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GLOBAL SOLUTIONS FOR A FAMILY OF GSQG FRONT EQUATIONS

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Abstract. We prove the global existence of solutions with small and smooth initial data of a nonlinear dispersive equation for the motion of generalized surface quasi-geostrophic (GSQG) fronts in a parameter regime $1 < \alpha < 2$, where $\alpha = 1$ corresponds to the SQG equation and $\alpha = 2$ corresponds to the incompressible Euler equations. This result completes previous global well-posedness results for $0 < \alpha \leq 1$. We also use contour dynamics to derive the GSQG front equations for $1 < \alpha < 2$.

1. **Introduction.**

The inviscid generalized surface quasi-geostrophic (GSQG) equation is a two-dimensional transport equation for an active scalar

$$\theta_t(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla\theta(\mathbf{x}, t) = 0, \tag{1}$$

$$(-\Delta)^{\alpha/2}\mathbf{u}(\mathbf{x}, t) = \nabla^{\perp}\theta(\mathbf{x}, t), \tag{2}$$

where $0 < \alpha \leq 2$ is a parameter. Here, the scalar field $\theta\colon \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ is transported by the divergence-free velocity $\mathbf{u}\colon \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2$, which is determined nonlocally from θ by (2), $\mathbf{x} = (x, y)$ is the spatial variable, $\nabla^{\perp} = (-\partial_y, \partial_x)$, and $(-\Delta)^{\alpha/2}$ is a fractional Laplacian. When $\alpha = 2$, equations (1)–(2) correspond to the streamfunction-vorticity formulation of the two-dimensional incompressible Euler equations [41]. When $\alpha = 1$, these equations are the surface quasi-geostrophic (SQG) equation, which arises from oceanic and atmospheric science [40, 44] and has mathematical similarities with the three-dimensional incompressible Euler equations [8, 9].

The transport equation (1) is compatible with piecewise-constant solutions for θ, and the simplest class of such solutions is obtained when θ takes on two distinct values. As in [28], we distinguish between different geometries. We refer to patch solutions when one of the values is 0 and the support of θ is a simply connected, bounded set whose boundary is a simple closed curve; we refer to front solutions when the two different values of θ are taken on in half-spaces whose common boundary is a curve, or front, with infinite length. In this paper we consider front solutions.

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The advantage of these solutions over patch solutions is that their boundary geometry is simpler, especially when the front can be represented as a graph, although the lack of compact support in θ introduces additional complications in the formulation of front equations.

Smooth and analytic solutions for spatially periodic SQG fronts are proved to be locally well-posed in [18, 45], and almost-sharp SQG fronts are studied in [10, 17, 19, 20]. Global $C^{1,\varepsilon}$ regularity of the spatially periodic Euler fronts is proved in [4]. Contour dynamics equations for GSQG fronts with $0 < \alpha < 1$ are straightforward to derive because the standard potential representation for $\mathbf{u}$ converges even though θ does not have compact support, and Cordoba et. al. [11] prove that flat planar fronts are asymptotically stable in that case. However, the same derivation does not work when $1 \le \alpha \le 2$ because the Riesz potential [46, 47] for $(-\Delta)^{-\alpha/2}$ decays too slowly at infinity for the straightforward potential representation of $\mathbf{u}$ to converge.

A derivation of front contour dynamics equations for $1 \le \alpha \le 2$ by a regularization procedure is given in [25], and the same equations are derived in [29] for SQG fronts with $\alpha = 1$ by decomposing the velocity field into a planar shear flow and a perturbation due to the front motion with an absolutely convergent potential representation. We provide a similar derivation of GSQG front equations with $1 < \alpha < 2$ in the appendix of the present paper. A derivation for Euler fronts with $\alpha = 2$ is given in [24]. The local well-posedness of a cubically nonlinear approximation for SQG fronts is proved in [26], and flat planar SQG fronts are shown to be globally asymptotically stable in [1, 27]. A similar idea is also used in proving globally asymptotically stability of the quasi-geostrophic shallow water front [48]. A related two-front GSQG problem is studied in [28].

This paper is concerned with the regime $1 < \alpha < 2$. We assume that the front is a graph $y = \varphi(x,t)$ and study piecewise-constant distributional solutions of the GSQG equations (1)–(2) with

$$\theta(\mathbf{x},t) = \begin{cases} \theta_+ & \text{if } y > \varphi(x,t), \\ \theta_- & \text{if } y < \varphi(x,t), \end{cases} \tag{3}$$

where θ_+, θ_- are distinct constants. The graph assumption greatly simplifies the evolution equation for the front, but it does not describe wave-breaking or filamentation of the front. However, for the small-slope fronts we study in this paper, we will prove that wave-breaking never occurs.

In Appendix A, we show that for $1 < \alpha < 2$ the front location satisfies the evolution equation

$$\begin{aligned} &\varphi_t(x,t) + \Theta A|\partial_x|^{1-\alpha}\varphi_x(x,t) = \\ &- \Theta \int_{\mathbb{R}} [\varphi_x(x,t) - \varphi_x(x+\zeta,t)] \left\{ \frac{1}{|\zeta|^{2-\alpha}} - \frac{1}{(\zeta^2 + [\varphi(x,t) - \varphi(x+\zeta,t)]^2)^{(2-\alpha)/2}} \right\} \mathrm{d}\zeta, \end{aligned} \tag{4}$$

where $|\partial_x|^{1-\alpha}$ denotes the Fourier multiplier with symbol $|\xi|^{1-\alpha}$, and

$$\Theta = g_\alpha(\theta_+ - \theta_-), \qquad A = 2\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha-1), \qquad g_\alpha = \frac{\Gamma(1-\alpha/2)}{2^\alpha \pi \Gamma(\alpha/2)}. \tag{5}$$

Our main result, stated in Theorem 4.1, is that the Cauchy problem on $\mathbb{R}$ for the GSQG front equation (4) with sufficiently small and smooth initial data has smooth solutions globally in time. Together with [11] for $0 < \alpha < 1$ and [27] for $\alpha = 1$, this completes the proof of asymptotic stability of planar GSQG fronts in the entire range $0 < \alpha < 2$.

Our proof follows the ones in [11, 27] with improvements on the regularity requirement of the initial data by means of a more detailed analysis of the nonresonant and resonant interactions of the high-frequency components in Section 7.3 and Section 7.5. These high-frequency components were previously controlled using a cruder high-order Sobolev energy estimate, which required a large value of s.

A similar reduction in s could be made to our previous result for SQG fronts [27], where we took $s = 1200$. These results are still not optimal, and Ai and Avadanei have recently proved low-regularity results for the SQG [2] and GSQG [3] front equations.

The GSQG front equation is of order $2-\alpha$ for $\alpha \neq 1$, and in making standard Sobolev energy estimates we can gain one derivative by use of a Kato-Ponce commutator estimate [38]. For $\alpha \in (0,1)$ the fractional derivative loss is greater than 1, and for the SQG case $\alpha = 1$ there is a logarithmic loss of derivatives. To control this loss, we need to use a weighted energy, which can be constructed by means of para-differential calculus. In the current case, when $\alpha \in (1,2)$, the order of derivative loss is less than one, and standard Sobolev energy estimates suffice. Thus, in contrast to [11, 27] with $\alpha \in (0,1]$, we do not need to construct a weighted energy, and para-differential calculus is not required.

The linear part of the equation provides $t^{-1/2}$ decay for the L^∞-norm of the solution, but this is not sufficient to close the global energy estimates for the full equation, since the $O(t^{-1})$ contribution from the cubically nonlinear term is not integrable in time. We therefore use the method of space-time resonances introduced by Germain, Masmoudi and Shatah [21, 22, 23], together with estimates for weighted L^∞_ξ-norms — the so-called Z-norms — developed by Ionescu and his collaborators [11, 12, 13, 33, 34, 35, 36].

We remark that the Euler front equations with $\alpha = 2$ are nondispersive [5, 25], and — in the absence of dispersive decay — one cannot expect to get global smooth solutions. In that case, numerical solutions of the full contour dynamics equations [14] indicate that the graphical description of the front may fail in finite time, after which the front breaks and forms extremely thin filaments, similar to the ones that are observed in patches [15].

The rest of the paper is organized as follows. In Section 2, we review results from Fourier analysis and state some estimates for multilinear Fourier integral operators. In Section 3, we carry out a multilinear expansion of the nonlinearity in (4) and discuss the structure of the equation, which enables us to derive improved energy estimates. In Section 4, we state the main global theorem and outline the steps in the proof of the theorem. Finally, in Sections 5–7 we provide the proofs of each step.

2. **Preliminaries.**

In this section, we summarize some notations and lemmas that we will use below.

We denote the Fourier transform of $f\colon \mathbb{R} \to \mathbb{C}$ by $\hat{f}\colon \mathbb{R} \to \mathbb{C}$, where $\hat{f} = \mathcal{F}f$ is given by

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x}\,\mathrm{d}\xi, \qquad \hat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x) e^{-i\xi x}\,\mathrm{d}x.$$

For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R})$ the space of Schwartz distributions f with $\|f\|_{H^s} < \infty$, where

$$\|f\|_{H^s} = \left[\int_{\mathbb{R}} (1+\xi^2)^s |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right]^{1/2}.$$

Throughout this paper, we use $A \lesssim B$ to mean there is a constant C such that $A \leq CB$, and $A \gtrsim B$ to mean there is a constant C such that $A \geq CB$. We remark that the constant C may depend on α. We use $A \approx B$ to mean that $A \lesssim B$ and $B \lesssim A$. The notation $\text{Ø}(f)$ denotes a term satisfying

$$\|\text{Ø}(f)\|_{H^s} \lesssim \|f\|_{H^s}$$

whenever there exists $s \in \mathbb{R}$ such that $f \in H^s$. We also use $O(f)$ to denote a term satisfying $|O(f)| \lesssim |f|$ pointwise.

Let $\psi\colon \mathbb{R} \to [0,1]$ be a smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For any $k \in \mathbb{Z}$, we define

$$\begin{aligned}
&\psi_k(\xi) = \psi(\xi/2^k) - \psi(\xi/2^{k-1}), \qquad \psi_{\leq k}(\xi) = \psi(\xi/2^k), \qquad \psi_{\geq k}(\xi) = 1 - \psi(\xi/2^{k-1}),\\
&\tilde{\psi}_k(\xi) = \psi_{k-1}(\xi) + \psi_k(\xi) + \psi_{k+1}(\xi),
\end{aligned} \tag{6}$$

and denote by P_k, $P_{\leq k}$, $P_{\geq k}$, and $\tilde{P}_k$ the Fourier multiplier operators with symbols $\psi_k, \psi_{\leq k}, \psi_{\geq k}$, and $\tilde{\psi}_k$, respectively. Notice that $\psi_k(\xi) = \psi_0(\xi/2^k)$, $\tilde{\psi}_k(\xi) = \tilde{\psi}_0(\xi/2^k)$, and

$$\|\psi_k\|_{L^2} \approx 2^{k/2}, \qquad \|\psi_k'\|_{L^2} \approx 2^{-k/2}. \tag{7}$$

Using this notation, we define the dyadic components of a function φ by

$$\varphi_k = P_k\varphi, \quad \hat{\varphi}_k = \psi_k\hat{\varphi}. \tag{8}$$

The proof of the following interpolation lemma can be found in [36].

Lemma 2.1. *For any $k \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$, we have*

$$\|\widehat{P_k f}\|_{L^\infty}^2 \lesssim \|P_k f\|_{L^1}^2 \lesssim 2^{-k}\|\hat{f}\|_{L^2_\xi}\left[2^k\|\partial_\xi \hat{f}\|_{L^2_\xi} + \|\hat{f}\|_{L^2_\xi}\right].$$

Next, we state an estimate for multilinear Fourier multipliers proved in [35]. Define a class S^∞ of symbols by

$$S^\infty = \{\kappa\colon \mathbb{R}^m \to \mathbb{C} \mid \kappa \text{ is continuous and } \|\kappa\|_{S^\infty} < \infty\} \qquad \|\kappa\|_{S^\infty} = \|\mathcal{F}^{-1}(\kappa)\|_{L^1}, \tag{9}$$

and given $\kappa \in S^\infty$, define a multilinear operator M_κ acting on Schwartz functions $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R})$ by

$$M_\kappa(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^m} e^{ix(\xi_1 + \dots + \xi_m)}\kappa(\xi_1, \dots, \xi_m)\hat{f}_1(\xi_1)\cdots\hat{f}_m(\xi_m)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_m.$$

Lemma 2.2. *(i) If $\kappa_1, \kappa_2 \in S^\infty$, then $\kappa_1\kappa_2 \in S^\infty$.*

(ii) Suppose that $1 \leq p_1, \dots, p_m \leq \infty$, $1 \leq p \leq \infty$, satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} = \frac{1}{p}.$$

If $\kappa \in S^\infty$, then

$$\|M_\kappa\|_{L^{p_1}\times\dots\times L^{p_m} \to L^p} \lesssim \|\kappa\|_{S^\infty}.$$

(iii) Assume $p, q, r \in [1, \infty]$ satisfy $1/p + 1/q + 1/r = 1$, and $m \in S^\infty_{\eta_1,\eta_2}L^\infty_\xi$. Then, for any $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, and $h \in L^r(\mathbb{R})$,

$$\left\| \int_{\mathbb{R}^2} m(\eta_1, \eta_2, \xi) \hat{f}(\eta_1) \hat{g}(\eta_2) \hat{h}(\xi - \eta_1 - \eta_2) \, \mathrm{d}\eta_1 \, \mathrm{d}\eta_2 \right\|_{L^\infty_\xi}$$
$$\lesssim \|m\|_{S^\infty_{\eta_1,\eta_2} L^\infty_\xi} \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

We define a $B^{a,b}$ semi-norm by

$$\|f\|_{B^{a,b}} = \sum_{j\in\mathbb{Z}} (2^{aj} + 2^{bj}) \|P_j f\|_{L^\infty}. \tag{10}$$

Here $0 \le a \le b$ and a, b are indices of higher and lower frequencies. Compared with the L^∞ norm, $B^{a,b}$ satisfies one property not shared with the L^∞ norm, that is

$$\|f\|_{B^{a,b}} \approx \sum_{j\in\mathbb{Z}} \|P_j f\|_{B^{a,b}}.$$

When $a = b$, we write the semi-norm as $B^a := B^{a,a}$ for short.

3. **Structure of the equation and energy estimate.** In this section, we derive an *a priori* energy estimate for the GSQG front equation. Local well-posedness then follows by standard arguments for quasi-linear evolution equations [32, 37], similar to the ones in [27] for local solutions of the SQG front equation.

When $1 < \alpha < 2$, standard Sobolev H^s-energy estimates for (4) follow in a similar way to the ones proved in [28] for the two-front GSQG equations. The result is that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(t)\|^2_{H^s} \lesssim P(\|\varphi(t)\|^2_{H^s})$$

for $s \ge 4$, where P is a smooth, nonnegative, increasing function.

In this section, we use a dyadic frequency decomposition to prove an improved energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\varphi(t)\|^2_{H^s} \lesssim \|\varphi(t)\|^2_{H^s} \sum_{n=1}^{\infty} \left(c_n \|\varphi\|^{2n}_{B^{2-\alpha,2+\delta}} \right),$$

where $s > 5/2$ and $0 < \delta < s - 5/2$, in which case H^s is continuously embedded in $B^{2-\alpha,2+\delta}$. We will prove the time decay of the $B^{2-\alpha,2+\delta}$-norm in the following sections, and use it to prove global existence.

Without loss of generality, we fix $\Theta = -1$ in the following.

Assuming that $|\varphi_x| \ll 1$, we first carry out a multilinear expansion of the nonlinear term, in a similar way to [11, 27]. Omitting the details, we find by a Taylor expansion that (4) can be written as

$$\varphi_t(x, t) - A|\partial_x|^{1-\alpha} \varphi_x(x, t) + \partial_x \mathcal{N}(\varphi) = 0, \tag{11}$$

where $\mathcal{N}(\varphi) = \sum\limits_{n=1}^{\infty} \mathcal{N}_{2n+1}(\varphi)$, with

$$\mathcal{F}\mathcal{N}_{2n+1}(\varphi)(\xi) = \frac{c_n}{2n+1} \iint_{\mathbb{R}^{2n}} \mathbf{T}_n(\eta_1, \cdots, \eta_{2n}, \xi - \eta_1 - \cdots - \eta_{2n}) \hat{\varphi}(\eta_1) \cdots \hat{\varphi}(\eta_{2n}) \hat{\varphi}(\xi - \eta_1 - \cdots - \eta_{2n}) \, \mathrm{d}\boldsymbol{\eta}_n.$$

Here $\boldsymbol{\eta}_n = (\eta_1, \eta_2, \dots, \eta_{2n})$, and the symbol $\mathbf{T}_n$ is given by

$$\mathbf{T}_n(\eta_1, \cdots, \eta_{2n+1}) = \int_{\mathbb{R}} \frac{\prod_{j=1}^{2n+1} \left(1 - e^{i\eta_j \zeta}\right)}{\zeta^{2n+1}} |\zeta|^{\alpha-1} \operatorname{sgn} \zeta \, \mathrm{d}\zeta, \;\; c_n = \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma(n+1)\Gamma\left(\frac{\alpha}{2} - n\right)}. \tag{12}$$

We have suppressed the dependence of $\hat{\varphi}(\xi,t)$ on t in the expression for $\mathcal{F}\mathcal{N}_{2n+1}$ in order to save space. We will use this convention in the whole paper.

By taking the dyadic decomposition of φ in (8) we get

$$\begin{aligned}\mathcal{F}\mathcal{N}_{2n+1}(\varphi)(\xi) =&\frac{c_n}{2n+1}\sum_{j_1,\cdots,j_{2n+1}\in\mathbb{Z}}\iint_{\mathbb{R}^{2n}}\mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\\ &\cdot\hat{\varphi}_{j_1}(\eta_1)\cdots\hat{\varphi}_{j_{2n}}(\eta_{2n})\hat{\varphi}_{j_{2n+1}}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n}.\end{aligned}$$

Since the symbol $\mathbf{T}_n$ is a symmetric function, we can assume by a change of variables that $j_1\geq j_2\geq\cdots\geq j_{2n+1}$. We denote the summation over these ordered indices by $\sum\limits_{Q}$, so that

$$\begin{aligned}\mathcal{F}\mathcal{N}_{2n+1}(\varphi)(\xi) =&\frac{c_n}{(2n+1)}\sum_{Q}\iint_{\mathbb{R}^{2n}}\mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\\ &\cdot\hat{\varphi}_{j_1}(\eta_1)\cdots\hat{\varphi}_{j_{2n}}(\eta_{2n})\hat{\varphi}_{j_{2n+1}}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n}.\end{aligned}$$

For the higher-order energy estimate, we multiply the Fourier transform of (11) by $(1+|\xi|^2)^s\bar{\hat{\varphi}}(\xi,t)$, take the real part, and integrate with respect to ξ, to get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{1}{2}(1+|\xi|^2)^s|\hat{\varphi}(\xi,t)|^2\,\mathrm{d}\xi\\ &=-\sum_{n=1}^{\infty}\frac{c_n}{2n+1}i\sum_{Q}\iint_{\mathbb{R}^{2n+1}}\xi(1+|\xi|^2)^s\mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\\ &\qquad\cdot\hat{\varphi}_{j_1}(\eta_1)\hat{\varphi}_{j_2}(\eta_2)\cdots\hat{\varphi}_{j_{2n}}(\eta_{2n})\hat{\varphi}_{j_{2n+1}}(\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(-\xi)\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n}\,\mathrm{d}\xi.\end{aligned}$$

Furthermore, making the change of variables $\eta_{2n+1}=\eta_{2n+1}(\eta_1)=\xi-\eta_1-\cdots-\eta_{2n}$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{1}{2}(1+|\xi|^2)^s|\hat{\varphi}(\xi,t)|^2\,\mathrm{d}\xi\\ &=-\sum_{n=1}^{\infty}\frac{c_n}{2n+1}i\sum_{Q}\iint_{\mathbb{R}^{2n+1}}\xi(1+|\xi|^2)^s\,\mathbf{T}_n(\xi-\eta_2-\cdots-\eta_{2n+1},\eta_{2n},\cdots,\eta_{2n+1})\\ &\quad\hat{\varphi}_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})\hat{\varphi}_{j_2}(\eta_2)\cdots\hat{\varphi}_{j_{2n+1}}(\eta_{2n+1})\hat{\varphi}(-\xi)\,\mathrm{d}\eta_2\cdots\mathrm{d}\eta_{2n+1}\,\mathrm{d}\xi.\end{aligned}\tag{13}$$

When $j_1\leq 0$, all the frequencies are bounded. By Proposition B.1, the nonlinear terms satisfy

$$\begin{aligned}&\left|\iint_{\mathbb{R}^{2n+1}}\xi(1+|\xi|^2)^s\,\mathbf{T}_n(\xi-\eta_2-\cdots-\eta_{2n+1},\eta_{2n},\cdots,\eta_{2n+1})\right.\\ &\qquad\left.\hat{\varphi}_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})\hat{\varphi}_{j_2}(\eta_2)\cdots\hat{\varphi}_{j_{2n+1}}(\eta_{2n+1})\hat{\varphi}(-\xi)\,\mathrm{d}\eta_2\cdots\mathrm{d}\eta_{2n+1}\,\mathrm{d}\xi\right|\\ &\lesssim 2^{j_1+(2-\alpha)j_2+j_3+\cdots+j_{2n+1}}(1+\ln n)\|\varphi_{j_1}\|_{L^2}\|\varphi\|_{L^2}\|\varphi_{j_2}\|_{L^\infty}\cdots\|\varphi_{j_{2n+1}}\|_{L^\infty}.\end{aligned}\tag{14}$$

When $j_1>0$, we rewrite the symbol as

$$\begin{aligned}&\mathbf{T}_n(\eta_1,\cdots,\eta_{2n+1})\\ &=-\int_{\mathbb{R}}|\zeta|^{\alpha-1}\mathrm{sgn}\,\zeta\int_{[0,1]^{2n+1}}\prod_{j=1}^{2n+1}i\eta_j e^{i\eta_j s_j\zeta}\,\mathrm{d}\mathbf{s}_n\,\mathrm{d}\zeta\end{aligned}$$

$$
\begin{aligned}
&= - 2i\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)\prod_{j=1}^{2n+1}(i\eta_j)\int_{[0,1]^{2n+1}}\left|\sum_{j=1}^{2n+1}\eta_j s_j\right|^{-\alpha}\operatorname{sgn}\left(\sum_{j=1}^{2n+1}\eta_j s_j\right)\mathrm{d}\mathbf{s}_n\\
&= - 2\Gamma(\alpha-1)\sin\left(\frac{\pi\alpha}{2}\right)|\eta_1|^{1-\alpha}\prod_{j=2}^{2n+1}(i\eta_j)\int_{[0,1]^{2n}}\left|1+\sum_{j=2}^{2n+1}\frac{\eta_j}{\eta_1}s_j\right|^{1-\alpha}\\
&\quad - \left|\sum_{j=2}^{2n+1}\frac{\eta_j}{\eta_1}s_j\right|^{1-\alpha}\mathrm{d}\hat{\mathbf{s}}_n\\
&=\mathbf{T}_n^{1-\alpha}(\eta_1,\cdots,\eta_{2n+1})+\mathbf{T}_n^{\leq -1}(\eta_1,\cdots,\eta_{2n+1})
\end{aligned}
$$

where $\mathbf{s}_n=(s_1,\hat{\mathbf{s}}_n)=(s_1,s_2,\ldots,s_{2n+1})$,

$$
\mathbf{T}_n^{1-\alpha}(\eta_1,\cdots,\eta_{2n+1})=-2\Gamma(\alpha-1)\sin\left(\frac{\pi\alpha}{2}\right)|\eta_1|^{1-\alpha}\prod_{j=2}^{2n+1}(i\eta_j),
$$

$$
\begin{aligned}
\mathbf{T}_n^{\leq -1}(\eta_1,\cdots,\eta_{2n+1})=&- 2\Gamma(\alpha-1)\sin\left(\frac{\pi\alpha}{2}\right)|\eta_1|^{1-\alpha}\prod_{j=2}^{2n+1}(i\eta_j)\\
&\cdot\int_{[0,1]^{2n}}\left|1+\sum_{j=2}^{2n+1}\frac{\eta_j}{\eta_1}s_j\right|^{1-\alpha}-1-\left|\sum_{j=2}^{2n+1}\frac{\eta_j}{\eta_1}s_j\right|^{1-\alpha}\mathrm{d}\hat{\mathbf{s}}_n.
\end{aligned}
$$

