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Adaptive Cluster-Dynamic Mode Decomposition with Application to the Burgers' Equation

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***Abstract*— This paper proposes a new model reduction method that improves the prediction accuracy of dynamic modes decomposition with control (DMDc). DMD is a data-driven technique that extracts low-order models from high-dimensional complex dynamic systems with actuation. With DMDc, an input-output reduced-order model can be obtained for system identification and prediction. In this work, in order to better capture the nonlinear behavior, an adaptive clustering method is introduced to group the snapshots obtained by experimental data or numerical simulation into several sub-regions that display similar behavior to construct a reduced-order model. Cluster methods with DMDc are combined to construct the local reduced-order model. Furthermore, with the prediction process, new incoming data is fed into the clusters to update the cluster-DMDc reduced order model to obtain more accurate predictions. Time clustering is applied to the snapshots generated by the Burgers' equations with boundary actuation, and the adaptive cluster DMDc reduced-order model outperforms the standard DMDc.**

***Index Terms*— Reduced order modeling, Dynamic mode decomposition with control, and fluid flow systems.**

I. Introduction

Dynamic mode decomposition (DMD) [1] is often used in the field of fluid dynamics since it is capable of capturing the spatial-temporal structure of the dynamic flows. DMD has also been applied in other fields such as neuroscience, load forecasting, parameter estimation, and image processing [2]. DMD can be considered as a finite-dimensional approximation of the infinite-dimensional, linear Koopman operator which describes the evolution of state measurements [3] [4] [5] [6]. In recent years, many extensions of DMD have been proposed. For instance, [2] directly imposes the sparsity regularization and the nonnegativity constraint on the structures of the DMD modes. In [7], an online DMD framework was proposed. As time evolves, online DMD updates the approximation of the system dynamics using incoming data and computes the DMD matrix with rank-1 updates without the need to store all of the data. In [8], DMD with control (DMDc) was proposed to take actuation into account. The general notion of clustering is also used in various domains such as data compressing, image processing, etc. In [9], [10], [11] [12], Centroid Voronoi Tessellations (CVT), a clustering method based on k-means algorithm, is used to construct a reduced order model by choosing the cluster centroids as the reduced basis. In [13], the authors group snapshots into several clusters, and then cluster centroids partition the state space in complementary non-overlapping regions. Based on these regions, the state transition matrix is constructed using a Markov process and finally applied to a mixing layer problem. In [14], differential geometry properties of Riemannian manifolds are used to produce suitable reduced-order bases for nonlinear dynamical systems. The authors of [15] [16] (based on the work of [17]) used the fact that snapshots generated by nonlinear systems belong to nonlinear manifolds, however, popular reduced order models such as DMD and Proper Orthogonal Decomposition (POD) [11] are inherently linear, that is, they assume that the snapshots belong to a linear vector space. Then in [15] [16] [18] [19], the authors propose to quantify the global nonlinear manifold geodesic by partitioning the manifold into several regions and then approximating using local Euclidean distances. Snapshot data were grouped into several sub-regions, and reduced-order POD models were constructed in each sub-region. The results show a significant improvement over conventional POD. In [19] offline cluster DMD and DMDc algorithms are proposed with application to the predictive control of hydraulic fracturing. The clustering method is based on an algorithm proposed in [20]. In contrast, here the k-means clustering algorithm is used in conjunction with DMD and DMDc to yield online cluster-DMD and DMDc algorithms that adapt to incoming data. More specifically, this paper presents an extension of the work in [16] [8]. We propose the use of a clustering algorithm to group the system snapshots obtained through experimental data or numerical simulation into several sub-regions that display similar behavior to construct a reduced-order model for each sub-region. The k-means algorithm [21] is used to group the snapshots that have similar behavior into clusters. In particular, time clustering is applied to the snapshots generated by the Burgers' equation with boundary actuation resulting in a cluster DMDc reduced order method that yields better results than standard DMDc. Next, in the prediction task, rather than group the clusters all at once, we sequentially group the clusters and adaptively update the DMDc reduced order model, consequently transforming the

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offline data-driven technique into an online technique. The Burgers' equation is used as a surrogate to the Navier-Stokes equation since it has the same nonlinearity as the latter. It is a fundamental partial differential equation occurring in several fields including fluid mechanics, acoustics, gas dynamics, and traffic flow [22]. Finally, cluster DMDc is shown to outperform standard DMDc in general by exploiting a property of the Frobenius matrix norm in Lemma 2.1. This paper is organized as follows: Section II provides a brief background on DMDc and the k-means clustering algorithm, followed by the proposed adaptive cluster DMDc reduced order modeling method. Section III presents a numerical study that illustrates the adaptive cluster DMDc reduced order model with a comparison with conventional DMDc. Concluding remarks are provided in Section IV.

