t_0)Pe^{-At_0}\left[B\ AB\ \cdots\ A^{n-1}B\right] = 0,$$

which, under Assumption 2, implies that

$$C^\mathsf{T}(t_0)Pe^{-At_0} = 0.$$

Since $P \succ 0$ and e^{-At_0} is invertible, $C^\mathsf{T}(t_0) = 0$ and $C(t) = C(t_0) = 0$ for all $t \geq t_0$. Thus $\mathcal{M} \subseteq \{(C, z) :\ C = 0\}$ and $C(t) \to 0$ as $t \to \infty$.

Under Assumption 1, there exist $\alpha_0 > 0$ such that

$$\|e^{At}\| \leq \alpha_0,\ \|e^{-At}\| \leq \alpha_0 \text{ for all } t \geq 0.$$

Note that for any trajectory of the closed-loop system,

$$\|x(t)\| \leq \|e^{At}\|\|C(t)\| \leq \alpha_0\|C(t)\|$$

and hence $x(t) \to 0$ as $t \to \infty$. As $\lambda_{\mathtt{max}}(P)I_{n_x} \succeq P \succeq \lambda_{\mathtt{min}}(P)I_{n_x}$, where $\lambda_{\mathtt{max}}(P)$, $\lambda_{\mathtt{min}}(P)$ denote maximum and minimum eigenvalues of P, respectively, and since $V(t) \leq V(0)$, it follows that $\|C(t)\| \leq \sqrt{\kappa(P)}\|C(t_0)\|$ ($\kappa(P) = \lambda_{\mathtt{max}}(P)/\lambda_{\mathtt{min}}(P)$ is the condition number of P). Thus $\|x(t)\| \leq \alpha_0^2\sqrt{\kappa(P)}\|x(t_0)\|$, showing uniform stability. Since (10), (11) is an autonomous system and $\mathcal{Z}$ is compact, the time for $C(t)$ to enter and remain thereafter in a ball $\mathcal{B}_{\epsilon_2}$ around the origin of radius ϵ_2 if $C(t_0) \in \mathcal{B}_{\epsilon_1}$, $\epsilon_1 \geq \epsilon_2$ can be upper bounded by a finite upper bound, $\tau(\epsilon_1, \epsilon_2)$, which is independent of t_0 and $z(t_0)$. Since $\|x(t)\| \leq \alpha_0\|C(t)\|$, $\|C(t_0)\| \leq \alpha_0\|x(t_0)\|$, $x(t)$ inherits the same property implying global uniform asymptotic stability at the origin. □

Remark 1: Closed-loop stability holds for any $P \succ 0$, $K > 0$ and P and K can be used as tuning parameters. Since $K > 0$ can be arbitrary, the above results imply that the control law provides an infinite (i.e., $(0, \infty)$) gain margin.

Remark 2: Theorem 1 still holds if the minimum 2-norm projection $\mathtt{Proj}_\mathcal{U}$ in (6) is replaced by a *radial* projection,

$$u = \overline{\mathtt{Proj}}_\mathcal{U}[u_{\mathtt{nom}}] = \max\{\rho :\ 0 \leq \rho \leq 1,\ \rho u_{\mathtt{nom}} \in \mathcal{U}\} \cdot u_{\mathtt{nom}}.$$

The following result pertains to a more general case when 0 is permitted to lie on the boundary of $\mathcal{U}$, i.e., "one-sided" constraints on u are allowed.

Theorem 2: Suppose in Assumption 3 the condition $0 \in \operatorname{Int}\mathcal{U}$ is replaced by $0 \in \mathcal{U}$. Let $N \in \mathbb{R}^{n_q \times n_u}$ be such that $\mathtt{Proj}_\mathcal{U}[u] = 0 \Rightarrow Nu \leq 0$ while the pair (A, BN^T) is controllable. If Assumption 1 is modified so that A has no eigenvalues at the origin, then the conclusion of Theorem 1 holds.

