Even singular integral operators that are well behaved on a purely unrectifiable set

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Abstract: We prove the existence of a $(d-2)$-dimensional purely unrectifiable set upon which a family of *even* singular integral operators is bounded.

1 Introduction

Understanding the geometry of a measure μ for which an associated *odd* singular integral operator is bounded in $L^2(\mu)$ is a heavily studied problem in harmonic analysis. For instance, the question of whether the $L^2(\mu)$ boundedness for s-dimensional Riesz transforms implies the rectifiability (and closely related conditions) of the measure μ, is known as the David-Semmes question [DS]. This is only known when $s = 1$ [MMV, Dav, Le] and $s = d-1$ [ENV2, NToV, DT, To]. There are several positive and negative results that have been proved for a wide variety of odd kernels, see e.g. [Ch, CMPT, Hu, JN, MP].

Taking our inspiration from [MOV, MOV2], the goal of this note is to take a first step in the study of the problem for *even kernels* by characterizing the even kernels that can be bounded in L^2 on a particular class of purely unrectifiable sets, namely, the natural analogues of the sets first considered in $\mathbb{R}^2$ in [JN] and recently generalized by [MP] to co-dimension one sets in $\mathbb{R}^d$.

Let us recall that a set $A \subset \mathbb{R}^d$ is called *m-rectifiable* if there are Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^d$ for all $i = 1, 2, ...$, such that

$$\mathscr{H}^m(A \setminus \cup f_i(\mathbb{R}^m)) = 0.$$

(Here $\mathscr{H}^m$ denotes the m-dimensional Hausdorff measure.) In contrast, a set B is *m-purely unrectifiable* if $\mathscr{H}^m(B \cap A) = 0$ for every m-rectifiable set $A \subset \mathbb{R}^d$.

A measure μ is said to have *m-growth* if there exists a constant $C > 0$, such that $\mu(B(x,R)) \leq Cr^m$ for every ball $B(x,r) \subset \mathbb{R}^d$. Finally, we say μ is an *m-dimensional measure* if $\mathscr{H}^m(\operatorname{supp}\mu) < \infty$ and μ has m-growth.

Throughout the paper, we will denote by Ω an even Hölder continuous function on $\mathbb{S}^{d-1}$ with $\int_{\mathbb{S}^{d-1}} \Omega(\xi) d\mathscr{H}^{d-1}(\xi) = 0$. For a finite measure μ, the m-dimensional SIO associated to Ω is bounded in $L^2(\mu)$ if there is a constant $C > 0$ such that for every $f \in L^2(\mu)$

$$\sup_{\varepsilon > 0} \int_{\mathbb{R}^d} \Big| \int_{y:|x-y|>\varepsilon} \frac{\Omega\big(\frac{x-y}{|x-y|}\big)}{|x-y|^m} f(y) d\mu(y) \Big|^2 d\mu(x) \leq C\|f\|^2_{L^2(\mu)}.$$

Employing the $T(1)$-theorem for spaces of non-homogeneous type [NTV], provided that μ has m-growth one can reduce the study of operator boundedness to understanding whether the potential, defined for $x \notin \operatorname{supp}(\mu)$ by

$$T(\mu)(x) = \int_{\mathbb{R}^d} \frac{\Omega\big(\frac{x-y}{|x-y|}\big)}{|x-y|^m} d\mu(y),$$

belongs to $L^\infty(\mathbb{R}^d \setminus \operatorname{supp}(\mu))$.

Now let us state the main result of this paper, which concerns even SIOs of co-dimension 2.

Theorem 1.1. *Let $d \geq 3$. There exists a $(d-2)$-purely unrectifiable set E and a $(d-2)$-dimensional probability measure supported on E such that the $(d-2)$-dimensional potential associated to Ω belongs to $L^\infty(\mathbb{R}^d \setminus \operatorname{supp}(\mu))$ if and only if*

$$\int_{\mathbb{S}^{d-1}} \xi_i \cdot \xi_j \cdot \Omega(\xi)\, d\mathscr{H}^{d-1}(\xi) = 0 \text{ for all } i,j \in \{1,\dots,d\}. \tag{1.1}$$

Additionally, unless Ω vanishes identically, the SIO associated to Ω fails to exist in the sense of principal value μ-almost everywhere.

It is interesting to note that for the singular integral operator associated to the measure μ we construct in Theorem 1.1, the two properties of $L^2(\mu)$-boundedness and existence in the principle value are quite distinct.