Here $\mathbf{T}_n^{\leq -1}(\eta_1,\cdots,\eta_{2n+1})$ is the lower-order symbol satisfying

$$
\|\mathbf{T}_n^{\leq -1}(\eta_1,\cdots,\eta_{2n+1})\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\|_{S^\infty}\lesssim 2^{(-\alpha)j_1+2j_2+j_3+\cdots+j_{2n+1}},
$$

and $\mathbf{T}_n^{1-\alpha}(\eta_1,\cdots,\eta_{2n+1})$ is a symbol of order $(1-\alpha)$ with respect the highest frequency

$$
\|\mathbf{T}_n^{1-\alpha}(\eta_1,\cdots,\eta_{2n+1})\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\|_{S^\infty}\lesssim 2^{(1-\alpha)j_1+j_2+j_3+\cdots+j_{2n+1}}. \tag{15}
$$

Therefore

$$
\begin{aligned}
&\Bigg|\iint_{\mathbb{R}^{2n+1}}\xi(1+|\xi|^2)^s\,\mathbf{T}_n^{\leq -1}(\xi-\eta_2-\cdots-\eta_{2n+1},\eta_{2n},\cdots,\eta_{2n+1})\\
&\quad\hat{\varphi}_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})\hat{\varphi}_{j_2}(\eta_2)\cdots\hat{\varphi}_{j_{2n+1}}(\eta_{2n+1})\hat{\varphi}(-\xi)\,\mathrm{d}\eta_2\cdots\mathrm{d}\eta_{2n+1}\,\mathrm{d}\xi\Bigg|\\
&\lesssim 2^{(2s+1-\alpha)j_1+2j_2+j_3+\cdots+j_{2n+1}}\|\varphi_{j_1}\|_{L^2}\|\varphi\|_{L^2}\|\varphi_{j_2}\|_{L^\infty}\cdots\|\varphi_{j_{2n+1}}\|_{L^\infty}.
\end{aligned}
$$

To estimate

$$
\begin{aligned}
&\Bigg|\iint_{\mathbb{R}^{2n+1}}\xi(1+|\xi|^2)^s\,\mathbf{T}_n^{1-\alpha}(\xi-\eta_2-\cdots-\eta_{2n+1},\eta_{2n},\cdots,\eta_{2n+1})\\
&\quad\hat{\varphi}_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})\hat{\varphi}_{j_2}(\eta_2)\cdots\hat{\varphi}_{j_{2n+1}}(\eta_{2n+1})\hat{\varphi}(-\xi)\,\mathrm{d}\eta_2\cdots\mathrm{d}\eta_{2n+1}\,\mathrm{d}\xi\Bigg|,
\end{aligned}
$$

we use a symmetry argument, which is equivalent to a Kato-Ponce commutator estimate carried out in frequency space. Interchanging the variables $-\xi$ and $\xi-\eta_2-\cdots-\eta_{2n+1}$, and then taking the average, we have

$$
\iint_{\mathbb{R}^{2n+1}}\xi(1+|\xi|^2)^s\,\mathbf{T}_n^{1-\alpha}(\xi-\eta_2-\cdots-\eta_{2n+1},\eta_2,\cdots,\eta_{2n+1})
$$

$$
\begin{aligned}
&\hat{\varphi}_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})\hat{\varphi}_{j_2}(\eta_2)\cdots\hat{\varphi}_{j_{2n+1}}(\eta_{2n+1})\hat{\varphi}(-\xi)\,\mathrm{d}\eta_2\cdots\mathrm{d}\eta_{2n+1}\,\mathrm{d}\xi\\
=&\iint_{\mathbb{R}^{2n+1}} -(\xi-\eta_2-\cdots-\eta_{2n+1})(1+|\xi-\eta_2-\cdots-\eta_{2n+1}|^2)^s\\
&\quad\cdot \mathbf{T}_n^{1-\alpha}(-\xi,\eta_2,\cdots,\eta_{2n+1})\\
&\quad\cdot \hat{\varphi}_{j_1}(-\xi)\hat{\varphi}_{j_2}(\eta_2)\cdots\hat{\varphi}_{j_{2n+1}}(\eta_{2n+1})\hat{\varphi}(\xi-\eta_2-\cdots-\eta_{2n+1})\,\mathrm{d}\eta_2\cdots\mathrm{d}\eta_{2n+1}\,\mathrm{d}\xi\\
=&\frac{1}{2}\iint_{\mathbb{R}^{2n+1}}\Big[\xi(1+|\xi|^2)^s\mathbf{T}_n^{1-\alpha}(\xi-\eta_2-\cdots-\eta_{2n+1},\eta_2,\cdots,\eta_{2n+1})\\
&\cdot\psi_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})-(\xi-\eta_2-\cdots-\eta_{2n+1})(1+|\xi-\eta_2-\cdots-\eta_{2n+1}|^2)^s\\
&\cdot\mathbf{T}_n^{1-\alpha}(-\xi,\eta_2,\cdots,\eta_{2n+1})\psi_{j_1}(-\xi)\Big]\\
&\cdot\hat{\varphi}_{j_2}(\eta_2)\cdots\hat{\varphi}_{j_{2n+1}}(\eta_{2n+1})\hat{\varphi}(-\xi)\hat{\varphi}(\xi-\eta_2-\cdots-\eta_{2n+1})\,\mathrm{d}\eta_2\cdots\mathrm{d}\eta_{2n+1}\,\mathrm{d}\xi.
\end{aligned}
$$

We can split the symbol into three parts

$$
\begin{aligned}
&\Big[\xi(1+|\xi|^2)^s\mathbf{T}_n^{1-\alpha}(\xi-\eta_2-\cdots-\eta_{2n+1},\eta_2,\cdots,\eta_{2n+1})\psi_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})\\
&-(\xi-\eta_2-\cdots-\eta_{2n+1})(1+|\xi-\eta_2-\cdots-\eta_{2n+1}|^2)^s\mathbf{T}_n^{1-\alpha}(-\xi,\eta_2,\cdots,\eta_{2n+1})\\
&\psi_{j_1}(-\xi)\Big]\\
=&\Big(\xi(1+|\xi|^2)^s-(\xi-\eta_2-\cdots-\eta_{2n+1})(1+|\xi-\eta_2-\cdots-\eta_{2n+1}|^2)^s\Big)\\
&\cdot\mathbf{T}_n^{1-\alpha}(\xi-\eta_2-\cdots-\eta_{2n+1},\eta_2,\cdots,\eta_{2n+1})\psi_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})\\
&+(\xi-\eta_2-\cdots-\eta_{2n+1})(1+|\xi-\eta_2-\cdots-\eta_{2n+1}|^2)^s\psi_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})\\
&\cdot\Big(\mathbf{T}_n^{1-\alpha}(\xi-\eta_2-\cdots-\eta_{2n+1},\eta_2,\cdots,\eta_{2n+1})-\mathbf{T}_n^{1-\alpha}(-\xi,\eta_2,\cdots,\eta_{2n+1})\Big)\\
&+(\xi-\eta_2-\cdots-\eta_{2n+1})(1+|\xi-\eta_2-\cdots-\eta_{2n+1}|^2)^s\mathbf{T}_n^{1-\alpha}(-\xi,\eta_2,\cdots,\eta_{2n+1})\\
&\cdot\Big(\psi_{j_1}(\xi-\eta_2-\cdots-\eta_{2n+1})-\psi_{j_1}(-\xi)\Big).
\end{aligned}
$$

When writing it as the symbol of a multilinear Fourier integral operator, we use η_1 to replace $\xi-\eta_2-\cdots-\eta_{2n+1}$. By (15) and the algebraic property of S^∞ norm,

$$
\begin{aligned}
&\left\|\left(\left(\sum_{i=1}^{2n+1}\eta_i\right)\left(1+\left|\sum_{i=1}^{2n+1}\eta_i\right|^2\right)^s-\eta_1\left(1+|\eta_1|^2\right)^s\right)\cdot\mathbf{T}_n^{1-\alpha}(\eta_1,\cdots,\eta_{2n},\eta_{2n+1})\right.\\
&\qquad\left.\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n}}(\eta_{2n})\psi_{j_{2n+1}}(\eta_{2n+1})\right\|_{S^\infty}\\
&\lesssim(1+2^{2sj_1})2^{j_2}2^{(2-\alpha)j_2+j_3+\cdots+j_{2n+1}}(1+\ln n).
\end{aligned}
$$

Since the symbol $\mathbf{T}_n^{1-\alpha}$ is real, and it is an even function with η_1, i.e.

$$
\mathbf{T}_n^{1-\alpha}(\eta_1,\eta_2,\cdots,\eta_{2n+1})=\mathbf{T}_n^{1-\alpha}(-\eta_1,\eta_2,\cdots,\eta_{2n+1}),
$$

we have

$$
\begin{aligned}
&\left\|\eta_1\left(1+|\eta_1|^2\right)^s\cdot\Big(\mathbf{T}_n^{1-\alpha}(\eta_1,\eta_2,\cdots,\eta_{2n+1})-\mathbf{T}_n^{1-\alpha}(-\sum_{i=1}^{2n+1}\eta_i,\eta_2,\cdots,\eta_{2n+1})\Big)\right.\\
&\qquad\left.\cdot\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n}}(\eta_{2n})\psi_{j_{2n+1}}(\eta_{2n+1})\right\|_{S^\infty}
\end{aligned}
$$

$$= \Big\| \eta_1(1+|\eta_1|^2)^s \cdot \Big(\mathbf{T}_n^{1-\alpha}(\eta_1,\eta_2,\cdots,\eta_{2n+1}) - \mathbf{T}_n^{1-\alpha}(\sum_{i=1}^{2n+1}\eta_i,\eta_2,\cdots,\eta_{2n+1}) \Big)$$
$$\cdot \psi_{j_1}(\eta_1)\cdots\psi_{j_{2n}}(\eta_{2n})\psi_{j_{2n+1}}(\eta_{2n+1}) \Big\|_{S^\infty}$$
$$\lesssim n(1+2^{2sj_1}) \prod_{i=2}^{2n+1} 2^{j_i}(1+2^{(1+\delta)j_2}),$$

where $\delta > 0$ is arbitrarily small. Also, using $\psi'_{j_1}(\xi) = \frac{1}{2^{j_1}}\psi'(\xi/2^{j_1})$ and Proposition B.1 we get

$$\Big\| \eta_1(1+|\eta_1|^2)^s \mathbf{T}_n^{1-\alpha}(-\sum_{i=1}^{2n+1}\eta_i,\eta_2\cdots,\eta_{2n+1})\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})$$
$$\cdot \big(\psi_{j_1}(\eta_1) - \psi_{j_1}(-\sum_{i=1}^{2n+1}\eta_i)\big) \Big\|_{S^\infty}$$
$$\lesssim 2^{2sj_1+((2-\alpha)j_2+j_3+\cdots+j_{2n+1})}(1+\ln n).$$

Combining the above estimates, we have

$$\Bigg\| \left(\sum_{i=1}^{2n+1}\eta_i\right)\left(1+\left|\sum_{i=1}^{2n+1}\eta_i\right|^2\right)^s \mathbf{T}_n(\eta_1,\eta_2,\cdots,\eta_{2n+1})\psi_{j_1}(\eta_1)$$
$$-(\eta_1)(1+|\eta_1|^2)^s\mathbf{T}_n\left(-\sum_{i=1}^{2n+1}\eta_i,\eta_2,\cdots,\eta_{2n+1}\right)\psi_{j_1}\left(-\sum_{i=1}^{2n+1}\eta_i\right)\Bigg\|_{S^\infty}$$
$$\lesssim (1+2^{2sj_1})(2^{(2-\alpha)j_2}+2^{(2+\delta)j_2})2^{(j_3+\cdots+j_{2n+1})}n. \tag{16}$$

Therefore,

$$\Bigg| \iint_{\mathbb{R}^{2n+1}} \xi(1+|\xi|^2)^s\mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})$$
$$\hat{\varphi}_{j_1}(\eta_1)\hat{\varphi}_{j_2}(\eta_2)\cdots\hat{\varphi}_{j_{2n}}(\eta_{2n})\hat{\varphi}_{j_{2n+1}}(\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(-\xi)\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n}\,\mathrm{d}\xi\Bigg|$$
$$\lesssim n\|\varphi_{j_1}\|_{H^s}^2\|(|\partial_x|^{2-\alpha}+|\partial_x|^{2+\delta})\varphi_{j_2}\|_{L^\infty}\prod_{i=3}^{2n+1}\|\partial_x\varphi_{j_i}\|_{L^\infty}.$$

After taking the summation for integers $j_1 \geq \cdots \geq j_{2n+1}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|_{H^s}^2$$
$$\lesssim \|\varphi(t)\|_{H^s}^2\sum_{n=1}^{\infty}\left(c_n\sum_{\substack{j_i\in\mathbb{Z}\\ i=2,\cdots,2n+1}}\|(|\partial_x|^{2-\alpha}+|\partial_x|^{2+\delta})\varphi_{j_2}\|_{L^\infty}\prod_{i=3}^{2n+1}\|\partial_x\varphi_{j_i}\|_{L^\infty}\right)$$
$$\lesssim \|\varphi(t)\|_{H^s}^2\sum_{n=1}^{\infty}\left(c_n\|\varphi\|_{B^{2-\alpha,2+\delta}}^{2n}\right), \tag{17}$$

where $\delta > 0$ is arbitrarily small.

Based on this improved energy estimate, we can summarize the theorem of local well-posedness as following.

Theorem 3.1. *Let $\alpha \in (1, 2)$. Assume the initial data $\varphi_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$, and $\|\varphi_0\|_{B^{2-\alpha,2+\delta}} < 1$ for any $0 < \delta < s - 5/2$. Then there exists a positive number T, such that the Cauchy problem (4) with the initial data φ_0 has a unique solution $\varphi(x, t)$ in $C([0, T); H^s(\mathbb{R}))$.*

4. **Global solution for small initial data.** From now on, we choose the following parameter values

$$
\begin{aligned}
&1 < \alpha < 2, \quad s > \frac{7}{2} + \frac{3}{2}\alpha, \quad s - 1 \geq r \geq 0, \\
&\text{positive numbers } p_0 \text{ and } \delta \text{ satisfying:} \\
&0 < \delta \leq \frac{1}{3}(s - \frac{7}{2} - \frac{3}{2}\alpha) - \frac{4}{3}p_0(s + \frac{3}{4}\alpha - \frac{1}{2}), p_0 < \frac{0.005}{\frac{2\alpha}{s-5} + \frac{\alpha}{4(\alpha-1)}}, \text{ and } p_0 < \frac{1}{3}\frac{\alpha - 1}{\alpha + 1}, \\
&\gamma_l = \frac{\alpha}{2}, \qquad \gamma_h = 2 + \frac{\alpha}{2} + 2\delta.
\end{aligned} \tag{18}
$$

One possible choice, for example, is $s = 7$, $r = 0$, $\delta = 10^{-4}$, $p_0 = 10^{-4}(\alpha - 1)$.

We denote by

$$
\mathcal{S} = (2 - \alpha)t\partial_t + x\partial_x \tag{19}
$$

the scaling vector field that commutes with the linearization of the GSQG front equation (11), $\varphi_t = A\partial_x|\partial_x|^{1-\alpha}\varphi$.

Theorem 4.1 (Main theorem)**.** *Let s, r, p_0 be defined as in (18). There exists $0 < \varepsilon \ll 1$ such that if $0 < \varepsilon_0 \leq \varepsilon$ and $\varphi_0 \in H^s(\mathbb{R})$ satisfies*

$$
\|\varphi_0\|_{H^s} + \|x\partial_x\varphi_0\|_{H^r} \leq \varepsilon_0,
$$

then there exists a unique global solution $\varphi \in C([0, \infty); H^s(\mathbb{R}))$ of (4) with initial condition $\varphi|_{t=0} = \varphi_0$. Moreover, this solution satisfies

$$
\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^r} \lesssim \varepsilon_0(t + 1)^{p_0},
$$

where $\mathcal{S}$ is the vector field in (19).

This theorem is a consequence of local existence and the following bootstrap result involving a Z-norm of the solution, which we define for a function $f \in H^s(\mathbb{R})$ by

$$
\|f\|_Z = \left\|(|\xi|^{\gamma_l} + |\xi|^{\gamma_h})\hat{f}(\xi)\right\|_{L^\infty_\xi}, \tag{20}
$$

where γ_l and γ_h are defined in (18). This definition of the Z-norm comes from the $B^{2-\alpha,2+\delta}$ norm in the energy estimates and the linear dispersive estimate (21) below. In the definition, we choose $|\xi|^{\gamma_l}$ as a weight for low frequencies and $|\xi|^{\gamma_h}$ for high frequencies.

Proposition 4.2 (Bootstrap)**.** *Let $T > 1$ and suppose that $\varphi \in C([0, T]; H^s(\mathbb{R}))$ is a solution of (4), where the initial data satisfies*

$$
\|\varphi_0\|_{H^s} + \|x\partial_x\varphi_0\|_{H^r} \leq \varepsilon_0
$$

for some $0 < \varepsilon_0 \ll 1$. If there exists $\varepsilon_0 \ll \varepsilon_1 \lesssim \varepsilon_0^{1/3}$ such that the solution satisfies

$$
(t + 1)^{-p_0} (\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^r}) + \|\varphi(t)\|_Z \leq \varepsilon_1
$$

for every $t \in [0, T]$, then the solution satisfies an improved bound

$$
(t + 1)^{-p_0} (\|\varphi(t)\|_{H^s} + \|\mathcal{S}\varphi(t)\|_{H^r}) + \|\varphi(t)\|_Z \leq \varepsilon_0.
$$

We call the assumptions in Proposition 4.2 the *bootstrap assumptions*. To prove Proposition 4.2, we need the following lemmas, most of whose proofs are deferred to the next sections.

Lemma 4.3 (Sharp pointwise decay)**.** *Under the bootstrap assumptions,*

$$\|\varphi(t)\|_{B^{2-\alpha,2+\delta}} \lesssim \varepsilon_1 (t+1)^{-1/2} \qquad \text{for } 0 \le t \le T.$$

Lemma 4.4 (Scaling vector field estimate)**.** *Under the bootstrap assumptions,*

$$(t+1)^{-p_0}\|\mathcal{S}\varphi(t)\|_{H^r} \lesssim \varepsilon_0 \qquad \text{for } 0 \le t \le T.$$

Lemma 4.5 (Nonlinear dispersive estimate)**.** *Under the bootstrap assumptions, the solutions of* (11) *satisfies*

$$\|\varphi(t)\|_Z \lesssim \varepsilon_0 \qquad \text{for } 0 \le t \le T.$$

Proposition 4.2 then follows by combining the energy estimate (17) and Lemmas 4.3–4.5.

5. Sharp dispersive estimate.

In this section, we prove Lemma 4.3. We first state a dispersive estimate for the linearized evolution operator $e^{At\partial_x|\partial_x|^{1-\alpha}}$. This estimate is similar to ones in [11, 27] and we omit the proof. We recall that P_k is the frequency-localization operator with symbol ψ_k defined in (6).

Lemma 5.1. *Let $t > 0$ and $f \in L^2(\mathbb{R})$. Then*

$$\begin{aligned}\|e^{At\partial_x|\partial_x|^{1-\alpha}} P_k f\|_{L^\infty} &\lesssim (t+1)^{-1/2} 2^{\alpha k/2} \|\widehat{P_k f}\|_{L^\infty_\xi} \\ &\quad + (t+1)^{-3/4} 2^{(3\alpha/4-1)k} \left[\|P_k(x\partial_x f)\|_{L^2} + \|\tilde{P}_k f\|_{L^2}\right].\end{aligned} \tag{21}$$

Using this lemma, we are able to complete the proof of Lemma 4.3.