Proof: The proof is similar to Theorem 1 with several modifications of the analysis of the largest invariant set contained in $M = \{(C, z) : \dot{V} = 0\}$. Under assumptions of Theorem 2, $\dot{V}(t) = 0$ for all $t \geq t_0$ still implies that $C(t) = C$ is a constant, $u(t) = Proj_\mathcal{U}[u_{nom}(t)] = 0$ and hence $Nu_{nom}(t) \leq 0$. If n_i^T denotes the ith row of N, $i = 1, \cdots, n_q$, then $f_i(t) = n_i^\mathsf{T}B^\mathsf{T}e^{-A^\mathsf{T}t}PC \leq 0$ for all $t \geq t_0$. Note that under assumptions of Theorem 2, f_i is a linear combination of sine functions of different non-zero frequencies and phases. The first and second indefinite integrals of f_i are then also linear combinations of sine functions and hence must be bounded. From this it follows (proof by contradiction) that $f_i(t)$ must be strictly positive on some time interval of nonzero length unless $f_i(t) = 0$ for all $t \geq t_0$. Thus $(BN^T)^\mathsf{T}(e^{-A^\mathsf{T}t}PC) = 0$ for all $t \geq t_0$. Then, by the same controllability-based argument as in Theorem 1, it follows that $C(t) = C(t_0) = 0$ for all $t \geq t_0$. The rest of the proof is the same as of Theorem 1. □

As an illustration, consider the case $n_u = 2$ and $\mathcal{U} = [-a, 0] \times [-b, b]$ where $a, b > 0$. Then $Proj_\mathcal{U}[u] = 0$ if and only if $u = [u_1\ \ 0]^\mathsf{T}$, $u_1 > 0$, and we can choose $N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. As N is square and invertible, controllability of (A, B) and of (A, BN^T) are equivalent.

By extending the above observation to more than two dimensions, we obtain the following result:

Theorem 3: Suppose either $U = [-a_1, 0] \times \cdots \times [-a_{n_u}, 0]$ or $U = [0, a_1] \times \cdots \times [0, a_{n_u}]$ where $a_i > 0$, $i = 1, \cdots, n_u$. If Assumptions 2 and 3 hold, Assumption 1 is modified so that A has no eigenvalues at the origin, then the conclusion of Theorem 1 holds.

Proof: Let $U = [-a_1, 0] \times \cdots \times [-a_{n_u}, 0]$ (the other case is analogous). Consider any u for which $\mathtt{Proj}_\mathcal{U}[u] = 0$. Clearly, this implies $u_i \geq 0$ for all $i = 1, \cdots, n_u$. Picking $N = -I_{n_u}$, then $Nu \leq 0$ for such u. As N is invertible, controllability of (A, B) implies controllability of (A, BN^T). The result now follows by applying Theorem 2. □

B. Eigenvalues on imaginary axis and in open-left half plane

Suppose we now allow eigenvalues in the open-left half plane replacing Assumption 1 with:

Assumption 1′: The eigenvalues of A are on $j\omega$-axis or in open-left half plane. The Jordan canonical form blocks corresponding to purely imaginary eigenvalues are diagonal.

Remark 3: Note that under Assumption 1′, e^{-At} can grow unbounded as $t \to \infty$ due to eigenvalues of A in the open left half plane; hence the trajectories of the generator subsystem (11) may not be bounded and LaSalle's Theorem cannot be applied to show stability of (9) at the origin. Note that $\dot{V}(t) \leq 0$ still holds for all $t \geq 0$ and hence $C(t)$ remains bounded.

Remark 4: In the special case, when A has *all* eigenvalues in the open half plane, $e^{At} \to 0$ as $t \to \infty$ and $x(t) = e^{At}C(t) \to 0$ as $t \to \infty$ even if the convergence of $C(t)$ to zero cannot be concluded. Note that the growing terms with time in e^{-At}, which complicate control law (6) implementation, can be mitigated by the periodic reset of initial time (current time becomes 0 time) and of $C(0)$.