The class of kernels that satisfy the hypothesis for our main result above is non-empty. To construct an example, fix a non-negative function $\varphi \in C^1([0,1])$ with $\varphi(1) = 0$. For $\xi = (\xi_1, \xi_2, \xi') \in \mathbb{R}^d$ first define

$$a(\xi_1, \xi_2, \xi') = (\xi_2^2 - \xi_1^2)\varphi(\xi_1^2 + |\xi'|^2)\varphi(\xi_2^2 + |\xi'|^2).$$

For any $\xi' \in \mathbb{R}^{d-2}$ with $|\xi'| \leq 1$, the integral of $a(\cdot,\cdot,\xi')$ over the quarter circle $\{(\xi_1,\xi_2) : \xi_1 \geq 0,\, \xi_2 \geq 0,\, |\xi_1|^2 + |\xi_2|^2 = 1 - |\xi'|^2\}$ equals 0, and a vanishes if either $|\xi_1|^2$ or $|\xi_2|^2$ equals $1 - |\xi'|^2$ For $\xi = (\xi_1, \xi_2, \xi') \in \mathbb{S}^{d-1}$ set

$$\begin{aligned} &\Omega(\xi_1,\xi_2,\xi') = a(\xi_1,\xi_2,\xi') \quad \text{if} \quad \{\xi_1,\xi_2 \geq 0\} \cup \{\xi_1,\xi_2 \leq 0\} \\ &\Omega(\xi_1,\xi_2,\xi') = a(\xi_2,\xi_1,\xi') = -a(\xi_1,\xi_2,\xi') \quad \text{if} \quad \{\xi_1 \geq 0, \xi_2 < 0\} \cup \{\xi_1 \leq 0, \xi_2 \geq 0\}. \end{aligned}$$

The function Ω is a Hölder continuous mean-zero even function on the sphere that satisfies (1.1).

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2 The reflectionless property

We will denote by m_d the d-dimensional Lebesgue measure, and we often denote the surface area measure on $\mathbb{S}^{d-1}$ by σ. The next lemma is the key to our construction, and is based around the proof of Lemma 3 of [MOV2].

Lemma 2.1 (The Reflectionless Property)**.** *Let $x_0 \in \mathbb{R}^d$, $r > 0$. The condition (1.1) holds if and only if*

$$\int_{B(x_0,r)} K(x-y)\, dm_d(y) = 0 \text{ for any } x \in B(x_0,r). \tag{2.1}$$

Proof. Without loss of generality we may assume that $x_0 = 0$ and $r = 1$. Employing the mean zero property of Ω, first observe that

$$\int_{B(0,1)} K(x-y)dm_d(y) = \int_{B(x,1+|x|)\setminus B(0,1)} K(x-y)dm_d(y).$$

Express this integral in terms of the polar coordinates $y = x + r\zeta$ centered at x. Setting $r(x,\xi)$ to be the (smallest) solution of $|x + r(x,\xi)\cdot\xi| = 1$, we get

$$\begin{aligned}\int_{B(0,1)} K(x-y)dm_d(y) &= \int_{|\zeta|=1}\int_{r(x,\zeta)}^{1+|x|} \frac{1}{r^{d-2}}\Omega(\zeta)r^{d-1}drd\sigma(\xi)\\ &= \frac{1}{2}\int_{|\zeta|=1}(1+|x|)^2\Omega(\zeta)d\sigma(\zeta) - \frac{1}{2}\int_{|\zeta|=1}(r(x,\zeta))^2\Omega(\zeta)d\sigma(\zeta)\\ &=: I - II\end{aligned}$$

The term I vanishes as $\int_{\mathcal{S}^{d-1}}\Omega(x)d\sigma(x) = 0$.

Set U^+ be the half of the unit sphere above the hyperplane $\{x_d = 0\}$. Since Ω is an even function,

$$II = \frac{1}{2}\int_{|\zeta|=1}(r(x,\zeta))^2\Omega(\zeta)d\sigma(\zeta) = \frac{1}{2}\int_{U^+}\big[r(x,\zeta)^2 + r(x,-\zeta)^2\big]\Omega(\zeta)d\sigma(\zeta)$$

A simple computation yields $(r(x,\zeta)^2 + r(x,-\zeta)^2) = 4(x\cdot\zeta)^2 - 2(|x|^2-1)$, which leads to

$$\begin{aligned}II &= \frac{1}{2}\int_{U^+}(r(x,\zeta)^2 + r(x,-\zeta)^2)\Omega(\zeta)d\sigma(\zeta)\\ &= \frac{1}{2}\int_{U^+}4(x\cdot\zeta)^2\Omega(\zeta)d\sigma(\zeta) - \frac{1}{2}\int_{U^+}2(|x|^2-1)\Omega(\zeta)d\sigma(\zeta) =: III + IV.\end{aligned}$$

Since Ω has mean-zero over U_+, the term IV vanishes.

We conclude that (2.1) holds if and only if the term III vanishes for every $x \in B(0,1)$, which is in turn equivalent to the condition (1.1). This completes the proof of the lemma. □

3 Construction of the zero lower density set and the associated measure

The construction our zero lower density set follows along the same lines as the papers [JN] and [MP].

3.1 The set E

Set $\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ to be the volume of the d-dimensional unit ball.