II. Background: Dynamic mode decomposition

A. DMD and DMDc

DMD aims to extract information about a dynamical system from an ensemble of experimental or numerical snapshot data. Letting $x_k \in \mathbb{R}^n$ be a measurement made at time k, DMD assumes that (x_k, x_{k+1}) can be approximated by a linear operator A as [8]:

$$x_{k+1} \approx Ax_k \tag{1}$$

Represented in matrix form:

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_{N-1} \\ | & | & & | \end{bmatrix} \tag{2}$$

$$X' = \begin{bmatrix} | & | & & | \\ x_2 & x_3 & \dots & x_N \\ | & | & & | \end{bmatrix} \tag{3}$$

Where m is the total number of the snapshots and X' is the time shift snapshots matrix of X, the goal of DMD is to identify a matrix $A \in \mathbb{R}^{n\times n}$ that best approximates $X' \approx AX$. Such an estimate can be obtained according to [23]:

$$A = X'X^{\dagger} \tag{4}$$

Where $X^{\dagger}$ is the Moore-Penrose pseudoinverse of X. For an underdetermined system, DMD finds the matrix A that solves:

$$\min_A \|X' - AX\|_F, \tag{5}$$

i.e., A can be regarded as a least-squares solution relating X' and X. Singular value decomposition (SVD) can also be used to compute a lower-rank approximation for A,

$$\begin{aligned} X = U\Sigma V^* &= \begin{bmatrix} \tilde{U}_r & \tilde{U}_{\text{rem}} \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_r & 0 \\ 0 & \Sigma_{\text{rem}} \end{bmatrix} \begin{bmatrix} \tilde{V}_r{}^* \\ \tilde{V}_{\text{rem}}^* \end{bmatrix} \\ &\approx \tilde{U}_r \tilde{\Sigma}_r \tilde{V}_r{}^* \end{aligned} \tag{6}$$

Where $U \in \mathbb{R}^{n\times n}$, $\tilde{U}_r \in \mathbb{R}^{n\times r}$ is the first r columns of U, $\Sigma \in \mathbb{R}^{n\times m-1}$, $\tilde{\Sigma} \in \mathbb{R}^{r\times r}$ is $r \times r$ matrix extracted from the left upper corner of Σ, $V^* \in \mathbb{R}^{m-1\times m-1}$ and $\tilde{V}_r{}^* \in \mathbb{R}^{r\times m-1}$ is the first r row of V^*, $*$ denotes the complex conjugate transpose. r is the truncation value, rem are the remaining singular values. When constructing a reduced order model, an appropriate r should be chosen to retain important system information while avoiding overfitting. With the snapshots in hand, the approximated DMD model A using SVD can be computed as [8]:

$$A \approx X'\tilde{V}_r \tilde{\Sigma}_r{}^{-1} \tilde{U}_r{}^* = \bar{A} \tag{7}$$

where $\bar{A}$ is an approximation of operator A. The dynamical system can be represented by $x_{k+1} = \bar{A}x_k$. Reduced order model can be constructed by the transform $x = \tilde{U}\tilde{x}$, then [8] :

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k \tag{8}$$

In a similar vein, DMDc is a variant of the DMD algorithm that characterizes the relationship between a future measurement x_{k+1} with the current measurement x_k and current control input u_k. With all pairs of state measurements and control signals at hand, the system is represented by

$$x_{k+1} = Ax_k + Bu_k \tag{9}$$

Where $u_k \in \mathbb{R}^l$ and $B \in \mathbb{R}^{n\times l}$. To handle the incorporation of control input, consider a new matrix of control input snapshots defined as:

$$\Gamma = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_{m-1} \\ | & | & & | \end{bmatrix} \tag{10}$$