To avoid issues highlighted in Remark 3, we proceed by applying a similarity transformation,

$$\bar{x} = T^{-1}x = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

to system (1) so that

$$\dot{\bar{x}} = \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & \bar{A}_{22} \end{bmatrix} \bar{x} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u, \tag{13}$$

where all eigenvalues of $\bar{A}_{11}$ are in the open-left half plane and all eigenvalues of $\bar{A}_{22}$ are on the imaginary axis. An example of such a transformation is a transformation into the real Jordan canonical form. The control law is now defined for $\bar{x}_2$-subsystem based on the procedure in Section II-A:

$$u_{\texttt{nom}}(t) = -K\bar{B}_2^{\mathsf{T}} e^{-\bar{A}_{22}^{\mathsf{T}} t} \bar{P}_{22} \bar{C}_2(t), \;\; u(t) = \texttt{Proj}_{\mathcal{U}}\left[u_{\texttt{nom}}(t)\right], \tag{14}$$

where $K > 0$ is a scalar gain, $\bar{P}_{22} \succ 0$, and

$$\bar{C}_2(t) = e^{-\bar{A}_{22} t} \bar{x}_2(t). \tag{15}$$

Assumption 2′: The pair $(\bar{A}_{22}, \bar{B}_2)$ is controllable (or equivalently (1) under Assumption 1' is stabilizable).

Per Theorem 1, the control law (14)-(15) is stabilizing for $\bar{x}_2$ subsystem, and $\bar{C}_2(t) \to 0$ as $t \to \infty$; hence, $u(t) \to 0$ as $t \to \infty$. Since $\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + \bar{B}_1 u$, $u(t) \to 0$ as $t \to \infty$, and $\bar{A}_{11}$ is a Hurwitz matrix, it follows that $\bar{x}_1(t) \to 0$ as $t \to \infty$. We summarize the above observations formally as

Theorem 4: Under Assumptions $1'$, $2'$ and 3, the time-dependent dynamic feedback control law given by (14)-(15), globally uniformly asymptotically stabilizes system (1) at the origin while satisfying the control constraints (2).

The results of Theorem 2 and 3 are extended in the same way; we leave out the details.

III. Examples of Constrained Stabilization of Linear Systems

We set $P = I_{n_x}$, the $n_x \times n_x$ identity matrix in all the subsequent examples.

A. Example 1

Consider an undamped mass-spring system with the actuator dynamics represented by the third state and with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$\mathcal{U} = [-0.1, 0.1]$ and $K = 2$. The eigenvalues of A are at $\pm j$ and -0.3. The similarity transformation puts A into the real Jordan canonical form in (13) where $\bar{x}_1 = x_3$, $\bar{x}_2 = \begin{bmatrix} \bar{x}_{21} & \bar{x}_{22} \end{bmatrix}^{\mathsf{T}}$, $\bar{x}_{21} = 0.7071x_2 + 0.1946x_3$, $\bar{x}_{31} = 0.7071x_1 - 0.6487x_3$. The time histories of states and control input resulting from applying (14)-(15) and initial condition $x(0) = (1, 1, 1)$ are shown in Figure 1. The time histories of $\bar{C}_2(t) = \begin{bmatrix} \bar{C}_{21} & \bar{C}_{22} \end{bmatrix}^{\mathsf{T}}$ and $V(t) = \frac{1}{2}\bar{C}_2^{\mathsf{T}}\bar{C}_2$ are illustrated in Figure 2.

B. Example 2

Consider two masses interconnected by a spring where one of the masses is connected with another spring to the

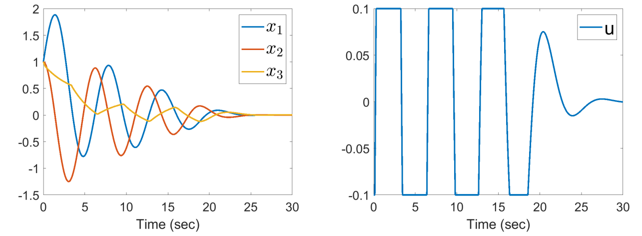

Fig. 1: Left: Time histories of states in Example 2. Right: Time history of control in Example 1.