Lemma 3.1. *One can pack* $(\frac{R}{r})^{d-2}$ *pairwise essentially disjoint cubes of side length* $\sqrt[d]{\kappa_d r^{d-2}R^2}$ *into a ball of radius* $R\left(1 + \sqrt{d}\sqrt[d]{\kappa_d}\sqrt[d]{\frac{r^{d-2}}{R^{d-2}}}\right)$.

Proof. Without loss of generality we can assume that our ball is centered at the origin. Now we will consider a cubic grid of mesh size $\sqrt[d]{\kappa_d r^{d-2}R^2}$. Suppose now that the cubes $Q_1, Q_2, ..., Q_M$

intersect $B(0,R)$. These cubes are contained in the ball centred at 0 with radius $R\big(1+\sqrt{d}\sqrt[d]{\kappa_d}\sqrt[d]{\frac{r^{d-2}}{R^{d-2}}}\big)$. Finally, since

$$M\kappa_d r^{d-2}R^2 = \sum_{j=1}^{M} m_d(Q_j) > m_d(B(0,R)) = \kappa_d R^d,$$

we have $M > \frac{R^{d-2}}{r^{d-2}}$, and the lemma follows. □

To begin let us consider a sequence $\{r_k\}_{k\geq 0}$ that tends to zero quickly and such that $r_0 = 1$ and $r_{k+1} < \frac{r_k}{B}$ for an absolute constant B which will be chosen later. Additionally shall assume that $\frac{r_k}{r_{k+1}} \in \mathbb{N}$ and $\frac{1}{r_k} \in \mathbb{N}$.

Set $\widetilde{B}_1^0 = B(0,1)$. We will construct the set iteratively. Given the k-th generation of $\frac{1}{r_k^{d-2}}$ balls $\tilde{B}_j^k$ of radius r_k, we proceed to the $(k+1)$-st generation as follows: for each ball $\widetilde{B}_j^k$ we apply Lemma 3.1 with $R = r_k$ and $r = r_{k+1}$, so we find $(\frac{r_k}{r_{k+1}})^{d-2}$ of pairwise disjoint cubes Q_l^{k+1} balls of sidelength $\sqrt[d]{\kappa_d r_{k+1}^{d-2} r_k^2}$ contained in ball $(1 + A\sqrt[d]{\frac{r^{d-2}}{R^{d-2}}})\widetilde{B}_j^k$ where $A = \sqrt{d}\sqrt[d]{\kappa_d}$.

Set $\widetilde{B}_j^{k+1} = B(z_l^{k+1}, r_{k+1})$ where z_l^{k+1} denotes the center of the cube Q_l^{k+1}. We carry out this process for each ball $\widetilde{B}_j^k$ from the k-th generation. In total, we get $\frac{1}{r_{k+1}^{d-2}}$ balls $\widetilde{B}_j^{k+1}$ in the $(k+1)$-st level.

Set $\delta_{k+1} = A\sqrt[d]{\frac{r_{k+1}^{d-2}}{r_k^{d-2}}}$ and

$$B_j^k = (1+\delta_{k+1})\widetilde{B}_j^k \quad \textit{and} \quad E^k = \bigcup_{j\geq 1} B_j^k.$$

We will frequently make use of the following properties of the construction

(i) For each $k \geq 1$,

$$\bigcup_l Q_l^{k+1} \subset E^k.$$

(ii) For each $k \geq 1$, we have $B_j^k \subset Q_j^k$, and moreover (provided that B is chosen appropriately)

$$\operatorname{dist}(B_j^k, \partial Q_j^k) \geq \frac{1}{4}\sqrt[d]{\kappa_d r_k^{d-2} r_{k-1}^2}.$$

(iii) For each $k \geq 1$ and for $i \neq j$, $\operatorname{dist}(B_j^k, B_i^k) \geq \frac{1}{4}\sqrt[d]{\kappa_d r_k^{d-2} r_{k-1}^2}$.

Observe that for each $k \geq 0$ we have $E^{k+1} \subset E^k$ and now set $E = \bigcap_{k\geq 0} E^k$. It is not hard to check that the set E satisfies $0 < \mathscr{H}^{d-2}(E) < \infty$, and $\liminf_{r\to 0} \frac{\mathscr{H}^{d-2}(E\cap B(x,r))}{r^{d-2}} = 0$ for every $x \in E$. Consequently the set E is $(d-2)$-purely unrectifiable [Mat].

It will be convenient to use the following notation: Each $x \in E^k$ is contained in a some unique ball B_j^k and in a unique cube Q_j^k, we will denote these by $B^k(x)$ and $Q^k(x)$ respectively.

3.2 The measure μ

Set

$$\mu_j^k = \frac{1}{r_k^2}\chi_{\tilde{B}_j^k} m_d \quad \textit{and} \quad \mu^k = \sum_j \mu_j^k.$$

Observe that $\operatorname{supp}(\mu^k) \subset E^k$ and $\mu^k(\mathbb{R}^d) = 1$ for all k. The following properties hold for the measures μ^k:

(a) $\operatorname{supp}(\mu^k) \subset \bigcup_{j\geq 1} B_j^m$ if $k \geq m$.