In conjunction with the previously defined matrices X and $X^{'}$, the dynamical system can be represented according to [8]

$$X' \approx AX + B\Gamma \tag{11}$$

The goal of DMDc is to find the best linear operators A and B to approximate the dynamical system with actuation. In this case, to construct the relationship between unknown A, B with known data X and Γ, (11) is rewritten as [8]:

$$X' \approx \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X \\ \Gamma \end{bmatrix} = C\Omega \tag{12}$$

where $C = [A\ B]$, $\Omega = [X\ \Gamma]^T$. With a similar form as DMD in (4), DMDc is defined as $C = X'\Omega^{\dagger}$, where C can be found by minimizing the Frobenius norm $\|C - X'\Omega^{\dagger}\|_F$. As in DMD, SVD is used on the augmented input data matrix Ω, that is, $\Omega = U\Sigma V^* \approx \tilde{U}\tilde{\Sigma}\tilde{V}^*$, with the truncation value r. Then, the approximated linear operators A and B can be found as [8]:

$$\begin{aligned} C &\approx X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^* \\ [A\ \ B] &\approx [X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}_1^*\ \ X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}_2^*] \end{aligned} \tag{13}$$

where $C \in \mathbb{R}^{n\times(n+l)}, \tilde{U}_1 \in \mathbb{R}^{n\times r}, \tilde{U}_2 \in \mathbb{R}^{l\times r}, \tilde{U} = [\tilde{U}_1\ \tilde{U}_2]$. In order to find a reduced order representation of dynamic systems, a second SVD on the output matrix X' is used

[8], where $X' \approx \hat{U}\hat{\Sigma}\hat{V}$ with the truncation value p, here $\hat{U} \in \mathbb{R}^{n\times p}, \hat{\Sigma} \in \mathbb{R}^{p\times p}, \hat{V}^* \in \mathbb{R}^{p\times m-1}$. Then the reduced order system matrices for A and B can be realized with the linear transformation $x = \hat{U}\tilde{x}$:

$$\begin{aligned} \tilde{A} &= \hat{U}^* A\hat{U} = \hat{U}^* X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}_1^*\hat{U} \\ \tilde{B} &= \hat{U}^* A\hat{U} = \hat{U}^* X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}_2^* \end{aligned} \tag{14}$$

The reduced order dynamic system with the given control inputs can be constructed as:

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k \tag{15}$$

After presenting the background for standard DMD and DMDc, next, clustering is introduced.

B. k-means clustering

In this work, we consider clustering in conjunction with the implementation of DMDc. Specifically, we implement the k-means clustering method which groups the snapshots together by calculating the minimal distance between each point and the kth cluster centroid according to the Euclidean distance $d(x_i, x_j) = \sqrt{(x_i - x_j)^T(x_i - x_j)}$, where d is the Euclidean distance between two distinct snapshots x_i and x_j, [21]. Let c_i be the argument of the minimum distance between x_i and x_{c_j}, i.e., $c_i = \arg\min_{j=1,\ldots,K} d(x_i, x_{c_j})$. The new centroids are $x_{c_j} = \frac{\sum_{i=1}^{N} 1_{c_i=j} x_i}{\sum_{i=1}^{N} 1_{c_i=j}}$, where $j = 1, \ldots, K$, K is the number of clusters, and $1_{c_j=j} = 1$ if $c_i = j$ and $1_{c_j=j} = 0$ otherwise. If $\tilde{S}$ is the set of all the snapshots, and if $\tilde{S}_i$ represents the i^{th} cluster with center x_{c_i}, then $\tilde{S} = \cup_{i=1}^{k}\tilde{S}_i$, where the union is a union of disjoint sets. In the next section, the clusters $\tilde{S}_i$ are used to derive cluster DMDc.

C. Cluster DMDc

Each cluster of snapshots $\tilde{S}_i$ is used to form the matrix of snapshots X_i at time k and X_i' at time $k+1$. In this case, since for any $m \times n$ matrix $M = (m_{ij}), \|M\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}|m_{ij}|^2$, i.e., the Frobenius norm of a matrix is equal to the sum of the Frobenius norms of any partition. This observation results in the following Lemma.