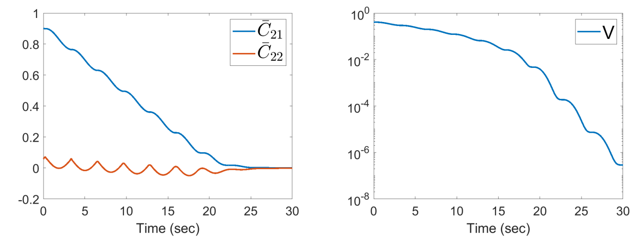

Fig. 2: Left: Time histories of $\bar{C}_{21}(t)$ and $\bar{C}_{22}(t)$ in Example 2. Right: Time history of Lyapunov function, V, in Example 2.

wall, and assume there is no damping. The model is given by (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ a_{41} & 0 & a_{43} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ -1/m_1 & 1/m_2 \end{bmatrix},$$

$a_{21} = -k_1/m_1 - 2k_2/m_1$, $a_{23} = 2k_2/m_1$, $a_{41} = k_2/m_2 + k_1/m_1 + 2k_2/m_1$, $a_{43} = -k_2/m_2 - 2k_2/m_1$, $m_1 = 1, m_2 = 1, k_1 = 5, k_2 = 1$. The eigenvalues of A are at $\pm 3.0777j$, $\pm 0.7265j$. The control inputs are forces applied to each mass. The simulation results for $K = 10$, $\mathcal{U} = [-0.1, 0.1] \times [-0.1, 0.1]$ and $x(0) = (1, 0, 0, 0)$ are shown in Figure 3.

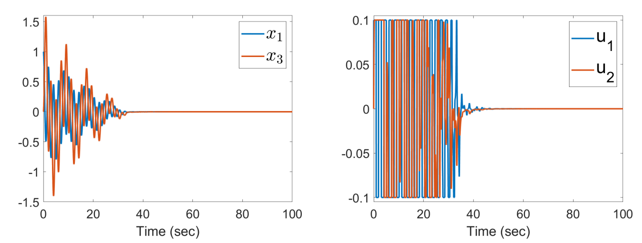

Fig. 3: Left: Time histories of mass position states x_1 and x_3 with both inputs in Example 2. Right: Time history of controls with both inputs in Example 2.

The same controller with just the first input and one sided control constraints such that $\mathcal{U} = [-0.1, 0]$ produces responses in Figure 4.

IV. Extension to Nonlinear Systems

We now consider an extension of the approach to a class of nonlinear systems that can be represented by a model,

$$\dot{x} = F(x) + B(x)u. \tag{16}$$

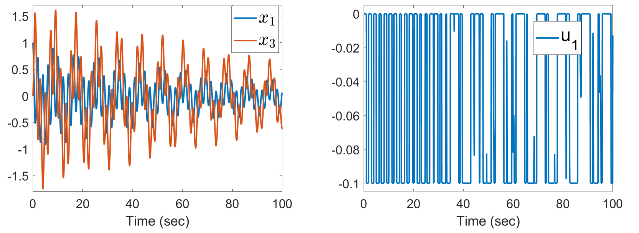

Fig. 4: Left: Time histories of mass position states x_1 and x_3 with only the first input in Example 2 with $\mathcal{U} = [-0.1, 0]$. Right: Time history of controls with only the first input in Example 2 with $\mathcal{U} = [-0.1, 0]$.

Let $\bar{x}(C,t)$ denote the solution of the unforced system,

$$\dot{x}(t) = F(x(t)), \quad t \geq 0, \tag{17}$$

where C is an n_x-vector of constants that the solution depends on. For instance, C can be chosen as a vector of initial conditions, i.e., $C = x(0)$. Note that

$$\frac{\partial}{\partial t}\bar{x}(C,t) = F(\bar{x}(C,t)), \ t \geq 0. \tag{18}$$

Suppose we now seek the solution of (16) in the form $x(t) = \bar{x}(C(t),t)$, where C is made time-dependent. Then,

$$\begin{aligned}\frac{d}{dt}\bar{x}(C(t),t) &= \frac{\partial \bar{x}(C,t)}{\partial C}\dot{C} + \frac{\partial \bar{x}(C,t)}{\partial t} \\ &= F(\bar{x}(C,t)) + B(\bar{x}(C,t))u,\end{aligned}$$

and, taking into account (18), this leads to the variational differential equation,

$$\dot{C} = \Phi(t,C)u, \ \Phi(t,C) = \left[\frac{\partial \bar{x}}{\partial C}\right]^{-1} B, \tag{19}$$

where the sensitivity matrix in (19) is assumed to be invertible (which is a reasonable assumption given the solution uniqueness). Given (19), and a Lyapunov function (5), we can define a control law,

$$u_{\mathtt{nom}}(t) = -K\Phi^{\mathsf{T}}(t,C(t))PC(t), \ u(t) = \mathtt{Proj}_{\mathcal{U}}\left[u_{\mathtt{nom}}(t)\right], \tag{20}$$

where $K > 0$ is a scalar gain. With V given by (5), this leads to (8).