Proof of Lemma 4.3. We observe that it suffices to bound the terms

$$\sum_{k\le 0} \left\|2^{k(2-\alpha)} P_k\varphi\right\|_{L^\infty} \quad \text{and} \quad \sum_{k>0} \left\|2^{k(2+\delta)} P_k\varphi\right\|_{L^\infty}.$$

We also introduce the notation

$$h(x,t) = e^{-At\partial_x|\partial_x|^{1-\alpha}}\varphi(x,t), \qquad \hat{h}(\xi,t) = e^{-iAt\xi|\xi|^{1-\alpha}}\hat{\varphi}(\xi,t) \tag{22}$$

for the function h obtained by removing the action of the linearized evolution group on φ. By (11),

$$\begin{aligned}\mathcal{F}[x\partial_x h](\xi,t) &= -\hat{h}(\xi,t) - \xi\partial_\xi \hat{h}(\xi,t), \\ \xi\partial_\xi\hat{h}(\xi,t) &= \xi e^{-iAt\xi|\xi|^{1-\alpha}}\left(-i(2-\alpha)At|\xi|^{1-\alpha}\hat{\varphi}(\xi,t) + \partial_\xi\hat{\varphi}(\xi,t)\right) \\ &= e^{-iAt\xi|\xi|^{1-\alpha}}\Big(-(2-\alpha)t\hat{\varphi}_t(\xi,t) - (2-\alpha)it\xi\hat{\mathcal{N}}(\xi,t) + \xi\partial_\xi\hat{\varphi}(\xi,t)\Big) \\ &= e^{-iAt\xi|\xi|^{1-\alpha}}\Big(-\widehat{\mathcal{S}\varphi}(\xi,t) - \hat{\varphi}(\xi,t) - (2-\alpha)it\xi\hat{\mathcal{N}}(\xi,t)\Big),\end{aligned} \tag{23}$$

where $\mathcal{S}$ is defined in (19), and

$$\begin{aligned}&\mathcal{N}(x,t) \\ &= \sum_{n=1}^\infty \frac{c_n}{2n+1}\int_{\mathbb{R}^{2n+1}} \mathbf{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1,t)\hat{\varphi}(\eta_2,t)\cdots\hat{\varphi}(\eta_{2n+1},t)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n\end{aligned}$$

denotes the nonlinear term of (11). It follows from (23) that

$$\mathcal{F}_x[x\partial_x h](\xi,t) = -e^{-iAt\xi|\xi|^{1-\alpha}}\left(\widehat{\mathcal{S}\varphi}(\xi,t) + (2-\alpha)it\xi\mathcal{N}(\hat{\varphi}(\xi,t))\right).$$

We observe that in view of Proposition B.1, the nonlinear term $\mathcal{N}$ satisfies

$$\left\||\partial_x|^j\partial_x\mathcal{N}\right\|_{L^2} \lesssim \sum_{n=1}^{\infty} C(n,s)\|\varphi\|_{B^{2-\alpha,1}}\|\varphi\|_{B^1}^{2n-1}\|\varphi\|_{H^{j+1}} \qquad \text{for } j=0,\dots,r. \tag{24}$$

Since $e^{At\partial_x|\partial_x|^{1-\alpha}}$ and P_k commute, and $x\partial_x h = \partial_x(x\partial_x h) - \partial_x h$, by Lemma 5.1 we have that

$$\begin{aligned}\|P_k|\partial_x|^{2-\alpha}\varphi\|_{L^\infty} \lesssim (t+1)^{-1/2}2^{(2-\frac{\alpha}{2}-\gamma_l)k}\|\mathcal{F}(P_k|\partial_x|^{\gamma_l}\varphi)\|_{L^\infty_\xi} \\ + (t+1)^{-3/4}2^{(1-\frac{\alpha}{4})k}\left[\|P_k(x\partial_x h)\|_{L^2} + \|\tilde{P}_k(\varphi)\|_{L^2}\right].\end{aligned} \tag{25}$$

It follows from (23) that

$$\|P_k(x\partial_x h)\|_{L^2} \lesssim \|P_k\varphi\|_{L^2} + \|P_k\mathcal{S}\varphi\|_{L^2} + t\|P_k\mathcal{N}\|_{L^2}.$$

We first observe that $k \le 0$ automatically implies $(t+1)^{-1/4+p_0}2^{(1-\frac{\alpha}{4})k} \lesssim 1$, and then we have

$$\begin{aligned}\|P_k|\partial_x|^{2-\alpha}\varphi\|_{L^\infty} \lesssim &(t+1)^{-1/2}2^{\delta k}\|\psi_k(\xi)|\xi|^{\gamma_l}\hat{\varphi}(\xi)\|_{L^\infty_\xi} \\ &+ (t+1)^{-1/2-p_0}[\|\tilde{P}_k\varphi\|_{L^2} + \|P_k\mathcal{S}\varphi\|_{L^2} + t\|P_k\mathcal{N}\|_{L^2}].\end{aligned}$$

Thus, summing over $k \le 0$, using (24) and the bootstrap assumption, we find that

$$\sum_{k\le 0}\left\||\partial_x|^{2-\alpha}P_k\varphi\right\|_{L^\infty} \lesssim \varepsilon_1(t+1)^{-1/2}.$$

Now we turn to $\|P_k|\partial_x|^{2+\delta}\varphi\|_{L^\infty}$ when $k>0$. It follows from Lemma 5.1 that

$$\begin{aligned}\|P_k|\partial_x|^{2+\delta}\varphi\|_{L^\infty} \lesssim (t+1)^{-1/2}2^{-\delta k}\|\mathcal{F}(P_k|\partial_x|^{\gamma_h}\varphi)\|_{L^\infty_\xi} \\ + (t+1)^{-3/4}2^{(1+\delta+3\alpha/4)k}[\|P_k(x\partial_x h)\|_{L^2} + \|P_k(\varphi)\|].\end{aligned}$$

If $k\in\mathbb{N}$ and $(t+1)^{-1/4+p_0}2^{(1+\delta+3\alpha/4)k} \lesssim 1$, then

$$\begin{aligned}\|P_k|\partial_x|^{2+\delta}\varphi\|_{L^\infty} \lesssim (t+1)^{-1/2}2^{-\delta k}\|\psi_k(\xi)|\xi|^{\gamma_h}\hat{\varphi}(\xi)\|_{L^\infty_\xi} \\ + (t+1)^{-1/2-p_0}[\|\tilde{P}_k\varphi\|_{L^2} + \|P_k\mathcal{S}\varphi\|_{L^2} + t\|P_k\mathcal{N}\|_{L^2}].\end{aligned} \tag{26}$$

Finally, if $k\in\mathbb{N}$ and $(t+1)^{-1/4+p_0}2^{(1+\delta+3\alpha/4)k} \gtrsim 1$, then $2^{-(1+\delta+3\alpha/4)k} \lesssim (t+1)^{-1/4+p_0}$. In this case, we have

$$\begin{aligned}\|P_k|\partial_x|^{2+\delta}\varphi\|_{L^\infty} &\lesssim \||\xi|^{2+\delta}\psi_k(\xi)\hat{\varphi}(\xi)\|_{L^1_\xi} \lesssim \||\xi|^{2+\delta-s}\psi_k(\xi)\|_{L^2}\|P_k\varphi\|_{H^s} \\ &\lesssim 2^{-(s-\frac{3}{2}-\delta)k}\|P_k\varphi\|_{H^s} \lesssim (t+1)^{-\frac{1}{2}-p_0}\|P_k\varphi\|_{H^s},\end{aligned} \tag{27}$$

where we have used the fact that $\frac{s-\frac{3}{2}-\delta}{1+\delta+3\alpha/4}(\frac{1}{4}-p_0) \ge \frac{1}{2}+p_0$, which can be verified from (18).

Using the bootstrap assumptions, the estimate (24), and summing (26) and (27) in the corresponding ranges of k, we obtain that

$$\sum_{k>0}\left\|P_k|\partial_x|^{2+\delta}\varphi\right\|_{L^\infty} \lesssim \varepsilon_1(t+1)^{-1/2},$$

which concludes the proof. $\square$

6. **Scaling vector field estimate.** In this section, we prove the scaling vector field estimate in Lemma 4.4. A direct calculation gives the following commutator identities.

Lemma 6.1. *Let $\varphi(x,t)$ be a Schwartz distribution on $\mathbb{R}^2$ such that $|\partial_x|^{1-\alpha}\varphi(x,t)$ is a Schwartz distribution and $\mathcal{S}$ the vector field (19). Then*

$$[\mathcal{S},\partial_x]\varphi = -\partial_x\varphi, \qquad [\mathcal{S},|\partial_x|^{1-\alpha}\partial_x]\varphi = -(2-\alpha)|\partial_x|^{1-\alpha}\partial_x\varphi,$$
$$[\mathcal{S},\partial_t]\varphi = -(2-\alpha)\partial_t\varphi, \qquad [\mathcal{S},\partial_t - A|\partial_x|^{1-\alpha}\partial_x]\varphi = -(2-\alpha)(\partial_t - A|\partial_x|^{1-\alpha}\partial_x)\varphi.$$

We now prove the scaling vector field estimate.

Proof of Lemma 4.4. Act $\mathcal{S}$ to equation (11), by Lemma 6.1,

$$\begin{aligned}\partial_t\mathcal{S}\varphi - A|\partial_x|^{1-\alpha}\partial_x\mathcal{S}\varphi + \partial_x\mathcal{S}\mathcal{N}(\varphi) - \partial_x\mathcal{N}(\varphi) &= (2-\alpha)(\partial_t - A|\partial_x|^{1-\alpha}\partial_x)\varphi\\ &= -(2-\alpha)\partial_x\mathcal{N}(\varphi).\end{aligned}$$

To estimate $\|\mathcal{S}\varphi\|_{H^r}$, we need to estimate

$$\int i\xi(1+|\xi|^2)^r\mathcal{F}[\mathcal{N}(\varphi)](\xi,t)\mathcal{F}[\mathcal{S}\varphi](-\xi,t)\,\mathrm{d}\xi$$

and

$$\int i\xi(1+|\xi|^2)^r\mathcal{F}[\mathcal{S}\mathcal{N}(\varphi)](\xi,t)\mathcal{F}[\mathcal{S}\varphi](-\xi,t)\,\mathrm{d}\xi.$$

By the symbol estimates Proposition B.1, we obtain the estimate of the first integral:

$$\begin{aligned}&\left|\int(1+|\xi|^2)^r\mathcal{F}[\partial_x\mathcal{N}(\varphi)](\xi,t)\mathcal{F}[\mathcal{S}\varphi](-\xi,t)\,\mathrm{d}\xi\right|\\ &\lesssim \sum_{n=1}^{\infty}c_n\left(\sum_{\substack{j_i\,\in\,\mathbb{Z}\\ i\,=\,2,\cdots,2n+1}}\||\partial_x|^{2-\alpha}\varphi_{j_2}\|_{L^\infty}\prod_{i=3}^{2n+1}\|\partial_x\varphi_{j_i}\|_{L^\infty}\right)\|\varphi\|_{H^{r+1}}\|\mathcal{S}\varphi\|_{H^r}.\end{aligned}\tag{28}$$

To estimate the second integral, we notice

$$\begin{aligned}&\int i\xi(1+|\xi|^2)^r\mathcal{F}[\mathcal{S}\mathcal{N}(\varphi)](\xi,t)\mathcal{F}[\mathcal{S}\varphi](-\xi,t)\,\mathrm{d}\xi\\ &=\int i\xi(1+|\xi|^2)^r[(2-\alpha)t\partial_t\widehat{\mathcal{N}} - \partial_\xi(\xi\widehat{\mathcal{N}})]\mathcal{F}[\mathcal{S}\varphi](-\xi,t)\,\mathrm{d}\xi\\ &=\int i\xi(1+|\xi|^2)^r[(2-\alpha)t\partial_t\widehat{\mathcal{N}} - \xi\partial_\xi\widehat{\mathcal{N}}]\mathcal{F}[\mathcal{S}\varphi](-\xi,t)\,\mathrm{d}\xi\\ &\quad-\int i\xi(1+|\xi|^2)^r\widehat{\mathcal{N}}\mathcal{F}[\mathcal{S}\varphi](-\xi,t)\,\mathrm{d}\xi.\end{aligned}$$

Denote the operator $\hat{\mathcal{S}} := (2-\alpha)t\partial_t - \xi\partial_\xi$. Notice that the Fourier transform of $\mathcal{S}\varphi$ is

$$\mathcal{F}\mathcal{S}\varphi = (2-\alpha)t\partial_t\hat{\varphi} + i\partial_\xi(i\xi\hat{\varphi}) = (2-\alpha)t\partial_t\hat{\varphi} - \partial_\xi(\xi\hat{\varphi}) = (\hat{\mathcal{S}}-1)\hat{\varphi}.$$

So the integral to be estimated is written as

$$\begin{aligned}&\int i\xi(1+|\xi|^2)^r\mathcal{F}[\mathcal{S}\mathcal{N}(\varphi)](\xi,t)\mathcal{F}[\mathcal{S}\varphi](-\xi,t)\,\mathrm{d}\xi\\ &=\int i\xi(1+|\xi|^2)^r\hat{\mathcal{S}}\widehat{\mathcal{N}}(\xi)\left([\hat{\mathcal{S}}\hat{\varphi}](-\xi) - \hat{\varphi}(-\xi)\right)\,\mathrm{d}\xi - \int i\xi(1+|\xi|^2)^r\widehat{\mathcal{N}}\mathcal{F}[\mathcal{S}\varphi](-\xi)\,\mathrm{d}\xi,\end{aligned}$$

where the second term satisfies the same estimate as (28). Then we estimate the first term.

Since

$$
\begin{aligned}
&\xi\partial_\xi \iint_{\mathbb{R}^{2n}} \mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(\eta_1)\cdots\hat{\varphi}(\eta_{2n})\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n\\
=&\iint_{\mathbb{R}^{2n}} \xi\partial_\xi\mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(\eta_1)\cdots\hat{\varphi}(\eta_{2n})\\
&\cdot\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n\\
&+\iint_{\mathbb{R}^{2n}} \mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(\eta_1)\cdots\hat{\varphi}(\eta_{2n})(\xi-\eta_1-\cdots-\eta_{2n})\\
&\qquad\cdot\partial_\xi\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n\\
&+\iint_{\mathbb{R}^{2n}} \mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\eta_1\partial_{\eta_1}\hat{\varphi}(\eta_1)\cdots\hat{\varphi}(\eta_{2n})\\
&\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n+\cdots\\
&+\iint_{\mathbb{R}^{2n}} \xi\partial_\xi\mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(\eta_1)\cdots\eta_{2n}\partial_{\eta_{2n}}\hat{\varphi}(\eta_{2n})\\
&\cdot\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n\\
&+\iint_{\mathbb{R}^{2n}} (\eta_1\partial_{\eta_1}+\cdots+\eta_{2n}\partial_{\eta_{2n}})\mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(\eta_1)\cdots\hat{\varphi}(\eta_{2n})\\
&\qquad\cdot\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n,
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\hat{\mathcal{S}}\widehat{\mathcal{N}}_{2n+1}(\varphi)(\xi)\\
=&\frac{c_n}{2n+1}\iint_{\mathbb{R}^{2n}} \mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})[\hat{\mathcal{S}}\hat{\varphi}(\eta_1)]\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\\
&\cdot\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n+\cdots\\
&+\frac{c_n}{2n+1}\iint_{\mathbb{R}^{2n}} \mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\\
&\qquad\cdot[\hat{\mathcal{S}}\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})]\,\mathrm{d}\boldsymbol{\eta}_n\\
&+\frac{c_n}{2n+1}\iint_{\mathbb{R}^{2n}} [\eta_{2n+1}\partial_{2n+1}\mathbf{T}_n](\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\\
&\cdot\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n\\
=&c_n\iint_{\mathbb{R}^{2n}} \mathbf{T}_n(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})[\hat{\mathcal{S}}\hat{\varphi}(\eta_1)]\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\\
&\cdot\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n\\
&+\frac{c_n}{2n+1}\iint_{\mathbb{R}^{2n}} [\eta_{2n+1}\partial_{2n+1}\mathbf{T}_n](\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\\
&\cdot\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})\,\mathrm{d}\boldsymbol{\eta}_n,
\end{aligned}
\tag{29}
$$

where the last equality is from the symmetry of $\mathbf{T}_n$.

Then we estimate

$$
\int i\xi(1+|\xi|^2)^r\hat{\mathcal{S}}\widehat{\mathcal{N}}_{2n+1}(\varphi)(\xi)[\hat{\mathcal{S}}\hat{\varphi}](-\xi)\,\mathrm{d}\xi,
$$

which can be written as two parts by (29). The first part is similar as the estimate of (13). We split the symbol $\mathbf{T}_n$ as $\mathbf{T}_n^{1-\alpha}(\eta_1,\cdots,\eta_{2n+1})+\mathbf{T}_n^{\le -1}(\eta_1,\cdots,\eta_{2n+1})$.

The lower-order term satisfies

$$\begin{aligned}&\Bigg|\iint_{\mathbb{R}^{2n+1}} i\xi(1+|\xi|^2)^r \mathbf{T}_n^{\le -1}(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\\&\qquad\cdot[\hat{\mathcal{S}}\hat{\varphi}(\eta_1)]\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})[\hat{\mathcal{S}}\hat{\varphi}](-\xi)\,\mathrm{d}\boldsymbol{\eta}_n\,\mathrm{d}\xi\Bigg|\\&\lesssim \|\hat{\mathcal{S}}\hat{\varphi}\|_{H^r}^2\sum_{n=1}^{\infty}\left(c_n\sum_{\substack{j_i\in\mathbb{Z}\\ i=2,\cdots,2n+1}}\|\partial_x^2\varphi_{j_2}\|_{L^\infty}\prod_{i=3}^{2n+1}\|\partial_x\varphi_{j_i}\|_{L^\infty}\right).\end{aligned}$$

By symmetrization of $\mathbf{T}_n^{1-\alpha}$ and the fact that $\mathbf{T}_n^{1-\alpha}$ is even with respect to its highest frequency, the integral related to the first part of (29) can be written as

$$\begin{aligned}&\iint_{\mathbb{R}^{2n+1}} i\xi(1+|\xi|^2)^r \mathbf{T}_n^{1-\alpha}(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\\&\qquad\cdot[\hat{\mathcal{S}}\hat{\varphi}(\eta_1)]\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})[\hat{\mathcal{S}}\hat{\varphi}](-\xi)\,\mathrm{d}\boldsymbol{\eta}_n\,\mathrm{d}\xi\\&= -\iint_{\mathbb{R}^{2n+1}} i\eta_1(1+|\eta_1|^2)^r \mathbf{T}_n^{1-\alpha}(-\xi,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\\&\qquad\cdot[\hat{\mathcal{S}}\hat{\varphi}](-\xi)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})[\hat{\mathcal{S}}\hat{\varphi}](\eta_1)\,\mathrm{d}\boldsymbol{\eta}_n\,\mathrm{d}\xi\\&= \frac{i}{2}\iint_{\mathbb{R}^{2n+1}}\Big[\xi(1+|\xi|^2)^r \mathbf{T}_n^{1-\alpha}(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\\&\qquad-\eta_1(1+|\eta_1|^2)^r \mathbf{T}_n^{1-\alpha}(\xi,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\Big]\\&\qquad\cdot[\hat{\mathcal{S}}\hat{\varphi}(\eta_1)]\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})[\hat{\mathcal{S}}\hat{\varphi}](-\xi)\,\mathrm{d}\boldsymbol{\eta}_n\,\mathrm{d}\xi.\end{aligned}$$

After taking dyadic decomposition and using the symbol estimate (16), we obtain

$$\begin{aligned}&\Bigg|\iint_{\mathbb{R}^{2n+1}} i\xi(1+|\xi|^2)^r \mathbf{T}_n^{1-\alpha}(\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\\&\qquad\cdot[\hat{\mathcal{S}}\hat{\varphi}(\eta_1)]\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})[\hat{\mathcal{S}}\hat{\varphi}](-\xi)\,\mathrm{d}\boldsymbol{\eta}_n\,\mathrm{d}\xi\Bigg|\\&\lesssim \|\hat{\mathcal{S}}\hat{\varphi}\|_{H^r}^2\sum_{n=1}^{\infty}\left(c_n\sum_{\substack{j_i\in\mathbb{Z}\\ i=2,\cdots,2n+1}}\||\partial_x|^{2-\alpha}\varphi_{j_2}\|_{L^\infty}\prod_{i=3}^{2n+1}\|\partial_x\varphi_{j_i}\|_{L^\infty}\right).\end{aligned}$$

Then we estimate the second part of (29). By Proposition B.1, the integral related to the second part of (29) is bounded by

$$\begin{aligned}&\Bigg|\iint_{\mathbb{R}^{2n+1}} i\xi(1+|\xi|^2)^r [\eta_{2n+1}\partial_{2n+1}\mathbf{T}_n](\eta_1,\cdots,\eta_{2n},\xi-\eta_1-\cdots-\eta_{2n})\\&\qquad\cdot\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\cdots\hat{\varphi}(\eta_{2n})\hat{\varphi}(\xi-\eta_1-\cdots-\eta_{2n})[\hat{\mathcal{S}}\hat{\varphi}](-\xi)\,\mathrm{d}\boldsymbol{\eta}_n\,\mathrm{d}\xi\Bigg|\end{aligned}$$

$$\lesssim (1+\ln n)\|\hat{\mathcal{S}}\hat{\varphi}\|_{H^r}\|\varphi\|_{H^{r+2}}\left(\sum_{\substack{j_i\,\in\,\mathbb{Z}\\ i\,=\,2,\cdots,2n+1}}\||\partial_x|^{2-\alpha}\varphi_{j_2}\|_{L^\infty}\prod_{i=3}^{2n+1}\|\partial_x\varphi_{j_i}\|_{L^\infty}\right).$$

Combining the above estimate, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{S}\varphi\|_{H^r}^2 \lesssim (\|\hat{\mathcal{S}}\hat{\varphi}\|_{H^r}+\|\varphi\|_{H^{r+1}})\|\mathcal{S}\varphi\|_{H^r}\sum_{n=1}^{\infty}\left(c_n\|\varphi\|_{B^{2-\alpha,2+\delta}}\|\varphi\|_{B^1}^{2n-1}\right).$$

By Gronwall's inequality, under the bootstrap assumption, this leads to Lemma 4.4. □

7. Nonlinear dispersive estimate.

In this section, we prove the estimate

$$(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})|\hat{\varphi}(\xi,t)| \lesssim \varepsilon_0 \qquad \text{for all } \xi\in\mathbb{R}, \tag{30}$$

which establishes Lemma 4.5 for the Z-norm $\|\varphi\|_Z$ defined in (20). We recall that we use the parameter values given in (18).

This section is organized as follows. Using Lemma 2.1, we first prove in Section 7.1 that the estimate (30) holds for sufficiently large and small $|\xi|$. In Section 7.2, we introduce a logarithmic phase shift into the solution which is used later to absorb the effects of the space-time resonances. The main part of the section is a detailed analysis of the nonresonant and resonant interactions between different Fourier components of the solution, which is carried out in Sections 7.3–7.6.

To classify the cubic resonances between frequencies $\xi-\eta_1-\eta_2$, η_1, η_2 into ξ, where $\xi,\eta_1,\eta_2\in\mathbb{R}$, we introduce the phase

$$\Phi(\xi,\eta_1,\eta_2)=(\xi-\eta_1-\eta_2)|\xi-\eta_1-\eta_2|^{1-\alpha}+\eta_1|\eta_1|^{1-\alpha}+\eta_2|\eta_2|^{1-\alpha}-\xi|\xi|^{1-\alpha}. \tag{31}$$

The space resonances satisfy $\partial_{\eta_1}\Phi=\partial_{\eta_2}\Phi=0$, which implies that the frequencies $\xi-\eta_1-\eta_2$, η_1, η_2 have the same linearized group velocity. It is straightforward to check that the only space resonances are

$$(\xi-\eta_1-\eta_2,\eta_1,\eta_2)=(-\xi,\xi,\xi),\ (\xi,-\xi,\xi),\ \text{or } (\xi,\xi,-\xi), \tag{32}$$

$$(\xi-\eta_1-\eta_2,\eta_1,\eta_2)=\left(\frac{\xi}{3},\frac{\xi}{3},\frac{\xi}{3}\right). \tag{33}$$

The time resonances satisfy $\Phi=0$, which implies that the time-frequencies of $\xi-\eta_1-\eta_2$, η_1, η_2 are in resonance with the time-frequency of ξ. This condition is satisfied by the resonances (32), which are space-time resonances, but not by (33), which is a space resonance. There are additional time resonances of the form $\xi=\xi-\eta+\eta$, but they are not space resonances for $\xi\neq\eta$, so they require no further analysis.