Lemma 2.1: For any matrix $X', X \in R^{n\times(m-1)}$, and any K partitions of compatible dimensions X_i' and X_i, of X' and X, respectively, the following holds:

$$\min_A \|X' - AX\|_F^2 \geq \sum_{i=1}^{K} \min_{A_i} \|X_i' - A_i X_i\|_F^2 \tag{16}$$

Proof:

$$\begin{aligned} \min_A \|X' - AX\|_F^2 &= \min_A \sum_{i=1}^{K} \|X_i' - AX_i\|_F^2 \\ &\geq \sum_{i=1}^{K} \min_A \|X_i' - AX_i\|_F^2 \\ &= \sum_{i=1}^{K} \|X_i' - A_i X_i\|_F^2 \end{aligned} \tag{17}$$

since the minimum with respect to the matrix A of the sum of the norms is always greater or equal to the sum of the minimum of each norm with respect to A. And the minimum is achieved for each cluster separately for some matrix $A_i, i = 1, \ldots, K$, where $A_i = X_i' X_i^{\dagger}$. This clearly shows that cluster DMDc will provide a better fit to the data than conventional DMDc in general.

As seen in algorithm 1, after obtaining the snapshots, and the number of clusters is chosen, the k-means algorithm groups the snapshot data into several sub-regions. Following section II, a reduced order model is constructed for each cluster as follows:

$$\begin{aligned} \tilde{A}_i &= \hat{U}_i{}^* X_i'\tilde{V}_i\tilde{\Sigma}_i{}^{-1}\tilde{U}_{i_1}^*\hat{U}_i \\ \tilde{B}_i &= \hat{U}_i{}^* X_i'\tilde{V}_i\tilde{\Sigma}_i{}^{-1}\tilde{U}_{i_2} \end{aligned} \tag{18}$$

When the control inputs are zeros, the DMDc is identical to the DMD.

Algorithm 1: Cluster-DMD reduced order model

Input: Snapshots dataset X, X' and Γ

1 Feed X to the k-means algorithm for k clusters with the corresponding X_i, Γ_i, **1-time shift** X_i';

2 **for** $i = 1 \quad to \quad k$ **do**

3 Construct the input matrix $\Omega_i = [X_i\ \Gamma_i]^T$ and find SVD of $\Omega_i, \Omega_i \approx [\tilde{U}_{i_1}\ \tilde{U}_{i_2}]^T\tilde{\Sigma}_i\tilde{V}_i$;

4 Find the SVD of 1-time shift matrix, $X_i' \approx \hat{U}_i\hat{\Sigma}_i\hat{V}_i$;

5 Compute the reduced order model $\tilde{A}_i = \hat{U}_i{}^* X_i'\tilde{V}_i\tilde{\Sigma}_i{}^{-1}\tilde{U}_{i_1}^*\hat{U}_i, \tilde{B}_i = \hat{U}_i{}^* X_i'\tilde{V}_i\tilde{\Sigma}_i{}^{-1}\tilde{U}_{i_2}$

Output: Reduced order model $\tilde{A}_i, \tilde{B}_i$ for each cluster

D. Cluster DMDc with adaptation

DMDc is a powerful technique for system identification. Inequality (16) shows cluster-DMDc has generally better performance in representing data, i.e. system identification. Furthermore, DMDc is also a potential predictive tool, to improve its prediction accuracy a new method is proposed. DMDc is an offline data-driven method, in [7] an online scheme for DMD is proposed for online system identification for large-scale time-varying systems. The scheme updates the system's dynamics as new data becomes available transforming the off-line technique to an online technique. Borrowing the idea as in [7], to obtain more accurate predictions, we propose an online cluster DMDc method which results in an adaptive cluster DMDc reduced model as new prediction data are gathered. The key idea here is that with the prediction process, we are adding the predicted solution to the nearest cluster and updating the DMDc reduced order models for the corresponding cluster.

Given the initial state x_0, with the boundary control sequence $u_k, k = 1\ldots, n$, the prediction process is as follows: $x_{k+1} = \tilde{A}_i x_k + \tilde{B}_i u_k, i = \arg\min_{j=1,\ldots,K} |x_{k+1} - x_{c_j}|_2^2$, where i is the index of the i^{th} cluster, add x_{k+1} to the i^{th} cluster obtaining a new cluster X_i, then SVD is applied to obtain

the updated $\tilde{A}_i, \tilde{B}_i$ as seen in Algorithm 2.
In the next section, this method is applied to the Burgers' equation, in particular, the method is shown to be more effective at predicting the dynamics than the offline DMDc and cluster-DMDc methods.