We sketch the closed-loop stability analysis procedure that could be applied to classes of systems with specific properties and enables us to highlight needed assumptions below. The LaSalle's-Yoshizawa's theorem (Theorem 4.7 in [1]) can be exploited to show $\Phi^{\mathsf{T}}(t,C(t))PC(t) \to 0$ as $t \to \infty$. Assuming that the matrix function $\Phi(t,C)$ in (19) is persistently exciting for any *constant* C, i.e., there exists $\Delta > 0$ and $\epsilon > 0$ such that for all $t \geq 0$ and $C \in \mathbb{R}^{n_x}$,

$$\int_t^{t+\Delta} \Phi(\tau,C)\Phi^{\mathsf{T}}(\tau,C)d\tau > \epsilon I_{n_x},$$

it can be shown, applying Theorem 14 in [2], that $C(t) \to 0$ as $t \to \infty$. Finally, assuming the solution mapping $\bar{x}(C,t)$ satisfies $\bar{x}(0,t) = 0$ for all $t \geq 0$ and $\sup_{t\geq 0}\|\bar{x}(C,t)\| \to 0$ as $C \to 0$ it follows that $x(t) \to 0$ as $t \to \infty$.

Remark 5: In the linear system case in Section II-A with $C = x(0)$, $\bar{x}(C,t) = e^{At}C$. The persistence of excitation condition on $\Phi(t,C) = e^{-At}B$ is satisfied since the controllability gramian of $(-A,B)$ is nonsingular over any nonzero length time interval as (A,B) (and hence $(-A,B)$) is, under Assumption 2, controllable. The condition $\sup_{t\geq 0}\|\bar{x}(C,t)\| \to 0$ as $C \to 0$ follows by Assumption 1 on eigenvalues of A.

Remark 6: If an explicit functional representation of the solution $\bar{x}(C,t)$ is not available, the sensitivity matrix needed for control implementation can be computed by numerical integration of the sensitivity differential equations. For instance, let $C = x(0)$ be the vector of initial conditions of (17), and

$$S(t) = \frac{\partial \bar{x}(C,t)}{\partial C}|_{C(t),t},$$

then

$$\frac{dS}{d\tau} = \left[\frac{\partial F}{\partial x}(\bar{x}(C(\tau),\tau))\right]S, \quad S(0) = I_{n_x}, \quad 0 \leq \tau \leq t, \tag{21}$$

where (21) and (18) with $\bar{x}(C,0) = C$ are integrated together. Note that $C(t)$ in (20) needs to be consistent with $x(t)$ so that $x(t) = \bar{x}(C(t),t)$. A correction, based on one iteration of Newton's method,

$$\hat{C}(t) = C(t) + S^{-1}(t)(x(t) - \bar{x}(C(t),t)), \tag{22}$$

and with $\hat{C}(t)$ replacing $\hat{C}(t)$ in (20) helps ensure this consistency and feedback from the actual measurements of $x(t)$.

V. Examples of Constrained Stabilization of Nonlinear Systems

The implementation based on Remark 6 is used in the subsequent examples.

A. *Example 3*

We consider predator-prey dynamics, represented by the following model (in scaled units),

$$\dot{x}_1 = -x_2(x_1+2), \quad \dot{x}_2 = x_1(x_2+1) + u,$$

where x_1 and x_2 are the deviations of prey and predator populations, respectively, from their equilibrium values at $(2,1)$, u is the control input corresponding to adding predators to the population, $\mathcal{U} = [0, 0.1]$ (i.e., predators once added cannot be removed), and $K = 10$. The eigenvalues of A matrix of the linearized system at the equilibrium are at $\pm 1.4142j$. The responses to $x(0) = (2.0, -0.8)$, $C(0) = x(0)$ are shown in Figure 5.