(b) $\mu^k(B_j^m) = r_m^{d-2}$ for $k \geq m$.

(c) There exists a constant $C > 0$ such that for any k and ball $B(x,r)$,

$$\mu^k(B(x,r)) \leq Cr^{d-2}.$$

Properties (a) and (b) follow immediately from construction. For (c), first note that for $r \geq 1$, this property is clear as μ^k is a probability measure. If $0 < r < 1$, then $r \in (r_{m+1}, r_m)$ for some $m \in \mathbb{N}$. In the case when $m \geq k$, $r_m \leq r_k$, and hence the ball $B(z,r)$ intersects with at most one B_j^k. Therefore

$$\mu^k(B(z,r)) = \frac{1}{r_k^2} m_d(B(z,r) \cap \widetilde{B}_j^k) \leq \frac{r^d}{r_k^2} \leq r^{d-2}.$$

On the other hand, if $k \geq m$, then property (iii) of the construction ensures that $B(z,r)$ intersects at most $1 + C\frac{r^d}{r_{m+1}^{d-2} r_m^2}$ balls B_j^{m+1}. Property (b), then ensures that

$$\mu^k(B(z,r)) = \sum_j \mu^k(B(z,r) \cap B_j^{m+1}) \leq \left(1 + C\frac{r^d}{r_{m+1}^{d-2} r_m^2}\right) r_{m+1}^{d-2} \leq Cr^{d-2},$$

and (c) is proved.

Finally, passing to a subsequence if necessary, the measures μ^k converge weakly to a $(d-2)$-dimensional measure μ supported on E.

4 The boundedness of the potential associated to Ω

Fix $\alpha \in (0,1]$ to be the Hölder exponent of Ω. All absolute constants in this section may depend on dimension, and the quantity

$$\|\Omega\|_{C^\alpha(\mathbb{S}^{d-1})} := \sup_{\omega,\xi\in\mathbb{S}^{d-1}} \frac{|\Omega(\omega) - \Omega(\xi)|}{|\omega - \xi|^\alpha}$$

without further mention. We will write $P \lesssim Q$ to mean that $P \leq CQ$ for an absolute constant $C > 0$.

We shall henceforth assume that E is constructed so that

$$\sum_{k\geq 1} \delta_k^{2\alpha/d} < \infty. \tag{4.1}$$

Our first goal will be to show that the property (1.1) will ensure that $\|T_\mu(1)\|_{L^\infty(\mathbb{R}^d \setminus \operatorname{supp}\mu)} < \infty$. This will in turn follow from the weak convergence of μ^k to μ and the following proposition.

Proposition 4.1. *There is a constant $C > 0$ such that the following holds: Provided* $\operatorname{dist}(x, \operatorname{supp}(\mu)) = \epsilon > 0$, *then for any $m \in \mathbb{N}$ with $r_m < \frac{\epsilon}{2d}$,*

$$\left| \int K(x-\zeta) d\mu^m(\zeta) \right| \leq C.$$

Proof. To begin the proof of Proposition 4.1, fix $x^* \in \operatorname{supp}(\mu)$ with $\operatorname{dist}(x, x^*) = \epsilon$. Select m satisfying $r_m < \frac{\epsilon}{2d}$, and let n be the least integer such that $r_n \leq \epsilon$ (hence $m \geq n$). Observe that

$$\begin{aligned} \int K(x-\zeta) d\mu^m(\zeta) &= \int_{B^n(x^*)} K(x-\zeta) d\mu^m(\zeta) + \sum_{k=1}^{n} \int_{B^{k-1}(x^*) \setminus B^k(x^*)} K(x-\zeta) d\mu^m(\zeta) \\ &= A_1 + A_2. \end{aligned}$$

To estimate A_1 observe that every $\zeta \in \operatorname{supp}(\mu^m)$ is contained in a ball B_j^m of radius $(1+\delta_{m+1})r_m$. Therefore

$$\operatorname{dist}(x, \operatorname{supp}(\mu^m)) \geq \epsilon - (1+\delta_{m+1}) r_m \geq \frac{\epsilon}{2}. \tag{4.2}$$

Consequently, using property (b) of the measure μ^m,

$$|A_1| \leq \int_{B^n(x^*)} |K(x-\zeta)| d\mu^m(\zeta) \lesssim \frac{\mu^m(B^n(x^*))}{\epsilon^{d-2}} \lesssim \frac{r_n^{d-2}}{\epsilon^{d-2}} \lesssim 1.$$

To estimate the term A_2 we make the following claim: For some constant $C > 0$ such that for any $k \in \{1, 2..., n\}$,

$$\left| \int_{B^{k-1}(x^*) \setminus B^k(x^*)} K(x-\zeta) d\mu^m(\zeta) \right| \lesssim \delta_k^{\alpha 2/d} + \sqrt[d]{\frac{\epsilon^2}{r_{k-1}^2}}. \tag{4.3}$$

Employing (4.3) yields (recalling the assumption (4.1))

$$|A_2| \lesssim \sum_{k=1}^{n} \Big(\delta_k^{2\alpha/d} + \sqrt[d]{\frac{\epsilon^2}{r_{k-1}^2}} \Big) \lesssim 1,$$

and Proposition 4.1 follows. Therefore our goal will be to prove our claim (4.3). To do so we will appeal to the following comparison lemma.