III. Application to the Burgers' Equation

A. 1-D Burgers' equation

DMD is often used in fluid dynamics applications [23] [1] [24]. The Burgers' equation is a partial differential equation (PDE) that is a simplified version of the Navier-Stokes equations when neglecting the pressure term and external force. It takes the following form [22]:

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \frac{1}{\mathrm{Re}}\frac{\partial^2 w}{\partial x^2}, \quad x \in \Omega \tag{19}$$

Where $\Omega = [l, r]$, $t \in [0, T]$. The initial and boundary conditions are specified as follows:

$$\begin{aligned} w(x, t_0) &= w_0(x) \\ w(\partial\Omega, t) &= \begin{bmatrix} w(l,t) \\ w(r,t) \end{bmatrix} = \begin{bmatrix} u_l(t) \\ u_r(t) \end{bmatrix} = u(t) \end{aligned} \tag{20}$$

The term $\frac{1}{\mathrm{Re}}$ is kinematic viscosity [22]. Re is the counterpart to the Reynolds number from the Navier-Stokes equation, and x and t are the space and time variables, respectively. $\partial\Omega$ is the boundary of Ω. u_l, u_r are left and right boundary inputs. Finite difference approximation is applied:

$$\frac{dw_i}{dt} + w_i\frac{w_i - w_{i-1}}{\Delta x} = \frac{1}{\mathrm{Re}}\frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta x^2} \tag{21}$$

Where $i = 0, 1, \ldots, N$. Equation (21) can be written in the matrix form as:

$$\dot{w} = Aw + Bu + F(w) \tag{22}$$

In (22), Aw is the linear term $\frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta x^2}$ and $F(w)$ is the nonlinear term $w_i\frac{w_i - w_{i-1}}{\Delta x}$ in (21).

B. Numerical Snapshots

We utilize a finite difference approximation in Matlab to numerically solve the initial boundary value problem with and without (open loop) control to obtain the snapshots. In the simulation, we choose $Re = 520$, $N = 800$, $\Omega = [l, r] = [0, 1]$, $t \in [0, T], T = 10s$ with time step size $\Delta t = 0.00125s$ and space step size 0.004. The full-order model consists of 251 states. Next, the DMDc, cluster DMDc, and adaptive cluster DMDc results will be presented.

C. Identification

1) Cluster DMDc with boundary inputs: In this case, boundary inputs are given as:

$$\begin{aligned} u_l &= -0.2 \times \cos(3\pi t/T) \\ u_r &= \quad 0.8 \times \cos(\pi t/T) \end{aligned} \tag{23}$$

Where u_l and u_r are the left and right boundary inputs, respectively.

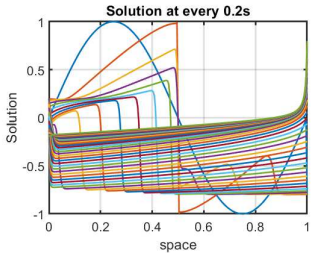

Fig. 1. Snapshots are generated by the Burgers' equation with non-zero boundary input as in (23).

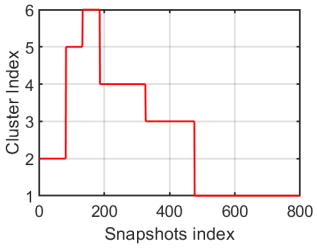

Fig. 2. Snapshots are generated by the Burgers' equation with non-zero boundary input. The figure shows the distribution of clusters along the snapshots index.

With the snapshots at hand, the k-mean algorithm groups the snapshots into 6 sub-regions. The 2 truncation numbers for the constructed reduced order model are chosen to be $r = 8$, $p = 5$. In Fig. 2 the distribution of cluster index along time steps is shown. Then the reduced order models $\tilde{A}_i, \tilde{B}_i, i = 1, \ldots, 6$ of order 5 for each cluster are constructed. As seen in (1), DMDc reduced order model $\tilde{A}$ and cluster DMDc $\tilde{A}_i, \tilde{B}_i$ are used to reconstruct the dynamical system. In Fig. 3, three snapshots are shown. One can see that given the same initial state and actuation data, the DMDc and cluster DMDc reduced order model can represent the system accurately. In the right panel of Fig. 3 and Fig. 6, the proposed cluster DMDc method outperforms the standard DMDc [8]. Next, we will use the resulting models for prediction.