7.1. Large and small frequencies.

When $|\xi|<(t+1)^{-\frac{p_0}{\alpha-1}}$, Lemma 2.1, the bootstrap assumptions, Lemma 4.4, and the conservation of the L^2-norm of φ give

$$\begin{aligned}|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\hat{\varphi}(\xi,t)|^2 &\lesssim (|\xi|^{\gamma_l}+|\xi|^{\gamma_h})^2|\xi|^{-1}\|\hat{\varphi}\|_{L^2_\xi}(\||\xi|\partial_\xi\hat{h}\|_{L^2_\xi}+\|\hat{\varphi}\|_{L^2_\xi})\\ &\lesssim |\xi|^{\alpha-1}\|\varphi\|_{L^2}(\|\mathcal{S}\varphi\|_{L^2}+\|\varphi\|_{L^2})\\ &\lesssim \varepsilon_0^2.\end{aligned}$$

Let $p_1 = \frac{2p_0}{s+1-2\gamma_h}$. When $|\xi| > (t+1)^{p_1}$, Lemma 2.1 and the bootstrap assumptions give

$$\begin{aligned}|(|\xi|^{\gamma_l} + |\xi|^{\gamma_h})\hat{\varphi}(\xi,t)|^2 &\lesssim \frac{(|\xi|^{\gamma_l} + |\xi|^{\gamma_h})^2}{|\xi|^{s+1}}\|\varphi\|_{H^s}(\|\mathcal{S}\varphi\|_{L^2} + \|\varphi\|_{L^2})\\ &\lesssim |\xi|^{2\gamma_h - s-1}\varepsilon_0^2(t+1)^{2p_0}\\ &\lesssim \varepsilon_0^2.\end{aligned}$$

Thus, we only need to consider the frequency range

$$(t+1)^{-p_0} \le |\xi| \le (t+1)^{p_1}. \tag{34}$$

In the following, we fix ξ in this range and denote by $\mathfrak{d}(\xi,t)$ a smooth cut-off function such that

$$\begin{aligned}&\mathfrak{d}(\xi,t) = 1 \text{ on } \big\{(\xi,t) \mid (t+1)^{-p_0} \le |\xi| \le (t+1)^{p_1}\big\},\\ &\mathfrak{d}(\xi,t) \text{ is supported on a small neighborhood of}\\ &\big\{(\xi,t) \mid (t+1)^{-p_0} \le |\xi| \le (t+1)^{p_1}\big\}.\end{aligned} \tag{35}$$

7.2. **Modified scattering.** In this subsection, we introduce a phase shift to account for the modified scattering of the solution and carry out a dyadic decomposition of the cubic term. Taking the Fourier transform of (11), we obtain that

$$\begin{aligned}\hat{\varphi}_t(\xi,t) + iA'\xi \iint_{\mathbb{R}^2} &\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{\varphi}(\xi-\eta_1-\eta_2,t)\hat{\varphi}(\eta_1,t)\hat{\varphi}(\eta_2,t)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\\ &+\widehat{\mathcal{N}_{\ge 5}(\varphi)}(\xi,t) = iA\xi|\xi|^{1-\alpha}\hat{\varphi}(\xi,t),\end{aligned} \tag{36}$$

where $A' = -A/(6(3-\alpha))$ and

$$\begin{aligned}&\mathbf{T}_1'(\eta_1,\eta_2,\eta_3) = \frac{(2-\alpha)(3-\alpha)}{A}\mathbf{T}_1(\eta_1,\eta_2,\eta_3)\\ =&|\eta_1|^{3-\alpha} + |\eta_2|^{3-\alpha} + |\eta_3|^{3-\alpha} + |\eta_1+\eta_2+\eta_3|^{3-\alpha} - |\eta_1+\eta_2|^{3-\alpha} - |\eta_1+\eta_3|^{3-\alpha}\\ &- |\eta_2+\eta_3|^{3-\alpha},\end{aligned} \tag{37}$$

$$\begin{aligned}&\mathcal{N}_{\ge 5}(\varphi)(x,t)\\ =&\sum_{n=2}^{\infty}\frac{c_n}{2n+1}\partial_x\int_{\mathbb{R}^{2n+1}}\mathbf{T}_n(\boldsymbol{\eta}_n)\hat{\varphi}(\eta_1,t)\hat{\varphi}(\eta_2,t)\cdots\hat{\varphi}(\eta_{2n+1},t)e^{i(\eta_1+\eta_2+\cdots+\eta_{2n+1})x}\,\mathrm{d}\boldsymbol{\eta}_n.\end{aligned}$$

Here

$$\begin{aligned}&\mathbf{T}_1(\eta_1,\eta_2,\eta_3) = \int_{\mathbb{R}}\frac{\prod_{j=1}^3\left(1-e^{i\eta_j\zeta}\right)}{\zeta^3}|\zeta|^{\alpha-1}\operatorname{sgn}\zeta\,\mathrm{d}\zeta\\ =&\int_{\mathbb{R}}\frac{1}{|\zeta|^{4-\alpha}}\Big(1 - e^{i\eta_1\zeta} - e^{i\eta_2\zeta} - e^{i\eta_3\zeta} + e^{i(\eta_1+\eta_2)\zeta} + e^{i(\eta_1+\eta_3)\zeta} + e^{i(\eta_2+\eta_3)\zeta}\\ &\qquad - e^{i(\eta_1+\eta_2+\eta_3)\zeta}\Big)\,\mathrm{d}\zeta\\ =&\int_{\mathbb{R}}\frac{1}{|\zeta|^{4-\alpha}}\Big[(1-e^{i\eta_1\zeta}) + (1-e^{i\eta_2\zeta}) + (1-e^{i\eta_3\zeta}) + (e^{i(\eta_1+\eta_2)\zeta}-1)\\ &\qquad + (e^{i(\eta_1+\eta_3)\zeta}-1) + (e^{i(\eta_2+\eta_3)\zeta}-1) + (1-e^{i(\eta_1+\eta_2+\eta_3)\zeta})\Big]\,\mathrm{d}\zeta.\end{aligned} \tag{38}$$

To evaluate these integrals, we consider

$$\int_{\mathbb{R}} \frac{1}{|\zeta|^{4-\alpha}}(1 - e^{ib\zeta})\,\mathrm{d}\zeta = |b|^{3-\alpha} \int_{\mathbb{R}} \frac{1 - \cos\zeta}{|\zeta|^{4-\alpha}}\,\mathrm{d}\zeta. \tag{39}$$

The integral on the right-hand side of (39) is given by

$$\int_{\mathbb{R}} \frac{1 - \cos\zeta}{|\zeta|^{4-\alpha}}\,\mathrm{d}\zeta = \frac{2}{\alpha - 3}\int_0^\infty (1 - \cos\zeta)\,\mathrm{d}\zeta^{\alpha-3}.$$

Since $\alpha \in (1, 2)$, an integration by parts and use of the expression in (5) for A gives

$$\begin{aligned} \int_{\mathbb{R}} \frac{1 - \cos\zeta}{|\zeta|^{4-\alpha}}\,\mathrm{d}\zeta &= -\frac{2}{\alpha - 3}\int_0^\infty \zeta^{\alpha-3}\sin\zeta\,\mathrm{d}\zeta \\ &= -\frac{2}{\alpha - 3}\Gamma(\alpha - 2)\sin\left(\frac{1}{2}\pi(\alpha - 2)\right) \\ &= \frac{2}{(\alpha - 3)(\alpha - 2)}\Gamma(\alpha - 1)\sin\left(\frac{1}{2}\pi\alpha\right) \\ &= \frac{A}{(\alpha - 3)(\alpha - 2)}, \end{aligned} \tag{40}$$

where the second step follows from the formula ([43] Chapter 5, equation (5.9.7))

$$\int_0^\infty t^{z-1}\sin t\,\mathrm{d}t = \Gamma(z)\sin\left(\frac{1}{2}\pi z\right), \quad -1 < \Re z < 1.$$

Then (37) follows from (38)–(40).

From (36), we find that $\hat{h}(\xi, t) = e^{-iAt\xi|\xi|^{1-\alpha}}\hat{\varphi}(\xi, t)$ defined in (22) satisfies

$$\begin{aligned} &\hat{h}_t(\xi, t) + \\ &iA'\xi \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}(\xi - \eta_1 - \eta_2, t)\hat{h}(\eta_1, t)\hat{h}(\eta_2, t)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\ &\qquad + e^{-iAt\xi|\xi|^{1-\alpha}}\widehat{\mathcal{N}_{\geq 5}(\varphi)}(\xi, t) = 0, \end{aligned} \tag{41}$$

where Φ is defined in (31).

We define the following phase correction

$$\begin{aligned} &\Theta(\xi, t) \\ &= \frac{2\pi A'\xi|\xi|^\alpha}{(\alpha - 1)(2 - \alpha)A}[\mathbf{T}_1'(\xi, \xi, -\xi) + \mathbf{T}_1'(\xi, -\xi, \xi) + \mathbf{T}_1'(-\xi, \xi, \xi)]\int_0^t \frac{|\hat{\varphi}(\xi, \tau)|^2}{\tau + 1}\,\mathrm{d}\tau, \end{aligned} \tag{42}$$

which accounts for a cumulative frequency shift from the nonlinearity in the long-time behavior of the Fourier components of the solution due to space-time resonances of the form $\xi = \xi + \xi - \xi$. This phase correction is generic in cubically nonlinear dispersive equations and grows logarithmically in time (*cf.* [11, 30, 31, 35]). We then let

$$\hat{v}(\xi, t) = e^{i\Theta(\xi,t)}\hat{h}(\xi, t).$$

Using this expression in (41) and (42), we obtain that

$$\begin{aligned} &\hat{v}_t(\xi, t) \\ &= e^{i\Theta(\xi,t)}[\hat{h}_t(\xi, t) + i\Theta_t(\xi, t)\hat{h}(\xi, t)] = \hat{U}(\xi, t) - e^{-iAt\xi|\xi|^{1-\alpha}}e^{i\Theta(\xi,t)}\widehat{\mathcal{N}_{\geq 5}(\varphi)}(\xi, t), \end{aligned} \tag{43}$$

where

$$\begin{aligned}&\hat{U}(\xi,t) \qquad (44)\\ =&e^{i\Theta(\xi,t)}\Bigg\{-iA'\xi e^{-iAt\xi|\xi|^{1-\alpha}}\iint_{\mathbb{R}^2}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{\varphi}(\xi-\eta_1-\eta_2,t)\hat{\varphi}(\eta_1,t)\\ &\quad\hat{\varphi}(\eta_2,t)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2+\\ &\frac{2\pi iA'\xi|\xi|}{(\alpha-1)(2-\alpha)A(t+1)}\left[\mathbf{T}_1'(\xi,\xi,-\xi)+\mathbf{T}_1'(\xi,-\xi,\xi)+\mathbf{T}_1'(-\xi,\xi,\xi)\right]|\hat{h}(\xi,t)|^2\hat{h}(\xi,t)\Bigg\}.\end{aligned}$$

Then we get from (43) that

$$\|\varphi(t)\|_Z=\|v(0)\|_Z+\left\|\int_0^t U(\xi,\tau)\,\mathrm{d}\tau\right\|_Z+\left\|\int_0^t \mathcal{N}_{\geq 5}(\xi,\tau)\,\mathrm{d}\tau\right\|_Z. \qquad (45)$$

We estimate U in Sections 7.3–7.5 and take care of the term involving $\widehat{\mathcal{N}_{\geq 5}(\varphi)}$ in Section 7.6.

Suppressing the dependence of φ and h on the time variable t, we write the first integral in U in terms of h as

$$\begin{aligned}&e^{-iAt\xi|\xi|^{1-\alpha}}\iint_{\mathbb{R}^2}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{\varphi}(\xi-\eta_1-\eta_2)\hat{\varphi}(\eta_1)\hat{\varphi}(\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\\ =&\iint_{\mathbb{R}^2}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}(\xi-\eta_1-\eta_2)\hat{h}(\eta_1)\hat{h}(\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.\end{aligned}$$

To carry out the dyadic decomposition, we let $h_j=P_jh$ and $\varphi_j=P_j\varphi$, where P_j is the projection onto frequencies of order 2^j with symbol ψ_j defined in (6), and rewrite the integral in each dyadic block as

$$\iint_{\mathbb{R}^2}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2. \qquad (46)$$

Without loss of generality, we assume $j_1\geq j_2\geq j_3$. Denote the set $\mathcal{P}$ as all the indices $j_1,j_2,j_3\in\mathbb{Z}$, such that $j_1\geq j_2\geq j_3$ with possible repetition. We split the index set $\mathcal{P}$ into $\mathcal{P}_1\bigcup\mathcal{P}_2$, where $\mathcal{P}_1$ includes the indices satisfying $j_1\geq j_2\geq j_3$ and $j_1-j_3>1$, which correspond to nonresonant frequencies. $\mathcal{P}_2$ includes indices satisfying $j_1\geq j_2\geq j_3$ and $j_1-j_3\leq 1$, which contains the resonant frequencies.

In the following subsections, we estimate this integral in various regions of frequency-space. In Section 7.3, we estimate the non-resonant frequencies by integrating by part with respect to time variable. In Section 7.4, we use oscillatory integral estimates to estimate the integral for near-resonant frequencies. In Section 7.5, we estimate resonant frequencies, and, finally, in Section 7.6 we estimate the higher-degree remainder term in (41).

7.3. **Non-resonant frequencies.** The frequencies are non-resonant if $|j_1-j_3|>1$. We recall that h is defined in (22) and the symbol $\tilde{\psi}_k$ is defined in (6). We will estimate

$$\begin{aligned}\Bigg\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\xi\int_0^t\iint_{\mathbb{R}^2}&\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iA\tau\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\\ &\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau\Bigg\|_{L^\infty_\xi} \qquad (47)\end{aligned}$$

when $|j_1-j_3|>1$.

Since

$$e^{iA\tau\Phi(\xi,\eta_1,\eta_2)} = \frac{1}{iA\Phi(\xi,\eta_1,\eta_2)}\left[\partial_\tau e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\right],$$

we can integrate by parts with respect to τ,

$$\begin{aligned}
&\int_0^t \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iA\tau\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau \\
&= \int_0^t \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\frac{1}{iA\Phi(\xi,\eta_1,\eta_2)}\left[\partial_\tau e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\right]\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2) \\
&\quad \cdot \hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau \\
&= \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\frac{1}{iA\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2) \\
&\quad \cdot \hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\Big|_{\tau=0}^{t} \\
&\quad - \int_0^t \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\frac{1}{iA\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\partial_\tau[\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2) \\
&\quad \cdot \hat{h}_{j_3}(\xi-\eta_1-\eta_2)]\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau.
\end{aligned}$$

Since

$$\left\|\frac{1}{\Phi(\eta_1+\eta_2+\eta_3,\eta_1,\eta_2)}\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)\right\|_{S^\infty} \lesssim 2^{(\alpha-1)j_1}2^{-j_2},$$

we have

$$\begin{aligned}
&\Bigg\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\xi\iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\frac{1}{iA\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\hat{h}_{j_1}(\eta_1) \\
&\quad \cdot \hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\Bigg\|_{L^\infty_\xi} \\
&\lesssim (2^{\gamma_l j_1}+2^{\gamma_h j_1})2^{\alpha j_1+(1-\alpha)j_2+j_3}\|\varphi_{j_1}\|_{L^2}\|\varphi_{j_2}\|_{L^\infty}\|\varphi_{j_3}\|_{L^2} \\
&\lesssim \|(2^{\gamma_l j_1}+2^{\gamma_h j_1})2^{\alpha j_1}\varphi_{j_1}\|_{L^2}\|2^{(2-\alpha)j_2}\varphi_{j_2}\|_{L^\infty}\|\varphi_{j_3}\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\xi\int_0^t \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\frac{e^{i\tau\Phi(\xi,\eta_1,\eta_2)}}{iA\Phi(\xi,\eta_1,\eta_2)} \\
&\qquad \partial_\tau[\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)]\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau\Bigg\|_{L^\infty_\xi} \\
&\lesssim \int_0^t (2^{\gamma_l j_1}+2^{\gamma_h j_1})2^{\alpha j_1+(1-\alpha)j_2+j_3}\big(\|\partial_\tau h_{j_1}\|_{L^2}\|\varphi_{j_2}\|_{L^\infty}\|\varphi_{j_3}\|_{L^2} \\
&\qquad + \|\varphi_{j_1}\|_{L^2}\|\partial_\tau h_{j_2}\|_{L^2}\|\varphi_{j_3}\|_{L^\infty} + \|\varphi_{j_1}\|_{L^2}\|\varphi_{j_2}\|_{L^\infty}\|\partial_\tau h_{j_3}\|_{L^2}\big)\,\mathrm{d}\tau.
\end{aligned}$$

Notice that for any $j \in \mathbb{Z}$,

$$\|\partial_t h_j\|_{L^2} \lesssim \|\psi_j(D)\partial_x\mathcal{N}\|_{L^2} \lesssim \sum_{n=1}^\infty \|\varphi\|^{2n}_{B^{(2-\alpha,1)}}\|2^j\varphi_j\|_{L^2} \lesssim \varepsilon_1^2(t+1)^{-1}\|2^j\varphi_j\|_{L^2}.$$

Therefore

$$\Bigg\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\xi\int_0^t \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\frac{1}{iA\Phi(\xi,\eta_1,\eta_2)}e^{i\tau\Phi(\xi,\eta_1,\eta_2)}$$

$$\partial_\tau[\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)]\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau\Big\|_{L^\infty_\xi}$$

$$\lesssim \int_0^t \varepsilon_1^2(\tau+1)^{-1}(2^{\gamma_l j_1}+2^{\gamma_h j_1})2^{\alpha j_1}\big(\|2^{j_1}\varphi_{j_1}\|_{L^2}\|2^{(2-\alpha)j_2}\varphi_{j_2}\|_{L^\infty}\|\varphi_{j_3}\|_{L^2}$$
$$+\|\varphi_{j_1}\|_{L^2}\|2^{(2-\alpha)j_2}\varphi_{j_2}\|_{L^2}\|2^{j_3}\varphi_{j_3}\|_{L^\infty}+\|\varphi_{j_1}\|_{L^2}\|2^{(2-\alpha)j_2}\varphi_{j_2}\|_{L^\infty}\|2^{j_3}\varphi_{j_3}\|_{L^2}\big)\,\mathrm{d}\tau.$$

Taking a summation with respect to all non-resonant frequencies (j_1, j_2, j_3), we obtain

$$\sum_{\mathcal{P}_1}(47)\lesssim \|\varphi\|^2_{L^\infty_t H^s}\|\varphi\|_{L^\infty_t B^{(2-\alpha,1)}}+\int_0^t \varepsilon_1^3(\tau+1)^{-1}\|\varphi(\tau)\|^2_{H^s}\|\varphi(\tau)\|_{B^{(2-\alpha,1)}}\,\mathrm{d}\tau$$
$$\lesssim \varepsilon_1^3\lesssim\varepsilon_0.$$

7.4. **Near-resonant frequencies.** The remaining dyadic blocks to consider are when $(j_1, j_2, j_3) \in \mathcal{P}_2$, i.e.

$$|j_3-j_2|\le 1,\qquad |j_3-j_1|\le 1. \tag{48}$$

In this case, the integration region is divided into four disjoint sets in the (η_1, η_2)-plane. We define cut-off functions $\upsilon_\pm(\eta)$ by

$$\upsilon_+(\eta)=\begin{cases}1 & \text{if } \eta\ge 0,\\ 0 & \text{if } \eta<0,\end{cases}\qquad \upsilon_-(\eta)=\begin{cases}0 & \text{if } \eta\ge 0,\\ 1 & \text{if } \eta<0.\end{cases}$$

The set of discontinuities of $\upsilon_\pm(\eta_1)\upsilon_\pm(\eta_2)$, which is $\{(\eta_1,\eta_2)\mid \eta_1=0 \text{ or } \eta_2=0\}$, is disjoint from the integration region. Thus, the functions $\upsilon_\pm(\eta_1)\upsilon_\pm(\eta_2)$ are equal to 1 on each connected branch of the support of the integrand in (46). Therefore, for each j_1, j_2, and j_3, we only need to estimate four integrals with different combinations of $+$ and $-$ signs

$$\iint_{\mathbb{R}^2}\mathbf{T}_1(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{it\Phi(\eta_1,\eta_2,\xi)}\hat{h}^\pm_{j_1}(\eta_1)\hat{h}^\pm_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2, \tag{49}$$

where $\hat{h}^\pm_j(\eta)=\hat{h}_j(\eta)\upsilon_\pm(\eta)$.

In the following, we assume that $\xi > 0$. The case when $\xi < 0$ can be discussed in the similar way.

The case of $(-,-)$**.** When $\eta_1 < 0, \eta_2 < 0, \xi > 0$, we have $|\xi-\eta_1-\eta_2| = \xi-\eta_1-\eta_2 > 0$, thus

$$|\Phi|\gtrsim 2^{(2-\alpha)j_1}.$$

This is the case away from resonances, whose estimate can be achieved in the same way as in Section 7.3.

The case of $(+,+)$**.** The space-time resonance $(\eta_1,\eta_2) = (\xi,\xi)$ and the space resonance $(\eta_1,\eta_2) = (\xi/3,\xi/3)$ are both in the support of the integrand. By introducing an additional cut-off function $\psi_{j_1-3}(\eta_1+\eta_2-2\xi)$, we write the integral into two parts:

$$\iint_{\mathbb{R}^2}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)$$
$$\psi_{j_1-3}(\eta_1+\eta_2-2\xi)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,$$
$$\iint_{\mathbb{R}^2}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)$$

$$[1-\psi_{j_1-3}(\eta_1+\eta_2-2\xi)]\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,$$

where the first integral contains the space-time resonance $(\eta_1,\eta_2)=(\xi,\xi)$ and the second integral contains the space resonance $(\eta_1,\eta_2)=(\xi/3,\xi/3)$. Then we estimate these two integrals in the following.

We consider the partition of unity

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2}\psi_{k_1}(\eta_1-\xi)\psi_{k_2}(\eta_2-\xi)=1\quad\text{and}\quad\sum_{(k_3,k_4)\in\mathbb{Z}^2}\psi_{k_3}\left(\eta_1-\frac{\xi}{3}\right)\psi_{k_4}\left(\eta_2-\frac{\xi}{3}\right)=1.$$

Then we write the integrals with a finer dyadic decomposition as

$$\begin{aligned}&\iint_{\mathbb{R}^2}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\\&\qquad\psi_{j_1-3}(\eta_1+\eta_2-2\xi)\psi_{k_1}(\eta_1-\xi)\psi_{k_2}(\eta_2-\xi)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,\end{aligned}\tag{50}$$

$$\begin{aligned}&\iint_{\mathbb{R}^2}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\\&\qquad[1-\psi_{j_1-3}(\eta_1+\eta_2-2\xi)]\psi_{k_3}\left(\eta_1-\frac{\xi}{3}\right)\psi_{k_4}\left(\eta_2-\frac{\xi}{3}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2.\end{aligned}\tag{51}$$

In this subsection, we introduce a decay function

$$\varrho_1(t)=(t+1)^{-0.49},\tag{52}$$

and restrict our attention to the following two near-resonant cases:

$$\text{(i) }\max\{k_1,k_2\}\geq\log_2[\varrho_1(t)];$$
$$\text{(ii) }\max\{k_3,k_4\}\geq\log_2[\varrho_1(t)].$$

The cases of $\max\{k_1,k_2\}<\log_2[\varrho_1(t)]$ or $\max\{k_3,k_4\}<\log_2[\varrho_1(t)]$, which include the resonant frequencies, will be discussed in Section 7.5.

Near space-time resonance.