Lemma 4.2. *Let $x_0 \in \mathbb{R}^d$. Fix $r, R \in (0,1]$ with r smaller than R. Let $Q \in \mathbb{R}^d$ be some cube centered at x_0 with the sidelength $l(Q) = \sqrt[d]{\kappa_d r^{d-2} R^2}$ and let $B = B(x_0, 2r)$. Suppose that ν_1, ν_2 are Borel measures with $\operatorname{supp}(\nu_1) \subset Q$, $\operatorname{supp}(\nu_2) \subset B$ and $\nu_1(\mathbb{R}^d) = \nu_2(\mathbb{R}^d)$. Then for any $x \in \mathbb{R}^d$ with $\operatorname{dist}(x, Q) \geq \frac{\sqrt[d]{\kappa_d r^{d-2} R^2}}{8}$, we have*

$$\left| \int_Q K(x-\zeta) d\nu_1(\zeta) - \int_B K(x-\zeta) d\nu_2(\zeta) \right| \lesssim \int_Q \frac{(r^{d-2}R^2)^{\alpha/d} d\nu_1(\zeta)}{|x-\zeta|^{d-2+\alpha}} + \int_B \frac{r^\alpha d\nu_2(\zeta)}{|x-\zeta|^{d-2+\alpha}}. \tag{4.4}$$

Proof. We can set $x_0 = 0$, without loss of generality. For any $\zeta \in Q$ and for any x such that $\operatorname{dist}(x, Q) \geq \frac{\sqrt[d]{\kappa_d r^{d-2} R^2}}{8}$, we have $|x| \approx |x - t\xi|$ for any $t \in [0,1]$, and so we have the standard kernel estimate

$$|K(x-\zeta)-K(x)| \lesssim \frac{|\zeta|^\alpha}{|x-\zeta|^{d-2+\alpha}}, \tag{4.5}$$

Therefore

$$\begin{aligned}
&\Big|\int_Q K(x-\zeta)d\nu_1(\zeta) - \int_B K(x-\zeta)d\nu_2(\zeta)\Big| \\
&\le \Big|\int_Q [K(x-\zeta)-K(x)]d\nu_1(\xi)\Big| + \Big|\int_B [K(x-\zeta)-K(x)]d\nu_2(\zeta)\Big| \\
&\lesssim \int_Q \frac{|\zeta|^\alpha}{|x-\zeta|^{d-2+\alpha}}d\nu_1(\zeta) + \int_B \frac{|\zeta|^\alpha}{|x-\zeta|^{d-2+\alpha}}d\nu_2(\zeta).
\end{aligned}$$

Observing that $|\zeta| \lesssim \sqrt[d]{r^{d-2}R^2}$ for $\xi \in Q$, while $|\zeta| \lesssim r$ for $\xi \in B$, completes the proof of lemma. □

Now we will proceed to prove the claim (4.3). Denote by $\mathcal{S}$ the collection

$$\mathcal{S} = \{j : B_j^k \neq B^k(x^*) \text{ and } B_j^k \subset B^{k-1}(x^*)\}.$$

First consider $j \in \mathcal{S}$ satisfying $\operatorname{dist}(x, Q_j^k) \ge \frac{\sqrt[d]{\kappa_d r_k^{d-2} r_{k-1}^2}}{8}$. In this case we apply Lemma 4.2 with $\nu_1 = \chi_{Q_j^k}\frac{m_d}{r_{k-1}^2}$ and $\nu_2 = \chi_{\widetilde{B}_j^k}\mu^m$, $R = r_{k-1}$, $r = r_k$ and $x_0 = x_{Q_j^k}$. This gives

$$\Big|\int_{Q_j^k} K(x-\zeta)\frac{dm_d(\zeta)}{r_{k-1}^2} - \int_{B_j^k} K(x-\zeta)d\mu^m(\zeta)\Big| \lesssim \frac{(r_k^{d-2}r_{k-1}^2)^{\alpha/d}}{r_{k-1}^2}\int_{Q_j^k}\frac{dm_d(\zeta)}{|x-\zeta|^{d-2+\alpha}} + \int_{B_j^k}\frac{r_k^\alpha d\mu^m(\zeta)}{|x-\zeta|^{d-2+\alpha}}, \tag{4.6}$$

There can be at most C indices $j \in \mathcal{S}$ satisfying $\operatorname{dist}(x, Q_j^k) \le \frac{\sqrt[d]{\kappa_d r_k^{d-2} r_{k-1}^2}}{8}$. For such a j we employ the elementary fact that for any set S with finite measure

$$\int_S |K(\zeta)|dm_d(\zeta) \lesssim \int_S \frac{1}{|\zeta|^{d-2}}dm_d(\zeta) \lesssim \sqrt[d]{(m_d(S))^2}. \tag{4.7}$$