D. Prediction

Cluster DMDc is implemented as an offline data-driven method. Once a representative model is obtained, the underlying governing equation is no longer needed. However, off-

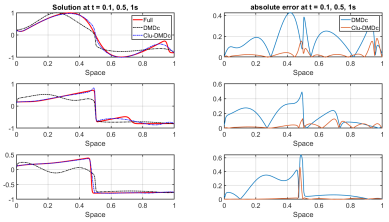

Fig. 3. Snapshots are generated by the Burgers' equation with non-zeros boundary inputs, DMDc and cluster DMDc reduced order models are used to reconstruct the snapshot data.

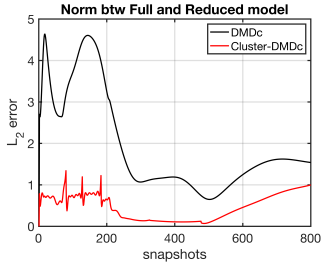

Fig. 4. At each time step, the norm of the errors of DMDc and cluster DMDc reconstructed solutions and snapshots, respectively, are shown. That is, $\|\boldsymbol{x_i^{DMDc}} - \boldsymbol{x_i}\|$ v.s. $\|\boldsymbol{x_i^{Clu-DMDc}} - \boldsymbol{x_i}\|$ are plotted.

line prediction models need to be trained with large datasets, the new proposed adaptive Cluster DMDc avoids using pre-processed data. First, the following boundary inputs are given below to construct the DMDc reduced order models for each cluster as seen in Algorithm (1):

$$u_l = 0.2\cos(3\pi t/T) \quad u_r \quad = -0.8\cos(\pi t/T) \tag{24}$$

1) Algorithm for Cluster DMDc: Next, $K = 6$ is chosen to generate clusters, then we feed the training data into algorithm 1 to obtain a cluster DMDc reduced model for each cluster. For prediction, given the initial condition and actuation data for the system, then at the next time step, the state solution belongs to a specific cluster, that is, in the next step for prediction, the reduced order model should switch to the corresponding cluster of the current state solution, meanwhile, the incoming state data is fed into the specific cluster to update the DMDc reduced order model to accommodate general boundary control inputs. Below is the algorithm for the prediction process.

Algorithm 2: adaptive Cluster DMD with control

Input: Initial state x_1, Snapshots X

Apply Algorithm 1 to get $\tilde{A}_i, \tilde{B}_i$ based on X

for $i = 1 : \quad n(\textit{time steps}) : \quad \textit{end}$ **do**

  compute the distance between the x_{i+1} and all k clusters centroids $c_i, \ldots, c_k$;

  find $j = \arg\min_j \|x_i - c_j\|$;

  add new data x_i into j^{th} cluster and update $\tilde{A}_j$, $\tilde{B}_j$ using algorithm (1);

  compute the next states solution $\tilde{x}_{k+1}$ based on (15) ;

Output: Prediction of $x_2, \ldots, x_m$

2) Prediction: The adaptive cluster DMDc model is used to make predictions for the dynamic system. Here, 4 separate trials are considered with outputs generated with the following boundary inputs. For the comparison, some boundary inputs are set at different magnitudes and frequencies as follows: $u_{l_1} = 0.2\cos(3\pi t/T), u_{r_1} = -0.8\cos(\pi t/T), u_{l_2} = 0.18\cos(3\pi t/T), u_{r_2} = -0.8\cos(\pi t/T), u_{l_3} = 0.19\cos(3\pi t/T), u_{r_3} = -0.81\cos(\pi t/T), u_{l_4} = 0.19\cos(3\pi t/T), u_{r_4} = -0.81\cos(1.5\pi t/T)$.

The first boundary inputs considered, u_{l_1}, u_{r_1} are used to construct the basic cluster-DMDc reduced order model, then we predict the system governed by the Burgers' equation with different boundary inputs $u_{l_2,r_2}, u_{l_3,r_3}, u_{l_4,r_4}$. As seen in the 2^{nd} trial, boundary inputs u_{l_2,r_2}, u_{l_2} are different from u_{l_1} in magnitude, in the 3^{rd} trial, the left and right boundary inputs are different from u_{l_1}, u_{r_1} in magnitude. For the 4^{th} trial, the magnitude of boundary inputs is the same as in the 3^{rd} trial, but the frequency of the right boundary input is different. That is, the generality of the cluster-DMDc is tested and an adaptive Cluster-DMDc reduced order model is proposed. In Fig. 5, at each prediction iteration, the L_2 errors are calculated and shown. The blue, red, and black curves are prediction performance using DMDc, cluster DMDc and adaptive cluster DMDc, respectively. In the table below Fig. 5, the overall measured square error between the real system and the predicted system is calculated. Combining the information in the figure and table, the proposed adaptive cluster DMDc has the best prediction performance.