B. *Example 4*

We consider spacecraft stabilization to L_4 Lagrange point of Earth-Moon system based on the *nonlinear* Circular Restricted Three Body Problem (CR3BP) dynamics model [4], where $n_x = 6$, $n_u = 3$, distances and time are in standard scaled units, $K = 10$, and $\mathcal{U} = \{u : \ \|u\| \leq 5 \times 10^{-5}\}$.

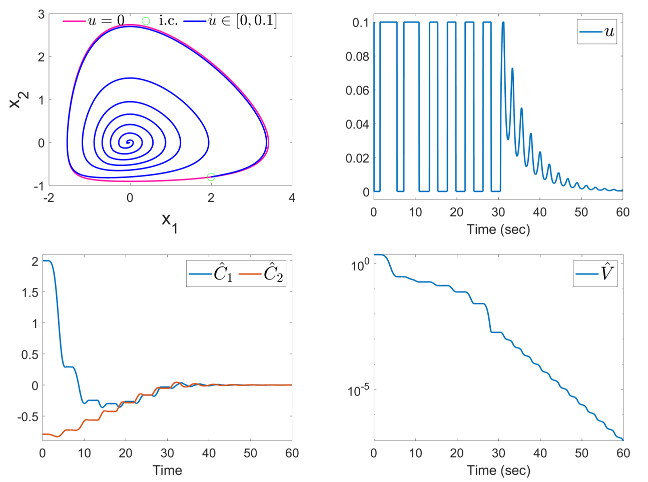

Fig. 5: Left: Closed-loop phase plane trajectories (x_2 vs. x_1) compared with the ones for $u = 0$ in Example 3. Right: Time history of control input. Bottom Left: Time history of $\hat{C}_1$ and $\hat{C}_2$. Bottom Right: Time history of $\hat{V}(t) = \frac{1}{2}\hat{C}^{\mathsf{T}}(t)\hat{C}(t)$.

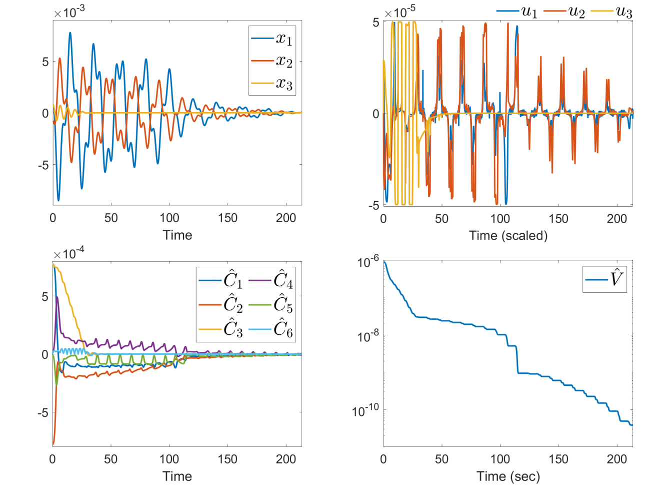

Fig. 6: Top Left: Time history of position coordinates relative to L_4 in Example 4. Top Right: Time history of controls. Bottom Left: Time history of $\hat{C}_i$, $i = 1, \cdots, 6$. Bottom Right: Time history of $\hat{V}(t) = \frac{1}{2}\hat{C}^{\mathsf{T}}(t)\hat{C}(t)$.

The equations of motion for the CR3BP with the origin shifted to L_4 Lagrange point are given as

$$
\begin{aligned}
&\dot{x}_1 = x_4,\ \ \dot{x}_2 = x_5,\ \ \dot{x}_3 = x_6,\\
&\dot{x}_4 = 2x_5 + \tilde{x}_1 - \frac{(1-\mu)(\tilde{x}_1+\mu)}{r_1^3}\\
&\quad - \frac{\mu(-1+\tilde{x}_1+\mu)}{r_2^3} + u_1,\\
&\dot{x}_5 = -2x_4 + \tilde{x}_2 - \frac{\tilde{x}_2(1-\mu)}{r_1^3} - \frac{\mu\tilde{x}_2}{r_2^3} + u_2,\\
&\dot{x}_6 = -\frac{x_3(1-\mu)}{r_1^3} - \frac{\mu x_3}{r_2^3} + u_3,
\end{aligned} \tag{23}
$$