Combined with (4.2) this results in

$$\Big|\int_{Q_j^k} K(x-\zeta)\frac{dm_d(\zeta)}{r_{k-1}^2} - \int_{B_j^k} K(x-\zeta)d\mu^m(\zeta)\Big| \lesssim \frac{\sqrt[d]{(m_d(Q_j^k))^2}}{r_{k-1}^2} + \frac{\mu^m(B_j^k)}{\epsilon^{d-2}}. \tag{4.8}$$

But now notice that

$$\frac{\sqrt[d]{\kappa_d r_k^{d-2} r_{k-1}^2}}{8} \ge \operatorname{dist}(x, Q_j^k) \ge \operatorname{dist}(x^*, Q_j^k) - d(x, x^*) \ge \frac{\sqrt[d]{\kappa_d r_k^{d-2} r_{k-1}^2}}{4} - \epsilon,$$

and hence we have $\epsilon \gtrsim \sqrt[d]{r_k^{d-2} r_{k-1}^2}$. Therefore we can bound

$$\frac{\sqrt[d]{(m_d(Q_j^k))^2}}{r_{k-1}^2} + \frac{\mu^m(B_j^k)}{\epsilon^{d-2}} \lesssim \Big(\frac{r_k}{r_{k-1}}\Big)^{2(d-2)/d} \lesssim \delta_k^2.$$

Altogether, we can therefore estimate

$$A_3 = \Big| \int_{\bigcup_{j\in\mathcal{S}} Q_j^k} K(x-\zeta)\frac{dm_d(\zeta)}{r_{k-1}^2} - \int_{\bigcup_{j\in\mathcal{S}} Q_j^k} K(x-\zeta)d\mu^m(\zeta)\Big|,$$

by a sum of two terms: The contribution to A_3 from j which satisfy $\text{dist}\,(x, Q_j^k) \geq \frac{\sqrt[d]{\kappa_d r_k^{d-2} r_{k-1}^2}}{8}$ is at most a constant multiple of

$$\frac{(r_k^{d-2}r_{k-1}^2)^{\alpha/d}}{r_{k-1}^2}\int_{B(x,2r_{k-1}\setminus B(x,\frac{\sqrt[d]{\kappa_d r_k^{d-2}r_{k-1}^2}}{8})} \frac{dm_d(\zeta)}{|x-\zeta|^{d-2+\alpha}} + \int_{\mathbb{R}^d\setminus B(x,\frac{\sqrt[d]{\kappa_d r_k^{d-2}r_{k-1}^2}}{8})} \frac{r_k^\alpha d\mu^m(\zeta)}{|x-\zeta|^{d-2+\alpha}},$$

which is in turn bounded by a constant multiple of $\big(\frac{r_k}{r_{k-1}}\big)^{\alpha(d-2)/d} + \big(\frac{r_k}{r_{k-1}}\big)^{2\alpha/d} \lesssim \delta_k^{2\alpha/d}$. The contribution from the remaining j is at most a constant multiple of δ_k^2, and so we arrive at

$$A_3 \lesssim \delta_k^{2\alpha/d}.$$

Now write

$$\Big| \int_{B^{k-1}(x^*)\setminus B^k(x^*)} K(x-\zeta)d\mu^m(\zeta)\Big| \leq A_3 + A_4,$$

where

$$A_4 = \Big| \int_{\bigcup_{j\in\mathcal{S}} Q_j^k} K(x-\zeta)\frac{dm_d(\zeta)}{r_{k-1}^2}\Big|.$$

Observe that

$$A_4 \leq \Big| \int_{\cup_{j\in\mathcal{S}} Q_j^k} K(x-\zeta)\frac{dm_d(\zeta)}{r_{k-1}^2} - \int_{B^{k-1}(x^*)} K(x-\zeta)\frac{dm_d(\zeta)}{r_{k-1}^2}\Big| + \Big| \int_{B^{k-1}(x^*)} K(x-\zeta)\frac{dm_d(\zeta)}{r_{k-1}^2}\Big|$$
$$:= A_5 + A_6.$$

To estimate the term A_5 we use the equation (4.7):

$$A_5 \lesssim \frac{1}{r_{k-1}^2} m_d(\cup_{j\in\mathcal{S}} Q_j^k \triangle B^{k-1}(x^*))^{\frac{2}{d}} \lesssim \sqrt[d]{\delta_k^2}. \tag{4.9}$$

Finally, to estimate the term A_6, notice that $x \in (1+\frac{\epsilon}{r_{k-1}^2})B^{k-1}(x^*)$. Using the reflectionless property in Lemma 2.1 and the inequality (4.7), we get

$$A_6 = \Big| \int_{(1+\frac{\epsilon}{r_{k-1}})B^{k-1}(x^*)\setminus \tilde{B}^{k-1}(x^*)} K(x-\zeta)\frac{dm_d(\zeta)}{r_{k-1}^2}\Big| \lesssim \frac{1}{r_{k-1}^2} m_d\bigg((1+\frac{\epsilon}{r_{k-1}^2})B^{k-1}(x^*)\setminus \tilde{B}^{k-1}(x^*)\bigg)^{\frac{2}{d}}$$
$$\lesssim \sqrt[d]{\Big(\delta_k + \frac{\epsilon}{r_{k-1}^2}\Big)^2}.$$

This gives the desired claim (4.3) hence finishes the proof for the proposition.