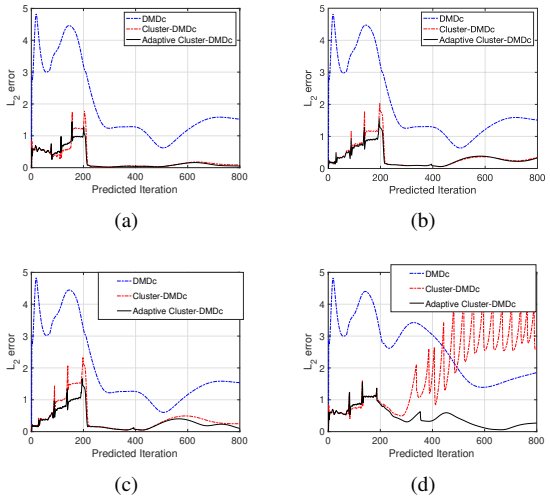

Fig. 5. a) $\boldsymbol{u_l = 0.2 \times \cos(3\pi t/T), u_r = -0.8 \times \cos(\pi t/T)}$, b) $\boldsymbol{u_l = 0.18 \times \cos(3\pi t/T), u_r = -0.8 \times \cos(\pi t/T)}$, c) $\boldsymbol{u_l = 0.19 \times \cos(3\pi t/T), u_r = -0.81 \times \cos(\pi t/T)}$, d) $\boldsymbol{u_l = 0.19 \times \cos(3\pi t/T), u_r = -0.81 \times \cos(1.5\pi t/T)}$

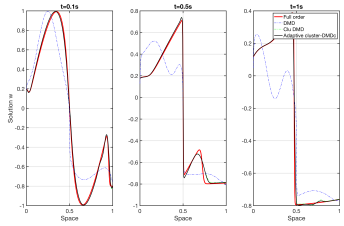

Fig. 6. This figure shows the prediction details in trial 3 at t= 0.1s, t=0.5s, and t =1s, respectively.

Overall MSE	DMDc	Clu-DMDc	Adaptive Clu-DMDc
Trial 1	0.0206	0.00079	0.00064
Trial 2	0.0206	0.0013	0.00083
Trial 3	0.0222	0.0016	0.0012
Trial 4	0.0310	0.0484	0.0015

TABLE I
PREDICTION PERFORMANCE USING DMDC, CLUSTER DMDC, AND ADAPTIVE CLUSTER DMDC

IV. CONCLUSION

In this paper, a cluster DMDc reduced-order modeling with an adaptation strategy is proposed. This technique extends DMDc [8] which extracts a low-order model with actuation. The clustered DMDc presented here is based on the fact that nonlinear system snapshot solutions belong to nonlinear manifolds for which geodesics can be approximated locally, i.e., in clusters, by the Euclidean distance. By grouping the snapshots that have similar features the clusters are obtained. DMDc reduced-order modeling is then applied to each cluster to obtain the cluster DMDc. It is shown that cluster DMDc provides better fits to model data than standard DMDc in general. By updating the cluster DMDc reduced-order model sequentially, an online data-driven technique for predictive purposes is obtained. This is illustrated in simulation results applied to the Burgers' equation. A number of open questions and possible extensions remain. For each cluster, the number of dynamic modes is fixed, while for some clusters there would possibly exist unstable modes so that the prediction error would increase when the dynamic system moves into these clusters. In [25], a user-defined low-order rank DMD model is introduced which consistently provides a more accurate approximation of the true system eigenvalues than DMD. In addition, how to choose the optimal number of clusters is not considered in this work. The latter may further improve the accuracy of the proposed adaptive Cluster-DMDc. In [26], a quantization error model was proposed to generate optimal clusters for a given dataset. Future work will focus on finding the optimal cluster number and selecting the best modes to improve performance. Further comparison, in particular, with the Extended DMD (EDMD), a data-driven method to approximate the Koopman operator [27], will be carried out.

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