We first estimate (50). Since the integrals are symmetric in η_1 and η_2, we can assume without loss of generality that $k_2\geq k_1$. Using integration by parts, we write the integral as

$$\begin{aligned}&\iint_{\mathbb{R}^2}\frac{\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{2(2-\alpha)iAt(|\eta_1|^{1-\alpha}-|\xi-\eta_1-\eta_2|^{1-\alpha})}\partial_{\eta_1}e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)\\&\qquad\cdot\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\psi_{j_1-3}(\eta_1+\eta_2-2\xi)\Big[\psi_{\leq k_2}(\eta_1-\xi)\psi_{k_2}(\eta_2-\xi)\Big]\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,\\&=-\frac{V_1+V_2+V_3+V_4}{2i(2-\alpha)At},\end{aligned}$$

where

$$\begin{aligned}V_1(\xi,t)=&\iint_{\mathbb{R}^2}\partial_{\eta_1}\left[\frac{\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{|\eta_1|^{1-\alpha}-|\xi-\eta_1-\eta_2|^{1-\alpha}}\right]e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)\\&\cdot\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\psi_{j_1-3}(\eta_1+\eta_2-2\xi)\psi_{\leq k_2}(\eta_1-\xi)\psi_{k_2}(\eta_2-\xi)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,\end{aligned}$$

$$\begin{aligned}V_2(\xi,t)=&\iint_{\mathbb{R}^2}\left[\frac{\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{|\eta_1|^{1-\alpha}-|\xi-\eta_1-\eta_2|^{1-\alpha}}\right]e^{iAt\Phi(\xi,\eta_1,\eta_2)}\partial_{\eta_1}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)\\&\cdot\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\psi_{j_1-3}(\eta_1+\eta_2-2\xi)\psi_{\leq k_2}(\eta_1-\xi)\psi_{k_2}(\eta_2-\xi)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,\end{aligned}$$

$$V_3(\xi,t)=\iint_{\mathbb{R}^2}\left[\frac{\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{|\eta_1|^{1-\alpha}-|\xi-\eta_1-\eta_2|^{1-\alpha}}\right]e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)\partial_{\eta_1}$$
$$\cdot\,\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\psi_{j_1-3}(\eta_1+\eta_2-2\xi)\psi_{\le k_2}(\eta_1-\xi)\psi_{k_2}(\eta_2-\xi)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2,$$

$$V_4(\xi,t)=\iint_{\mathbb{R}^2}\left[\frac{\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{|\eta_1|^{1-\alpha}-|\xi-\eta_1-\eta_2|^{1-\alpha}}\right]e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)$$
$$\cdot\,\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\partial_{\eta_1}\left[\psi_{j_1-3}(\eta_1+\eta_2-2\xi)\psi_{\le k_2}(\eta_1-\xi)\psi_{k_2}(\eta_2-\xi)\right]\mathrm{d}\eta_1\,\mathrm{d}\eta_2.$$

Then we estimate V_1 to V_4. We first denote the symbol of V_1 by

$$\begin{aligned}
&m(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\\
&=\partial_{\eta_1}\left[\frac{\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{|\eta_1|^{1-\alpha}-|\xi-\eta_1-\eta_2|^{1-\alpha}}\right]\\
&=\frac{\partial_{\eta_1}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{|\eta_1|^{1-\alpha}-|\xi-\eta_1-\eta_2|^{1-\alpha}}\\
&\quad-\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)(1-\alpha)\frac{|\eta_1|^{-\alpha}-|\xi-\eta_1-\eta_2|^{-\alpha}}{(|\eta_1|^{1-\alpha}-|\xi-\eta_1-\eta_2|^{1-\alpha})^2},
\end{aligned}$$

where we used the fact that $\eta_1>0,\eta_2>0$ and $\xi-\eta_1-\eta_2<0$.

Noticing the cancellation of the factor $(|\eta_1|-|\xi-\eta_1-\eta_2|)$ in the fraction $\frac{|\eta_1|^{-\alpha}-|\xi-\eta_1-\eta_2|^{-\alpha}}{(|\eta_1|^{1-\alpha}-|\xi-\eta_1-\eta_2|^{1-\alpha})^2}$, we have the symbol estimate

$$\begin{aligned}
&\left\|\frac{|\eta_1|^{-\alpha}-|\eta_3|^{-\alpha}}{(|\eta_1|^{1-\alpha}-|\eta_3|^{1-\alpha})^2}\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)\upsilon_+(\eta_1)\upsilon_+(\eta_2)\right.\\
&\qquad\left.\cdot\,\psi_{\le k_2}(-\eta_2-\eta_3)\psi_{k_2}(-\eta_1-\eta_3)\right\|_{S^\infty}\\
&\lesssim\frac{2^{(-\alpha-1)j_1}2^{k_2}}{[2^{-\alpha j_1}2^{k_2}]^2}=2^{(\alpha-1)j_1-k_2}.
\end{aligned}$$

By Proposition B.1, we have

$$\begin{aligned}
&\left\|m(\eta_1,\eta_2,\eta_3)\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)\upsilon_+(\eta_1)\upsilon_+(\eta_2)\psi_{\le k_2}(-\eta_2-\eta_3)\psi_{k_2}(-\eta_1-\eta_3)\right\|_{S^\infty}\\
&\lesssim 2^{(3-\alpha)j_1}[2^{-\alpha j_1}2^{k_2}]^{-1}+2^{(3-\alpha)j_1}2^{(\alpha-1)j_1-k_2}\\
&=2^{-k_2}(2^{3j_1}+2^{2j_1}).
\end{aligned}$$

Therefore,

$$\|V_1\|_{L^\infty_\xi}\lesssim(2^{3j_1}+2^{2j_1})2^{-k_2}\|\varphi_{j_1}\|_{L^2}\|\varphi_{j_2}\|_{L^2}\|\varphi_{j_3}\|_{L^\infty}. \tag{53}$$

Similarly we obtain the estimates for V_2–V_4 by using the above symbol estimates.

$$\begin{aligned}
\|V_2\|_{L^\infty_\xi}&\lesssim 2^{3j_1-k_2}\|\partial_{\eta_1}\hat{h}_{j_1}\|_{L^2}\|\varphi_{j_2}\|_{L^2}\|\varphi_{j_3}\|_{L^\infty},\\
\|V_3\|_{L^\infty_\xi}&\lesssim 2^{3j_1-k_2}\|\varphi_{j_1}\|_{L^\infty}\|\varphi_{j_2}\|_{L^2}\|\partial_{\eta_3}\hat{h}_{j_3}\|_{L^2},\\
\|V_4\|_{L^\infty_\xi}&\lesssim 2^{3j_1-k_2}(2^{-k_2}+2^{-j_1})\|\psi_{\le k_2}(\eta_1-\xi)\hat{\varphi}_{j_1}(\eta_1)\|_{L^2_{\eta_1}L^\infty_\xi}\\
&\qquad\cdot\|\psi_{k_2}(\eta_2-\xi)\hat{\varphi}_{j_2}(\eta_2)\|_{L^2_{\eta_2}L^\infty_\xi}\|\varphi_{j_3}\|_{L^\infty}\\
&\lesssim 2^{3j_1-k_2}(2^{-k_2}+2^{-j_1})2^{k_2}\|\hat{\varphi}_{j_1}\|_{L^\infty_\xi}\|\hat{\varphi}_{j_2}\|_{L^\infty_\xi}\|\varphi_{j_3}\|_{L^\infty}
\end{aligned}$$

$$\lesssim [2^{3j_1-k_2} + 2^{2j_1}]\|\hat{\varphi}_{j_1}\|_{L^\infty_\xi}\|\hat{\varphi}_{j_2}\|_{L^\infty_\xi}\|\varphi_{j_3}\|_{L^\infty}.$$

Finally, we take the summation over $\log_2[\varrho_1(t)] \le k_2 \le j_1 + 5$ to get

$$\begin{aligned}
&\Bigg\|\xi(|\xi|^{\gamma_l} + |\xi|^{\gamma_h}) \int_0^t \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{iAt\Phi(\xi,\eta_1,\eta_2)} \hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2) \\
&\quad \cdot \hat{h}_{j_3}(\xi - \eta_1 - \eta_2)\psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi) \\
&\quad \Bigg[\sum_{k_2=\log_2[\varrho_1(\tau)]}^{j_1+5} \psi_{\le k_2}(\eta_1 - \xi)\psi_{k_2}(\eta_2 - \xi)\Bigg] \,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau\Bigg\|_{L^\infty_\xi} \qquad (54)\\
&\lesssim \int_0^t \frac{1}{\tau+1}\Bigg[\max\{2^{j_1+5}, \varrho_1(\tau)\}(\|\varphi_{j_1}\|_{H^s} + \|\partial_\xi \hat{h}_{j_1}\|_{L^2_\xi})\|\varphi_{j_2}\|_{L^2}\|\varphi_{j_3}\|_{B^{(2-\alpha),1}} \\
&\quad + \max\{2^{j_1+5}, \varrho_1(\tau)\}\|\varphi_{j_1}\|_Z\|\varphi_{j_2}\|_Z\|\varphi_{j_3}\|_{B^{2-\alpha,2+\delta}}\Bigg]\,\mathrm{d}\tau.
\end{aligned}$$

Notice that $\|\varphi(t)\|_{B^{2-\alpha,2+\delta}} \lesssim \varepsilon_1(t+1)^{-1/2}$ from Lemma 4.3, so the right-hand-side of above inequality is integrable. It is also summable with respect to j_1, j_2, j_3 under $|j_3 - j_2| \le 1$ and $|j_3 - j_1| \le 1$. By using the bootstrap assumption, we get

$$\sum_{\mathcal{P}_2} (54) \lesssim \varepsilon_1^3 \lesssim \varepsilon_0.$$

Near space resonance. Next we estimate (51). Since the integrals are symmetric in η_1 and η_2, we can assume without loss of generality that $k_4 \ge k_3$. Using integration by parts, we write the integral as

$$\begin{aligned}
&\iint_{\mathbb{R}^2} \frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{2(2-\alpha)iAt(|\eta_1|^{1-\alpha} - |\xi - \eta_1 - \eta_2|^{1-\alpha})}\partial_{\eta_1} e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2) \\
&\cdot \hat{h}_{j_3}(\xi - \eta_1 - \eta_2)[1 - \psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi)]\left[\psi_{\le k_4}\left(\eta_1 - \frac{\xi}{3}\right)\psi_{k_4}\left(\eta_2 - \frac{\xi}{3}\right)\right]\mathrm{d}\eta_1\,\mathrm{d}\eta_2, \\
= &- \frac{W_1 + W_2 + W_3 + W_4}{2i(2-\alpha)At},
\end{aligned}$$

where

$$\begin{aligned}
&W_1(\xi, t) \\
&= \iint_{\mathbb{R}^2} \partial_{\eta_1}\left[\frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{|\eta_1|^{1-\alpha} - |\xi - \eta_1 - \eta_2|^{1-\alpha}}\right] e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2) \\
&\quad \cdot \hat{h}_{j_3}(\xi - \eta_1 - \eta_2)[1 - \psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi)]\psi_{\le k_4}\left(\eta_1 - \frac{\xi}{3}\right)\psi_{k_4}\left(\eta_2 - \frac{\xi}{3}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2,
\end{aligned}$$

$$\begin{aligned}
&W_2(\xi, t) \\
&= \iint_{\mathbb{R}^2} \left[\frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{|\eta_1|^{1-\alpha} - |\xi - \eta_1 - \eta_2|^{1-\alpha}}\right] e^{iAt\Phi(\xi,\eta_1,\eta_2)}\partial_{\eta_1}\hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2) \\
&\quad \cdot \hat{h}_{j_3}(\xi - \eta_1 - \eta_2)[1 - \psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi)]\psi_{\le k_4}\left(\eta_1 - \frac{\xi}{3}\right)\psi_{k_4}\left(\eta_2 - \frac{\xi}{3}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2,
\end{aligned}$$

$W_3(\xi, t)$

$$= \iint_{\mathbb{R}^2} \left[\frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{|\eta_1|^{1-\alpha} - |\xi - \eta_1 - \eta_2|^{1-\alpha}} \right] e^{iAt\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}^+(\eta_1) \hat{h}_{j_2}^+(\eta_2) \partial_{\eta_1}$$
$$\cdot \hat{h}_{j_3}(\xi - \eta_1 - \eta_2)[1 - \psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi)] \psi_{\le k_4}\left(\eta_1 - \frac{\xi}{3}\right) \psi_{k_4}\left(\eta_2 - \frac{\xi}{3}\right) \mathrm{d}\eta_1 \, \mathrm{d}\eta_2,$$

$W_4(\xi, t)$

$$= \iint_{\mathbb{R}^2} \left[\frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{|\eta_1|^{1-\alpha} - |\xi - \eta_1 - \eta_2|^{1-\alpha}} \right] e^{iAt\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}^+(\eta_1) \hat{h}_{j_2}^+(\eta_2)$$
$$\cdot \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \partial_{\eta_1} \left[(1 - \psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi)) \psi_{\le k_4}\left(\eta_1 - \frac{\xi}{3}\right) \psi_{k_4}\left(\eta_2 - \frac{\xi}{3}\right) \right]$$
$$\mathrm{d}\eta_1 \, \mathrm{d}\eta_2.$$

Then we estimate W_1 to W_4. We first denote the symbol of W_1 by

$$\begin{aligned}
&\mu(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \\
=&\partial_{\eta_1} \left[\frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{|\eta_1|^{1-\alpha} - |\xi - \eta_1 - \eta_2|^{1-\alpha}} \right] \\
=&\frac{\partial_{\eta_1} \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{|\eta_1|^{1-\alpha} - |\xi - \eta_1 - \eta_2|^{1-\alpha}} \\
&- \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)(1 - \alpha) \frac{|\eta_1|^{-\alpha} + |\xi - \eta_1 - \eta_2|^{-\alpha}}{(|\eta_1|^{1-\alpha} - |\xi - \eta_1 - \eta_2|^{1-\alpha})^2},
\end{aligned}$$

where we used the fact that $\eta_1 > 0, \eta_2 > 0$ and $\xi - \eta_1 - \eta_2 > 0$.

By Proposition B.1, we have

$$\begin{aligned}
&\left\| \mu(\eta_1, \eta_2, \eta_3) \psi_{j_1}(\eta_1) \psi_{j_2}(\eta_2) \psi_{j_3}(\eta_3) \upsilon_+(\eta_1) \upsilon_+(\eta_2) \psi_{\le k_4}\left(\eta_1 - \frac{\eta_1 + \eta_2 + \eta_3}{3}\right) \right. \\
&\left. \cdot \psi_{k_4}\left(\eta_2 - \frac{\eta_1 + \eta_2 + \eta_3}{3}\right) \right\|_{S^\infty} \\
\lesssim\ & 2^{(3-\alpha)j_1} [2^{-\alpha j_1} 2^{k_4}]^{-1} + 2^{(3-\alpha)j_1} 2^{-\alpha j_1} [2^{-\alpha j_1} 2^{k_4}]^{-2} \\
=\ & 2^{3j_1 - k_4} + 2^{3j_1 - 2k_4} = 2^{3j_1 - k_4}(1 + 2^{-k_4}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|W_1\|_{L_\xi^\infty} &\lesssim 2^{3j_1 - k_4}(1 + 2^{-k_4}) 2^{k_4} \|\hat{\varphi}_{j_1}\|_{L_\xi^\infty} \|\hat{\varphi}_{j_2}\|_{L_\xi^\infty} \|\varphi_{j_3}\|_{L^\infty} \\
&\lesssim 2^{3j_1}(1 + 2^{-k_4}) \|\hat{\varphi}_{j_1}\|_{L_\xi^\infty} \|\hat{\varphi}_{j_2}\|_{L_\xi^\infty} \|\varphi_{j_3}\|_{L^\infty}, \\
\|W_2\|_{L_\xi^\infty} &\lesssim 2^{3j_1 - k_4} \|\partial_\xi \hat{h}_{j_1}\|_{L^2} \|\varphi_{j_2}\|_{L^2} \|\varphi_{j_3}\|_{L^\infty}, \\
\|W_3\|_{L_\xi^\infty} &\lesssim 2^{3j_1 - k_4} \|\partial_\xi \hat{h}_{j_3}\|_{L^2} \|\varphi_{j_2}\|_{L^2} \|\varphi_{j_1}\|_{L^\infty}, \\
\|W_4\|_{L_\xi^\infty} &\lesssim 2^{3j_1 - k_4}(2^{-k_4} + 2^{-j_1}) 2^{k_4} \|\hat{\varphi}_{j_1}\|_{L_\xi^\infty} \|\hat{\varphi}_{j_2}\|_{L_\xi^\infty} \|\varphi_{j_3}\|_{L^\infty} \\
&\lesssim (2^{3j_1 - k_4} + 2^{2j_1}) \|\hat{\varphi}_{j_1}\|_{L_\xi^\infty} \|\hat{\varphi}_{j_2}\|_{L_\xi^\infty} \|\varphi_{j_3}\|_{L^\infty}.
\end{aligned}$$

Finally, we take the summation over $\log_2[\varrho_1(t)] \le k_4 \le j_1 + 5$ to get

$$
\begin{aligned}
&\bigg\| \xi(|\xi|^{\gamma_l} + |\xi|^{\gamma_h}) \int_0^t \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{iAt\Phi(\xi,\eta_1,\eta_2)} \hat{h}^+_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2) \\
&\hat{h}_{j_3}(\xi - \eta_1 - \eta_2)[1 - \psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi)] \\
&\cdot \Big[\sum_{k_4=\log_2[\varrho_1(\tau)]}^{j_1+5} \psi_{k_3}\Big(\eta_1 - \frac{\xi}{3}\Big)\psi_{k_4}\Big(\eta_2 - \frac{\xi}{3}\Big)\Big] \,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\tau \bigg\|_{L^\infty_\xi} \qquad (55)\\
&\lesssim \int_0^t \frac{1}{\tau+1} \bigg[\max\{2^{j_1+5}, \varrho_1(\tau)\}(\|\varphi_{j_1}\|_{H^s} + \|\partial_\xi \hat{h}_{j_1}\|_{L^2_\xi})\|\varphi_{j_2}\|_{L^2}\|\varphi_{j_3}\|_{B^{(2-\alpha),1}} \\
&\qquad + \max\{2^{j_1+5}, \varrho_1(\tau)\}\|\varphi_{j_1}\|_Z\|\varphi_{j_2}\|_Z\|\varphi_{j_3}\|_{B^{2-\alpha,2+\delta}} \bigg] \,\mathrm{d}\tau.
\end{aligned}
$$

From Lemma 4.3, $\|\varphi(t)\|_{B^{2-\alpha,2+\delta}} \lesssim \varepsilon_1(t+1)^{-1/2}$, so the right-hand-side of above inequality is integrable. It is also summable with respect to j_1, j_2, j_3 under $|j_3 - j_2| \le 1$ and $|j_3 - j_1| \le 1$. By using the bootstrap assumption, we get

$$\sum_{\mathcal{P}_2} (55) \lesssim \varepsilon_1^3 \lesssim \varepsilon_0.$$

The case of $(+,-)$ **or** $(-,+)$**.** If $\eta_1 > 0, \eta_2 < 0$, in the support of the integrand, then there is space-time resonance $(\xi, -\xi)$. We use the unit decomposition

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \psi_{k_1}(\eta_1 - \xi)\psi_{k_2}(\eta_2 + \xi) = 1,$$

and estimate

$$
\begin{aligned}
&\iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{iAt\Phi(\xi,\eta_1,\eta_2)} \hat{h}^+_{j_1}(\eta_1)\hat{h}^-_{j_2}(\eta_2)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \\
&\qquad\qquad \sum_{(k_1,k_2)\in\mathbb{Z}^2} \psi_{k_1}(\eta_1 - \xi)\psi_{k_2}(\eta_2 + \xi)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.
\end{aligned}
$$

If $\eta_1 < 0, \eta_2 > 0$, in the support of the integrand, then there is space-time resonance $(-\xi, \xi)$. We use the unit decomposition

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} \psi_{k_1}(\eta_1 + \xi)\psi_{k_2}(\eta_2 - \xi) = 1,$$

and estimate

$$
\begin{aligned}
&\iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{iAt\Phi(\xi,\eta_1,\eta_2)} \hat{h}^-_{j_1}(\eta_1)\hat{h}^+_{j_2}(\eta_2)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \\
&\qquad\qquad \sum_{(k_1,k_2)\in\mathbb{Z}^2} \psi_{k_1}(\eta_1 + \xi)\psi_{k_2}(\eta_2 - \xi)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.
\end{aligned}
$$

We can integrate by parts and estimate each integral in the same way as above, and get the same estimate as in (54).

7.5. **Resonant frequencies.** In this section, we estimate the integral

$$\iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{iAt\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \cdot \psi_{k_1}(\eta_1 - \xi_1)\psi_{k_2}(\eta_2 - \xi_2)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \tag{56}$$

under the conditions that

$$|j_1 - j_3| \le 1, \qquad |j_2 - j_3| \le 1, \qquad k_1 < \log_2[\varrho_1(t)], \qquad k_2 < \log_2[\varrho_1(t)],$$

where $\varrho_1(t)$ is defined in (52), then sum the result over $k_1, k_2 < \log_2(\varrho_1(t))$. Notice that when t is large enough, $\psi_{k_1}(\eta_1 - \xi_1)\psi_{k_2}(\eta_2 - \xi_2)$ will be supported on the set where $\psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi) = 1$ when $(\xi_1, \xi_2) = (\xi, \xi)$; and $\psi_{k_1}(\eta_1 - \xi_1)\psi_{k_2}(\eta_2 - \xi_2)$ will be supported on the set where $1 - \psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi) = 1$ when $(\xi_1, \xi_2) = (\frac{\xi}{3}, \frac{\xi}{3})$. So we can ignore the cut-off function $\psi_{j_1-3}(\eta_1 + \eta_2 - 2\xi)$ in the above integral.

If $m < \log_2 \varrho_1(t) \le m + 1$ for $m \in \mathbb{Z}$, then $-\infty < k_i \le m$ for $i = 1, 2$, and

$$\sum_{k_i=-\infty}^{m} \psi_{k_i}(\xi) \quad \text{is supported in} \quad \left\{\xi \in \mathbb{R} \mid |\xi| < \frac{8}{5} \cdot 2^m \right\}.$$

Thus, after summing over (k_1, k_2), we only need to consider (56) with a cutoff function in the integrand of the form

$$\mathfrak{b}(\xi, \eta_1, \eta_2, t) := \psi\left(\frac{\eta_1 - \xi_1}{\varrho(t)}\right) \cdot \psi\left(\frac{\eta_2 - \xi_2}{\varrho(t)}\right), \tag{57}$$

where the function $\varrho(t)$ is defined by

$$\varrho(t) = 2^m \text{ if } 2^m < \varrho_1(t) \le 2^{m+1}.$$

To make the expressions shorter, we write $\mathfrak{b}(\xi, \eta_1, \eta_2, t)$ as $\mathfrak{b}$ when there is no ambiguity.

From (52), we have

$$\frac{1}{2}(t + 1)^{-0.49} \le \varrho(t) < (t + 1)^{-0.49}. \tag{58}$$

The points $(\xi_1, \xi_2) \in \{(\xi/3, \xi/3), (\xi, \xi), (\xi, -\xi), (-\xi, \xi)\}$ are the space and space-time resonances in (32)–(33). We therefore need to estimate

$$\iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{iAt\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \mathfrak{b}(\xi, \eta_1, \eta_2, t)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2,$$

with the cutoff function (57) replacing $\psi_{k_1}(\eta_1 - \xi_1)\psi_{k_2}(\eta_2 - \xi_2)$, in which case the integral is taken over one of the following four disjoint sets

$$A_1 = \left\{(\eta_1, \eta_2) \,\middle|\, \left|\eta_1 - \frac{\xi}{3}\right| < \frac{8}{5}\varrho(t), \quad \left|\eta_2 - \frac{\xi}{3}\right| < \frac{8}{5}\varrho(t)\right\},$$

$$A_2 = \left\{(\eta_1, \eta_2) \,\middle|\, |\eta_1 - \xi| < \frac{8}{5}\varrho(t), \quad |\eta_2 - \xi| < \frac{8}{5}\varrho(t)\right\},$$

$$A_3 = \left\{(\eta_1, \eta_2) \,\middle|\, |(\eta_1 - \xi)| < \frac{8}{5}\varrho(t), \quad |\eta_2 - (-\xi)| < \frac{8}{5}\varrho(t)\right\},$$

$$A_4 = \left\{(\eta_1, \eta_2) \,\middle|\, |\eta_1 - (-\xi)| < \frac{8}{5}\varrho(t), \quad |\eta_2 - \xi| < \frac{8}{5}\varrho(t)\right\}.$$

The regions A_1, A_2, A_3, A_4 are discs centered at $(\xi/3, \xi/3)$, (ξ, ξ), $(\xi, -\xi)$, and $(-\xi, \xi)$, respectively. The region A_1 corresponds to space resonances $\xi = \xi/3 + \xi/3 + \xi/3$, while A_2, A_3, A_4 correspond to space-time resonances $\xi = \xi + \xi - \xi$.