□

4.1 The potential is unbounded when the condition (1.1) fails

Now we will show that the potential associated to Ω is unbounded if the reflectionless property (2.1) (or equivalently the condition (1.1)) fails. So suppose that there exists some $x_0 \in B(0,1)$ such that $\int_{B(0,1)} K(x_0 - y) dm_d(y) > 0$. Hence, there exists some $r_0 > 0$ such that

$$\int_{B(0,1)} K(x-y) dm_d(y) > c_0 > 0, \quad \text{for every } x \in B(x_0, r_0) \subset B(0,1).$$

Recalling the notation of our set, put $G_j^n = x_j^n + r_n B(x_0, r_0)$ for all n. So we have

$$\int_{B_j^n} K(x-y) \frac{dm_d(y)}{r_n^2} > c_0 > 0, \text{ for all } x \in G_j^n.$$

Fix $N \in \mathbb{N}$, fix some $x \in E_{n+1} \cap \bigcup_j G_j^n$. A straightforward modification of the analysis of the previous section leads to

$$\int_{B^n(x) \setminus B^{n+1}(x)} K(x-y) d\mu(y) > c_0 - C\delta_n^{\frac{2\alpha}{d}},$$

and so, if $x \in \cap_{n=1}^N \bigcup_j G_n^j$, then by summing over all $n = 1, \ldots, N$ we get

$$\int_{\mathbb{R}^d \setminus B^{n+1}(x)} K(x-y) d\mu(y) > Nc_0 - C \sum_{n=1}^N \delta_n^{\frac{2\alpha}{d}}.$$

Consequently, the potential associated to Ω does not belong to in $L^\infty(\mathbb{R}^d \setminus \operatorname{supp}(\mu))$.

5 On the non-existence of the principal value integral

The result of this section will be rather general and be valid for all odd and even kernels (it will also not depend on the particular co-dimension, but we only state the result for co-dimension 2).

Theorem 5.1. *Suppose that $\Omega : \mathbb{S}^{d-1} \to \mathbb{R}$ is α-Hölder continuous and does not vanish identically on the sphere. Then provided the sequence $\frac{r_{n+1}}{r_n}$ converges to zero sufficiently quickly, the potential associated to Ω fails to exist in the sense of principal value, meaning that*

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B(x,\varepsilon)} \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^{d-2}} d\mu(y),$$

fails to exist for μ-almost every $x \in \mathbb{R}^d$.

The main estimate is the following lemma:

Lemma 5.2. *There exists $c_0 > 0$ denpending on Ω such that, provided n is sufficiently large, there is a ball $D_j^n \subset B_j^n$ with $m_d(D_j^n) \geq c_0 r_n^d$ such that if $z \in D_j^n$, then*

$$\left| \int_{B(z, tr_n) \setminus B(z, sr_n)} K(x-\zeta) d\mu(\zeta) \right| \geq c_0.$$

Before we exhibit the proof of this lemma, let us see how we get the Proposition 5.1 using this. To observe this, set $F = \{z \in E : z \notin \cup_j D_j^n \quad \text{for all but finitely many n}\}$ and then it suffices to show $\mu(F) = 0$. Also observe that by denoting by $F_n = \{z \in E : z \notin \cup_j D_j^m \quad \textit{for all} \quad m \geq n\}$, we have that $F \subset F_n$, so it will suffice to show that $\mu(F_n) = 0$ for all n.

To show this, note that there exists $d_0 > 0$ such that for all sufficiently large $m \geq 0$, at most $(1-d_0)\big(\frac{r_m}{r_{m+1}}\big)^{d-2}$ cubes Q_l^{m+1} fail to intersect D_j^m, thus

$$\mu\bigg(\bigcup_l \Big\{B_l^{m+1} : B_l^{m+1} \cap D_j^m = \emptyset\Big\}\bigg) \leq (1-d_0)\bigg(\frac{r_m}{r_{m+1}}\bigg)^{d-2} r_{m+1}^{d-2} \leq (1-d_0)\mu(B_j^m).$$

Whenever n is large enough, this inequality can be iterated to get

$$\mu(\{z \in E : z \notin D_j^{n+k} \quad \text{for} \quad k = 1, 2, ..., m\}) \leq (1-d_0)^m,$$

which shows that $\mu(F_n) = 0$.