where $r_1 = ((\tilde{x}_1+\mu)^2 + \tilde{x}_2^2 + x_3^2)^{1/2}$, $r_2 = ((\tilde{x}_1 - 1 + \mu)^2 + \tilde{x}_2^2 + x_3^2)^{1/2}$, $\tilde{x}_1 = x_1 + \frac{1}{2} - \mu$, $\tilde{x}_2 = x_2 + \frac{\sqrt{3}}{2}$. and $\mu = m_m/(m_e + m_m) = 1.215059 \times 10^{-2}$ is the mass ratio, with m_e being the mass of the Earth and m_m being the mass of the Moon.

As this example is higher dimensional, to reduce the computation time, equation (21) is solved every 0.05 scaled time units and $S(t)$ is maintained constant between such updates. Asymptotically stable responses to the initial condition $x(0) = (300/D, -300/D, 300/D, 0, 0, 0)$ where $D = 384748$ km are shown in Figure 6. Note that L_4 is destabilized by adding energy dissipation as it is at the maximum of pseudo-potential [4]; hence our control laws are different from "simply adding damping."

VI. Concluding Remarks

Linear time invariant systems with controllable eigenvalues on the imaginary axis that have equal geometric and algebraic multiplicity and other eigenvalues in open left half plane are globally stabilizable subject to arbitrary tight (and under assumption of no zero eigenvalue, even "one-sided") control constraints using a feedback law derived from variational equations and a quadratic Lyapunov function. The method can be extended to certain classes of nonlinear systems.

Acknowledgement

The first author expresses his gratutude to Professor Dennis Bernstein from the University of Michigan for encouraging to write this paper.

References

[1] W.M. Haddad and V. Chellaboina. *Nonlinear dynamical systems and control: a Lyapunov-based approach*. Princeton university press, 2008.

[2] E.J. Hancock and A. Papachristodoulou. An invariance principle for time-varying systems. In *51st IEEE Conference on Decision and Control*, pages 4597–4602, 2012.

[3] T. Hu and Z. Lin. *Control systems with actuator saturation: Analysis and design*. Springer Science & Business Media, 2001.

[4] W.S. Koon, M.W. Lo, J.E. Marsden, and S.D. Ross. *Dynamical systems, the three-body problem and space mission design*. Marsden Books, 2011.

[5] V.I. Korobov. A general approach to the solution of the bounded control synthesis problem in a controllability problem. *Matematicheskii Sbornik*, 151(4):582–606, 1979.

[6] J.P. La Salle. *The stability of dynamical systems*. SIAM, 1976.

[7] V. Lakshmikantham and S.G. Deo. *Method of variation of parameters for dynamic systems*. CRC Press, 1998.

[8] A. Saberi, A.A. Stoorvogel, and P. Sannuti. *Control of linear systems with regulation and input constraints*. Springer Science & Business Media, 2012.

[9] H. Sussmann, E. Sontag, and Y. Yang. A general result on the stabilization of linear systems using bounded controls. In *Proceedings of 32nd IEEE Conference on Decision and Control*, pages 1802–1807. IEEE, 1993.

[10] S. Tarbouriech, G. Garcia, J.M.G. da Silva Jr, and I. Queinnec. *Stability and stabilization of linear systems with saturating actuators*. Springer Science & Business Media, 2011.

[11] A.R. Teel. Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems & control letters*, 18(3):165–171, 1992.

[12] X. Wang, H. Grip, A. Saberi, and T.A. Johansen. A new low-and-high gain feedback design using MPC for global stabilization of linear systems subject to input saturation. In *2012 American Control Conference (ACC)*, pages 2337–2342. IEEE, 2012.

[13] B. Zhou and J. Lam. Global stabilization of linearized spacecraft rendezvous system by saturated linear feedback. *IEEE Transactions on Control Systems Technology*, 25(6):2185–2193, 2017.