7.5.1. *Space resonances.* When $(\eta_1, \eta_2) \in A_1$, we can expand $\mathbf{T}_1/\Phi$ around $(\xi, \xi/3, \xi/3)$ as

$$\frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} = \frac{3^{2-\alpha} - 2^{3-\alpha} + 1}{3 - 3^{2-\alpha}}\xi + O\left(\left|\eta_1 - \frac{\xi}{3}\right|^2 + \left|\eta_2 - \frac{\xi}{3}\right|^2 \right). \tag{59}$$

For $m \in \mathbb{Z}$, let $t_m = 2^{-m/0.49} - 1$ denote the time such that $\log_2 \varrho_1(t_m) = m$, and for $t \in [0, \infty)$, let $M(t) \in \mathbb{Z}$ be the negative integer such that $M(t) < \log_2 \varrho_1(t) \leq M(t) + 1$. Then $\varrho(t)$ and the cut-off function $\mathfrak{b}(\xi, \eta_1, \eta_2, t)$ in (57) are discontinuous at $t = t_m$. After writing

$$e^{iA\tau\Phi(\xi,\eta_1,\eta_2)} = \frac{1}{iA\Phi(\xi, \eta_1, \eta_2)} \left[\partial_\tau e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\right],$$

and integrating by parts with respect to τ in each time interval between the time discontinuities, we obtain

$$\begin{aligned}
&\int_1^t i\xi \iint_{\mathbb{R}^2} \mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{iA\tau\Phi(\xi,\eta_1,\eta_2)} \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau) \\
&\qquad \mathfrak{b}\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2\, \mathrm{d}\tau \\
&\quad = \frac{1}{A}\left(J_1 - \int_1^t J_2(\tau)\, \mathrm{d}\tau\right),
\end{aligned}$$

where

$$\begin{aligned}
&J_1 \\
&= \iint_{\mathbb{R}^2} \xi \frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau) e^{iA\tau\Phi(\xi,\eta_1,\eta_2)} \\
&\quad \cdot \mathfrak{b}(\tau)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \Big|_{\tau = t_{M(t)}}^{\tau = t} \\
&\quad + \sum_{m=M(t)+1}^{0} \iint_{\mathbb{R}^2} \xi \frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau) \\
&\quad e^{iA\tau\Phi(\xi,\eta_1,\eta_2)} \mathfrak{b}(\tau)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2 \Big|_{\tau = t_m}^{\tau = t_{m-1}}, \\
&J_2(\tau) \\
&= \xi \iint_{\mathbb{R}^2} \frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} e^{iA\tau\Phi(\xi,\eta_1,\eta_2)} \\
&\quad \cdot \partial_\tau \left[\hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau)\right] \mathfrak{b}(\tau)\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2.
\end{aligned}$$

For J_1, we have from (59) that

$$\begin{aligned}
&\Bigg|(|\xi|^{\gamma_l} + |\xi|^{\gamma_h}) \iint_{\mathbb{R}^2} \mathfrak{b}(\tau)\xi \frac{\mathbf{T}_1'(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1, \tau)\hat{h}_{j_2}(\eta_2, \tau)\hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau) \\
&\quad \cdot e^{iA\tau\Phi(\xi,\eta_1,\eta_2)}\, \mathrm{d}\eta_1\, \mathrm{d}\eta_2\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\lesssim \left|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\iint_{\mathbb{R}^2}\mathfrak{b}(\tau)\xi^2\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)e^{i\tau\Phi(\xi,\eta_1,\eta_2)}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right| \\
&\quad+(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\iint_{\mathbb{R}^2}\mathfrak{b}(\tau)[\varrho(\tau)]^2\left|\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)\right|\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\
&\lesssim \|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\hat{h}_{j_1}\|_{L^\infty_\xi}\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\hat{h}_{j_2}\|_{L^\infty_\xi}\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\hat{h}_{j_3}\|_{L^\infty_\xi}\left([\varrho(\tau)]^2+[\varrho(\tau)]^4\right).
\end{aligned}$$

If $(\eta_1,\eta_2)\in A_1$, then the number of terms in the sum over j_1, j_2, j_3 is of the order $\log(t+1)$, so the right-hand side of this inequality is uniformly bounded for $\tau\geq 0$ after summing over j_1, j_2, j_3.

After taking the time derivative ∂_τ, the term J_2 can be written as a sum of three terms:

$$\begin{aligned}
&\xi\iint_{\mathbb{R}^2}\frac{\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}e^{iA\tau\Phi(\xi,\eta_1,\eta_2)}\left[\partial_\tau\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)\right] \\
&\cdot\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2, \\
&\xi\iint_{\mathbb{R}^2}\frac{\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}e^{iA\tau\Phi(\xi,\eta_1,\eta_2)}\left[\hat{h}_{j_1}(\eta_1,\tau)\partial_\tau\hat{h}_{j_2}(\eta_2,\tau)\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)\right] \\
&\cdot\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2, \\
&\xi\iint_{\mathbb{R}^2}\frac{\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)}{\Phi(\xi,\eta_1,\eta_2)}e^{iA\tau\Phi(\xi,\eta_1,\eta_2)}\left[\hat{h}_{j_1}(\eta_1,\tau)\hat{h}_{j_2}(\eta_2,\tau)\partial_\tau\hat{h}_{j_3}(\xi-\eta_1-\eta_2,\tau)\right] \\
&\cdot\mathfrak{b}(\xi,\eta_1,\eta_2,\tau)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2.
\end{aligned}$$

Using equation (41), the bootstrap assumptions, and Lemma 4.3, we have

$$\begin{aligned}
&\|\partial_\tau\hat{h}\|_{L^\infty_\xi} \\
\lesssim&\|\xi\iint_{\mathbb{R}^2}\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iA\tau\Phi(\xi,\eta_1,\eta_2)}\hat{h}(\xi-\eta_1-\eta_2)\hat{h}(\eta_1)\hat{h}(\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\|_{L^\infty_\xi} \\
&+\|\widehat{\mathcal{N}_{\geq 5}(\varphi)}\|_{L^\infty_\xi} \\
\lesssim&\|\varphi\|_{H^1}^2\cdot\sum_{j=0}^{\infty}\|\varphi\|_{B^1}^{2j+1} \\
\lesssim&\varepsilon_1^3(\tau+1)^{2p_0-\frac{1}{2}}.
\end{aligned}$$

Therefore, using this estimate in the J_2-terms and the fact that $\varepsilon_1^3\lesssim\varepsilon_0$, we get that

$$\begin{aligned}
|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})J_2(\tau)|&\lesssim\sum\|h_{\ell_1}\|_Z\|\partial_\tau\hat{h}_{\ell_2}\|_{L^\infty_\xi}\|h_{\ell_3}\|_Z[\varrho(\tau)]^2 \\
&\lesssim\varepsilon_0(\tau+1)^{p_0-\frac{1}{2}}[\varrho(\tau)]^2\sum\|h_{\ell_1}\|_Z\|h_{\ell_3}\|_Z,
\end{aligned}$$

where the summation is taken over permutations ℓ_1, ℓ_2, ℓ_3 of j_1, j_2, j_3 for (η_1,η_2) in the space-resonance region A_1. Again, since the number of summations is of the order $\log(\tau+1)$, the resulting sum is integrable over $\tau\in(1,\infty)$.

7.5.2. *Space-time resonances.* We now use modified scattering to control U in (44). We need to estimate

$$\begin{aligned}
&A'\iint_{A_2\cup A_3\cup A_4}i\xi\mathfrak{b}(\xi,\eta_1,\eta_2,t)\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)e^{iAt\Phi(\xi,\eta_1,\eta_2)}\hat{h}(\xi-\eta_1-\eta_2) \\
&\qquad\cdot\hat{h}(\eta_1)\hat{h}(\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\
&-\frac{2\pi iA'\xi|\xi|^\alpha}{(\alpha-1)(2-\alpha)A(t+1)}\left[\mathbf{T}_1'(\xi,\xi,-\xi)+\mathbf{T}_1'(\xi,-\xi,\xi)+\mathbf{T}_1'(-\xi,\xi,\xi)\right]|\hat{h}(\xi,t)|^2\hat{h}(\xi,t).
\end{aligned} \tag{60}$$

The estimates for A_2, A_3, and A_4 are similar, so we only present the details for the A_2 integral. The corresponding integral for A_2 in (60) can be decomposed into

$$A' i\xi \iint_{A_2} \mathfrak{b}(\xi,\eta_1,\eta_2,t) e^{iAt\Phi(\xi,\eta_1,\eta_2)} \cdot \Big[\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{h}(\xi-\eta_1-\eta_2)\hat{h}(\eta_1)\hat{h}(\eta_2) - \mathbf{T}_1'(\xi,\xi,-\xi)|\hat{h}(\xi,t)|^2\hat{h}(\xi,t)\Big]\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \tag{61}$$

and

$$A' i\xi \mathbf{T}_1'(\xi,\xi,-\xi)|\hat{h}(\xi,t)|^2\hat{h}(\xi,t) \left[\iint_{A_2} e^{it\Phi(\xi,\eta_1,\eta_2)}\psi\left(\frac{\eta_1-\xi}{\varrho(t)}\right)\cdot\psi\left(\frac{\eta_2-\xi}{\varrho(t)}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2 - \frac{2\pi|\xi|^\alpha}{(\alpha-1)(2-\alpha)A(t+1)}\right]. \tag{62}$$

The estimates for (61) are achieved by a Taylor expansion

$$\begin{aligned}
&A'\Big|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})i\xi\iint_{A_2} e^{it\Phi(\xi,\eta_1,\eta_2)}\mathfrak{b}(\xi,\eta_1,\eta_2,t)\cdot\Big[\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{h}_{j_1}(\eta_1)\\
&\quad\cdot\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2) - \mathbf{T}_1'(\xi,\xi,-\xi)|\hat{h}(\xi,t)|^2\hat{h}(\xi,t)\Big]\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\Big|\\
&\lesssim (|\xi|^{\gamma_l}+|\xi|^{\gamma_h})|\xi|\iint_{A_2}\Big|\partial_{\eta_1}\Big[\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\\
&\quad\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\Big]\Big|_{\eta_1=\eta_1'}\cdot(\xi-\eta_1)\Big|\\
&\quad+\Big|\partial_{\eta_2}\Big[\mathbf{T}_1'(\eta_1,\eta_2,\xi-\eta_1-\eta_2)\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)\hat{h}_{j_3}(\xi-\eta_1-\eta_2)\Big]\Big|_{\eta_2=\eta_2'}(\xi-\eta_2)\Big|\\
&\quad\mathrm{d}\eta_1\,\mathrm{d}\eta_2\\
&\lesssim \|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\hat{\varphi}_{j_1}\|_{L^\infty_\xi}\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\hat{\varphi}_{j_2}\|_{L^\infty_\xi}\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\hat{\varphi}_{j_3}\|_{L^\infty_\xi}[\varrho(t)]^3\\
&\qquad+\sum\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\hat{\varphi}_{\ell_1}\|_{L^\infty_\xi}\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\hat{\varphi}_{\ell_2}\|_{L^\infty_\xi}\|\mathcal{S}\varphi_{\ell_3}\|_{H^r}[\varrho(t)]^{5/2}\\
&\lesssim \varepsilon_0(t+1)^{-1.1},
\end{aligned}$$

where (η_1',η_2') in the first inequality is some point on the line segment connecting (ξ,ξ) and (η_1,η_2), and the summation in the second inequality is over permutations (ℓ_1,ℓ_2,ℓ_3) of (j_1,j_2,j_3). Taking a summation over j_1,j_2,j_3 and using the estimates in the above subsections together with the time-decay of $\varrho(t)$ in (58), we see that this term is integrable in time and is bounded by a constant multiple of ε_0^2.

As for (62), it suffices to estimate

$$\begin{aligned}
&\Big|A' i\xi(|\xi|+|\xi|^{r+4})\mathbf{T}_1'(\xi,\xi,-\xi)|\hat{h}(\xi,t)|^2\hat{h}(\xi,t)\\
&\quad\cdot\left[\iint_{A_2} e^{iAt\Phi(\xi,\eta_1,\eta_2)}\psi\left(\frac{\eta_1-\xi}{\varrho(t)}\right)\cdot\psi\left(\frac{\eta_2-\xi}{\varrho(t)}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2-\frac{2\pi|\xi|^\alpha}{(\alpha-1)(2-\alpha)At}\right]\Big|\\
&\lesssim \|(|\xi|+|\xi|^{r+4})\hat{\varphi}(\xi)\|_{L^\infty_\xi}\||\xi|\hat{\varphi}(\xi)\|_{L^\infty_\xi}\|(|\xi|+|\xi|^3)\hat{\varphi}(\xi)\|_{L^\infty_\xi}\\
&\quad\cdot\left\|\iint_{A_2} e^{iAt\Phi(\xi,\eta_1,\eta_2)}\psi\left(\frac{\eta_1-\xi}{\varrho(t)}\right)\cdot\psi\left(\frac{\eta_2-\xi}{\varrho(t)}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2-\frac{2\pi|\xi|^\alpha}{(\alpha-1)(2-\alpha)At}\right\|_{L^\infty_\xi}\\
&\lesssim \|\varphi\|_Z^3\left\|\iint_{A_2} e^{iAt\Phi(\xi,\eta_1,\eta_2)}\psi\left(\frac{\eta_1-\xi}{\varrho(t)}\right)\cdot\psi\left(\frac{\eta_2-\xi}{\varrho(t)}\right)\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right.
\end{aligned}$$

$$- \frac{2\pi|\xi|^\alpha}{(\alpha-1)(2-\alpha)At}\Bigg\|_{L^\infty_\xi}.$$

Writing $(\eta_1, \eta_2) = (\xi + \zeta_1, \xi + \zeta_2)$, we find from (31) that

$$\begin{aligned}\Phi(\xi, \eta_1, \eta_2) &= (\alpha-1)(2-\alpha)\frac{\xi}{|\xi|^{1+\alpha}}\zeta_1\zeta_2 + O\left(\frac{\zeta_1^3+\zeta_2^3}{|\xi|^{1+\alpha}}\right)\\ &= (\alpha-1)(2-\alpha)\frac{\xi}{|\xi|^{1+\alpha}}\zeta_1\zeta_2 + O\left([\varrho(t)]^3(t+1)^{(1+\alpha)p_0}\right).\end{aligned}$$

By the assumption of p_0 in (18) and $\varrho(t)$ satisfies (58), the error term is integrable in time, so we now only need to estimate

$$\begin{aligned} J_3 = \Bigg\| \iint_{\mathbb{R}^2} & e^{iAt(\alpha-1)(2-\alpha)\frac{\xi}{|\xi|^{1+\alpha}}\zeta_1\zeta_2}\psi\left(\frac{\zeta_1}{\varrho(t)}\right)\cdot\psi\left(\frac{\zeta_2}{\varrho(t)}\right)\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2 \\ & - \frac{2\pi|\xi|^\alpha}{(\alpha-1)(2-\alpha)At}\Bigg\|_{L^\infty_\xi}. \end{aligned} \tag{63}$$

Making the change of variables

$$\zeta_1 = x_1\sqrt{\frac{|\xi|^\alpha}{(\alpha-1)(2-\alpha)At}}, \qquad \zeta_2 = x_2\sqrt{\frac{|\xi|^\alpha}{(\alpha-1)(2-\alpha)At}},$$

in (63) and using the fact that $|\xi| \le (t+1)^{p_1}$, we find that

$$\begin{aligned} J_3 &\lesssim \frac{(t+1)^{\alpha p_1}}{t} \\ &\cdot\Bigg\|\iint_{\mathbb{R}^2} e^{ix_1x_2}\psi\left(\sqrt{\frac{|\xi|^\alpha}{(\alpha-1)(2-\alpha)At}}\frac{x_1}{\varrho(t)}\right)\cdot\psi\left(\sqrt{\frac{|\xi|^\alpha}{(\alpha-1)(2-\alpha)At}}\frac{x_2}{\varrho(t)}\right)\mathrm{d}x_1\,\mathrm{d}x_2 \\ &- 2\pi\Bigg\|_{L^\infty_\xi}. \end{aligned} \tag{64}$$

The integral identity

$$\int_{\mathbb{R}} e^{-ax^2-bx}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}} \qquad \text{for all } a, b \in \mathbb{C} \text{ with } \Re a > 0$$

gives that

$$\iint_{\mathbb{R}^2} e^{ix_1x_2}e^{-\frac{x_1^2}{B^2}}e^{-\frac{x_2^2}{B^2}}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \sqrt{\pi}B\int_{\mathbb{R}} e^{-\frac{x_2^2}{B^2}}e^{-\frac{B^2x_2^2}{4}}\,\mathrm{d}x_2 = 2\pi + O(B^{-1})$$

as $B \to \infty$, and therefore

$$\iint_{\mathbb{R}^2} e^{ix_1x_2}\psi\left(\frac{x_1}{B}\right)\psi\left(\frac{x_2}{B}\right)\mathrm{d}x_1\,\mathrm{d}x_2 = 2\pi + O(B^{-1/2}) \qquad \text{as } B \to \infty. \tag{65}$$

Using (65) with

$$B = \varrho(t)\sqrt{\frac{(\alpha-1)(2-\alpha)At}{|\xi|^\alpha}} = O(t^{0.01-\frac{\alpha p_0}{2(\alpha-1)}})$$

in (64) then yields

$$J_3 \lesssim (t+1)^{\alpha p_1 - 1}(t+1)^{-\frac{1}{2}[0.01-\frac{\alpha p_0}{2(\alpha-1)}]}.$$

By (18), the right-hand side decays faster in time than $1/t$, which implies that (62) is integrable in time and bounded by a constant multiple of ε_0^3.

Putting all the above estimates together, we conclude that

$$\int_0^\infty \|(|\xi|^{\gamma_l} + |\xi|^{\gamma_h}) U_1(\xi, t)\|_{L^\infty_\xi} \, \mathrm{d}t \lesssim \varepsilon_0. \tag{66}$$

7.6. **Higher-degree terms.** In this subsection, we prove $\|(|\xi|^{\gamma_l}+|\xi|^{\gamma_h})\widehat{\mathcal{N}_{\geq 5}(\varphi)}\|_{L^\infty_\xi}$ is integrable in time.

By Lemma 2.2 and Proposition B.1, we have

$$\left\|(|\xi|^{\gamma_l} + |\xi|^{\gamma_h})\widehat{\mathcal{N}_{\geq 5}(\varphi)}\right\|_{L^\infty_\xi} \lesssim \|\varphi\|_{H^s}^2 \sum_{n=2}^\infty \|\varphi\|_{B^{2-\alpha,2+\delta}}^{2n-1}.$$

Using the dispersive estimate in Lemma 4.3, we see the right-hand side is integrable in t, and

$$\int_0^\infty \|(|\xi|^{\gamma_l} + |\xi|^{\gamma_h})\widehat{\mathcal{N}_{\geq 5}(\varphi)}\|_{L^\infty_\xi} \, \mathrm{d}t \lesssim \varepsilon_0. \tag{67}$$

Finally, the use of (66) and (67) in (45) completes the proof of Lemma 4.5, and therefore the proof of Theorem 4.1.

Appendix A. Contour dynamics derivation for the GSQG front equation ($1 < \alpha < 2$).

In this appendix, we derive the GSQG front equation (4) for $1 < \alpha < 2$ by the method used in [29] for SQG fronts.

The front-solutions considered here have unbounded velocity fields as $|y| \to \infty$, and we interpret (2) in a distributional sense. Let $L^1_\alpha(\mathbb{R}^n)$ denote the space of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|}{1 + |\mathbf{x}|^{n+\alpha}} \, \mathrm{d}\mathbf{x} < \infty,$$

and let $\mathcal{D}'(\mathbb{R}^n)$ denote the usual space of Schwartz distributions. Then the fractional Laplacian

$$(-\Delta)^{\alpha/2} : L^1_\alpha(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$$

can be defined by [6]

$$\left\langle (-\Delta)^{\alpha/2} f, \phi \right\rangle = \int_{\mathbb{R}^n} f(\mathbf{x}) \cdot (-\Delta)^{\alpha/2} \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n).$$

Here, $(-\Delta)^{\alpha/2}\phi$ is defined as a Fourier multiplier or singular integral [39]. For $1 < \alpha < 2$, the only α-harmonic solutions $f \in L^1_\alpha(\mathbb{R}^n)$ of $(-\Delta)^{\alpha/2} f = 0$ are the affine functions [7, 16], so if we require that $\mathbf{u}(\mathbf{x})$ has sublinear growth in $\mathbf{x}$, then (2) determines $\mathbf{u}$ uniquely up to a spatially uniform constant. This constant can be removed by a transformation into a suitable reference frame, and for definiteness we set it equal to zero.

To start with, we consider the shear flow solutions $\bar{\mathbf{u}}(y) = (U(y), 0)$ associated with a planar front

$$\bar{\theta}(y) = \begin{cases} \theta_+ & \text{if } y > 0, \\ \theta_- & \text{if } y < 0. \end{cases} \tag{68}$$

In that case, (2) reduces to the equation

$$|\partial_y|^\alpha U(y) = -\frac{\Theta}{g_\alpha} \delta(y), \tag{69}$$

where δ is the delta-distribution and Θ, g_α are defined in (5). Equation (69) has the sublinear solution

$$U(y) = \Theta C'_\alpha |y|^{\alpha-1}, \qquad \bar{\mathbf{u}}(y) = (U(y), 0), \tag{70}$$

where

$$C'_\alpha = -\frac{1}{\sqrt{\pi}} \sin\left(\frac{\pi\alpha}{2}\right) \Gamma\left(\frac{1-a}{2}\right) \Gamma\left(\frac{a}{2}\right).$$

We now derive contour dynamics equations for GSQG ($1 < \alpha < 2$) front solutions (3) whose velocity field has the asymptotic behavior

$$\mathbf{u}(\mathbf{x}, t) = (\Theta C'_\alpha |y|^{\alpha-1}, 0) + o(1) \quad \text{as } |y| \to \infty$$

by decomposing the solutions into a planar shear flow and a perturbation whose velocity field approaches zero as $|y| \to \infty$.

We denote the front $y = \varphi(x, t)$ by $\Gamma(t) = \partial\Omega(t)$, and consider its motion on a time interval $0 \leq t \leq T$ for some $T > 0$. We assume that:

(i) $\varphi(\cdot, t) \in C^1(\mathbb{R})$ and $\varphi(x, t)$ is bounded on $\mathbb{R} \times [0, T]$;

(ii) $\varphi_x(x, t) = O(|x|^{-(\alpha-1+\beta)})$ as $|x| \to \infty$ for some $\beta > 0$.

In that case, all of the integrals in the following converge.

We choose $h > 0$ such that $-h < \inf\{\varphi(x, t) : (x, t) \in \mathbb{R} \times [0, T]\}$, and let

$$\tilde{\theta}(y) = \begin{cases} \theta_+ & \text{if } y > -h, \\ \theta_- & \text{if } y < -h, \end{cases}, \qquad \tilde{\mathbf{u}}(y) = \left(\Theta C'_\alpha |y + h|^{\alpha-1}, 0\right), \tag{71}$$

be the planar front solution (68), (70) translated to $y = -h$. This front is an artificial front which does not intersect the actual front. We introduce it to obtain absolutely convergent potential representations, and the final solution is smooth across $y = -h$.