In order to prove the Lemma 5.2, we use the following result, which relies on an application of Lemma 4.2.

Claim 5.3. *For $r, s \in (0,2)$. For $n \in \mathbb{Z}_+$ sufficiently large and any disc B_j^n, $z \in \mathbb{R}^d$, we have*

$$\bigg|\int_{B_j^n \cap [B(z,tr_n)\setminus B(z,sr_n)]} K(x-\zeta)d(\mu - \frac{m_d}{r_n^2})(\zeta)\bigg| \lesssim \delta_{n+1}^{\alpha}.$$

Proof. To prove this, let us first denote by $A(z,r_n) = B(z,tr_n) \setminus B(z,sr_n)$. Next we suppose that some cube $Q_l^{n+1} \in A(z,r_n)$. Then using Lemma 4.2, we get

$$\bigg|\int_{Q_l^{n+1}} K(x-\zeta)d(\mu - \frac{m_d}{r_n^2})(\zeta)\bigg| \lesssim \Big(\frac{r_{n+1}}{r_n}\Big)^{d-2+\alpha} + \Big(\frac{r_{n+1}}{r_n}\Big)^{(d-2)(1+\alpha/d)} \lesssim \Big(\frac{r_{n+1}}{r_n}\Big)^{(d-2)(1+\alpha/d)}.$$

If we instead have $Q_l^{n+1} \cap \partial A(z,r_n) \neq \emptyset$, then we have the crude estimate

$$\bigg|\int_{Q_l^{n+1}\cap A(z,r_n)} K(x-\zeta)d(\mu - \frac{m_d}{r_n^2})(\zeta)\bigg| \lesssim \Big(\frac{r_{n+1}}{r_n}\Big)^{d-2}.$$

There are at most $(\frac{r_n}{r_{n+1}})^{d-2}$ squares Q_l^{n+1} contained in $A(z,r_n)$ and observe that no more than $C\big(\frac{r_n}{r_{n+1}}\big)^{(d-2)(d-1)/d}$ cubes Q_l^{n+1} intersect $\partial A(z,r_n)$.

On the other hand observe that the set $\tilde{A}$ consisting of points in $A(z,r_n) \cap B_j^n$ that are not contained in any cube Q_l^{n+1} has measure that is no greater than

$$m_d\Big(\Big[\bigcup\{Q_\ell^{n+1} : Q_\ell^{n+1} \subset (1+\delta_{n+1})B_j^n\}\Big] \triangle B_j^n\Big) \lesssim \delta_{n+1} r_n^d.$$

Whence, $\int_{\tilde{A}} |K(z-\zeta)| \frac{dm_d(\zeta)}{r_n^2} \lesssim \delta_{n+1}$.

Bringing all these estimates together yields

$$\bigg|\int_{B_j^n \cap [B(z,tr_n)\setminus B(z,sr_n)]} K(x-\zeta)d(\mu - \frac{m_d}{r_n^2})(\zeta)\bigg| \lesssim \Big(\frac{r_{n+1}}{r_n}\Big)^{\alpha(d-2)/d} + \Big(\frac{r_{n+1}}{r_n}\Big)^{(d-2)/d} + \delta_{n+1} \lesssim \delta_{n+1}^{\alpha},$$

as required. □

Proof. If Ω does not vanish identically, then there exists $c > 0$, $z_0 \in B(0,1)$ and $r, s \in (0,1)$ such that

$$\int_{B(0,1)\cap[B(z_0,r)\setminus B(z_0,s)]} K(z-\zeta)dm_d(\zeta) \geq c.$$

(One can select z_0 close enough to 0 in the correct direction, $r = 2$ and $s = 1$.) Now observe next that have $\int_{B(0,1)\cap[B(z,tr_n)\setminus B(z,sr_n)]} K(z-\zeta)dm_d(\zeta)$ is a continuous function. Hence, then there exists some $\alpha > 0$, such that if $z \in B(z_0,\alpha) \subset B(0,1)$ we have

$$\left| \int_{B(0,1)\cap[B(z,t)\setminus B(z,s)]} K(z-\zeta)dm_d(\zeta) \right| \geq \frac{c}{2}.$$

Therefore, with $D_n^j = B(z_j^n + r_n z, \alpha r_n)$ we have

$$\left| \int_{B_j^n\cap B(z,tr_n)\setminus B(z,sr_n)} K(x-\zeta)\frac{dm_d(\zeta)}{r_n^2} \right| \geq \frac{c}{2} \text{ for every } z \in D_j^n.$$

Now applying the Claim 5.3, we deduce for all $z \in D_j^n$, $|\int_{B(z,tr_n)\setminus B(z,sr_n)} K(z-\zeta)d\mu(\zeta)| \geq c/2 - C\delta_{n+1}^{\alpha}$. Finally, note that the right hand side is atleast $\frac{c}{4}$ for n sufficiently large, and Lemma 5.2 is proved for a suitable choice of c_0. □

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