We decompose the front solution (3) as

$$\theta(\mathbf{x}, t) = \tilde{\theta}(y) + \theta^*(\mathbf{x}, t),$$

where $\tilde{\theta}$ is defined in (71), and

$$\theta^*(\mathbf{x}, t) = \begin{cases} -\Theta/g_\alpha & \text{if } -h < y < \varphi(x, t), \\ 0 & \text{otherwise.} \end{cases}$$

We denote the support of $\theta^*(\cdot, t)$ by $\Omega^*(t)$. The corresponding decomposition of the velocity field is

$$\mathbf{u}(\mathbf{x}, t) = \tilde{\mathbf{u}}(y) + \mathbf{u}^*(\mathbf{x}, t),$$

where $\tilde{\mathbf{u}}$ is defined in (71). By use of a Riesz potential representation [42], we find that $\mathbf{u}^* = \nabla^\perp(-\Delta)^{-\alpha/2}\theta^*$ is given by

$$\mathbf{u}^*(\mathbf{x}, t) = \Theta \, \mathrm{p.v.} \int_{\Omega^*(t)} \nabla^\perp_{\mathbf{x}'} \frac{1}{|\mathbf{x} - \mathbf{x}'|^{2-\alpha}} \, \mathrm{d}\mathbf{x}'.$$

Writing $\mathbf{x}' = (x', y')$, we see that the integrand is $O\left(|x'|^{-(3-\alpha)}\right)$ as $|x'| \to \infty$ and compactly supported in y', so this principal value integral converges absolutely at infinity for $\alpha < 2$.

Applying Green's theorem on a truncated region with $|x - x'| < \lambda$, and taking the limit $\lambda \to \infty$ (as in [29]), we get that

$$\mathbf{u}^*(\mathbf{x}, t) = \left(u^*(x, y, t), v^*(x, y, t)\right),$$

$$u^*(x,y,t) = -\Theta \int_{\mathbb{R}} \frac{1}{\left[(x-x')^2 + (y-\varphi(x',t))^2\right]^{\frac{2-\alpha}{2}}} - \frac{1}{\left((x-x')^2 + (y+h)^2\right)^{\frac{2-\alpha}{2}}} \,\mathrm{d}x',$$
$$v^*(x,y,t) = -\Theta \int_{\mathbb{R}} \frac{\varphi_{x'}(x',t)}{\left[(x-x')^2 + (y-\varphi(x',t))^2\right]^{\frac{2-\alpha}{2}}} \,\mathrm{d}x'.$$

The integral for u^* converges since the integrand is $O(|x'|^{3-\alpha})$ as $|x'| \to \infty$, while the integral for v^* converges since we assume that $\varphi_{x'}(x',t) = O(|x'|^{-(\alpha-1+\beta)})$ as $|x'| \to \infty$ for some $\beta > 0$.

Let $\mathbf{x} = (x, \varphi(x,t))$ be a point on the front and denote by

$$\mathbf{n}(\mathbf{x},t) = \frac{1}{\sqrt{1+\varphi_x^2(x,t)}}(-\varphi_x(x,t), 1)$$

the unit upward normal to $\Gamma(t)$ at $\mathbf{x}$. The motion of the front is determined by the normal velocity $\mathbf{u}\cdot\mathbf{n}$, so the front $y = \varphi(x,t)$ moves with the upward normal velocity

$$\mathbf{u}\cdot\mathbf{n} = \frac{\varphi_t}{\sqrt{1+\varphi_x^2}}.$$

Using the previous expressions for $\mathbf{u}$, we therefore get that

$$\begin{aligned}
&\varphi_t(x,t) \\
&= \Theta \int_{\mathbb{R}} \frac{\varphi_x(x,t) - \varphi_{x'}(x',t)}{\left[(x-x')^2 + (y-\varphi(x',t))^2\right]^{\frac{2-\alpha}{2}}} - \frac{\varphi_x(x,t)}{\left[(x-x')^2 + (\varphi(x,t)+h)^2\right]^{\frac{2-\alpha}{2}}} \,\mathrm{d}x' \\
&\quad - \Theta C'_\alpha \left|\varphi(x,t)+h\right|^{\alpha-1} \varphi_x(x,t) \\
&= \Theta(I_1(x,t) + I_2(x,t) + I_3(x,t)),
\end{aligned}$$

where

We can express I_2 as a Fourier multiplier. Note that $I_2 = 0$ if $\varphi = 1$, and if $\varphi(x) = e^{i\xi x}$ with $\xi \neq 0$, then

$$\begin{aligned}
I_2(x) &= i\xi e^{i\xi x} \int_{\mathbb{R}} \left[\frac{1-e^{i\xi(x'-x)}}{|x-x'|^{2-\alpha}} - \frac{1}{((x')^2+1)^{\frac{2-\alpha}{2}}}\right] \mathrm{d}x' \\
&= 2i\xi e^{i\xi x} \int_0^\infty \left[\frac{1-\cos\xi s}{s^{2-\alpha}} - \frac{1}{(s^2+1)^{\frac{2-\alpha}{2}}}\right] \mathrm{d}s.
\end{aligned}$$

Using the identity

$$\int_0^\infty \left[\frac{1}{(s^2+1)^{\frac{2-\alpha}{2}}} - \frac{1}{(s^2+c^2)^{\frac{2-\alpha}{2}}}\right] \mathrm{d}s = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1-\alpha}{2}\right)}{2\Gamma\left(1-\frac{\alpha}{2}\right)}\left(1-|c|^{\alpha-1}\right), \tag{72}$$

with $c = 1/|\xi|$, the change of variable $s' = |\xi|s$, and identities for generalized hypergeometric functions, we get that

$$I_2(x) = -\frac{\sqrt{\pi}\,\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)} i\xi e^{i\xi x} - iA\xi|\xi|^{1-\alpha}e^{i\xi x},$$

where A is given in (5). It follows that

$$I_2 = -\frac{\sqrt{\pi}\,\Gamma\left(\frac{1-\alpha}{2}\right)}{2\Gamma\left(1-\frac{\alpha}{2}\right)}\varphi_x - A|\partial_x|^{1-\alpha}\varphi_x.$$

For I_3, we find after some algebra and the use of(72) that

$$I_3 = \frac{\sqrt{\pi}\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}\varphi_x.$$

Putting everything together, we get the contour dynamics equation for GSQG $(1<\alpha<2)$ fronts

$$\begin{aligned}
&\varphi_t(x,t) + \Theta A|\partial_x|^{1-\alpha}\varphi_x(x,t) \\
&\quad - \Theta\int_{\mathbb{R}}\left\{\frac{\varphi_x(x,t)-\varphi_{x'}(x',t)}{\left[(x-x')^2+(y-\varphi(x',t))^2\right]^{1-\alpha/2}} - \frac{\varphi_x(x,t)-\varphi_{x'}(x',t)}{|x-x'|^{2-\alpha}}\right\}\mathrm{d}x' = 0,
\end{aligned}$$

which is equivalent to (4). This equation also agrees with the result obtained in [25] by a regularization method.

Appendix B. **Symbol estimates.** The symbol $\mathbf{T}_n$ is defined in (12).

Proposition B.1. *Assume $n \geq 1$, $j_1, j_2, \cdots, j_{2n+1} \in \mathbb{Z}$, and $j_1 \geq \cdots \geq j_{2n+1}$.*

$$\begin{aligned}
&\|\mathbf{T}_n(\eta_1,\eta_2,\cdots,\eta_{2n+1})\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\|_{S^\infty} \\
&\qquad \lesssim 2^{(2-\alpha)j_2+j_3+\cdots+j_{2n+1}}(1+\ln n).
\end{aligned}\tag{73}$$

$$\begin{aligned}
&\|\partial_{\eta_l}\mathbf{T}_n(\eta_1,\eta_2,\cdots,\eta_{2n+1})\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\|_{S^\infty} \\
&\lesssim \begin{cases} 2^{(2-\alpha)j_2+j_3+\cdots+j_{2n+1}}(1+\ln n), & l=1, \\ 2^{(2-\alpha)j_3+j_4+\cdots+j_{2n+1}}(1+\ln n), & l=2, \\ 2^{(2-\alpha)j_2+j_3+\cdots+j_{2n+1}-j_l}(1+\ln n), & l=3,\cdots,2n+1. \end{cases}
\end{aligned}\tag{74}$$

When $j_1 \geq j_2 \geq \cdots \geq j_{2n+1}$, and $j_1 > 0$,

$$\begin{aligned}
&\|[\mathbf{T}_n(\eta_1,\eta_2,\cdots,\eta_{2n+1}) - \mathbf{T}_n(\eta_1+\cdots+\eta_{2n+1},\eta_2,\cdots,\eta_{2n+1})] \\
&\qquad \cdot\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\|_{S^\infty} \\
&\qquad\qquad \lesssim n2^{-j_1}\prod_{i=2}^{2n+1}2^{j_i}(1+2^{(1+\delta)j_2}),
\end{aligned}\tag{75}$$

where $\delta > 0$ is a small constant.

Proof. To prove (73), we have

$$\begin{aligned}
&\mathcal{F}^{-1}\left[\mathbf{T}_n(\eta_1,\cdots,\eta_{2n+1})\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\right] \\
&= \iint_{\mathbb{R}^{2n+1}} e^{i(y_1\eta_1+\cdots+y_{2n+1}\eta_{2n+1})}\left[\int_{\mathbb{R}}\frac{\prod_{j=1}^{2n+1}(1-e^{i\eta_j\zeta})}{\zeta^{2n+1}}|\zeta|^{\alpha-1}\operatorname{sgn}\zeta\,\mathrm{d}\zeta\right] \\
&\quad \cdot\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1} \\
&= \iint_{\mathbb{R}^{2n+1}}\left[\int_{\mathbb{R}}\frac{(e^{iy_1\eta_1}-e^{i\eta_1(\zeta+y_1)})\cdots(e^{iy_{2n+1}\eta_{2n+1}}-e^{i\eta_{2n+1}(\zeta+y_{2n+1})})}{|\zeta|^{2n+2-\alpha}}\,\mathrm{d}\zeta\right] \\
&\quad \cdot\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1} \\
&= \int_{\mathbb{R}}\frac{1}{|\zeta|^{2n+2-\alpha}}\left[\mathcal{F}^{-1}[\psi_{j_1}](y_1)-\mathcal{F}^{-1}[\psi_{j_1}](\zeta+y_1)\right]\cdots\left[\mathcal{F}^{-1}[\psi_{j_{2n+1}}](y_{2n+1})\right. \\
&\qquad \left.-\mathcal{F}^{-1}[\psi_{j_{2n+1}}](\zeta+y_{2n+1})\right]\mathrm{d}\zeta.
\end{aligned}$$

We observe that for any $l = 1, \cdots, 2n+1$,

$$\left|\mathcal{F}^{-1}[\psi_{j_l}](y)-\mathcal{F}^{-1}[\psi_{j_l}](\zeta+y)\right| = 2^{j_l}\left|\mathcal{F}^{-1}[\psi_0](2^{j_l}y)-\mathcal{F}^{-1}[\psi_0](2^{j_l}(\zeta+y))\right|,$$

and

$$\int_{\mathbb{R}} \left|\mathcal{F}^{-1}[\psi_{j_l}](2^{j_l}y_l) - \mathcal{F}^{-1}[\psi_{j_l}](2^{j_l}(\zeta + y_l))\right| \,\mathrm{d}y_l \lesssim \min\{2^{j_l}|\zeta|, 1\}. \tag{76}$$

Taking the L^1-norm, we obtain

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}[\mathbf{T}_n(\eta_1, \cdots, \eta_{2n+1})\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})]\right\|_{L^1}\\
\lesssim\ &\int_{\mathbb{R}} \frac{1}{|\zeta|^{2n+2-\alpha}} \min\{2^{j_1}|\zeta|, 1\}\cdots\min\{2^{j_{2n+1}}|\zeta|, 1\}\,\mathrm{d}\zeta\\
\lesssim\ &\sum_{l=1}^{2n}\int_{2^{-j_l}<|\zeta|<2^{-j_{l+1}}} \frac{1}{|\zeta|^{2n+2-\alpha}}(2^{j_{l+1}}|\zeta|)\cdots(2^{j_{2n+1}}|\zeta|)\,\mathrm{d}\zeta\\
&+\int_{|\zeta|<2^{-j_1}} \frac{1}{|\zeta|^{2n+2-\alpha}}(2^{j_1}|\zeta|)\cdots(2^{j_{2n+1}}|\zeta|)\,\mathrm{d}\zeta + \int_{|\zeta|>2^{-j_{2n+1}}} \frac{1}{|\zeta|^{2n+2-\alpha}}\,\mathrm{d}\zeta\\
\lesssim\ &2^{j_2+\cdots+j_{2n+1}}\frac{1}{\alpha-1}2^{(1-\alpha)j_2} + \sum_{l=2}^{2n} 2^{j_{l+1}+\cdots+j_{2n+1}}\frac{1}{l-\alpha}(2^{(l-\alpha)j_l} - 2^{(l-\alpha)j_{l+1}})\\
&+\frac{1}{2n+1-\alpha}2^{j_{2n+1}(2n+1-\alpha)}\\
\lesssim\ &2^{(2-\alpha)j_2+\cdots+j_{2n+1}}(1+\ln n),
\end{aligned}$$

where $\ln n$ comes from the estimate of $\sum_{l=2}^{2n}\frac{1}{l-\alpha}$, which grows as the same rate as harmonic series.

To prove (74), we have

$$\begin{aligned}
&\mathcal{F}^{-1}\left[\partial_{\eta_l}\mathbf{T}_n(\eta_1, \cdots, \eta_{2n+1})\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\right]\\
=\ &\iint_{\mathbb{R}^{2n+1}} e^{i(y_1\eta_1+\cdots+y_{2n+1}\eta_{2n+1})}\left[\int_{\mathbb{R}}\frac{\prod_{j=1,j\neq l}^{2n+1}(1-e^{i\eta_j\zeta})}{\zeta^{2n+1}}(-i\zeta e^{i\eta_l\zeta})|\zeta|^{\alpha-1}\mathrm{sgn}\,\zeta\,\mathrm{d}\zeta\right]\\
&\cdot\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\,\mathrm{d}\eta_{2n+1}\\
=&-i\iint_{\mathbb{R}^{2n+1}}\int_{\mathbb{R}}(e^{iy_1\eta_1}-e^{i\eta_1(\zeta+y_1)})\cdots e^{i\eta_l(\zeta+y_l)}\cdots(e^{iy_{2n+1}\eta_{2n+1}}-e^{i\eta_{2n+1}(\zeta+y_{2n+1})})\\
&\frac{1}{|\zeta|^{2n+1-\alpha}}\,\mathrm{d}\zeta\cdot\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\,\mathrm{d}\eta_{2n+1}\\
=\ &\int_{\mathbb{R}}\frac{1}{|\zeta|^{2n+1-\alpha}}\left[\mathcal{F}^{-1}[\psi_{j_1}](y_1)-\mathcal{F}^{-1}[\psi_{j_1}](\zeta+y_1)\right]\cdots\left(\mathcal{F}^{-1}[\psi_l](\zeta+y_l)\right)\cdots\\
&\cdot\left[\mathcal{F}^{-1}[\psi_{j_{2n+1}}](y_{2n+1})-\mathcal{F}^{-1}[\psi_{j_{2n+1}}](\zeta+y_{2n+1})\right]\,\mathrm{d}\zeta.
\end{aligned}$$

Observe that

$$\left|\mathcal{F}^{-1}[\psi_{j_l}](y_l+\zeta)\right| = 2^{j_l}\left|\mathcal{F}^{-1}[\psi_0](2^{j_l}(y_l+\zeta))\right|,$$

and

$$\int_{\mathbb{R}}\left|\mathcal{F}^{-1}[\psi_{j_l}](2^{j_l}(\zeta+y_l))\right|\,\mathrm{d}y_l \lesssim 1.$$

Therefore, by using (76), we obtain

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}[\partial_{\eta_l}\mathbf{T}_n(\eta_1, \cdots, \eta_{2n+1})\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})]\right\|_{L^1}\\
\lesssim\ &\int_{\mathbb{R}}\frac{1}{|\zeta|^{2n+1-\alpha}}\prod_{k=1,k\neq l}^{2n+1}\min\{2^{j_k}|\zeta|, 1\}\,\mathrm{d}\zeta
\end{aligned}$$

$$\lesssim \begin{cases} 2^{(2-\alpha)j_2+\cdots+j_{2n+1}}(1+\ln n), & l=1, \\ 2^{(2-\alpha)j_3+\cdots+j_{2n+1}}(1+\ln n), & l=2, \\ 2^{(2-\alpha)j_2+\cdots+j_{2n+1}-j_l}(1+\ln n), & l\neq 1,2, \end{cases}$$

which leads to (74).

To prove (75), we have

$$\begin{aligned}
&\mathcal{F}^{-1}\left[(\mathbf{T}_n(\eta_1,\cdots,\eta_{2n+1})-\mathbf{T}_n(\eta_1+\cdots+\eta_{2n+1},\cdots,\eta_{2n+1}))\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\right] \\
&=\iint_{\mathbb{R}^{2n+1}} e^{i(y_1\eta_1+\cdots+y_{2n+1}\eta_{2n+1})} \\
&\quad\cdot\left[\int_{\mathbb{R}} \frac{\prod_{j=2}^{2n+1}(1-e^{i\eta_j\zeta})(e^{i\zeta(\eta_1+\cdots+\eta_{2n+1})}-e^{i\eta_1\zeta})}{\zeta^{2n+1}}|\zeta|^{\alpha-1}\operatorname{sgn}\zeta\,\mathrm{d}\zeta\right] \\
&\quad\cdot\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1} \\
&=\iint_{\mathbb{R}^{2n+1}}\Big[\int_{\mathbb{R}}\frac{1}{|\zeta|^{2n+2-\alpha}}(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})\cdots(e^{iy_{2n+1}\eta_{2n+1}}-e^{i\eta_{2n+1}(\zeta+y_{2n+1})}) \\
&\quad\cdot(e^{i\zeta(\eta_2+\cdots+\eta_{2n+1})}-1)e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\Big]\cdot\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1} \\
&=\iint_{\mathbb{R}^{2n+1}}\Big[\int_{\mathbb{R}}(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})\cdots(e^{iy_{2n+1}\eta_{2n+1}}-e^{i\eta_{2n+1}(\zeta+y_{2n+1})})\frac{1}{|\zeta|^{2n+2-\alpha}} \\
&\quad\cdot(e^{i\zeta(\eta_2+\cdots+\eta_{2n+1})}-1)\frac{1}{i\eta_1}\partial_\zeta e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\Big]\psi_{j_1}(\eta_1)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1} \\
&=\iint_{\mathbb{R}^{2n+1}}\Big[\int_{\mathbb{R}}\partial_\zeta\left(\frac{1}{|\zeta|^{2n+2-\alpha}}(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})\cdots(e^{iy_{2n+1}\eta_{2n+1}}-e^{i\eta_{2n+1}(\zeta+y_{2n+1})})\right) \\
&\quad\cdot(e^{i\zeta(\eta_2+\cdots+\eta_{2n+1})}-1)e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\Big]\frac{i\psi_{j_1}(\eta_1)}{\eta_1}\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1},
\end{aligned}$$

where

$$\begin{aligned}
&\partial_\zeta\frac{(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})\cdots(e^{iy_{2n+1}\eta_{2n+1}}-e^{i\eta_{2n+1}(\zeta+y_{2n+1})})(e^{i\zeta(\eta_2+\cdots+\eta_{2n+1})}-1)}{|\zeta|^{2n+2-\alpha}} \\
&=\sum_{l=2}^{2n+1}\frac{1}{|\zeta|^{2n+2-\alpha}}(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})\cdots(-i\eta_l e^{i\eta_l(\zeta+y_l)})\cdots \\
&\quad\cdot(e^{iy_{2n+1}\eta_{2n+1}}-e^{i\eta_{2n+1}(\zeta+y_{2n+1})})(e^{i\zeta(\eta_2+\cdots+\eta_{2n+1})}-1) \\
&\quad+\frac{1}{|\zeta|^{2n+2-\alpha}}(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)})\cdots(e^{iy_{2n+1}\eta_{2n+1}}-e^{i\eta_{2n+1}(\zeta+y_{2n+1})}) \\
&\quad\cdot(i(\eta_2+\cdots+\eta_{2n+1})e^{i\zeta(\eta_2+\cdots+\eta_{2n+1})}) \\
&\quad-(2n+2-\alpha)\frac{1}{\zeta|\zeta|^{2n+2-\alpha}}(e^{iy_2\eta_2}-e^{i\eta_2(\zeta+y_2)}) \\
&\quad\cdots\cdot(e^{iy_{2n+1}\eta_{2n+1}}-e^{i\eta_{2n+1}(\zeta+y_{2n+1})})(e^{i\zeta(\eta_2+\cdots+\eta_{2n+1})}-1) \\
&=\sum_{l=2}^{2n+1}\mathrm{I}_l+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

Then we estimate each term separately. We split each term into the large $|\zeta|$ part and the small $|\zeta|$ part. For $l=2,\cdots,2n+1$,

$$\iint_{\mathbb{R}^{2n+1}}\left[\int_{\mathbb{R}}\mathrm{I}_l e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right]\cdot\frac{i\psi_{j_1}(\eta_1)}{\eta_1}\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1}$$

$$= \iint_{\mathbb{R}^{2n+1}} \left[\int_{\mathbb{R}} \mathrm{I}_l \psi(\zeta) e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right] \cdot \frac{i\psi_{j_1}(\eta_1)}{\eta_1} \psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots \mathrm{d}\eta_{2n+1}$$

$$+ \iint_{\mathbb{R}^{2n+1}} \left[\int_{\mathbb{R}} \mathrm{I}_l (1-\psi(\zeta)) e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right] \cdot \frac{i\psi_{j_1}(\eta_1)}{\eta_1} \psi_{j_2}(\eta_2) \cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots \mathrm{d}\eta_{2n+1}.$$

For the large $|\zeta|$ part, when the integral is on the support of $1-\psi(\zeta)$, we don't need to use the cancellation in the numerator of I_l to estimate the symbol. So, we have

$$\iint_{\mathbb{R}^{2n+1}} \left[\int_{\mathbb{R}} \mathrm{I}_l (1-\psi(\zeta)) e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right] \cdot \frac{i\psi_{j_1}(\eta_1)}{\eta_1} \psi_{j_2}(\eta_2)\cdots \cdot\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots \mathrm{d}\eta_{2n+1}$$

$$= \iint_{\mathbb{R}^{2n+1}} \Bigg[\int_{\mathbb{R}} \frac{1}{|\zeta|^{2n+2-\alpha}} (e^{i\eta_2(y_2+\zeta)} - e^{i\eta_2(2\zeta+y_2)})\cdots(-i\eta_l e^{i\eta_l(2\zeta+y_l)})\cdots$$
$$(e^{i\eta_{2n+1}(y_{2n+1}+\zeta)} - e^{i\eta_{2n+1}(2\zeta+y_{2n+1})})\cdot(1-\psi(\zeta))e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\Bigg] \frac{i\psi_{j_1}(\eta_1)}{\eta_1}$$
$$\cdot\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots \mathrm{d}\eta_{2n+1}$$
$$- \iint_{\mathbb{R}^{2n+1}} \Big[\int_{\mathbb{R}} \frac{1}{|\zeta|^{2n+2-\alpha}} (e^{iy_{2n+1}\eta_{2n+1}} - e^{i\eta_{2n+1}(\zeta+y_{2n+1})})(e^{iy_2\eta_2} - e^{i\eta_2(\zeta+y_2)})$$
$$\cdots(-i\eta_l e^{i\eta_l(\zeta+y_l)})\cdots(e^{iy_{2n+1}\eta_{2n+1}} - e^{i\eta_{2n+1}(\zeta+y_{2n+1})})\cdot(1-\psi(\zeta))e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\Big]$$
$$\cdot\frac{i\psi_{j_1}(\eta_1)}{\eta_1}\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots \mathrm{d}\eta_{2n+1}$$
$$= \int \frac{(1-\psi(\zeta))}{|\zeta|^{2n+2-\alpha}} \mathcal{F}^{-1}[\frac{\psi_{j_1}(\eta_1)}{\eta_1}](\zeta+y_1)\left(\mathcal{F}^{-1}[\psi_{j_2}](y_2+\zeta) - \mathcal{F}^{-1}[\psi_{j_2}](y_2+2\zeta)\right)\cdots$$
$$\mathcal{F}^{-1}[\eta_l\psi_{j_l}(\eta_l)](2\zeta+y_l)\cdots$$
$$\left(\mathcal{F}^{-1}[\psi_{j_{2n+1}}](y_{2n+1}+\zeta) - \mathcal{F}^{-1}[\psi_{j_{2n+1}}](y_{2n+1}+2\zeta)\right)d\zeta$$
$$- \int \frac{(1-\psi(\zeta))}{|\zeta|^{2n+2-\alpha}} \mathcal{F}^{-1}[\frac{\psi_{j_1}(\eta_1)}{\eta_1}](\zeta+y_1)\left(\mathcal{F}^{-1}[\psi_{j_2}](y_2) - \mathcal{F}^{-1}[\psi_{j_2}](y_2+\zeta)\right)\cdots$$
$$\mathcal{F}^{-1}[\eta_l\psi_{j_l}(\eta_l)](\zeta+y_l)\cdots\left(\mathcal{F}^{-1}[\psi_{j_{2n+1}}](y_{2n+1}) - \mathcal{F}^{-1}[\psi_{j_{2n+1}}](y_{2n+1}+\zeta)\right)d\zeta.$$

By (76),

$$\Big\| \iint_{\mathbb{R}^{2n+1}} \left[\int_{\mathbb{R}} \mathrm{I}_l (1-\psi(\zeta)) e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right] \cdot \frac{i\psi_{j_1}(\eta_1)}{\eta_1} \psi_{j_2}(\eta_2)\cdots \cdot\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots \mathrm{d}\eta_{2n+1}\Big\|_{L^1}$$
$$\lesssim 2^{-j_1+j_l}\int\left(\prod_{i=2,i\neq l}^{2n+1}\min\{2^{j_i}|\zeta|,1\}\right)\cdot\frac{(1-\psi(\zeta))}{|\zeta|^{2n+2-\alpha}}d\zeta$$
$$\lesssim 2^{-j_1+j_2+\cdots+j_{2n+1}}.$$

Similarly,

$$\Bigg\| \iint_{\mathbb{R}^{2n+1}} \left[\int_{\mathbb{R}} \mathrm{II} (1-\psi(\zeta)) e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right] \cdot \frac{i\psi_{j_1}(\eta_1)}{\eta_1} \psi_{j_2}(\eta_2)\cdots$$

$$\cdot \psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\,\mathrm{d}\eta_{2n+1}\Bigg\|_{L^1}$$
$$\lesssim 2^{-j_1+j_2+\cdots+j_{2n+1}}.$$

$$\Bigg\|\iint_{\mathbb{R}^{2n+1}}\left[\int_{\mathbb{R}}\mathrm{III}(1-\psi(\zeta))e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right]\cdot\frac{i\psi_{j_1}(\eta_1)}{\eta_1}\psi_{j_2}(\eta_2)\cdots$$
$$\cdot \psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\,\mathrm{d}\eta_{2n+1}\Bigg\|_{L^1}$$
$$\lesssim 2^{-j_1+2j_2+j_3+\cdots+j_{2n+1}}.$$

For the small $|\zeta|$ part,

$$\Bigg\|\iint_{\mathbb{R}^{2n+1}}\left[\int_{\mathbb{R}}\mathrm{I}_l\psi(\zeta)e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right]\cdot\frac{i\psi_{j_1}(\eta_1)}{\eta_1}\psi_{j_2}(\eta_2)\cdots$$
$$\cdot \psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\,\mathrm{d}\eta_{2n+1}\Bigg\|_{L^1}$$
$$=\Bigg\|\iint_{\mathbb{R}^{2n+1}}\Bigg[\int_{\mathbb{R}}\frac{1}{|\zeta|^{2n+2-\alpha}}e^{i(x_1\xi_1+\cdots+x_{2n+1}\xi_{2n+1})}(1-e^{i2^{j_2}\xi_2\zeta})\cdots(-i2^{j_l}\xi_l e^{i2^{j_l}\xi_l\zeta})\cdots$$
$$\cdot(1-e^{i2^{j_{2n+1}}\xi_{2n+1}\zeta})(e^{i\zeta(2^{j_2}\xi_2+\cdots+2^{j_{2n+1}}\xi_{2n+1})}-1)\psi(\zeta)e^{i2^{j_1}\xi_1\zeta}\,\mathrm{d}\zeta\Bigg]$$
$$\frac{i\psi_0(\xi_1)}{2^{j_1}\xi_1}\psi_0(\xi_2)\cdot\cdots\psi_0(\xi_{2n+1})\,\mathrm{d}\xi_1\cdots\,\mathrm{d}\xi_{2n+1}\Bigg\|_{L^1_x}$$
$$=2^{-j_1}\Bigg\|\iint_{\mathbb{R}^{2n}}\Bigg[\frac{1}{|\zeta|^{2n+2-\alpha}}e^{i(x_2\xi_2+\cdots+x_{2n+1}\xi_{2n+1})}(1-e^{i2^{j_2}\xi_2\zeta})\cdots(-i2^{j_l}\xi_l e^{i2^{j_l}\xi_l\zeta})\cdots$$
$$\cdot(1-e^{i2^{j_{2n+1}}\xi_{2n+1}\zeta})(e^{i\zeta(2^{j_2}\xi_2+\cdots+2^{j_{2n+1}}\xi_{2n+1})}-1)\psi(\zeta)\Bigg]$$
$$\cdot\mathcal{F}^{-1}_{\xi_1}\left[\frac{\psi_0(\xi_1)}{\xi_1}\right](x_1+2^{j_1}\zeta)\cdot\psi_0(\xi_2)\cdots\psi_0(\xi_{2n+1})\,\mathrm{d}\zeta\,\mathrm{d}\xi_2\cdots\,\mathrm{d}\xi_{2n+1}\Bigg\|_{L^1_x}.$$

Denote the cone K_l as $\{(x_1,\cdots,x_{2n+1}\mid |x_l|\geq|(x_1,\cdots,x_{l-1},0,x_{l+1},\cdots,x_{2n+1})|\}$. Then $\mathbb{R}^{2n+1}=\bigcup_{l=1}^{2n+1}K_l$. For L^1 norm of the function $f(x_1,\cdots,x_{2n+1})$, we have the following bound.

$$\begin{aligned}\|f\|_{L^1}&=\int_{\mathbb{R}^{2n+1}}|f(x_1,\cdots,x_{2n+1})|\,\mathrm{d}x_1\cdots\,\mathrm{d}x_{2n+1}\\&=\sum_{l=1}^{2n+1}\int_{K_l}\frac{\sqrt{(1+x_1^2)\cdots(1+x_{2n+1}^2)}\cdot|x_l|^\delta}{\sqrt{(1+x_1^2)\cdots(1+x_{2n+1}^2)}\cdot|x_l|^\delta}|f(x_1,\cdots,x_{2n+1})|\,\mathrm{d}x_1\cdots\,\mathrm{d}x_{2n+1}\\&\leq\sum_{l=1}^{2n+1}\left\|\sqrt{(1+x_1^2)\cdots(1+x_{2n+1}^2)}\cdot|x_l|^\delta|f(x_1,\cdots,x_{2n+1})|\right\|_{L^\infty}\end{aligned}$$

$$\cdot \int_{K_l} \frac{1}{\sqrt{(1+x_1^2)\cdots(1+x_{2n+1}^2)}\cdot|x_l|^\delta}\,\mathrm{d}x_1\cdots\mathrm{d}x_{2n+1}$$
$$\lesssim \sum_{l=1}^{2n+1}\left\|\sqrt{(1+x_1^2)\cdots(1+x_{2n+1}^2)}\cdot|x_l|^\delta|f(x_1,\cdots,x_{2n+1})|\right\|_{L^\infty},$$

where $\delta>0$ is a small constant. Therefore,

$$\left\|\iint_{\mathbb{R}^{2n+1}}\left[\int_{\mathbb{R}}\mathrm{I}_l\psi(\zeta)e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right]\cdot\frac{i\psi_{j_1}(\eta_1)}{\eta_1}\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1}\right\|_{L^1}$$
$$\lesssim 2^{-j_1}\sum_{k=2}^{2n+1}\int_{\mathbb{R}}\Bigg\|(1-\partial_{\xi_2}^2)^{1/2}\cdots(1-\partial_{\xi_{2n+1}}^2)^{1/2}|\partial_{\xi_k}|^\delta\Bigg[(1-e^{i2^{j_2}\xi_2\zeta})\cdots(-i2^{j_l}\xi_l e^{i2^{j_l}\xi_l\zeta})$$
$$\cdots(1-e^{i2^{j_{2n+1}}\xi_{2n+1}\zeta})\frac{1}{|\zeta|^{2n+2-\alpha}}\cdot(e^{i\zeta(2^{j_2}\xi_2+\cdots+2^{j_{2n+1}}\xi_{2n+1})}-1)\psi_0(\xi_2)\cdots$$
$$\cdot\psi_0(\xi_{2n+1})\Bigg]\Bigg\|_{L^1_{\xi'}}\cdot\left\|\mathcal{F}^{-1}_{\xi_1}\left[\frac{\psi_0(\xi_1)}{\xi_1}\right](x_1+2^{j_1}\zeta)\right\|_{L^1_{x_1}}\psi(\zeta)\,\mathrm{d}\zeta$$

where $\xi'=(\xi_2,\cdots,\xi_{2n+1})$. Since $L^1_{x_1}$ norm is translation invariant,

$$\left\|\mathcal{F}^{-1}_{\xi_1}\left[\frac{\psi_0(\xi_1)}{\xi_1}\right](x_1+2^{j_1}\zeta)\right\|_{L^1_{x_1}}$$

is bounded by a constant independent of ζ and j_1. Considering the support of $\psi(\zeta)$, we have

$$\Bigg\|(1-\partial_{\xi_2}^2)^{1/2}\cdots(1-\partial_{\xi_{2n+1}}^2)^{1/2}|\partial_{\xi_k}|^\delta\Bigg[(1-e^{i2^{j_2}\xi_2\zeta})\cdots(-i2^{j_l}\xi_l e^{i2^{j_l}\xi_l\zeta})\cdots$$
$$\cdot(1-e^{i2^{j_{2n+1}}\xi_{2n+1}\zeta})\frac{1}{|\zeta|^{2n+2-\alpha}}(e^{i\zeta(2^{j_2}\xi_2+\cdots+2^{j_{2n+1}}\xi_{2n+1})}-1)\psi_0(\xi_2)\cdots\psi_0(\xi_{2n+1})\Bigg]\Bigg\|_{L^1_{\xi'}}$$
$$\lesssim\prod_{i=2}^{2n+1}2^{j_i}(1+2^{j_l})(1+2^{\delta j_2})(|\zeta|^{\alpha-2}+1).$$

Thus we obtain

$$\left\|\iint_{\mathbb{R}^{2n+1}}\left[\int_{\mathbb{R}}\mathrm{I}_l\psi(\zeta)e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right]\cdot\frac{i\psi_{j_1}(\eta_1)}{\eta_1}\psi_{j_2}(\eta_2)\cdots\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1}\right\|_{L^1}$$
$$\lesssim 2^{-j_1}\prod_{i=2}^{2n+1}2^{j_i}(1+2^{(1+\delta)j_2}).$$

Similarly,

$$\Bigg\|\iint_{\mathbb{R}^{2n+1}}\left[\int_{\mathbb{R}}\mathrm{II}\psi(\zeta)e^{i\eta_1(\zeta+y_1)}\,\mathrm{d}\zeta\right]\cdot\frac{i\psi_{j_1}(\eta_1)}{\eta_1}\psi_{j_2}(\eta_2)\cdots$$
$$\cdot\psi_{j_{2n+1}}(\eta_{2n+1})\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_{2n+1}\Bigg\|_{L^1}$$
$$\lesssim 2^{-j_1}\prod_{i=2}^{2n+1}2^{j_i}(1+2^{(1+\delta)j_i}),$$

$$
\begin{aligned}
&\Bigg\| \iint_{\mathbb{R}^{2n+1}} \left[\int_{\mathbb{R}} \mathrm{III}\psi(\zeta) e^{i\eta_1(\zeta+y_1)} \,\mathrm{d}\zeta \right] \cdot \frac{i\psi_{j_1}(\eta_1)}{\eta_1} \psi_{j_2}(\eta_2) \cdots \\
&\cdot \psi_{j_{2n+1}}(\eta_{2n+1}) \,\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_{2n+1} \Bigg\|_{L^1} \\
\lesssim\ & 2^{-j_1} \prod_{i=2}^{2n+1} 2^{j_i}(1 + 2^{(1+\delta)j_i}).
\end{aligned}
$$

This leads to (75).

□

Lemma B.2. *If* $|j_1 - j_3| > 1$ *and* $m \in \mathbb{N}$, *then*

$$
\left\| \frac{1}{(|\eta_1|^{1-\alpha} - |\eta_3|^{1-\alpha})^m} \psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 2^{m(\alpha-1)\min\{j_1,j_3\}}.
$$

Proof. Notice that on the support of $\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)$,

$$
\big| |\eta_1|^{1-\alpha} - |\eta_3|^{1-\alpha} \big| \gtrsim 2^{(1-\alpha)\min\{j_1,j_3\}}.
$$

By the definition of the S^∞-norm (9) and the definition of ψ_k (6), we have that

$$
\begin{aligned}
&\left\| \frac{\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)}{(|\eta_1|^{1-\alpha} - |\eta_3|^{1-\alpha})^m} \right\|_{S^\infty} \\
=\ & \left\| \iint_{\mathbb{R}^3} \frac{\psi_{j_1}(\eta_1)\psi_{j_2}(\eta_2)\psi_{j_3}(\eta_3)}{(|\eta_1|^{1-\alpha} - |\eta_3|^{1-\alpha})^m} e^{i(y_1\eta_1 + y_2\eta_2 + y_3\eta_3)} \,\mathrm{d}\eta_1 \,\mathrm{d}\eta_2 \,\mathrm{d}\eta_3 \right\|_{L^1} \\
=\ & \iint_{\mathbb{R}^3} \left| \iint_{\mathbb{R}^3} \frac{\psi_0(2^{-j_1}\eta_1)\psi_0(2^{-j_2}\eta_2)\psi_0(2^{-j_3}\eta_3)}{(|\eta_1|^{1-\alpha} - |\eta_3|^{1-\alpha})^m} e^{i(y_1\eta_1 + y_2\eta_2 + y_3\eta_3)} \,\mathrm{d}\eta_1 \,\mathrm{d}\eta_2 \,\mathrm{d}\eta_3 \right| \mathrm{d}\mathbf{y} \\
\lesssim\ & 2^{m(\alpha-1)\min\{j_1,j_3\}},
\end{aligned}
$$

where the last inequality comes from using oscillatory integral estimates, together with the facts that the support of ψ_0 is $(-8/5, -5/8) \cup (5/8, 8/5)$ and $|j_1 - j_3| > 1$. □

Conflicts and interests. Opinions expressed in this paper are those of the author, and do not necessarily reflect the view of J.P.Morgan.

REFERENCES

[1] A. Ai and O.-N. Avadanei, Well-posedness for the surface quasi-geostrophic front equation, preprint, 2022, arXiv:2212.00117.

[2] A. Ai and O.-N. Avadanei, Low regularity solutions for the surface quasi-geostrophic front equation, preprint, 2023, arXiv:2310.20143.

[3] A. Ai and O.-N. Avadanei, Low regularity well-posedness for the generalized surface quasi-geostrophic front equation, preprint, 2022, arXiv:2311.07551.

[4] D. M. Ambrose, F. Hadadifard and J. P. Kelliher, Contour dynamics and global regularity for periodic vortex patches and layers, *SIAM J. Math. Anal.*, **56** (2024), 2286–2311, preprint, 2022, arXiv:2209.14481.

[5] J. Biello and J. K. Hunter, Nonlinear Hamiltonian waves with constant frequency and surface waves on vorticity discontinuities, *Comm. Pure Appl. Math.*, **63** (2010), 303-336.

[6] K. Bogdan and T. Byczkowski, Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains, *Studia Mathematica*, **133** (1999), 53-92.

[7] W. Chen, L. D'Ambrosio and Y. Li, Some Liouville theorems for the fractional Laplacian, *Nonlinear Anal.*, **121** (2015), 370-381.

[8] P. Constantin, A. J. Majda and E. Tabak, Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar, *Nonlinearity*, **7** (1994), 1495-1533.

[9] P. Constantin, A. J. Majda and E. G. Tabak, Singular front formation in a model for quasi-geostrophic flow, *Phys. Fluids*, **6** (1994), 9-11.

[10] D. Córdoba, C. Fefferman and J. L. Rodrigo, Almost sharp fronts for the surface quasi-geostrophic equation, *Proc. Natl. Acad. Sci. USA*, **101** (2004), 2687-2691.

[11] D. Córdoba, J. Gómez-Serrano and A. D. Ionescu, Global solutions for the generalized SQG patch equation, *Arch. Ration. Mech. Anal.*, **233** (2019), 1211-1251.

[12] Y. Deng, A. D. Ionescu and B. Pausader, The Euler-Maxwell system for electrons: Global solutions in 2D, *Arch. Rational Mech. Anal.*, **225** (2017), 771-871.

[13] Y. Deng, A. D. Ionescu, B. Pausader and F. Pusateri, Global solutions of the gravity-capillary water wave system in 3 dimensions, *Acta Mathematica*, **219** (2017), 213-402.

[14] D. G. Dritschel, The repeated filamentation of two-dimensional vorticity interfaces, *J. Fluid Mech.* **194** (1988), 511-547.

[15] D. G. Dritschel, Contour dynamics and contour surgery: Numerical algorithms for extended, high-resolution modelling of vortex dynamics in two-dimensional, inviscid, incompressible flows, *Comput. Phys. Rep.* **10**, (1989), 77-146.

[16] M. M. Fall, Entire s-harmonic functions are affine, *P*roc. Amer. Math. Soc., **144** (2016), 2587-2592.

[17] C. Fefferman, G. Luli and J. Rodrigo, The spine of an SQG almost-sharp front, *N*onlinearity, **25** (2012), 329-342.

[18] C. Fefferman and J. L. Rodrigo, Analytic sharp fronts for the surface quasi-geostrophic equation, *Comm. Math. Phys.*, **303** (2011), 261-288.

[19] C. Fefferman and J. L. Rodrigo, Almost sharp fronts for SQG: The limit equations, *Comm. Math. Phys.*, **313** (2012), 131-153.

[20] C. L. Fefferman and J. L. Rodrigo, Construction of almost-sharp fronts for the surface quasi-geostrophic equation, *Arch. Rational Mech. Anal.*, **218** (2015), 123-162.

[21] P. Germain, Space-time resonances, *Journées Équations aux DÉrivÉes Partielles*, **8** (2010), 1-10.

[22] P. Germain, N. Masmoudi and J. Shatah, Global solutions for 3d quadratic Schrodinger equations, *Int. Math. Res. Not.*, **2009** (2009), 414-432.

[23] P. Germain, N. Masmoudi and J. Shatah, Global solutions for the gravity water waves equation in dimension 3, *Ann. Math.*, **175** (2012), 691-754.

[24] J. K. Hunter, R. C. Moreno-Vasquez, J. Shu and Q. Zhang, On the approximation of vorticity fronts by the Burgers-Hilbert equation, *Asymptot. Anal.*, **129** (2022), 141-177.

[25] J. K. Hunter and J. Shu, Regularized and approximate equations for sharp fronts in the surface quasi-geostrophic equation and its generalization, *Nonlinearity*, **31** (2018), 2480-2517.

[26] J. K. Hunter, J. Shu and Q. Zhang, Local well-posedness of an approximate equation for SQG fronts, *J. Math. Fluid Mech.*, **20** (2018), 1967-1984.

[27] J. K. Hunter, J. Shu and Q. Zhang, Global solutions of a surface quasi-geostrophic front equation, *Pure Appl. Anal.*, **3** (2021), 403-472.

[28] J. K. Hunter, J. Shu and Q. Zhang, Two-front solutions of the SQG equation and its generalizations, *Commun. Math. Sci.*, **18** (2020), 1685-1741.

[29] J. K. Hunter, J. Shu and Q. Zhang, Contour dynamics for surface quasi-geostrophic fronts, *Nonlinearity*, **33** (2020), 4699-4714.

[30] M. Ifrim and D. Tataru, Global bounds for the cubic nonlinear Schrödinger equation (NLS) in one space dimension, *Nonlinearity*, **28** (2015), 2661-2675.

[31] M. Ifrim and D. Tataru, Two dimensional water waves in holomorphic coordinates II: Global solutions, *Bull. Soc. Math. France*, **144** (2016), 369-394.

[32] M. Ifrim and D. Tataru, Local well-posedness for quasi-linear problems: A primer, *Bull. Amer. Math. Soc.*, **60** (2023), 167-194.

[33] A. D. Ionescu and B. Pausader, The Euler-Poisson system in 2d: Global stability of the constant equilibrium solution, *Int. Math. Res. Not.*, **2013** (2013), 761-826.

[34] A. D. Ionescu and F. Pusateri, *Global Regularity for 2D Water Waves with Surface Tension*, Memoirs of the AMS, **256** (2018), v+124.

[35] A. D. Ionescu and F. Pusateri, Global solutions for the gravity water waves system in 2D, *Invent. Math.*, **199** (2015), 653-804.

[36] A. D. Ionescu and F. Pusateri, Global analysis of a model for capillary water waves in two dimensions, *Comm. Pure Appl. Math.*, **69** (2016), 2015-2071.

[37] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, *Spectral Theory and Differential Equations: Proceedings of the Symposium Held at*

Dundee, Scotland, 1-19 July, 1974. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, 25-70.

[38] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Commun.Pure Appl. Math.*, **41** (1988), 891-907.

[39] M. Kwaśnicki, Ten equivalent definitions of the fractional laplace operator, *Fractional Calculus and Applied Analysis*, **20** (2017), 7-51.

[40] G. Lapeyre, Surface quasi-geostrophy, *Fluids*, **2** (2017), 7.

[41] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, 2002.

[42] J. Malý and W. P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Mathematical Surveys and Monographs, **51**. American Mathematical Society, Providence, R.I., 1997.

[43] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *The NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, NY, 2010.

[44] J. Pedlosky, *Geophysical Fluid Dynamics*, 2^{nd} ed. Springer-Verlag, New York, N. Y., 1987.

[45] J. L. Rodrigo, On the evolution of sharp fronts for the quasi-geostrophic equation, *Comm. Pure and Appl. Math.*, **58** (2005), 821-866.

[46] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, **30**, Princeton University Press, Princeton, N.J., 1970.

[47] E. M. Stein, *Harmonic analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, **43**. Monographs in Harmonic Analysis, III. Princeton University, Princeton, N.J., 1993.

[48] F. Yan and Q. Zhang, Global solutions of quasi-geostrophic shallow-water fronts. preprint, 2022, `arXiv:2203.01967`.

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